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Pairs of foliations on timelike surfaces in the de Sitter space S^3_1

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Abstract

We define in this paper the asymptotic, characteristic and principal directions associated to the de Sitter Gauss map on a smooth timelike surface M in the de Sitter space S^3_1. We study their properties and determine the local topological configurations of their integral curves. These curves form pairs of foliations on some regions of M and are defined in an analogous way to their classical contrepart on surfaces in the Euclidean 3-space. However, we show that their behaviour is distinct from that of their analogue on surface in the Euclidean 3-space.

1 Introduction

Let M be a smooth and oriented surface in the Euclidean space $\mathbb{R}^3$. The shape operator S (or the Weingarten map) is a self-adjoint operator defined on each tangent plane of M and describes the shape of M in $\mathbb{R}^3$. It also determines on M three pairs of foliations as follows. At each point $p \in M$, the shape operator S_p is a symmetric operator on T_pM. As this vector space inherits the Euclidean scalar product ".", it has a basis of orthonormal vectors given by the eigenvectors of S_p. The directions of these vectors are called principal directions and their integral curves the lines of principal curvature. These form a pair of foliation on M away from umbilic points. A direction $u \in T_pM$ is asymptotic if $S_p(u).u = 0$. There are two asymptotic directions at each hyperbolic point and their integral curves are called the asymptotic curves. At an elliptic point there is a unique pair of conjugate directions for which the included

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angle is extremal ([10]). These directions are called the characteristic directions and their integral curves the characteristic curves.

The above three pairs of foliations are given, in a local chart, by binary differential equations (BDEs), also know as quadratic differential equations. These are equations in the form

$$a(u,v)dv^2 + 2b(u,v)dvdu + c(u,v)du^2 = 0,$$

where the coefficients a, b, c are smooth functions on some open set $U \subset \mathbb{R}^2$ (here smooth means C^∞). A BDE defines no directions at points where $\delta(u,v) = (b^2 - ac)(u,v) < 0$, two directions at points in the region where $\delta(u,v) > 0$, and a double direction at points on the set $\Delta = \{(u,v) \in U : \delta(u,v) = 0\}$ when the coefficients of the equation do not all vanish at a given point. At such points, every direction is a solution of the BDE. The set Δ is called the *discriminant* of the equation and the function δ the *discriminant function*. BDEs are well studied; see for example [9, 21] for references.

It is shown in [3] that the equations of the asymptotic, characteristic and principal curves are related. A BDE can be viewed as a quadratic form and represented at each point in the plane by a point in the projective plane. If Γ denotes the set of degenerate quadratic forms, then the asymptotic, characteristic and principal BDEs represent a self-polar triangle with respect to Γ ([3]). In particular, any two of them determine the third one. (See [20] for a generalisation of these results.)

We consider in this paper timelike surfaces M in the de Sitter space S^3_1. Given $p \in M$, there is a well defined unit normal vector $\boldsymbol{e}(p)$ to M at p ([12]), see §2. (If M is orientable, then $\boldsymbol{e}(p)$ is globally defined. However, it is always locally defined and our investigation here is local in nature.) The vector $\boldsymbol{e}(p)$ is in the de Sitter space S^3_1 and we have the de Sitter Gauss map

$$\begin{array}{rccc} \mathbb{E}: & M & \to & S^3_1 \\ & p & \to & \boldsymbol{e}(p) \end{array}$$

The Weingarten map $-d\mathbb{E}_p$ is a self-adjoint operator on T_pM ([12]). However, as M is timelike, the restriction of the pseudo-scalar product in the Minkowski space to T_pM is also a pseudo scalar product. Therefore $-d\mathbb{E}_p$ does not always have real eigenvalues. We can still define the concepts of principal/asymptotic/characteristic directions at p using $-d\mathbb{E}_p$ and the pseudo-scalar product on T_pM. The difference here to the Euclidean case is that we may not have principal directions (which are along the eigenvectors of $-d\mathbb{E}_p$) at every point on M.

The principal/asymptotic/characteristic directions determine pairs of foliations on some regions of the surface. We study in this paper the local behaviour of these foliations. We deal with the lines of principal curvature in §3, the asymptotic curves in §4 and the characteristic curves in §5. We carry out in the Appendix (§6) a topological classification of BDEs with 1-jet equivalent to $(\alpha x + y)dy^2 \pm 2xdxdy$ and with discriminant having an A_3-singularity. The models in §6 are used to determine the

configuration of the characteristic curves at timelike umbilic points. For the figures in this paper, we draw one foliation determined by a given BDE in blue and the other in red. The discriminant is drawn in thick black.

The results in this paper are true for self-adjoint operators defined on a two dimensional Laurentzian manifold (compare with [20]). This is the case, for example, for 2-dimensional timelike surfaces in the Minkowski 3-space ([16]) and in the Anti de Sitter 4-space ([8]). In each of these cases a self-adjoint operator is derived from the differential of a well defined Gauss map.

2 Preliminaries

The *Minkowski space* $(\mathbb{R}^4_1, \langle , \rangle)$ is the 4-dimensional vector space $\mathbb{R}^4$ endowed with the pseudo scalar product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + \sum_{i=1}^3 x_iy_i$, where $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_0, y_1, y_2, y_3)$ in $\mathbb{R}^4_1$. We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^4_1 \setminus \{0\}$ is *spacelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, *lightlike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$, *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$.

The norm of a vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Given a vector $\boldsymbol{v} \in \mathbb{R}^4_1$ and a real number c, the hyperplane with pseudo normal $\boldsymbol{v}$ is defined by

$$HP(v, c) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}.$$

We say that $HP(\boldsymbol{v}, c)$ is a *spacelike*, *timelike* or *lightlike hyperplane* if v is timelike, spacelike or lightlike respectively. We also say that a two dimensional vector space is *spacelike* if all its vectors are spacelike, *timelike* if it has a spacelike and a timelike vector and *lightlike* otherwise.

The de Sitter space is the pseudo-sphere $S^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$.

The wedge product of three vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \in \mathbb{R}^4_1$ is given by

$$\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a^1_0 & a^1_1 & a^1_2 & a^1_3 \\ a^2_0 & a^2_1 & a^2_2 & a^2_3 \\ a^3_0 & a^3_1 & a^3_2 & a^3_3 \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^4_1$ and $\boldsymbol{a}_i = (a^i_0, a^i_1, a^i_2, a^i_3)$, $i = 1, 2, 3$. We have $\langle \boldsymbol{a}, \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3 \rangle = \det(\boldsymbol{a}, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$, so the vector $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3$ is pseudo orthogonal to all the vectors $\boldsymbol{a}_i$, $i = 1, 2, 3$, that is, $\langle \boldsymbol{a}_i, \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3 \rangle = 0$ for $i = 1, 2, 3$.

Definition 2.1 *A smooth surface M embedded in S^3_1 is timelike if its tangent space T_pM at any point $p \in M$ is a timelike vector space.*

Given a local chart $\boldsymbol{i} : U \to M$, where U is an open subset of $\mathbb{R}^2$, we denote by $\boldsymbol{x} : U \to S^3_1$ such embedding, identify $\boldsymbol{x}(U)$ with U through the embedding $\boldsymbol{x}$ and write $M = \boldsymbol{x}(U)$. The first fundamental form of M at a point p is the quadratic

form $\mathrm{I}_p : T_pM \to \mathbb{R}$ given by $\mathrm{I}_p(\boldsymbol{v}) = \langle \boldsymbol{v}, \boldsymbol{v} \rangle$. If $\boldsymbol{v} = a\boldsymbol{x}_u + b\boldsymbol{x}_v \in T_pM$, then $\mathrm{I}_p(\boldsymbol{v}) = Ea^2 + 2Fab + Gb^2$, where

$$E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle, \quad F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle, \quad G = \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle$$

are the coefficients of the first fundamental form at p. Because M is timelike, we have $EG - F^2 < 0$, so at any point $p \in M$ there are two lightlike directions in T_pM. These are the solutions of $I_p(\boldsymbol{v}) = 0$.

The lightlike directions can be given locally on $\boldsymbol{x}(U)$ by two smooth direction fields. One can show the following (the proof is identical to the Euclidean case; see for example [7] page 216).

Theorem 2.2 *At any point $p \in M$, there is a local parametrisation of a neighbourhood V of p, such that for any $p' \in V$, the coordinate curves through p' are tangent to the lightlike directions. Equivalently, there exist a local parametrisation $\boldsymbol{x} : U \to V \subset M$ with $E = G = 0$ in U.*

Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \equiv 1$, we have $\langle \boldsymbol{x}_u, \boldsymbol{x} \rangle \equiv 0$ and $\langle \boldsymbol{x}_v, \boldsymbol{x} \rangle \equiv 0$. Let $q = (u, v) \in U$. We define the spacelike unit normal vector $\boldsymbol{e}(q)$ to M at $p = \boldsymbol{x}(q)$ by

$$\boldsymbol{e}(q) = \frac{\boldsymbol{x}(q) \wedge \boldsymbol{x}_u(q) \wedge \boldsymbol{x}_v(q)}{\|\boldsymbol{x}(q) \wedge \boldsymbol{x}_u(q) \wedge \boldsymbol{x}_v(q)\|}.$$

This *de Sitter Gauss map* of M is the map

$$\begin{array}{rccc} \mathbb{E}: & U & \to & S_1^3 \\ & q & \mapsto & \boldsymbol{e}(q) \end{array}$$

(see [12]). For any $p = \boldsymbol{x}(q) \in M$ and $\boldsymbol{v} \in T_pM$, one can show that $D_{\boldsymbol{v}}\mathbb{E} \in T_pM$, where $D_{\boldsymbol{v}}$ denotes the covariant derivative with respect to the tangent vector $\boldsymbol{v}$.

The linear transformation $A_p = -d\mathbb{E}(q)$ is called the *de Sitter shape operator*. One can show that this is a self-adjoint operator, that is, a linear and symmetric operator on T_pM.

Remark 2.3 There is a crucial fact here. The surface M is timelike, so the restriction of the pseudo scalar product in $\mathbb{R}^4_1$ to M is a pseudo scalar product. Therefore A_p does not always have real eigenvalues.

The second fundamental form of M at a point p is the quadratic form $\mathrm{II}_p : T_pM \to \mathbb{R}$ given by $\mathrm{II}_p(\boldsymbol{v}) = \langle A_p(\boldsymbol{v}), \boldsymbol{v} \rangle$. If $\boldsymbol{v} = a\boldsymbol{x}_u + b\boldsymbol{x}_v \in T_pM$, then $\mathrm{II}_p(\boldsymbol{v}) = la^2 + 2mab + nb^2$, where

$$\begin{array}{rcl} l & = & -\langle \boldsymbol{e}_u, \boldsymbol{x}_u \rangle = \langle \boldsymbol{e}, \boldsymbol{x}_{uu} \rangle \\ m & = & -\langle \boldsymbol{e}_u, \boldsymbol{x}_v \rangle = \langle \boldsymbol{e}, \boldsymbol{x}_{uv} \rangle = \langle \boldsymbol{e}, \boldsymbol{x}_{vu} \rangle = -\langle \boldsymbol{e}_v, \boldsymbol{x}_u \rangle \\ n & = & -\langle \boldsymbol{e}_v, \boldsymbol{x}_v \rangle = \langle \boldsymbol{e}, \boldsymbol{x}_{vv} \rangle \end{array}$$

are the coefficients of the second fundamental form at p. The matrix of the de Sitter shape operator A_p with respect to the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ is given by

$$A_p = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

We call

$$K(p) = \det(A_p) = \frac{ln - m^2}{EG - F^2}$$

the (*de Sitter*) *Gauss-Kronecker curvature* of M at p. The set of points where this curvature vanishes is labeled the (*de Sitter*) *parabolic set* of M. We also call

$$H(p) = \frac{lG - 2mF + nE}{2(EG - F^2)}$$

the (*de Sitter*) *mean curvature* of M at p.

We say that a property is *generic* if it holds for a residual set of embeddings of M in S^3_1.

We consider here topological equivalence between BDEs and say that two BDEs are topologically equivalent if there is a local homeomorphism in the plane taking the integral curves of one to those of the other.

3 Lines of principal curvature

Following Remark 2.3, the de Sitter shape operator A_p does not always have real eigenvalues. When it does, we denote by κ_i, $i = 1, 2$, these eigenvalues and call them the (*de Sitter*) *principal curvature*. Then $K(p) = \kappa_1(p)\kappa_2(p)$ and $H(p) = \frac{1}{2}(\kappa_1(p) + \kappa_2(p))$. The eigenvectors of A_p are called the (*de Sitter*) *principal directions*.

The equation for the lines of principal curvature (which are the integral curves of the line fields determined by the principal directions) is given by the same formula of their contrepart for surfaces in the Euclidean 3-space, namely by

$$\begin{vmatrix} dv^2 & -dvdu & du^2 \\ E & F & G \\ l & m & n \end{vmatrix} = 0,$$

equivalently,

$$(Gm - Fn)dv^2 + (Gl - En)dvdu + (Fl - Em)du^2 = 0. \tag{1}$$

The discriminant function of this equation is

$$\delta(u, v) = \left((Gl - En)^2 - 4(Gm - Fn)(Fl - Em)\right)(u, v).$$

When $\delta(u,v) > 0$, there are two distinct principal directions at $\boldsymbol{x}(u,v)$. These coincide at points where $\delta(u,v) = 0$. There are no principal directions at points where $\boldsymbol{x}(u,v) < 0$. We labeled in [14] the locus of points where $\delta(u,v) = 0$ the *Lightlike Principal Locus* (LPL). We have the following from [14].

Proposition 3.1 (1) *For a generic timelike surface M in the de Sitter space, the LPL is a curve on M. It can be characterised as the set of points on M where the two principal directions coincide and become a lightlike direction.*

(2) *The LPL divides the surface into two regions. In one of them there are no principal directions and in the other there are two distinct principal directions at each point. In the later case, the principal directions are orthogonal and one is spacelike while the other is timelike.*

We have an extra information about the LPL.

Proposition 3.2 *For a generic timelike surface M in the de Sitter space, the LPL curve is a smooth curve with possible Morse singularities of type node. The singular points are the points where the de Sitter shape operator is a multiple of the identity. For this reason, they are labeled "timelike umbilic points".*

Proof. We take a local parametrisation $\boldsymbol{x} : U \to V \subset M$ with $E = G = 0$ in U, that is, the images of $u = constant$ and $v = constant$ have lightlike tangent directions at all points (Theorem 2.2). Then the equation of the lines of principal curvature becomes

$$ndv^2 - ldu^2 = 0.$$

The discriminant function of this equation is $\delta(u,v) = (ln)(u,v)$. The LPL is the zero set of δ. It is clear that this is a curve when M is a generic surface. In general, this curve is smooth unless $l(q) = n(q) = 0$ at some point $q \in U$. These points can occur generically at isolated points on the LPL (we have two independent conditions). At such points the singularity of the LPL is generically a Morse A_1^--singularity, i.e., it is equivalent by smooth changes of coordinates in the source to $\pm(u^2 - v^2)$.

The matrix of A_p, with $p = \boldsymbol{x}(q)$, with respect to the above parametrisation is

$$-\frac{1}{F^2}\begin{pmatrix} 0 & -F \\ -F & 0 \end{pmatrix}\begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

It is a multiple of the identity at q if and only if $l(q) = n(q) = 0$ □

We are interested in the local topological configurations of the lines of principal curvature. Away from the LPL, we either have locally a pair of transverse foliations or no lines of principal curvature. So we analyse the configurations at points on the LPL. The lines of principal curvature are given by a BDE. These equations are well studied (see for example [9, 21] for references). The terminology used in the following result is clarified in its proof and in Figures 1 and 2.

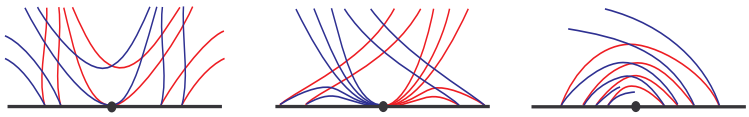

Figure 1: Folded singularities: saddle (left), node (centre), focus (right).

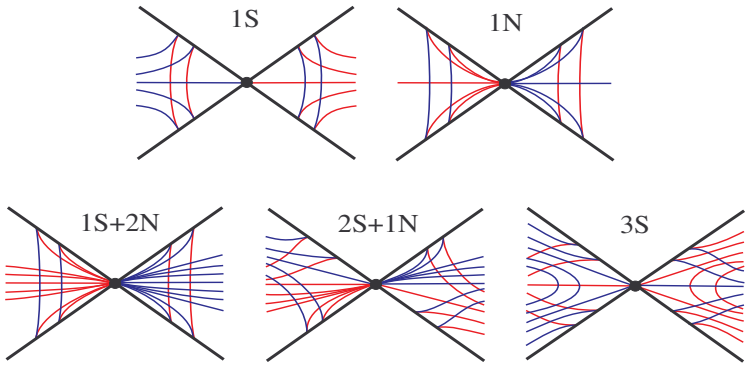

Figure 2: Topological configurations of the solutions of a BDE at a Morse Type 2 singularity when the discriminant has an A_1^--singularity, [6].

Theorem 3.3 (1) *At smooth points on the LPL, the lines of principal curvature form a family of cusps with the cusps tracing the LPL, except maybe at some isolated points on this curve. At such points, the equation of the lines of principal curvature has generically a folded singularity of type saddle, node or focus (Figure* 1*).*

(2) *At a timelike umbilic point, the equation of the lines of principal curvature has generically a Morse Type 2 singularity of type* A_1^-*. All the five generic cases of such singularities can occur (Figure* 2*).*

Proof. We take a local parametrisation $\boldsymbol{x} : U \to V \subset M$ with $E = G = 0$ in U (Theorem 2.2), so that the equation of the lines of principal curvature is given by $ndv^2 - ldu^2 = 0$.

(1) At a smooth point $p_0 = \boldsymbol{x}(q_0)$ on the LPL, we have either $l(q_0)$ or $n(q_0)$ non zero. Assume that $n(q_0) \neq 0$ and write the equation locally at q_0 in the form

$$dv^2 - \frac{l}{n}du^2 = 0.$$

We can now use the criteria for recognition of the singularities of a BDE (see for example [4] and [9] for terminology). The equation is locally smoothly equivalent to $dv^2 - udu^2 = 0$ if and only if $\partial l/\partial u(q_0) \neq 0$. In this case the configuration is a family

of cusps. When $\partial l/\partial u(q_0) = 0$, the BDE has a singularity. Well folded singularities are topologically equivalent to

$$dv^2 + (v - \lambda u^2)du^2 = 0,$$

with λ any fixed value in the open intervals determined by the exceptional values 0 and $1/16$. The singularity is of type folded saddle if $\lambda < 0$, folded node if $0 < \lambda < 1/16$ and folded focus if $1/16 < \lambda$ (Figure 1). For the equation of the lines of principal curvature, a necessary condition to have a folded singularity is $(\partial l/\partial v)(q_0).(\partial^2 l/\partial u^2)(q_0) \neq 0$. Then

$$\lambda = n(q_0)\frac{\partial^2 l}{\partial u^2}(q_0)/\frac{\partial l}{\partial v}(q_0),$$

so all the folded singularities can occur.

(2) If p_0 is a timelike umbilic point, $l(q_0) = n(q_0) = 0$. So the coefficients of the BDE of the lines of principal curvature vanish at q_0. As the LPL, which is the discriminant of the equation of the lines of principal curvature, has a Morse singularity of type A_1^-, the equation has generically a Morse Type 2 singularity of type A_1^- ([6]).

We assume that q_0 is the origin. The bi-valued field determined by the BDE can be lifted to a single direction field ξ on a surface

$$N = \{(u, v, [\alpha : \beta]) \in \mathbb{R}^2, 0 \times \mathbb{R}P^1 : a\beta^2 + 2b\alpha\beta + c\alpha^2 = 0\},$$

where a, b, c are the coefficients of the lines of principal curvature BDE. If we consider the affine chart $p = \beta/\alpha$ and set $F(x, y, p) = a(x, y)p^2 + 2b(x, y)p + c(x, y)$, then the lifted direction field is parallel to the vector field $\xi = F_p\partial/\partial u + pF_p\partial/\partial v - (F_u + pF_v)\partial/\partial p$. The topological type of the BDE is determined by the number and type of the singularities of the vector field ξ on the exceptional fibre $(0, 0) \times \mathbb{R}P^1$ (see [6] for the analytic case and Remark 2 in [21] for the smooth case). If we write $j^1 l = l_1 u + l_2 v$ and $j^1 n = n_1 u + n_2 v$, then the singularities of ξ are at the roots of the cubic $\phi = (F_u + pF_v)(0, 0, p) = n_2 p^3 + n_1 p^2 - l_2 p - l_1$. We need ϕ to have simple roots, that is

$$4n_1^3 l_1 + n_1^2 l_2^2 + 18n_1 n_2 l_1 l_2 - 27n_2^2 l_1^2 + 4n_2 l_2^3 \neq 0.$$

The eigenvalues of the linear part of ξ at the roots of ϕ are $-\phi'(p)$ and $\alpha_1(p)$, where $\alpha_1(p) = 2p(n_2 p + n_1)$ (see [6]). So we also need α_1 not to vanish at a root of ϕ, that is,

$$l_1(n_2 l_1 - n_1 l_2) \neq 0.$$

When the above conditions are satisfied, which is the case at generic timelike umbilic points, ξ has either saddle or node singularities at the roots of ϕ. The number and type of these singularities, which is determined by the sign of $-\phi'(p)\alpha_1(p)$, determines the topological configuration of the BDE. It is clear that we can have all the five possible generic cases in [6], see Figure 2 where the labeling refers to the number and type of the zeros of ξ on the exceptional fibre. □

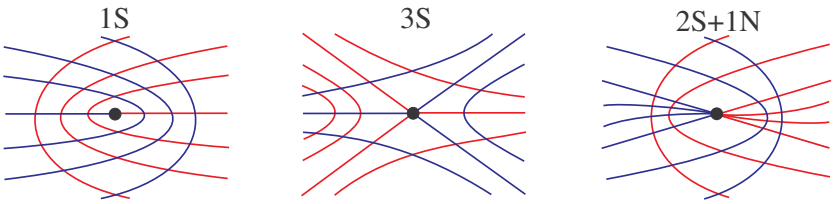

Figure 3: Topological configurations of the solutions of a BDE at a Morse Type 2 singularity when the discriminant has an A_1^+-singularity.

Remark 3.4 (1) The principal directions can be detected via singularity theory. It is shown in [14] that, away from the LPL, the folding map of the surface with respect to a hyperplane has a degenerate singularity if and only if the normal to the hyperplane is parallel to a principal direction. The LPL and the folded singularities of the lines of principal curvature can also be detected via the singularities of projections along lightlike geodesics [15].

(2) For surfaces in the Euclidean 3-space (or two dimensional Riemannian surfaces with a self-adjoint operator [20]), the lines of principal curvature have Morse type 2 singularities of type A_1^+ at umbilic points ([17, 2], see Figure 3). Their configurations are distinct from those of timelike surfaces at timelike umbilic points.

4 Asymptotic curves

We define an asymptotic direction via the de Sitter shape operator.

Definition 4.1 *A direction $\boldsymbol{v} \in T_pM$ is called asymptotic if* $\mathrm{II}_p(\boldsymbol{v}) = \langle A_p(\boldsymbol{v}), \boldsymbol{v}\rangle = 0$.

It follows from the above definition that the asymptotic directions are given by

$$ndv^2 + 2mdudv + ldu^2 = 0. \tag{2}$$

The discriminant of equation (2) is the set of points where $m^2 - nl$ vanishes. This is exactly the de Sitter parabolic set given by $K = (nl - m^2)/(EG - F^2) = 0$. As $EG - F^2 < 0$, there are two distinct asymptotic directions in the region $K > 0$ and no asymptotic directions in the region $K < 0$ (the opposite happens in the Euclidean case). On the de Sitter parabolic set there is a unique double asymptotic direction. The de Sitter parabolic set of a generic surface, when not empty, is a smooth curve.

Proposition 4.2 *The de Sitter parabolic set and the LPL are tangential at their points of intersection. At such points the unique asymptotic direction on the de Sitter parabolic set is lightlike; on one side of such points it is spacelike and on the other it is timelike.*

Proof. We take a local parametrisation $\boldsymbol{x} : U \to V \subset M$ with $E = G = 0$ in U (Theorem 2.2). Then the LPL is given by $nl = 0$ and the de Sitter parabolic set by $m^2 - nl = 0$. We observe first that the singular points of the LPL are not generically de Sitter parabolic points (otherwise $n = l = m = 0$). Suppose, without loss of generality, that $n(q) = 0$ and $l(q) \neq 0$ at the point in consideration. So the LPL is given locally by $n = 0$. Then the point $p = \boldsymbol{x}(q)$ is also a parabolic point if and only if $m(q) = 0$. Now the gradient of $m^2 - nl$ at q is $(-n_u l, -n_v l)(q)$ and is parallel to the gradient of n at q. So the two curves are tangential at their meeting point.

At q, equation (2) becomes $l(q)du^2 = 0$, so the unique asymptotic direction on the surface is along $\boldsymbol{x}_v(q)$ which is lightlike. Near the point q on the de Sitter parabolic set, the double asymptotic direction is along $(du, dv) = (m, -n)$ in the parameter space, so is along $m\boldsymbol{x}_u - n\boldsymbol{x}_v$ on the surface. We have $\langle m\boldsymbol{x}_u - n\boldsymbol{x}_v, m\boldsymbol{x}_u - n\boldsymbol{x}_v\rangle = -2nmF$, and this changes sign generically at q. □

Proposition 4.3 *An asymptotic direction at a point $p \in M$ is also a principal direction at p if and only if p is a de Sitter parabolic point or a point on the LPL. On the LPL, the asymptotic direction is lightlike.*

Proof. We take a local parametrisation as in the proof of Proposition 4.2. Then the equation of the lines of principal curvature is given by $ndv^2 - ldu^2 = 0$. Subtracting this from equation (2) yields $du(ldu + mdv) = 0$.

The lightlike direction $du = 0$ is both asymptotic and principal if and only if $n(q) = 0$, if and only if $p = \boldsymbol{x}(q)$ is on the LPL. The direction $ldu + mdv = 0$ is both asymptotic and principal if and only if $(m^2 - nl)(q) = 0$, if and only if p is a de Sitter parabolic point. □

Theorem 4.4 *On the de Sitter parabolic curve, the asymptotic curves form a family of cusps with the cusps tracing the de Sitter parabolic curve, except may be at some isolated points on this curve. At such points, the equation of the asymptotic curves has generically a folded singularity of type saddle, node or focus (Figure 1).*

Proof. We can suppose, without loss of generality, that the point in consideration is $p_0 = (0, 1, 0, 0)$ and take the surface in de Sitter Monge form

$$\phi(x, y) = (f(x, y), \sqrt{1 + f^2(x, y) - x^2 - y^2}, x, y),$$

with

$$f(x, y) = a_{10}x + a_{11}y + a_{20}x^2 + a_{21}xy + a_{22}y^2 + a_{30}x^3 + a_{31}x^2y + a_{32}xy^2 + a_{33}y^3 + h.o.t.$$

and (x, y) in some neighbourhood of the origin. We require $a_{10}^2 + a_{11}^2 > 1$ for the surface to be timelike near the p_0. We calculate the 2-jet of the coefficients of the asymptotic BDE and find

$$
\begin{aligned}
j^2 l &= 2a_{20} + 6a_{30}x + 2a_{31}y + 12a_{40}x^2 + (-2a_{20}a_{10}a_{11} + 6a_{41} + a_{21}(a_{10}^2 - 1))xy + \\
&\quad (2a_{42} + a_{20}(1 - a_{11}^2) + a_{22}(a_{10}^2 - 1))y^2, \\
j^2 m &= -a_{21} - 2a_{31}x - 2a_{32}y + (-3a_{41} + \tfrac{1}{2}a_{21}(a_{10}^2 - 1) + a_{20}a_{10}a_{11})x^2 + \\
&\quad 2(a_{21}a_{10}a_{11} - 2a_{42})xy + (-3a_{43} + \tfrac{1}{2}a_{21}(-1 + a_{11}^2) + a_{11}a_{10}a_{22})y^2, \\
j^2 n &= 2a_{22} + 2a_{32}x + 6a_{33}y + (2a_{42} - a_{20} + a_{22}(1 + a_{11}^2 - a_{10}^2))x^2 + \\
&\quad (-2a_{11}a_{10}a_{22} + 6a_{43} + a_{21}(a_{11}^2 - 1))xy + 12a_{44}y^2.
\end{aligned}
$$

The origin is a de Sitter parabolic point (i.e., a point on the discriminant of the asymptotic curves BDE) if and only if $a_{21}^2 - 4a_{20}a_{22} = 0$. We can suppose, without loss of generality, that $\phi_x(0,0)$ is an asymptotic direction so that $a_{20} = 0$. Then the origin is a parabolic point if $a_{21} = 0$. We have generically $a_{22} \neq 0$, otherwise the parabolic set is singular. The origin is a singularity of the asymptotic BDE if $a_{30} = 0$. The singularity is in general a folded one and its type is determined by

$$\lambda_{as} = \frac{1}{2}\frac{(12a_{22}a_{40} - 6a_{30}a_{32} - a_{31}^2)a_{22}^2}{(3a_{22}a_{41} - a_{31}a_{32} - 9a_{30}a_{33})^2}$$

(see the criteria in [4]). We have a folded saddle if $\lambda_{as} < 0$, a folded node if $0 < \lambda_{as} < \frac{1}{16}$ and a folded focus $\lambda_{as} > \frac{1}{16}$. It is clear that the three cases can occur for generic surfaces. □

Remark 4.5 (1) At the point of tangency of the de Sitter parabolic curve and the LPL the equation of the asymptotic curves is in general not singular. So in a neighbourhood of such points the asymptotic curves form a family of cusps.

(2) Asymptotic directions can be detected via singularity theory. It is shown in [15] that a projection along parallel geodesics to an orthogonal quadric has a singularity of type cusp or worse at p if and only if the tangent direction to the geodesic at p is along an asymptotic direction.

5 Characteristic curves

We use the results in [3] to define the characteristic directions. A BDE can be viewed as a quadratic form and represented at each point in the plane by a point in the projective plane. If Γ denotes the set of degenerate quadratic forms, then the asymptotic and principal BDEs of a timelike surface determine a unique BDE, which we call *the characteristic curves BDE*, in such a way that the three equations represent at each point on M a self-polar triangle in the projective plane with respect to Γ ([3]). The characteristic curves BDE is given as the jacobian of the asymptotic and principal curves BDE and has the following expression

$$
\begin{aligned}
(2m(mG - nF) - n(lG - nE))dv^2 + 2(m(lG + nE) - 2lnF)dvdu + \\
(l(lG - nE) - 2m(lF - mE))du^2 = 0. \qquad (3)
\end{aligned}
$$

If we take a parametrisation of the surface with $E = G = 0$ (Theorem 2.2), then equation (3) becomes

$$mndv^2 + 2lndvdu + mldu^2 = 0.$$

The discriminant of this equation is $ln(ln - m^2) = 0$, which is the union of the de Sitter parabolic set and the LPL.

We say that two tangent directions $\boldsymbol{u}, \boldsymbol{v} \in T_pM$ are *conjugate*, and write $\boldsymbol{v} = \bar{\boldsymbol{u}}$, if $\langle A_p(\boldsymbol{u}), \boldsymbol{v}\rangle = 0$. For instance, a direction $\boldsymbol{u} \in T_pM$ is asympotic if and only if $\bar{\boldsymbol{u}} = \boldsymbol{u}$. If we take a local parametrisation as in Theorem 2.2, then $\boldsymbol{u} = a\boldsymbol{x}_u + b\boldsymbol{x}_v$ and $\boldsymbol{v} = c\boldsymbol{x}_u + d\boldsymbol{x}_v$ are conjugate if and only if

$$acl + (ad + bc)m + bdn = 0.$$

We are interested in directions whose conjugate is a lightlike direction. Suppose that $\boldsymbol{u}$ is a lightlike direction at $p = \boldsymbol{x}(q)$, and take $\boldsymbol{u} = \boldsymbol{x}_u(q)$ with $\boldsymbol{x}$ as in Theorem 2.2. Then $\bar{\boldsymbol{u}} = (m\boldsymbol{x}_u - l\boldsymbol{x}_v)(q)$. So $\bar{\boldsymbol{u}}$ is lightlike if and only if $l(q) = 0$ or $m(q) = 0$. The condition $l(q) = 0$ means that p is on the LPL and $\bar{\boldsymbol{u}} = \boldsymbol{u}$. The condition $m(q) = 0$ is equivalent to $H(p) = 0$. Then $\bar{\boldsymbol{u}}$ is the other lightlike direction. Observe that when $m(q) = 0$ characteristic directions at p are both lightlike and are along $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$. So we have the following result.

Proposition 5.1 *Let M be a timelike surface and $p \in M$.*

(1) A lightlike direction $\boldsymbol{u}$ at p is self-conjugate if and only if p is on the LPL and $\boldsymbol{u}$ is the unique principal direction at p (which is also an asymptotic direction, see Proposition 4.3).

(2) The conjugate of a lightlike direction $\boldsymbol{u}$ at p is the other lightlike direction at p if and only if p is on the curve $H = 0$. At such points both lightlike directions are characteristic directions.

We analyse now to the configuration of the characteristic curves.

Theorem 5.2 (1) *On the de Sitter parabolic curve and away from its points of tangency with the LPL, the characteristic curves form a family of cusps with the cusps tracing the de Sitter parabolic curve, except maybe at some isolated points on this curve. At such points, the equation of the characteristic curves has generically a folded singularity of type saddle, node or focus (Figure 1). The folded singularities of the characteristic and asymptotic BDEs coincide. The indices of the two equations at such points are not related in general.*

(2) *At a point of tangency of the de Sitter parabolic curve and LPL, the characteristic BDE is topologically equivalent to*

$$vdv^2 + 2udvdu + v^3du^2 = 0 \quad \textit{Figure 4 left, or to}$$
$$vdv^2 - 2udvdu + v^3du^2 = 0 \quad \textit{Figure 4 right.}$$

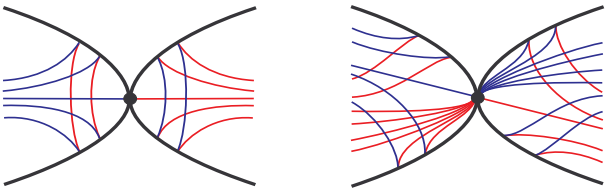

Figure 4: Configurations of the characteristic curves at the point of tangency of the de Sitter parabolic set and the LPL.

(3) *At a timelike umbilic point the characteristic BDE has generically a Morse Type 2 singularity with discriminant of type A_1^-. All the generic five topological models of such singularities can occur (Figure 2).*

Proof. (1) We follow the same setting as in the proof of Theorem 4.4. The 2-jets the origin of the coefficients of the characteristic BDE are too lengthy to reproduce here. The calculations show that if p_0 is not on the LPL, then it is a point on the discriminant (i.e., the de Sitter parabolic set) if and only if $a_{21}^2 - 4a_{20}a_{22} = 0$. We can take without loss of generality $a_{20} = 0$. Then p_0 is a parabolic point if $a_{21} = 0$ and in this case $\phi_x(0,0)$ is a characteristic direction. (It is also an asymptotic direction. In fact on the parabolic set, the characteristic and asymptotic directions coincide and become a principal direction.) We have generically $a_{22} \neq 0$, otherwise the de Sitter parabolic set is singular, and $a_{10}^2 - 1 \neq 0$ otherwise p_0 is also on the LPL. Then the origin is a singularity of the characteristic BDE if $a_{30} = 0$. Using the criteria in [4], we find that the singularity is in general a folded one and its type is determined by

$$\lambda_{char} = \frac{1}{2} \frac{a_{22}^2(a_{10}^2-1)^2(a_{31}^2 - 12a_{22}a_{40})}{((5a_{31}a_{32} - 3a_{22}a_{41})(a_{10}^2-1) + 4a_{10}a_{11}a_{31}^2)^2}.$$

We have a folded saddle if $\lambda_{char} < 0$, a folded node if $0 < \lambda_{char} < \frac{1}{16}$ and a folded focus $\lambda_{char} > \frac{1}{16}$. In general $\lambda_{char} \neq \lambda_{as}$ (see the proof of Theorem 4.4 for λ_{as}), so the type of the folded singularities of the asymptotic and characteristic BDEs are not related in general. (We remark that for surfaces in the Euclidean 3-space the two BDEs have opposite indices at their folded singularities, [3].)

(2) We take the surface in de Sitter Monge form (see the proof of Theorem 4.4) and let $a_{10} = a_{11} = 1$. Then $\phi_x(0,0)$ and $\phi_y(0,0)$ are lightlike directions. The point p_0 is on both the LPL and the de Sitter parabolic set if and only if $l(0) = m(0) = 0$, that is $a_{20} = a_{21} = 0$, or $m(0) = n(0) = 0$, that is $a_{21} = a_{22} = 0$. We assume that $a_{20} = a_{21} = 0$ and $a_{22} \neq 0$. Then the discriminant of the BDE of the characteristic curves has an A_3^- singularity (i.e., equivalent to $\pm(x^2 - y^4)$) provided $3a_{30}a_{32} - a_{31}^2 \neq 0$, which is satisfied generically.

Let (a, b, c) denote the coefficients of the BDE the characteristic curves. Then we have at the origin, after scaling, $j^1a = -a_{31}x - a_{32}y$, $j^1b = 3a_{30}x + a_{31}y$ and $j^1c = 0$. As $3a_{30}a_{32} - a_{31}^2 \neq 0$, we can make change of coordinates in the source and multiply by non-zero constant so that $j^1a = \alpha x + y$, $j^1b = \pm x$ and $j^1c = 0$, with $\alpha = 1/(8|a_{22}|\sqrt{|3a_{30}a_{32} - a_{31}^2|})$. The result now follows from Theorem 6.1 in the Appendix and the fact that the discriminant has an A_3^--singularity.

(3) We return here to the parametrisation in Theorem 2.2. If $p_0 = \boldsymbol{x}(q_0)$ is a timelike umbilic point, then $l(q_0) = n(q_0) = 0$ (we have generically $m(q_0) \neq 0$). So the coefficients of the BDE of the characteristic curves vanish at q_0. As the LPL, which is locally the discriminant of the equation, has a Morse singularity of type A_1^-, the equation has generically a Morse Type 2 singularity ([6]) with a discriminant having an A_1^--singularity. We assume that q_0 is the origin. The 1-jet of the BDE is $j^1ndv^2 + j^1ldu^2$. Following the calculation and notation in the proof of Theorem 3.3, the singularities of the lifted field ξ are the roots of the cubic $\phi = n_2p^3 + n_1p^2 + l_2p + l_1$. We need ϕ to have simple roots, that is

$$4n_1^3l_1 - n_1^2l_2^2 - 18n_1n_2l_1l_2 + 27n_2^2l_1^2 + 4n_2l_2^3 \neq 0.$$

The eigenvalues of the linear part of ξ at the roots of ϕ are $-\phi'(p)$ and $\alpha_1(p)$, where $\alpha_1(p) = 2p(n_2p + n_1)$ (see [6]). So we also need α_1 not to vanish at the roots of ϕ, that is

$$l_1(n_2l_1 - n_1l_2) \neq 0.$$

Then ξ has either saddle or node singularities at the roots of ϕ. The number and type of these singularities (which is determined by the sign of $-\phi'(p)\alpha_1(p)$) determine the topological configuration of the BDE. It is clear that we can have all the five possible generic cases in [6], see Figure 2. We observe here that, in general, there is no relation between the type of the singularity of the BDE of the lines of principal curvature and that of the characteristic curves. □

Remark 5.3 Let $\mathcal{C}(M, S_1^3)$ denotes the space of embeddings of an orientable timelike surface M in the de Sitter space, endowed with the Whitney topology. It follows by Thom's transversality theorem that the set of embeddings with the properties (a)-(f) below form a residual subset of $\mathcal{C}(M, S_1^3)$.

(a) The de Sitter parabolic set, when not empty, is a smooth curve.

(b) The LPL, when not empty, is a smooth curve or has isolated singularities of type A_1^-.

(c) The de Sitter parabolic set and the LPL have ordinary tangency at their points of intersection.

(d) The singularities of the BDE of the lines of principal curvature are those described in Theorem 3.3.

(e) The singularities of the BDE of the asymptotic curves are those described in Theorem 4.4.

(f) The singularities of the BDE of the characteristic curves are those described in Theorem 5.2.

One can always construct a patch of a timelike surface in S^1_3 with one of the pairs of foliations described in this paper having one of its possible stable local singularities. In the following example, we find explicit expressions for the above foliations on a special surface. Consider the timelike surface M given by $\boldsymbol{x} : \mathbb{R}^2 \to S^3_1$ with

$$\boldsymbol{x}(u,v) = \frac{1}{\sqrt{2}}(\sinh(u), \cosh(u), \cos(v), \sin(v)).$$

We have $\boldsymbol{x}_u = \frac{1}{\sqrt{2}}(\cosh(u), \sinh(u), 0, 0)$ and $\boldsymbol{x}_v = \frac{1}{\sqrt{2}}(0, 0, -\sin(v), \cos(v))$, so the coefficients of the first fundamental form are given by

$$E = -\frac{1}{2}, \quad F = 0, \quad G = \frac{1}{2}.$$

The de Sitter Gauss map is given by

$$\boldsymbol{e}(u,v) = \frac{1}{\sqrt{2}}(\sinh(u), \cosh(u), -\cos(v), -\sin(v)).$$

Therefore the coefficients of the second fundamental form are given by

$$l = \frac{1}{2}, \quad m = 0, \quad n = \frac{1}{2}.$$

It follows that the equations for the pairs of foliations described in this paper are as follows.

Lines of principal curvature:	$dvdu = 0,$
Asymptotic curves:	$dv^2 + du^2 = 0,$
Characteristic curves:	$dv^2 - du^2 = 0.$

So the lines of principal curvature are given $u = constant$ and $v = constant$, there are no asymptotic curves and the characteristic curves are given by $v \pm u = constant$.

6 Appendix

We are interested in BDEs with coefficients with 1-jets $(\alpha x + y, \pm x, 0)$. By equivalent we mean taking one BDE to another by smooth changes of coordinates in the source and multiplication by nowhere zero functions.

Theorem 6.1 *A BDE with 1-jet equivalent to* $(\alpha x+y, \pm x, 0)$ *and with a discriminant with* A_3*-singularity is topologically equivalent to one of the following cases.*

(i) Discriminant has an A_3^-*-singularity:*

$$\begin{array}{ll} ydy^2 + 2xdydx + y^3dx^2 = 0 & \textit{Figure 5 A, or to} \\ ydy^2 - 2xdydx + y^3dx^2 = 0 & \textit{Figure 5 B} \end{array}$$

(ii) Discriminant has an A_3^+*-singularity*

$$\begin{array}{ll} ydy^2 + 2xdydx - y^3dx^2 = 0 & \textit{Figure 5 C, or to} \\ ydy^2 - 2xdydx - y^3dx^2 = 0 & \textit{Figure 5 D} \end{array}$$

Proof. We take the coefficients of the BDE $\omega = (a, b, c)$ in the form

$$\begin{array}{l} a = \alpha x + y + M_1(x, y) \\ b = \epsilon x + M_2(x, y) \\ a = \lambda y^3 + M_3(x, y) \end{array}$$

where M_i are germs, at the origin, of smooth functions with $j^1M_1 = j^1M_2 = 0$, $j^3M_3 = 0$, $\epsilon = \pm 1$ and $\lambda = \pm 1$.

We consider a blowing up of the BDE following the method introduced in [17, 11] for BDEs whose discriminants are isolated points, and extended in [19, 21] for general BDEs.

Following the notation in [11], let $f_i(w)$, $i = 1, 2$ denote the foliation associated to the BDE $\omega = (a, b, c)$, which is tangent to the vector field $a\frac{\partial}{\partial x} + (-b + (-1)^i\sqrt{b^2 - ac}\ )\frac{\partial}{\partial y}$. If ψ is a diffeomorphism and $\lambda(x, y)$ is a non-vanishing real valued function, then ([11]) for $k = 1, 2$

1. $\psi(f_k(w)) = f_k(\psi^*(\omega))$, if ψ is orientation preserving;
2. $\psi(f_k(w)) = f_{3-k}(\psi^*(\omega))$, if ψ is orientation reversing;
3. $f_k(\lambda w) = f_k(\omega)$, if $\lambda(x, y)$ is positive;
4. $f_k(\lambda w) = f_{3-k}(\omega)$, if $\lambda(x, y)$ is negative.

We consider the directional blowing-up

$$(1) : \left\{ \begin{array}{l} x = v^2 \\ y = uv \end{array} \right. \quad (2) : \left\{ \begin{array}{l} x = -v^2 \\ y = uv \end{array} \right. \quad (3) : \left\{ \begin{array}{l} x = uv^2 \\ y = v \end{array} \right.$$

The blowing up (1) (resp. (2)) is a diffeomorphism from the region $v > 0$ (resp. $v < 0$) in the (u, v)-plane to the region $x > 0$ (resp. $x < 0$) in the (x, y)-plane and is orientation reversing (resp. preserving) in this region. Applying the blowing up (2) gives similar results to applying (1). So we deal in more details with blowing up (1). We also consider the blowing up (3) but this does not reveal any extra information.

Consider the blowing-up (1). Then the new BDE $\omega_0 = (u, v)^*\omega$ has coefficients

$$\begin{aligned}\bar{a} &= v(u(u^2+auv+4\epsilon v^2)+N_1(u,v)),\\ \bar{b} &= v^2((u^2+auv+2\epsilon v^2)+N_2(u,v)),\\ \bar{c} &= v^3(u+av+N_3(u,v)),\end{aligned}$$

where N_i, $i=1,2,3$, are smooth functions with $j^3N_1=0$, $j^2N_2=0$ and $j^1N_3=0$. We can write $(\bar{a},\bar{b},\bar{c})=v(A_1,vB_1,v^2C_1)$ with

$$\begin{aligned}A_1 &= u(u^2+auv+4\epsilon v^2)+N_1(u,v),\\ B_1 &= u^2+auv+2\epsilon v^2+N_2(u,v),\\ C_1 &= u+av+N_3(u,v).\end{aligned}$$

and consider the quadratic form $\omega_1=(A_1,vB_1,v^2C_1)$.

The discriminant of ω_1, which is the blowing up of the discriminant of ω, is the exceptional fibre $v=0$ when $\lambda=-1$ and the union of the exceptional fibre with two smooth curves C_i, $i=1,2$, meeting transversaly the exceptional fibre at $(u_i,0)$, where u_i are solutions of $(b_{22}^2-1)u^4+2\epsilon b_{22}u^2+1=0$.

We decompose ω_1 into two 1-forms, and to these 1-forms are associated the vector fields

$$X_i=A_1\frac{\partial}{\partial u}+(-vB_1+(-1)^i\sqrt{v^2(B_1^2-A_1C_1)}\,)\frac{\partial}{\partial v},\quad i=1,2.$$

These vector fields are tangent to the foliations defined by ω_1 and have the exceptional fibre $v=0$ (or part of it in the case $\lambda=+1$) as an integral curve. We deal now with the cases A_3^+ and A_3^- separately.

The case A_3^- ($\lambda=+1$)

We have $A_1(u,0)=u^3$, so $A(u_i,0)\neq 0$, $i=1,2$ with u_i as above. This means that near $(u_i,0)$, $i=1,2$, the integral curves of X_i form segments of smooth curves ending transeversaly on C_i, $i=1,2$. It also means that both vector fields X_1 and X_2 are smooth away from $(0,0)$ and the points $(u_i,0)$, $i=1,2$. So we need to analyse the configurations of these vector fields at $(0,0)$, with $v\geq 0$ (we are dealing with the blowing up (1) which is a diffeoemorphism from the region $v>0$ delimited by the exceptional fibre and the curves C_1 and C_2 to the region in the (x,y)-plane delimited by the discriminant and with $x>0$). We have, at the origin,

$$X_1=(u(u^2+auv+4\epsilon v^2)+h.o.t.)\frac{\partial}{\partial u}+(-uv(u+av)+h.o.t.)\frac{\partial}{\partial v}.$$

The first component A_1 of X_1 has a D_4-singularity provided $a^2-16\epsilon\neq 0$ (i.e., equivalent to $u(u^2\pm v^2)$). So $A_1=(u-g_1(u,v))(u^2+auv+4\epsilon v^2+g_2(u,v))$ for some germ of smooth functions g_1 and g_2 with $j^1g_1=j^2g_2=0$. We observe that when $u-g_1(u,v)=0$ the second component of X_1 also vanishes. Therefore $X_1=(u-g_1(u,v))\bar{X}_1$ with

$$\bar{X}_1=(u^2+auv+4\epsilon v^2+h.o.t.)\frac{\partial}{\partial u}+(-v(u+av)+h.o.t.)\frac{\partial}{\partial v}.$$

The 2-jet of the vector field $\bar{X}_1$ satisfies the general position condition in Proposition 3.5 in [18]. Therefore, by Takens' result, $\bar{X}_1$ is topologically equivalent to $(u^2+auv+4\epsilon v^2)\frac{\partial}{\partial u}-v(u+av)\frac{\partial}{\partial v}$ near the origin. Using a polar blowing up, we find that the configuration of $\bar{X}_1$ at the origin and hence of X_1 is as in Figure 5. (The singularities of the blown up field are either saddles or nodes and are as shown in Figure 5.)

We now consider the vector field X_2. We have, at the origin,

$$X_2=(u(u^2+auv+4\epsilon v^2)+h.o.t.)\frac{\partial}{\partial u}+(-v(u^2+auv+4\epsilon v^2)+h.o.t.)\frac{\partial}{\partial v}.$$

If $a^2-16\epsilon>0$, we can proceed as for X_1 above. We can factor out a term $u^2+auv+4\epsilon v^2+h.o.t$ from both component of X_2 and consider the vector field $\bar{X}_2=(u+h.o.t.)\frac{\partial}{\partial u}+(-v+h.o.t.)\frac{\partial}{\partial v}$ which has a saddle-singularity at the origin. So X_2 has a saddle singularity at the origin.

If $a^2-16\epsilon<0$, we consider a blowing up of the singularity of X_2 and find again that it has a saddle singularity at the origin.

Applying the blowing up (2) gives similar result, with the position of the nodes and saddles of X_1 inverted when $a^2-8\epsilon>0$. So we have the configuration of the solution curves of ω in region (x,y)-plane delimited by the discriminant and with $x<0$ as shown in Figure 5. We also need to invert the colours of the foliations as we factored out a term v in ω (see comments at the beginning of the proof).

Therefore the configurations of the solutions of the BDE ω are as in shown in Figure 5.

The case A_3^+ $(\lambda=-1)$

The situation here is similar to the case A_3^- (the 3-jets of X_1 and X_2 do not depend on λ). The difference is that the whole exceptional fibre is an integral curve of both X_1 and X_2. The configurations are as in Figure 5. □

Remarks 6.2 (1) The configurations in Figure 5 are topologically equivalent to Morse type 2 singularities (Figure 2 and Figure 3).

(2). A BDE with 1-jet $(\alpha x+y,\pm x,0)$ is equivalent to one with a 1-jet $(y,\pm x+\beta y,0)$ which is studied in [5]. A result in [5] states that we can reduce formally such BDEs, for almost all values of α, by a formal change of coordinates in the source and multiplication by non-zero formal power series, to one in the form

$$(\alpha x+y)dy^2\pm 2xdxdy+c(y)dx^2=0,$$

where $c(y)$ is a formal power series with a zero 2-jet. Theorem 6.1 gives the topological models for the case when $j^3c=\lambda y^3$, with $\lambda\neq 0$.

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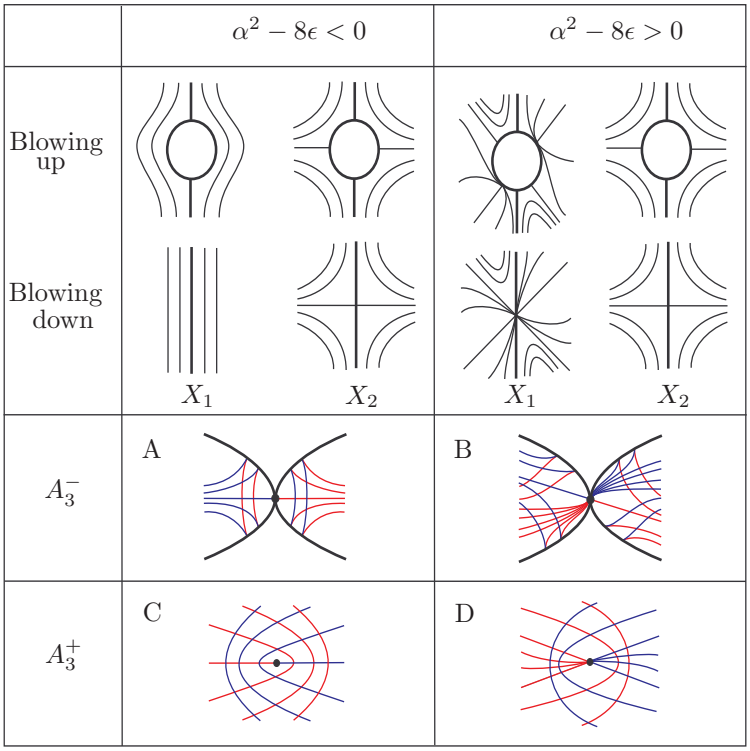

Figure 5: Topological models of BDEs with 1-jet equivalent to $(\alpha x + y, \epsilon x, 0)$, $\epsilon = \pm 1$, and with discriminant with an A_3-singularity (bottom two rows). The figures in the top row give the configurations of the vector fields X_1 and X_2 at the origin and those of their blowing up.

References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol. I.* Birkhäuser, 1986.

[2] J. W. Bruce and D. Fidal, On binary differential equations and umbilics. *Proc. Royal Soc. Edinburgh*, 111A (1989), 147–168.

[3] J. W. Bruce and F. Tari, Dupin indicatrices and families of curve congruences. *Trans. Amer. Math. Soc.* 357 (2005), 267–285.

[4] J. W. Bruce and F. Tari, Duality and implicit differential equations. *Nonlinearity* 13 (2000), 791–812.

[5] J.W. Bruce and F. Tari, On the multiplicity of implicit differential equations. *J. Differential Equations* 148 (1998), 122–147.

[6] J. W. Bruce and F. Tari, On binary differential equations. *Nonlinearity* 8 (1995), 255–271.

[7] M. do Carmo, *Geometria diferential de curvas e superfícies*. Textos Universitários, Sociedade Brasileira de Matemática, 2006.

[8] L. Chen and S. Izumiya, Singularities of Anti de Sitter torus Gauss map. Preprint, 2008.

[9] A.A. Davydov, *Qualitative control theory*. Translations of Mathematical Monographs 142, AMS, Providence, R.I., 1994.

[10] L.P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*. Ginn and Company, 1909.

[11] V. Guíñez, Locally stable singularities for positive quadratic differential forms. *J. Differential Equations* 110 (1994), 1–37.

[12] S. Izumiya, Timelike hypersurfaces in de Sitter space and Legendrian singularities. *J. Math. Sciences*. 144 (2007), 3789–3803.

[13] S. Izumiya, D-H. Pei and T. Sano, Singularities of hyperbolic Gauss maps. *Proc. London Math. Soc.* 86 (2003), 485–512.

[14] S. Izumiya, M. Takahashi and F. Tari, Folding maps on spacelike and timelike surfaces and duality. Preprint, 2007.

[15] S. Izumiya and F. Tari, Projections along geodesics of surfaces in the de Sitter space. Preprint, 2008.

[16] D. Pei, Singularities of $\mathbb{R}P^2$-valued Gauss maps of surfaces in Minkowski 3-space. *Hokkaido Math. J.* 28 (1999), 97–115.

[17] J. Sotomayor and C. Gutierrez, Structurally stable configurations of lines of principal curvature, Bifurcation, ergodic theory and applications (Dijon, 1981), 195–215, Astérisque, 98–99, Soc. Math. France, Paris, 1982.

[18] F. Takens, Singularities of vector fields. *Inst. Hautes Études Sci. Publ. Math.* 43 (1974), 47–100.

[19] F. Tari, On pairs of geometric foliations on a cross-cap. *Tohoku Math. J.* 59 (2007), 233–258.

[20] F. Tari, Self-adjoint operators on surfaces in $\mathbb{R}^n$. To appear in *Differential Geom. Appl.*

[21] F. Tari, Codimension 2 singularities and their bifurcations in binary differential equations. To appear in *Discrete and Continuous Dynamical Systems*.

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