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A SET OF INTEGRAL ELEMENTS OF HIGHER ORDER JET SPACES

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1. Introduction

In this paper, we will consider extension of "Monster Goursat manifold" in [MZ] to multi independent variables case. We denote by (R, D) a pair of a manifold R and a distribution D on R.

Historically, the original problem is to characterize the canonical systems $(J^k(M, n), C^k)$ on jet spaces. This problem was studied by Engel, Goursat and E.Cartan and they showed that, generically, the contact systems on k-jet spaces of 1 independent and 1 dependent variable were characterized as the so-called "Goursat flags". But A.Giaro, A. Kumpera and C. Ruiz explicitly pointed out that a Goursat flag of length 3 has singularities in [GKR], that is, they proved the existence of the (R, D) which is a Goursat flag and is not locally isomorphic to $(J^3(M, 1), C^3)$. The correct characterization of the canonical (contact) systems on jet spaces was given by R. Bryant in [B] for the first order systems and K.Yamaguchi in [Y1] and [Y2] for higher order systems for n independent and m dependent variables.

To this situation, R. Montgomery and M. Zhitomirskii constructed the "Monster Goursat manifold" by successive applications of the "Cartan prolongation of rank 2 distributions [BH]" to a surface and showed that Goursat flag (R, D) of length k is locally isomorphic to this "Monster Goursat manifold" in [MZ]

After that, "Monster Goursat manifolds" are extended to multi dependent variables cases, that is, successive applications of the "generalized Cartan prolongation" to the space of 1-jets of 1 independent and m dependent variables in [M2] or "rank 1 prolongation" in [SY]. And the characterization of the extended Monster Goursat manifolds was given in [SY], [M1].

"Monster manifolds(multi independent variables)" should be the sets of integral elements of higher order jet spaces;

Let $(J^k(M, n), C^k)$ be the k-jet space and its canonical system($\dim M = m+n$). We define $\Sigma(J^k(M, n))$ as follows:

$$\Sigma(J^k(M, n)) := \bigcup_{x \in J^k} \Sigma_x$$

where $\Sigma_x = \{\text{n-dim integral elements of } (J^k(M, n), C^k)\}$.

The first obstruction to form Monster manifolds is that Σ may not be a manifold. This phenomenon does not appear for 1 independent variable case, due to the lack of the 2-form condition. So the main purpose of this paper is to clarify when a set of integral elements of higher order jet spaces becomes a manifold. In fact we shall prove

Theorem 4.1. *$\Sigma(J^k(M^{m+n}, n))$ are not manifolds except for $\Sigma(J^2(M^{1+2}, 2))$ and trivial cases.*

Here trivial cases are when $n = 1$ and when $k = m = 1$. In the first case, $\Sigma(J^k(M^{m+1}, 1))$ is a projective bundle over $J^k(M^{m+1}, 1)$ by a rank 1 prolongation. In the second case, $\Sigma(J^1(M^{1+n}, n))$ is the Lagrange-Grassmann bundle over a contact manifold $J^1(M^{1+n}, n)$. We will prove $\Sigma(J^2(M^{1+2}, 2))$ becomes a manifold in [S].

2. Geometric construction of Jet Spaces

In this section, we will briefly recall the geometric construction of jet bundles in general, following [Y1] and [Y2], which is our basis for the later considerations.

Let M be a manifold of dimension $m + n$. Fixing the number n, we form the space of n-dimensional *contact elements* to M, i.e., the Grassmann bundle $J(M, n)$ over M consisting of n-dimensional subspaces of tangent spaces to M. Namely, $J(M, n)$ is defined by

$$J(M, n) = \bigcup_{x \in M} J_x, \qquad J_x = \mathrm{Gr}(T_x(M), n),$$

where $\mathrm{Gr}(T_x(M), n)$ denotes the Grassmann manifold of n-dimensional subspaces in $T_x(M)$. Let $\pi : J(M, n) \to M$ be the bundle projection. The *canonical system* C on $J(M, n)$ is, by definition, the differential system of codimension m on $J(M, n)$ defined by

$$C(u) = \pi_*^{-1}(u) = \{v \in T_u(J(M, n)) \mid \pi_*(v) \in u\} \subset T_u(J(M, n)) \xrightarrow{\pi_*} T_x(M),$$

where $\pi(u) = x$ for $u \in J(M, n)$.

Let us describe C in terms of a canonical coordinate system in $J(M, n)$. Let $u_o \in J(M, n)$. Let $(x_1, \ldots, x_n, z^1, \ldots, z^m)$ be a coordinate system on a neighborhood U' of $x_o = \pi(u_o)$ such that $dx_1, \ldots, dx_n$ are linearly independent when restricted to $u_o \subset T_{x_o}(M)$. We put $U = \{\ u \in \pi^{-1}(U') | dx_1|_u, \ldots, dx_n|_u$ are linearly independent $\}$. Then U is a neighborhood of u_o in $J(M, n)$. Here $dz^\alpha|_u$ is a linear combination of $dx_i|_u$'s,i.e., $dz^\alpha|_u = \sum_{i=1}^n p_i^\alpha(u) dx_i|_u$. Thus, there exist unique functions p_i^α on U such that C is defined on U by the following 1-forms;

$$\varpi^\alpha = dz^\alpha - \sum_{i=1}^n p_i^\alpha dx_i \qquad (\alpha = 1, \ldots, m),$$

where we identify z^α and x_i on U' with their lifts on U. The system of functions $(x_i, z^\alpha, p_i^\alpha)$ $(\alpha = 1, \ldots, m, i = 1, \ldots, n)$ on U is called a *canonical coordinate system* of $J(M, n)$ subordinate to (x_i, z^α).

$(J(M, n), C)$ is the (geometric) 1-jet space and especially, in case $m = 1$, is the so-called contact manifold. Let M, $\hat{M}$ be manifolds of dimension $m+n$ and $\varphi : M \to \hat{M}$ be a diffeomorphism. Then φ induces the isomorphism $\varphi_* : (J(M, n), C) \to (J(\hat{M}, n), \hat{C})$, i.e., the differential map $\varphi_* : J(M, n) \to J(\hat{M}, n)$ is a diffeomorphism sending C onto $\hat{C}$. The reason why the case $m = 1$ is special is explained by the following theorem of Bäcklund.

Theorem(Bäcklund) *Let M and $\hat{M}$ be manifolds of dimension $m+n$. Assume $m \geq 2$. Then, for an isomorphism $\Phi : (J(M,n), C) \to (J(\hat{M},n), \hat{C})$, there exists a diffeomorphism $\varphi : M \to \hat{M}$ such that $\Phi = \varphi_*$.*

The essential part of this theorem is to show that $F = \operatorname{Ker} \pi_*$ is the covariant system of $(J(M,n), C)$ for $m \geq 2$. Namely an isomorphism Φ sends F onto $\hat{F} = \operatorname{Ker} \hat{\pi}_*$ for $m \geq 2$. For the proof, we refer the reader to Theorem 1.4 in [Y2].

In case $m = 1$, it is a well known fact that the group of isomorphisms of $(J(M,n), C)$, i.e., the group of contact transformations, is larger than the group of diffeomorphisms of M. Therefore, when we consider the geometric 2-jet spaces, the situation differs according to whether the number m of dependent variables is 1 or greater.

(1) Case $m = 1$. We should start from a contact manifold (J, C) of dimension $2n+1$, which is locally a space of 1-jet for one dependent variable by Darboux's theorem. Then we can construct the geometric second order jet space $(L(J), E)$ as follows: We consider the Lagrange-Grassmann bundle $L(J)$ over J consisting of all n-dimensional integral elements of (J, C);

$$L(J) = \bigcup_{u \in J} L_u \subset J(J,n),$$

where L_u is the Grassmann manifolds of all Lagrangian (or Legendrian) subspaces of the symplectic vector space $(C(u), d\varpi)$. Here ϖ is a local contact form on J. Namely, $v \in J(J,n)$ is an integral element if and only if $v \subset C(u)$ and $d\varpi|_v = 0$, where $u = \pi(v)$. Let $\pi : L(J) \to J$ be the projection. Then the canonical system E on $L(J)$ is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J),$$

where $\pi(v) = u$ for $v \in L(J)$. We have $\partial E = \pi_*^{-1}(C)$ and $\operatorname{Ch}(C) = \{0\}$ (cf.[Y1]). Hence we get $\operatorname{Ch}(\partial E) = \operatorname{Ker} \pi_*$, which implies the Bäcklund theorem for $(L(J), E)$ (cf. [Y1]).

Now we put

$$(J^2(M,n), C^2) = (L(J(M,n)), E),$$

where M is a manifold of dimension $n+1$.

(2) Case $m \geq 2$. Since $F = \operatorname{Ker} \pi_*$ is a covariant system of $(J(M,n), C)$, we define $J^2(M,n) \subset J(J(M,n),n)$ by

$$J^2(M,n) = \{n\text{-dim. } \textit{integral elements} \text{ of } (J(M,n), C), \text{ transversal to } F\},$$

C^2 is defined as the restriction to $J^2(M,n)$ of the canonical system on $J(J(M,n),n)$.

Now the higher order (geometric) jet spaces $(J^{k+1}(M,n), C^{k+1})$ for $k \geq 2$ are defined (simultaneously for all m) by induction on k. Namely, for $k \geq 2$, we define $J^{k+1}(M,n) \subset J(J^k(M,n),n)$ and C^{k+1} inductively as follows:

$$J^{k+1}(M,n) = \{n\text{-dim. } \textit{integral elements} \text{ of } (J^k(M,n), C^k), \text{ transversal to } \operatorname{Ker}\ (\pi^k_{k-1})_*\ \},$$

where $\pi^k_{k-1} : J^k(M,n) \to J^{k-1}(M,n)$ is the projection. Here we have

$$\operatorname{Ker}\ (\pi^k_{k-1})_* = \operatorname{Ch}(\partial C^k),$$

and C^{k+1} is defined as the restriction to $J^{k+1}(M,n)$ of the canonical system on $J(J^k(M,n),n)$.

where the derived system and Cauchy characteristic system of (R, D) are generally defined as follows;

The derived system ∂D of D is defined, in terms of sections, by

$$\partial \mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

where $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D. In general ∂D is obtained as a subsheaf of the tangent sheaf of R (for the precise argument, see e.g.[Y1], [BCG3]). Moreover higher derived systems $\partial^i D$ are defined successively by

$$\partial^i D = \partial(\partial^{i-1} D),$$

where we put $\partial^0 D = D$ by convention.

the Cauchy characteristic system $\mathrm{Ch}\,(D)$ of a differential system (R, D) is defined by

$$\mathrm{Ch}\,(D)(x) = \{X \in D(x) \mid X \rfloor d\omega_i \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \quad \text{for } i = 1, \ldots, s\},$$

where $D = \{\, \omega_1 = \cdots = \omega_s = 0 \,\}$ is defined locally by defining 1-forms $\{\omega_1, \ldots, \omega_s\}$.

Here we observe that, if we drop the transversality condition in our definition of $J^k(M, n)$ and collect all n-dimensional integral elements, we may have some singularities in $J^k(M, n)$ in general. However, since every 2-form vanishes on 1-dimensional subspaces, in case $n = 1$, the integrability condition for $v \in J(J^{k-1}(M, 1), 1)$ reduces to $v \subset C^{k-1}(u)$ for $u = \pi^k_{k-1}(v)$. Hence, in this case, we can safely drop the transversality condition in the above construction as in the next section.

3. Rank 1 Prolongation

Let (R, D) be a differential system, i.e., R is a manifold of dimension $s + m + 1$ and D is a subbundle of $T(R)$ of rank $m + 1$. Starting from (R, D), we define $(P(R), \widehat{D})$ as follows (cf. [BH]):

$$P(R) = \bigcup_{x \in R} P_x \subset J(R, 1),$$

where

$$P_x = \{\text{1-dim. integral elements of } (R, D)\} = \{u \subset D(x) \mid \text{1-dim. subspaces}\} \cong \mathbb{P}^m.$$

Let $p : P(R) \to R$ be the projection. We define the canonical system $\widehat{D}$ on $P(R)$ by

$$\widehat{D}(u) = p_*^{-1}(u) = \{v \in T_u(P(R)) \mid p_*(v) \in u\} \subset T_u(P(R)) \xrightarrow{p_*} T_x(R),$$

where $p(u) = x$ for $u \in P(R)$.

We call $(P(R), \widehat{D})$ the *prolongation of rank 1* (or *Rank* 1 *Prolongation* for short) of (R, D). Then $P(R)$ is a manifold of dimension $2m + s + 1$ and $\widehat{D}$ is a differential system of rank $m+1$. In case $(R, D) = (M, T(M))$, we have $(P(M), \widehat{D}) = (J(M, 1), C)$. Moreover

$$J^2(M, 1) \subset P(J(M, 1)) \subset J(J(M, 1), 1)$$

As for the prolongation of rank 1, we have

Proposition([SY]) *Let (R, D) be a differential system of rank $m + 1$ and let $(P(R), \widehat{D})$ be the prolongation of rank 1 of (R, D). Then $\widehat{D}$ is of rank $m + 1$, $\partial \widehat{D} = p_*^{-1}(D)$ and*

$\mathrm{Ch}\,(\widehat{D})$ *is trivial. Moreover, if* $\mathrm{Ch}\,(D)$ *is trivial, then* $\mathrm{Ch}\,(\partial\widehat{D})$ *is a subbundle of* $\widehat{D}$ *of corank* 1.

This proposition implies that, starting from any differential system (R, D), we can repeat the procedure of Rank 1 Prolongation. Let $(P^1(R), D^1)$ be the prolongation of rank 1 of (R, D). Then $(P^k(R), D^k)$ is defined inductively as the prolongation of rank 1 of $(P^{k-1}(R), D^{k-1})$, which is called *k-th prolongation of rank 1* of (R, D). Moreover, starting from a manifold M of dimension $m+1$, we put

$$(P^k(M), C^k) = (P(P^{k-1}(M)), \widehat{C}^{k-1})$$

where $(P^1(M), C^1) = (J(M, 1), C)$. When $m = 1$, $(P^k(M), C^k)$ are called "monster Goursat manifolds" in [MZ].

4. Main theorem

In this section, we extend "Monster Goursat manifolds" to multi independent variables cases, that is, consider geometric construction of the jet spaces without transversality conditions.

Let $(J^k(M, n), C^k)$ be a k-jet space and its canonical system($\dim M = m + n$). We define $\Sigma(J^k(M, n))$ as follows:

$$\Sigma(J^k(M, n)) := \bigcup_{x \in J^k} \Sigma_x,$$

where $\Sigma_x = \{\text{n-dim integral element of } (J^k(M, n), C^k)\}$. By definition,

$$J^{k+1}(M, n) \subset \Sigma(J^k(M, n)) \subset J(C^k, n) \subset J(J^k(M, n), n),$$

where $J(C^k, n)$ is defined by

$$J(C^k, n) = \bigcup_{x \in J^k} C_x, \qquad C_x = \mathrm{Gr}(C^k(x), n),$$

i.e. C_x is the Grassmann manifold consisting of n-dimensional subspaces in $C^k(x)$. Note that

$$\Sigma(J^k(M, 1)) = J(C^k, 1) = P(J^k(M, 1)).$$

$J^{k+1}(M, n)$ is an open dense subset in $\Sigma(J^k(M, n))$, so the codimension of $\Sigma(J^k(M, n))$ equals to the codimension of $J^{k+1}(M, n)$ in $J(C^k, n)$.

Generally $\Sigma(J^k(M, n))$ is a variety and is not a submanifold in $J(C^k, n)$. In fact we have

Theorem 4.1. $\Sigma(J^k(M^{m+n}, n))$ *are not manifolds except for* $\Sigma(J^2(M^{1+2}, 2))$ *and trivial cases.*

Remark 4.2. $\Sigma(J^k(M^{m+1}, 1))$ are manifolds, because $n = 1$ case is nothing but a rank 1 prolongation. And $\Sigma(J^1(M^{1+n}, n))$ are Lagrange-Grassmann bundle $L(J)$, by definition. Hence $\Sigma(J^1(M^{1+n}, n))$ are also manifolds. we call these cases *trivial cases*. $\Sigma(J^2(M^{1+2}, 2))$ is shown to be a manifold in [S].

Proof. We will divide the proof into several cases.

(1) $n = 2, m = 2, k = 1$ case;

For $w \in \Sigma(J^1(M^{2+2}, 2))$, let $p(w) = v \in J^1(M^{2+2}, 2)$, where $p : \Sigma(J^1(M^{2+2}, 2)) \to J^1(M^{2+2}, 2)$ is the projection. And let $(U, (x_1, x_2, y_1, y_2, p_1^1, p_1^2, p_2^1, p_2^2))$ be a canonical coordinate in $J^1(M^{2+2}, 2)$ around v. Then the canonical system C^1 is expressed by $\{\varpi_1 = \varpi_2 = 0\}$, where $\varpi_1 = dy_1 - p_1^1 dx_1 - p_2^1 dx_2, \varpi_2 = dy_2 - p_1^2 dx_1 - p_2^2 dx_2$.
We select two 1-forms among the 1-forms $\{dx_1, dx_2, dp_1^1, dp_1^2, dp_2^1, dp_2^2\}$ to cover an open subset $\pi^{-1}(U)$ in $J(C^1, 2)$, where $\pi : J(C^1, 2) \to J^1(M, 2)$ is the projection. As an example, we consider an open subset $U_{p_1^1, p_2^1} \subset \pi^{-1}(U)$, where

$$U_{p_1^1, p_2^1} := \{w \in \pi^{-1}(U) \mid dp_1^1 \wedge dp_2^1|_w \neq 0\}.$$

Then, for $w \in U_{p_1^1, p_2^1}$, restricting $dx_1, dx_2, dp_1^2, dp_2^2$ to w, we can introduce the inhomogeneous coordinate in $U_{p_1^1, p_2^1}$ of $J(C^1, 2)$ around w as follows;

$$\begin{cases} dx_1|_w &= A(w)dp_1^1|_w + B(w)dp_2^1|_w \\ dx_2|_w &= C(w)dp_1^1|_w + D(w)dp_2^1|_w \\ dp_1^2|_w &= E(w)dp_1^1|_w + F(w)dp_2^1|_w \\ dp_2^2|_w &= G(w)dp_1^1|_w + H(w)dp_2^1|_w. \end{cases}$$

Moreover a 2-dim integral element w satisfies $d\varpi_1|_w = 0$, $d\varpi_2|_w = 0$, that is $d\varpi_1|_w = (-B(w)+C(w))dp_1^1 \wedge dp_2^1|_w$, $d\varpi_2|_w = (-E(w)B(w)+F(w)A(w)-G(w)D(w)+C(w)H(w))dp_1^1 \wedge dp_2^1|_w$. In this way, we obtain the defining equations $f_1 = f_2 = 0$ of $\Sigma(J^1(M^{2+2}, 2))$ in the inhomogeneous coordinate $U_{p_1^1, p_2^1}$ of $J(C^1, 2)$, where

$$\begin{cases} f_1 &= -B + C, \\ f_2 &= -EB + FA - GD + CH. \end{cases}$$

In this case the defining functions f_1 and f_2 degenerate at $S = \{A = B = C = D = F = G = 0, E = F\}$. Thus S is the singular locus of $\Sigma(J^1(M^{2+2}, 2))$ in $U_{p_1^1, p_2^1}$.

Now we count the number of defining equations and the codimension of $\Sigma(J^k(M, n))$ in $J(C^k, n)$ for general cases before the proof of the other cases.

First, we precisely recall the construction of $\Sigma(J^k(M, n))$ from $J^k(M, n)$.

For $w_0 \in \Sigma(J^k(M, n))$, let $p(w_0) = v_0 \in J^k(M, n)$, where $p : \Sigma(J^k(M, n)) \to J^k(M, n)$ is the projection. And let $(U, (x_i, y^j, p_I^j)$ $(1 \leq |I| \leq k, 1 \leq i \leq n, 1 \leq j \leq m)$ be a canonical coordinate in $J^k(M, n)$ around v_0, where I is a multi-index, that is $I = (i_1, \cdots, i_r), i_j \in \{1, \cdots, n\} (j = 1, \cdots, r), |I| = r$. And $p_{I_1} = p_{I_2}$, if $|I_1| = |I_2|$ and there exists $\sigma \in S_{|I_1|}$(symmetric group) such that $\sigma(I_1) = I_2$.

Then the canonical system C^k is expressed by $\{\varpi_I^j = 0 \ (0 \leq |I| \leq k-1)\}$, where $\varpi_0^j = dy^j - \Sigma_i p_i^j dx_i, \varpi_I^j = dp_I^j - \Sigma_i p_{Ii}^j dx_i$. From $\operatorname{rank} C^k = n + m \cdot {}_nH_k$, we have

$$\dim \mathrm{Gr}(C^k(v_0), n) = \{(n + m \cdot {}_nH_k) - n\} \cdot n = m \cdot n \cdot \begin{pmatrix} n+k-1 \\ k \end{pmatrix}.$$

This is the fiber dimension of $\pi : J(C^k, n) \to J^k(M, n)$, where π is the projection.

In fact, we select n 1-forms $\eta_1, \cdots, \eta_n$ among 1-forms $\{dx_i, dp_I^j\}$ $(|I| = k)$ such that $\eta_1 \wedge \cdots \wedge \eta_n|_{w_0} \neq 0$. Put

$$U_{\eta_1,\cdots,\eta_n} := \{w \in \pi^{-1}(U) \mid \eta_1 \wedge \cdots \wedge \eta_n|_w \neq 0\}.$$

Then, for $w \in U_{\eta_1,\cdots,\eta_n}$, restricting each 1-form ξ_α of $\{dx_i, dp_I^j\}$ other than $\{\eta_1, \cdots, \eta_n\}$ to w, we can introduce the inhomogeneous coordinate in $U_{\eta_1,\cdots,\eta_n}$ of $J(C^k, n)$ around w_0 as follows;

$$\xi_\alpha|_w = \sum_{l=1}^{n} a_l^\alpha(w)\eta_l|_w \qquad (\alpha = 1, 2, \cdots, m \cdot {}_nH_k).$$

Moreover an n-dim integral element w satisfies $d\varpi_I^j|_w = 0$ $(|I| = k-1)$(the number of $d\varpi_I^j|_w$ equals to $m \cdot {}_nH_{k-1}$);

$$d\varpi_I^j|_w = \sum_{i<l} f_{il}(w)\eta_i \wedge \eta_l|_w.$$

So we get the defining equations $f_{il} = 0$ of $\Sigma(J^k(M, n))$ in $J(C^k, n)$. Thus the number of the defining equations is

$$(m \cdot {}_nH_{k-1}) \cdot \frac{n(n-1)}{2}.$$

(The precise expressions of defining equations in the selected inhomogeneous coordinate will appear in each case below.)

Here the fiber dimension of $J^{k+1}(M, n) \to J^k(M, n)$ is

$$m \cdot {}_nH_{k+1} = m \cdot \begin{pmatrix} n+k \\ k+1 \end{pmatrix}.$$

Note that $\Sigma(J^k(M, n))$ is a subvariety of $J(C^k, n)$ containing $J^{k+1}(M, n)$ as an open dense subset. Therefore, the codimension of $\Sigma(J^k(M, n))$ in $J(C^k, n)$ is

$$m \cdot n \cdot \begin{pmatrix} n+k-1 \\ k \end{pmatrix} - m \cdot \begin{pmatrix} n+k \\ k+1 \end{pmatrix} = \frac{mk}{(k+1)!}(n+k-1) \cdot \cdots \cdot n \cdot (n-1).$$

Then, the difference of the number of defining equations and the codimension is

$$\frac{mk(k-1)}{2(k+1)!}(n+k-2) \cdot \cdots \cdot n \cdot (n-1) \cdot (n-2) \geq 0.$$

Hence, the codimension is less than the number of defining equations, in general. So we divide the proof into the cases that the number of the defining equations equals to the codimension or not. That is, we divide into the cases (2) $k = 1$, (3) $n = 2$ and (4) others.

(2) $n \geq 3, m \geq 2, k = 1$ cases;

For $w \in \Sigma(J^1(M^{m+n}, n))$, let $p(w) = v \in J^1(M^{m+n}, n)$, where $p : \Sigma(J^1(M^{m+n}, n) \to J^1(M^{m+n}, n)$ is the projection. And let $(U, (x_i, y_j, p_i^j))$ be a canonical coordinate in $J^1(M^{m+n}, n)$ around v. Then the canonical system C^1 is expressed by $\{\varpi^j = 0\}$, where $\varpi^j = dy^j - \sum_{l=1}^n p_l^j dx_l$.

We select n 1-forms among the 1-forms $\{dx_i, dp_i^j\}$ to cover an open set $\pi^{-1}(U)$ in $J(C^1, n)$, where $\pi : J(C^1, n) \to J^1(M^{m+n}, n)$ is the projection. Especially, we consider an open set $U_{p_1^1,\cdots,p_n^1} \subset \pi^{-1}(U)$, where

$$U_{p_1^1,\cdots,p_n^1} := \{w \in \pi^{-1}(U) \mid dp_1^1 \wedge \cdots \wedge dp_n^1|_w \neq 0\}.$$

For $w \in U_{p_1^1,\cdots,p_n^1}$, restricting dx_i, dp_i^j $(j \neq 1)$ to w, we have the inhomogeneous coordinate in $U_{p_1^1,\cdots,p_n^1}$ of $J(C^1, n)$ around w as follows;

$$\begin{cases} dx_i|_w &= \sum_{l=1}^n a_l^i(w) dp_l^1|_w \\ dp_i^j|_w &= \sum_{l=1}^n b_i{}^j{}_l(w) dp_l^1|_w \quad (j \neq 1). \end{cases}$$

Moreover an n-dim integral element w satisfies $d\varpi^j|_w = 0$;

$$\begin{cases} d\varpi^1|_w &= -\sum_{h=1}^n (dp_h^1|_w \wedge \sum_{l=1}^n a_l^h(w) dp_l^1|_w) \\ d\varpi^j|_w &= -\sum_{h=1}^n \{(\sum_{l=1}^n b_h{}^j{}_l(w) dp_l^1|_w) \wedge (\sum_{l=1}^n a_l^h(w) dp_l^1|_w)\} \quad (j \neq 1). \end{cases}$$

In this case, the number of defining equations of $\Sigma(J^1(M, n))$ in $J(C^1, n)$ coincides with the codimension of $\Sigma(J^1(M, n))$ in $J(C^1, n)$. So the appearance of the homogeneous quadratic relations from the second group of the above defining equations implies the drop of rank of defining equations at the origin of the inhomogeneous coordinate. Hence $\Sigma(J^1(M, n))$ is not a manifold, by the implicit function theorem.

(3) $n = 2$ cases;
we divide the proof according to $m = 1$ or greater.

(3)-(i) $n = 2, m = 1, k \geq 3$ cases;

For $w \in \Sigma(J^k(M^{1+2}, 2))$, let $p(w) = v \in J^k(M^{1+2}, 2)$, where $p : \Sigma(J^k(M^{1+2}, 2) \to J^k(M^{1+2}, 2)$ is the projection. And let $(U, (x_1, x_2, y, p_1, p_2, \cdots, p_I)$ $(|I| \leq k)$ be a canonical coordinate in $J^k(M^{1+2}, 2)$ around v, where $I = (i_1, \cdots, i_r), i_j \in \{1, 2\} (j = 1, \cdots, r)$. Then the canonical system C^k is expressed by $\{\varpi_I = 0 \ (0 \leq |I| \leq k-1)\}$, where $\varpi_0 = dy - p_1 dx_1 - p_2 dx_2, \varpi_I = dp_I - p_{I1} dx_1 - p_{I2} dx_2$.

We select two 1-forms among the 1-forms $\{dx_1, dx_2, dp_I\} (|I| = k)$ to cover an open set $\pi^{-1}(U)$ in $J(C^k, 2)$, where $\pi : J(C^k, 2) \to J^k(M^{1+2}, 2)$. Especially, we consider an open set $U_{p_{1\cdots 1}, p_{1\cdots 12}} \subset \pi^{-1}(U)$, where

$$U_{p_{1\cdots 1}, p_{1\cdots 12}} := \{w \in \pi^{-1}(U) \mid dp_{1\cdots 1} \wedge dp_{1\cdots 12}|_w \neq 0\}.$$

For $w \in U_{p_{1\cdots 1}, p_{1\cdots 12}}$, restricting dx_1, dx_2, dp_I $(|I| = k, I \neq (1, \cdots, 1), (1, \cdots, 1, 2))$ to w, we have the inhomogeneous coordinate;

$$\begin{cases} dx_1|_w &= a_1(w) dp_{1\cdots 1}|_w + a_2(w) dp_{1\cdots 12}|_w \\ dx_2|_w &= b_1(w) dp_{1\cdots 1}|_w + b_2(w) dp_{1\cdots 12}|_w \\ dp_I|_w &= c_I^1(w) dp_{1\cdots 1}|_w + c_I^2(w) dp_{1\cdots 12}|_w. \end{cases}$$

Moreover a 2-dim integral element w satisfies $d\varpi_I|_w = 0$ $(|I| = k-1)$;

$$\left\{\begin{array}{ll} d\varpi_{1\cdots 1}|_w & = (-a_2(w)+b_1(w))dp_{1\cdots 1}\wedge dp_{1\cdots 12}|_w \\ d\varpi_{1\cdots 12}|_w & = (a_1(w)-c^1_{1\cdots 122}(w)b_2(w)+b_1(w)c^2_{1\cdots 122}(w))dp_{1\cdots 1}\wedge dp_{1\cdots 12}|_w \\ d\varpi_{1\cdots 122}|_w & = (-c^1_{1\cdots 122}(w)a_2(w)+c^2_{1\cdots 122}(w)a_1(w)-c^1_{1\cdots 1222}(w)b_2(w)+ \\ & \qquad\qquad c^2_{1\cdots 1222}(w)b_1(w))dp_{1\cdots 1}\wedge dp_{1\cdots 12}|_w \\ & \vdots \\ d\varpi_{2\cdots 2}|_w & = (-c^2_{12\cdots 2}(w)a_2(w)+c^1_{12\cdots 2}(w)a_1(w)-c^1_{2\cdots 2}(w)b_2(w)+ \\ & \qquad\qquad c^2_{2\cdots 2}(w)b_1(w))dp_{1\cdots 1}\wedge dp_{1\cdots 12}|_w. \end{array}\right.$$

Hence $\Sigma(J^k(M^{1+2},2))$ is not a manifold by the same reasoning as in (2).

(3)-(ii) $n=2, m\geq 2, k\geq 2$ cases;

For $w\in\Sigma(J^k(M^{m+2},2))$, let $p(w)=v\in J^k(M^{m+2},2)$, where $p:\Sigma(J^k(M^{m+2},2)\to J^k(M^{m+2},2)$ is the projection. And let $(U,(x_1,x_2,y^i,p^i_1,p^i_2,\cdots,p^i_I)$ $(|I|\leq k,\ 1\leq i\leq m)$ be a canonical coordinate in $J^k(M^{m+2},2)$ around v, where $I=(i_1,\cdots,i_r), i_j\in\{1,2\}(j=1,\cdots,r)$. Then the canonical system C^k is expressed by $\{\varpi^i_I=0\ (0\leq|I|\leq k-1,\ 1\leq i\leq m)\}$, where $\varpi^i_0=dy^i-p^i_1dx_1-p^i_2dx_2, \varpi^i_I=dp^i_I-p^i_{I1}dx_1-p^i_{I2}dx_2$.

We select two 1-forms among the 1-forms $\{dx_1,dx_2,dp^i_I\}(|I|=k)$ to cover an open set $\pi^{-1}(U)$ in $J(C^k,2)$, where $\pi:J(C^k,2)\to J^k(M^{m+2},2)$ is the projection. Especially, we consider an open set $U_{p^1_{1\cdots 1},p^2_{1\cdots 1}}\subset\pi^{-1}(U)$, where

$$U_{p^1_{1\cdots 1},p^2_{1\cdots 1}}:=\{w\in\pi^{-1}(U)\mid dp^1_{1\cdots 1}\wedge dp^2_{1\cdots 1}|_w\neq 0\}.$$

For $w\in U_{p^1_{1\cdots 1},p^2_{1\cdots 1}}$, restricting dx_1,dx_2,dp^i_I $(|I|=k,(i,I)\neq(1,1,\cdots,1),(2,1,\cdots,1))$ to w, we have

$$\left\{\begin{array}{ll} dx_1|_w & = a_1(w)dp^1_{1\cdots 1}|_w+a_2(w)dp^2_{1\cdots 1}|_w \\ dx_2|_w & = b_1(w)dp^1_{1\cdots 1}|_w+b_2(w)dp^2_{1\cdots 1}|_w \\ dp^i_I|_w & = c^i_I(w)dp^1_{1\cdots 1}|_w+d^i_I(w)dp^2_{1\cdots 1}|_w. \end{array}\right.$$

Moreover a 2-dim integral element w satisfies $d\varpi^i_I|_w=0$ $(|I|=k-1)$;

$$\left\{\begin{array}{ll} d\varpi^1_{1\cdots 1}|_w & = (-a_2(w)-(c^1_{1\cdots 12}(w)b_2(w)-d^1_{1\cdots 12}(w)b_1(w)))dp^1_{1\cdots 1}\wedge dp^2_{1\cdots 1}|_w \\ d\varpi^1_I|_w & = (-(c^1_{I1}(w)a_2(w)-d^1_{I1}(w)a_1(w))-(c^1_{I2}(w)b_2(w)-d^1_{I2}(w)b_1(w))) \\ & \qquad dp^1_{1\cdots 1}\wedge dp^2_{1\cdots 1}|_w \quad (|I|=k-1, I\neq(1\cdots 1)) \\ d\varpi^2_{1\cdots 1}|_w & = (-a_1(w)-(c^2_{1\cdots 12}(w)b_2(w)-d^2_{1\cdots 12}(w)b_1(w)))dp^1_{1\cdots 1}\wedge dp^2_{1\cdots 1}|_w \\ d\varpi^2_I|_w & = (-(c^2_{I1}(w)a_2(w)-d^2_{I1}(w)a_1(w))-(c^2_{I2}(w)b_2(w)-d^2_{I2}(w)b_1(w))) \\ & \qquad dp^1_{1\cdots 1}\wedge dp^2_{1\cdots 1}|_w \quad (|I|=k-1, I\neq(1\cdots 1)) \\ d\varpi^i_I|_w & = (-(c^i_{I1}(w)a_2(w)-d^i_{I1}(w)a_1(w))-(c^i_{I2}(w)b_2(w)-d^i_{I2}(w)b_1(w))) \\ & \qquad dp^1_{1\cdots 1}\wedge dp^2_{1\cdots 1}|_w \quad (|I|=k-1, 3\leq i\leq m). \end{array}\right.$$

Hence $\Sigma(J^k(M^{m+2},2))$ is not a manifold by the same reasoning as in (2).

(4) $n\geq 3$ cases;

In these cases, we also divide the proof according to $m=1$ or greater.

(4)-(i) $n\geq 3, m=1, k\geq 2$ cases;

For $w \in \Sigma(J^k(M^{1+n}, n))$, let $p(w) = v \in J^k(M^{1+n}, n)$, where $p : \Sigma(J^k(M^{1+n}, n) \to J^k(M^{1+n}, n)$ is the projection. And let $(U, (x_i, y, p_I)$ $(|I| \leq k)$ be a canonical coordinate in $J^k(M^{1+n}, n)$ around v, where $I = (i_1, \cdots, i_r), i_j \in \{1, \cdots, n\}(j = 1, \cdots, r)$. Then the canonical system C^k is expressed by $\{\varpi_I = 0 \ (0 \leq |I| \leq k-1)\}$, where $\varpi_0 = dy - \sum_{i=1}^n p_i dx_i, \varpi_I = dp_I - \sum_{i=1}^n p_{Ii} dx_i$.

We select n 1-forms among the 1-forms $\{dx_i, dp_I\}(|I| = k)$ to cover an open set $\pi^{-1}(U)$ in $J(C^k, n)$, where $\pi : J(C^k, n) \to J^k(M^{1+n}, n)$ is the projection. Especially, we consider an open set $U_{p_{1\cdots 1}, \cdots, p_{n\cdots n}} \subset \pi^{-1}(U)$, where

$$U_{p_{1\cdots 1}, \cdots, p_{n\cdots n}} := \{w \in \pi^{-1}(U) \mid dp_{1\cdots 1} \wedge \cdots \wedge dp_{n\cdots n}|_w \neq 0\}.$$

For $w \in U_{p_{1\cdots 1}, \cdots, p_{n\cdots n}}$, restricting dx_i, dp_I $(|I| = k,\ I \neq (i, \cdots, i))$ to w, we have

$$\begin{cases} dx_i|_w &= \sum_{j=1}^n a_j^i(w) dp_{j\cdots j}|_w \\ dp_I|_w &= \sum_{j=1}^n b_j^I(w) dp_{j\cdots j}|_w \quad (|I| = k,\ I \neq (i, \cdots, i)). \end{cases}$$

Moreover an n-dim integral element w satisfies $d\varpi_I|_w = 0$ $(|I| = k-1)$;

$$\begin{cases} d\varpi_{1\cdots 1}|_w &= -dp_{1\cdots 1}|_w \wedge (\sum_{j=1}^n a_j^1(w) dp_{j\cdots j}|_w) \\ & \qquad - \sum_{i=2}^n \{(\sum_{j=1}^n b_j^{1\cdots 1i}(w) dp_{j\cdots j}|_w) \wedge (\sum_{j=1}^n a_j^i dp_{j\cdots j}|_w)\} \\ & \vdots \\ d\varpi_{i\cdots i}|_w &= -dp_{i\cdots i}|_w \wedge (\sum_{j=1}^n a_j^i(w) dp_{j\cdots j}|_w) \\ & \qquad - \sum_{l\neq i} \{(\sum_{j=1}^n b_j^{i\cdots il}(w) dp_{j\cdots j}|_w) \wedge (\sum_{j=1}^n a_j^l dp_{j\cdots j}|_w)\} \\ & \vdots \\ d\varpi_I|_w &= -\sum_{i=1}^n \{(\sum_{j=1}^n b_j^{Ii}(w) dp_{j\cdots j}|_w) \wedge (\sum_{j=1}^n a_j^i(w) dp_{j\cdots j}|_w)\} \\ & \qquad\qquad (|I| = k,\ I \neq (i, \cdots, i)). \end{cases}$$

In this case, the number of defining equations exceeds the codimension of $\Sigma(J^k(M^{1+n}, n))$. But the rank of defining equations at the origin is $n(n-1)$, which easily follows from the above equations. Then the difference of the codimension and the rank of defining equations at the origin is

$$\frac{1 \cdot k}{(k+1)!}(n+k-1) \cdot \cdots \cdot n \cdot (n-1) - n(n-1).$$

We can check the value is positive by induction on k. Thus the rank at the origin is strictly less than the codimension of $\Sigma(J^k(M^{1+n}, n))$. Hence $\Sigma(J^k(M^{1+n}, n))$ is not a manifold.

(4)-(ii) $n \geq 3, m \geq 2, k \geq 1$ cases;

For $w \in \Sigma(J^k(M^{m+n}, n))$, let $p(w) = v \in J^k(M^{m+n}, n)$, where $p : \Sigma(J^k(M^{m+n}, n) \to J^k(M^{m+n}, n)$ is the projection. And let $(U, (x_i, y_j, p_I^j)$ $(|I| \leq k)$ be a canonical coordinate in $J^k(M^{m+n}, n)$ around v, where $I = (i_1, \cdots, i_r), i_j \in \{1, \cdots, n\}(j = 1, \cdots, r)$. Then the canonical system C^k is expressed by $\{\varpi_I^j = 0 \ (0 \leq |I| \leq k-1),\ 1 \leq j \leq m\}$, where $\varpi_0^j = dy^j - \sum_{l=1}^n p_l^j dx_l, \varpi_I^j = dp_I^j - \sum_{l=1}^n p_{Il}^j dx_l$.

We select n 1-forms among the 1-forms $(dx_i, dp_I^j)(|I| = k)$ to cover an open set $\pi^{-1}(U)$ in $J(C^k, n)$, where $\pi : J(C^k, n) \to J^k(M^{m+n}, n)$ is the projection. Especially,

we consider an open set $U_{p^1_{1\cdots1},\cdots,p^1_{n\cdots n}} \subset \pi^{-1}(U)$, where

$$U_{p^1_{1\cdots1},\cdots,p^1_{n\cdots n}} := \{w \in \pi^{-1}(U) \mid dp^1_{1\cdots1} \wedge \cdots \wedge dp^1_{n\cdots n}|_w \neq 0\}.$$

For $w \in U_{p^1_{1\cdots1},\cdots,p^1_{n\cdots n}}$, restricting dx_i, dp^j_I $(|I| = k,\ (j,I) \neq (1,(i,\cdots,i)))$ to w, we have

$$\begin{cases} dx_i|_w &= \sum_{l=1}^n a^i_l(w) dp^1_{l\cdots l}|_w \\ dp^1_I|_w &= \sum_{l=1}^n b_l{}^1{}_I(w) dp^1_{l\cdots l}|_w \quad (|I| = k,\ I \neq (i,\cdots,i)) \\ dp^j_I|_w &= \sum_{l=1}^n b_l{}^j{}_I(w) dp^1_{l\cdots l}|_w \quad (|I| = k,\ 2 \leq j \leq m). \end{cases}$$

Moreover an n-dim integral element w satisfies $d\varpi_I|_w = 0$ $(|I| = k-1)$;

$$\begin{cases} d\varpi^1_{1\cdots1}|_w &= -dp^1_{1\cdots1}|_w \wedge (\sum_{l=1}^n a^1_l(w) dp^1_{l\cdots l}|_w) \\ &\qquad - \sum_{i=2}^n \{(\sum_{l=1}^n b_l{}^1{}_{1\cdots1i}(w) dp^1_{l\cdots l}|_w) \wedge (\sum_{l=1}^n a^i_l(w) dp^1_{l\cdots l}|_w)\} \\ &\vdots \\ d\varpi^1_{i\cdots i}|_w &= -dp^1_{i\cdots i}|_w \wedge (\sum_{l=1}^n a^i_l(w) dp^1_{l\cdots l}|_w) \\ &\qquad - \sum_{j\neq i} \{(\sum_{l=1}^n b_l{}^1{}_{i\cdots ij}(w) dp^1_{l\cdots l}|_w) \wedge (\sum_{l=1}^n a^j_l(w) dp^1_{l\cdots l}|_w)\} \\ &\vdots \\ d\varpi^1_I|_w &= -\sum_{h=1}^n \{(\sum_{l=1}^n b_l{}^l{}_{Ih}(w) dp_{l\cdots l}|_w) \wedge (\sum_{l=1}^n a^1_l(w) dp_{l\cdots l}|_w)\} \\ &\qquad\qquad (|I| = k-1,\ I \neq (i,\cdots,i)) \\ d\varpi^j_I|_w &= -\sum_{h=1}^n \{(\sum_{l=1}^n b_l{}^l{}_{Ih}(w) dp_{l\cdots l}|_w) \wedge (\sum_{l=1}^n a^j_l(w) dp_{l\cdots l}|_w)\} \\ &\qquad\qquad (|I| = k-1,\ 2 \leq j \leq m). \end{cases}$$

In this case, the number of defining equations exceeds the codimension of $\Sigma(J^k(M^{m+n}, n))$. The rank of defining equations at the origin is $n(n-1)$, which follows from the above equations. Thus the rank at the origin is strictly less than the codimension of $\Sigma(J^k(M^{m+n}, n))$. Hence $\Sigma(J^k(M^{m+n}, n))$ is not a manifold.

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