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ON A BOUND FOR AMPLITUDES OF NAVIER-STOKES FLOW WITH ALMOST PERIODIC INITIAL DATA

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Abstract. For any bounded (real) initial data it is known that there is a unique global solution to the two dimensional Navier-Stokes equations. This paper is concerned with a bound for the sum of the modulus of amplitudes when initial velocity is spatially almost periodic in 2D. In the case of general dimension, it is bounded on local time of existence shown by Giga, Inui, Mahalov and Matsui in 2005. A class of initial data is given such that the sum of the modulus of amplitudes of a solution is bounded on any finite time interval. It is shown by an explicit example that such a bound may diverge to infinity as the time goes to infinity at least for complex initial data.

AMS Subject Classification(2000):76D05,35Q30,42B05

Key words: Navier-Stokes equations, almost periodic initial data, frequency sets and amplitudes

1. Introduction

A unique local-in-time classical solution of the Cauchy problem for the Navier-Stokes equations in any dimension is known to exist for any bounded (real) initial data (see [9, 19]). It was shown in [16] that the solution is always spatially almost periodic at any time provided that the smooth solution exists since the solution continuously depends on initial data. This persistency property of almost periodicity also holds for the three-dimensional Navier-Stokes equations in a rotating frame with almost periodic initial data even if they are complex. It was shown in [15] for $n \geq 2$ that the local-in-time solution is analytic in time. When the initial data is almost periodic, this implies that the complex amplitude is analytic in time. In particular, a new mode cannot be created at any positive time. Although in 2D case the solution is bounded in space and any finite time interval, it is an open problem whether or not the sum of modulus of amplitudes is bounded in any finite time interval provided that the corresponding quantity of almost periodic (real) initial data is finite.

In this paper we study bounds of the sum of modulus of amplitudes of a global-in-time solution of the 2D-Navier-Stokes equations:

$$\begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla \pi = 0, \quad t > 0, \quad x \in \mathbb{R}^2, \\ \text{div } u = 0, \quad x \in \mathbb{R}^2, \\ u|_{t=0} = u_0, \end{cases} \tag{1}$$

when initial data is a real, spatially almost periodic function of the form

$$u_0(x) = \sum_{m \in \mathbb{Z}^\ell} c_m \exp\left(i \sum_{j=1}^{\ell} m_j (r_j \cdot x) \right), \quad x \in \mathbb{R}^2, \tag{2}$$

where $\{c_m\}_{m\in\mathbb{Z}^\ell} = \{c_m^1, c_m^2\}_{m\in\mathbb{Z}^\ell} \subset \mathbb{C}^2$, $r_j \in \mathbb{R}^2$ $(j = 1, \cdots, \ell)$ and $\{r_j\}_{j=1}^\ell$ is linearly independent over the field of all rational number $\mathbb{Q}$. Here $u = u(t,x) = (u^1(t,x), u^2(t,x))$ and $\pi = \pi(t,x)$ are unknown functions of the velocity of the fluid and its pressure respectively; $u_0 = u_0(x) = (u_0^1(x), u_0^2(x))$ is a given initial velocity vector field. In the case of general dimension, the sum of the modulus of amplitudes of a solution is bounded on local time of existence (see [11]).

As we mentioned in the first paragraph, it is known that the problem (1) admits a unique global-in-time smooth solution for any bounded measurable initial data ([9]); see also [23] for improvement and [24] for extension to Boussinesq flow. Uniqueness is guaranteed by imposing structure or growth of the pressure field (see [9, 22]).

We are interested in obtaining bounds of the sum of the modulus of amplitudes

$$\sum_{m\in\mathbb{Z}^\ell} |c_m(t)|$$

for any bounded time interval $[0,T]$. Note that this quantity equals to the FM norm of $u(t)$, Fourier preimage of all finite Radon measures proposed by [11]. Our goal is to find a class of real initial data such that the above quantity $\|u(t)\|_{FM}$ is bounded in $[0,T]$ for any $T > 0$. For this purpose we extend an almost periodic function u_0 to a periodic function of more variables by adding parameters. This device is proposed by [8]. We impose that the initial data is in the periodic Sobolev space and prove that the solution belongs to this Sobolev space by deriving a new a priori estimate including additional parameters. Applying the Sobolev embedding theorem we get a desired bound on the sum of modulus of amplitudes. It is not clear whether this bound is uniform with respect to real initial data if one merely bounds the sum of modulus of initial amplitudes. If an initial data is allowed to be complex, we can show that the FM norm of the solution may diverge to infinity as the time goes to infinity by giving an explicit example (see Theorem 3.1).

As an application we strengthen the result of [25]. In [25] the third author showed that existence on long time intervals of regular solutions to the 3D-Navier-Stokes equations in a rotating frame with spatially almost periodic data provided that the Coriolis force is large enough by assuming that there is a bounded for $\|u(t)\|_{FM}$ in $[0,T]$ for the sum of modulus of amplitudes of a solution $u(t)$, i.e., the 2D-Navier-Stokes equations. However, existence of such a bound is not given unless initial data is periodic. The 3D-Navier-Stokes equations with the Coriolis force are described as follows:

$$\begin{cases} \partial_t v + (v,\nabla)v + \Omega e_3 \times v - \Delta v = -\nabla p, \\ \nabla \cdot v = 0, \ v|_{t=0} = v_0, \end{cases} \tag{3}$$

where $v = v(x,t) = (v^1(x,t), v^2(x,t), v^3(x,t))$ is the unknown velocity vector field and $p = p(x,t)$ is the unknown scalar pressure at the point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ in space and time $t > 0$ while $v_0 = v_0(x)$ is the given initial velocity field. Here $\Omega \in \mathbb{R}$ is the Coriolis parameter, which is twice the angular velocity of the rotation around the vertical unit vector $e_3 = (0,0,1)$, the kinematic viscosity coefficient in normalized by one. By $\times$ we denote the exterior product, and hence, the Coriolis term is represented by $e_3 \times u = Ju$ with the corresponding skew-symmetric 3×3 matrix J.

We note that Babin, Mahalov and Nicolaenko [2] considered global solvability of (3) with periodic initial data (see also [1, 3, 4, 5, 21]). They proved existence on infinite intervals of regular solutions to the 3D-Navier-Stokes equations with a large (but finite)

Coriolis parameter with no smallness assumption on the initial data. Chemin, Desjardins, Gallagher and Grenier [7] considered the initial data decaying at space infinity and proved a global existence result. The almost periodic case is very different from periodic case and the case of decaying at the space infinity.

If initial data is allowed to be complex, results related to existence of global-in-time solution is quite different since energy estimate is no longer available. In fact, Li and Sinai [20] constructed a finite time blow-up solution of (1) for $n = 3$; however, their proof still works for two-dimensional problem. More precisely, let

$$X := \{u \in L^2 : \text{supp } \hat{u} \text{ is compact.}\}. \tag{4}$$

Note that $FM \subset X$. They showed that there exists $u_0 \in X$ such that $\|u(t)\|_{L^2} \to \infty$ and $\int_0^t \|\nabla u(\tau)\|_{L^2} d\tau \to \infty$ as $t \to T$ for some $T < \infty$. However, it seems not clear whether or not FM-norm of their solution blows up.

2. Function spaces and Main result

In this section, we define function spaces for almost periodic functions, and state a main result. Let us define FM and FM_0 spaces as follows (see also [11]).

Definition 2.1. Let

$$FM := \{u \in \mathcal{S}' : \|u\|_{FM} < \infty\},$$
$$\|u\|_{FM} := \sup\{|\langle u, \phi\rangle| : \phi \in C_\infty(\mathbb{R}^n) \quad \text{and} \quad \|\phi\|_{L^\infty} \le 1\},$$

where C_∞ is the space of all (complex-valued) continuous functions that converge to zero at the space infinity (equipped with the supremum norm $\|\cdot\|_{L^\infty}$). Let

$$FM_0 := \{u \in FM : \langle u, \phi_j\rangle \to 0 \ (j \to \infty)\},$$

where $\{\phi_j\}_{j=1}^\infty \subset C_\infty(\mathbb{R}^n)$ with supp $\phi_j \subset [-1/j, 1/j)^n$ and $\|\phi_j\|_{L^\infty} \le 1$ for $j = 1, 2, \cdots$.

Remark 2.2. FM_0 is a closed subspace of FM with the norm $\|\cdot\|_{FM}$.

Remark 2.3. If u is an almost periodic function in FM_0, namely, if u is expressed as $u = \sum_{\lambda\in\Lambda\setminus\{0\}} c_\lambda e^{i\lambda\cdot x}$ by using a countable set Λ in $\mathbb{R}^3$, then

$$\|u\|_{FM} = \sum_{\lambda\in\Lambda\setminus\{0\}} |c_\lambda|.$$

Definition 2.4. For $\{r_j\}_{j=1}^\ell \subset \mathbb{R}^2$ (linearly independent over $\mathbb{Q}$), let

$$D^\ell := \{\vec{s} \in \mathbb{R}^{2\ell} : 0 \le r_j \cdot s_j \le 2\pi \quad \text{for all} \quad j = 1, 2, \cdots, \ell\}.$$

Remark 2.5. The set D^ℓ is fundamental domain of $\mathbb{T}^\ell \times \mathbb{R}^\ell = \mathbb{R}^{2\ell}/(2\pi\mathbb{Z}\mathbf{e}_1 + \cdots 2\pi\mathbb{Z}\mathbf{e}_\ell)$, where we set $\mathbf{e}_j := \hat{r}_j/|r_j|$ and $\hat{r}_j := (\underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{j-1}, r_j, \underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{\ell-j}) \in \mathbb{R}^{2\ell}$, $\mathbf{0} = (0, 0) \in \mathbb{R}^2$. Note that $\{\mathbf{e}_j\}_{j=1}^\ell$ is an orthonormal system. We choose $\{\mathbf{e}_j\}_{j=\ell+1}^{2\ell}$ such that $\{\mathbf{e}_j\}_{j=1}^{2\ell}$ become orthonormal basis of $\mathbb{R}^{2\ell}$. For $\vec{s} = (s_1, \cdots, s_\ell, s_{\ell+1}, \cdots, s_{2\ell}) \in \mathbb{R}^{2\ell}$, we easily see that $\vec{s}$ is expressed as $\sum_{j=1}^{2\ell} a_j \mathbf{e}_j$, where $a_j := (\vec{s}, \mathbf{e}_j) = \hat{r}_j \cdot \vec{s}/|r_j| = r_j \cdot s_j/|r_j|$. For $f : \mathbb{R}^{2\ell} \to \mathbb{C}^2$ we define $\bar{f} : \mathbb{R}^{2\ell} \to \mathbb{C}^2$ as $\bar{f}(a_1, \cdots, a_{2\ell}) := f(\vec{s})$ for $\vec{s} \in \mathbb{R}^{2\ell}$. We say that the function f is D^ℓ-periodic if

$$\bar{f}(a_1 + 2\pi m_1, \cdots, a_\ell + 2\pi m_\ell, a_{\ell+1}, \cdots, a_{2\ell}) = \bar{f}(a_1, \cdots, a_\ell, a_{\ell+1}, \cdots, a_{2\ell})$$

for any $m = (m_1, \cdots, m_\ell) \in \mathbb{Z}^\ell$ and $(a_1, \cdots, a_{2\ell}) \in \mathbb{R}^{2\ell}$.

Definition 2.6. Let us define the function spaces as follows,

$$L_x^\infty(W_{\vec{s}}^{\eta,\infty})(\mathbb{R}^2\times D^\ell) := \{f(\cdot,\cdot) : \mathbb{R}^2\times D^\ell \to \mathbb{C}^2 : f \text{ is } D^\ell-\text{periodic}, \sum_{|k|\le\eta} \|\nabla_{\vec{s}}^k f(x,\vec{s})\|_{L^\infty_{x,\vec{s}}} < \infty\},$$

$$L_x^\infty(W_{\vec{s}}^{\eta,2})(\mathbb{R}^2\times D^\ell) := \{f(\cdot,\cdot) : \mathbb{R}^2\times D^\ell \to \mathbb{C}^2 : f \text{ is } D^\ell-\text{periodic}, \sum_{|k|\le\eta} \|\nabla_{\vec{s}}^k f(x,\vec{s})\|_{L^\infty_x(L^2_{\vec{s}})} < \infty\}$$

and

$$X^\eta(\mathbb{R}^2\times D^\ell) := \{f \in L_x^\infty(W_{\vec{s}}^{\eta,\infty})(\mathbb{R}^2\times D^\ell) \cap L_x^\infty(W_{\vec{s}}^{\eta,2})(\mathbb{R}^2\times D^\ell) : \|f\|_{X^\eta} := \sum_{|k|\le\eta} \|\nabla_{\vec{s}}^k f(x,\vec{s})\|_{L^\infty_{x,\vec{s}}} + \sum_{|k|\le\eta} \|\nabla_{\vec{s}}^k f(x,\vec{s})\|_{L^\infty_x(L^2_{\vec{s}})} < \infty\}.$$

Remark 2.7. It is easy to see that

$$L_x^\infty(W_{\vec{s}}^{\eta,\infty})(\mathbb{R}^2\times D^\ell) \subset L_x^\infty(W_{\vec{s}}^{\eta,2})(\mathbb{R}^2\times D^\ell).$$

Definition 2.8. We usually denote the following equality:

$$u^{\vec{s}}(x) = \sum_{m\in\mathbb{Z}^\ell} c_m \exp\left(i\sum_{j=1}^{\ell} m_j(r_j\cdot(x-s_j))\right)$$

if the function $u^{\vec{s}}$ satisfies the following convergence:

$$\lim_{N\to\infty} \left\| u^{\vec{s}}(x) - \sum_{m\in\mathbb{Z}^\ell,|m|\le N} c_m \exp\left(i\sum_{j=1}^{\ell} m_j(r_j\cdot(x-s_j))\right)\right\|_{L^2_{\vec{s}}} = 0 \text{ for } a.e.\ x\in\mathbb{R}^{2\ell}.$$

Remark 2.9. If there is x such that $u^{\vec{s}}(x) = \sum_{m\in\mathbb{Z}^\ell} c_m \exp\left(i\sum_{j=1}^{\ell} m_j(r_j\cdot(x-s_j))\right) \in L^2_{\vec{s}}(D^\ell)$, then $\|u^{\vec{s}}(y)\|_{L^2_{\vec{s}}(D^\ell)} = \|u^{\vec{s}}(y')\|_{L^2_{\vec{s}}(D^\ell)}$ for any y, $y'\in\mathbb{R}^2$.

Remark 2.10. Let $u(x) = \sum_{m\in\mathbb{Z}^\ell} c_m \exp\left(i\sum_{j=1}^{\ell} m_j(r_j\cdot x)\right)$ and $u^{\vec{s}}(x) = \sum_{m\in\mathbb{Z}^\ell} c_m \exp\left(i\sum_{j=1}^{\ell} m_j(r_j\cdot(x-s_j))\right)$. Since the function $u^{\vec{s}}(0)$ is periodic in $\vec{s}\in D^\ell$, we can use Sobolev's embedding theorem. Then we have

$$\begin{aligned}\|u(x)\|_{FM_x(\mathbb{R}^2)} &= \|u^{\vec{s}}(0)\|_{FM_{\vec{s}}(\mathbb{R}^{2\ell})} = \sum_{m\in\mathbb{Z}^\ell} |c_m| \\ &\le \left(\sum_{m\in\mathbb{Z}^\ell}(1+|m|^2)^\eta |c_m|^2\right)^{1/2}\left(\sum_{m\in\mathbb{Z}^\ell}(1+|m|^2)^{-\eta}\right)^{1/2} \le C_\eta \sum_{k\le\eta}\|\nabla_{\vec{s}}^k u^{\vec{s}}(0)\|_{L^2_{\vec{s}}(D^\ell)} \\ &\le C_\eta \sum_{k\le\eta}\|\nabla_{\vec{s}}^k u^{\vec{s}}(x)\|_{L^\infty_x(L^2_{\vec{s}})} \le C_\eta \|u^{\vec{s}}\|_{X^\eta(\mathbb{R}^2\times D^\ell)} \quad \text{for} \quad \eta > \ell/2.\end{aligned}$$

Remark 2.11. If the frequency set Λ of the function u has only finite element, then $u^{\vec{s}} \in X^k(\mathbb{R}^2\times D^\ell)$ for any $k\in\mathbb{Z}_+$.

Now we state the main theorem.

Theorem 2.12. *Let* $u_0^{\vec{s}} = \sum_{m\in\mathbb{Z}^\ell} c_m \exp\left(i\sum_{j=1}^{\ell} m_j(r_j\cdot(x-s_j))\right) \in X^\eta(\mathbb{R}^2\times D^\ell)$ *(*$\eta > \ell/2$*) with* $c_0 = 0$ *(note that* $u_0(x) := u_0^{\vec{s}}(x)|_{\vec{s}=(\mathbf{0},\cdots,\mathbf{0})} \in FM_0(\mathbb{R}^2)$*). Then the unique (mild) solution* $u^{\vec{s}}(t,x)$ *to the equation (1) (with initial data* $u_0^{\vec{s}}(x)$*) is expressed as*

$$u^{\vec{s}}(t,x) = \sum_{m\in\mathbb{Z}^\ell\setminus\{0\}} c_m(t)\exp\left(i\sum_{j=1}^{\ell} m_j(r_j\cdot(x-s_j))\right).$$

Moreover the solution $u^{\vec{s}}$ *is in* $L^\infty_{loc}([0,\infty):X^\eta(\mathbb{R}^2\times D^\ell))$ *if* $u_0^{\vec{s}}$ *is real. More precisely,*

$$\sup_{0<t\le T}\|u^{\vec{s}}\|_{X^\eta(\mathbb{R}^2\times D^\ell)}(t)\le C,$$

where $C>0$ *is depending only on* $T>0$ *and* $\|u_0^{\vec{s}}\|_{X^\eta(\mathbb{R}^2\times D^\ell)}$.

Remark 2.13. Let u be the solution to the equation (1).

If $u_0^{\vec{s}} = \sum_{m\in\mathbb{Z}^\ell} c_m \exp\left(i\sum_{j=1}^{\ell} m_j(r_j\cdot(x-s_j))\right) \in X^\eta(\mathbb{R}^2\times D^\ell)$ with $c_0=0$, we have

$$u(\cdot,t)\in C([0,\infty):FM_0(\mathbb{R}^2)) \quad \text{for} \quad u(x,0)=u_0(x)=u_0^{\vec{s}}(x)|_{\vec{s}=(\mathbf{0},\cdots,\mathbf{0})}$$

by Remark 2.10. In particular, the sum $\sum_{m\in\mathbb{Z}^\ell\setminus\{0\}}|c_m(t)|$ is bounded in any finite time interval $[0,T]$.

Remark 2.14. We give an example. The almost periodic (not periodic) function $u_0 = \alpha\cos(x_1+x_2+x_3)+\alpha\cos\left(\sqrt{2}(x_1+x_2+x_3)\right)$, $\alpha=(1,1,-2)$ satisfies Remark 2.11. Thus the solution $u(\cdot,t)$ to the equation (1) is in $C([0,\infty):FM_0(\mathbb{R}^2))$ by Remark 2.13.

Proof of Theorem 2.12. Let $S(t)u_0^{\vec{s}} = u^{\vec{s}}$ and $U(x,\vec{y},t) := I_{\vec{s}\to\vec{y}-\rho(x)}S(t)u^{\vec{s}}(x)$, where $I_{\vec{x}\to\vec{y}}u(\vec{x}) = u(\vec{y})$ and $\rho(x) = (x,x,\cdots,x)$. Now we show that the function $U(s,\vec{y},t)$ is expressed as the following periodic function:

$$U(x,\vec{y},t) = U(\vec{y},t) = \sum_{m\in\mathbb{Z}^\ell} c_m(t)\exp\left(i\sum_{j=1}^{\ell} m_j(r_j\cdot y_j)\right),\ c_m(0)=c_m. \tag{5}$$

First we show that the function $U(x,\vec{y},t)$ is periodic with respect to $\overrightarrow{y}$. By using the uniqueness result of the mild solutions (see [9, 18, 22]), we have

$$I_{\vec{s}\to\vec{w}}S(t)u_0^{\vec{s}}(x) = S(t)I_{\vec{s}\to\vec{w}}u_0^{\vec{s}}(x). \tag{6}$$

Since $u_0^{\vec{s}+\vec{w}}(x) = u_0^{\vec{s}}(x)$ for $\vec{w} = (w_1\tilde{m}_1,\cdots,w_\ell\tilde{m}_\ell)$, where $w_j = 2\pi r_j/|r_j|^2$ and $\tilde{m}_j\in\mathbb{Z}$, we see that

$$\begin{aligned} U(x,\vec{y}+\vec{w}) &= I_{\vec{s}\to\vec{y}+\vec{w}-\rho(x)}S(t)u_0^{\vec{s}}(x)\\ &= I_{\vec{s}\to\vec{y}-\rho(x)}I_{\vec{s}\to\vec{s}+\vec{w}}S(t)u_0^{\vec{s}}(x) = I_{\vec{s}\to\vec{y}-\rho(x)}S(t)I_{\vec{s}\to\vec{s}+\vec{w}}u_0^{\vec{s}}(x) = I_{s\to y-\rho(x)}S(t)u_0^{\vec{s}+\vec{w}}(x)\\ &= I_{s\to y-\rho(x)}S(t)u_0^{\vec{s}}(x) = U(x,\vec{y}). \end{aligned}$$

Next we show that $U(x,\vec{y},t) = U(\vec{y},t)$. By (6), we also see that

$$\begin{aligned} U(x+c,\vec{y},t) &= I_{x\to x+c}I_{\vec{s}\to\vec{y}-\rho(x)}S(t)u_0^{\vec{s}}(x) = I_{\vec{s}\to\vec{y}-(\rho(x)+\rho(c))}S(t)u_0^{\vec{s}}(x+c)\\ &= I_{\vec{s}\to\vec{y}-\rho(x)}I_{\vec{s}\to\vec{s}-\rho(c)}S(t)u_0^{\vec{s}}(x+c) = I_{\vec{s}\to\vec{y}-\rho(x)}S(t)I_{\vec{s}\to\vec{s}-\rho(c)}u_0^{\vec{s}}(x+c)\\ &= I_{\vec{s}\to\vec{y}-\rho(x)}S(t)u_0^{\vec{s}-\rho(c)}(x+c) \quad \text{for all} \quad c\in\mathbb{R}^2, \end{aligned}$$

where we used $S(t)u_0^{\vec{s}}(x+c) = S(t)u_0^{\vec{s}+\rho(c)}(x)$ since the equation (1) is translation invariant. Thus,

$$U(x+c,\vec{y},t) = I_{\vec{s}\to\vec{y}-\rho(x)}S(t)u^{\vec{s}-\rho(c)}(x+c) = I_{\vec{s}\to\vec{y}-\rho(x)}S(t)u^{\vec{s}}(x) = U(x,\vec{y},t).$$

Therefore we can rewrite $U(x,\vec{y},t) = U(\vec{y},t)$, the function $U(\vec{y},t)$ is periodic with values $\vec{y}$ and is expressed as (5). Since $I_{\vec{y}\to\rho(x)-\vec{s}}U(\vec{y},t) = u^{\vec{s}}$, we have the following expression:

$$u^{\vec{s}}(x) = \sum_{m\in\mathbb{Z}^\ell} c_m(t)\exp\left(i\sum_{j=1}^{\ell} m_j(r_j\cdot(x-s_j))\right).$$

We now derive an a priori estimate. First we take existence time T arbitrarily. Since D^ℓ is compact and the solution is real-valued,

$\sup_{0<t<T}\|u^{\vec{s}}\|_{L^\infty_{x,\vec{s}}(\mathbb{R}^2\times D^\ell)} = \sup_{0<t\le T}\sup_{\vec{s}\in D^\ell}\|u^{\vec{s}}(t)\|_{L^\infty(\mathbb{R}^2)}$ is always bounded (see [17, 23, 24]). Now we use the mild solution to the equation (1) for each $\vec{s}$ as follows:

$$u^{\vec{s}}(t) = e^{t\Delta}u_0^{\vec{s}} - \int_0^t e^{(t-\tau)\Delta}\mathbf{P}\nabla\cdot(u^{\vec{s}}\otimes u^{\vec{s}})(\tau)d\tau,$$

and

(7)
$$\begin{cases} \nabla_{\vec{s}}u^{\vec{s}}(t) = e^{t\Delta}\nabla_{\vec{s}}u_0^{\vec{s}} - \int_0^t e^{(t-\tau)\Delta}\mathbf{P}\nabla\cdot(\nabla_{\vec{s}}u^{\vec{s}}\otimes u^{\vec{s}} + u^{\vec{s}}\otimes\nabla_{\vec{s}}u^{\vec{s}})(\tau)d\tau, \\ \nabla^2_{\vec{s}}u^{\vec{s}}(t) = e^{t\Delta}\nabla^2_{\vec{s}}u_0^{\vec{s}} - \int_0^t e^{(t-\tau)\Delta}\mathbf{P}\nabla\cdot(\nabla^2_{\vec{s}}u^{\vec{s}}\otimes u^{\vec{s}} + 2\nabla_{\vec{s}}u^{\vec{s}}\otimes\nabla_{\vec{s}}u^{\vec{s}} + u^{\vec{s}}\otimes\nabla^2_{\vec{s}}u^{\vec{s}})(\tau)d\tau, \\ \qquad\qquad\vdots \end{cases}$$

where $\mathbf{P}$ is Helmholtz projection. A direct calculation yields

$$\|\nabla_{\vec{s}}u^{\vec{s}}(t)\|_{L^\infty_{x,\vec{s}}} \le \|\nabla_{\vec{s}}u_0^{\vec{s}}\|_{L^\infty_{x,\vec{s}}} + Ct^{1/2}\sup_{0<\tau\le t}\|u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}}\sup_{0<\tau\le t}\|\nabla_{\vec{s}}u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}}.$$

Let us set $t := 1/(2C\sup_{0<\tau\le T}\|u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}})^2$. By absorbing argument, we have

$$\|\nabla_{\vec{s}}u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}} \le 2\|\nabla_{\vec{s}}u_0^{\vec{s}}\|_{L^\infty_{x,\vec{s}}} \quad \text{for} \quad 0\le\tau\le t.$$

Repeating these arguments for several times, we have

$$\sup_{0<t<T}\|\nabla_{\vec{s}}u^{\vec{s}}(t)\|_{L^\infty_{x,\vec{s}}} \le C_T^1$$

where C_T^1 is depending only on $\sup_{0<\tau<T}\|u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}}$ and $\|\nabla_{\vec{s}}u_0^{\vec{s}}\|_{L^\infty_{x,\vec{s}}}$. Repeating the above arguments, we also have

$$\sup_{0<t<T}\|\nabla^\eta_{\vec{s}}u^{\vec{s}}(t)\|_{L^\infty_{x,\vec{s}}} \le C_T^2,$$

where C_T^2 is depending only on $\sup_{0<\tau<T}\|u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}}$,$\sup_{0<\tau<T}\|\nabla^{j-1}_{\vec{s}}u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}}$ $(1\le j\le\eta)$ and $\|\nabla^k_{\vec{s}}u_0^{\vec{s}}\|_{L^\infty_{x,\vec{s}}}$ $(1\le k\le\eta)$.

Next we estimate $\nabla_{\vec{s}}u^{\vec{s}}$ with the norm $L^\infty_x(L^2_{\vec{s}})$. By (7), we easily have

$$\|\nabla_{\vec{s}}u^{\vec{s}}(t)\|_{L^\infty_x(L^2_{\vec{s}})} \le \|\nabla_{\vec{s}}u_0^{\vec{s}}\|_{L^\infty_x(L^2_{\vec{s}})} + t^{1/2}\sup_{0<\tau\le t}\|u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}}\sup_{0<\tau\le t}\|\nabla_{\vec{s}}u^{\vec{s}}(\tau)\|_{L^\infty_x(L^2_{\vec{s}})}.$$

By absorbing argument as we calculated before, we have

$$\|\nabla_{\vec{s}}u^{\vec{s}}(t)\|_{L^\infty_x(L^2_{\vec{s}})} \le C_T^3,$$

where C_T^3 is depending only on $\sup_{0<\tau<T}\|u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}}$ and $\|u_0^{\vec{s}}\|_{L^\infty_x(L^2_{\vec{s}})}$. Repeating these arguments, we have

$$\|\nabla^\eta_{\vec{s}} u^{\vec{s}}(t)\|_{L^\infty_x(L^2_{\vec{s}})} \le C_T^4,$$

where C_T^4 is depending only on $\sup_{0<\tau<T}\|u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}}$, $\sup_{0<\tau<T}\|\nabla^j_{\vec{s}} u^{\vec{s}}(\tau)\|_{L^\infty_x(L^2_{\vec{s}})}$, $\sup_{0<\tau<T}\|\nabla^j_{\vec{s}} u^{\vec{s}}(\tau)\|_{L^\infty_{x,\vec{s}}}$ $(1 \le j \le \eta-1)$ and $\|\nabla^k_{\vec{s}} u_0^{\vec{s}}\|_{L^\infty_x(L^2_{\vec{s}})}$ $(1 \le k \le \eta)$. Combining these estimates, we see that

$$\sup_{0<\tau\le T}\|u(\tau)||_{X^\eta} \le C_T,$$

where C_T is depending only on T and $\|u_0\|_{X_\eta}$. This is the desired estimate. ∎

3. An example of complex unbounded FM solutions

The FM norm dominates L^∞ norm. For L^∞ norm it is known that

$$\|u(t)\|_\infty \le C\|u_0\|_\infty \exp(C\|\omega_0\|_\infty t)$$

for all $t > 0$ (see [24]), where ω_0 is the initial vorticity. By the regularizing effect the vorticity and velocity at some positive time t_0 is dominated by L^∞ norm of initial data u_0 and t_0 which can be chosen based on $\|u_0\|_\infty$ (see [9]). Thus we have a bound for $\|u(t)\|_\infty$ on $[0,T]$ depending only on T and $\|u_0\|_\infty$ provided that u_0 is real. The existence of a uniform bound in L^∞ in $\mathbb{R}^2\times(0,\infty)$ is a still open problem. In this section we shall prove that $\|u(t)\|_{FM}$ may diverge to infinity as $t\to\infty$ at least for some complex initial data. To construct such solutions, we use second approximation of the solution to (1) and extract inflation term from low frequency part. This device is proposed by [6, 26].

Theorem 3.1. *Let us set initial value as*

$$u_0 := \sum_{j=1}^\infty c_j\left(a^0 e^{ik^0_{\alpha_j}\cdot x} + a^1 e^{ik^1_{\alpha_j}\cdot x}\right),$$

where $\{c_j\}_{j=1}^\infty$, $\{\alpha_j\}_{j=1}^\infty \subset \mathbb{R}_+$, $\sum_{j=1}^\infty c_j \le 1$, $\{\alpha_j\}$ *is linearly independent over* $\mathbb{Q}$, $\alpha_j/(c_j)^2 \to 0$ $(j\to\infty)$,

$$a^0 = \begin{pmatrix}1\\0\end{pmatrix}, a^1 = \begin{pmatrix}-1\\1\end{pmatrix}, k^0_\alpha = \begin{pmatrix}0\\\alpha\end{pmatrix} \quad \textit{and} \quad k^1_\alpha = \begin{pmatrix}\alpha\\\alpha\end{pmatrix}.$$

(Note that $\|u_0\|_{FM} \le 3$ *for any* $\{\alpha\}_j$, *and* div $u_0 = 0$.*) Assume that the solution* $u(t)$ *to the equation (1) is in* $C([0,\infty) : FM_0)$. *Then we have*

$$\|u(t)\|_{FM} \to \infty \quad (t\to\infty).$$

Proof of Thorem 3.1. Let us define

$$u_\eta := \sum_{m+m'=\eta, 1\le m,m'\le\eta-1}\int_0^t e^{(t-\tau)\Delta}\mathbf{P}\nabla\cdot(u_m\otimes u_{m'})(\tau)d\tau, \quad \eta = 2,3,\cdots$$

and $u_1 := e^{t\Delta}u_0$. We easily see that the solution u to the equation (1) is expressed as $u = \sum_{\eta=1}^\infty u_\eta$. Since the solution is complex valued function, we see that

$$\text{supp }\hat{u}_\eta(\cdot,t) \subset \bigcup_{(j_1,\cdots,j_\eta)\in\{1,2,\cdots\}^\eta}[0,\infty)\times(\{\sum_{m=1}^\eta \alpha_{j_m}\}) \quad \text{for} \quad t>0$$

and $\{\alpha_j\}_{j=1}^{\infty}$ is linearly independent over the field of all rational number $\mathbb{Q}$. For example,

$$\text{supp } \hat{u}_2(\cdot,t) \subset [0,\infty) \times [(\{2\alpha_1\}) \cup (\{\alpha_1+\alpha_2\}) \cup (\{2\alpha_2\}) \cup (\{\alpha_1+\alpha_3\}) \cdots] \quad \text{for} \quad t>0.$$

Thus we have $\text{supp } \hat{u}_\eta \cap \text{supp } \hat{u}_{\eta'} = \emptyset$ $(\eta \neq \eta')$. Therefore we can estimate the FM norm of u as follows:

$$\|u\|_{FM} = \sum_{\eta=1}^{\infty} \|u_\eta\|_{FM} \geq \|u_2\|_{FM}.$$

We now only calculate the FM norm of u_2. This is the inflation term controlled by the parameters $\{\alpha_j\}_j$. Since $a^0 \cdot k^0_{\alpha_j} = a^1 \cdot k^1_{\alpha_j} = 0$ for $j = 1, 2, \cdots$, we have

$$\begin{aligned}
\mathbf{P}\nabla \cdot (u_1 \otimes u_1) &= \mathbf{P}\nabla \cdot \sum_{\sigma_1,\sigma_2 \in \{0,1\}} c_j c_{j'} (a^{\sigma_1} \otimes a^{\sigma_2}) \sum_{j,j'=1}^{\infty} e^{i(k^{\sigma_1}_{\alpha_j} + k^{\sigma_2}_{\alpha_{j'}})\cdot x} e^{-(|k^{\sigma_1}_{\alpha_j}|^2 + |k^{\sigma_2}_{\alpha_{j'}}|^2)t} = \\
&\sum_{\sigma_1\sigma_2 \in \{0,1\}} \sum_{j,j'=1}^{\infty} c_j c_{j'} \mathbf{P} e^{i(k^{\sigma_1}_{\alpha_j} + k^{\sigma_2}_{\alpha_{j'}})\cdot x} a^{\sigma_2} \left(a^{\sigma_1} \cdot (k^{\sigma_1}_{\alpha_j} + k^{\sigma_2}_{\alpha_{j'}}) \right) e^{-(|k^{\sigma_1}_{\alpha_j}|^2 + |k^{\sigma_2}_{\alpha_{j'}}|^2)t} = \\
&\sum_{j,j'=1}^{\infty} c_j c_{j'} e^{i(k^0_{\alpha_j} + k^1_{\alpha_{j'}})\cdot x} P_{0,1} \left((a^1 (a^0 \cdot k^1_{\alpha_j}) + a^0 (a^1 \cdot k^0_{\alpha_{j'}}) \right) e^{-(|k^0_{\alpha_j}|^2 + |k^1_{\alpha_{j'}}|^2)t} \\
&\quad + \sum_{j,j'=1}^{\infty} c_j c_{j'} e^{i(k^0_{\alpha_j} + k^0_{\alpha_{j'}})\cdot x} P_{0,0} a^0 (a^0 \cdot k^1_{\alpha_j}) e^{-(|k^0_{\alpha_j}|^2 + |k^0_{\alpha_{j'}}|^2)t} \\
&\quad + \sum_{j,j'=1}^{\infty} c_j c_{j'} e^{i(k^1_{\alpha_j} + k^1_{\alpha_{j'}})\cdot x} P_{1,1} a^1 (a^1 \cdot k^0_{\alpha_j}) e^{-(|k^1_{\alpha_j}|^2 + |k^1_{\alpha_{j'}}|^2)t},
\end{aligned}$$

where

$$P_{0,1} := I - \frac{1}{|\xi|} \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_2\xi_1 & \xi_2^2 \end{pmatrix} \Big|_{\xi=(\xi_1,\xi_2)=k^0_{\alpha_j}+k^1_{\alpha_{j'}}} = \frac{1}{5} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix},$$

(we define $P_{0,0}$ and $P_{1,1}$ in the same way) I is 2×2 identity matrix. Here we used $\mathbf{P} e^{i(k^0_{\alpha_j} + k^1_{\alpha_{j'}})x} = P_{0,1} e^{i(k^0_{\alpha_j} + k^1_{\alpha_{j'}})x}$. Thus

$$\begin{aligned}
\|u_2\|_{FM} &= \left\| \int_0^t e^{(t-\tau)\Delta} \mathbf{P}\nabla \cdot (u_1 \otimes u_1)(\tau) d\tau \right\|_{FM} \geq \\
&\sum_{j,j'=1}^{\infty} \left| P_{0,1}\alpha_j a^1 + P_{0,1}\alpha_{j'} a^0 \right| c_j c_{j'} e^{-|k^0_{\alpha_j} + k^1_{\alpha_{j'}}|^2 t} \int_0^t e^{2k^0_{\alpha_j} \cdot k^1_{\alpha_{j'}} \tau} d\tau \geq \\
&\sum_{j=1}^{\infty} \left| P_{0,1} a^1 + P_{0,1} a^0 \right| \alpha_j (c_j)^2 e^{-|k^0_{\alpha_j} + k^1_{\alpha_j}|^2 t} \int_0^t e^{2k^0_{\alpha_j} \cdot k^1_{\alpha_j} \tau} d\tau = (RHS).
\end{aligned}$$

Since $e^{2k^0_{\alpha_j} \cdot k^1_{\alpha_j}} = e^{2{\alpha_j}^2} \geq 1$, $|k^0_{\alpha_j} + k^1_{\alpha_j}|^2 = 5{\alpha_j}^2$ and $|P_{0,1}a^1 + P_{0,1}a^0| = 11/5$, we have

$$(RHS) \geq (11/5) \sum_{j=1}^{\infty} (c_j)^2 \alpha_j e^{-5{\alpha_j}^2 t} t.$$

For each $k > 0$ the right hand side is estimated from below by $(11/5)(c_k)^2 t\alpha_k e^{-5\alpha_k^2 t}$. Thus

$$(11/5)\sum_{j=1}^{\infty}(c_j)^2\alpha_j e^{-5{\alpha_j}^2 t}t \geq (11/(25e))(c_k)^2/\alpha_k.$$

Sending k to ∞ we observe that $(c_k)^2/\alpha_k$ tend to the infinity by assumption so we conclude that $\|u(t)\|_{FM} \to \infty$ as $t \to \infty$. Thus we have the desired estimate. ∎

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