Slant Geometry of Spacelike Hypersurfaces in the Lightcone

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Abstract

In this paper, we consider one-parameter families of new extrinsic differential geometries on spacelike hypersurfaces in the lightcone. These geometries are constructed by applying two of one-parameter families of Legendrian dualities between pseudo-spheres in Lorentz-Minkowski space. These Legendrian dualities have been recently given as a part of extensions of the mandala of Legendrian dualities in the previous research of the authors.

1 Introduction

In this paper, we construct one-parameter families of new extrinsic differential geometries on spacelike hypersurfaces in the lightcone. It was shown in [4] that a simply connected Riemannian manifold N with $\dim N \geq 3$ is conformally flat if and only if it can be embedded as a spacelike hypersurface in the lightcone. It is clearly seen that if an extrinsic differential geometry on spacelike hypersurfaces in the lightcone is studied, then the extrinsic invariants of conformally flat Riemannian manifolds may be obtained. This is one of the main motivations for the study of spacelike hypersurfaces in the lightcone. The lightcone is one of the pseudo-spheres in Lorentz-Minkowski space. The other pseudo-spheres are de Sitter space and Hyperbolic space. Although there are a lot of researches on submanifolds in de Sitter space and Hyperbolic space [2, 6, 7, 8, 9, 10, 11, 14, 19], there are not so many results on submanifolds in the lightcone. Since the induced metric on the lightcone is degenerate, we cannot define the unit normal vector field by the ordinary arguments for a spacelike hypersurface in the lightcone. In order to avoid this difficulty, the basic duality theorem for four Legendrian double fibrations has been given in [12]. As an application of the basic duality theorem, an extrinsic differential geometry on spacelike hypersurfaces in the lightcone has been presented in [12]. We call this geometry a *lightcone flat geometry* in the lightcone. We remark that a geometry of spacelike hypersurfaces in the lightcone has been independently constructed by Liu [15, 16] using the different idea from [12]. The Gauss-Kronecker curvature related with the lightcone flat geometry is called *lightcone Gauss-Kronecker curvature.* In [12], extra two different curvatures which are called

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hyperbolic Gauss-Kronecker curvature and *de Sitter Gauss-Kronecker curvature* of a spacelike hypersurface in the lightcone have been introduced. Here, we call the geometries related with those curvatures a *hyperbolic flat geometry* and a *de Sitter flat geometry* in the lightcone, respectively.

On the other hand, the dualities in [12] have been generalized into pseudo-spheres in general semi-Euclidean space in [5] which are called the *mandala of Legendrian dualities for pseudo spheres.* These Legendrian dualities have been also extended for one-parameter families of pseudo-spheres in Lorentz-Minkowski space in [13]. There are some new applications of such Legendrian dualities. Some basic results on these new applications have been given in [13]. In this paper, as one of the applications of the extended mandala of Legendrian dualities, we construct one-parameter families of new extrinsic differential geometries on spacelike hypersurfaces in the lightcone which include the results of [12] as a special case. Moreover, we construct a *ϕ-de Sitter flat geometry* and a *ϕ-hyperbolic flat geometry* in the lightcone for $\phi \in [0, \pi/2]$. If $\phi = 0$, both of the geometries are equal to the lightcone flat geometry (i.e., the *horizontal geometry*). If $\phi = \pi/2$, the ϕ-de Sitter flat geometry is equal to the de Sitter flat geometry and the ϕ-hyperbolic flat geometry is equal to the hyperbolic flat geometry (i.e, the *vertical geometries*). Therefore, we call each of the ϕ-de Sitter flat geometry and the ϕ-hyperbolic flat geometry a *slant geometry* in the lightcone.

In this paper, we only construct the basic framework on the slant geometry in the lightcone from a contact view point. Other results of this new geometry in the lightcone will belong to the future research projects. Another applications of the extended mandala of Legendrian dualities for spacelike hypersurfaces in Hyperbolic space and de Sitter space will be appeared in the forthcoming paper [3].

2 Basic notions

In this section, we give some basic notions related with Lorentz-Minkowski space and the contact geometry. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R},\ i = 0, \dots, n\}$ be an $(n+1)$-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_0, y_1, \dots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$. The space $(\mathbb{R}^{n+1}, \langle , \rangle)$ is called *Lorentz-Minkowski* $(n+1)$*-space* and denoted by $\mathbb{R}^{n+1}_1$. We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0, respectively. The norm of a vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by $HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}$. We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike, respectively. In $\mathbb{R}^{n+1}_1$, we have three kinds of pseudo-spheres which are called *Hyperbolic n-space*, *de Sitter n-space* and the (*open*) *lightcone* and defined respectively by

$$H^n(-c^2) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 |\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -c^2\},$$
$$S^n_1(c^2) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = c^2\ \}$$

and

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\ \},$$

for any real number c. Instead of $S^n_1(1)$, we usually write S^n_1. For $\phi \in [0, \pi/2]$, we call $H^n(-\sin^2\phi)$ (respectively, $S^n_1(\sin^2\phi)$) a *ϕ-hyperbolic space* (respectively, *ϕ-de Sitter space*).

We consider a spacelike hypersurface $HL(\boldsymbol{n}, c)$ in the lightcone LC^* defined by

$$HL(\boldsymbol{n}, c) = HP(\boldsymbol{n}, c) \cap LC^*,$$

for $c \neq 0$. We say that $HL(\boldsymbol{n}, c)$ is a quadric hypersurface (or briefly, hyperquadric) in the lightcone. $HL(\boldsymbol{n}, c)$ is called elliptic, hyperbolic or parabolic if $\boldsymbol{n}$ is timelike, spacelike or lightlike, respectively. These hyperquadrics are the candidates of totally umbilic spacelike hypersurfaces in the lightcone (cf., [12]).

Now, we briefly review some properties of contact manifolds and Legendrian submanifolds. Let N be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on N. Locally, such a field is defined as the field of zeros of a 1-form α. The tangent hyperplane field K is *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of N. We say that (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case, K is called a *contact structure* and α is a *contact form.* Let $\phi : N \longrightarrow N'$ be a diffeomorphism between contact manifolds (N, K) and (N', K'). We say that ϕ is a *contact diffeomorphism* if $d\phi(K) = K'$. Two contact manifolds (N, K) and (N', K') are *contact diffeomorphic* if there exists a contact diffeomorphism $\phi : N \longrightarrow N'$. A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if dim $L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. A smooth fiber bundle $\pi : E \longrightarrow M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : E \longrightarrow M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \longrightarrow M$ is called a *Legendrian map.* The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i which is denoted by $W(L)$. Here, L is called the *Legendrian lift* of $W(L)$. For any $z \in E$, it is known that there is a local coordinate system $(x, y, p) = (x_1, \dots, x_m, y, p_1, \dots, p_m)$ around z such that $\pi(x, y, p) = (x, y)$ and the contact structure is given by the 1-form $\alpha = dy - \sum_{i=1}^m p_i dx_i$ (cf. [1] , 20.3).

Throughout our study, we are interested in the following three double fibrations which have been given in [12, 13].

(1) (a) $H^n(-1) \times S_1^n \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \}$,

(b) $\pi_{11} : \Delta_1 \longrightarrow H^n(-1)$, $\pi_{12} : \Delta_1 \longrightarrow S_1^n$,

(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_1$.

(2) (a) $LC^* \times S_1^n(\sin^2 \phi) \supset \Delta_{43}^{\pm}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \pm(\cos \phi + 1) \}$,

(b) $\pi[\phi]_{(43)1}^{\pm} : \Delta_{43}^{\pm}(\phi) \longrightarrow LC^*$, $\pi[\phi]_{(43)2}^{\pm} : \Delta_{43}^{\pm}(\phi) \longrightarrow S_1^n(\sin^2 \phi)$,

(c) $\theta[\phi]_{(43)1}^{\pm} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_{43}^{\pm}(\phi)$, $\theta[\phi]_{(43)2}^{\pm} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_{43}^{\pm}(\phi)$.

(3) (a) $H^n(-\sin^2 \phi) \times LC^* \supset \Delta_{42}^{\pm}(\phi) = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \pm(\cos \phi + 1) \}$,

(b) $\pi[\phi]_{(42)1}^{\pm} : \Delta_{42}^{\pm}(\phi) \longrightarrow H^n(-\sin^2 \phi)$, $\pi[\phi]_{(42)2}^{\pm} : \Delta_{42}^{\pm}(\phi) \longrightarrow LC^*$,

(c) $\theta[\phi]_{(42)1}^{\pm} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_{42}^{\pm}(\phi)$, $\theta[\phi]_{(42)2}^{\pm} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_{42}^{\pm}(\phi)$.

Here, $\pi_{11}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{12}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$, $\pi[\phi]_{(ij)1}^{\pm}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$ and $\pi[\phi]_{(ij)2}^{\pm}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$ for $(i, j) = (4, 2), (4, 3)$. Moreover, $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = -w_0 dv_0 + \sum_{i=1}^n w_i dv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = -v_0 dw_0 + \sum_{i=1}^n v_i dw_i$ are one-forms on $\mathbb{R}_1^{n+1} \times \mathbb{R}_1^{n+1}$. We remark that $\theta_{11}^{-1}(0)$ and $\theta_{12}^{-1}(0)$ (respectively, ${\theta[\phi]_{(ij)1}^{\pm}}^{-1}(0)$ and ${\theta[\phi]_{(ij)2}^{\pm}}^{-1}(0)$) define the same tangent hyperplane field denoted by K_1 (respectively, $K[\phi]_{ij}^{\pm}$) over Δ_1 (respectively, $\Delta_{ij}^{\pm}(\phi)$). In [13], the following theorem has been shown:

Theorem 2.1 *Under the same notations as those of the previous paragraph,* (Δ_1, K_1) *and* $(\Delta_{ij}^{\pm}(\phi), K[\phi]_{ij}^{\pm})$ $((i, j) = (4, 2), (4, 3))$ *are contact manifolds such that* π_{1k} *and* $\pi[\phi]_{(ij)k}^{\pm}$ $(k = 1, 2)$

are Legendrian fibrations. Moreover, these contact manifolds are contact diffeomorphic to each other.

This theorem is a part of the assertions of Theorem 3.2 in [13]. Actually, we also have contact manifolds $(\Delta^{\pm}_{ij}(\phi), K[\phi]^{\pm}_{ij})$ for $(i,j) = (1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),$ $(4,1),(4,2),(4,3)$ in [13]. We remark that $S^n_1(\sin^2 0) \setminus \{\mathbf{0}\} = H^n(-\sin^2 0) \setminus \{\mathbf{0}\} = LC^*$ and we write that $\Delta^{\pm}_4 = \Delta^{\pm}_{43}(0) = \Delta^{\pm}_{42}(0)$. Suppose that we have a Legendrian immersion $\mathcal{L}_{ij}[\phi] : U \longrightarrow \Delta^{\pm}_{ij}(\phi)$ with the form $\mathcal{L}_{ij}[\phi](u) = (L_1(u), L_2(u))$. Then we say that $L_1(u)$ and $L_2(u)$ are the $\Delta^{\pm}_{ij}(\phi)$-dual. Especially, we say that $L_2(u)$ is the ϕ-de Sitter dual of $L_1(u)$ if $L_1(u)$ and $L_2(u)$ are the $\Delta^{-}_{43}(\phi)$-dual and $L_1(u)$ is the ϕ-hyperbolic dual of $L_2(u)$ if $L_1(u)$ and $L_2(u)$ are the $\Delta^{-}_{42}(\phi)$-dual.

3 Slant geometry with respect to the ϕ-de Sitter duals

In this section, we establish a new extrinsic differential geometry on spacelike hypersurfaces in the lightcone with respect to the ϕ-de Sitter duals as an application of the extended mandala of Legendrian dualities. We call this geometry a *ϕ-de Sitter flat geometry*. It has been known that the induced metric on the lightcone is degenerate. So, we cannot apply ordinary submanifold theory of semi-Riemannian geometry. In this case, the Δ^{-}_4-duality is very useful. Let $\mathcal{L}_4 : U \longrightarrow \Delta^{-}_4$ be a Legendrian embedding with $\mathcal{L}_4(u) = (\boldsymbol{X}^{\ell}_{+}(u), \boldsymbol{X}^{\ell}_{-}(u))$ for an open subset $U \subset \mathbb{R}^{n-1}$. Assume that $\boldsymbol{X}^{\ell}_{+} : U \longrightarrow LC^*$ is a spacelike embedding. In [12], the Legendrian embedding $\mathcal{L}_4$ has been used for the construction of the extrinsic differential geometry on the spacelike hypersurface $M^L_{+} = \boldsymbol{X}^{\ell}_{+}(U)$ in the lightcone. It has been shown that for any spacelike embedding $\boldsymbol{X}^{\ell}_{+} : U \longrightarrow LC^*$, there exists a unique Legendrian embedding $\mathcal{L}_4 : U \longrightarrow \Delta^{-}_4$ such that $\pi_{41} \circ \mathcal{L}_4 = \boldsymbol{X}^{\ell}_{+}$. Since $\mathcal{L}_4$ is Legendrian, $\boldsymbol{X}^{\ell}_{-}(u)$ is a lightlike normal vector which is called *lightcone normal vector* of M^L_{+} at $p = \boldsymbol{X}^{\ell}_{+}(u)$. We also define the following two vector fields

$$\boldsymbol{X}^h(u) = \frac{\boldsymbol{X}^{\ell}_{-}(u) + \boldsymbol{X}^{\ell}_{+}(u)}{2} \quad \text{and} \quad \boldsymbol{X}^d(u) = \frac{\boldsymbol{X}^{\ell}_{-}(u) - \boldsymbol{X}^{\ell}_{+}(u)}{2}.$$

Then $\boldsymbol{X}^h(u) \in H^n(-1)$ and $\boldsymbol{X}^d(u) \in S^n_1$ such that $\mathcal{L}_1(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^d(u))$ gives a Legendrian embedding into Δ_1.

Let us consider the contact manifold $(\Delta^{-}_{43}(\phi), K[\phi]^{-}_{43})$ and the contact diffeomorphism $\Psi^{-}_{4(43)} : \Delta^{-}_4 \longrightarrow \Delta^{-}_{43}(\phi)$ defined by

$$\Psi^{-}_{4(43)}(\boldsymbol{v}, \boldsymbol{w}) = \left(\boldsymbol{v}, \frac{1}{2}\left((\cos\phi - 1)\boldsymbol{v} + (\cos\phi + 1)\boldsymbol{w}\right)\right).$$

Let us also define a map $\mathbb{N}^d_{\ell}[\phi] : U \longrightarrow S^n_1(\sin^2\phi)$ by

$$\mathbb{N}^d_{\ell}[\phi](u) = \frac{1}{2}((\cos\phi - 1)\boldsymbol{X}^{\ell}_{+}(u) + (\cos\phi + 1)\boldsymbol{X}^{\ell}_{-}(u))$$

and an embedding $\mathcal{L}_{43}[\phi] : U \longrightarrow \Delta^{-}_{43}(\phi)$ by

$$\mathcal{L}_{43}[\phi](u) = (\boldsymbol{X}^{\ell}_{+}(u), \mathbb{N}^d_{\ell}[\phi](u)),$$

for $\phi \in [0, \pi/2]$. Then we have $\mathcal{L}_{43}[\phi] = \Psi^{-}_{4(43)} \circ \mathcal{L}_4$, so that $\mathcal{L}_{43}[\phi]$ is a Legendrian embedding. Consequently, we get $\langle d\boldsymbol{X}^{\ell}_{+}, \mathbb{N}^d_{\ell}[\phi]\rangle = \mathcal{L}_{43}[\phi]^*\theta[\phi]^{-}_{(43)1} = 0$. This means that $\mathbb{N}^d_{\ell}[\phi](u)$ can be considered as a normal vector of M_+^L at $p = \boldsymbol{X}_+^\ell(u)$. We remark that $\mathbb{N}_\ell^d[0](u) = \boldsymbol{X}_-^\ell(u)$ and $\mathbb{N}_\ell^d[\pi/2](u) = \boldsymbol{X}^d(u)$. If we have another Legendrian embedding

$$\mathcal{L}_{43}^1[\phi] : U \longrightarrow \Delta_{43}^-(\phi)$$

defined by $\mathcal{L}_{43}^1[\phi](u) = (\boldsymbol{X}_+^\ell(u),\ \mathbb{N}_\ell^{d1}[\phi](u))$, then $\mathbb{N}_\ell^d[\phi](u)$ and $\mathbb{N}_\ell^{d1}[\phi](u)$ are parallel. However, we obtain the relations $\langle \boldsymbol{X}_+^\ell(u), \mathbb{N}_\ell^d[\phi](u)\rangle = \langle \boldsymbol{X}_+^\ell(u), \mathbb{N}_\ell^{d1}[\phi](u)\rangle = -(\cos\phi + 1)$, so that $\mathbb{N}_\ell^{d1}[\phi](u) = \mathbb{N}_\ell^d[\phi](u)$. This means that $\mathcal{L}_{43}[\phi]$ is the unique Legendrian lift of $\boldsymbol{X}_+^\ell(u)$. Hence, $\mathbb{N}_\ell^d[\phi]$ is the *ϕ-de Sitter dual* of $\boldsymbol{X}_+^\ell(U) = M_+^L$.

We define a family of functions

$$H_\phi^d : U \times S_1^n(\sin^2\phi) \longrightarrow \mathbb{R}$$

by $H_\phi^d(u, \boldsymbol{v}) = \langle \boldsymbol{X}_+^\ell(u), \boldsymbol{v}\rangle + \cos\phi + 1$. We call H_ϕ^d a *ϕ-de Sitter height function* on $\boldsymbol{X}_+^\ell : U \longrightarrow LC^*$. Since $\boldsymbol{X}_+^\ell$ is a spacelike embedding and $\boldsymbol{X}_+^\ell(u)$ and $\boldsymbol{X}_-^\ell(u)$ are linearly independent lightlike vectors,

$$\left\{\boldsymbol{X}_+^\ell(u),\ \boldsymbol{X}_-^\ell(u),\ \boldsymbol{X}_{+u_1}^\ell(u), ..., \boldsymbol{X}_{+u_{n-1}}^\ell(u)\right\}$$

is a basis of $T_p\mathbb{R}_1^{n+1}$ for $p = \boldsymbol{X}_+^\ell(u)$.

Proposition 3.1 *Let $H_\phi^d : U \times S_1^n(\sin^2\phi) \longrightarrow \mathbb{R}$ be a ϕ-de Sitter height function on $\boldsymbol{X}_+^\ell : U \longrightarrow LC^*$. Then*
(1) $H_\phi^d(u, \boldsymbol{v}) = 0$ if and only if $(\boldsymbol{X}_+^\ell(u), \boldsymbol{v}) \in \Delta_{43}^-(\phi)$.
(2) $H_\phi^d(u, \boldsymbol{v}) = \dfrac{\partial H_\phi^d}{\partial u_i}(u, \boldsymbol{v}) = 0$ $(i = 1, ..., n-1)$ if and only if $\boldsymbol{v} = \mathbb{N}_\ell^d[\phi](u)$.

Proof. The assertion (1) follows from the definition of H_ϕ^d and $\Delta_{43}^-(\phi)$.
(2) There exist real numbers λ, μ, ξ_j $(j = 1, ..., n-1)$ such that $\boldsymbol{v} = \lambda\boldsymbol{X}_+^\ell + \mu\boldsymbol{X}_-^\ell + \sum_{j=1}^{n-1}\xi_j\boldsymbol{X}_{+u_j}^\ell$. Since $\left\langle \boldsymbol{X}_+^\ell, \boldsymbol{X}_+^\ell\right\rangle = 0$, we obtain $\left\langle \boldsymbol{X}_+^\ell, \boldsymbol{X}_{+u_j}^\ell\right\rangle = 0$. Consequently, $0 = H_\phi^d(u, \boldsymbol{v}) = \left\langle \boldsymbol{X}_+^\ell, \mu\boldsymbol{X}_-^\ell\right\rangle + \cos\phi + 1 = -2\mu + \cos\phi + 1$ if and only if $\mu = \frac{\cos\phi+1}{2}$. Since $\frac{\partial H_\phi^d}{\partial u_i}(u, \boldsymbol{v}) = \left\langle \boldsymbol{X}_{+u_i}^\ell, \boldsymbol{v}\right\rangle$, we have $0 = \left\langle \boldsymbol{X}_{+u_i}^\ell, \frac{\cos\phi+1}{2}\boldsymbol{X}_-^\ell\right\rangle + \sum_{j=1}^{n-1}\xi_j g_{ij}(u)$. The equation $\left\langle d\boldsymbol{X}_+^\ell, \boldsymbol{X}_-^\ell\right\rangle = 0$ means that $\left\langle \boldsymbol{X}_{+u_i}^\ell, \boldsymbol{X}_-^\ell\right\rangle = 0$. For this reason, it follows that $\sum_{j=1}^{n-1}\xi_j g_{ij}(u) = 0$. Since g_{ij} is positive definite, we get $\xi_j = 0$ $(j = 1, ..., n-1)$. Moreover, since we have $\sin^2\phi = \langle \boldsymbol{v}, \boldsymbol{v}\rangle = \lambda(\cos\phi + 1)\left\langle \boldsymbol{X}_+^\ell, \boldsymbol{X}_-^\ell\right\rangle = -2(\cos\phi + 1)\lambda$, it is obtained that $\lambda = \frac{\cos\phi - 1}{2}$. Thus, the proof is completed. □

Now, we study the extrinsic differential geometry of $\boldsymbol{X}_+^\ell$ by using $\mathbb{N}_\ell^d[\phi]$ like as the Gauss map of a hypersurface in Euclidean space. For our purpose, we have the following fundamental lemma:

Lemma 3.2 *For any $p = \boldsymbol{X}_+^\ell(u_0) \in M_+^L$ and $\boldsymbol{v} \in T_pM_+^L$, we have $D_v\mathbb{N}_\ell^d[\phi](u_0) \in T_pM_+^L$. Here, D_v denotes the covariant derivative with respect to the tangent vector $\boldsymbol{v}$.*

Proof. It has been proved in [12] that $D_v\boldsymbol{X}^{\ell}_{-}(u_0) \in T_pM^L_+$. Since

$$D_v\mathbb{N}^d_\ell[\phi](u_0) = \frac{1}{2}\left((\cos\phi - 1)\, D_v\boldsymbol{X}^{\ell}_{+}(u_0) + (\cos\phi + 1)\, D_v\boldsymbol{X}^{\ell}_{-}(u_0)\right),$$

$D_v\mathbb{N}^d_\ell[\phi](u_0) \in T_pM^L_+$. □

Under the identification of U and M^L_+ through the embedding $\boldsymbol{X}^{\ell}_{+}$, the derivative $d\boldsymbol{X}^{\ell}_{+}(u_0)$ is the identity mapping $\mathrm{id}_{T_pM^L_+}$ on $T_pM^L_+$, where $p = \boldsymbol{X}^{\ell}_{+}(u_0)$. Moreover by Lemma 3.2, $d\mathbb{N}^d_\ell[\phi](u_0)$ can be considered as a linear transformation on the tangent space $T_pM^L_+$. We have the following relation:

$$d\mathbb{N}^d_\ell[\phi](u_0) = \frac{1}{2}(\cos\phi - 1)\, id_{T_pM^L_+} + \frac{1}{2}(\cos\phi + 1)\, d\boldsymbol{X}^{\ell}_{-}(u_0).$$

We call the linear transformations $S^d[\phi]\,(p) = -d\mathbb{N}^d_\ell[\phi](p) : T_pM^L_+ \longrightarrow T_pM^L_+$ and $S^{\ell}_{-}(p) = -d\boldsymbol{X}^{\ell}_{-}(p) : T_pM^L_+ \longrightarrow T_pM^L_+$, the *$\phi$-de Sitter shape operator* and the *lightcone shape operator*, respectively. We denote the eigenvalues of $S^d[\phi]\,(p)$ and $S^{\ell}_{-}(p)$ by $\overline{\kappa}^d[\phi]\,(p)$ and $\kappa^{\ell}_{-}(p)$, respectively. Because of the relation $S^d[\phi]\,(p) = -\frac{1}{2}(\cos\phi - 1)\, id_{T_pM^L_+} + \frac{1}{2}(\cos\phi + 1)\, S^{\ell}_{-}(p)$, $S^d[\phi]\,(p)$ and $S^{\ell}_{-}(p)$ have the common eigen vectors. As a result, we get a relation $\overline{\kappa}^d[\phi]\,(p) = -\frac{1}{2}(\cos\phi - 1) + \frac{1}{2}(\cos\phi + 1)\,\kappa^{\ell}_{-}(p)$. We call $\overline{\kappa}^d[\phi]\,(p)$ and $\kappa^{\ell}_{-}(p)$, a *ϕ-de Sitter principal curvature* and a *lightcone principal curvature* of $M^L_+ = \boldsymbol{X}^{\ell}_{+}(U)$ at $p = \boldsymbol{X}^{\ell}_{+}(u_0)$, respectively. We also give the following definitions of the curvatures of $M^L_+ = \boldsymbol{X}^{\ell}_{+}(U)$ at $p = \boldsymbol{X}^{\ell}_{+}(u_0)$:

$K^d_\ell[\phi]\,(u_0) = \det S^d[\phi]\,(p)$; *$\phi$-de Sitter Gauss-Kronecker curvature,*

$H^d_\ell[\phi]\,(u_0) = \dfrac{1}{n-1}\mathrm{Trace}\, S^d[\phi]\,(p)$; *$\phi$-de Sitter mean curvature.*

We also define the *lightcone Gauss-Kronecker curvature* and the *lightcone mean curvature* of $M^L_+ = \boldsymbol{X}^{\ell}_{+}(U)$ at $p = \boldsymbol{X}^{\ell}_{+}(u_0)$ by $K^{\ell}_{-}(p) = \det S^{\ell}_{-}(p)$ and $H^{\ell}_{-}(p) = \frac{1}{n-1}\mathrm{Trace}\, S^{\ell}_{-}(p)$, respectively.

Since $\boldsymbol{X}^{\ell}_{+u_i}$ $(i = 1, ..., n-1)$ are spacelike vectors, the induced Riemannian metric (the *first fundamental form*) on $M^L_+ = \boldsymbol{X}^{\ell}_{+}(U)$ is given by $ds^2 = \sum_{i,j=1}^{n-1} g_{ij}du_idu_j$, where $g_{ij}(u) = \left\langle \boldsymbol{X}^{\ell}_{+u_i}(u), \boldsymbol{X}^{\ell}_{+u_j}(u)\right\rangle$ for any $u \in U$. We also define the *ϕ-de Sitter second fundamental invariant* by $h^d[\phi]_{ij}(u) = \left\langle -\left(\mathbb{N}^d_\ell[\phi]\right)_{u_i}(u), \boldsymbol{X}^{\ell}_{+u_j}(u)\right\rangle$ for any $u \in U$. If we denote $h^{\ell}_{-ij}(u) = \left\langle -\boldsymbol{X}^{\ell}_{-u_i}(u), \boldsymbol{X}^{\ell}_{+u_j}(u)\right\rangle$, then we have the following relation:

$$h^d[\phi]_{ij}(u) = -\frac{1}{2}(\cos\phi - 1)g_{ij}(u) + \frac{1}{2}(\cos\phi + 1)h^{\ell}_{-ij}(u).$$

Proposition 3.3 *Under the above notations, we have the following ϕ-de Sitter Weingarten formula:*

$$\left(\mathbb{N}^d_\ell[\phi]\right)_{u_i} = -\sum_{j=1}^{n-1} h^d[\phi]^j_i \boldsymbol{X}^{\ell}_{+u_j},$$

where $\left(h^d[\phi]^j_i\right) = \left(h^d[\phi]_{ik}\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = \left(g_{kj}\right)^{-1}$.

Proof. By Lemma 3.2, there exist real numbers Γ^j_i such that

$$\left(\mathbb{N}^d_\ell[\phi]\right)_{u_i} = \sum_{j=1}^{n-1} \Gamma^j_i \boldsymbol{X}^{\ell}_{+u_j}.$$

By definition, we have

$$-h^d[\phi]_{i\beta} = \sum_{\alpha=1}^{n-1} \Gamma_i^\alpha \left\langle \boldsymbol{X}^\ell_{+u_\alpha}, \boldsymbol{X}^\ell_{+u_\beta} \right\rangle = \sum_{\alpha=1}^{n-1} \Gamma_i^\alpha g_{\alpha\beta}.$$

Hence, we get

$$-h^d[\phi]_i^j = -\sum_{\beta=1}^{n-1} h^d[\phi]_{i\beta} g^{\beta j} = \sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{n-1} \Gamma_i^\alpha g_{\alpha\beta} g^{\beta j} = \Gamma_i^j.$$

This completes the proof of the ϕ-de Sitter Weingarten formula. $\square$

As a corollary of the above proposition, we obtain an explicit expression of the ϕ-de Sitter Gauss-Kronecker curvature by Riemannian metric and the ϕ-de Sitter second fundamental invariant.

Corollary 3.4 *Under the same notations as in the above proposition, the ϕ-de Sitter Gauss-Kronecker curvature is given by*

$$K^d_\ell[\phi] = \frac{\det\left(h^d[\phi]_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}.$$

Proof. By the ϕ-de Sitter Weingarten formula, the representation matrix of the ϕ-de Sitter shape operator with respect to the basis $\left\{\boldsymbol{X}^\ell_{+u_1}(u), ..., \boldsymbol{X}^\ell_{+u_{n-1}}(u)\right\}$ is $\left(h^d[\phi]_i^j\right) = \left(h^d[\phi]_{i\beta}\right)\left(g^{\beta j}\right)$. It is obvious from this fact that

$$K^d_\ell[\phi] = \det S^d[\phi] = \det\left(h^d[\phi]_i^j\right) = \det\left(\left(h^d[\phi]_{i\beta}\right)\left(g^{\beta j}\right)\right) = \frac{\det\left(h^d[\phi]_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}.$$

$\square$

It has been given in [12] that a point $u \in U$ or $p = \boldsymbol{X}^\ell_+(u)$ is a *lightcone umbilic point* if $S^\ell_-(p) = \kappa^\ell_-(p)\, id_{T_pM^L_+}$. Since $S^d[\phi](p)$ and $S^\ell_-(p)$ have the common eigenvectors, we say that a point $u \in U$ or $p = \boldsymbol{X}^\ell_+(u)$ is an *umbilic point* if it is a lightcone umbilic point which is equivalent to the condition $S^d[\phi](p) = \overline{\kappa}^d[\phi](p)\, id_{T_pM^L_+}$. We also say that $M^L_+ = \boldsymbol{X}^\ell_+(U)$ is totally umbilic if all points of M^L_+ are umbilic. Totally umbilic spacelike hypersurfaces have been classified in [12] by using the lightcone principal curvature. Here, we give a classification of totally umbilic spacelike hypersurfaces by using the ϕ-de Sitter principal curvature.

Proposition 3.5 *Suppose that $M^L_+ = \boldsymbol{X}^\ell_+(U)$ is totally umbilic and fix $\phi \in \left[0, \frac{\pi}{2}\right]$. Then $\overline{\kappa}^d[\phi](p)$ is constant $\overline{\kappa}^d[\phi]$. Under this condition, we have the following classifications:*

(1) *If $\overline{\kappa}^d[\phi] < 0$, then M^L_+ is a part of hyperbolic hyperquadric $HL(\boldsymbol{c}, -(\cos\phi + 1))$.*

(2) *If $\overline{\kappa}^d[\phi] = 0$:*

 (i) *If $\phi = 0$, then M^L_+ is a part of parabolic hyperquadric $HL(\boldsymbol{c}, -2)$.*

 (ii) *If $\phi \neq 0$, then M^L_+ is a part of hyperbolic hyperquadric $HL(\boldsymbol{c}, -(\cos\phi + 1))$.*

(3) *If $\overline{\kappa}^d[\phi] > 0$:*

 (i) *If $\phi = 0$, then M^L_+ is a part of elliptic hyperquadric $HL(\boldsymbol{c}, -2)$.*

 (ii) *If $\phi \neq 0$:*

(a) *If* $2\overline{\kappa}^d[\phi](\cos\phi+1)<\sin^2\phi$, *then* M_+^L *is a part of hyperbolic hyperquadric*

$$HL(\boldsymbol{c},-(\cos\phi+1)).$$

(b) *If* $2\overline{\kappa}^d[\phi](\cos\phi+1)=\sin^2\phi$, *then* M_+^L *is a part of parabolic hyperquadric*

$$HL(\boldsymbol{c},-(\cos\phi+1)).$$

(c) *If* $2\overline{\kappa}^d[\phi](\cos\phi+1)>\sin^2\phi$, *then* M_+^L *is a part of elliptic hyperquadric*

$$HL(\boldsymbol{c},-(\cos\phi+1)).$$

Proof. By definition, we get $S^d[\phi]\left(\boldsymbol{X}^\ell_{+u_i}\right)=-\left(\mathbb{N}^d_\ell[\phi]\right)_{u_i}=\overline{\kappa}^d[\phi]\boldsymbol{X}^\ell_{+u_i}$ for $i=1,...,n-1$. It follows that

$$-\left(\mathbb{N}^d_\ell[\phi]\right)_{u_iu_j}=\left(\overline{\kappa}^d_\phi\right)_{u_j}\boldsymbol{X}^\ell_{+u_i}+\overline{\kappa}^d_\phi\boldsymbol{X}^\ell_{+u_iu_j}.$$

On the other hand, we can write that $S^d[\phi]\left(\boldsymbol{X}^\ell_{+u_j}\right)=-\left(\mathbb{N}^d_\ell[\phi]\right)_{u_j}=\overline{\kappa}^d[\phi]\boldsymbol{X}^\ell_{+u_j}$ for $j=1,...,n-1$. Therefore, we have

$$-\left(\mathbb{N}^d_\ell[\phi]\right)_{u_ju_i}=\left(\overline{\kappa}^d[\phi]\right)_{u_i}\boldsymbol{X}^\ell_{+u_j}+\overline{\kappa}^d[\phi]\boldsymbol{X}^\ell_{+u_ju_i}.$$

Since $\left(\mathbb{N}^d_\ell[\phi]\right)_{u_iu_j}=\left(\mathbb{N}^d_\ell[\phi]\right)_{u_ju_i}$ and $\overline{\kappa}^d_\phi\boldsymbol{X}^\ell_{+u_iu_j}=\overline{\kappa}^d_\phi\boldsymbol{X}^\ell_{+u_ju_i}$, we obtain

$$\left(\overline{\kappa}^d[\phi]\right)_{u_j}\boldsymbol{X}^\ell_{+u_i}=\left(\overline{\kappa}^d[\phi]\right)_{u_i}\boldsymbol{X}^\ell_{+u_j}.$$

By definition, $\left\{\boldsymbol{X}^\ell_{+u_1},...,\boldsymbol{X}^\ell_{+u_{n-1}}\right\}$ is linearly independent. Hence, $\overline{\kappa}^d_\phi$ is constant. Since $\overline{\kappa}^d_\phi$ is constant and $d\mathbb{N}^d_\ell[\phi]=-\overline{\kappa}^d[\phi]d\boldsymbol{X}^\ell_+$, it is obvious that $d\left(\mathbb{N}^d_\ell[\phi]+\overline{\kappa}^d[\phi]\boldsymbol{X}^\ell_+\right)=0$. Consequently, there is a constant vector $\boldsymbol{c}$ such that

$$\begin{aligned}\boldsymbol{c}&=\mathbb{N}^d_\ell[\phi]\,(u)+\overline{\kappa}^d[\phi]\boldsymbol{X}^\ell_+\,(u)\\&=\left(\frac{1}{2}(\cos\phi-1)+\overline{\kappa}^d[\phi]\right)\boldsymbol{X}^\ell_+\,(u)+\frac{1}{2}(\cos\phi+1)\boldsymbol{X}^\ell_-(u).\end{aligned}$$

It is obvious that $\langle\boldsymbol{c},\boldsymbol{c}\rangle=\sin^2\phi-2\,(\cos\phi+1)\,\overline{\kappa}^d[\phi]$ and $\left\langle\boldsymbol{X}^\ell_+\,(u)\,,\boldsymbol{c}\right\rangle=-(\cos\phi+1)$. Here if $\phi=0$, then $\boldsymbol{c}=\overline{\kappa}^d[\phi]\boldsymbol{X}^\ell_+\,(u)+\boldsymbol{X}^\ell_-(u)$, $\langle\boldsymbol{c},\boldsymbol{c}\rangle=-4\overline{\kappa}^d_\phi$ and $\left\langle\boldsymbol{X}^\ell_+\,(u)\,,\boldsymbol{c}\right\rangle=-2$. Moreover, $\left\langle\boldsymbol{X}^\ell_+\,(u)\,,\boldsymbol{c}\right\rangle<0$ for $\phi\in[0,\pi/2]$. Now, we can write the following classifications:
(1) We suppose that $\overline{\kappa}^d[\phi]<0$: In this case, $\langle\boldsymbol{c},\boldsymbol{c}\rangle>0$ for $\phi\in[0,\pi/2]$. So, we have $M_+^L\subset HL\,(\boldsymbol{c},-(\cos\phi+1))$.
(2) We suppose that $\overline{\kappa}^d[\phi]=0$: By definition, we obtain $d\mathbb{N}^d_\ell[\phi]=0$, so that $\boldsymbol{c}=\mathbb{N}^d_\ell[\phi]\,(u)$ is constant. In this case, we also have $\langle\boldsymbol{c},\boldsymbol{c}\rangle=\sin^2\phi$ for $\phi\in[0,\pi/2]$. Here if $\phi=0$, then $\langle\boldsymbol{c},\boldsymbol{c}\rangle=0$. And also if $\phi\neq0$, then $\langle\boldsymbol{c},\boldsymbol{c}\rangle>0$. So, we get the following two classifications:
(i) If $\phi=0$, then $M_+^L\subset HL\,(\boldsymbol{c},-2)$.
(ii) If $\phi\neq0$, then $M_+^L\subset HL\,(\boldsymbol{c},-(\cos\phi+1))$.
(3) We suppose that $\overline{\kappa}^d[\phi]>0$: In this case, we obtain the following two classifications:
(i) If $\phi=0$, then $\langle\boldsymbol{c},\boldsymbol{c}\rangle<0$. So, $M_+^L\subset HL\,(\boldsymbol{c},-2)$.
(ii) If $\phi\neq0$, there are three classifications:
(a) If $2\,(\cos\phi+1)\,\overline{\kappa}^d[\phi]<\sin^2\phi$, then $\langle\boldsymbol{c},\boldsymbol{c}\rangle>0$. Thus, $M_+^L\subset HL\,(\boldsymbol{c},-(\cos\phi+1))$.

(b) If $2\,(\cos\phi+1)\,\overline{\kappa}^d[\phi]=\sin^2\phi$, then $\langle \boldsymbol{c},\boldsymbol{c}\rangle=0$. Hence, $M_+^L\subset HL\,(\boldsymbol{c},-(\cos\phi+1))$.

(c) If $2\,(\cos\phi+1)\,\overline{\kappa}^d[\phi]>\sin^2\phi$, then $\langle \boldsymbol{c},\boldsymbol{c}\rangle<0$. So, $M_+^L\subset HL\,(\boldsymbol{c},-(\cos\phi+1))$.

This completes the proof. □

In the above classification, for the case $\overline{\kappa}^d[\phi]=0$ the hyperquadric $HL(\boldsymbol{c},-(\cos\phi+1))$ ($\phi\in[0,\pi/2]$) plays the similar roles with a hyperplane in Euclidean space. We call it a *ϕ-de Sitter flat hyperquadric.* We say that $p=\boldsymbol{X}_+^\ell(u)$ is a *lightcone parabolic point* if $K_-^\ell(p)=0$ and a *lightcone flat point* if it is an umbilic point and $K_-^\ell(p)=0$ (cf., [12]). We also say that $p=\boldsymbol{X}_+^\ell(u)$ is a *ϕ-de Sitter parabolic point* if $K_\ell^d[\phi]\,(u)=0$ and a *ϕ-de Sitter flat point* if it is an umbilic point and $K_\ell^d[\phi]\,(u)=0$ which are equivalent to the condition that one of the conditions (2)(i) or (2)(ii) is satisfied.

On the other hand, we denote the Hessian matrix of the ϕ-de Sitter height function $h^d_{\phi,v_0}(u)=H^d_\phi\,(u,\boldsymbol{v}_0)$ at u_0 by $\mathrm{Hess}\left(h^d_{\phi,v_0}\right)(u_0)$.

Proposition 3.6 *Let $\boldsymbol{X}_+^\ell : U\longrightarrow LC^*$ be a spacelike hypersurface in the lightcone and $\boldsymbol{v}_0=\mathbb{N}^d_\ell[\phi](u_0)$. Then we have the following:*

(1) $p=\boldsymbol{X}_+^\ell(u_0)$ is a parabolic point if and only if $\det\mathrm{Hess}\left(h^d_{\phi,v_0}\right)(u_0)=0$.

(2) $p=\boldsymbol{X}_+^\ell(u_0)$ is a flat point if and only if $\mathrm{rank}\,\mathrm{Hess}\left(h^d_{\phi,v_0}\right)(u_0)=0$.

Proof. By definition, we have $h^d_{\phi,v_0}(u_0)=\left\langle \boldsymbol{X}_+^\ell(u_0),\boldsymbol{v}_0\right\rangle+\cos\phi+1$. Using this equation, we get

$$\begin{aligned}
\frac{\partial^2 h^d_{\phi,v_0}}{\partial u_i\partial u_j}(u_0) &= \left\langle \boldsymbol{X}^\ell_{+u_iu_j}(u_0),\boldsymbol{v}_0\right\rangle\\
&= \frac{1}{2}(\cos\phi-1)\left\langle \boldsymbol{X}^\ell_{+u_iu_j}(u_0),\boldsymbol{X}^\ell_{+}(u_0)\right\rangle+\frac{1}{2}(\cos\phi+1)\left\langle \boldsymbol{X}^\ell_{+u_iu_j}(u_0),\boldsymbol{X}^\ell_{-}(u_0)\right\rangle\\
&= -\frac{1}{2}(\cos\phi-1)\left\langle \boldsymbol{X}^\ell_{+u_i}(u_0),\boldsymbol{X}^\ell_{+u_j}(u_0)\right\rangle-\frac{1}{2}(\cos\phi+1)\left\langle \boldsymbol{X}^\ell_{+u_j}(u_0),\boldsymbol{X}^\ell_{-u_i}(u_0)\right\rangle\\
&= -\frac{1}{2}(\cos\phi-1)g_{ij}(u_0)+\frac{1}{2}(\cos\phi+1)h^\ell_{-ij}(u_0)\\
&= h^d[\phi]_{ij}(u_0).
\end{aligned}$$

This means that $\mathrm{Hess}\left(h^d_{\phi,v_0}\right)(u_0)=\left(h^d[\phi]_{ij}(u_0)\right)$. Hence, we obtain

$$K^d_\ell[\phi](u_0)=\frac{\det\left(h^d[\phi]_{ij}(u_0)\right)}{\det\left(g_{\alpha\beta}(u_0)\right)}=\frac{\det\mathrm{Hess}\left(h^d_{\phi,v_0}\right)(u_0)}{\det\left(g_{\alpha\beta}(u_0)\right)}.$$

As a result, the first assertion follows from this formula.

For the second assertion, by the ϕ-de Sitter Weingarten formula, $p=\boldsymbol{X}_+^\ell(u_0)$ is an umbilic point if and only if there exists an orthogonal matrix A such that $A^t\left(h^d[\phi]^\alpha_i\right)A=\overline{\kappa}^d[\phi]I$. Therefore, we have

$$\left(h^d[\phi]^\alpha_i\right)=A\,\overline{\kappa}^d[\phi]\,A^t=\overline{\kappa}^d[\phi]I,$$

so that

$$\mathrm{Hess}\left(h^d_{\phi,v_0}\right)=\left(h^d[\phi]_{ij}\right)=\left(h^d[\phi]^\alpha_i\right)(g_{\alpha j})=\overline{\kappa}^d[\phi]\,(g_{ij})\,.$$

Thus, p is a flat point (i.e., $\overline{\kappa}^d[\phi](u_0)=0$) if and only if $\mathrm{rank}\,\mathrm{Hess}\left(h^d_{\phi,v_0}\right)(u_0)=0$. □

Now, we consider the other curvatures of $M_+^L = \boldsymbol{X}_+^\ell(U)$. Let $\Psi^-_{(43)1} : \Delta^-_{43}(\phi) \longrightarrow \Delta_1$ be a diffeomorphism defined by

$$\Psi^-_{(43)1}(\boldsymbol{v},\boldsymbol{w}) = \frac{1}{\cos\phi + 1}(\boldsymbol{v}+\boldsymbol{w}, -\cos\phi\boldsymbol{v}+\boldsymbol{w}).$$

Then we can calculate that

$$\begin{aligned}
(\Psi^-_{(43)1})^*\theta_{11} &= \frac{1}{\cos\phi+1}\langle d\left(\boldsymbol{v}+\boldsymbol{w}\right), -\cos\phi\boldsymbol{v}+\boldsymbol{w}\rangle|\Delta^-_{43}(\phi)\\
&= \frac{1}{\cos\phi+1}\left(\langle d\boldsymbol{v},\boldsymbol{w}\rangle - \cos\phi\left\langle d\boldsymbol{w},\boldsymbol{v}\right\rangle\right)|\Delta^-_{43}(\phi)\\
&= \langle d\boldsymbol{v},\boldsymbol{w}\rangle|\Delta^-_{43}(\phi)\\
&= \theta[\phi]^-_{(43)1},
\end{aligned}$$

so that $\Psi^-_{(43)1}$ is a contact diffeomorphism. Consequently, we have a Legendrian embedding

$$\widetilde{\mathcal{L}}_1 : U \longrightarrow \Delta_1$$

defined by $\widetilde{\mathcal{L}}_1(u) = \Psi^-_{(43)1}\circ\mathcal{L}_{43}[\phi](u)$. If we denote that $\widetilde{\mathcal{L}}_1(u) = (\boldsymbol{X}_1(u), \boldsymbol{X}_2(u))$, then we get

$$\boldsymbol{X}_1(u) = \frac{1}{\cos\phi+1}\left(\boldsymbol{X}_+^\ell(u) + \mathbb{N}_\ell^d[\phi](u)\right) = \frac{1}{2}\left(\boldsymbol{X}_+^\ell(u) + \boldsymbol{X}_-^\ell(u)\right) = \boldsymbol{X}^h(u)$$

and

$$\boldsymbol{X}_2(u) = \frac{1}{\cos\phi+1}\left(-\cos\phi\boldsymbol{X}_+^\ell(u) + \mathbb{N}_\ell^d[\phi](u)\right) = \frac{1}{2}\left(-\boldsymbol{X}_+^\ell(u) + \boldsymbol{X}_-^\ell(u)\right) = \boldsymbol{X}^d(u).$$

So, we obtain $\widetilde{\mathcal{L}}_1(u) = (\boldsymbol{X}^h(u), \boldsymbol{X}^d(u)) = \mathcal{L}_1(u)$. As a consequence of Lemma 3.4 in [12], we can define the *hyperbolic shape operator* of $M_+^L = \boldsymbol{X}_+^\ell(U)$ at $p = \boldsymbol{X}_+^\ell(u_0)$ by

$$S^H(p) = -d\boldsymbol{X}^h(u_0) : T_pM_+^L \longrightarrow T_pM_+^L$$

and the *de Sitter shape operator* of $M_+^L = \boldsymbol{X}_+^\ell(U)$ at $p = \boldsymbol{X}_+^\ell(u_0)$ by

$$S^D(p) = -d\boldsymbol{X}^d(u_0) : T_pM_+^L \longrightarrow T_pM_+^L.$$

Moreover, we can define the other curvatures of $M_+^L = \boldsymbol{X}_+^\ell(U)$ at $p = \boldsymbol{X}_+^\ell(u_0)$ as follows:

$K^H(u_0) = \det S^H(p)$; the *hyperbolic Gauss-Kronecker curvature,*

$K^D(u_0) = \det S^D(p)$; the *de Sitter Gauss-Kronecker curvature,*

$H^H(u_0) = \dfrac{1}{n-1}\mathrm{Trace}\, S^H(p)$; the *hyperbolic mean curvature,*

$H^D(u_0) = \dfrac{1}{n-1}\mathrm{Trace}\, S^D(p)$; the *de Sitter mean curvature.*

We also have the following expressions of the hyperbolic Gauss-Kronecker curvature and the de Sitter Gauss-Kronecker curvature as a corollary of Proposition 3.3.

Proposition 3.7 *Under the same notations in Corollary 3.4, we have the following formulas:*

(1) $K^H = \dfrac{1}{(\cos\phi+1)^{n-1}}\dfrac{\det\left(h^d[\phi]_{ij} - g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)} = \dfrac{1}{2^{n-1}}\dfrac{\det\left(h^\ell_{-ij} - g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}$,

(2) $K^D = \dfrac{1}{(\cos\phi+1)^{n-1}}\dfrac{\det\left(h^d[\phi]_{ij} + \cos\phi g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)} = \dfrac{1}{2^{n-1}}\dfrac{\det\left(h^\ell_{-ij} + g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}$.

Proof. Since

$$\boldsymbol{X}^h(u)=\frac{1}{\cos\phi+1}\left(\boldsymbol{X}^\ell_+(u)+\mathbb{N}^d_\ell[\phi](u)\right)=\frac{1}{2}\left(\boldsymbol{X}^\ell_+(u)+\boldsymbol{X}^\ell_-(u)\right),$$

we have

$$\left(\boldsymbol{X}^h\right)_{u_i}=\sum_{j=1}^{n-1}\frac{\left(\delta^j_i-h^d[\phi]^j_i\right)}{\cos\phi+1}\boldsymbol{X}^\ell_{+u_j}=\sum_{j=1}^{n-1}\frac{\left(\delta^j_i-\left(h^\ell_-\right)^j_i\right)}{2}\boldsymbol{X}^\ell_{+u_j}.$$

Hence, it follows that

$$\begin{aligned}
K^H &= \det\left(\frac{\left(h^d[\phi]^j_i-\delta^j_i\right)}{\cos\phi+1}\right)\\
&= \det\left(\left(\frac{h^d[\phi]_{i\beta}-g_{i\beta}}{\cos\phi+1}\right)\left(g^{\beta j}\right)\right)\\
&= \frac{1}{\left(\cos\phi+1\right)^{n-1}}\frac{\det\left(h^d[\phi]_{ij}-g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}
\end{aligned}$$

and

$$\begin{aligned}
K^H &= \det\left(\frac{\left(h^\ell_-\right)^j_i-\delta^j_i}{2}\right)\\
&= \det\left(\left(\frac{h^\ell_{-i\beta}-g_{i\beta}}{2}\right)\left(g^{\beta j}\right)\right)\\
&= \frac{1}{2^{n-1}}\frac{\det\left(h^\ell_{-ij}-g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}.
\end{aligned}$$

If we use the following relations

$$\boldsymbol{X}^d(u)=\frac{1}{\cos\phi+1}\left(-\cos\phi\boldsymbol{X}^\ell_+(u)+\mathbb{N}^d_\ell[\phi](u)\right)=\frac{1}{2}\left(\boldsymbol{X}^\ell_-(u)-\boldsymbol{X}^\ell_+(u)\right),$$

then we obtain the formula (2) by the similar calculations to the case (1). □

Since $M^L_+=\boldsymbol{X}^\ell_+(U)$ is a Riemannian manifold, it makes sense to consider the Christoffel symbolls:

$$\left\{\begin{array}{c}k\\ ij\end{array}\right\}=\frac{1}{2}\sum_m g^{km}\left\{\frac{\partial g_{jm}}{\partial u_i}+\frac{\partial g_{im}}{\partial u_j}-\frac{\partial g_{ij}}{\partial u_m}\right\}.$$

Proposition 3.8 *Let $\boldsymbol{X}^\ell_+:U\longrightarrow LC^*$ be a spacelike hypersurface. Then we have the following lightcone Gauss equations:*

$$\boldsymbol{X}^\ell_{+u_iu_j}=\sum_k\left\{\begin{array}{c}k\\ ij\end{array}\right\}\boldsymbol{X}^\ell_{+u_k}+\frac{1}{2}\left(g_{ij}\boldsymbol{X}^\ell_--h^\ell_{-ij}\boldsymbol{X}^\ell_+\right).$$

Proof. Since $\{\boldsymbol{X}^\ell_+,\ \boldsymbol{X}^\ell_-,\ \boldsymbol{X}^\ell_{+u_1},...,\boldsymbol{X}^\ell_{+u_{n-1}}\}$ is a basis of $\mathbb{R}^{n+1}_1$, we can write that

$$\boldsymbol{X}^\ell_{+u_iu_j} = \sum_k \Gamma^k_{ij}\boldsymbol{X}^\ell_{+u_k} + \Gamma_{ij}\boldsymbol{X}^\ell_- + \Gamma^{ij}\boldsymbol{X}^\ell_+.$$

Because of the relations $\langle \boldsymbol{X}^\ell_-, \boldsymbol{X}^\ell_{+u_m}\rangle = \langle \boldsymbol{X}^\ell_+, \boldsymbol{X}^\ell_{+u_m}\rangle = 0$, it is obvious that

$$\left\langle \boldsymbol{X}^\ell_{+u_iu_j}, \boldsymbol{X}^\ell_{+u_m}\right\rangle = \sum_k \Gamma^k_{ij}\left\langle \boldsymbol{X}^\ell_{+u_k}, \boldsymbol{X}^\ell_{+u_m}\right\rangle = \sum_k \Gamma^k_{ij} g_{km}.$$

Moreover, since $\frac{\partial g_{i\ell}}{\partial u_j} = \left\langle \boldsymbol{X}^\ell_{+u_iu_j}, \boldsymbol{X}^\ell_{+u_\ell}\right\rangle + \left\langle \boldsymbol{X}^\ell_{+u_i}, \boldsymbol{X}^\ell_{+u_\ell u_j}\right\rangle$ and $\boldsymbol{X}^\ell_{+u_iu_j} = \boldsymbol{X}^\ell_{+u_ju_i}$, we get $\Gamma^k_{ij} = \Gamma^k_{ji}$, $\Gamma_{ij} = \Gamma_{ji}$ and $\Gamma^{ij} = \Gamma^{ji}$. By exactly the same calculations like as the case of the hypersurfaces in Euclidean space, $\Gamma^k_{ij} = \left\{ \begin{array}{c} k \\ ij \end{array} \right\}$.

On the other hand, we have $\langle \boldsymbol{X}^\ell_+, \boldsymbol{X}^\ell_+\rangle = \langle \boldsymbol{X}^\ell_-, \boldsymbol{X}^\ell_-\rangle = 0$ and $\langle \boldsymbol{X}^\ell_+, \boldsymbol{X}^\ell_-\rangle = -2$. It follows that $-2\Gamma^{ij} = \left\langle \boldsymbol{X}^\ell_{+u_iu_j}, \boldsymbol{X}^\ell_-\right\rangle = h^\ell_{-ij}$ and $\left\langle \boldsymbol{X}^\ell_{+u_iu_j}, \boldsymbol{X}^\ell_+\right\rangle = -2\Gamma_{ij}$. Furthermore, we obtain that $\left\langle \boldsymbol{X}^\ell_{+u_iu_j}, \boldsymbol{X}^\ell_+\right\rangle = -\left\langle \boldsymbol{X}^\ell_{+u_i}, \boldsymbol{X}^\ell_{+u_j}\right\rangle = -g_{ij}$ which implies that $2\Gamma_{ij} = g_{ij}$. □

Now, we can give the following corollary:

Corollary 3.9 *Under the same assumption as the above proposition, we have*

$$\boldsymbol{X}^\ell_{+u_iu_j} = \sum_k \left\{ \begin{array}{c} k \\ ij \end{array} \right\} \boldsymbol{X}^\ell_{+u_k} + \frac{1}{2}\left(g_{ij} - h^\ell_{-ij}\right)\boldsymbol{X}^h + \frac{1}{2}\left(g_{ij} + h^\ell_{-ij}\right)\boldsymbol{X}^d.$$

Now, we stick to the case $n = 3$. First of all, we need to make some local calculations. Let $\boldsymbol{X}^\ell_+ : U \longrightarrow LC^*$ be a spacelike surface, where $U \subset \mathbb{R}^2$ is an open region and consider the Riemannian curvature tensor

$$R^\delta_{\alpha\beta\gamma} = \frac{\partial}{\partial u_\gamma}\left\{ \begin{array}{c} \delta \\ \alpha\beta \end{array} \right\} - \frac{\partial}{\partial u_\beta}\left\{ \begin{array}{c} \delta \\ \alpha\gamma \end{array} \right\} + \sum_\epsilon \left\{ \begin{array}{c} \epsilon \\ \alpha\beta \end{array} \right\}\left\{ \begin{array}{c} \delta \\ \epsilon\gamma \end{array} \right\} - \sum_\epsilon \left\{ \begin{array}{c} \epsilon \\ \alpha\gamma \end{array} \right\}\left\{ \begin{array}{c} \delta \\ \epsilon\beta \end{array} \right\}$$

and the tensor $R_{\alpha\beta\gamma\delta} = \sum_\epsilon g_{\alpha\epsilon}R^\epsilon_{\beta\gamma\delta}$. In [12], it has been shown that

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left\{g_{\beta\gamma}h^\ell_{-\alpha\delta} - g_{\beta\delta}h^\ell_{-\alpha\gamma} + h^\ell_{-\beta\gamma}g_{\alpha\delta} - h^\ell_{-\beta\delta}g_{\alpha\gamma}\right\}.$$

Let K_S be the sectional curvature of $M^L_+ = \boldsymbol{X}^\ell_+(U)$ which is defined by $K_S = -R_{1212}/\det(g_{\alpha\beta})$. As a consequence of the above arguments, the following proposition has been given in [12]:

Proposition 3.10 ([12]) *Under the above notations, we have*

$$K^D - K^H = K_S.$$

It is obvious that

$$S^H(p) = \frac{1}{\cos\phi + 1}\left(-id_{T_pM^L_+} + S^d[\phi](p)\right) \text{ and } S^D(p) = \frac{1}{\cos\phi + 1}\left(\cos\phi\, id_{T_pM^L_+} + S^d[\phi](p)\right).$$

Let $\overline{\kappa}^d[\phi]_i$ $(i=1,2)$ be eigenvalues of $S^d[\phi]$ (i.e., ϕ-de Sitter principal curvatures of spacelike surface $\boldsymbol{X}^\ell_+$) and κ_i^H (respectively, κ_i^D) $(i=1,2)$ be hyperbolic (respectively, de Sitter) principal curvatures. Then we get the following relations:

$$\kappa_i^H = \frac{-1+\overline{\kappa}^d[\phi]_i}{\cos\phi+1} \text{ and } \kappa_i^D = \frac{\cos\phi+\overline{\kappa}^d[\phi]_i}{\cos\phi+1}.$$

By using these two relations, we obtain the following equations:

(1) $\kappa_i^H + \kappa_i^D = \dfrac{\cos\phi - 1 + 2\overline{\kappa}^d[\phi]_i}{\cos\phi+1}$,

(2) (i) $\overline{\kappa}^d[\phi]_i = (\cos\phi+1)\kappa_i^H + 1$,

(ii) $\overline{\kappa}^d[\phi]_i = (\cos\phi+1)\kappa_i^D - \cos\phi$.

Eventually, we have the following "Theorema Egregium":

Theorem 3.11 *The following relation holds:*

$$\frac{(\cos\phi+1)}{2}K_S - \frac{\cos\phi-1}{2} = H_\ell^d[\phi] = \frac{(\cos\phi+1)}{2}\left(H^H + H^D\right) - \frac{\cos\phi-1}{2}.$$

Proof. By definition and the above equations given in (2), we get

$$2H_\ell^d[\phi] = \overline{\kappa}^d[\phi]_1 + \overline{\kappa}^d[\phi]_2 = 2(\cos\phi+1)H^H + 2 = 2(\cos\phi+1)H^D - 2\cos\phi.$$

Therefore, we have

$$2H_\ell^d[\phi] = (\cos\phi+1)(H^H+H^D) + 1 - \cos\phi.$$

On the other hand, we obtain

$$\begin{aligned} K^H = \kappa_1^H\kappa_2^H &= \frac{1}{(\cos\phi+1)^2}(1-(\overline{\kappa}^d[\phi]_1+\overline{\kappa}^d[\phi]_2)+\overline{\kappa}^d[\phi]_1\overline{\kappa}^d[\phi]_2) \\ &= \frac{1}{(\cos\phi+1)^2}(1-2H_\ell^d[\phi]+K_\ell^d[\phi]) \end{aligned}$$

and

$$\begin{aligned} K^D = \kappa_1^D\kappa_2^D &= \frac{1}{(\cos\phi+1)^2}(\cos^2\phi+\cos\phi(\overline{\kappa}^d[\phi]_1+\overline{\kappa}^d[\phi]_2)+\overline{\kappa}^d[\phi]_1\overline{\kappa}^d[\phi]_2) \\ &= \frac{1}{(\cos\phi+1)^2}(\cos^2\phi+2\cos\phi H_\ell^d[\phi]+K_\ell^d[\phi]). \end{aligned}$$

It follows that

$$\begin{aligned} K^D - K^H &= \frac{1}{(\cos\phi+1)^2}(\cos^2\phi - 1 + 2(\cos\phi+1)H_\ell^d[\phi]) \\ &= \frac{1}{\cos\phi+1}(\cos\phi - 1 + 2H_\ell^d[\phi]). \end{aligned}$$

So, by Proposition 3.10, we get

$$K_S = \frac{1}{\cos\phi+1}(\cos\phi-1+2H_\ell^d[\phi]).$$

This completes the proof. $\square$

Remark 3.12 *The above theorem asserts that the ϕ-de Sitter mean curvature of a spacelike surface in the 3-dimensional lightcone coincides with the sectional (intrinsic Gauss) curvature if and only if $\phi = 0$. Consequently, the mean curvature is an intrinsic invariant when $\phi = 0$. By the above arguments, we also have*

$$\begin{aligned}\overline{\kappa}^d[\phi]_1\overline{\kappa}^d[\phi]_2 &= (\cos\phi+1)^2\kappa_1^H\kappa_2^H + (\cos\phi+1)(\kappa_1^H+\kappa_2^H)+1 \\ &= (\cos\phi+1)^2\kappa_1^D\kappa_2^D - \cos\phi(\cos\phi+1)(\kappa_1^D+\kappa_2^D)+\cos^2\phi,\end{aligned}$$

so that

$$\begin{aligned}K_\ell^d[\phi] &= (\cos\phi+1)^2K^H + 2(\cos\phi+1)H^H+1 \\ &= (\cos\phi+1)^2K^D - 2\cos\phi(\cos\phi+1)H^D+\cos^2\phi,\end{aligned}$$

which is an extrinsic invariant for any $\phi \in \left[0, \frac{\pi}{2}\right]$.

4 The ϕ-de Sitter dual as a wave front

In order to investigate the ϕ-de Sitter dual of a spacelike hypersurface in the lightcone as a wave front set, we give a quick review on the Legendrian singularity theory due to Arnol'd-Zakalyukin [1, 20]. Let $\pi : PT^*(M) \longrightarrow M$ be the projective cotangent bundle over an n-dimensional manifold M. This fibration can be considered as a Legendrian fibration with the canonical contact structure K on $PT^*(M)$. Now, we review geometric properties of this space. Let us consider the tangent bundle $\tau : TPT^*(M) \to PT^*(M)$ and the differential map $d\pi : TPT^*(M) \to TM$ of π. For any $X \in TPT^*(M)$, there exists an element $\alpha \in T^*(M)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(M)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus, we can define the canonical contact structure on $PT^*(M)$ by $K = \{X \in TPT^*(M) | \tau(X)(d\pi(X)) = 0\}$. For a local coordinate neighbourhood $(U, (x_1, \dots, x_n))$ on M, we have a trivialization $PT^*(U) \cong U \times P(\mathbb{R}^{n-1})^*$ and we call $((x_1, \dots, x_n), [\xi_1 : \cdots : \xi_n])$ *homogeneous coordinates*, where $[\xi_1 : \cdots : \xi_n]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^{n-1})^*$. It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=1}^n \mu_i\xi_i = 0$, where $d\pi(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i}$. It is known that any Legendrian fibration is locally equivalant to $\pi : PT^*(M) \to M$, (cf. [1], Part III).

The main tool of the theory of Legendrian singularities is the notion of generating families. Since we only consider local properties, we may assume that $M = \mathbb{R}^n$. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is a *Morse family of hypersurfaces* if the map germ

$$\Delta^*F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$. In this case, we have a smooth $(n-1)$-dimensional submanifold germ

$$\Sigma_*(F) = \left\{(q,x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \mid F(q,x) = \frac{\partial F}{\partial q_1}(q,x) = \cdots = \frac{\partial F}{\partial q_k}(q,x) = 0\right\} = (\Delta^*F)^{-1}(0)$$

and a map germ $\mathcal{L}_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\mathcal{L}_F(q,x) = \left(x, [\frac{\partial F}{\partial x_1}(q,x) : \cdots : \frac{\partial F}{\partial x_n}(q,x)]\right)$$

which is a Legendrian immersion germ. Then we have the following fundamental theorem of Arnol'd-Zakalyukin [1, 20].

Proposition 4.1 *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F a *generating family* of $\mathcal{L}_F(\Sigma_*(F))$. Consequently, the wave front is

$$W(\mathcal{L}_F)=\left\{x\in\mathbb{R}^n\ |\exists q\in\ \mathbb{R}^k \text{ such that } F(q,x)=\frac{\partial F}{\partial q_1}(q,x)=\cdots=\frac{\partial F}{\partial q_k}(q,x)=0\right\}.$$

We also denote that $\mathcal{D}_F=W(\mathcal{L}_F)$ and call it the *discriminant set* of F.

Let us consider a point $\boldsymbol{v}=(v_0,v_1,\dots,v_n)\in S^n_1(\sin^2\phi)$. Then we have $(v_1,\dots.v_n)\neq(0,\dots,0)$. Without the loss of generality, we suppose that $v_1>0$. We choose the local coordinate neighbourhood system (V^1_+,U^1,ψ), where

$$V^1_+=\{\boldsymbol{v}\in S^n_1(\sin^2\phi)\mid v_1>0\},\ U^1=\{(x_1,\dots,x_n)\in\mathbb{R}^n\mid x_1^2-\sum_{i=2}^n x_i^2+\sin^2\phi>0\}$$

and $\psi:V^1_+\longrightarrow U^1$ is induced by the canonical projection. We consider the projective cotangent bundle $\pi:PT^*(S^n_1(\sin^2\phi))\longrightarrow S^n_1(\sin^2\phi)$ with the canonical contact structure. By using the above coordinate system, we have a trivialization as follows:

$$\Phi:PT^*(V^1_+)\equiv V^1_+\times P(\mathbb{R}^{n-1})^*\ ;\ \Phi([\sum_{i=1}^n\xi_i dv_i])=((v_0,v_1,\dots,v_n),[\xi_1:\cdots:\xi_n]).$$

On the other hand, we define the mapping

$$\Psi:\Delta^-_{43}(\phi)|(LC^*\times V^1_+)\longrightarrow V^1_+\times P(\mathbb{R}^{n-1})^*$$

by

$$\Psi(\boldsymbol{v},\boldsymbol{w})=(\boldsymbol{w},[-v_0w_1+v_1w_0:v_2w_1-v_1w_2:\cdots:v_nw_1-v_1w_n]).$$

For the canonical contact form $\theta=\sum_{i=1}^n\xi_i dx_i$ on $PT^*(V^1_+)$, we get

$$\begin{aligned}\Psi^*\theta&=((-v_0w_1+v_1w_0)dw_0+(v_2w_1-v_1w_2)dw_2+\cdots+(v_nw_1-v_1w_n)dw_n)|\Delta^-_{43}(\phi)\\&=w_1(-v_0dw_0+v_1dw_1+\cdots+v_ndw_n)|\Delta^-_{43}(\phi)=w_1\langle\boldsymbol{v},d\boldsymbol{w}\rangle|\Delta^-_{43}(\phi)=w_1\theta[\phi]^-_{(43)2},\end{aligned}$$

where $w_1=\sqrt{w_0^2-\sum_{i=2}^n w_i^2+\sin^2\phi}$. Thus, Ψ is a contact morphism.

Proposition 4.2 *The ϕ-de Sitter height function $H^d_\phi:U\times S^n_1(\sin^2\phi)\longrightarrow\mathbb{R}$ is a Morse family of hypersurfaces.*

Proof. We consider the local coordinate neighborhood V^1_+. For any $\boldsymbol{v}=(v_0,v_1,\dots,v_n)\in V^1_+$, we have $v_1=\sqrt{v_0^2-\sum_{i=2}^n v_i^2+\sin^2\phi}$, so that

$$H^d_\phi(u,\boldsymbol{v})=-x_0(u)v_0+x_1(u)\sqrt{v_0^2-\sum_{i=2}^n v_i^2+\sin^2\phi}+x_2(u)v_2+\cdots+x_n(u)v_n+\cos\phi+1,$$

where $\boldsymbol{X}_+^\ell(u) = (x_0(u), \dots, x_n(u))$. We define a mapping

$$\Delta^* H_\phi^d : U \times S_1^n(\sin^2\phi) \longrightarrow \mathbb{R} \times \mathbb{R}^{n-1}$$

by $\Delta^* H_\phi^d = \left(H_\phi^d, \frac{\partial H_\phi^d}{\partial u_1}, \dots, \frac{\partial H_\phi^d}{\partial u_{n-1}} \right)$. We have to prove that $\Delta_* H_\phi^d$ is non-singular at any point on $\Sigma_*(H_\phi^d) = (\Delta^* H_\phi^d)^{-1}(0)$. If $(u, \boldsymbol{v}) \in \Sigma_*(H_\phi^d)$, then $\boldsymbol{v} = \mathbb{N}_\ell^d[\phi](u)$ by Proposition 3.1. The Jacobian matrix of $\Delta^* H_\phi^d$ is given as follows:

$$\begin{pmatrix} \langle \boldsymbol{X}^\ell_{+u_1}, \boldsymbol{v}\rangle & \cdots & \langle \boldsymbol{X}^\ell_{+u_{n-1}}, \boldsymbol{v}\rangle & \\ \langle \boldsymbol{X}^\ell_{+u_1u_1}, \boldsymbol{v}\rangle & \cdots & \langle \boldsymbol{X}^\ell_{+u_1u_{n-1}}, \boldsymbol{v}\rangle & \mathrm{A} \\ \vdots & \vdots & \vdots & \\ \langle \boldsymbol{X}^\ell_{+u_{n-1}u_1}, \boldsymbol{v}\rangle & \cdots & \langle \boldsymbol{X}^\ell_{+u_{n-1}u_{n-1}}, \boldsymbol{v}\rangle & \end{pmatrix},$$

where

$$\mathrm{A} = \begin{pmatrix} -x_0 + \frac{v_0}{v_1}x_1 & x_2 - \frac{v_2}{v_1}x_1 & \cdots & x_n - \frac{v_n}{v_1}x_1 \\ -x_{0u_1} + \frac{v_0}{v_1}x_{1u_1} & x_{2u_1} - \frac{v_2}{v_1}x_{1u_1} & \cdots & x_{nu_1} - \frac{v_n}{v_1}x_{1u_1} \\ \vdots & \vdots & \vdots & \vdots \\ -x_{0u_{n-1}} + \frac{v_0}{v_1}x_{1u_{n-1}} & x_{2u_{n-1}} - \frac{v_2}{v_1}x_{1u_{n-1}} & \cdots & x_{nu_{n-1}} - \frac{v_n}{v_1}x_{1u_{n-1}} \end{pmatrix}.$$

Let us show that $\det A$ does not vanish at $(u, \boldsymbol{v}) \in \Sigma_*(H_\phi^d)$. We denote that

$$\boldsymbol{a} = \begin{pmatrix} x_0 \\ x_{0u_1} \\ \vdots \\ x_{0u_{n-1}} \end{pmatrix}, \boldsymbol{b}_1 = \begin{pmatrix} x_1 \\ x_{1u_1} \\ \vdots \\ x_{1u_{n-1}} \end{pmatrix}, \dots, \boldsymbol{b}_n = \begin{pmatrix} x_n \\ x_{nu_1} \\ \vdots \\ x_{nu_{n-1}} \end{pmatrix}.$$

Then we obtain

$$\det A = \frac{v_0}{v_1}\det(\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_n) - \frac{v_1}{v_1}\det(\boldsymbol{a} \ \boldsymbol{b}_2 \ \dots \boldsymbol{b}_n) - \cdots - \frac{v_n}{v_1}\det(\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_{n-1} \ \boldsymbol{a}).$$

On the other hand, we have

$$\boldsymbol{X}_+^\ell \wedge \boldsymbol{X}^\ell_{+u_1} \wedge \cdots \wedge \boldsymbol{X}^\ell_{+u_{n-1}} = (-\det(\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_n), -\det(\boldsymbol{a} \ \boldsymbol{b}_2 \ \dots \boldsymbol{b}_n), \dots, -\det(\boldsymbol{b}_1 \ \dots \ \boldsymbol{b}_{n-1} \ \boldsymbol{a})).$$

Now, we consider a hyperplane $HP(\boldsymbol{c}, 0)$, where $\boldsymbol{c} = \boldsymbol{X}_+^\ell \wedge \boldsymbol{X}^\ell_{+u_1} \wedge \cdots \wedge \boldsymbol{X}^\ell_{+u_{n-1}}$. By definition, the basis of the vector subspace $HP(\boldsymbol{c}, 0)$ is $\{\boldsymbol{X}_+^\ell, \ \boldsymbol{X}^\ell_{+u_1}, \dots, \boldsymbol{X}^\ell_{+un-1}\}$. Since $\boldsymbol{X}_+^\ell, \ \boldsymbol{X}^\ell_{+u_i}$ $(i = 1, \dots, n-1)$ are tangent to the lightcone LC^*, the hyperplane $HP(\boldsymbol{c}, 0)$ is a lightlike hyperplane. Since $\langle \boldsymbol{c}, \boldsymbol{X}_+^\ell \rangle = 0$, $\boldsymbol{c}$ and $\boldsymbol{X}_+^\ell$ are linearly dependent, so that there exists a non-zero real number λ such that $\lambda \boldsymbol{X}_+^\ell = \boldsymbol{X}_+^\ell \wedge \boldsymbol{X}^\ell_{+u_1} \wedge \cdots \wedge \boldsymbol{X}^\ell_{+u_{n-1}}$. Therefore, we get

$$\begin{aligned} \det A &= \left\langle \left(\frac{v_0}{v_1}, \dots, \frac{v_n}{v_1} \right), \boldsymbol{X}_+^\ell \wedge \boldsymbol{X}^\ell_{+u_1} \wedge \cdots \wedge \boldsymbol{X}^\ell_{+u_{n-1}} \right\rangle \\ &= \frac{1}{v_1}\langle \mathbb{N}_\ell^d[\phi], \boldsymbol{X}_+^\ell \wedge \boldsymbol{X}^\ell_{+u_1} \wedge \cdots \wedge \boldsymbol{X}^\ell_{+u_{n-1}} \rangle = \frac{1}{v_1}\langle \mathbb{N}_\ell^d[\phi], \lambda \boldsymbol{X}_+^\ell \rangle = -\frac{\lambda(\cos\phi + 1)}{v_1} \neq 0. \end{aligned}$$

If we adopt the other local coordinates, we obtain the similar calculations to the above. This completes the proof. □

We have the following theorem:

Theorem 4.3 *For any spacelike hypersurface $\boldsymbol{X}_+^\ell : U \longrightarrow LC^*$, the ϕ-de Sitter height function $H_\phi^d : U \times S_1^n(\sin^2\phi) \longrightarrow \mathbb{R}$ of $M_+^L = \boldsymbol{X}_+^\ell(U)$ is a generating family of the Legendrian immersion $\mathcal{L}_{43}[\phi](U) \subset \Delta_{43}^-(\phi)$ with respect to the Legendrian fibration $\pi[\phi]_{(43)2}^- : \Delta_{43}^-(\phi) \longrightarrow S_1^n(\sin^2\phi)$.*

Proof. We remember the contact morphism

$$\Psi : \Delta_{43}^-(\phi)|(LC^* \times V_+^1) \longrightarrow V_+^1 \times P(\mathbb{R}^{n-1})^*.$$

Since the ϕ-de Sitter height function $H_\phi^d : U \times V_+^1 \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces, we have a Legendrian immersion

$$\mathcal{L}_{H_\phi^d} : \Sigma_*(H_\phi^d) \longrightarrow V_+^1 \times P(\mathbb{R}^{n-1})^*$$

defined by

$$\mathcal{L}_{H_\phi^d}(u, \boldsymbol{v}) = \left(\boldsymbol{v}, \left[\frac{\partial H_\phi^d}{\partial v_0} : \frac{\partial H_\phi^d}{\partial v_2} : \cdots : \frac{\partial H_\phi^d}{\partial v_n}\right]\right),$$

where $\boldsymbol{v} = (v_0, \dots, v_n)$ and $v_1 = \sqrt{v_0^2 - \sum_{i=2}^n v_i^2 + \sin^2\phi}$. By Proposition 3.1, we get

$$\Sigma_*(H_\phi^d) = \{(u, \mathbb{N}_\ell^d[\phi](u)) \in U \times V_+^1 \mid u \in U\}.$$

Since $\boldsymbol{v} = \mathbb{N}_\ell^d[\phi](u)$ and $v_1 = \sqrt{v_0^2 - \sum_{i=2}^n v_i^2 + \sin^2\phi}$, we obtain

$$\frac{\partial H_\phi^d}{\partial v_0}(u, \mathbb{N}_\ell^d[\phi](u)) = -x_0(u) + \frac{n_0^\ell(u)}{n_1^\ell(u)} x_1(u), \ \frac{\partial H_\phi^d}{\partial v_i}(u, \mathbb{N}_\ell^d[\phi](u)) = x_i(u) - \frac{n_i^\ell(u)}{n_1^\ell(u)} x_1(u),$$

where $i = 2, \dots n$, $\boldsymbol{X}_+^\ell(u) = (x_0(u), \dots, x_n(u))$ and $\mathbb{N}_\ell^d[\phi](u) = (n_0^\ell(u), \dots, n_n^\ell(u))$. It follows that

$$\mathcal{L}_{H_\phi^d}(u, \mathbb{N}_\ell^d[\phi](u)) = (\mathbb{N}_\ell^d[\phi](u), [\boldsymbol{\xi}]),$$

where

$$[\boldsymbol{\xi}] = [-x_0(u)n_1^\ell(u) + n_0^\ell(u)x_1(u) : x_2(u)n_1^\ell(u) - n_2^\ell(u)x_1(u) : \cdots : x_n(u)n_1^\ell(u) - n_n^\ell(u)x_1(u)]).$$

Therefore, we have $\Psi \circ \mathcal{L}_{43}[\phi](u) = \mathcal{L}_{H_\phi^d}(u)$. This means that H_ϕ^d is a generating family of $\mathcal{L}_{43}[\phi](U) \subset \Delta_{43}^-(\phi)$ with respect to the Legendrian fibration $\pi[\phi]_{(43)2}^- : \Delta_{43}^-(\phi) \longrightarrow S_1^n(\sin^2\phi)$. This completes the proof. $\square$

5 Contact with ϕ-de Sitter flat hyperquadrics

In this section, we consider the contact of spacelike hypersurfaces in the lightcone with ϕ-de Sitter flat hyperquadrics. For our purpose, we briefly review the theory of contact due to Montaldi [18]. Let X_i and Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact of* X_1 and Y_1 at y_1 is the same type as the *contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case, we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition, $\mathbb{R}^n$ can be replaced by any manifold. In [18], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory. Let $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ be map germs. We say that f and g are *$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ such that $I(f \circ \phi) = I(g)$, where $I(f) = \langle f_1, \dots, f_p \rangle_{\mathcal{E}_n}$ is the ideal generated by the component function germs $f_1, \dots, f_p$ of f (i.e., $f = (f_1, \dots, f_p)$) in the local ring $\mathcal{E}_n = \{h \mid h : (\mathbb{R}^n, 0) \longrightarrow \mathbb{R}\}$ of function germs at 0.

Theorem 5.1 *Let X_i and Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.*

Now, we consider a function $\mathcal{H}^d_\phi : LC^* \times S^n_1(\sin^2\phi) \longrightarrow \mathbb{R}$ defined by $\mathcal{H}^d_\phi(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \cos\phi + 1$. For any $\boldsymbol{v}_0 \in S^n_1(\sin^2\phi)$, we denote that $(\mathfrak{h}^d_\phi)_{v_0}(\boldsymbol{u}) = \mathcal{H}^d_\phi(\boldsymbol{u}, \boldsymbol{v}_0)$ and we have a ϕ-de Sitter flat hyperquadric $(\mathfrak{h}^d_\phi)^{-1}_{v_0}(0) = HP(\boldsymbol{v}_0, -(\cos\phi + 1)) \cap LC^* = HL(\boldsymbol{v}_0, -(\cos\phi + 1))$. For any $u_0 \in U$, we consider the spacelike vector $\boldsymbol{v}_0 = \mathbb{N}^d_\ell[\phi](u_0)$. Then we have

$$(\mathfrak{h}^d_\phi)_{v_0} \circ \boldsymbol{X}^\ell_+(u_0) = \mathcal{H}^d_\phi \circ (\boldsymbol{X}^\ell_+ \times id_{S^n_1(\sin^2\phi)})(u_0, \boldsymbol{v}_0) = H^d_\phi(u_0, \mathbb{N}^d_\ell[\phi](u_0)) = 0.$$

By Proposition 3.1, we also have the following relations for $i = 1, \dots, n-1$:

$$\frac{\partial(\mathfrak{h}^d_\phi)v_0 \circ \boldsymbol{X}^\ell_+}{\partial u_i}(u_0) = \frac{\partial H^d_\phi}{\partial u_i}(u_0, \mathbb{N}^d_\ell[\phi](u_0)) = 0.$$

This means that the ϕ-de Sitter flat hyperquadric

$$(\mathfrak{h}^d_\phi)^{-1}_{v_0}(0) = HL(\boldsymbol{v}_0, -(\cos\phi + 1))$$

is tangent to $M^L_+ = \boldsymbol{X}^\ell_+(U)$ at $p = \boldsymbol{X}^\ell_+(u_0)$. In this case, we call $HL(\boldsymbol{v}_0, -(\cos\phi + 1))$ the *tangent ϕ-de Sitter flat hyperquadric* of $M^L_+ = \boldsymbol{X}^\ell_+(U)$ at $p = \boldsymbol{X}^\ell_+(u_0)$ (or, u_0), which we write $TDH[\phi](M^L_+, p)$ (or, $TDH[\phi](\boldsymbol{X}^\ell_+, u_0)$).

Eventually, we have tools for the study of the contact between spacelike hypersurfaces and ϕ-de Sitter flat hyperquadrics. Let $(\mathbb{N}^d_\ell[\phi])_i : (U, u_i) \longrightarrow (S^n_1(\sin^2\phi), \boldsymbol{v}_i)$ $(i = 1, 2)$ be ϕ-de Sitter dual germs of spacelike hypersurface germs $(\boldsymbol{X}^\ell_+)_i : (U, u_i) \longrightarrow (LC^*, \boldsymbol{u}_i)$. We say that $(\mathbb{N}^d_\ell[\phi])_1$ and $(\mathbb{N}^d_\ell[\phi])_2$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\phi : (U, u_1) \longrightarrow (U, u_2)$ and $\Phi : (S^n_1(\sin^2\phi), \boldsymbol{v}_1) \longrightarrow (S^n_1(\sin^2\phi), \boldsymbol{v}_2)$ such that $\Phi \circ (\mathbb{N}^d_\ell[\phi])_1 = (\mathbb{N}^d_\ell[\phi])_2 \circ \phi$. We remark that the $\mathcal{A}$-equivalence preserve the singularities of the both map-germs. In order to understand the geometric meanings of the $\mathcal{A}$-equivalence among the ϕ-de Sitter dual germs, we need the theory of Legendrian equivalence [1, 20, 21]. Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that H preserves fibers of π and $H(L) = L'$. A Legendrian immersion germ $i : (L, p) \subset PT^*\mathbb{R}^n$ (or, a *Legendrian map* $\pi \circ i$) at a point is said to be *Legendrian stable* if for every map with the given germ there is a neighbourhood in the space of Legendrian immersions (in the Whitney C^∞ topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has a point in the second neighbourhood at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs [21]:

Proposition 5.2 *Let $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L',p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that the representatives of both of the germs are proper mappings and the regular sets of the projections $\pi \circ i$ and $\pi \circ i'$ are dense. Then i and i' are Legendrian equivalent if and only if wave front sets $W(i)$ and $W(i')$ are diffeomorphic as set germs.*

The assumption in the above proposition is a generic condition for i and i'. Specially, if i and i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We consider the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$ of the local ring $\mathcal{E}_n$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(x,u) = (\psi_1(q,x), \psi_2(x))$ for $(q,x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is a *$\mathcal{K}$-versal deformation of* $f = F| \mathbb{R}^k \times \{\mathbf{0}\}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1} | \mathbb{R}^k \times \{\mathbf{0}\}, \ldots, \frac{\partial F}{\partial x_n} | \mathbb{R}^k \times \{\mathbf{0}\} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k},$$

(See [17]). The main result in Arnol'd-Zakalyukin's theory [1, 20] is the following:

Theorem 5.3 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families of hypersurfaces. Then*
(1) $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent if and only if F and G are P-$\mathcal{K}$-equivalent.
(2) $\mathcal{L}_F$ is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $F \mid \mathbb{R}^k \times \{\mathbf{0}\}$.

Since F and G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$, we do not need the notion of stably P-$\mathcal{K}$-equivalences under this situation (cf., [1]).

If both of the regular sets of $(\mathbb{N}^d_\ell[\phi])_i$ $(i = 1, 2)$ are dense in (U, u_i), it follows from Proposition 5.2 that $(\mathbb{N}^d_\ell[\phi])_1$ and $(\mathbb{N}^d_\ell[\phi])_2$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian immersion germs $\mathcal{L}^1_{43}[\phi] : (U, u_1) \longrightarrow (\Delta^-_{43}(\phi), z_1)$ and $\mathcal{L}^2_{43}[\phi] : (U, u_2) \longrightarrow (\Delta^-_{43}(\phi), z_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families $(H^d_\phi)_1$ and $(H^d_\phi)_2$ are P-$\mathcal{K}$-equivalent by Theorem 5.3. Here, $(H^d_\phi)_i : (U \times S^n_1(\sin^2\phi), (u_i, \boldsymbol{v}_i)) \longrightarrow \mathbb{R}$ are the ϕ-de Sitter height function germs of $(\boldsymbol{X}^\ell_+)_i$.

On the other hand, if we denote that $(h^d_{\phi i, v_i})(u) = (H^d_\phi)_i(u, \boldsymbol{v}_i)$, then we have $(h^d_{\phi i, v_i})(u) = (\mathfrak{h}^d_\phi)_{v_i} \circ (\boldsymbol{X}^\ell_+)_i(u)$. By Theorem 5.1, for $p_i = (\boldsymbol{X}^\ell_+)(u_i)$

$$K((M^L_+)_1, TDH((M^L_+)_1, p_1), p_1) = K((M^L_+)_2, TDH((M^L_+)_2, p_2), p_2)$$

if and only if $(h^d_{\phi 1, v_1})$ and $(h^d_{\phi 1, v_2})$ are $\mathcal{K}$-equivalent.

Theorem 5.4 *Let $(\boldsymbol{X}^\ell_+)_i : (U, u_i) \longrightarrow (LC^*, p_i)$ $(i = 1, 2)$ be spacelike hypersurface germs such that the corresponding Legendrian map germs*

$$(\mathbb{N}^d_\ell[\phi])_i = \pi[\phi]^-_{(43)2} \circ \mathcal{L}^i_{43}[\phi] : (U, u_i) \longrightarrow (S^n_1(\sin^2\phi), \boldsymbol{v}_i)$$

are Legendrian stable. Then the following conditions are equivalent:
(1) $(\mathbb{N}^d_\ell[\phi])_1$ and $(\mathbb{N}^d_\ell[\phi])_2$ are $\mathcal{A}$-equivalent.
(2) $((\mathbb{N}^d_\ell[\phi])_1(U), z_1)$ and $((\mathbb{N}^d_\ell[\phi])_2(U), z_2)$ are diffeomorphic as set germs.
(3) $\mathcal{L}^1_{43}[\phi] : (U, u_1) \longrightarrow (\Delta^-_{43}(\phi), z_1)$ and $\mathcal{L}^2_{43}[\phi] : (U, u_2) \longrightarrow (\Delta^-_{43}(\phi), z_2)$ are Legendrian equivalent.
(4) $(H^d_\phi)_1$ and $(H^d_\phi)_2$ are P-$\mathcal{K}$-equivalent.
(5) $(h^d_{\phi 1,v_1})$ and $(h^d_{\phi 2,v_2})$ are $\mathcal{K}$-equivalent.
(6) $K((M^L_+)_1, TDH((M^L_+)_1, p_1), p_1) = K((M^L_+)_2, TDH((M^L_+)_2, p_2), p_2)$.

Proof. By the previous arguments (mainly from Theorem 5.1), it has been already shown that conditions (5) and (6) are equivalent. By Theorem 5.3, the conditions (3) and (4) are equivalent. By definition, the condition (4) implies the condition (5). Suppose that $(\mathbb{N}^d_\ell[\phi])_i$ are Legendrian stable. By the uniqueness result of the P-$\mathcal{K}$-versal deformation, the condition (5) implies the condition (4). Moreover, by Proposition 5.2 and Theorem 5.3, the conditions (1),(2) and (3) are equivalent. □

6 Slant geometry with respect to the ϕ-hyperbolic duals

In this section, we establish another new extrinsic differential geometry on spacelike hypersurfaces in the lightcone with respect to the *ϕ-hyperbolic duals* as an application of the extended mandala of Legendrian dualities. We call this geometry a *ϕ-hyperbolic flat geometry.* The results are analogous to those of the previous sections. So, from now on, we omit almost all of the proofs except some special cases for the assertions.

We consider the contact manifold $(\Delta^-_{42}(\phi), K[\phi]^-_{42})$ and the contact diffeomorphism $\Psi^-_{4(42)} : \Delta^-_4 \longrightarrow \Delta^-_{42}(\phi)$ defined by

$$\Psi^-_{4(42)}(\boldsymbol{v}, \boldsymbol{w}) = \left(\frac{1}{2}\left((1+\cos\phi)\boldsymbol{v} + (1-\cos\phi)\boldsymbol{w}\right), \boldsymbol{w}\right).$$

Suppose that $\boldsymbol{X}^\ell_- : U \longrightarrow LC^*$ is a spacelike embedding. Then we define a map $\mathbb{N}^h_\ell[\phi] : U \longrightarrow H^n(-\sin^2\phi)$ by

$$\mathbb{N}^h_\ell[\phi](u) = \frac{1}{2}((1+\cos\phi)\boldsymbol{X}^\ell_+(u) + (1-\cos\phi)\boldsymbol{X}^\ell_-(u))$$

for $\phi \in [0, \pi/2]$ and have a map $\mathcal{L}_{42}[\phi] : U \longrightarrow \Delta^-_{42}(\phi)$ defined by $\mathcal{L}_{42}[\phi](u) = (\mathbb{N}^h_\ell[\phi](u), \boldsymbol{X}^\ell_-(u))$. By exactly the same reason as the previous sections, $\mathcal{L}_{42}[\phi]$ is a Legendrian embedding, so that $\mathbb{N}^h_\ell[\phi](u)$ can be considered as a normal vector of M^L_- at $p = \boldsymbol{X}^\ell_-(u)$. We remark that $\mathbb{N}^h_\ell[0](u) = \boldsymbol{X}^\ell_+(u)$ and $\mathbb{N}^h_\ell[\pi/2](u) = \boldsymbol{X}^h(u)$, so that $\mathbb{N}^h_\ell[\phi](u)$ is the *ϕ-hyperbolic dual* of $\boldsymbol{X}^\ell_-(U) = M^L_-$. We call the geometry related to the Legendrian duals $\mathbb{N}^d_\ell[\phi]$ and $\mathbb{N}^h_\ell[\phi]$ a *slant geometry of spacelike hypersurfaces in the lightcone.*

We define a family of functions

$$H^h_\phi : U \times H^n(-\sin^2\phi) \longrightarrow \mathbb{R}$$

by $H^h_\phi(u, \boldsymbol{v}) = \langle \boldsymbol{X}^\ell_-(u), \boldsymbol{v}\rangle + 1 + \cos\phi$. We call H^h_ϕ a *ϕ-hyperbolic height function* on $\boldsymbol{X}^\ell_- : U \longrightarrow LC^*$. Since $\boldsymbol{X}^\ell_-$ is a spacelike embedding and $\boldsymbol{X}^\ell_-(u)$ and $\boldsymbol{X}^\ell_+(u)$ are linearly independent lightlike vectors,

$$\left\{\boldsymbol{X}^\ell_-(u),\ \boldsymbol{X}^\ell_+(u),\ \boldsymbol{X}^\ell_{-u_1}(u), ..., \boldsymbol{X}^\ell_{-u_{n-1}}(u)\right\}$$

is a basis of $T_p\mathbb{R}^{n+1}_1$ for $p = \boldsymbol{X}^\ell_-(u)$.

Proposition 6.1 *Let $H^h_\phi : U \times H^n(-\sin^2\phi) \longrightarrow \mathbb{R}$ be a ϕ-hyperbolic height function on $\boldsymbol{X}^\ell_- : U \longrightarrow LC^*$. Then*
(1) $H^h_\phi(u, \boldsymbol{v}) = 0$ if and only if $(\boldsymbol{v}, \boldsymbol{X}^\ell_-(u)) \in \Delta^-_{42}(\phi)$.
(2) $H^h_\phi(u, \boldsymbol{v}) = \dfrac{\partial H^h_\phi}{\partial u_i}(u, \boldsymbol{v}) = 0$ $(i = 1, ..., n-1)$ if and only if $\boldsymbol{v} = \mathbb{N}^h_\ell[\phi](u)$.

Now, we study the extrinsic differential geometry of $\boldsymbol{X}^\ell_-$ by using $\mathbb{N}^h_\ell[\phi]$ like as the Gauss map of a hypersurface in Euclidean space. For our purpose, we have the following fundamental lemma:

Lemma 6.2 *For any $p = \boldsymbol{X}^\ell_-(u_0) \in M^L_-$ and $\boldsymbol{v} \in T_pM^L_-$, we have $D_v\mathbb{N}^h_\ell[\phi](u_0) \in T_pM^L_-$. Here, D_v denotes the covariant derivative with respect to the tangent vector $\boldsymbol{v}$.*

Under the identification of U and M^L_- through the embedding $\boldsymbol{X}^\ell_-$, the derivative $d\boldsymbol{X}^\ell_-(u_0)$ is the identity mapping $\mathrm{id}_{T_pM^L_-}$ on $T_pM^L_-$, where $p = \boldsymbol{X}^\ell_-(u_0)$. Moreover by Lemma 6.2, $d\mathbb{N}^h_\ell[\phi](u_0)$ can be considered as a linear transformation on the tangent space $T_pM^L_-$. We have the following relation:

$$d\mathbb{N}^h_\ell[\phi](u_0) = \frac{1}{2}(1 - \cos\phi)\, id_{T_pM^L_-} + \frac{1}{2}(1 + \cos\phi)\, d\boldsymbol{X}^\ell_+(u_0).$$

We call the linear transformations $S^h[\phi](p) = -d\mathbb{N}^h_\ell[\phi](p) : T_pM^L_- \longrightarrow T_pM^L_-$ and $S^\ell_+(p) = -d\boldsymbol{X}^\ell_+(p) : T_pM^L_- \longrightarrow T_pM^L_-$, the *$\phi$-hyperbolic shape operator* and the *lightcone shape operator*, respectively. We denote the eigenvalues of $S^h[\phi](p)$ and $S^\ell_+(p)$ by $\overline{\kappa}^h[\phi](p)$ and $\kappa^\ell_+(p)$, respectively. Because of the relation $S^h[\phi](p) = -\frac{1}{2}(1 - \cos\phi)\, id_{T_pM^L_-} + \frac{1}{2}(1 + \cos\phi)\, S^\ell_+(p)$, $S^h[\phi](p)$ and $S^\ell_+(p)$ have the common eigen vectors. As a result, we get a relation $\overline{\kappa}^h[\phi](p) = -\frac{1}{2}(1 - \cos\phi) + \frac{1}{2}(1 + \cos\phi)\kappa^\ell_+(p)$. We call $\overline{\kappa}^h[\phi](p)$ and $\kappa^\ell_+(p)$, a *ϕ-hyperbolic principal curvature* and a *lightcone principal curvature* of $M^L_- = \boldsymbol{X}^\ell_-(U)$ at $p = \boldsymbol{X}^\ell_-(u_0)$, respectively. We give the following definitions of the curvatures of $M^L_- = \boldsymbol{X}^\ell_-(U)$ at $p = \boldsymbol{X}^\ell_-(u_0)$:

$K^h_\ell[\phi](u_0) = \det S^h[\phi](p)$; *ϕ-hyperbolic Gauss-Kronecker curvature,*

$H^h_\ell[\phi](u_0) = \dfrac{1}{n-1}\mathrm{Trace}\, S^h[\phi](p)$; *$\phi$-hyperbolic mean curvature.*

We also define the *lightcone Gauss-Kronecker curvature* and the *lightcone mean curvature* of $M^L_- = \boldsymbol{X}^\ell_-(U)$ at $p = \boldsymbol{X}^\ell_-(u_0)$ by $K^\ell_+(p) = \det S^\ell_+(p)$ and $H^\ell_+(p) = \frac{1}{n-1}Trace S^\ell_+(p)$, respectively.

Since $\boldsymbol{X}^\ell_{-u_i}$ $(i = 1, ..., n-1)$ are spacelike vectors, the induced Riemannian metric (the *first fundamental form*) on $M^L_- = \boldsymbol{X}^\ell_-(U)$ is given by $ds^2 = \sum_{i,j=1}^{n-1} g_{ij}du_idu_j$, where $g_{ij}(u) = \left\langle \boldsymbol{X}^\ell_{-u_i}(u), \boldsymbol{X}^\ell_{-u_j}(u)\right\rangle$ for any $u \in U$. We also define the *ϕ-hyperbolic second fundamental invariant* by $h^h[\phi]_{ij}(u) = \left\langle -\left(\mathbb{N}^h_\ell[\phi]\right)_{u_i}(u), \boldsymbol{X}^\ell_{-u_j}(u)\right\rangle$ for any $u \in U$. If we denote that $h^\ell_{+ij}(u) = \left\langle -\boldsymbol{X}^\ell_{+u_i}(u), \boldsymbol{X}^\ell_{-u_j}(u)\right\rangle$, then we have the following relation:

$$h^h[\phi]_{ij}(u) = -\frac{1}{2}(1 - \cos\phi)g_{ij}(u) + \frac{1}{2}(1 + \cos\phi)h^\ell_{+ij}(u).$$

Proposition 6.3 *Under the above notations, we have the following ϕ-hyperbolic Weingarten formula:*

$$\left(\mathbb{N}^h_\ell[\phi]\right)_{u_i} = -\sum_{j=1}^{n-1} h^h[\phi]^j_i \boldsymbol{X}^\ell_{-u_j},$$

where $\left(h^h[\phi]^j_i\right) = \left(h^h[\phi]_{ik}\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = \left(g_{kj}\right)^{-1}$.

As a corollary of the above proposition, we obtain an explicit expression of the ϕ-hyperbolic Gauss-Kronecker curvature by Riemannian metric and the ϕ-hyperbolic second fundamental invariant.

Corollary 6.4 *Under the same notations as in the above proposition, the ϕ-hyperbolic Gauss-Kronecker curvature is given by*

$$K^h_\ell[\phi] = \frac{\det\left(h^h[\phi]_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}.$$

We say that a point $u \in U$ or $p = \boldsymbol{X}^\ell_-(u)$ is an *umbilic point* if $S^h[\phi](p) = \overline{\kappa}^h[\phi](p)\, id_{T_pM^L_-}$. We also say that $M^L_- = \boldsymbol{X}^\ell_-(U)$ is *totally umbilic* if all points of M^L_- are umbilic. Here, we give a classification of totally umbilic spacelike hypersurfaces by using the ϕ-hyperbolic principal curvature.

Proposition 6.5 *Assume that $M^L_- = \boldsymbol{X}^\ell_-(U)$ is totally umbilic and fix $\phi \in \left[0, \frac{\pi}{2}\right]$. Then $\overline{\kappa}^h[\phi](p)$ is constant $\overline{\kappa}^h[\phi]$. Under this condition, we have the following classification:*

(1) If $\overline{\kappa}^h[\phi] < 0$:

(i) If $\phi = 0$, then M^L_- is a part of hyperbolic hyperquadric $HL(\boldsymbol{c}, -2)$.

(ii) If $\phi \neq 0$:

(a) If $2\left|\overline{\kappa}^h[\phi](1+\cos\phi)\right| > \sin^2\phi$, then M^L_- is a part of hyperbolic hyperquadric

$$HL(\boldsymbol{c}, -(1+\cos\phi)).$$

(b) If $2\left|\overline{\kappa}^h[\phi](1+\cos\phi)\right| = \sin^2\phi$, then M^L_- is a part of parabolic hyperquadric

$$HL(\boldsymbol{c}, -(1+\cos\phi)).$$

(c) If $2\left|\overline{\kappa}^h[\phi](1+\cos\phi)\right| < \sin^2\phi$, then M^L_- is a part of elliptic hyperquadric

$$HL(\boldsymbol{c}, -(1+\cos\phi)).$$

(2) If $\overline{\kappa}^h[\phi] = 0$:

(i) If $\phi = 0$, then M^L_- is a part of parabolic hyperquadric $HL(\boldsymbol{c}, -2)$.

(ii) If $\phi \neq 0$, then M^L_- is a part of elliptic hyperquadric $HL(\boldsymbol{c}, -(1+\cos\phi))$.

(3) If $\overline{\kappa}^h[\phi] > 0$, then M^L_- is a part of elliptic hyperquadric $HL(\boldsymbol{c}, -(1+\cos\phi))$.

In the above classification, the hyperquadric $HL(\boldsymbol{c}, -(1+\cos\phi))$ ($\phi \in [0, \pi/2]$) with $\overline{\kappa}^h[\phi] = 0$ is called a *ϕ-hyperbolic flat hyperquadric.* We say that $p = \boldsymbol{X}^\ell_-(u)$ is a *ϕ-hyperbolic parabolic point* if $K^h_\ell[\phi](u) = 0$ and a *ϕ-hyperbolic flat point* if it is an umbilic point and $K^h_\ell[\phi](u) = 0$ which are equivalent to the condition that one of the conditions (2)(i) or (2)(ii) is satisfied.

On the other hand, we denote the Hessian matrix of the ϕ-hyperbolic height function $h^h_{\phi,v_0}(u) = H^h_\phi(u, \boldsymbol{v}_0)$ at u_0 by Hess $\left(h^h_{\phi,v_0}\right)(u_0)$.

Proposition 6.6 *Let* $\boldsymbol{X}^{\ell}_{-} : U \longrightarrow LC^*$ *be a spacelike hypersurface in the lightcone and* $\boldsymbol{v}_0 = \mathbb{N}^h_\ell[\phi](u_0)$. *Then we have the following:*
(1) $p = \boldsymbol{X}^{\ell}_{-}(u_0)$ is a parabolic point if and only if $\det \mathrm{Hess}\left(h^h_{\phi,v_0}\right)(u_0) = 0$.
(2) $p = \boldsymbol{X}^{\ell}_{-}(u_0)$ is a flat point if and only if $\mathrm{rank}\,\mathrm{Hess}\left(h^h_{\phi,v_0}\right)(u_0) = 0$.

For the diffeomorphism

$$\Psi^-_{(42)1} : \Delta^-_{42}(\phi) \longrightarrow \Delta_1$$

defined by

$$\Psi^-_{(42)1}(\boldsymbol{v},\boldsymbol{w}) = \frac{1}{\cos\phi + 1}(\boldsymbol{v} + \cos\phi \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{v}),$$

we can calculate that

$$\begin{aligned}
(\Psi^-_{(42)1})^*\theta_{11} &= \frac{1}{\cos\phi+1}\langle d\left(\boldsymbol{v} + \cos\phi\boldsymbol{w}\right), \boldsymbol{w} - \boldsymbol{v}\rangle|\Delta^-_{42}(\phi) \\
&= \frac{1}{\cos\phi+1}\left(\langle d\boldsymbol{v}, \boldsymbol{w}\rangle - \cos\phi\,\langle d\boldsymbol{w}, \boldsymbol{v}\rangle\right)|\Delta^-_{42}(\phi) \\
&= \langle d\boldsymbol{v}, \boldsymbol{w}\rangle|\Delta^-_{42}(\phi) \\
&= \theta[\phi]^-_{(42)1},
\end{aligned}$$

so that $\Psi^-_{(42)1}$ is a contact diffeomorphism. Consequently, we have a Legendrian embedding

$$\overline{\mathcal{L}}_1 : U \longrightarrow \Delta_1$$

defined by $\overline{\mathcal{L}}_1(u) = \Psi^-_{(42)1} \circ \mathcal{L}_{42}[\phi](u)$. If we denote that $\overline{\mathcal{L}}_1(u) = (\boldsymbol{X}_1(u), \boldsymbol{X}_2(u))$, then we get

$$\boldsymbol{X}_1(u) = \frac{1}{\cos\phi+1}\left(\mathbb{N}^h_\ell[\phi](u) + \cos\phi\boldsymbol{X}^{\ell}_{-}(u)\right) = \frac{1}{2}\left(\boldsymbol{X}^{\ell}_{+}(u) + \boldsymbol{X}^{\ell}_{-}(u)\right) = \boldsymbol{X}^h(u)$$

and

$$\boldsymbol{X}_2(u) = \frac{1}{\cos\phi+1}\left(\boldsymbol{X}^{\ell}_{-}(u) - \mathbb{N}^h_\ell[\phi](u)\right) = \frac{1}{2}\left(\boldsymbol{X}^{\ell}_{-}(u) - \boldsymbol{X}^{\ell}_{+}(u)\right) = \boldsymbol{X}^d(u).$$

So, we obtain $\overline{\mathcal{L}}_1(u) = \left(\boldsymbol{X}^h(u), \boldsymbol{X}^d(u)\right) = \mathcal{L}_1(u)$. As a consequence of Lemma 3.4 in [12], we can define the *hyperbolic shape operator* of $M^L_- = \boldsymbol{X}^{\ell}_{-}(U)$ at $p = \boldsymbol{X}^{\ell}_{-}(u_0)$ by

$$S_H(p) = -d\boldsymbol{X}^h(u_0) : T_pM^L_- \longrightarrow T_pM^L_-$$

and the *de Sitter shape operator* of $M^L_- = \boldsymbol{X}^{\ell}_{-}(U)$ at $p = \boldsymbol{X}^{\ell}_{-}(u_0)$ by

$$S_D(p) = -d\boldsymbol{X}^d(u_0) : T_pM^L_- \longrightarrow T_pM^L_-.$$

Moreover, we can define the other curvatures of $M^L_- = \boldsymbol{X}^{\ell}_{-}(U)$ at $p = \boldsymbol{X}^{\ell}_{-}(u_0)$ as follows:

$K_H(u_0) = \det S_H(p)$; the *hyperbolic Gauss-Kronecker curvature,*
$K_D(u_0) = \det S_D(p)$; the *de Sitter Gauss-Kronecker curvature,*
$H_H(u_0) = \dfrac{1}{n-1}\mathrm{Trace}\, S_H(p)$; the *hyperbolic mean curvature,*
$H_D(u_0) = \dfrac{1}{n-1}\mathrm{Trace}\, S_D(p)$; the *de Sitter mean curvature.*

We also have the following expressions of the hyperbolic Gauss-Kronecker curvature and the de Sitter Gauss-Kronecker curvature as a corollary of Proposition 6.3.

Proposition 6.7 *Under the same notations in Corollary 6.4, we have the following formulas:*
(1) $K_H = \dfrac{1}{(\cos\phi+1)^{n-1}}\dfrac{\det\left(h^h[\phi]_{ij}-\cos\phi g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)} = \dfrac{1}{2^{n-1}}\dfrac{\det\left(h^\ell_{+ij}-g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}$,
(2) $K_D = \dfrac{1}{(\cos\phi+1)^{n-1}}\dfrac{\det\left(-h^h[\phi]_{ij}-g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)} = \dfrac{1}{2^{n-1}}\dfrac{\det\left(-h^\ell_{+ij}-g_{ij}\right)}{\det\left(g_{\alpha\beta}\right)}$.

Proof. Since

$$\boldsymbol{X}^h(u) = \frac{1}{\cos\phi+1}\left(\mathbb{N}^h_\ell[\phi](u)+\cos\phi\boldsymbol{X}^\ell_-(u)\right) = \frac{1}{2}\left(\boldsymbol{X}^\ell_+(u)+\boldsymbol{X}^\ell_-(u)\right),$$

we have

$$\left(\boldsymbol{X}^h\right)_{u_i} = \sum_{j=1}^{n-1}\frac{\left(-h^h[\phi]_i^j+\cos\phi\delta_i^j\right)}{\cos\phi+1}\boldsymbol{X}^\ell_{-u_j} = \sum_{j=1}^{n-1}\frac{\left(-\left(h^\ell_+\right)_i^j+\delta_i^j\right)}{2}\boldsymbol{X}^\ell_{-u_j}.$$

Using these two relations, we obtain the equations in (1).

The equations in (2) also follow from the following relations

$$\boldsymbol{X}^d(u) = \frac{1}{\cos\phi+1}\left(\boldsymbol{X}^\ell_-(u)-\mathbb{N}^h_\ell[\phi](u)\right) = \frac{1}{2}\left(\boldsymbol{X}^\ell_-(u)-\boldsymbol{X}^\ell_+(u)\right).$$

□

Since $M^L_- = \boldsymbol{X}^\ell_-(U)$ is a Riemannian manifold, it makes sense to consider the Christoffel symbolls. Therefore, we can give the following proposition:

Proposition 6.8 *Let $\boldsymbol{X}^\ell_- : U \longrightarrow LC^*$ be a spacelike hypersurface. Then we have the following lightcone Gauss equations:*

$$\boldsymbol{X}^\ell_{-u_iu_j} = \sum_k \left\{ \begin{array}{c} k \\ ij \end{array} \right\} \boldsymbol{X}^\ell_{-u_k} + \frac{1}{2}\left(g_{ij}\boldsymbol{X}^\ell_+ - h^\ell_{+ij}\boldsymbol{X}^\ell_-\right).$$

Corollary 6.9 *Under the same assumption as the above proposition, we have*

$$\boldsymbol{X}^\ell_{-u_iu_j} = \sum_k \left\{ \begin{array}{c} k \\ ij \end{array} \right\} \boldsymbol{X}^\ell_{-u_k} + \frac{1}{2}\left(g_{ij}-h^\ell_{+ij}\right)\boldsymbol{X}^h + \frac{1}{2}\left(g_{ij}+h^\ell_{+ij}\right)\boldsymbol{X}^d.$$

Now, we stick to the case $n=3$. Even if we change the hypersurface $M^L_+ = \boldsymbol{X}^\ell_+(U)$ into $M^L_- = \boldsymbol{X}^\ell_-(U)$, we have the same relation

$$K_D - K_H = K_S$$

among the curvatures as in Proposition 3.10, where K_S is the sectional curvature of $M^L_- = \boldsymbol{X}^\ell_-(U)$.

It is obvious that

$$S_H(p) = \frac{1}{\cos\phi+1}\left(S^h[\phi](p)-\cos\phi\, id_{T_pM^L_-}\right) \text{ and } S_D(p) = \frac{1}{\cos\phi+1}\left(-id_{T_pM^L_-}-S^h[\phi](p)\right).$$

Let $\overline{\kappa}^h[\phi]_i$ $(i = 1, 2)$ be eigenvalues of $S^h[\phi]$ (i.e., ϕ-hyperbolic principal curvatures of space-like surface $\boldsymbol{X}^\ell_-$) and κ_{Hi} (respectively, κ_{Di}) $(i = 1, 2)$ be hyperbolic (respectively, de Sitter) principal curvatures. Then we get the following relations:

$$\kappa_{Hi} = \frac{\overline{\kappa}^h[\phi]_i - \cos\phi}{\cos\phi + 1} \text{ and } \kappa_{Di} = \frac{-1 - \overline{\kappa}^h[\phi]_i}{\cos\phi + 1}.$$

By using these two relations, we obtain the following equations:
(1) $\kappa_{Hi} + \kappa_{Di} = -1$,
(2) (i) $\overline{\kappa}^h[\phi]_i = (\cos\phi + 1)\kappa_{Hi} + \cos\phi$
(ii)$\overline{\kappa}^h[\phi]_i = -\left(1 + (\cos\phi + 1)\kappa_{Di}\right)$.

Eventually, we have the following "Theorema Egregium":

Theorem 6.10 *The following relation holds:*

$$\frac{(\cos\phi + 1)}{2}K_S - \frac{1 - \cos\phi}{2} = H^h_\ell[\phi] = \frac{(\cos\phi + 1)}{2}(H_H - H_D) + \frac{\cos\phi - 1}{2}.$$

Proof. By definition and the above equations given in (2), we get

$$2H^h_\ell[\phi] = \overline{\kappa}^h[\phi]_1 + \overline{\kappa}^h[\phi]_2 = 2(\cos\phi + 1)H_H + 2\cos\phi = -2(\cos\phi + 1)H_D - 2.$$

Therefore, we have

$$2H^h_\ell[\phi] = (\cos\phi + 1)(H_H - H_D) + \cos\phi - 1.$$

On the other hand, we obtain

$$\begin{aligned} K_H = \kappa_{H1}\kappa_{H2} &= \frac{1}{(\cos\phi + 1)^2}(\overline{\kappa}^h[\phi]_1\overline{\kappa}^h[\phi]_2 - \cos\phi(\overline{\kappa}^h[\phi]_1 + \overline{\kappa}^h[\phi]_2) + \cos^2\phi) \\ &= \frac{1}{(\cos\phi + 1)^2}(K^h_\ell[\phi] - 2\cos\phi H^h_\ell[\phi] + \cos^2\phi) \end{aligned}$$

and

$$\begin{aligned} K_D = \kappa_{D1}\kappa_{D2} &= \frac{1}{(\cos\phi + 1)^2}(1 + \overline{\kappa}^h[\phi]_1 + \overline{\kappa}^h[\phi]_2 + \overline{\kappa}^h[\phi]_1\overline{\kappa}^h[\phi]_2) \\ &= \frac{1}{(\cos\phi + 1)^2}(1 + 2H^h_\ell[\phi] + K^h_\ell[\phi]). \end{aligned}$$

It follows that

$$\begin{aligned} K_D - K_H &= \frac{1}{(\cos\phi + 1)^2}(1 - \cos^2\phi + 2(1 + \cos\phi)H^h_\ell[\phi]) \\ &= \frac{1}{\cos\phi + 1}(1 - \cos\phi + 2H^h_\ell[\phi]). \end{aligned}$$

So, by the same reason as in Proposition 3.10, we get

$$K_S = \frac{1}{\cos\phi + 1}(1 - \cos\phi + 2H^h_\ell[\phi]).$$

This completes the proof. □

Remark 6.11 *The above theorem asserts that the ϕ-hyperbolic mean curvature of a spacelike surface in the 3-dimensional lightcone coincides with the sectional (intrinsic Gauss) curvature if and only if $\phi = 0$. Consequently, the mean curvature is an intrinsic invariant when $\phi = 0$. By the above arguments, we also have*

$$\begin{aligned}\overline{\kappa}^h[\phi]_1\overline{\kappa}^h[\phi]_2 &= (\cos\phi+1)^2\kappa_{H1}\kappa_{H2} + \cos\phi(\cos\phi+1)(\kappa_{H1}+\kappa_{H2}) + \cos^2\phi \\ &= 1 + (\cos\phi+1)(\kappa_{D1}+\kappa_{D2}) + (\cos\phi+1)^2\kappa_{D1}\kappa_{D2},\end{aligned}$$

so that

$$\begin{aligned}K^h_\ell[\phi] &= (\cos\phi+1)^2K_H + 2\cos\phi(\cos\phi+1)H_H + \cos^2\phi \\ &= 1 + 2(\cos\phi+1)H_D + (\cos\phi+1)^2K_D,\end{aligned}$$

which is an extrinsic invariant for any $\phi \in \left[0, \frac{\pi}{2}\right]$.

7 The ϕ-hyperbolic dual as a wave front

In this section, we naturally interpret the ϕ-hyperbolic dual of a spacelike hypersurface in the lightcone as a wave front set in the framework of contact geometry. We also need the theory of Legendrian singularities which has been reviewed in §4.

We consider a point $\boldsymbol{v} = (v_0, v_1, \dots, v_n) \in H^n(-\sin^2\phi)$. Then we have the relation $v_0 = \pm\sqrt{v_1^2 + \cdots + v_n^2 + \sin^2\phi}$. Without the loss of generality, we choose the local coordinate neighbourhood $V_+ = \{\, \boldsymbol{v} \in H^n(-\sin^2\phi) \mid v_0 > 0 \,\}$ and the canonical projection onto $\mathbb{R}^n$ as a local coordinate systeem. We consider the projective cotangent bundle $\pi : PT^*(H^n(-\sin^2\phi)) \longrightarrow H^n(-\sin^2\phi)$ with the canonical contact structure. By using the above coordinate system, we have a trivialization as follows:

$$\Phi : PT^*(V_+) \equiv V_+ \times P(\mathbb{R}^{n-1})^* \ ; \ \Phi([\sum_{i=1}^n \xi_i dv_i]) = ((v_0, v_1, \dots, v_n), [\xi_1 : \cdots : \xi_n]).$$

On the other hand, we define the following mapping:

$$\Psi : \Delta^-_{42}(\phi) \longrightarrow H^n(-\sin^2\phi) \times P(\ \mathbb{R}^{n-1})^* \ ; \ \Psi(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, [v_0w_1 - v_1w_0 : \cdots : v_0w_n - v_nw_0]).$$

By exactly the sam e calculations as those in §4, we have $\Psi^*\theta = v_0\theta[\phi]^-_{(42)1}$ for the canonical contact form $\theta = \sum_{i=1}^n \xi_i dv_i$ on $PT^*(V_+)$, so that Ψ is a contact morphism. Then we can give the following proposition:

Proposition 7.1 *The ϕ-hyperbolic height function $H^h_\phi : U \times H^n(-\sin^2\phi) \longrightarrow \mathbb{R}$ is a Morse family of hypersurfaces.*

We can also give the following theorem:

Theorem 7.2 *For any spacelike hypersurface $\boldsymbol{X}^\ell_- : U \longrightarrow LC^*$, the ϕ-hyperbolic height function $H^h_\phi : U \times H^n(-\sin^2\phi) \longrightarrow \mathbb{R}$ of $M^L_- = \boldsymbol{X}^\ell_-(U)$ is a generating family of the Legendrian immersion $\mathcal{L}_{42}[\phi](U)$ with respect to the Legendrian fibration $\pi[\phi]^-_{(42)1} : \Delta^-_{42}(\phi) \longrightarrow H^n(-\sin^2\phi)$.*

8 Contact with ϕ-hyperbolic flat hyperquadrics

In this section, we consider the contact of spacelike hypersurfaces in the lightcone with ϕ-hyperbolic flat hyperquadrics. We also use the theory of contact due to Montaldi [18].

Now, we consider a function $\mathcal{H}^h_\phi : LC^* \times H^n(-\sin^2\phi) \longrightarrow \mathbb{R}$ defined by $\mathcal{H}^h_\phi(\boldsymbol{u},\boldsymbol{v}) = \langle \boldsymbol{u},\boldsymbol{v}\rangle + \cos\phi + 1$. For any $\boldsymbol{v}_0 \in H^n(-\sin^2\phi)$, we denote that $(\mathfrak{h}^h_\phi)_{v_0}(\boldsymbol{u}) = \mathcal{H}^h_\phi(\boldsymbol{u},\boldsymbol{v}_0)$ and we have a ϕ-hyperbolic flat hyperquadric $(\mathfrak{h}^h_\phi)^{-1}_{v_0}(0) = HP(\boldsymbol{v}_0, -(\cos\phi+1)) \cap LC^* = HL(\boldsymbol{v}_0, -(\cos\phi+1))$. For any $u_0 \in U$, we consider the timelike vector $\boldsymbol{v}_0 = \mathbb{N}^h_\ell[\phi](u_0)$. Then we have

$$(\mathfrak{h}^h_\phi)_{v_0} \circ \boldsymbol{X}^\ell_-(u_0) = \mathcal{H}^h_\phi \circ (\boldsymbol{X}^\ell_- \times id_{H^n(-\sin^2\phi)})(u_0,\boldsymbol{v}_0) = H^h_\phi(u_0, \mathbb{N}^h_\ell[\phi](u_0)) = 0.$$

By Proposition 6.1, we also get the following relations for $i = 1, \ldots, n-1$:

$$\frac{\partial(\mathfrak{h}^h_\phi)_{v_0} \circ \boldsymbol{X}^\ell_-}{\partial u_i}(u_0) = \frac{\partial H^h_\phi}{\partial u_i}(u_0, \mathbb{N}^h_\ell[\phi](u_0)) = 0.$$

This means that the ϕ-hyperbolic flat hyperquadric

$$(\mathfrak{h}^h_\phi)^{-1}_{v_0}(0) = HL(\boldsymbol{v}_0, -(\cos\phi + 1))$$

is tangent to $M^L_- = \boldsymbol{X}^\ell_-(U)$ at $p = \boldsymbol{X}^\ell_-(u_0)$. In this case, we call $HL(\boldsymbol{v}_0, -(\cos\phi+1))$ the *tangent ϕ-hyperbolic flat hyperquadric* of $M^L_- = \boldsymbol{X}^\ell_-(U)$ at $p = \boldsymbol{X}^\ell_-(u_0)$ (or, u_0), which we write $THH[\phi](M^L_-, p)$ (or, $THH[\phi](\boldsymbol{X}^\ell_-, u_0)$).

Eventually, we have tools for the study of the contact between spacelike hypersurfaces and ϕ-hyperbolic flat hyperquadrics. Let $(\mathbb{N}^h_\ell[\phi])_i : (U, u_i) \longrightarrow (H^n(-\sin^2\phi), \boldsymbol{v}_i)$ $(i = 1, 2)$ be ϕ-hyperbolic dual germs of spacelike hypersurface germs $(\boldsymbol{X}^\ell_-)_i : (U, u_i) \longrightarrow (LC^*, \boldsymbol{u}_i)$. We ca also understand the geometric meanings of the $\mathcal{A}$-equivalence among the ϕ-hyperbolic dual germs as an application of the theory of Legendrian singularities [1, 20, 21].

If both of the regular sets of $(\mathbb{N}^h_\ell[\phi])_i$ $(i = 1, 2)$ are dense in (U, u_i), it follows from Proposition 5.2 that $(\mathbb{N}^h_\ell[\phi])_1$ and $(\mathbb{N}^h_\ell[\phi])_2$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian immersion germs $\mathcal{L}^1_{42}[\phi] : (U, u_1) \longrightarrow (\Delta^-_{42}(\phi), z_1)$ and $\mathcal{L}^2_{42}[\phi] : (U, u_2) \longrightarrow (\Delta^-_{42}(\phi), z_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families $(H^h_\phi)_1$ and $(H^h_\phi)_2$ are P-$\mathcal{K}$-equivalent by Theorem 5.3. Here, $(H^h_\phi)_i : (U \times H^n(-\sin^2\phi), (u_i, \boldsymbol{v}_i)) \longrightarrow \mathbb{R}$ is the ϕ-hyperbolic height function germ of $(\boldsymbol{X}^\ell_-)_i$.

On the other hand, if we denote that $(h^h_{\phi i,v_i})(u) = (H^h_\phi)_i(u, \boldsymbol{v}_i)$, then we have $(h^h_{\phi i,v_i})(u) = (\mathfrak{h}^h_\phi)_{v_i} \circ (\boldsymbol{X}^\ell_-)_i(u)$. By Theorem 5.1, for $p_i = (\boldsymbol{X}^\ell_-)(u_i)$

$$K((M^L_-)_1, THH((M^L_-)_1, p_1), p_1) = K((M^L_-)_2, THH((M^L_-)_2, p_2), p_2)$$

if and only if $(h^h_{\phi 1,v_1})$ and $(h^h_{\phi 1,v_2})$ are $\mathcal{K}$-equivalent.

Theorem 8.1 *Let $(\boldsymbol{X}^\ell_-)_i : (U, u_i) \longrightarrow (LC^*, p_i)$ $(i = 1, 2)$ be spacelike hypersurface germs such that the corresponding Legendrian map germs*

$$(\mathbb{N}^h_\ell[\phi])_i = \pi[\phi]^-_{(42)1} \circ \mathcal{L}^i_{42}[\phi] : (U, u_i) \longrightarrow (H^n(-\sin^2\phi), \boldsymbol{v}_i)$$

are Legendrian stable. Then the following conditions are equivalent:

(1) $(\mathbb{N}^h_\ell[\phi])_1$ and $(\mathbb{N}^h_\ell[\phi])_2$ are $\mathcal{A}$-equivalent.

(2) $((\mathbb{N}^h_\ell[\phi])_1(U), z_1)$ and $((\mathbb{N}^h_\ell[\phi])_2(U), z_2)$ are diffeomorphic as set germs.
(3) $\mathcal{L}^1_{42}[\phi] : (U, u_1) \longrightarrow (\Delta^-_{42}(\phi), z_1)$ and $\mathcal{L}^2_{42}[\phi] : (U, u_2) \longrightarrow (\Delta^-_{42}(\phi), z_2)$ are Legendrian equivalent.
(4) $(H^h_\phi)_1$ and $(H^h_\phi)_2$ are P-$\mathcal{K}$-equivalent.
(5) $(h^h_{\phi 1, v_1})$ and $(h^h_{\phi 2, v_2})$ are $\mathcal{K}$-equivalent.
(6) $K((M^L_-)_1, THH((M^L_-)_1, p_1), p_1) = K((M^L_-)_2, THH((M^L_-)_2, p_2), p_2)$.

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