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IKEDA'S CONJECTURE ON THE PERIOD OF THE DUKE-IMAMOḠLU-IKEDA LIFT

HIDENORI KATSURADA AND HISA-AKI KAWAMURA

ABSTRACT. Let k and n be positive even integers. For a primitive form f in $\mathfrak{S}_{2k-n}(SL_2(\mathbf{Z}))$, let $I_n(f)$ be the Duke-Imamoḡlu-Ikeda lift of f to $\mathfrak{S}_k(Sp_n(\mathbf{Z}))$, and $\widetilde{f}$ the cusp form in Kohnen's plus subspace of weight $k-n/2+1/2$ for $\Gamma_0(4)$ corresponding to f under the Shimura correspondence. We then express the ratio $\dfrac{\langle I_n(f), I_n(f)\rangle}{\langle \widetilde{f}, \widetilde{f}\rangle}$ of the period of $I_n(f)$ to that of $\widetilde{f}$ in terms of special values of certain L-functions of f. This proves the conjecture proposed by Ikeda [Ike06] concerning the period of the Duke-Imamoḡlu-Ikeda lift.

1. INTRODUCTION

One of the fascinating problems in the theory of modular forms is to find the relation between the periods (or the Petersson products) of cuspidal Hecke eigenforms which are related with each other through their L-functions. In particular, there are several important results concerning the relation between the period of a cuspidal Hecke eigenform f with respect to an elliptic modular group Γ and that of its lift $\widehat{f}$. Here we mean by the lift $\widehat{f}$ of f a cuspidal Hecke eigenform with respect to another modular group Γ' whose certain L-function can be expressed in terms of certain L-functions of f. Thus we propose the following problem:

Problem A. *Express the ratio* $\dfrac{\langle \widehat{f}, \widehat{f}\rangle}{\langle f, f\rangle^e}$ *in terms of arithmetic invariants of* f*, for example, the special values of certain* L*-functions* f *for some integer* e*.*

We also propose the following problem:

Problem A'. *In addition to the notation and the assumption as Problem A, consider another lift* $\widetilde{f}$ *of* f*. Then express the ratio* $\dfrac{\langle \widehat{f}, \widehat{f}\rangle}{\langle \widetilde{f}, \widetilde{f}\rangle}$ *in terms of arithmetic invariants of* f*.*

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As will be explained later, these two problems are closely related.

Zagier [Zag77] solved the Problem A for the Doi-Nagamnuma lift $\widehat{f}$ of f. Murase and Sugano [MS06] solved the Problem A for the Kudla lift $\widehat{f}$ of f. Kohnen and Skoruppa [KS89] solved the Problem B in the case $\widetilde{f}$ is the Hecke eigenform in Kohnen's plus subspace corresponding to f under the Shimura correspondence and $\widehat{f}$ is the Saito-Kurokawa lift of f (see also Oda [Oda81]). This result also solved the Problem A combined with the result of Kohnen-Zagier [KZ81]. (See also Theorem 2.2). We note that this type of period relation is not only interesting and important in its own right but also plays an important role in arithmetic theory of modular forms. For instance, by using Kohnen and Skoruppa's result, Brown [Bro07] and Katsurada [Kat08a] independently proved Harder's conjecture concerning congruence between Saito-Kurokawa lifts and non-Saito-Kurokawa lifts under mild conditions. Furthermore, by using this congruence, Brown costructed a non-trivial element of a certain Bloch-Kato Selmer group. We also note that this type of conguence relation was conjectured by Doi-Hida-Ishii [DHI98] in the case $\widehat{f}$ is the Doi-Naganuma lift of f.

Now let f be a primitive form, namely, a normalized Hecke eigtenform in $\mathfrak{S}_{2k-n}(SL_2(\mathbf{Z}))$ Then Duke and Imamo$\bar{\text{g}}$lu conjectured, in their unpublished paper, that there exists a cuspidal Hecke eigenform in $\mathfrak{S}_k(Sp_n(\mathbf{Z}))$ whose standard L-function can be expressed as $\zeta(s)\prod_{i=1}^{n} L(s+k-i,f)$, where $\zeta(s)$ is Riemann's zeta function and $L(s,f)$ is Hecke's L-function of f. Ikeda [Ike01] did construct such a modular form $I_n(f)$. We call $I_n(f)$ the Duke-Imamo$\bar{\text{g}}$lu-Ikeda lift of f. Let $\widetilde{f}$ be the cusp form in Kohnen's plus subspace of weight $k-n/2+1/2$ for $\Gamma_0(4)$ corresponding to f under the Shimura correspondence. In [Ike06], Ikeda among others conjectured that the ratio $\dfrac{\langle I_n(f), I_n(f)\rangle}{\langle \widetilde{f}, \widetilde{f}\rangle}$ should be expressed as $L(k,f)\zeta(n)\prod_{i=1}^{n/2-1} L(2i+1,f,\mathrm{Ad})\zeta(2i)$ up to elementary factor, where $L(s,f,\mathrm{Ad})$ is the adjoint L-function of f (cf. Conjecture A). This is a conjectural generalization of Kohnen and Skoruppa's result on the Saito-Kurokawa lift. The aim of this paper is to prove Ikeda's conjecture and to apply this to Problem A for the Duke-Imamo$\bar{\text{g}}$lu-Ikeda lift (cf. Theorems 2.1 and 2.2).

We note that $I_n(f)$ is not realized as a theta lift at present except in the case $n=2$. Therefore we cannot use a general method for inner product formula of theta lifts due to Rallis [Ral88]. The method we use is to give explicit formulas of several types of Dirichlet series of Rankin-Selberg type, and compare their residues. We explain it more precisely.

First let $\phi_{I_n(f),1}$ be the first Fourier-Jacobi coefficient of $I_n(f)$ and $\sigma_{n-1}(\phi_{I_n(f),1}) = \sum_A c(A)\mathbf{e}(\mathrm{Tr}(AZ))$ the element of generalized Kohnen's

plus subspace of weight $k-1/2$ with respect to $\Gamma_0^{(n-1)}(4)$ corresponding to $\phi_{I_n(f),1}$ under the Ibukiyama isomorphism σ_{n-1}. In Section 3, we consider the following Dirichlet series $R(s,\sigma_{n-1}(\phi_{I_n(f),1}))$ of Rankin-Selberg type associated with it:

$$R(s,\sigma_{n-1}(\phi_{I_n(f),1}))=\sum_A \frac{|c(A)|^2}{e(A)(\det A)^s},$$

where A runs over all the $SL_{n-1}(\mathbf{Z})$-equivalence classes of positive definite half-integral matrices of degree $n-1$ and $e(A)$ denotes the order of the unit group of A in $SL_{n-1}(\mathbf{Z})$. For the precise definition, see Section 3. This type of Dirichlet series was studied by many people in integral weight case, and its analytic properties are known (cf. Kalinin [Kal84]). In half-integral weight case, similarly to the integral weight case, we also get an analytic properties of $R(s,\sigma_{n-1}(\phi_{I_n(f),1}))$, and in particular we can express its residue at $k-1/2$ in terms of the period of $\phi_{I_n(f),1}$ (cf. Corollary to Proposition 3.1). We then rewrite Ikeda's conjecture in terms of the relation between the residue of $R(s,\sigma_{n-1}(\phi_{I_n(f),1}))$ at $s=k-1/2$ and the period of $\widetilde{f}$ (cf. Conjecture B). In order to prove Conjecture B, we have to get an explicit formula of $R(s,\sigma_{n-1}(\phi_{I_n(f),1}))$ in terms of $L(s,f,\mathrm{Ad})$ and $L(s,\widetilde{f})$. To get it, in Section 4, we reduce our computation to that of certain formal power series, which we call formal power series of Rankin-Selberg type, associated with local Siegel series similarly to [IK04] and [IK06] (cf. Theorem 4.2). Section 5 is devoted to the computation of them. This computation is similar to those in [IK04] and [IK06], but is more elaborate and longer than them. In particular we should be careful in dealing with the case $p=2$. After overcoming such obstacles we can get explicit formulas of formal power series of Rankin-Selberg type (cf. Theorem 5.5.1). In Section 6, by using Theorem 5.5.1, we immediately get an explicit formula of $R(s,\sigma_{n-1}(\phi_{I_n(f),1}))$ (cf. Theorem 6.2,) and by taking the residue of it at $k-1/2$ we prove Conjecture B, and thetrefore prove Conjecture A (cf. Theorem 6.3).

We note that we can also give an explicit formula of the Rankin-Selberg series of $I_n(f)$. However, it does not seem useful for proving Conjecture A directly from such a formula.

We also note that we can apply the above result to a problem concerning congruence between Duke-Imamoḡlu-Ikeda lifts and non-Duke-Imamoḡlu-Ikeda lifts. This was announced in [KK08b], and the detail will be discussed in [Kat08b].

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Notation. Let R be a commutative ring. We denote by $R^\times$ and R^* the semigroup of non-zero elements of R and the unit group of R, respectively. We also put $S^\square = \{a^2 \mid a \in S\}$ for a subset S of R. We denote by $M_{mn}(R)$ the set of $m \times n$-matrices with entries in R. In particular put $M_n(R) = M_{nn}(R)$. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix A. For an $m \times n$-matrix X and an $m \times m$-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, if R is an integral domain of characteristic different from 2, let $\mathcal{L}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{L}_n(R)$ is the subset of symmetric matrices of degree n whose (i,j)-component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. In particular, we put $\mathcal{L}_n = \mathcal{L}_n(\mathbf{Z})$, and $\mathcal{L}_{n,p} = \mathcal{L}_n(\mathbf{Z}_p)$ for a prime number p. For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. If S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. $GL_n(R)$ acts on the set $S_n(R)$ in the following way:

$$GL_n(R) \times S_n(R) \ni (g, A) \longmapsto {}^tgAg \in S_n(R).$$

Let G be a subgroup of $GL_n(R)$. For a subset $\mathcal{B}$ of $S_n(R)$ stable under the action of G we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ with respect to G. We sometimes identify $\mathcal{B}/G$ with a complete set of representatives of $\mathcal{B}/G$. We abbreviate $\mathcal{B}/GL_n(R)$ as $\mathcal{B}/\sim$ if there is no fear of confusion. Two symmetric matrices A and A' with entries in R are said to be equivalent over R' with each other and write $A \sim_{R'} A'$ if there is an element X of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

For an integer $D \in \mathbf{Z}$ such that $D \equiv 0$ or $\equiv 1 \bmod 4$, let $\mathfrak{d}_D$ be the discriminant of $\mathbf{Q}(\sqrt{D})$, and put $\mathfrak{f}_D = \sqrt{\frac{D}{\mathfrak{d}_D}}$. We call an integer D a fundamental discriminant if it is the discriminant of some quadratic extension of $\mathbf{Q}$ or 1. For a fundamental discriminant D, let $\left(\frac{D}{*}\right)$ be the character corresponding to $\mathbf{Q}(\sqrt{D})/\mathbf{Q}$. Here we make the convention that $\left(\frac{D}{*}\right) = 1$ if $D = 1$.

We put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbf{C}$. For a prime number p we denote by $\nu_p(*)$ the additive valuation of $\mathbf{Q}_p$ normalized so that $\nu_p(p) = 1$, and by $\mathbf{e}_p(*)$ the continuous additive character of $\mathbf{Q}_p$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for $x \in \mathbf{Q}$.

2. Ikeda's conjecture on the Period of the Duke-Imamoḡlu-Ikeda lift

Put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where 1_n and O_n denotes the unit matrix and the zero matrix of degree n, respectively. Furthermore, put

$$\Gamma^{(n)} = Sp_n(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \mid J_n[M] = J_n\}.$$

Let $\mathbf{H}_n$ be Siegel's upper half-space of degree n. Let l be an integer or half integer. For a congruence subgroup Γ of $\Gamma^{(n)}$, we denote by $\mathfrak{M}_l(\Gamma)$ the space of holomorphic modular forms of weight l with respect to Γ. We denote by $\mathfrak{S}_l(\Gamma)$ the subspace of $\mathfrak{M}_l(\Gamma)$ consisting of cusp forms. For two holomorphic cusp forms F and G of weight l with respect to Γ we define the Petersson product $\langle F, G\rangle$ by

$$\langle F, G\rangle = [\Gamma^{(n)} : \Gamma\{\pm 1_{2n}\}]^{-1} \int\limits_{\Gamma\backslash\mathbf{H}_n} F(Z)\overline{G(Z)} \det(\mathrm{Im}(Z))^l d^*Z,$$

where d^*Z denote the invariant volume element on $\mathbf{H}_n$ defined as usual. We call $\langle F, F\rangle$ the period of F. Let

$$\Gamma_0^{(m)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(m)} \;\middle|\; C \equiv O_m \bmod N \right\},$$

and in particular put $\Gamma_0(N) = \Gamma_0^{(1)}(N)$. Let p be a prime number. For a non-zero element $a \in \mathbf{Q}_p$ we put $\chi_p(a) = 1, -1$, or 0 according as $\mathbf{Q}_p(a^{1/2}) = \mathbf{Q}_p, \mathbf{Q}_p(a^{1/2})$ is an unramified quadratic extension of $\mathbf{Q}_p$, or $\mathbf{Q}_p(a^{1/2})$ is a ramified quadratic extension of $\mathbf{Q}_p$. We note that $\chi_p(D) = \left(\frac{D}{p}\right)$ if D is a fundamental discriminant. For an element T of $\mathcal{L}_{n,p}^{\times}$ with n even, put $\xi_p(T) = \chi_p((-1)^{n/2}\det T)$. Let T be an element of $\mathcal{L}_n^{\times}$. Then $(-1)^{n/2}\det(2T) \equiv 0$ or $\equiv 1 \bmod 4$, and we define $\mathfrak{d}_T$ and $\mathfrak{f}_T$ as $\mathfrak{d}_T = \mathfrak{d}_{(-1)^{n/2}\det(2T)}$ and $\mathfrak{f}_T = \mathfrak{f}_{(-1)^{n/2}\det(2T)}$, respectively. Let T be an element of $\mathcal{L}_{n,p}^{\times}$ there exists an element $\tilde{T}$ of $\mathcal{L}_n^{\times}$ such that $\widetilde{T} \sim_{\mathbf{Z}_p} T$. We then put $\mathfrak{d}_T = \mathfrak{d}_{\tilde{T}}$ and $\mathfrak{f}_T = \mathfrak{f}_{\tilde{T}}$. We note that $\mathfrak{d}_T$ and $\mathfrak{f}_T$ are uniquely determined by T up to $\mathbf{Z}_p^{*\square}$-multiple and $\mathbf{Z}_p^*$-multiple, respectively. We put $\mathfrak{e}_p(T) = \nu_p(\mathfrak{f}_T)$.

Now for $T \in \mathcal{L}_{n,p}^{\times}$ we define the local Siegel series $b_p(T, s)$ by

$$b_p(T, s) = \sum_{R \in S_n(\mathbf{Q}_p)/S_n(\mathbf{Z}_p)} \mathbf{e}_p(\mathrm{tr}(TR)) p^{-\nu_p(\mu_p(R))s},$$

where $\mu_p(R) = [R\mathbf{Z}_p^n + \mathbf{Z}_p^n : \mathbf{Z}_p^n]$. We remark that there exists a unique polynomial $F_p(T, X)$ in X such that

$$b_p(T, s) = F_p(T, p^{-s}) \frac{(1 - p^{-s})\prod_{i=1}^{n/2}(1 - p^{2i-2s})}{1 - \xi_p(T)p^{n/2-s}}$$

(cf. Kitaoka [Kit84]). We then define a Laurent polynomial $\widetilde{F}_p(T,X)$ as

$$\widetilde{F}_p(B,X)=X^{-\mathfrak{e}_p(T)}F_p(T,p^{-(n+1)/2}X).$$

We remark that $\widetilde{F}_p(B,X^{-1})=\widetilde{F}_p(B,X)$ (cf. [Kat99]). Now let k be a positive even integer. Let

$$f(z)=\sum_{m=1}^{\infty}a(m)\mathbf{e}(mz)$$

be a primitive form in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$. Let $\alpha_p\in\mathbf{C}$ such that $\alpha_p+\alpha_p^{-1}=p^{-k+n/2+1/2}a(p)$, which we call the Satake p-parameter of f. Then for a Dirichlet character χ we define Hecke's L-function $L(s,f,\chi)$ twisted by χ as

$$L(s,f,\chi)=\prod_p\{(1-\alpha_p p^{-s+k-n/2-1/2}\chi(p))(1-\alpha_p^{-1}p^{-s+k-n/2-1/2}\chi(p))\}^{-1}.$$

In particular, if χ is the principal character we write $L(s,f,\chi)$ as $L(s,f)$ as usual. Let

$$\widetilde{f}(z)=\sum_m c(m)\mathbf{e}(mz)$$

be a cuspidal Hecke eigenform in Kohnen's plus subspace $\mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$ corresponding to f under the Shimura correspondence (cf. Kohnen, [Koh80]). For the precise definition of Kohnen's plus subspace, we give it in Section 3 in more general setting. We define a Fourier series $I_n(f)(Z)$ in $Z\in\mathbf{H}_n$ by

$$I_n(f)(Z)=\sum_{T\in\mathcal{L}_{n>0}}a_{I_n(f)}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$a_{I_n(f)}(T)=c(|\mathfrak{d}_T|)\mathfrak{f}_T^{k-n/2-1/2}\prod_p\widetilde{F}_p(T,\alpha_p).$$

Then Ikeda [Ike01] showed the following:

$I_n(f)(Z)$ is a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$, and its standard L-function coincides with

$$\zeta(s)\prod_{i=1}^{n}L(s+k-i,f).$$

This was first conjectured by Duke and Imamoḡlu. We call $I_n(f)$ the Duke-Imamoḡlu-Ikeda lift of f as in Section 1. We note that $I_n(f)$ is uniquely determined by $\widetilde{f}$. We also note that $I_2(f)$ coincides with the Saito-Kurokawa lift of f.

To formulate Ikeda's conjecture, put

$$\Gamma_{\mathbf{R}}(s)=\pi^{-s/2}\Gamma(s/2)\quad\text{and}\quad\Gamma_{\mathbf{C}}(s)=\Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1).$$

We note that $\Gamma_{\mathbf{C}}(s)=2(2\pi)^{-s}\Gamma(s)$. Furthermore put

$$\xi(s) = \Gamma_{\mathbf{R}}(s)\zeta(s) \quad \text{and} \quad \widetilde{\xi}(s) = \Gamma_{\mathbf{C}}(s)\zeta(s).$$

For a Dirichlet character χ put

$$\Lambda(s,\, f, \chi) = \frac{\Gamma_{\mathbf{C}}(s) L(s,\, f, \chi)}{\tau(\chi)},$$

where $\tau(\chi)$ is the Gauss sum of χ. In particular, we simply write $\Lambda(s,\, f, \chi)$ as $\Lambda(s,\, f)$ if χ is the principal character. Furthermore, we define the adjoint L-function $L(s,\, f,\, \mathrm{Ad})$ as

$$L(s, f, \mathrm{Ad}) = \prod_p \{(1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}p^{-s})(1-p^{-s})\}^{-1},$$

and put

$$\Lambda(s,\, f,\, \mathrm{Ad}) = \Gamma_{\mathbf{R}}(s+1)\Gamma_{\mathbf{C}}(s+2k-n-1) L(s,\, f,\, \mathrm{Ad}),$$

and

$$\widetilde{\Lambda}(s,\, f,\, \mathrm{Ad}) = \Gamma_{\mathbf{R}}(s)\Lambda(s,\, f,\, \mathrm{Ad}).$$

We note that

$$\Lambda(1-s,\, f,\, \mathrm{Ad}) = \Lambda(s,\, f,\, \mathrm{Ad}),$$

and

$$\widetilde{\Lambda}(s,\, f,\, \mathrm{Ad}) = \Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s+2k-n-1) L(s,\, f,\, \mathrm{Ad}).$$

Now we have the following diagram of liftings:

$$\begin{array}{ccccc} \mathfrak{S}^+_{k-(n-1)/2}(\Gamma_0(4)) & \simeq & \mathfrak{S}_{2k-n}(\Gamma^{(1)}) & \to & \mathfrak{S}_k(\Gamma^{(n)}) \\ \widetilde{f} & \leftrightarrow & f & \mapsto & I_n(f) \end{array}$$

Then Ikeda [Ike06] among others proposed the following conjecture:

Conjecture A. *We have*

$$\frac{\langle I_n(f),\, I_n(f)\rangle}{\langle \widetilde{f},\, \widetilde{f}\rangle} = 2^{\alpha(n,k)} \Lambda(k,\, f) \widetilde{\xi}(n) \prod_{i=1}^{n/2-1} \widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad})\, \widetilde{\xi}(2i),$$

where $\alpha(n,k) = -(n-3)(k-n/2) - n + 1$.

Remark. When $n = 2$, Conjecture A holds true; It has been proved by Kohnen and Skoruppa [KS89] (see also Oda [Oda81]).

Now our main result in this paper is the following:

Theorem 2.1. *Conjecture A holds true for any positive even integer* n.

By the above theorem, we can solve the Problem A for the Duke-Imamoḡlu-Ikeda lift:

Theorem 2.2. *Let the notation be as above. Let D be a fundamental discriminant D such that $(-1)^{n/2}D > 0$ and suppose that $L(k - n/2, f, (\frac{D}{*})) \neq 0$. Then*

$$\frac{\langle I_n(f),\, I_n(f)\rangle}{\langle f,\, f\rangle^{n/2}} = \frac{\sqrt{-1}^{a_{n,k}} 2^{b_{n,k}} |c(|D|)|^2 \Lambda(k,\, f)}{|D|^{k-n/2}\Lambda(k - n/2, f, (\frac{D}{*}))} \widetilde{\xi}(n)$$

$$\times \prod_{i=1}^{n/2-1} \frac{\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad})}{\langle f,\, f\rangle} \widetilde{\xi}(2i),$$

where $a_{n,k} = 0$ or -1 according as $n \equiv 0 \bmod 4$ or $n \equiv 2 \bmod 4$, and $b_{n,k}$ is some integer depending only on n and k.

Proof. By Theorem 1 in [KZ81], for any such D we have

$$\frac{|c(|D|)|^2}{\langle \widetilde{f}, \widetilde{f}\rangle} = \frac{2^{k-n/2-1}|D|^{k-n/2}\Lambda(k - n/2, f, (\frac{D}{*}))}{\sqrt{-1}^{a_{n,k}}\langle f, f\rangle}.$$

Thus, by Theorem 2.1, the assertion holds. □

It is well-known that $\dfrac{(-1)^{n/4}\Lambda(k,\, f)}{\Lambda(k - n/2, f, (\frac{D}{*}))}$ and $\dfrac{\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad})}{\langle f,\, f\rangle}$ for $i = 1, ..., n/2 - 1$ are algebraic numbers and belong to the Hecke field $\mathbf{Q}(f)$ (cf. Shimura [Shi76], [Shi00]). Thus we obtain

Corollary. *If all the Fourier coefficients of $\widetilde{f}$ are algebraic, then the ratio $\dfrac{\langle I_n(f),\, I_n(f)\rangle}{\langle f,\, f\rangle^{n/2}}$ is algebraic.*

We note that we can multiply some non-zero complex number c with $\widetilde{f}$ so that all the Fourier coefficients of $c\widetilde{f}$ belong to $\mathbf{Q}(f)$. We also note that the above result has been proved by Furusawa [Fur84] in case $n = 2$, and by Y. Choie and Kohnen [CK03] in general case. Thus Theorem 2.2 can be regarded as a refinement of their results.

3. Rankin-Selberg convolution product of the image of the first Fourier-Jacobi coefficient of the Duke-Imamoḡlu-Ikeda lift under the Ibukiyama isomorphism

To prove Conjecture A, we rewrite it in terms of the residue of the Rankin-Selberg convolution product of a certain half-integral weight modular form. Let l be a positive integer. Let $F(Z)$ be an element of $\mathfrak{S}_{l-1/2}(\Gamma_0^{(m)}(4))$. Then $F(Z)$ has the following Fourier expansion:

$$F(Z) = \sum_{A \in \mathcal{L}_{m>0}} a_F(A)\mathbf{e}(\mathrm{tr}(AZ))$$

We define the Rankin-Selberg convolution product $R(s,F)$ of F as

$$R(s,F)=\sum_{A\in\mathcal{L}_{m>0}/SL_m(\mathbf{Z})}\frac{|a_F(A)|^2}{e(A)(\det A)^s},$$

where $e(A)=\#\{X\in SL_m(\mathbf{Z})\mid A[X]=A\}$. Put

$$\mathcal{L}'_{m>0}=\{A\in\mathcal{L}_{m>0}\mid A\equiv -\,{}^trr \bmod 4\mathcal{L}_m \text{ for some } r\in\mathbf{Z}^m\}.$$

We note the r in the above definition is uniquely determined modulo $2\mathbf{Z}^m$ by A, which will be denoted by r_A. Now we define generalized Kohnen's plus subspace of weight $l-1/2$ with respective to $\Gamma_0^{(m)}(4)$ as

$$\mathfrak{S}^+_{l-1/2}(\Gamma_0^{(m)}(4))=$$

$$\left\{F(Z)=\sum_{A\in\mathcal{L}_{m>0}}c(A)\mathbf{e}(\mathrm{tr}(AZ))\in\mathfrak{S}_{l-1/2}(\Gamma_0^{(m)}(4))\;\middle|\;\begin{array}{l}c(A)=0\\ \text{unless } A\in\mathcal{L}'_{m>0}\end{array}\right\}.$$

Then there exists a isomorphism from the space of Jacobi forms of index 1 to generalized Kohnen's plus space due to Ibukiyama. To explain this, let $\Gamma_J^{(m)}=\Gamma^{(m)}\ltimes \mathrm{H}_m(\mathbf{Z})$, where $\mathrm{H}_m(\mathbf{Z})$ is the subgroup of the Heisenberg group $\mathrm{H}_m(\mathbf{R})$ consisting of all elements with integral entries.

Let $J^{\mathrm{cusp}}_{l,\,N}(\Gamma_J^{(m)})$ denote the space of Jacobi cusp forms of weight l and index N with respect to the Jacobi group $\Gamma_J^{(m)}$. Let $\phi(Z,z)\in J^{\mathrm{cusp}}_{l,1}(\Gamma_J^{(m)})$. Then we have the following Fourier-Jacobi expansion:

$$\phi(Z,\,z)=\sum_{\substack{T\in\mathcal{L}_m,\,r\in\mathbf{Z}^m,\\ 4T-{}^trr>0}}c(T,r)\mathbf{e}(\mathrm{tr}(TZ)+r^tz).$$

We say that two elements (T,r) and (T',r') of $\mathcal{L}_m\times\mathbf{Z}^m$ are $SL_m(\mathbf{Z})$-equivalent and write $(T,r)\sim(T',r')$ if there exists an element $g\in SL_m(\mathbf{Z})$ such that $T'-{}^tr'r'/4=(T-{}^trr/4)[g]$. We then define a Dirichlet series $R(s,\phi)$ as

$$R(s,\phi)=\sum_{(T,r)}\frac{|c(T,r)|^2}{e(T-{}^trr/4)(\det(T-{}^trr/4))^s},$$

where (T,r) runs over a complete set of representatives of $SL_m(\mathbf{Z})$-equivalence classes of $\mathcal{L}_m\times\mathbf{Z}^m$ such that $T-{}^trr/4\in\mathcal{L}_{m>0}$. Now $\phi(Z,z)$ can also be expressed as follows:

$$\phi(Z,z)=\sum_{r\in\mathbf{Z}^m/2\mathbf{Z}^m}h_r(Z)\theta_r(Z,z),$$

where $h_r(Z)$ is a holomorphic function on $\mathbf{H}_m$, and

$$\theta_r(Z,z)=\sum_{\lambda\in M_{1,m}(\mathbf{Z})}\mathbf{e}(\mathrm{tr}(Z[{}^t(\lambda+2^{-1}r)])+2(\lambda+2^{-1}r)^tz).$$

We note that $h_r(Z)$ have the following Fourier expansion:

$$h_r(Z) = \sum_T c(T,r)\mathbf{e}(\mathrm{tr}((T - {}^t rr/4)Z)),$$

where T runs over all elements of $\mathcal{L}_m$ such that $T - {}^t rr/4$ is positive definite. Put $\mathbf{h}(Z) = (h_r(Z))_{r\in\mathbf{Z}^m/2\mathbf{Z}^m}$. Then $\mathbf{h}$ is a vector valued modular form of weight $l-1/2$ with respect to $\Gamma^{(m)}$, that is, for each $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{(m)}$ we have

$$\mathbf{h}(\gamma(Z)) = J(\gamma,Z)\mathbf{h}(\gamma(Z)).$$

Here $J(\gamma,Z)$ is an $m\times m$ matrix whose entries are holomorphic functions on $\mathbf{H}_m$ such that $\overline{{}^t J(\gamma,Z)}J(\gamma,Z) = |j(\gamma,Z)|^{2l-1}1_m$, where $j(\gamma,Z) = \det(CZ+D)$. In particular, we have

$$\sum_{r\in\mathbf{Z}^m/2\mathbf{Z}^m} h_r(\gamma(Z))\overline{h_r(\gamma(Z))} = |j(\gamma,Z)|^{2l-1}\sum_{r\in\mathbf{Z}^m/2\mathbf{Z}^m} h_r(Z)\overline{h_r(Z)}.$$

We then put

$$\sigma_m(\phi)(Z) = \sum_{r\in\mathbf{Z}^m/2\mathbf{Z}^m} h_r(4Z).$$

Then Ibukiyama [Ibu92] showed the following:

Let l be a positive even integer. Then σ_m gives a $\mathbf{C}$-linear isomorphism

$$\sigma_m : J^{\mathrm{cusp}}_{l,1}(\Gamma^{(m)}_J) \simeq \mathfrak{S}^+_{l-1/2}(\Gamma^{(m)}_0(4)),$$

which is compatible with the actions of Hecke operators.

We call σ_m the Ibukiyama isomorphism. We note that

$$\sigma_m(\phi) = \sum_{A\in S_m(\mathbf{Z})_{>0}} c((A + {}^t r_A r_A)/4, r_A)\mathbf{e}(\mathrm{tr}(AZ)),$$

where $r = r_A$ denote an element of $\mathbf{Z}^m$ such that $A + {}^t r_A r_A \in 4\mathcal{L}_m$. This r_A is uniquely determined up to modulo $2\mathbf{Z}^m$, and $c((A+{}^t r_A r_A)/4, r_A)$ does not depend on the choice of the representative of r_A mod $2\mathbf{Z}^m$. Furthermore, we have

$$R(s,\sigma_m(\phi)) = \sum_{A\in\mathcal{L}'_{m>0}/SL_m(\mathbf{Z})} \frac{|c((A+{}^t rr)/4, r)|^2}{e(A)\det A^s},$$

and hence

$$R(s,\phi) = 2^{2sm}R(s,\sigma_m(\phi)).$$

Now for $\phi,\ \psi \in J^{\mathrm{cusp}}_{l,1}(\Gamma^{(m)}_J)$ we define the Petersson product of ϕ and ψ by

$$\langle \phi,\ \psi\rangle = \int_{\Gamma^{(m)}_J\backslash(\mathbf{H}_m\times\mathbf{C}^m)} \phi(Z,\ z)\overline{\psi(Z,\ z)}\det(v)^{l-m-2}\exp(-4\pi v^{-1}[{}^t y])\, dudvdxdy,$$

where $Z = u + \sqrt{-1}v \in \mathbf{H}_m$, $z = x + \sqrt{-1}y \in \mathbf{C}^m$. Now we consider the analytic properties of $R(s, \phi)$.

Proposition 3.1. *Let l be a positive integer. Let $\phi(Z, z) \in J_{l,1}^{\mathrm{cusp}}(\Gamma_J^{(m)})$. Put*

$$\mathcal{R}(s, \phi) = \gamma_m(s)\xi(2s + m + 2 - 2l) \prod_{i=1}^{[m/2]} \xi(4s + 2m + 4 - 4l - 2i)R(s, \phi),$$

where

$$\gamma_m(s) = 2^{1-2sm} \prod_{i=1}^{m} \Gamma_{\mathbf{R}}(2s - i + 1).$$

Then the following assertions hold:

(1) *$\mathcal{R}(s, \phi)$ has a meromorphic continuation to the whole s-plane, and has the following functional equation:*

$$\mathcal{R}(2l - 3/2 - m/2 - s, \phi) = \mathcal{R}(s, \phi).$$

(2) *$\mathcal{R}(s, \phi)$ is holomorphic for $\mathrm{Re}(s) > l - 1/2$, and has a simple pole at $s = l - 1/2$ with the residue $2^{m+1} \prod_{i=1}^{[m/2]} \xi(2i + 1)\langle \phi, \phi \rangle$.*

Proof. The assertion can be proved in the same manner as in Kalinin [Kal84], but for the convenience of readers we here give an outline of the proof. We define the non-holomorphic Siegel Eisenstein series $E^{(m)}(Z, s)$ by

$$E^{(m)}(Z, s) = (\det \mathrm{Im}(Z))^s \sum_{M \in \Gamma_\infty^{(m)} \backslash \Gamma^{(m)}} |j(M, Z)|^{-2s},$$

where $\Gamma_\infty^{(m)} = \left\{ \begin{pmatrix} A & B \\ O_m & D \end{pmatrix} \in \Gamma^{(m)} \right\}$. For the $\phi(Z, z)$ let $\mathbf{h}(Z) = (h_r(Z))_{r \in \mathbf{Z}^m/2\mathbf{Z}^m}$ be as above. Since $\mathbf{h}$ is a vector valued modular form with respect to $\Gamma^{(m)}$, we can apply the Rankin-Selberg method and we obtain

$$\mathcal{R}(s, \phi) = \int_{\Gamma^{(m)} \backslash \mathbf{H}_m} \sum_{r \in \mathbf{Z}^m/2\mathbf{Z}^m} h_r(Z)\overline{h_r(Z)} \mathrm{Im}(Z)^{l-1/2} \mathcal{E}^{(m)}(Z, s) d^*Z,$$

where

$$\mathcal{E}^{(m)}(Z, s) = \xi(2s + m + 2 - 2l) \times \prod_{i=1}^{[m/2]} \xi(4s + 2m + 4 - 4l - 2i) E^{(m)}(Z, s + m/2 + 1 - l).$$

It is well-known that $\mathcal{E}^{(m)}(Z, s)$ has a meromorphic continuation to the whole s-plane, and has the following functional equation:

$$\mathcal{E}^{(m)}(Z, 2l - 3/2 - m/2 - s) = \mathcal{E}^{(m)}(Z, s).$$

Thus the first assertion (1) holds. Furthermore it is holomorphic for $\mathrm{Re}(s) > l - 1/2$, and has a simple pole at $s = l - 1/2$ with the residue $\prod_{i=1}^{[m/2]} \xi(2j+1)$. We note that

$$\langle \phi,\, \phi \rangle = 2^{-m-1} \int\limits_{\Gamma^{(m)} \backslash \mathbf{H}_m} \sum_{r \in \mathbf{Z}^m / 2\mathbf{Z}^m} h_r(Z) \overline{h_r(Z)} \mathrm{Im}(Z)^{l-1/2} d^* Z.$$

Thus the second assertion (2) holds. □

Now let l be a positive even integer. For $F \in \mathfrak{S}^+_{l-1/2}(\Gamma_0^{(m)}(4))$ put

$$\mathcal{R}(s, F) = \prod_{i=1}^{m} \Gamma_{\mathbf{R}}(2s - i + 1)$$
$$\times \quad \xi(2s + m + 2 - 2l) \prod_{i=1}^{[m/2]} \xi(4s + 2m + 4 - 4l - 2i) R(s, F).$$

We note that

$$\mathcal{R}(s, \sigma_m(\phi)) = 2^{-1} \mathcal{R}(s, \phi)$$

for $\phi \in J^{\mathrm{cusp}}_{l,1}(\Gamma_J^{(m)})$. Thus we obtain

Corollary. *In addition to the notation and the assumption as Proposition 3.1, suppose that l is even. Then $\mathcal{R}(s, \sigma_m(\phi))$ has a meromorphic continuation to the whole s-plane, and has the following functional equation:*

$$\mathcal{R}(2l - 3/2 - m/2 - s, \sigma_m(\phi)) = \mathcal{R}(s, \sigma_m(\phi)).$$

Furthermore it is holomorphic for $\mathrm{Re}(s) > l - 1/2$, *and has a simple pole at $s = l - 1/2$ with the residue $2^m \prod_{i=1}^{[m/2]} \xi(2i+1) \langle \phi, \phi \rangle$.*

Let f be a primitive form in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$, and $\widetilde{f}$ and $I_n(f)$ be as in Section 2. Write $Z \in \mathbf{H}_n$ as $Z = \begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}$ with $\tau \in \mathbf{H}_{n-1}$, $z \in \mathbf{C}^{n-1}$ and $\tau' \in \mathbf{H}_1$. Then we have the following Fourier-Jacobi expansion of $I_n(f)$:

$$I_n(f)\left(\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix} \right) = \sum_{N=0}^{\infty} \phi_{I_n(f),N}(\tau,\, z) \mathbf{e}(N\tau'),$$

where $\phi_{I_n(f),N}(\tau,\, z)$ is called the N-th Fourier-Jacobi coefficient of $I_n(f)$ and defined by

$$\phi_{I_n(f),N}(\tau,\, z) = \sum_{\substack{T \in \mathcal{L}_{n-1},\, r \in \mathbf{Z}^{n-1}, \\ 4NT - {}^t r r > 0}} a_{I_n(f)} \left(\begin{pmatrix} N & r/2 \\ {}^t r/2 & T \end{pmatrix} \right) \mathbf{e}(\mathrm{tr}(T\tau) + r\, {}^t z).$$

We easily see that $\phi_{I_n(f),N}$ belongs to $J^{\mathrm{cusp}}_{k,\,N}(\Gamma^{(n-1)}_J)$ for each $N \in \mathbf{Z}_{>0}$. Now we have the following diagram of liftings:

$$\begin{array}{ccc} \mathfrak{S}^+_{k-(n-1)/2}(\Gamma^{(1)}_0(4)) \ni \widetilde{f} & \longrightarrow & f \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \\ & & \downarrow \\ & & I_n(f) \in \mathfrak{S}_k(\Gamma^{(n)}) \\ & & \downarrow \\ \mathfrak{S}^+_{k-1/2}(\Gamma^{(n-1)}_0(4)) \ni \sigma_{n-1}(\phi_{I_n(f),1}) & \longleftarrow & \phi_{I_n(f),1} \in J^{\mathrm{cusp}}_{k,\,1}(\Gamma^{(n-1)}_J) \end{array}$$

Under the above notation, we propose the following conjecture:

Conjecture B.

$$\begin{aligned} &\mathrm{Res}_{s=k-1/2}\mathcal{R}(s, \sigma_{n-1}(\phi_{I_n(f),1})) \\ &\quad = \ 2^{\beta(n,k)}\langle \widetilde{f},\, \widetilde{f}\rangle \prod_{i=1}^{n/2-1} \widetilde{\xi}(2i)\xi(2i+1)\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad}), \end{aligned}$$

where $\beta(n,k) = -(n-4)k + (n^2-5n+2)/2$.

Then we can show the following:

Theorem 3.2. *Under the above notation and the assumption, Conjecture A is equivalent to Conjecture B.*

Proof. By Corollary to Main Theorem of [KK08a], we have

$$\frac{\langle I_n(f),\, I_n(f)\rangle}{\langle \phi_{I_n(f),1},\, \phi_{I_n(f),1}\rangle} = 2^{-k+n-1}\Lambda(k,\, f)\widetilde{\xi}(n)$$

(see the remark below). Thus Conjecture A holds true if and only if

$$\langle \phi_{I_n(f),1},\, \phi_{I_n(f),1}\rangle = 2^{-k(n-4)+n(n-7)/2+2}\langle \widetilde{f},\, \widetilde{f}\rangle \prod_{i=1}^{n/2-1} \widetilde{\xi}(2i)\widetilde{\Lambda}(2i+1,\, f,\, \mathrm{Ad}).$$

On the other hand, by Corollary to Proposition 3.1 we have

$$\mathrm{Res}_{s=k-1/2}\mathcal{R}(s, \sigma_{n-1}(\phi_{I_n(f),1})) = 2^{n-1}\langle \phi_{I_n(f),1},\, \phi_{I_n(f),1}\rangle \prod_{i=1}^{n/2-1} \xi(2i+1).$$

Thus the assertion holds. □

Remark. In [KK08a], we incorrectly quoted Yamazaki's result in [Yam90]. Indeed "$\langle F, G\rangle$" on the page 2026, line 14 of [KK08a] should read "$\frac{1}{2}\langle F, G\rangle$" (cf. Krieg [Kri91]) and therefore "2^{2k-n+1}" on the page 2027, line 7 of [KK08a] should read "2^{2k-n}".

4. Reduction to local computations

To prove Conjecture B, we give an explicit formula for $R(s, \sigma_{n-1}(\phi_{I_n(f),1}))$ for the first Fourier-Jacobi coefficient $\phi_{I_n(f),1}$ of $I_n(f)$. To do this, we reduce the problem to local computations. Put

$$\mathcal{L}'_{m,p} = \{A \in \mathcal{L}^{\times}_{m,p} \mid A \equiv -\,{}^t rr \bmod 4\mathcal{L}_{m,p} \text{ for some } r \in \mathbf{Z}_p^m\}.$$

Furthermore we put $S_m(\mathbf{Z}_p)_e = 2\mathcal{L}_{m,p}$ and $S_m(\mathbf{Z}_p)_o = S_m(\mathbf{Z}_p)\backslash S_m(\mathbf{Z}_p)_e$. We note that $\mathcal{L}'_{m,p} = \mathcal{L}^{\times}_{m,p} = S_m(\mathbf{Z}_p)^{\times}$ if $p \neq 2$.

First we can easily prove the following:

Lemma 4.1. *Let m be a positive even integer.*

(1) *Let A and B be elements of $\mathcal{L}'_{m-1,p}$. Then*

$$\begin{pmatrix} 1 & r_A/2 \\ {}^t r_A/2 & (A + {}^t r_A r_A)/4 \end{pmatrix} \sim \begin{pmatrix} 1 & r_B/2 \\ {}^t r_B/2 & (B + {}^t r_B r_B)/4 \end{pmatrix}$$

if $A \sim B$.

(2) *Let $A \in \mathcal{L}'_{m-1,p}$.*

(2.1) *Let $p \neq 2$. Then*

$$\begin{pmatrix} 1 & r_A/2 \\ {}^t r_A/2 & (A + {}^t r_A r_A)/4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

(2.2) *Let $p = 2$. If $r_A \equiv 0 \bmod 2$, then $A \sim 4B$ with $B \in \mathcal{L}_{m-1,2}$, and*

$$\begin{pmatrix} 1 & r_A/2 \\ {}^t r_A/2 & (A + {}^t r_A r_A)/4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}.$$

If $r_A \not\equiv 0 \bmod 2$, then $A \sim a \perp 4B$ with $a \equiv -1 \bmod 4$ and $B \in \mathcal{L}_{m-2,2}$, and we have

$$\begin{pmatrix} 1 & r_A/2 \\ {}^t r_A/2 & (A + {}^t r_A r_A)/4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & (a+1)/4 & 0 \\ 0 & 0 & B \end{pmatrix}.$$

Let m be a positive even integer. Let $T \in \mathcal{L}'_{m-1,p}$. Then there exists an element $r_T \in \mathbf{Z}_p^{m-1}$ such that $T^{(1)} := \begin{pmatrix} 1 & r_T/2 \\ {}^t r_T/2 & (T + {}^t r_T r_T)/4 \end{pmatrix}$ belongs to $\mathcal{L}_{m,p}$. Thus we can define $\mathfrak{d}_T^{(1)}$ and $\mathfrak{f}_T^{(1)}$ as $\mathfrak{d}_{T^{(1)}}$ and $\mathfrak{f}_{T^{(1)}}$, respectively. These do not depend on the choice of r_T. We note that $\det T = 2^{n-2}\mathfrak{d}_T^{(1)}(\mathfrak{f}_T^{(1)})^2$. We also put $\mathfrak{e}_p^{(1)}(T) = \nu_p(\mathfrak{f}_{T^{(1)}})$. We define a polynomial $F_p^{(1)}(T, X)$ and a Laurent polynomial $\widetilde{F}_p^{(1)}(T, X)$ by

$$F_p^{(1)}(T, X) = F_p(T^{(1)}, X),$$

and

$$\widetilde{F}_p^{(1)}(T, X) = X^{-\mathfrak{e}_p^{(1)}(T)} F_p^{(1)}(T, p^{-(n+1)/2} X).$$

Let B be a half-integral matrix B over $\mathbf{Z}_p$ of degree n. Let $p \neq 2$. Then

$$\widetilde{F}_p^{(1)}(B, X) = \widetilde{F}_p(1 \perp B, X).$$

Let $p = 2$. Then

$$\widetilde{F}_2^{(1)}(B, X) = \begin{cases} \widetilde{F}_2(\begin{pmatrix} 1 & 1/2 \\ 1/2 & (a+1)/4 \end{pmatrix} \perp B', X) & \text{if } B = a \perp 4B' \\ & \text{with } a \equiv -1 \bmod 4, \\ \widetilde{F}_2(1 \perp B', X) & \text{if } B = 4B'. \end{cases}$$

Furthermore, for each $T \in S_m(\mathbf{Z}_p)_e^\times$ put $F_p^{(0)}(T, X) = F_p(2^{-1}T, X)$ and $\widetilde{F}_p^{(0)}(T, X) = \widetilde{F}_p(2^{-1}T, X)$.

Now let m and l be positive integers such that $m \geq l$. Then for non-degenerate symmetric matrices A and B of degree m and l respectively with entries in $\mathbf{Z}_p$ we define the local density $\alpha_p(A, B)$ and the primitive local density $\beta_p(A, B)$ representing B by A as

$$\alpha_p(A, B) = 2^{-\delta_{m,l}} \lim_{a \to \infty} p^{a(-ml + l(l+1)/2)} \# \mathcal{A}_a(A, B),$$

and

$$\beta_p(A, B) = 2^{-\delta_{ml}} \lim_{a \to \infty} p^{a(-ml + l(l+1)/2)} \# \mathcal{B}_a(A, B),$$

where

$$\mathcal{A}_a(A, B) = \{X \in M_{ml}(\mathbf{Z}_p)/p^a M_{ml}(\mathbf{Z}_p) \mid A[X] - B \in p^a S_l(\mathbf{Z}_p)_e\},$$

and

$$\mathcal{B}_a(A, B) = \{X \in \mathcal{A}_a(A, B) \mid \operatorname{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} X = l\}.$$

In particular we write $\alpha_p(A) = \alpha_p(A, A)$. Furthermore put

$$M(A) = \sum_{A' \in \mathcal{G}(A)} \frac{1}{e(A')}$$

for a positive definite symmetric matrix A of degree $n - 1$ with entries in $\mathbf{Z}$, where $\mathcal{G}(A)$ denotes the set of $SL_{n-1}(\mathbf{Z})$-equivalence classes belonging to the genus of A. Then by Siegel's main theorem on the quadratic forms, we obtain

$$M(A) = e_{n-1} \kappa_{n-1} \det A^{n/2} \prod_p \alpha_p(A)^{-1}$$

where $e_{n-1} = 1$ or 2 according as $n = 2$ or not, and

$$\kappa_{n-1} = 2^{2-n} \prod_{i=1}^{(n-2)/2} \Gamma_{\mathbf{C}}(2i)$$

(cf. Theorem 6.8.1 in [Kit93]). Put

$$\mathcal{F}_p = \{d_0 \in \mathbf{Z}_p \mid \nu_p(d_0) \leq 1\}$$

if p is an odd prime, and

$$\mathcal{F}_2 = \{d_0 \in \mathbf{Z}_2 \mid d_0 \equiv 1 \bmod 4 \text{ or } d_0/4 \equiv -1 \bmod 4 \text{ or } \nu_2(d_0) = 3\}.$$

For $d_0 \in \mathcal{F}_p$ and a $GL_{n-1}(\mathbf{Z})_p$-invariant function ω_p on $\mathcal{L}^{\times}_{n-1,p}$ we define a formal power series $H_{n-1,p}(d_0, \omega_p, X, Y, t)$ by

$$H_{n-1,p}(d_0, \omega_p, X, Y, t)$$
$$:= \sum_{A \in \mathcal{L}'_{n-1,p}(d_0)/GL_{n-1}(\mathbf{Z}_p)} \frac{\widetilde{F}^{(1)}_p(A, X)\widetilde{F}^{(1)}_p(A, Y)}{\alpha_p(A)} \omega_p(A) t^{\nu_p(\det A)},$$

where $\mathcal{L}'_{n-1,p}(d_0) = \{A \in \mathcal{L}'_{n-1,p} \mid \mathfrak{d}^{(1)}_A = d_0\}$. Let $\iota_{m,p}$ be the constant function on $\mathcal{L}^{\times}_{m,p}$ taking the value 1, and $\varepsilon_{m,p}$ the function on $\mathcal{L}^{\times}_{m,p}$ assigning the Hasse invariant of A for $A \in \mathcal{L}^{\times}_{m,p}$. For the definition of the Hasse invarinat, see Kitaoka [Kit93]. We sometimes drop the suffix and write $\iota_{m,p}$ as ι_p or ι and the others if there is no fear of confusion. We call $H_{n-1,p}(d_0, \omega_p, X, Y, t)$ a formal power series of Rankin-Selberg type. An explicit formula for $H_{n-1,p}(d_0, \omega_p, X, Y, t)$ will be given in the next section for $\omega_p = \iota_{n-1,p}$ and $\varepsilon_{n-1,p}$. Let $\mathcal{F}$ denote the set of fundamental discriminants, and for $l = \pm 1$, put $\mathcal{F}^{(l)} = \{d_0 \in \mathcal{F} \mid ld_0 > 0\}$.

Now let f be a primitive form in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$, and $\widetilde{f}, I_n(f), \phi_{I_n(f),1}$ and $\sigma_{n-1}(\phi_{I_n(f),1})$ be as in Section 3. It follows from Lemma 4.1 that the Fourier coefficient $c_{\sigma_{n-1}(\phi_{I_n(f),1})}(T)$ of $\sigma_{n-1}(\phi_{I_n(f),1})$ is uniquely determined by the genus to which T belongs. Thus, by using the same method as in Proposition 2.2 of [IS95], similarly to [IK03], Theorem 3.3, (1), and [IK04], Theorem 3.2, we obtain

Theorem 4.2. *Let the notation and the assumption be as above. Then for* $\mathrm{Re}(s) \gg 0$, *we have*

$$R(s, \sigma_{n-1}(\phi_{I_n(f),1}))$$
$$= \frac{e_{n-1}}{2} \kappa_{n-1} 2^{-(k-n/2-1/2)(n-2)} \sum_{d_0 \in \mathcal{F}^{((-1)^{n/2})}} |c(|d_0|)|^2 |d_0|^{n/2-k+1/2}$$
$$\times \left\{ \prod_p H_{n-1,p}(d_0, \iota_p, \alpha_p, \alpha_p, p^{-s+k-1/2}) + \prod_p H_{n-1,p}(d_0, \varepsilon_p, \alpha_p, \alpha_p, p^{-s+k-1/2}) \right\},$$

where $c(|d_0|)$ *is the* $|d_0|$*-th Fourier coefficient of* $\widetilde{f}$, *and* α_p *is the Satake p-parameter of* f.

5. Formal power series associated with local Siegel series

Throughout this section we fix a positive even integer n. We also simply write ν_p as ν and the others if the prime number p is clear from the context.

In this section we give an explicit formula of $H_{n-1}(d_0, \omega, X, Y, t) = H_{n-1,p}(d_0, \omega, X, Y, t)$ for $\omega = \iota, \varepsilon$ (cf. Theorem 5.5.1). For the convenience of readers, we here give an outline of the proof. First we rewrite

$H_{n-1}(d_0,\omega,X,Y,t)$ in terms of another power series. For $d\in\mathbf{Z}_p$ put

$$S_m(\mathbf{Z}_p,d)=\{T\in S_m(\mathbf{Z}_p)\mid (-1)^{[(m+1)/2]}\det T=p^{2i}d \text{ with some } i\in\mathbf{Z}\},$$

and $S_m(\mathbf{Z}_p,d)_x=S_m(\mathbf{Z}_p,d)\cap S_m(\mathbf{Z}_p)_x$ for $x=e$ or o. We note that $S_m(\mathbf{Z}_p,d)=S_m(\mathbf{Z}_p,p^jd)$ for any even integer j. In particular, if m is even, put $\mathcal{L}^{(0)}_{m,p}=S_m(\mathbf{Z}_p)^\times_e$ and $\mathcal{L}^{(1)}_{m-1,p}=\mathcal{L}'_{m-1,p}$. We also define $\mathcal{L}^{(l)}_{m-l,p}(d)=S_{m-l}(\mathbf{Z}_p,d)\cap\mathcal{L}^{(l)}_{m-l,p}$ for $l=0,1$. We note that $\mathcal{L}^{(l)}_{m-l,p}(d)=\mathcal{L}'_{m-l,p}(d)$ for $d\in\mathcal{F}_p$. Let $\mathcal{D}_{m,i}=GL_m(\mathbf{Z}_p)\begin{pmatrix}1_{m-i}&0\\0&p1_i\end{pmatrix}GL_m(\mathbf{Z}_p)$. Henceforth, for a $GL_m(\mathbf{Z}_p)$-stable subset $\mathcal{B}$ of $S_m(\mathbf{Q}_p)$, we simply write $\sum_{T\in\mathcal{B}}$ instead of $\sum_{T\in\mathcal{B}/\sim}$ if there is no fear of confusion.

Suppose that m is a positive even integer. For $j=0,1$ and an element $T\in\mathcal{L}^{(j)}_{m-j,p}$, we define a polynomial $\widetilde{G}^{(j)}_p(T,X,t)$ in X and t by

$$\widetilde{G}^{(j)}_p(T,X,t)=\sum_{i=0}^{m-j}(-1)^ip^{i(i-1)/2}t^i\sum_{D\in GL_{m-j}(\mathbf{Z}_p)\backslash\mathcal{D}_{m-j,i}}\widetilde{F}^{(j)}_p(T[D^{-1}],X).$$

We also define a polynomial $G^{(j)}_p(T,X)$ in X by

$$G^{(j)}_p(T,X)=\sum_{i=0}^{m-j}(-1)^ip^{i(i-1)/2}(X^2p^{m+1-j})^i\sum_{D\in GL_{m-j}(\mathbf{Z}_p)\backslash\mathcal{D}_{m-j,i}}F^{(j)}_p(T[D^{-1}],X).$$

For $d_0\in\mathcal{F}_p$ and $l=0,1$ put

$$\kappa(d_0,m-1,l,t)=\{(-1)^{lm(m-2)/8}t^{m-2}2^{-(m-2)(m-1)/2}\}^{\delta_{2p}}\times((-1)^{m/2},(-1)^{m/2}d_0)_p^l\,p^{-(m/2-1)l\nu(d_0)},$$

and

$$\kappa(d_0,m,l,t)=\{(-1)^{m(m+2)/8}\,((-1)^{m/2}2,\,d_0)_2\}^{l\delta_{2p}}.$$

Furthermore for an element $T\in\mathcal{L}^{(1)}_{m-1,p}$ we define a polynomial $B^{(1)}_p(T,t)$ in t by

$$B^{(1)}_p(T,t)=\frac{(1-\xi_p(T^{(1)})p^{-m/2+1/2}t)\prod_{i=1}^{(m-2)/2}(1-p^{-2i+1}t^2)}{G^{(1)}_p(T,p^{-m+1/2}t)},$$

and for $\omega=\varepsilon^l$ define a formal power series $\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)$ in t by

$$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)=\kappa(d_0,n-1,l,t)^{-1}\sum_{B'\in\mathcal{L}^{(1)}_{n-1,p}(d_0)}\frac{\widetilde{G}^{(1)}_p(B',X,p^{-n}Yt^2)}{\alpha_p(B')}$$

$$\times Y^{-\mathfrak{e}^{(1)}(B')}\,t^{\nu(\det B')}B^{(1)}_p(B',p^{-n/2-1}Yt^2)G^{(1)}_p(B',p^{-(n+1)/2}Y)\omega(B').$$

Then

$$H_{n-1}(d_0,\omega,X,Y,t)=\frac{\kappa(d_0,n-1,l,t)\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)}{\prod_{j=1}^{n}(1-p^{j-1-n}XYt^2)(1-p^{j-1-n}X^{-1}Yt^2)}$$

for $\omega=\varepsilon^l$ (cf. Theorem 5.2.6). The polynomials $G_p^{(1)}(T,X)$ and $B_p^{(1)}(T,t)$ are expressed explicitly, and in particular they are determined by $\mathfrak{d}_T$ and the p-rank of T (cf. Lemmas 5.2.1 and 5.2.3). Thus we can rewrite the above in more concise form. To explain this, we generalize the polynomials $\widetilde{F}_p^{(j)}(T,X)$ and $\widetilde{G}_p^{(j)}(T,X,t)$ for $T\in\mathcal{L}_{m-j,p}^{(j)}$ and we put $\widetilde{F}_p^{(j)}(T,\xi,X)=X^{-\mathfrak{e}^{(j)}(T)}F_p^{(j)}(T,\xi X)$, and

$$\widetilde{G}_p^{(j)}(T,\xi,X,t)=\sum_{i=0}^{m-j}(-1)^i p^{i(i-1)/2}t^i\sum_{D\in GL_{m-j}(\mathbf{Z}_p)\backslash\mathcal{D}_{m-j,i}}\widetilde{F}_p^{(j)}(T[D^{-1}],\xi,X)$$

for $\xi=\pm1$, where $\mathfrak{e}^{(0)}(T)=\mathfrak{e}_p(T)$ for $T\in\mathcal{L}_{m,p}^{(0)}$. Then we define a formal power series $\widetilde{P}_{m-j}^{(j)}(n;d_0,\omega,\xi,X,Y,t)$ in t by

$$\begin{aligned}&\widetilde{P}_{m-j}^{(j)}(n;d_0,\omega,\xi,X,Y,t)\\ &=\ \kappa(d_0,m-j,l,t)^{-1}\sum_{B'\in\mathcal{L}_{m,p}^{(j)}(d_0)}\frac{\widetilde{G}_p^{(j)}(B',\xi,X,p^{-n}t^2Y)}{\alpha_p(B')}\omega(B')Y^{-\mathfrak{e}^{(j)}(B')}t^{\nu(\det(B'))}\end{aligned}$$

for $\omega=\varepsilon^l$. Here we make the convention that $\widetilde{P}_0^{(0)}(n;d_0,\omega,\xi,X,Y,t)=1$ or 0 according as $\nu(d_0)=0$ or not. An explicit formula of $\widetilde{P}_{m-j}^{(j)}(n;d_0,\omega,\xi,X,Y,t)$ for $j=0,1$ will be given (cf. Proposition 5.3.1, and Theorems 5.4.1 and 5.4.2). For simplicity suppose that $\nu(d_0)=0$ or $\omega=\iota$. Then we can rewrite $\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)$ in terms of $\widetilde{P}_{m-j}^{(j)}(n;d_0,\omega,\xi,X,Y,t)$ in the following way:

$$\begin{aligned}&\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)=(1-p^{-n}t^2)\\ &\times\{\sum_{l=0}^{(n-2)/2}\sum_{d\in\mathcal{U}(n-1,n-1-2l,d_0)}\widetilde{P}_{2l}^{(0)}(n;d_0d,\omega,\chi(d),X,Y,t)\\ &\times\prod_{i=1}^{(n-2-2l)/2}(1-p^{-2l-n-2i}t^4)T_{2l}(d_0,d,Y)\\ &+\sum_{l=0}^{(n-2)/2}\widetilde{P}_{2l+1}^{(1)}(n;d_0,\omega,1,X,Y,t)\prod_{i=2}^{(n-2-2l)/2}(1-p^{-2l-n-2i}t^4)T_{2l+1}(d_0,Y,t)\},\end{aligned}$$

where $\mathcal{U}(n-1,n-1-2l,d_0)$ is a certain finite subset of $\mathbf{Z}_2^*$, which will be defined in Subsection 5.3, and $T_{2r}(d_0,d,Y)$ is a polynomial in Y, and $T_{2r+1}(d_0,Y,t)$ is a polynomial in Y and t (cf. Theorem 5.3.10). Here the set $\mathcal{U}(n-1,n-1-2l,d_0)$ and the polynomials $T_{2r}(d_0,d,Y)$ and $T_{2r+1}(d_0,Y,t)$ will be explicitly given. Thus we get an explicit formula

for $H_{n-1}(d_0, \omega, X, Y, t)$ in this case. Similarly we get an explicit formula of $H_{n-1}(d_0, \omega, X, Y, t)$ for other cases. Each step is elementary, but rather elaborate. In particular we need a careful analysis for dealing with the case of $p = 2$.

5.1. **Preliminaries.**

For two elements S and T of $S_m(\mathbf{Z}_p)^\times$ and a nonnegative integer $i \leq m$, we introduce a modification $\alpha_p(S, T, i)$ of the local densitiy as follows:

$$\alpha_p(S, T, i) = 2^{-1} \lim_{e \to \infty} p^{(-m^2 + m(m+1)/2)e} \mathcal{A}_e(S, T, i),$$

where

$$\mathcal{A}_e(S, T, i) = \{\overline{X} \in \mathcal{A}_e(S, T) \mid X \in \mathcal{D}_{m,i}\}.$$

Lemma 5.1.1. *Let S and T be elements of $S_m(\mathbf{Z}_p)^\times$.*
(1) *Let $\Omega(S, T) = \{w \in M_m(\mathbf{Z}_p) \mid S[w] \sim T\}$, and $\Omega(S, T, i) = \Omega(S, T) \cap \mathcal{D}_{m,i}$. Then*

$$\frac{\alpha_p(S, T)}{\alpha_p(T)} = \#(\Omega(S, T)/GL_m(\mathbf{Z}_p)) p^{-m(\nu(\det T) - \nu(\det S))/2},$$

and

$$\frac{\alpha_p(S, T, i)}{\alpha_p(T)} = \#(\Omega(S, T, i)/GL_m(\mathbf{Z}_p)) p^{-m(\nu(\det T) - \nu(\det S))/2}.$$

(2) *Let $\widetilde{\Omega}(S, T) = \{w \in M_m(\mathbf{Z}_p) \mid S \sim T[w^{-1}]\}$, and $\widetilde{\Omega}(S, T, i) = \widetilde{\Omega}(S, T) \cap \mathcal{D}_{m,i}$. Then*

$$\frac{\alpha_p(S, T)}{\alpha_p(S)} = \#(GL_m(\mathbf{Z}_p) \backslash \widetilde{\Omega}(S, T)) p^{(\nu(\det T) - \nu(\det S))/2},$$

and

$$\frac{\alpha_p(S, T, i)}{\alpha_p(S)} = \#(GL_m(\mathbf{Z}_p) \backslash \widetilde{\Omega}(S, T, i)) p^{(\nu(\det T) - \nu(\det S))/2}.$$

Proof. The assertion (1) follows from Lemma 2.2 of [BS87]. Now by Proposition 2.2 of [Kat99] we have

$$\alpha_p(S, T) = \sum_{W \in GL_m(\mathbf{Z}_p) \backslash \widetilde{\Omega}(S,T)} \beta_p(S, T[W^{-1}]) p^{\nu(\det W)}.$$

Then $\beta_p(S, T[W^{-1}]) = \alpha_p(S)$ or 0 according as $S \sim T[W^{-1}]$ or not. Thus the assertion (2) holds. □

A non-degenerate square matrix $D = (d_{ij})_{m \times m}$ with entries in $\mathbf{Z}_p$ is said to be reduced if D satisfies the following two conditions:

(a) For $i = j$, $d_{ii} = p^{e_i}$ with a non-negative integer e_i;
(b) For $i \neq j$, d_{ij} is a non-negative integer satisfying $d_{ij} \leq p^{e_j} - 1$ if $i < j$ and $d_{ij} = 0$ if $i > j$.

It is well known that we can take the set of all reduced matrices as a complete set of representatives of $GL_m(\mathbf{Z}_p)\backslash M_m(\mathbf{Z}_p)^\times$. Let $l = 0$ or 1 according as m is even or odd. For $B \in \mathcal{L}_{m,p}^{(l)}$ put

$$\widetilde{\Omega}^{(l)}(B) = \{W \in GL_m(\mathbf{Q}_p) \cap M_m(\mathbf{Z}_p) \mid B[W^{-1}] \in \mathcal{L}_{m,p}^{(l)}\}.$$

Furthermore put $\widetilde{\Omega}^{(l)}(B,i) = \widetilde{\Omega}^{(l)}(B) \cap \mathcal{D}_{m.i}$. Let $n_0 \leq m$, and $\psi_{n_0,m}$ be the mapping from $GL_{n_0}(\mathbf{Q}_p)$ into $GL_m(\mathbf{Q}_p)$ defined by $\psi_{n_0,m}(D) = 1_{m-n_0} \perp D$.

Lemma 5.1.2. (1) *Let $p \neq 2$. Let $\Theta \in GL_{n_0}(\mathbf{Z}_p) \cap S_{n_0}(\mathbf{Z}_p)$, and $B_1 \in S_{m-n_0}(\mathbf{Z}_p)^\times$.*

(1.1) *Let n_0 be even. Then $\psi_{m-n_0,m}$ induces a bijection*

$$GL_{m-n_0}(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(l)}(pB_1) \simeq GL_m(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(l)}(\Theta\perp pB_1),$$

where $l = 0$ or 1 according as m is or even or odd.

(1.2) *Let n_0 be odd. Then $\psi_{m-n_0,m}$ induces a bijection*

$$GL_{m-n_0}(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(l)}(pB_1) \simeq GL_m(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(l')}(\Theta\perp pB_1),$$

where $l = 0$ or 1 according as m is or even or odd, and $l' = 1$ or 0 according as m is or even or odd.

(2) *Let $p = 2$. Let m be a positive integer, and n_0 an even integer not greater than m, and $\Theta \in GL_{n_0}(\mathbf{Z}_2) \cap S_{n_0}(\mathbf{Z}_2)_e$.*

(2.1) *Let $B_1 \in S_{m-n_0}(\mathbf{Z}_2)^\times$. Then $\psi_{m-n_0,m}$ induces a bijection*

$$GL_{m-n_0}(\mathbf{Z}_2)\backslash\widetilde{\Omega}^{(l)}(2^{l+1}B_1) \simeq GL_m(\mathbf{Z}_2)\backslash\Omega^{(l)}(2^l\Theta\perp 2^{l+1}B_1),$$

where $l = 0$ or 1 according as m is or even or odd.

(2.2) *Suppose that m is even. Let $a \in \mathbf{Z}_2$ such that $a \equiv -1 \bmod 4$, and $B_1 \in S_{m-n_0-2}(\mathbf{Z}_2)^\times$. Then $\psi_{m-n_0-1,m}$ induces a bijection*

$$\begin{aligned} &GL_{m-n_0-1}(\mathbf{Z}_2)\backslash\widetilde{\Omega}^{(1)}(a\perp 4B_1) \\ &\quad\simeq\ GL_m(\mathbf{Z}_2)\backslash\widetilde{\Omega}^{(0)}(\Theta\perp \begin{pmatrix} 2 & 1 \\ 1 & \frac{1+a}{2} \end{pmatrix} \perp 2B_1). \end{aligned}$$

(2.3) *Suppose that m is even, and let $B_1 \in S_{m-1-n_0}(\mathbf{Z}_2)^\times$. Then $\psi_{m-n_0-1,m}$ induces a bijection*

$$GL_{m-n_0-1}(\mathbf{Z}_2)\backslash\widetilde{\Omega}^{(1)}(4B_1) \simeq GL_m(\mathbf{Z}_2)\backslash\widetilde{\Omega}^{(0)}(\Theta\perp 2\perp 2B_1).$$

(3) *The assertions (1),(2) remain valid if one replaces $\widetilde{\Omega}(B)$ by $\widetilde{\Omega}(B,i)$.*

Proof. (1) Clearly the mapping $\psi_{m-n_0,m}$ induces an injection from $GL_{m-n_0}(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(l)}(pB_1)$ to $GL_m(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(l)}(\Theta\perp pB_1)$. To prove the surjectivity of ϕ, take a representative D of an element of $GL_m(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(l)}(\Theta\perp pB_1)$. Without loss of generality we may suppose that D is a reduced matrix. Since $(\Theta\perp pB_1)[D^{-1}] \in S_m(\mathbf{Z}_p)$, we have $D = \begin{pmatrix} 1_{n_0} & 0 \\ 0 & D_1 \end{pmatrix}$ with $D_1 \in \widetilde{\Omega}^{(l)}(pB_1)$. This proves the assertion (1.1). The assertion (1.2) can be proved in the same way as above.

(2) As in (1), the mapping $\psi_{m-n_0,m}$ induces an injection from $GL_{m-n_0}(\mathbf{Z}_2)\backslash\widetilde{\Omega}^{(l)}(2^{l+1}B_1)$ to $GL_m(\mathbf{Z}_2)\backslash\Omega^{(l)}(2^l\Theta\perp 2^{l+1}B_1)$. Then the surjectivity of ϕ in case $l=0$ can be proved in the same manner as (1). To prove the surjectivity of ϕ in case $l=1$, take a reduced matrix $D=\begin{pmatrix} D_1 & D_{12} \\ 0 & D_2 \end{pmatrix}$ with $D_1\in M_{n_0}(\mathbf{Z}_2)^\times, D_2\in M_{m-n_0}(\mathbf{Z}_2)^\times, D_{12}\in M_{n_0,m-n_0}(\mathbf{Z}_2)$. Then $(2\Theta\perp 4B_1)[D^{-1}]\in\mathcal{L}'_{m,2}$ if and only if $2\Theta[D_1^{-1}]\in 4\mathcal{L}_{n_0,2}$. In this case we can take D as $D=\begin{pmatrix} 1_{n_0} & 0 \\ 0 & D_2 \end{pmatrix}$. Thus the surjectivity of ϕ can be proved in the same as above.

The assertion (2.2) can be proved in the same way as above.

To prove (2.3), we may suppose that $n_0=0$ in view of (2.1). Let $D\in\widetilde{\Omega}^{(1)}(4B_1)$. Then

$$4B_1[D^{-1}] = {}^t r_0 r_0 + 4B'$$

with $r_0\in\mathbf{Z}_2^{m-1}$ and $B'\in\mathcal{L}_{m-1,2}$. Then we can take $r\in\mathbf{Z}_2^{m-1}$ such that

$$4\,{}^tD^{-1}\,{}^t r r D^{-1} \equiv {}^t r_0 r_0 \bmod 4\mathcal{L}_{m-1,2}.$$

Furthermore, $2rD^{-1}$ is uniquely determined modulo $2\mathbf{Z}_2^{m-1}$ by r_0. Put $\widetilde{D}=\begin{pmatrix} 1 & r \\ 0 & D \end{pmatrix}$. Then $\widetilde{D}$ belongs to $\widetilde{\Omega}^{(0)}(2\perp 2B_1)$, and the mapping $D\mapsto\widetilde{D}$ induces a bijection in question. □

Corollary. *Suppose that m is even. Let $B\in\mathcal{L}^{(1)}_{m-1,p}$. Then there exists a bijection*

$$\psi : GL_{m-1}(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(1)}(B)\simeq GL_m(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(0)}(\begin{pmatrix} 2 & {}^t r_B \\ r_B & (B+{}^t r_B r_B)/2 \end{pmatrix})$$

such that $\nu(\det(\psi(W)))=\nu(\det(W))$ for any $W\in GL_{m-1}(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(1)}(B)$. This induces a bijection ψ_i from $GL_{m-1}(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(1)}(B,i)$ to $GL_m(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(0)}(\begin{pmatrix} 2 & {}^t r_B \\ r_B & (B+{}^t r_B r_B)/2 \end{pmatrix}, i)$ for $i=0,\cdots,m-1$.

Proof. Let $p\neq 2$. Then we may suppose $r_B=0$, and the assertion follows from (1.2). Let $p=2$. If $r_B\equiv 0 \bmod 2$ we may suppose that $r_B=0$, and the assertion follows from (2.3). If $r_B\not\equiv 0 \bmod 4$, we may suppose that $B=a\perp 4B_1$ with $B_1\in\mathcal{L}_{m-2,2}$ and $r_B=(1,0,\ldots,0)$. Thus the assertion follows from (2.2). □

Lemma 5.1.3. *Suppose that $p\neq 2$.*
(1) *Let $B\in S_m(\mathbf{Z}_p)^\times$. Then*

$$\alpha_p(p^r dB)=p^{rm(m+1)/2}\alpha_p(B)$$

for any non-negative integer r and $d \in \mathbf{Z}_p^*$.
(2) *Let $U_1 \in GL_{n_0}(\mathbf{Z}_p) \cap S_{n_0}(\mathbf{Z}_p)$ and $B_1 \in S_{m-n_0}(\mathbf{Z}_p)^\times$. Then*

$$\alpha_p(pB_1 \perp U_1) = \alpha_p(pB_1) \times \begin{cases} 2\prod_{i=1}^{n_0/2}(1-p^{-2i})(1+\chi((-1)^{n_0/2}\det U_1)p^{-n_0/2})^{-1} & \text{if } n_0 \text{ even}, \\ 2\prod_{i=1}^{(n_0-1)/2}(1-p^{-2i}) & \text{if } n_0 \text{ odd}. \end{cases}$$

Proof. The assertions follow from the proof of Theorem 5.6.3 and Theorem 5.6.4, (a) of Kitaoka [Kit93]. □

Lemma 5.1.4. (1) *Let $B \in S_m(\mathbf{Z}_2)^\times$. Then*

$$\alpha_2(2^r dB) = 2^{rm(m+1)/2}\alpha_2(B)$$

for any non-negative integer r and $d \in \mathbf{Z}_2^*$.
(2) *Let n_0 be even, and $U_1 \in GL_{n_0}(\mathbf{Z}_2) \cap S_{n_0}(\mathbf{Z}_2)_e$. Then for $B_1 \in S_{m-n_0}(\mathbf{Z}_2)^\times$ we have*

$$\alpha_2(U_1 \perp 2B_1) = \alpha_2(2B_1) \times \begin{cases} 2\prod_{i=1}^{n_0/2}(1-2^{-2i})(1+\chi((-1)^{n_0/2}\det U_1)p^{-n_0/2})^{-1} & \text{if } B_1 \in S_{m-n_0}(\mathbf{Z}_2)_e, \\ 2\prod_{i=1}^{(n_0-1)/2}(1-2^{-2i}) & \text{if } B_1 \in S_{m-n_0}(\mathbf{Z}_2)_o, \end{cases}$$

and for $u_0 \in \mathbf{Z}_2^*$ and $B_2 \in S_{m-n_0-1}(\mathbf{Z}_2)^\times$ we have

$$\alpha_2(u_0 \perp 2U_1 \perp 4B_2) = \alpha_2(2B_2)2^{(m-2)(m-1)/2+1}\prod_{i=1}^{n_0/2}(1-2^{-2i}).$$

Proof. The assertions follow from the proof of Theorem 5.6.3 and Theorem 5.6.4, (a) of Kitaoka [Kit93]. □

Now let R be a commutative ring. Then the group $GL_m(R) \times R^*$ acts on $S_m(R)$. We write $B_1 \approx_R B_2$ if $B_2 \sim_R \xi B_1$ with some $\xi \in R^*$. Let m be a positive integer. Then for $B \in S_m(\mathbf{Z}_p)$ let $\widetilde{\mathcal{S}}_{m,p}(B)$ denote the set of elements of $S_m(\mathbf{Z}_p)$ such that $B' \approx_{\mathbf{Z}_p} B$, and let $\mathcal{S}_{m-1,p}(B)$ denote the set of elements of $S_{m-1}(\mathbf{Z}_p)$ such that $1 \perp B' \approx_{\mathbf{Z}_p} B$.

Lemma 5.1.5. *Let m be a positive even integer. Let $B \in S_m(\mathbf{Z}_2)_o^\times$. Then*

$$\sum_{B' \in \mathcal{S}_{m-1,2}(B)/\sim} \frac{1}{\alpha_2(B')} = \frac{\#(\widetilde{\mathcal{S}}_{m,2}(B)/\sim)}{2\alpha_2(B)}.$$

Proof. For a positive integer l let $l = l_1 + \cdots + l_r$ be the partition of l by positive integers, and $\{s_i\}_{i=1}^r$ the set of non-negative integers such that $0 \le s_1 < \cdots < s_r$. Then for a positive integer e let $S_l^0(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{l_i\}, \{s_i\})$ be the subset of $S_l(\mathbf{Z}_2/2^e\mathbf{Z}_2)$ consisting of symmetric matrices of the form $2^{s_1}U_1 \perp 2^{s_2}U_2 \perp \cdots \perp 2^{s_r}U_r$ with $U_i \in$

$S_{l_i}(\mathbf{Z}_2/2^e\mathbf{Z}_2)$ unimodular. Let $B \in S_m(\mathbf{Z}_2)_o$ and $\det B = (-1)^{m/2}d$. Then B is equivalent, over $\mathbf{Z}_2$, to a matrix of the following form:

$$2^{t_1}W_1 \perp 2^{t_2}W_2 \perp \cdots \perp 2^{t_r}W_r,$$

where $0 = t_1 < t_1 < \cdots < t_r$ and $W_1, ..., W_{r-1}$, and W_r are unimodular matrices of degree $n_1, ..., n_{r-1}$, and n_r, respectively, and in particular, W_1 is odd unimodular. Then by Lemma 3.2 of [IS95], similarly to (3.5) of [IS95], for a sufficiently large integer e, we have

$$\frac{\#(\widetilde{\mathcal{S}}_{m,2}(B)/\sim)}{\alpha_2(B)} = \sum_{\widetilde{B}\in\widetilde{\mathcal{S}}_{m,2}(B)/\sim} \frac{1}{\alpha_2(\widetilde{B})}$$

$$= 2^{m-1}2^{-\nu(d)+\sum_{i=1}^r n_i(n_i-1)e/2-(r-1)(e-1)-\sum_{1\le j<i\le r} n_in_jt_j}$$
$$\times \prod_{i=1}^r \#(SL_{n_i}(\mathbf{Z}_2/2^e\mathbf{Z}_2))^{-1}\#\widetilde{S}_m^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i\}, \{t_i\}, B),$$

where $\widetilde{S}_m^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i\}, \{t_i\}, B)$ is the subset of $S_m^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i\}, \{t_i\})$ consisting of matrices A such that $A \approx_{\mathbf{Z}_2/2^e\mathbf{Z}_2} B$. We note that our local density $\alpha_2(\widetilde{B})$ is 2^{-m} times that in [IS95] for $\widetilde{B} \in S_m(\mathbf{Z}_2)$. If $n_1 \ge 2$, put $r' = r, n_1' = n_1 - 1, n_2' = n_2, .., n_r' = n_r$, and $t_i' = t_i$ for $i = 1, ..., r'$, and if $n_1 = 1$, put $r' = r-1, n_i' = n_{i+1}$ and $t_i' = t_{i+1}$ for $i = 1, ..., r'$. Let $S_{m-1}^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i'\}, \{t_i'\}, B)$ be the subset of $S_{m-1}^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i'\}, \{t_i'\})$ consisting of matrices $B' \in S_{m-1}(\mathbf{Z}_2/2^e\mathbf{Z}_2)$ such that $1\perp B' \approx_{\mathbf{Z}_2/2^e\mathbf{Z}_2} B$. Then, similarly, we obtain

$$\sum_{B'\in\mathcal{S}_{m-1,2}(B)/\sim} \frac{1}{\alpha_2(B')}$$
$$= 2^{m-2}2^{-\nu(d)+\sum_{i=1}^{r'} n_i'(n_i'-1)e/2-(r'-1)(e-1)-\sum_{1\le j<i\le r'} n_i'n_j't_j'}$$
$$\times \prod_{i=1}^{r'} \#(SL_{n_i'}(\mathbf{Z}_2/2^e\mathbf{Z}_2))^{-1}\#S_{m-1}^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i'\}, \{t_i'\}, B).$$

Take an element A of $\widetilde{S}_m^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i\}, \{t_i\}, B)$. Then $A = 2^{s_1}U_1 \perp 2^{s_2}U_2 \perp \cdots \perp 2^{s_r}U_r$ with $U_i \in S_{n_i}(\mathbf{Z}_2/2^e\mathbf{Z}_2)$ unimodular. Put $U_1 = (u_{\lambda\mu})_{n_1\times n_1}$. Then by the assumption there exists an integer $1 \le \lambda \le n_1$ such that $u_{\lambda\lambda} \in \mathbf{Z}_2^*$. Let λ_0 be the least integer such that $u_{\lambda_0\lambda_0} \in \mathbf{Z}_2^*$, and V_1 be the matrix obtained from U_1 by interchanging the first and λ_0-th lows and the first and λ_0-th columns. Write V_1 as $V_1 = \begin{pmatrix} v_1 & \mathbf{v}_1 \\ {}^t\mathbf{v}_1 & V' \end{pmatrix}$ with $v_1 \in \mathbf{Z}_2^*, \mathbf{v}_1 \in M_{1,n_1-1}(\mathbf{Z}_2)$, and $V' \in S_{n_1-1}(\mathbf{Z}_2)$. Here we understand that $V' - {}^t\mathbf{v}_1\mathbf{v}_1$ is the empty matrix if $n_1 = 1$. Then

$$V_1 \sim \begin{pmatrix} v_1 & 0 \\ 0 & V' - {}^t\mathbf{v}_1v_1^{-1}\mathbf{v}_1 \end{pmatrix}.$$

Then the map $A \mapsto {v_1}^{-1}(2^{t_1}(V' - {}^t\mathbf{v}_1 v_1^{-1}\mathbf{v}_1) \perp 2^{t_2}U_2 \perp \cdots \perp 2^{t_r}U_r)$ induces a map Υ from $\widetilde{S}_m^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i\}, \{t_i\}, B)$ to $S_{m-1}^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i'\}, \{t_i'\}, B)$. By a simple calculation, we obtain

$$\#\Upsilon^{-1}(B') = 2^{(e-1)n_1}(2^{n_1} - 1)$$

for any $B' \in S_{m-1}^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i'\}, \{t_i'\}, B)$. We also note that

$$\#SL_{n_1}(\mathbf{Z}_2/2^e\mathbf{Z}_2) = 2^{(e-1)(2n_1-1)}2^{n_1-1}(2^{n_1}-1)\#(SL_{n_1-1}(\mathbf{Z}_2/2^e\mathbf{Z}_2)) \text{ or } 1$$

according as $n_1 \geq 2$ or $n_1 = 1$, and

$$\begin{aligned}
&\sum_{i=1}^{r} n_i(n_i-1)e/2 - (r-1)(e-1) - \sum_{1\leq j<i\leq r} n_i n_j t_j \\
&= \ e_{n_1} + \sum_{i=1}^{r'} n_i'(n_i'-1)e/2 - (r'-1)(e-1) + \sum_{1\leq j<i\leq r'} n_i' n_j' t_j',
\end{aligned}$$

where $e_{n_1} = (n_1 - 1)e$ or $e_{n_1} = 1 - e$ according as $n_1 \geq 2$ or $n_1 = 1$. Hence

$$\begin{aligned}
&2^{m-1}2^{-\nu(d)+\sum_{i=1}^{r} n_i(n_i-1)e/2-(r-1)(e-1)-\sum_{1\leq j<i\leq r} n_i n_j t_j} \\
&\quad \times \prod_{i=1}^{r} \#(SL_{n_i}(\mathbf{Z}_2/2^e\mathbf{Z}_2))^{-1} \#\widetilde{S}_m^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i\}, \{t_i\}, B) \\
&= 2 \cdot 2^{m-2}2^{-\nu(d)+\sum_{i=1}^{r'} n_i'(n_i'-1)e/2-(r'-1)(e-1)-\sum_{1\leq j\leq i\leq r'} n_i' n_j' t_j'} \\
&\quad \times \prod_{i=1}^{r} \#(SL_{n_i'}(\mathbf{Z}_2/2^e\mathbf{Z}_2))^{-1} \#S_{m-1}^{(0)}(\mathbf{Z}_2/2^e\mathbf{Z}_2, \{n_i'\}, \{t_i'\}, B).
\end{aligned}$$

This proves the assertion. □

The following lemma follows from [[IK06], Lemma 3.4]:

Lemma 5.1.6. *Let l be a positive integer, and q, U and Q variables. Put $\phi_r(q) = \prod_{i=1}^{m}(1-q^i)$ for a nonzero integer r. Then*

$$\begin{aligned}
&\prod_{i=1}^{l}(1 - U^{-1}Qq^{-i+1})U^l \\
&= \ \sum_{m=0}^{l} \frac{\phi_l(q^{-1})}{\phi_{l-m}(q^{-1})\phi_m(q^{-1})} \prod_{i=1}^{l-m}(1 - Qq^{-i+1}) \prod_{i=1}^{m}(1 - Uq^{i-1})(-1)^m q^{(m-m^2)/2}.
\end{aligned}$$

5.2. **Formal power series of Andrianov type.**

Let $\widetilde{G}_p^{(l)}(T,X,t)$ be the polynomial and in X and t, and $G_p^{(l)}(T,X)$ the polynomial in X defined at the beginning of Section 5. We note that

$$\widetilde{G}_p^{(l)}(T,X,1) = X^{-\mathfrak{e}^{(l)}(T)} G_p^{(l)}(T, Xp^{-(n+1)/2}).$$

For a $m \times m$ half-integral matrix B over $\mathbf{Z}_p$, let $(\overline{W}, \overline{q})$ denote the quadratic space over $\mathbf{Z}_p/p\mathbf{Z}_p$ defined by the quadratic form $\overline{q}(\mathbf{x}) = B[\mathbf{x}] \bmod p$, and define the radical $R(\overline{W})$ of $\overline{W}$ by

$$R(\overline{W}) = \{\mathbf{x} \in \overline{W} \mid \overline{B}(\mathbf{x}, \mathbf{y}) = 0 \text{ for any } \mathbf{y} \in \overline{W}\},$$

where $\overline{B}$ denotes the associated symmetric bilinear form of $\overline{q}$. We then put $l_p(B) = \operatorname{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} R(\overline{W})^{\perp}$, where $R(\overline{W})^{\perp}$ is the orthogonal complement of $R(\overline{W})^{\perp}$ in $\overline{W}$. Furthermore, in case $l_p(B)$ is even, put $\overline{\xi}_p(B) = 1$ or -1 according as $R(\overline{W})^{\perp}$ is hyperbolic or not. In case $l_p(B)$ is odd, we put $\overline{\xi}_p(B) = 0$. Here we make the convention that $\xi_p(B) = 1$ if $l_p(B) = 0$. We note that $\overline{\xi}_p(B)$ is different from the $\xi_p(B)$ in general, but they coincide if $B \in \mathcal{L}_{m,p} \cap \frac{1}{2} GL_m(\mathbf{Z}_p)$.

Let m be a positive even integer. For $B \in \mathcal{L}_{m-1,p}^{(1)}$ put $B^{(1)} = \begin{pmatrix} 1 & r/2 \\ {}^t r/2 & (B + {}^t rr)/4 \end{pmatrix}$, where r is an element of $\mathbf{Z}_p^{m-1}$ such that $B + {}^t rr \in 4\mathcal{L}_{m-1,p}$. Then we put $\xi^{(1)}(B) = \xi(B^{(1)})$ and $\overline{\xi}^{(1)}(B) = \overline{\xi}(B^{(1)})$. These do not depend on the choice of r, and we have $\xi^{(1)}(B) = \xi(B)$. Let $p \neq 2$. Then an element B of $\mathcal{L}_{m-1,p}^{(1)}$ is equivalent, over $\mathbf{Z}_p$, to $\Theta \perp pB_2$ with $\Theta \in GL_{m-n_1-1}(\mathbf{Z}_p) \cap S_{m-n_1-1}(\mathbf{Z}_p)$ and $B_2 \in S_{n_1}(\mathbf{Z}_p)$. Then $\overline{\xi}(B) = 0$ if n_1 is odd, and $\overline{\xi}^{(1)}(B) = \chi((-1)^{(m-n_1)/2} \det \Theta)$ if n_1 is even. Let $p = 2$. Then an element $B \in \mathcal{L}_{m-1,2}^{(1)}$ is equivalent, over $\mathbf{Z}_2$, to a matrix of the form $2\Theta \perp B_1$, where $\Theta \in GL_{m-n_1-2}(\mathbf{Z}_2) \cap S_{m-n_1-2}(\mathbf{Z}_2)_e$ and B_1 is one of the following three types:

(I) $B_1 = a \perp 4B_2$ with $a \equiv -1 \bmod 4$, and $B_2 \in S_{n_1}(\mathbf{Z}_2)_e$;

(II) $B_1 \in 4S_{n_1+1}(\mathbf{Z}_2)$;

(III) $B_1 = a \perp 4B_2$ with $a \equiv -1 \bmod 4$, and $B_2 \in S_{n_1}(\mathbf{Z}_2)_o$.

Then $\overline{\xi}^{(1)}(B) = 0$ if B_1 is of type (II) or type (III). Let B_1 be of type (I). Then $(-1)^{(m-n_1)/2} a \det \Theta \bmod (\mathbf{Z}_2^*)^{\square}$ is uniquely detemined by B, as will be shown in Lemma 5.3.2, and we have

$$\overline{\xi}^{(1)}(B) = \chi((-1)^{(m-n_1)/2} a \det \Theta).$$

Suppose that $p \neq 2$, and let $\mathcal{U} = \mathcal{U}_p$ be a complete set of representatives of $\mathbf{Z}_p^*/(\mathbf{Z}_p^*)^{\square}$. Then, for each positive integer m and $d \in \mathcal{U}_p$, there exists a unique, up to $\mathbf{Z}_p$-equivalence, element of $S_m(\mathbf{Z}_p) \cap GL_m(\mathbf{Z}_p)$ such that whose determinant is $(-1)^{[(m+1)/2]} d$, which will be denoted by $\Theta_{m,d}$. Suppose that $p = 2$, and put $\mathcal{U} = \mathcal{U}_2 = \{1, 5\}$. Then for

each positive even integer m and $d \in \mathcal{U}_2$ there exists a unique, up to $\mathbf{Z}_2$-equivalence, element of $S_m(\mathbf{Z}_2)_e \cap GL_m(\mathbf{Z}_2)$ whose determinant is $(-1)^{m/2}d$, which will be also denoted by $\Theta_{m,d}$. In particular, if p is any prime number and m is even, we put $\Theta_m = \Theta_{m,1}$ We make the convention that $\Theta_{m,d}$ is the empty matrix if $m = 0$. For an element $d \in \mathcal{U}$ we use the same symbol d to denote the coset $d \bmod (\mathbf{Z}_p^*)^{\square}$.

Lemma 5.2.1. *Let n be the fixed positive even integer. Let $B \in \mathcal{L}^{(1)}_{n-1,p}$ and put $\xi_0 = \chi((-1)^{n/2} \det B)$.*
(1) Let $p \neq 2$, and supposse that $B = \Theta_{n-n_1-1,d} \perp pB_1$ with $d \in \mathcal{U}$ and $B_1 \in \mathcal{L}_{n_1,p}$. Then

$$G_p^{(1)}(B,Y) = \begin{cases} 1 & \text{if } n_1 = 0, \\ (1-\xi_0 p^{n/2}Y) \displaystyle\prod_{i=1}^{n_1/2-1} (1-p^{2i+n}Y^2)(1+p^{n_1/2+n/2}\overline{\xi}^{(1)}(B)Y) & \text{if } n_1 \text{ is positive and even,} \\ (1-\xi_0 p^{n/2}Y) \displaystyle\prod_{i=1}^{(n_1-1)/2} (1-p^{2i+n}Y^2) & \text{if } n_1 \text{ is odd.} \end{cases}$$

(2) Let $p = 2$, and supposoe that $B = 2\Theta \perp B_1$ with $\Theta \in S_{n-n_1-2}(\mathbf{Z}_2)_e \cap GL_{n-n_1-2}(\mathbf{Z}_2)$ and $B_1 \in S_{n_1+1}(\mathbf{Z}_2)$. Then

$$G_2^{(1)}(B,Y) \times \begin{cases} 1 & \text{if } n_1 = 0, \\ (1-\xi_0 2^{n/2}Y) \displaystyle\prod_{i=1}^{n_1/2-1} (1-2^{2i+n}Y^2)(1+2^{n_1/2+n/2}\overline{\xi}^{(1)}(B)Y) & \text{if } n_1 \text{ is positive and } B_1 \text{ is of type (I),} \\ (1-\xi_0 2^{n/2}Y) \displaystyle\prod_{i=1}^{n_1/2} (1-2^{2i+n}Y^2) & \text{if } B_1 \text{ is of type (II) or (III).} \end{cases}$$

Here we remark that n_1 is even.

Proof. By Corollary to Lemma 5.1.2 and by definition we have $G_p^{(1)}(B,Y) = G_p(B^{(1)},Y)$. Thus the assertion follows from Lemma 9 of [Kit84]. $\square$

Lemma 5.2.2. *Let m be a positive even integer, and $l = 0$ or 1. Let $B \in \mathcal{L}^{(l)}_{m-l,p}$. Then*

$$\widetilde{F}^{(l)}(B,X) = \sum_{B' \in \mathcal{L}^{(l)}_{m-l,p}/GL_{m-l}(\mathbf{Z}_p)} X^{-\mathfrak{e}^{(l)}(B')} \frac{\alpha_p(B',B)}{\alpha_p(B)} \times G^{(l)}(B', p^{(-m-1)/2}X)(p^{-1}X)^{(\nu(\det B)-\nu(\det B'))/2}.$$

Proof. We have

$$\begin{aligned}
&\widetilde{F}^{(l)}(B,X)\\
&= \sum_{W\in GL_{m-l}(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(l)}(B)} X^{-\mathfrak{e}^{(l)}(B)}G^{(l)}(B[W^{-1}],p^{(-m-1)/2}X)X^{2\nu(\det W)}\\
&= \sum_{B'\in\mathcal{L}^{(l)}_{m-l,p}/GL_{m-l}(\mathbf{Z}_p)}\ \sum_{W\in GL_{m-l}(\mathbf{Z}_p)\backslash\widetilde{\Omega}^{(l)}(B',B)} X^{-\mathfrak{e}^{(l)}(B)}G^{(l)}(B',p^{(-m-1)/2}X)X^{2\nu(\det W)}\\
&= \sum_{B'\in\mathcal{L}^{(l)}_{m-l,p}/GL_{m-l}(\mathbf{Z}_p)} X^{-\mathfrak{e}^{(l)}(B')}\#(GL_{m-l}(\mathbf{Z}_p)\backslash\Omega(B',B))p^{(\nu(\det B)-\nu(\det B'))/2}\\
&\qquad\times G^{(l)}(B',p^{(-m-1)/2}X)(p^{-1}X)^{(\nu(\det B)-\nu(\det B'))/2}.
\end{aligned}$$

Thus the assertion follows from (2) of Lemma 5.1.1. □

Now let $B_p^{(1)}(B,t)$ be the polynomial in t defined at the beginning of Section 5. Then by Lemma 5.2.1 we have the following:

Lemma 5.2.3. *Let n be the fixed positive even integer. Let $B\in\mathcal{L}^{(1)}_{n-1,p}$.*

(1) *Let $p\neq 2$, and supposse that $B=\Theta_{n-n_1-1,d}\perp pB_1$ with $d\in\mathcal{U}$ and $B_1\in\mathcal{L}_{n_1,p}$. Then*

$$B_p^{(1)}(B,t)=\begin{cases}(1-\overline{\xi}^{(1)}(B)p^{(n_1-n+1)/2}t)\displaystyle\prod_{i=1}^{(n-n_1-2)/2}(1-p^{-2i+1}t^2) & \text{if } n_1 \text{ even},\\ \displaystyle\prod_{i=1}^{(n-n_1-1)/2}(1-p^{-2i+1}t^2) & \text{if } n_1 \text{ odd}.\end{cases}$$

(2) *Let $p=2$, and supposoe that $B=2\Theta\perp B_1\in\mathcal{L}'_{n-1,2}$ with $\Theta\in S_{n-n_1-2}(\mathbf{Z}_2)_e\cap GL_{n-n_1-2}(\mathbf{Z}_2)$ and $B_1\in S_{n_1+1}(\mathbf{Z}_2)$. Then*

$$\begin{aligned}
&B_p^{(1)}(B,t)\\
&=\begin{cases}(1-\overline{\xi}^{(1)}(B)p^{(n_1-n+1)/2}t)\displaystyle\prod_{i=1}^{(n-n_1-2)/2}(1-p^{-2i+1}t^2) & \text{if } B_1 \text{ is of type (I)},\\ \displaystyle\prod_{i=1}^{(n-n_1-2)/2}(1-p^{-2i+1}t^2) & \text{if } B_1 \text{ is of type (II) or (III)}.\end{cases}
\end{aligned}$$

For a non-degenerate half-integral matrix T over $\mathbf{Z}_p$ of degree n, put

$$R^{(l)}(T,X,t)=\sum_w \widetilde{F}_p^{(l)}(T[w],X)t^{\nu(\det w)}.$$

This type of formal power series was first introduced by Andrianov [And87] to study the standard L-function of Siegel modular form of integral weight. Therefore we call it the formal power series of Andrianov

type. (See also Böcherer [Böc86].) The following proposition follows from (1) of Lemma 5.1.1.

Proposition 5.2.4. *Let m be a positive even integer and $l = 0$ or 1. Let $T \in \mathcal{L}^{(l)}_{m-l,p}$. Then*

$$\sum_{B \in \mathcal{L}^{(1)}_{m-l,p}} \frac{\widetilde{F}^{(l)}_p(B,X)\alpha_p(T,B)}{\alpha_p(B)} t^{\nu(\det B)} = t^{\nu(\det T)} R^{(l)}(T,X,p^{-m+l}t^2).$$

The following theorem is due to [KK09].

Theorem 5.2.5. *Let T be an element of $\mathcal{L}^{(1)}_{n-1,p}$. Then*

$$R^{(1)}(T,X,t) = \frac{B^{(1)}_p(T,p^{n/2-1}t)\widetilde{G}^{(1)}_p(T,X,t)}{\prod_{j=1}^{n-1}(1-p^{j-1}X^{-1}t)(1-p^{j-1}Xt)}.$$

In [BS87], Böcherer and Sato got a similar formula for $T \in \mathcal{L}_{n,p}$. We note that the above formula for $p \neq 2$ can be derived directly from Theorem 20.7 in [Shi00] (see also Zhuravlev [Zhu85]). However, we note that we cannot use their results to prove the above formula for $p = 2$. Now by Theorem 5,2,5, we can rewrite $H_{n-1}(\omega, d_0, X, Y, t)$ in terms of $\widetilde{R}_{n-1}(d_0, \omega, X, Y, t)$ in the following way:

Theorem 5.2.6. *We have*

$$H_{n-1}(d_0,\omega,X,Y,t) = \frac{\kappa(d_0,n-1,l,t)\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)}{\prod_{j=1}^{n}(1-p^{j-1-n}XYt^2)(1-p^{j-1-n}X^{-1}Yt^2)}$$

for $\omega = \varepsilon^l$.

Proof. By Lemma 5.2.2 and Proposition 5.2.4, we have

$$H_{n-1}(d_0,\omega,X,Y,t) = \sum_{B \in \mathcal{L}^{(1)}_{n-1,p}(d_0)} \frac{\widetilde{F}^{(1)}_p(B,X)}{\alpha_p(B)} \omega(B) t^{\nu(\det B)}$$
$$\times \sum_{B' \in \mathcal{L}^{(1)}_{n-1,p}} \frac{Y^{-\mathfrak{e}^{(1)}(B')} G^{(1)}_p(B', p^{-(n+1)/2}Y)\alpha_p(B',B)}{\alpha_p(B')} (p^{-1}Y)^{(\nu(\det B)-\nu(\det B'))/2}.$$

Let B and B' be elements of $\mathcal{L}^{(1)}_{n-1,p}$, and suppose that $\alpha_p(B',B) \neq 0$. Then we note that $B \in \mathcal{L}^{(1)}_{n-1,p}(d_0)$ if and only if $B' \in \mathcal{L}^{(1)}_{n-1,p}(d_0)$. Hence

by Theorem 5.2.2 we have

$$\begin{aligned}
&H_{n-1}(d_0,\omega,X,Y,t)\\
&= \sum_{B'\in\mathcal{L}^{(1)}_{n-1,p}(d_0)} \frac{G^{(1)}_p(B',p^{-(n+1)/2}Y)Y^{-\mathfrak{e}^{(1)}(B')}}{\alpha_p(B')}(pY^{-1})^{\nu(\det B')/2}\omega(B')\\
&\quad\times \sum_{B\in\mathcal{L}^{(1)}_{n-1,p}} \frac{\widetilde{F}^{(1)}_p(B,X)\alpha_p(B',B)}{\alpha_p(B)}(t^2p^{-1}Y)^{\nu(\det B)/2}\\
&= \sum_{B'\in\mathcal{L}^{(1)}_{n-1,p}(d_0)} \frac{G^{(1)}_p(B',p^{-(n+1)/2}Y)Y^{-\mathfrak{e}^{(1)}(B')}}{\alpha_p(B')}t^{\nu(\det B')}\omega(B')R(B',X,t^2Yp^{-n})\\
&= \sum_{B'\in\mathcal{L}^{(1)}_{n-1,p}(d_0)} \frac{\widetilde{G}^{(1)}_p(B',X,p^{-n}Yt^2)}{\alpha_p(B')}\omega(B')Y^{-\mathfrak{e}^{(1)}(B')}t^{\nu(\det B')}\\
&\quad\times\frac{B^{(1)}_p(B',p^{-n/2-1}Yt^2)G^{(1)}_p(B',p^{-(n+1)/2}Y)}{\prod_{j=1}^n(1-p^{j-1-n}XYt^2)(1-p^{j-1-n}X^{-1}Yt^2)}.
\end{aligned}$$

□

5.3. **Formal power series of modified Koecher-Maass type.**

For $a,b\in\mathbf{Q}_p^\times$ let $(a,b)_p$ the Hilbert symbol on $\mathbf{Q}_p$. Let r be an even integer. Then for $d_0\in\mathcal{F}_p$ and $l=0,1$ let $\kappa(d_0,r-1,l,t)$ and $\kappa(d_0,r,l,t)$ be as those defined at the beginning of Section 5. We note that $\kappa(d_0,r,l,t)=1$ and

$$\kappa(d_0,r-1,l,t)=((-1)^{r/2},(-1)^{r/2}d_0)_p^l\ p^{-(r/2-1)l\nu(d_0)}$$

if $p\neq 2$. Let $j=0,1$, and $d_0\in\mathcal{F}_p$. We then define a formal power series $P^{(j)}_{r-j}(d_0,\omega,\xi,X,t)$ in t by

$$P^{(j)}_{r-j}(d_0,\omega,\xi,X,t)=\kappa(d_0,r-j,l_\omega,t)^{-1}\sum_{B\in\mathcal{L}^{(j)}_{r-j,p}(d_0)}\frac{\widetilde{F}^{(j)}_p(B,\xi,X)}{\alpha_p(B)}\omega(B)t^{\nu(\det B)}$$

for $\omega=\iota$ or ε, where $l_\omega=0$ or 1 according as $\omega=\iota$ or ε. In particular we put $P^{(j)}_{r-j}(d_0,\omega,X,t)=P^{(j)}_{r-j}(d_0,\omega,1,X,t)$. This type of formal power series appears in an explicit formula of the Koecher-Maass series associated with the Siegel Eisenstein series and the Duke-Imamoḡlu-Ikeda lift (cf. [IK04], [IK06]). Therefore we say that this formal power series is of Koecher-Maass type. For $T\in\mathcal{L}^{(j)}_{r-j,p}$ let $\widetilde{G}^{(j)}_p(T,\xi,X,t)$ be the polynomial and for $\xi=\pm1$, $j=0,1$ and $\omega=\iota,\varepsilon$, let $\widetilde{P}^{(j)}_{r-j}(n;d_0,\omega,\xi,X,Y,t)$ be the formal power series in t as defined at the beginning of Section 5, which will be said to be of modified Koecher-Maass type.

Remark. For a variable X we introduce the symbol $X^{1/2}$ so that $(X^{1/2})^2 = X$, and for an integer a write $X^{a/2} = (X^{1/2})^a$. Under this convention, we can write $X^{-\mathfrak{e}^{(1)}(T)}t^{\nu(\det T)}$ as $X^{\delta_{2p}(n-2)/2}X^{\nu(d_0)}(X^{-1/2}t)^{\nu(\det T)}$ if $T \in \mathcal{L}'_{m-1,p}(d_0)$ with a positive even integer m.

The relation between $\widetilde{P}^{(j)}_{r-j}(n; d_0, \omega, \xi, X, Y, t)$ and $P^{(j)}_{r-j}(d_0, \omega, \xi, X, t)$ will be given in the following proposition:

Proposition 5.3.1. *Let r be a positive even integer. Let $\omega = \varepsilon^l$ with $l = 0, 1$, and $j = 0, 1$. Then*

$$\widetilde{P}^{(j)}_{r-j}(n; d_0, \omega, \xi, X, Y, t) = P^{(j)}_{r-j}(d_0, \omega, \xi, X, tY^{-1/2}) \prod_{i=1}^{r-j}(1-t^4p^{-n-r+j-2+i}).$$

Proof. For $i = 0, ..., r-j$ put

$$\widetilde{P}^{(j)}_{r-j,i}(d_0, \omega, \xi, X, t) = \sum_{B \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \sum_{D \in \mathcal{D}_{r-j,i}} \frac{\widetilde{F}^{(j)}_p(B[D^{-1}], \xi, X)}{\alpha_p(B)} \omega(B) t^{\nu(\det B)}.$$

Then by (2) of Lemma 5.1.1 we have

$$\begin{aligned}
&\widetilde{P}^{(j)}_{r-j,i}(d_0, \omega, \xi, X, t) \\
&= \sum_{B \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \frac{1}{\alpha_p(B)} \sum_{B' \in \mathcal{L}^{(j)}_{r-j,p}} \frac{\widetilde{F}^{(j)}_p(B', \xi, X)\alpha_p(B', B, i)}{\alpha_p(B')} \omega(B) \\
&\qquad \times p^{-(\nu(\det B)-\nu(\det B'))/2} t^{\nu(\det B)}.
\end{aligned}$$

Let B and B' elements of $\mathcal{L}^{(j)}_{r-j,p}$, and suppose that $\alpha_p(B', B, i) \neq 0$. Then we note that $B \in \mathcal{L}^{(j)}_{r-j,p}(d_0)$ if and only if $B' \in \mathcal{L}^{(j)}_{r-j,p}(d_0)$. Hence by (1) of Lemma 5.1.1 we have

$$\begin{aligned}
&\widetilde{P}^{(j)}_{r-j,i}(d_0, \omega, \xi, X, t) \\
&= \sum_{B' \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \frac{\widetilde{F}^{(j)}_p(B', \xi, X)}{\alpha_p(B')} p^{\nu(\det B')/2} \omega(B') \sum_{B \in \mathcal{L}^{(j)}_{r-j,p}} (tp^{-1/2})^{\nu(\det B)} \frac{\alpha_p(B', B, i)}{\alpha_p(B)} \\
&= \sum_{B' \in \mathcal{L}^{(j)}_{r-j,p}(d_0)} \frac{\widetilde{F}^{(j)}_p(B', \xi, X)}{\alpha_p(B')} p^{\nu(\det B')/2} (tp^{-1/2})^{\nu(\det B')} (t^2p^{-r+j-1})^i \#\mathcal{D}_{r-j,i}.
\end{aligned}$$

By Lemma 3.2.18 in [And87], we have

$$\#\mathcal{D}_{r-j,i} = \frac{\phi_{r-j}(p)}{\phi_i(p)\phi_{r-j-i}(p)}.$$

Hence

$$\begin{aligned}
&\widetilde{P}^{(j)}_{r-j,i}(d_0,\omega,\xi,X,t)\\
&\quad= \sum_{B'\in\mathcal{L}^{(j)}_{r-j,p}(d_0)} \frac{\widetilde{F}^{(j)}_p(B',\xi,X)}{\alpha_p(B')}\omega(B')t^{\nu(\det B')}\frac{\phi_{r-j}(p)}{\phi_i(p)\phi_{r-j-i}(p)}(t^2p^{-r+j-1})^i\\
&\quad= \frac{\phi_{r-j}(p)}{\phi_i(p)\phi_{r-j-i}(p)}\kappa(d_0,r-j,l_\omega,t)P^{(j)}_{r-j}(d_0,\omega,\xi,X,t)(t^2p^{-r+j-1})^i.
\end{aligned}$$

Then by the remark just before this proposition we obtain

$$\begin{aligned}
&\widetilde{P}^{(j)}_{r-j}(n;d_0,\omega,\xi,X,Y,t)\\
&\quad= \sum_{i=0}^{r-j}(-1)^ip^{i(i-1)/2}(p^{-n}t^2Y)^i\kappa(d_0,r-j,l_\omega,t)^{-1}\widetilde{P}^{(j)}_{r-j,i}(d_0,\omega,\xi,X,tY^{-1/2}).
\end{aligned}$$

Thus, by (3.2.34) of [And87], we have

$$\begin{aligned}
&\widetilde{P}^{(j)}_{r-j}(n;d_0,\omega,\xi,X,t)\\
&\quad= \sum_{i=0}^{r-j}(-1)^ip^{i(i+1)/2}(p^{-n-r+j-2}t^4)^i\frac{\phi_{r-j}(p)}{\phi_i(p)\phi_{r-j-i}(p)}P^{(j)}_{r-j}(d_0,\omega,\xi,X,tY^{-1/2})\\
&\quad= P^{(j)}_{r-j}(d_0,\omega,\xi,X,tY^{-1/2})\prod_{i=1}^{r-j}(1-t^4p^{-n-r+j-2+i}).
\end{aligned}$$

□

Now we consider a partial series of $\widetilde{P}^{(j)}_{r-j}(n;d_0,\omega,\xi,X,Y,t)$. Let r be an even integer. First let $p\neq 2$. Then put

$$\begin{aligned}
&Q^{(0)}_r(n;d_0,\varepsilon^l,\xi,X,Y,t)\\
&\quad= \sum_{B'\in S_r(\mathbf{Z}_p,d_0)\cap S_r(\mathbf{Z}_p)} \frac{\widetilde{G}^{(l)}_p(pB',\xi,X,p^{-n}t^2Y)}{\alpha_p(pB')}\varepsilon(pB')^l(tY^{-1/2})^{\nu(\det pB')},
\end{aligned}$$

and

$$\begin{aligned}
&Q^{(1)}_{r-1}(n;d_0,\varepsilon^l,\xi,X,Y,t) = ((d_0,(-1)^{r/2})_p p^{(r-2)\nu(d_0)/2})^l\\
&\quad\times \sum_{B'\in p^{-1}S_{r-1}(\mathbf{Z}_p,d_0)\cap S_{r-1}(\mathbf{Z}_p)} \frac{\widetilde{G}^{(l)}_p(pB',\xi,X,p^{-n}t^2Y)}{\alpha_p(pB')}\varepsilon(pB')^l(tY^{-1/2})^{\nu(\det pB')}.
\end{aligned}$$

Next let $p=2$. Then put

$$\begin{aligned}
&Q^{(1)}_{r-1}(n;d_0,\varepsilon^l,\xi,X,Y,t) = \kappa(d_0,r-1,l,t)^{-1}\\
&\quad\times \sum_{B'\in S_{r-1}(\mathbf{Z}_2,d_0)\cap S_{r-1}(\mathbf{Z}_p)} \frac{\widetilde{G}^{(1)}_2(4B',\xi,X,2^{-n}t^2Y)}{\alpha_2(4B')}\varepsilon(4B')^l(tY^{-1/2})^{\nu_2(\det(4B'))},
\end{aligned}$$

and

$$Q_r^{(0)}(n; d_0, \varepsilon^l, \xi, X, Y, t) = \kappa(d_0, r, l, t)^{-1}$$
$$\times \sum_{B' \in S_r(\mathbf{Z}_2, d_0) \cap S_r(\mathbf{Z}_2)_e} \frac{\widetilde{G}_2^{(0)}(2B', \xi, X, 2^{-n} t^2 Y)}{\alpha_2(2B')} \varepsilon(B')^l (tY^{-1/2})^{\nu(\det(2B'))}.$$

Here we make the convention that $Q_0^{(0)}(n; d_0, \varepsilon^l, \xi, X, Y, t) = 1$ or 0 according as $\nu(d_0) = 0$ or not.

To consider the relation between

$$\widetilde{P}_{r-j}^{(j)}(n; d_0, \varepsilon^l, \xi, X, Y, t) \text{ and } Q_{r-j}^{(j)}(n; d_0, \varepsilon^l, \xi, X, Y, t),$$

and to express $\widetilde{R}_{n-1}(d_0, \varepsilon^l, X, Y, t)$ in terms of $\widetilde{P}_{r-j}^{(j)}(n; d_0, \varepsilon^l, \xi, X, Y, t)$, we provide some more preliminary results. First we review the canonical forms of the quadratic forms over $\mathbf{Z}_2$ following Watson [Wat76].

Lemma 5.3.2. *Let $B \in \mathcal{L}_{m,2}^{\times}$. Then B is equivalent, over $\mathbf{Z}_2$, to a matrix of the following form:*

$$\perp_{i=0}^{r} 2^i (V_i \perp U_i),$$

where $V_i = \perp_{j=1}^{k_i} c_{ij}$ with $0 \le k_i \le 2, c_{ij} \in \mathbf{Z}_2^*$ and $U_i = \frac{1}{2}\Theta_{m_i,d}$ with $0 \le m_i, d \in \mathcal{U}$. The degrees k_i and m_i of the matrices are uniquely determined by B. Furthermore we can take the matrix $\perp_{i=0}^{r} 2^i (V_i \perp U_i)$ uniquely so that it satisfies the following conditions:

(c.1) $c_{i1} = \pm 1$ *or* ± 3 *if* $k_i = 1$ *and* $(c_{i1}, c_{i2}) = (1, \pm 1), (1, \pm 3), (-1, -1)$, *or* $(-1, 3)$ *if* $k_i = 2$;

(c.2) $k_{i+2} = k_i = 0$ *if* $U_{i+2} = \frac{1}{2}\Theta_{m_{i+2},5}$ *with* $m_{i+2} > 0$;

(c.3) $-\det V_i \equiv 1 \bmod 4$ *if* $k_i = 2$ *and* $U_{i+1} = \frac{1}{2}\Theta_{m_{i+1},5}$ *with* $m_{i+1} > 0$;

(c.4) $(-1)^{k_i - 1} \det V_i \equiv 1 \bmod 4$ *if* $k_i, k_{i+1} > 0$;

(c.5) $V_i \ne \begin{pmatrix} -1 & 0 \\ 0 & c_{i2} \end{pmatrix}$ *if* $k_{i-1} > 0$;

(c.6) $V_i = \phi, (\pm 1), \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, *or* $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *if* $k_{i+2} > 0$.

The matrix satisfying the conditions (c.1) $\sim$ (c.6) is called the canonical form of B.

The following lemma follows from [[Kit93], Theorem 3.4.2].

Lemma 5.3.3. *Let m and r be integers such that $0 \le r \le m$, and $d_0 \in \mathbf{Z}_p^{\times}$.*

(1) *Let $p \ne 2$, and $T \in S_r(\mathbf{Z}_p, d_0)$. Then for any $d \in \mathcal{U}$ we have*

$$\varepsilon(\Theta_{m-r,d} \perp T) = ((-1)^{[(m-r+1)/2]} d, d_0)_p \varepsilon(T).$$

Furthermore we have

$$\varepsilon(pT) = \begin{cases} (p, d_0)_p \varepsilon(T) & \text{if } r \text{ even}, \\ (p, (-1)^{(r+1)/2})_p \varepsilon(T) & \text{if } r \text{ odd}, \end{cases}$$

and

$$\varepsilon(aT) = (a, d_0)_p^{r+1}\varepsilon(T)$$

for any $a \in \mathbf{Z}_p^*$.
(2) *Let* $p = 2$, *and* $T \in S_r(\mathbf{Z}_2, d_0)$. *Suppose that* $m - r$ *is even, and let* $d \in \mathcal{U}$. *Then for* $\Theta = 2\Theta_{m-r,d}$ *or* $\Theta = 2\Theta_{m-r-2}\perp(-d)$ *we have*

$$\varepsilon(\Theta\perp T) = (-1)^{(m-r)(m-r+2)/8}((-1)^{(m-r)/2}d, (-1)^{[(r+1)/2]}d_0)_2\varepsilon(T),$$

and

$$\varepsilon(\Theta_{m-r,d}\perp T) = (-1)^{(m-r)(m-r+2)/8}(2, d)_2((-1)^{(m-r)/2}d, (-1)^{[(r+1)/2]}d_0)_2\epsilon(T).$$

Furthermore we have

$$\varepsilon(2T) = (2, d_0)_2^{r+1}\varepsilon(T),$$

and

$$\varepsilon(a\perp T) = (a, (-1)^{[(r+1)/2]+1}d_0)_2\varepsilon(T)$$

for any $a \in \mathbf{Z}_2^*$, *and*

$$\varepsilon(aT) = \begin{cases} (a, d_0)_2\varepsilon(T) \text{ if } r \text{ even}, \\ (a, (-1)^{(r+1)/2})_2\varepsilon(T) \qquad \text{if } r \text{ odd} \end{cases}$$

for any $a \in \mathbf{Z}_2^*$.

Henceforth, for a while, we abbreviate $S_r(\mathbf{Z}_p)$ and $S_r(\mathbf{Z}_p, d)$ as $S_{r,p}$ and $S_{r,p}(d)$, respectively. Furthermore we abbreviate $S_r(\mathbf{Z}_2)_x$ and $S_r(\mathbf{Z}_2, d)_x$ as $S_{r,2;x}$ and $S_{r,2}(d)_x$, respectively, for $x = e, o$.

Let m be an even integer. Let $p \neq 2$. For $\xi = \pm 1$ let $H^{(0)}_{m,\xi}$ and $H^{(1)}_{m-1,\xi}$ be functions on $S_m(\mathbf{Z}_p)^\times$ and on $S_{m-1}(\mathbf{Z}_p)^\times$, respectively satisfying the following conditions:

(H-p-1) $H^{(0)}_{m,\xi}(\Theta_{m-2r,d}\perp pB) = H^{(0)}_{2r,\xi\chi(d)}(pB)$ for any $\xi = \pm 1, d \in \mathcal{U}$ and $B \in S_{2r}(\mathbf{Z}_p)$;
(H-p-2) $H^{(1)}_{m-1,\xi}(\Theta_{m-2r-2,d}\perp pB) = H^{(1)}_{2r+1,\xi}(pdB)$ for any $\xi = \pm 1, d \in \mathcal{U}$ and $B \in S_{2r+1}(\mathbf{Z}_p)$;
(H-p-3) $H^{(0)}_{m,\xi}(\Theta_{m-2r-1}\perp pB) = H^{(1)}_{2r+1,\xi}(-pB)$ for any $\xi = \pm 1$, and $B \in S_{2r+1}(\mathbf{Z}_p)$;
(H-p-4) $H^{(1)}_{m-1,\xi}(\Theta_{m-2r-1,d}\perp pB) = H^{(0)}_{2r,\xi\chi(d)}(pB)$ for any $\xi = \pm 1, d \in \mathcal{U}$ and $B \in S_{2r}(\mathbf{Z}_p)$;
(H-p-5) $H^{(0)}_{m,\xi}(dB) = H^{(0)}_{m,\xi}(B)$ for any $\xi = \pm 1, d \in \mathbf{Z}_p^*$ and $B \in S_m(\mathbf{Z}_p)$.

Let $d_0 \in \mathcal{F}_p$. Then we put

$$Q^{(1)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \epsilon^l, t) = \kappa(d_0, m-1, l, t)^{-1}$$
$$\times \sum_{d\in\mathcal{U}} \sum_{B\in p^{-1}S_{2r+1,p}(d_0 d)\cap S_{2r+1,p}} \frac{H^{(1)}_{m-1,\xi}(\Theta_{m-2r-2,d}\perp pB)\epsilon(\Theta_{m-2r-2,d}\perp pB)^l}{\alpha_p(\Theta_{m-2r-2,d}\perp pB)} t^{\nu(\det(pB))}.$$

For any $d \in \mathcal{U}$ we put

$$Q^{(1)}(d_0, d, H^{(1)}_{m-1,\xi}, 2r, \epsilon^l, t) = \kappa(d_0, m-1, l, t)^{-1}$$
$$\times \sum_{B \in S_{2r,p}(d_0 d) \cap S_{2r,p}} \frac{H^{(1)}_{m-1,\xi}((-\Theta_{m-2r-1,d}) \perp pB) \epsilon((-\Theta_{m-2r-1,d}) \perp pB)^l}{\alpha_p((-\Theta_{m-2r-1,d}) \perp pB)} t^{\nu(\det(pB))},$$

and

$$Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \epsilon^l, t)$$
$$= \sum_{B \in S_{2r,p}(d_0 d) \cap S_{2r,p}} \frac{H^{(0)}_{m,\xi}(\Theta_{m-2r,d} \perp pB) \epsilon(\Theta_{m-2r,d} \perp pB)^l}{\alpha_2(\Theta_{m-2r,d} \perp pB)} t^{\nu(\det(pB))}.$$

Here we make the convention that

$$Q^{(0)}(d_0, 1, H^{(0)}_{m,\xi}, m, \epsilon^l, t)$$
$$= \sum_{B \in S_{m,p}(d_0) \cap S_{m,p}} \frac{H^{(0)}_{m,\xi}(pB) \epsilon(pB)^l}{\alpha_2(pB)} t^{\nu(\det(pB))}.$$

Furthermore put

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \epsilon^l, t)$$
$$= \sum_{d \in \mathcal{U}} \sum_{B \in p^{-1} S_{2r+1,p}(d_0 d) \cap S_{2r+1,p}} \frac{H^{(0)}_{m,\xi}(\Theta_{m-2r-1,d} \perp pB) \epsilon(\Theta_{m-2r-1,d} \perp pB)^l}{\alpha_p(\Theta_{m-2r-1,d} \perp pB)} t^{\nu(\det(pB))}.$$

Let $p = 2$. Let $H^{(0)}_{m,\xi}$ and $H^{(1)}_{m-1,\xi}$ be functions on $S_m(\mathbf{Z}_p)^\times$ and on $S_{m-1}(\mathbf{Z}_2)^\times$, respectively satisfying the following conditions:

(H-2-1) $H^{(0)}_{m,\xi}(\Theta_{m-2r,d} \perp 2B) = H^{(0)}_{2r,\xi\chi(d)}(2B)$ for any $\xi = \pm 1, d \in \mathcal{U}$ and $B \in S_{2r}(\mathbf{Z}_2)$;
(H-2-2) $H^{(1)}_{m-1,\xi}(2\Theta_{m-2r-2,d} \perp 4B) = H^{(1)}_{2r+1,\xi}(4dB)$ for any $\xi = \pm 1, d \in \mathcal{U}$ and $B \in S_{2r+1}(\mathbf{Z}_2)$;
(H-2-3) $H^{(0)}_{m,\xi}(2 \perp \Theta_{m-2r-2} \perp 2B) = H^{(1)}_{2r+1,\xi}(-4B)$ for any $\xi = \pm 1$, and $B \in S_{2r+1}(\mathbf{Z}_2)$;
(H-2-4) $H^{(1)}_{m-1,\xi}(-a \perp 2\Theta_{m-2r-2,d} \perp 4B) = H^{(0)}_{2r,\xi\chi(a)}(2B)$ for any $\xi = \pm 1, a \in \mathcal{U}$ and $B \in S_{2r}(\mathbf{Z}_2)$,
(H-2-5) $H^{(0)}_{m,\xi}(dB) = H^{(0)}_{m,\xi}(B)$ for any $\xi = \pm 1, d \in \mathbf{Z}_2^*$ and $B \in S_{2r}(\mathbf{Z}_2)$.

Let $d_0 \in \mathcal{F}_2$. Then we put

$$\begin{aligned} Q^{(1)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \epsilon^l, t) &= \kappa(d_0, m-1, l, t)^{-1} \\ \times \Bigg\{ &\sum_{d \in \mathcal{U}} \sum_{B \in S_{2r+1,2}(d_0 d) \cap S_{2r+1,2;e}} H^{(1)}_{m-1,\xi}(2\Theta_{m-2r-2,d} \perp 4B) \\ &\times \frac{\epsilon(2\Theta_{m-2r-2,d} \perp 4B)^l}{\alpha_2(2\Theta_{m-2r-2,d} \perp 4B)} t^{m-2r-2+\nu(\det(4B))} \\ + &\sum_{B \in S_{2r+1,2}(d_0) \cap S_{2r+1,2;o}} H^{(1)}_{m-1,\xi}(2\Theta_{m-2r-2} \perp 4B) \\ &\times \frac{\epsilon(2\Theta_{m-2r-2} \perp 4B)^l}{\alpha_2(2\Theta_{m-2r-2} \perp 4B)} t^{m-2r-2+\nu(\det(4B))} \\ + &\sum_{B \in S_{2r+2,2}(d_0) \cap S_{2r+2,2;o}} H^{(1)}_{m-1,\xi}(-1 \perp 2\Theta_{m-2r-4} \perp 4B) \\ &\times \frac{\epsilon(-1 \perp 2\Theta_{m-2r-4} \perp 4B)^l}{\alpha_2(-1 \perp 2\Theta_{m-2r-4} \perp 4B)} t^{m-2r-4+\nu(\det(4B))} \Bigg\}. \end{aligned}$$

We note that

$$\begin{aligned} Q^{(1)}(d_0, H^{(1)}_{m-1,\xi}, m-1, \epsilon^l, t) &= \kappa(d_0, m-1, l, t)^{-1} \\ \times &\sum_{B \in S_{m-1,2}(d_0) \cap S_{m-1,2}} H^{(1)}_{m-1,\xi}(4B) \frac{\epsilon(4B)^l}{\alpha_2(4B)} t^{\nu(\det(4B))}. \end{aligned}$$

For any $d \in \mathcal{U}$ put

$$\begin{aligned} Q^{(1)}(d_0, d, H^{(1)}_{m-1,\xi}, 2r, \epsilon^l, t) &= \kappa(d_0, m-1, l, t)^{-1} \\ \times &\sum_{B \in S_{2r,2}(d_0 d) \cap S_{2r,2;e}} H^{(1)}_{m-1,\xi}(-d \perp 2\Theta_{m-2r-2} \perp 4B) \\ \times &\frac{\epsilon(-d \perp 2\Theta_{m-2r-2} \perp 4B)^l}{\alpha_2(-d \perp 2\Theta_{m-2r-2} \perp 4B)} t^{m-2r-2+\nu(\det(4B))}, \end{aligned}$$

and

$$\begin{aligned} Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \epsilon^l, t) &= \kappa(d_0, m, l, t)^{-1} \\ \times &\sum_{B \in S_{2r,2}(d_0 d) \cap S_{2r,2;e}} \frac{H^{(0)}_{m,\xi}(\Theta_{m-2r,d} \perp 2B) \epsilon(\Theta_{m-2r,d} \perp 2B)^l}{\alpha_2(\Theta_{m-2r,d} \perp 2B)} t^{\nu(\det(2B))}. \end{aligned}$$

Here we make the convention that

$$\begin{aligned} Q^{(0)}(d_0, 1, H^{(0)}_{m,\xi}, m, \epsilon^l, t) &= \kappa(d_0, m, l, t)^{-1} \\ \times &\sum_{B \in S_{m,2}(d_0) \cap S_{m,2;e}} \frac{H^{(0)}_{m,\xi}(2B) \epsilon(2B)^l}{\alpha_2(2B)} t^{\nu(\det(2B))}. \end{aligned}$$

Furthermore put

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \epsilon^l, t) = \kappa(d_0, m, l, t)^{-1}$$
$$\times \sum_{B \in S_{2r+2,2}(d_0) \cap S_{2r+2,2;o}} \frac{H^{(0)}_{m,\xi}(\Theta_{m-2r-2} \perp 2B)\epsilon(\Theta_{m-2r-2} \perp 2B)^l}{\alpha_2(\Theta_{m-2r-2} \perp 2B)} t^{\nu(\det(2B))}.$$

Proposition 5.3.4. (1) *Let $p \neq 2$.*

(1.1) *We have*

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \varepsilon^l, t) = \frac{Q^{(1)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{\phi_{(m-2r-2)/2}(p^{-2})}$$

if $l\nu(d_0) = 0$, and

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \varepsilon, t) = 0$$

if $\nu(d_0) = 1$.

(1.2) *Let $d \in \mathcal{U}$. Then*

$$Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \varepsilon^l, t)$$
$$= \frac{(1 + p^{-(m-2r)/2}\chi(d))Q^{(0)}(d_0 d, 1, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t)}{2\phi_{(m-2r)/2}(p^{-2})}$$

if $l\nu(d_0) = 0$, and

$$Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \varepsilon, t) = 0$$

if $\nu(d_0) = 1$.

(2) *Let $p = 2$.*

(2.1) *We have*

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \varepsilon^l, t) = \frac{Q^{(1)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{\phi_{(m-2r-2)/2}(2^{-2})}$$

if $l\nu(d_0) = 0$, and

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \varepsilon, t) = 0$$

if $\nu(d_0) > 0$.

(2.2) *Let $d \in \mathcal{U}$. Then*

$$Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \varepsilon^l, t)$$
$$= \frac{(1 + 2^{-(m-2r)/2}\chi(d))Q^{(0)}(d_0 d, 2r, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t)}{2\phi_{(m-2r)/2}(2^{-2})}$$

if $l\nu(d_0) = 0$, and

$$Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \varepsilon, t) = 0$$

if $\nu(d_0) > 0$.

Proof. (1) We note that

$$(-\Theta_{m-2r-1,d})\perp pB \sim d(-\Theta_{m-2r-1})\perp pB \approx (-\Theta_{m-2r-1})\perp dpB$$

for $d \in \mathcal{U}$ and $B \in p^{-1}S_{2r+1,p}(d_0d)$, and the mapping

$$S_{2r+1,p}(d_0d) \ni B \mapsto dB \in S_{2r+1,p}(d_0)$$

is a bijection. Furthermore by Lemma 5.3.3 we have

$$\varepsilon((-\Theta_{m-2r-1,d})\perp pB) = (d, d_0)_p\varepsilon(pB),$$

and $\varepsilon(dpB) = \varepsilon(pB)$. Thus the assertion (1.1) follows from (H-p-3),(H-p-5) and Lemma 5.1.3. Now by (H-p-2) and Lemmas 5.1.3 and 5.3.3, we have

$$Q^{(0)}(d_0, d, H^{(0)}_{m,\xi}, 2r, \varepsilon^l, t) = \frac{(1 + p^{-(m-2r)/2}\chi(d))((-1)^{(m-2r)/2}d, d_0)^l_p}{2\phi_{(m-2r)/2}(p^{-2})}$$

$$\times Q^{(0)}(d_0d, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t).$$

Thus the assertion (1.2) immediately follows in case $l\nu(d_0) = 0$. Now suppose that $l = 1$ and $\nu(d_0) = 1$. Take an element $a \in \mathbf{Z}^*_p$ such that $(a, p)_p = -1$. Then the mapping $S_{2r}(\mathbf{Z}_p) \ni B \mapsto aB \in S_{2r}(\mathbf{Z}_p)$ induces a bijection from $S_{2r,p}(dd_0)$ to itself, and $\varepsilon(apB) = -\varepsilon(pB)$ and $\alpha_p(apB) = \alpha_p(pB)$ for $B \in S_{2r,p}(dd_0)$. Furthermore by (H-p-5) we have

$$Q^{(0)}(d_0d, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t) = \sum_{B\in p^{-1}S_{2r}(dd_0)\cap S_{2r}} \frac{H^{(0)}_{2r,\xi\chi(d)}(apB)\varepsilon(apB)}{\alpha_p(apB)}$$

$$= -Q^{(0)}(d_0d, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t).$$

Hence $Q^{(0)}(d_0d, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t) = 0$. This completes the assertion.

(2) We prove (2.1). First suppose that $l = 0$, or $l = 1$ and $d_0 \equiv 1 \bmod 4$. Fix a complete set $\mathcal{B}$ of representatives of $(S_{2r+2,2}(d_0) \cap S_{2r+2,2;o})/\approx$. Then $2^{-1}S_{2r+1,2}(d_0) \cap S_{2r+1,2} = \cup_{B\in\mathcal{B}}\mathcal{S}_{2r+1,2}(B)$. We note that for any $B' \in \mathcal{S}_{2r+1,2}(B)$, we have $1\perp B' \approx B$, and hence

$$H^{(0)}_{2r+2,\xi}(2B) = H^{(0)}_{2r+2,\xi}(2\perp 2B') = H^{(1)}_{2r+1,\xi}(4B').$$

Thus, similarly to (1.1) we have

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r + 1, \iota, t)$$

$$= \sum_{B\in\mathcal{B}} \frac{H^{(0)}_{2r+2,\xi}(2B)}{\phi_{(m-2r-2)/2}(2^{-2})2^{(r+1)(2r+3)}\alpha_2(B)} \#(\widetilde{\mathcal{S}}_{2r+2,2}(B)/\sim)t^{\nu(\det(2B))}.$$

Hence by Lemma 5.1.5 we have

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r + 1, \varepsilon^l, t) = 2^{(2r+1)r}t^{-2r}$$

$$\times \sum_{B'\in 2^{-1}S_{2r+1,2}(d_0)\cap S_{2r+1,2}} \frac{H^{(0)}_{2r+1,\xi}(4B')}{\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(4B')} t^{\nu(\det(4B'))}.$$

This proves the assertion for $l = 0$. Now let $d_0 \equiv 1 \bmod 4$, and put $\xi_0 = (2, d_0)_2$. Then by Lemma 5.3.3 we have

$$\varepsilon(\Theta_{m-2r-2} \perp 2B) = (-1)^{m(m+2)/8+r(r+1)/2+(r+1)^2}\xi_0\varepsilon(B).$$

Furthermore for any $a \in \mathbf{Z}_2^*$ we have $\varepsilon(aB)^l = \varepsilon(B)^l$, and $\alpha_2(aB) = \alpha_2(B)$. Thus, by using the same argument as above we obtain

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \varepsilon, t) = (-1)^{m(m+2)/8}\xi_0$$

$$\times \sum_{B \in \mathcal{B}} \frac{H^{(0)}_{2r+2,\xi}(2B)(-1)^{m(m+2)/8+r(r+1)/2+(r+1)^2}\xi_0\varepsilon(B)}{\phi_{(m-2r-2)/2}(2^{-2})2^{(r+1)(2r+3)}\alpha_2(B)} \#(\widetilde{\mathcal{S}}_{2r+2,2}(B)/\sim)t^{\nu(\det(2B))}.$$

We note that $\varepsilon(1 \perp B') = \varepsilon(4B')$ for $B' \in S_{2r+1,2}$. Hence, again by Lemma 5.1.5, we have

$$Q^{(0)}(d_0, H^{(0)}_{m,\xi}, 2r+1, \varepsilon^l, t) = (-1)^{r(r+1)/2}((-1)^{r+1}, (-1)^{r+1})_2 2^{(2r+1)r}t^{-2r}$$

$$\times \sum_{B' \in 2^{-1}S_{2r+1,2}(d_0) \cap S_{2r+1,2}} \frac{H^{(0)}_{2r+1,\xi}(4B')\varepsilon(B)}{\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(4B')} t^{\nu(\det(4B'))}.$$

This proves the assertion for $l = 1$ and $d_0 \equiv 1 \bmod 4$.

Next suppose that $l = 1$ and $4^{-1}d_0 \equiv -1 \bmod 4$, or $l = 1$ and $8^{-1}d_0 \in \mathbf{Z}_2^*$. Then there exists an element $a \in \mathbf{Z}_2^*$ such that $(a, d_0)_2 = -1$. Then the map $2B \mapsto 2aB$ induces a bijection of $2S_{2r+2}(\mathbf{Z}_2, d_0)_o$ to itself. Furthermore $H^{(0)}_{2r+2,\xi}(2aB) = H^{(0)}_{2r+2,\xi}(2B), \varepsilon(2aB) = -\varepsilon(2B)$, and $\alpha_2(2aB) = \alpha_2(2B)$. Thus the assertion can be proved by using the same argument as in the proof of (1.2). The assertion (2.2) can be proved by (H-2-1) and Lemmas 5.1.4 and 5.3.3 similarly to (1.2). □

Proposition 5.3.5. (1) *Let $p \neq 2$.*

(1.1) *We have*

$$Q^{(1)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = \frac{Q^{(1)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{\phi_{(m-2r-2)/2}(p^{-2})}.$$

(1.2) *Let $d \in \mathcal{U}$. Then*

$$Q^{(1)}(d_0, d, H^{(0)}_{m-1,\xi}, 2r, \varepsilon^l, t) = \frac{Q^{(0)}(d_0 d, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t)}{2\phi_{(m-2r-2)/2}(p^{-2})}$$

if $l\nu(d_0) = 0$, and

$$Q^{(1)}(d_0, d, H^{(0)}_{m-1,\xi}, 2r, \varepsilon^l, t) = 0$$

otherwise.

(2) *Let $p = 2$.*

(2.1) *We have*

$$Q^{(1)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = \frac{Q^{(1)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{\phi_{(m-2r-2)/2}(2^{-2})}.$$

(2.2) *Let $d \in \mathcal{U}$. Then*

$$Q^{(1)}(d_0, d, H^{(0)}_{m-1,\xi}, 2r, \varepsilon^l, t) = \frac{Q^{(0)}(d_0 d, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t)}{2\phi_{(m-2r-2)/2}(2^{-2})}$$

if $l\nu(d_0) = 0$, and we have

$$Q^{(1)}(d_0, d, H^{(0)}_{m-1,\xi}, 2r, \varepsilon^l, t) = 0$$

otherwise.

Proof. (1.1) We may suppose that $r < (m-2)/2$. We note the mapping $S_{2r+1}(\mathbf{Z}_p) \ni B \mapsto dB \in S_{2r+1}(\mathbf{Z}_p)$ induces a bijection of $S_{2r+1}(\mathbf{Z}_p)(d_0 d)$ to $S_{2r+1}(\mathbf{Z}_p)(d_0)$. We also note that $\varepsilon(dB) = \varepsilon(B)$, and $\alpha_p(dB) = \alpha_p(B)$. Hence, by (H-$p$-2), Lemmas 5.1.3 and 5.3.3, similarly to (1.2) of Proposition 5.3.4, we have

$$Q^{(1)}(d_0, H^{(1)}_{m,\xi}, 2r+1, \varepsilon^l, t) = p^{(m/2-1)l\nu(d_0)}((-1)^{m/2}d_0, (-1)^{lm/2})_p$$

$$\times \sum_{B \in p^{-1}S_{2r+1,p}(d_0) \cap S_{2r+1,p}} \frac{H^{(1)}_{2r+1,\xi}(pB)\varepsilon(pB)^l}{2\phi_{(m-2r-2)/2}(p^{-2})\alpha_p(pB)} t^{\nu(\det(pB))}$$

$$\times \sum_{d \in \mathcal{U}} (1 + p^{-(m-2r-2)/2}\chi(d))((-1)^{(m-2r-2)/2}d, (-1)^{r+1}d_0 d)_p^l.$$

Thus the assertion clearly holds if $l\nu(d_0) = 0$. Suppose that $l = 1$ and $\nu(d_0) = 1$. Then

$$\begin{aligned} &((-1)^{(m-2r-2)/2}d, (-1)^{r+1}d_0 d)_p \\ &\quad = \chi(d)((-1)^{r+1}, (-1)^{r+1}d_0 d)_p((-1)^{m/2}, (-1)^{m/2}d_0)_p, \end{aligned}$$

and therefore

$$\begin{aligned} &\sum_{d \in \mathcal{U}} (1 + p^{-(m-2r-2)/2}\chi(d))((-1)^{(m-2r-2)/2}d, (-1)^{r+1}d_0)_p \\ &\quad = 2p^{-(m-2r-2)/2}((-1)^{r+1}, (-1)^{r+1}d_0 d)_p((-1)^{m/2}, (-1)^{m/2}d_0)_p. \end{aligned}$$

This completes the assertion.

(1.2) By (H-p-4) and by Lemmas 5.1.3 and 5.3.3, we have

$$\begin{aligned} &Q^{(1)}(d_0, d, H^{(1)}_{m-1,\xi}, 2r, \varepsilon^l, t) \\ &\quad = \frac{Q^{(0)}(d_0 d, H^{(0)}_{2r,\xi\chi(d)}, 2r, \varepsilon^l, t)}{2\phi_{(m-2r-2)/2}(p^{-2})}((-1)^{(m-2r)/2}d, d_0)_p^l. \end{aligned}$$

Thus the assertion (1.2) immediately follows if $l\nu(d_0) = 0$. The assertion for $l = 1$ and $\nu(d_0) = 1$ can also be proved by using the same argument as the latter half of (1.2) of Proposition 5.3.4.

(2.1) We may suppose that $r < (m-2)/2$. Put

$$Q^{(11)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = \kappa(d_0, m-1, l, t)^{-1}$$
$$\times \sum_{d\in\mathcal{U}} \sum_{B\in S_{2r+1,2}(d_0d)\cap S_{2r+1,2;e}} \frac{H^{(1)}_{m-1,\xi}(2\Theta_{m-2r-2,d}\perp 4B)\varepsilon^l(2\Theta_{m-2r-2,d}\perp 4B)}{\alpha_2(2\Theta_{m-2r-2,d}\perp 4B)}$$
$$\times t^{m-2r-2+\nu(\det(4B))},$$

$$Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = \kappa(d_0, m-1, l, t)^{-1}$$
$$\times \sum_{B\in S_{2r+1,2}(d_0)\cap S_{2r+1,2;o}} \frac{H^{(1)}_{m-1,\xi}(2\Theta_{m-2r-2}\perp 4B)\varepsilon^l(2\Theta_{m-2r-2}\perp 4B)}{\alpha_2(2\Theta_{m-2r-2}\perp 4B)}$$
$$\times t^{m-2r-2+\nu(\det(4B))},$$

and

$$Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = \kappa(d_0, m-1, l, t)^{-1}$$
$$\times \sum_{B\in S_{2r+2,2}(d_0)\cap S_{2r+2,2;o}} H^{(1)}_{m-1,\xi}(-1\perp 2\Theta_{m-2r-4}\perp 4B)$$
$$\times \frac{\varepsilon^l(-1\perp 2\Theta_{m-2r-4}\perp 4B)}{\alpha_2(-1\perp 2\Theta_{m-2r-4}\perp 4B)}\ t^{m-2r-4+\nu(\det(4B))}.$$

Then

$$Q^{(1)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = Q^{(11)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t)$$
$$+\ \ Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) + Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t).$$

We have

$$\varepsilon(2\Theta_{m-2r-2,d}\perp 4B) = (-1)^{m(m-2)/8}(-1)^{r(r+1)/2}((-1)^{m/2}, (-1)^{m/2}d_0)_2$$
$$\times((-1)^{r+1}, (-1)^{r+1}d_0d)_2(d_0, d)_2\ \varepsilon(4B)$$

for $d \in \mathcal{U}$ and $B \in S_{2r+1}(\mathbf{Z}_2, dd_0)$. Thus, similarly to (1.1), we obtain

$$Q^{(11)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = (-1)^{r(r+1)l/2}t^{-2r}((-1)^{r+1}, (-1)^{r+1}d_0)_2^l$$
$$\times\ \ 2^{(m/2-1)l\nu(d_0)} \sum_{B\in S_{2r+1,2}(d_0)\cap S_{2r+1,2;e}} \frac{2^{r(2r+1)}H^{(1)}_{2r+1,\xi}(4B)\varepsilon(4B)^l}{2\cdot 2^{m-2r-2}\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(4B)}t^{\nu(\det(4B))}$$
$$\times\ \ \sum_{d\in\mathcal{U}}(1+2^{-(m-2r-2)/2}\chi(d))(d, d_0)_2^l.$$

It follows from Lemma 5.3.2, (c.2) that $Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t)$ can also be expressed as

$$\begin{aligned} Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) &= \kappa(d_0, m-1, l, t)^{-1} \\ \times \quad & \frac{1}{2}\sum_{d\in\mathcal{U}} \sum_{B\in S_{2r+1,2}(d_0 d)\cap S_{2r+1,2;o}} \frac{H^{(1)}_{m-1,\xi}(2\Theta_{m-2r-2,d}\perp 4B)\varepsilon^l(2\Theta_{m-2r-2,d}\perp 4B)}{\alpha_2(2\Theta_{m-2r-2,d}\perp 4B)} \\ & \times t^{m-2r-2+\nu(\det(4B))}. \end{aligned}$$

Hence, in the same manner as above, we obtain

$$\begin{aligned} Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) &= (-1)^{r(r+1)l/2} t^{-2r} ((-1)^{r+1}, (-1)^{r+1} d_0)_2^l \\ \times \quad & 2^{(m/2-1)l\nu(d_0)} \sum_{B\in S_{2r+1,2}(d_0)\cap S_{2r+1,2;o}} \frac{2^{r(2r+1)} H^{(1)}_{2r+1,\xi}(4B)\varepsilon(4B)^l}{2\cdot 2^{m-2r-2}\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(4B)} t^{\nu(\det(4B))} \\ \times \quad & \sum_{d\in\mathcal{U}} (d, d_0)_2^l. \end{aligned}$$

Furthermore we have

$$\begin{aligned} \varepsilon(-1\perp 2\Theta_{m-2r-4}\perp 4B) &= (-1)^{m(m-2)/8}(-1)^{r(r+1)/2}((-1)^{m/2}, (-1)^{m/2} d_0)_2 \\ &\quad \times ((-1)^{r+1}, (-1)^{r+1} d_0)_2 (2, d_0)_2 \varepsilon(2B) \end{aligned}$$

for $d \in \mathcal{U}$ and $B \in S_{2r+2}(\mathbf{Z}_2, dd_0) \cap S_{2r+2,2;o}$. Hence

$$\begin{aligned} Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) &= (-1)^{r(r+1)l/2} t^{-2r-2} ((-1)^{r+1}, (-1)^{r+1} d_0)_2^l \\ & \quad \times (2, d_0)_2^l \; 2^{(m/2-1)l\nu(d_0)} \sum_{B\in S_{2r+2,2}(d_0)\cap S_{2r+2,2;o}} \frac{H^{(0)}_{2r+2,\xi}(2B)\varepsilon(4B)^l}{\phi_{(m-2r-4)/2}(2^{-2})\alpha_2(2B)} t^{\nu(\det(4B))} \\ &= (((-1)^{r+1}2, d_0)_2 (-1)^{(r+1)(r+2)/2})^l 2^{(m/2-1)l\nu(d_0)} \\ & \quad \times \sum_{B\in S_{2r+2,2}(d_0)\cap S_{2r+2,2;o}} \frac{H^{(0)}_{2r+2,\xi}(2B)\varepsilon(2B)^l}{\phi_{(m-2r-4)/2}(2^{-2})\alpha_2(2B)} t^{\nu(\det(2B))} \\ &= \frac{Q^{(0)}(d_0, H^{(0)}_{2r+2,\xi}, 2r+1, \varepsilon^l, t)}{\phi_{(m-2r-4)/2}(2^{-2})} 2^{(m/2-1)l\nu(d_0)}. \end{aligned}$$

First suppose that $l = 0$ or $\nu(d_0)$ is even. Then $(d, d_0)_2^l = 1$. Hence

$$\begin{aligned} & Q^{(11)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) + Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) \\ &= \frac{Q^{(1)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{2^{(m-2r-2)(1-\nu(d_0)/2)}\phi_{(m-2r-2)/2}(2^{-2})}. \end{aligned}$$

Furthermore by (2.1) of Proposition 5.3.4, we have

$$Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = \frac{Q^{(1)}(d_0, H^{(1)}_{2r+1,\xi}, 2r+1, \varepsilon^l, t)}{\phi_{(m-2r-4)/2}(2^{-2})}$$

if $d_0 \equiv 1 \bmod 4$, and

$$Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = 0$$

if $4^{-1}d_0 \equiv -1 \bmod 4$. Thus summing up these two quantities, we prove the assertion. Next suppose that $l = 1$ and $\nu(d_0) = 3$. Then, by using the same argument as above we obtain

$$Q^{(13)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon^l, t) = 0.$$

We also note that $(d, d_0) = \chi(d)$. Hence

$$Q^{(12)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon, t) = 0,$$

and therefore

$$\begin{aligned} Q^{(11)}(d_0, H^{(1)}_{m-1,\xi}, 2r+1, \varepsilon, t) &= (-1)^{r(r+1)/2} t^{-2r} 2^{r(2r+1)} ((-1)^{r+1}, (-1)^{r+1}d_0)_2 \\ &\quad \times 2^{3(m/2-1)} \sum_{B \in S_{2r+1,2}(d_0) \cap S_{2r+1,2;e}} \frac{H^{(1)}_{2r+1,\xi}(4B)\varepsilon(4B)}{2 \cdot 2^{m-2r-2}\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(4B)} t^{\nu(\det(4B))} \\ &\quad \times \sum_{d \in \mathcal{U}} (1 + 2^{-(m-2r-2)/2}\chi(d))\chi(d)) \\ &= (-1)^{r(r+1)/2} t^{-2r} 2^{r(2r+1)} ((-1)^{r+1}, (-1)^{r+1}d_0)_2 \; 2^{3r} \\ &\quad \times \sum_{B \in S_{2r+1,2;e}(d_0)} \frac{H^{(1)}_{2r+1,\xi}(4B)\varepsilon(4B)}{\phi_{(m-2r-2)/2}(2^{-2})\alpha_2(4B)} t^{\nu(\det(4B))}. \end{aligned}$$

This proves the assertion.

(2.2) The assertion can be proved in the same manner as in (1.2). □

Now to apply Propositions 5.3.4 and 5.3.5 to the formal power series $\widetilde{R}_{n-1}(d_0, \omega, X, Y, t)$ and $Q^{(1)}_{2r+1}(d_0, \omega, \eta, X, Y, t)$ we give some more lemmas.

Lemma 5.3.6. *Let m be an even integer, and r an integer such that $r \leq m$. Let $d \in \mathcal{U}$ and $\xi_0 = \pm 1$.*

(1) *Suppose that r is even.*

(1.1) *Let $B' \in S_r(\mathbf{Z}_p)$. Then*

$$\widetilde{G}^{(0)}_p(\Theta_{m-r,d} \perp pB', \xi_0, X, t) = \widetilde{G}^{(0)}_p(pB', \xi_0\chi(d), X, t).$$

(1.2) *Let $B' \in S_{r-1}(\mathbf{Z}_p)$. Then*

$$\widetilde{G}^{(1)}_p(\Theta_{m-r,d} \perp pB', \xi_0, X, t) = \widetilde{G}^{(1)}_p(pdB', \xi_0, X, t).$$

(2) *Suppose that r is odd.*

(2.1) *Let $B' \in S_r(\mathbf{Z}_p)$. Then*

$$\widetilde{G}^{(0)}_p(\Theta_{m-r,d} \perp pB', \xi_0, X, t) = \widetilde{G}^{(1)}_p(-pdB', \xi_0, X, t).$$

(2.2) *Let* $B' \in S_{r-1}(\mathbf{Z}_p)$. *Then*

$$\widetilde{G}_p^{(1)}(\Theta_{m-r,d}\perp pB', \xi_0, X, t) = \widetilde{G}_p^{(0)}(pB', \xi_0\chi(d), X, t).$$

Proof. Let $m-r$ be even. Then by Lemma 9 of [Kit84], we have

$$G_p^{(0)}(\Theta_{m-r,d}\perp pB', \xi_0, X) = G_p^{(0)}(pB', \xi_0\chi(d), X)$$

for $B' \in S_r(\mathbf{Z}_p)$. Hence by Lemma 5.2.2 we have

$$\widetilde{F}_p^{(0)}(\Theta_{m-r,d}\perp pB', \xi_0, X) = \widetilde{F}_p^{(0)}(pB', \xi_0\chi(d), X)$$

for $B' \in S_r(\mathbf{Z}_p)$. Thus the assertion (1.1) follows from (1.1) of Lemma 5.1.2. Furthermore we have

$$\widetilde{F}_p^{(1)}(\Theta_{m-r,d}\perp pB', \xi_0, X) = \widetilde{F}_p(1\perp\Theta_{m-r,d}\perp pB', \xi_0, X)$$

$$= \widetilde{F}_p(d\perp\Theta_{m-r}\perp pB', \xi_0, X) = \widetilde{F}_p(1\perp\Theta_{m-r}\perp pdB', \xi_0, X)$$

$$= \widetilde{F}_p(1\perp pdB', \xi_0, X) = \widetilde{F}_p^{(1)}(pdB', \xi_0, X)$$

for $B' \in S_{r-1}(\mathbf{Z}_p)$. Thus the assertion (1.2) follows from (1.2) of Lemma 5.1.2. The other assertions can be proved in a similar way. □

Lemma 5.3.7. *Let* $p = 2$. *Let* m *and* r *be even integers, and* $\xi_0 = \pm 1$.
(1) *Let* $d \in \mathcal{U}$.

(1.1) *Let* $B' \in S_r(\mathbf{Z}_2)_e$. *Then*

$$\widetilde{G}_2^{(0)}(\Theta_{m-r,d}\perp 2B', \xi_0, X, t) = \widetilde{G}_2^{(0)}(2B', \xi_0\chi(d), X, t),$$

(1.2) *Let* $B' \in S_{r-1}(\mathbf{Z}_2)$. *Then*

$$\widetilde{G}_2^{(1)}(2\Theta_{m-r,d}\perp 4B', \xi_0, X, t) = \widetilde{G}_2^{(1)}(4dB', \xi_0, X, t).$$

(2)

(2.1) *Let* $a \in \mathcal{U}$ *and* $B' \in S_r(\mathbf{Z}_2)$. *Then*

$$\widetilde{G}_2^{(1)}(-a\perp 2\Theta_{m-r-2}\perp 4B', \xi_0, X, t) = \widetilde{G}_2^{(0)}(2B', \xi_0\chi(a), X, t).$$

(2.2) *Let* $B' \in S_{r-1}(\mathbf{Z}_2)_e$ *and* $a \in \mathbf{Z}_2^*$. *Then*

$$\widetilde{G}_2^{(0)}(\Theta_{m-r}\perp 2a\perp 2B', \xi_0, X, t) = \widetilde{G}_2^{(1)}(4aB', \xi_0, X, t).$$

Proof. The assertion can be proved in a way similar to Lemma 5.3.6. □

Let $\widetilde{R}_{n-1}(d_0, \omega, X, Y, t)$ be the formal power series defined at the beginning of Section 5. We express $\widetilde{R}_{n-1}(d_0, \omega, X, Y, t)$ in terms of $\widetilde{Q}_{2r}^{(0)}(d_0 d, \omega, \chi(d), X, Y, t)$ and $\widetilde{Q}_{2r+1}^{(1)}(d_0, \omega, 1, X, Y, t)$. Henceforth, for $d_0 \in \mathcal{F}_p$ and non-negative integers m, r such that $r \leq m$, put $\mathcal{U}(m, r, d_0) = \{1\}, \mathcal{U} \cap \{d_0\}$, or $\mathcal{U}$ according as $r = 0, r = m$, or $1 \leq r \leq m-1$.

Theorem 5.3.8. *Let $d_0 \in \mathcal{F}_p$, and $\xi_0 = \chi(d_0)$. For $d \in \mathcal{U}(n-1, n-2r-1, d_0)$ put*

$$D_{2r}(d_0, d, Y, t) = \begin{cases} (1 - \xi_0 p^{-n-1/2} Y t^2) & \text{if } r = 0, \\ (1 - \xi_0 p^{-1/2} Y)(1 + p^{r-1/2}\chi(d) Y)(1 - p^{-n-1/2+r}\chi(d) Y t^2) & \text{if } r > 0. \end{cases}$$

(1) *Suppose that $p \neq 2$.*

(1.1) *Let $\omega = \iota$, or $\omega = \varepsilon$ and $\nu(d_0) = 0$. Then*

$$\widetilde{R}_{n-1}(d_0, \omega, X, Y, t)$$

$$= \sum_{r=0}^{(n-2)/2} \frac{\prod_{i=1}^{r-1}(1 - p^{2i-1}Y^2) \prod_{i=1}^{(n-2r-2)/2}(1 - p^{-2i-n-1}Y^2 t^4)}{2\phi_{(n-2r-2)/2}(p^{-2})}$$

$$\times \sum_{d \in \mathcal{U}(n-1, n-2r-1, d_0)} D_{2r}(d_0, d, Y, t) \widetilde{Q}_{2r}^{(0)}(n; d_0 d, \omega, \chi(d), X, Y, t)$$

$$+ \sum_{r=0}^{(n-2)/2} \frac{\prod_{i=1}^{r}(1 - p^{2i-1}Y^2) \prod_{i=1}^{(n-2r-2)/2}(1 - p^{-2i-n-1}Y^2 t^4)}{\phi_{(n-2r-2)/2}(p^{-2})}$$

$$\times (1 - \xi_0 p^{-1/2} Y) \widetilde{Q}_{2r+1}^{(1)}(n; d_0, \omega, 1, X, Y, t).$$

(1.2) *Let $\nu(d_0) = 1$. Then*

$$\widetilde{R}_{n-1}(d_0, \varepsilon, X, Y, t)$$

$$= \sum_{r=0}^{(n-2)/2} \frac{\prod_{i=1}^{r}(1 - p^{2i-1}Y^2) \prod_{i=1}^{(n-2r-2)/2}(1 - p^{-2i-n-1}Y^2 t^4)}{\phi_{(n-2r-2)/2}(p^{-2})}$$

$$\times (1 - \xi_0 p^{-1/2} Y) \widetilde{Q}_{2r+1}^{(1)}(n; d_0, \varepsilon, 1, X, Y, t).$$

(2) *Suppose that $p = 2$.*

(2.1) *Let $\omega = \iota$, or $\omega = \varepsilon$ and $\nu(d_0) = 0$. Then*

$$\widetilde{R}_{n-1}(d_0, \omega, X, Y, t)$$

$$= \sum_{r=0}^{(n-2)/2} \frac{\prod_{i=1}^{r-1}(1 - 2^{2i-1}Y^2) \prod_{i=1}^{(n-2r-2)/2}(1 - 2^{-2i-n-1}Y^2 t^4)}{2\phi_{(n-2r-2)/2}(2^{-2})}$$

$$\times \sum_{d \in \mathcal{U}(n-1, n-2r-1, d_0)} D_{2r}(d_0, d, Y, t) \widetilde{Q}_{2r}^{(0)}(n; d_0 d, \omega, \chi(d), X, Y, t)$$

$$+ \sum_{r=0}^{(n-2)/2} \frac{\prod_{i=1}^{r}(1 - 2^{2i-1}Y^2) \prod_{i=1}^{(n-2r-2)/2}(1 - 2^{-2i-n-1}Y^2 t^4)}{\phi_{(n-2r-2)/2}(2^{-2})}$$

$$\times (1 - \xi_0 p^{-1/2} Y) \widetilde{Q}_{2r+1}^{(1)}(n; d_0, \omega, 1, X, Y, t).$$

(2.2) *Let* $4^{-1}d_0 \equiv 1 \bmod 4$, *or* $8^{-1}d_0 \in \mathbf{Z}_2^*$. *Then*

$$\widetilde{R}_{n-1}(d_0,\varepsilon,X,Y,t)$$
$$= \sum_{r=0}^{(n-2)/2} \frac{\prod_{i=1}^{r}(1-2^{2i-1}Y^2)\prod_{i=1}^{(n-2r-2)/2}(1-2^{-2i-n-1}Y^2t^4)}{\phi_{(n-2r-2)/2}(2^{-2})}$$
$$\times(1-\xi_0 p^{-1/2}Y)\widetilde{Q}^{(1)}_{2r+1}(n;d_0,\varepsilon,1,X,Y,t).$$

Proof. Let $p \neq 2$. Let B be a symmetric matrix of degree $2r$ or $2r+1$ with entries in $\mathbf{Z}_p$. Then we note that $\Theta_{n-2r-2,d}\perp pB$ belongs to $\mathcal{L}_{n-1,p}(d_0)$ if and only if $B \in S_{2r+1,p}(p^{-1}d_0 d)\cap S_{2r+1,p}$, and that $\Theta_{n-2r-1,d}\perp pB$ belongs to $\mathcal{L}_{n-1,p}(d_0)$ if and only if $B \in S_{2r,p}(d_0 d)\cap S_{2r,p}$. Thus by the theory of Jordan forms, we have

$$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)$$
$$= \sum_{r=0}^{(n-2)/2} \sum_{d\in\mathcal{U}(n-1,n-2r-2,d_0)} \sum_{B'\in p^{-1}S_{2r+1,p}(d_0 d)\cap S_{2r+1,p}} \frac{G_p^{(1)}(\Theta_{n-2r-2,d}\perp pB', p^{-(n+1)/2}Y)}{\alpha_p(\Theta_{n-2r-2,d}\perp pB')}$$
$$\times B_p^{(1)}(\Theta_{n-2r-2,d}\perp pB', p^{-n/2-1}Yt^2)\widetilde{G}_p^{(1)}(\Theta_{n-2r-2,d}\perp pB', 1, X, p^{-n}t^2Y)$$
$$\times\omega(\Theta_{n-2r-2,d}\perp pB')Y^{-\mathfrak{e}^{(1)}(pB')}t^{\nu(\det(pB'))}$$
$$+ \sum_{r=0}^{(n-2)/2} \sum_{d\in\mathcal{U}(n-1,n-2r-1,d_0)} \sum_{B'\in S_{2r,p}(d_0 d)\cap S_{2r,p}} \frac{G_p^{(1)}(\Theta_{n-2r-1,d}\perp pB', p^{-(n+1)/2}Y)}{\alpha_p(\Theta_{n-2r-1,d}\perp pB')}$$
$$\times B_p^{(1)}(\Theta_{n-2r-1,d}\perp pB', p^{-n/2-1}Yt^2)\widetilde{G}_p^{(1)}(\Theta_{n-2r-1,d}\perp pB', 1, X, p^{-n}t^2Y)$$
$$\times\omega(\Theta_{n-2r-1,d}\perp pB')Y^{-\mathfrak{e}^{(1)}(pB')}t^{\nu(\det(pB'))}.$$

By Lemmas 5.2.1 and 5.2.3 we have

$$G_p^{(1)}(\Theta_{n-2r-2,d}\perp pB', p^{-(n+1)/2}Y)B_p^{(1)}(\Theta_{n-2r-2,d}\perp pB', p^{-n/2-1}Yt^2)$$
$$= \prod_{i=1}^{r}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2r-2)/2}(1-p^{-2i-n-1}Y^2t^4)(1-\xi_0 p^{-1/2}Y),$$

and

$$G_p^{(1)}(\Theta_{n-2r-1,d}\perp pB', p^{-(n+1)/2}Y)B_p^{(1)}(\Theta_{n-2r-1,d}\perp pB', p^{-n/2-1}Yt^2)$$
$$= \prod_{i=1}^{r-1}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2r-2)/2}(1-p^{-2i-n-1}Y^2t^4)D_{2r}(d_0,d,Y,t).$$

Put $H^{(1)}_{2i-1,\xi}(B) = \widetilde{G}_p^{(1)}(B,\xi,X,p^{-n}t^2Y)$ for $B \in S_{2i-1}(\mathbf{Z}_p)$, and $H^{(0)}_{2i,\xi}(B) = \widetilde{G}_p^{(0)}(B,\xi,X,p^{-n}t^2Y)$ for $B \in S_{2i}(\mathbf{Z}_p)$ and $\xi = \pm 1$. Then $H^{(1)}_{2i-1,\xi}$ and $H^{(0)}_{2i,\xi}$ satisfy the conditions (H-p-1) $\sim$ (H-p-5) by Lemmas 5.3.6 and 5.3.7. Thus the assertion (1) in case $p \neq 2$ follows from Propositions 5.3.4 and 5.3.5.

Next let $p = 2$. Let B be a symmetric matrix of degree $2r$ or $2r+1$ with entries in $\mathbf{Z}_2$, and $d \in \mathcal{U}$. We note that $2\Theta_{n-2r-2,d}\perp 4B$

belongs to $\mathcal{L}_{n-1,2}(d_0)$ if and only if $B \in S_{2r+1,2}(d_0 d) \cap S_{2r+1,2}$, and that $-d \perp 2\Theta_{n-2r-2} \perp 4B$ belongs to $\mathcal{L}_{n-1,2}(d_0)$ if and only if $B \in S_{2r+2,2}(d_0 d) \cap S_{2r+2,2}$. Then by the theory of canonical forms, we have

$$\widetilde{R}_{n-1}(d_0, \omega, X, Y, t)$$

$$= \sum_{r=0}^{(n-2)/2} \{ \sum_{d \in \mathcal{U}(n-1,n-2r-2,d_0)} \sum_{B' \in S_{2r+1,2}(d_0 d) \cap S_{2r+1,2;e}} G_p^{(1)}(2\Theta_{n-2r-2,d} \perp 4B', 2^{-(n+1)/2} Y)$$

$$\times B_p^{(1)}(2\Theta_{n-2r-2,d} \perp 4B', p^{-n/2-1} Y t^2) \frac{\widetilde{G}_2^{(1)}(2\Theta_{n-2r-2,d} \perp 4B', 1, X, 2^{-n} t^2 Y)}{\alpha_2(2\Theta_{n-2r-2,d} \perp 4B')}$$

$$\times \omega(2\Theta_{n-2r-2,d} \perp 4B') Y^{-\mathfrak{e}^{(1)}(4B') - (n-2r-2)} t^{\nu(\det(4B'))+n-2r-2}$$

$$+ \sum_{B' \in S_{2r+1,2}(d_0) \cap S_{2r+1,2;o}} G_p^{(1)}(2\Theta_{n-2r-2} \perp 4B', 2^{-(n+1)/2} Y)$$

$$\times B_p^{(1)}(2\Theta_{n-2r-2} \perp 4B', p^{-n/2-1} Y t^2) \frac{\widetilde{G}_2^{(1)}(2\Theta_{n-2r-2} \perp 4B', 1, X, 2^{-n} t^2 Y)}{\alpha_2(2\Theta_{n-2r-2} \perp 4B')}$$

$$\times \omega(2\Theta_{n-2r-2} \perp 4B') Y^{-\mathfrak{e}^{(1)}(4B') - (n-2r-2)} t^{\nu(\det(4B'))+n-2r-2}$$

$$+ \sum_{B' \in S_{2r+2,2}(d_0) \cap S_{2r+2,2;o}} G_p^{(1)}(-1 \perp 2\Theta_{n-2r-4} \perp 4B', 2^{-(n+1)/2} Y)$$

$$\times B_p^{(1)}(-1 \perp 2\Theta_{n-2r-4} \perp 4B', p^{-n/2-1} Y t^2) \frac{\widetilde{G}_2^{(1)}(-1 \perp 2\Theta_{n-2r-4} \perp 4B', 1, X, 2^{-n} t^2 Y)}{\alpha_2(-1 \perp 2\Theta_{n-2r-4} \perp 4B')}$$

$$\times \omega(-1 \perp 2\Theta_{n-2r-4} \perp 4B') Y^{-\mathfrak{e}^{(1)}(4B') - (n-2r-4)} t^{\nu(\det(4B'))+n-2r-4} \}$$

$$+ \sum_{r=0}^{(n-2)/2} \sum_{d \in \mathcal{U}(n-1,n-2r-1,d_0)} \sum_{B' \in S_{2r,2}(d_0 d) \cap S_{2r,2;e}} G_p^{(1)}(-d \perp 2\Theta_{n-2r-2} \perp 4B', 2^{-(n+1)/2} Y)$$

$$\times B_p^{(1)}(-d \perp 2\Theta_{n-2r-2} \perp 4B', p^{-n/2-1} Y t^2) \frac{\widetilde{G}_2^{(1)}(-d \perp 2\Theta_{n-2r-2} \perp 4B', 1, X, 2^{-n} t^2 Y)}{\alpha_2(-d \perp 2\Theta_{n-2r-2} \perp 4B')}$$

$$\times \omega(-d \perp 2\Theta_{n-2r-2} \perp 4B') Y^{-\mathfrak{e}^{(1)}(4B') - (n-2r-2)} t^{\nu(\det(4B'))+n-2r-2}.$$

Thus the assertion (1) in case $p = 2$ can be proved in the same way as above. Similarly the assertion (2) can be proved. □

Now to rewrite the above theorem, first we express $\widetilde{P}_{m-1}^{(1)}(n; d_0, \omega, \eta, X, Y, t)$ in terms of $Q_{2r+1}^{(1)}(n; d_0, \omega, \eta, X, Y, t)$ and $Q_{2r}^{(0)}(n; d_0 d, \omega, \eta, X, Y, t)$.

Proposition 5.3.9. *Let m be an even integer. Let $d_0 \in \mathcal{F}_p$, and $\eta = \pm 1$.*

(1) (1.1) *Let $l = 0$ or $\nu(d_0) = 0$. Then*

$$\widetilde{P}_{m-1}^{(1)}(n; d_0, \varepsilon^l, \eta, X, Y, t)$$

$$= \sum_{r=0}^{(m-2)/2} \frac{1}{\phi_{(m-2-2r)/2}(p^{-2})} Q_{2r+1}^{(1)}(n; d_0, \varepsilon^l, \eta, X, Y, t)$$

$$+\sum_{r=0}^{(m-2)/2}\sum_{d\in\mathcal{U}(m-1,m-1-2r,d_0)}\frac{1}{2\phi_{(m-2-2r)/2}(p^{-2})}Q_{2r}^{(0)}(n;d_0d,\varepsilon^l,\eta\chi(d),X,Y,t).$$

(1.2) *Let* $\nu(d_0)=1$. *Then*

$$Q_{2r}^{(0)}(n;d_0d,\varepsilon,\eta\chi(d),X,Y,t)=0$$

for any d *and*

$$\widetilde{P}_{m-1}^{(1)}(n;d_0,\varepsilon,\eta,X,Y,t)$$
$$=\sum_{r=0}^{(m-2)/2}\frac{1}{\phi_{(m-2-2r)/2}(p^{-2})}Q_{2r+1}^{(1)}(n;d_0,\varepsilon,\eta,X,Y,t).$$

(2) (2.1) *Let* $l=0$ *or* $\nu(d_0)=0$. *Then*

$$P_m^{(0)}(n;d_0,\varepsilon^l,\eta,X,Y,t)$$
$$=\sum_{r=0}^{m/2}\sum_{d\in\mathcal{U}(m,m-2r,d_0)}\frac{1+p^{(-m+2r)/2}\chi(d)}{2\phi_{(m-2r)/2}(p^{-2})}Q_{2r}^{(0)}(n;d_0d,\varepsilon^l,\eta\chi(d),X,Y,t)$$
$$+\sum_{r=0}^{(m-2)/2}\frac{1}{\phi_{(m-2r)/2}(p^{-2})}Q_{2r+1}^{(1)}(n;d_0,\varepsilon^l,\eta,X,Y,t).$$

(2.2) *Let* $\nu(d_0)\geq 1$. *Then*

$$\widetilde{P}_m^{(0)}(n;d_0,\varepsilon,\eta,X,Y,t)=0.$$

Proof. The assertion can be proved in a way similar to Theorem 5.3.8. □

Corollary. *Let* r *be a non-negative integer. Let* d_0 *be an element of* $\mathcal{F}_p$ *and* $\xi=\pm 1$.
(1) *Let* $l=0$ *or* $\nu(d_0)=0$. *Then*

$$Q_{2r}^{(0)}(n;d_0,\varepsilon^l,\xi,X,Y,t)$$
$$=\sum_{m=0}^{r}\sum_{d\in\mathcal{U}(2r,2m,d_0)}\frac{(-1)^m(\chi(d)+p^{-m})p^{-m^2}}{2\phi_m(p^{-2})}\widetilde{P}_{2r-2m}^{(0)}(n;d_0d,\varepsilon^l,\xi\chi(d),X,Y,t)$$
$$+\sum_{m=0}^{r-1}\frac{(-1)^{m+1}p^{-m-m^2}}{\phi_m(p^{-2})}\widetilde{P}_{2r-2m-1}^{(1)}(n;d_0,\varepsilon^l,\xi,X,Y,t)),$$

and

$$Q_{2r+1}^{(1)}(n;d_0,\varepsilon^l,\xi,X,Y,t)$$
$$=\sum_{m=0}^{r}\frac{(-1)^m p^{-m-m^2}}{\phi_m(p^{-2})}\widetilde{P}_{2r+1-2m}^{(1)}(n;d_0,\varepsilon^l,\xi,X,Y,t)$$
$$+\sum_{m=0}^{r}\sum_{d\in\mathcal{U}(2r+1,2m+1,d_0)}\frac{(-1)^{m+1}p^{-m-m^2}}{2\phi_m(p^{-2})}\widetilde{P}_{2r-2m}^{(0)}(n;d_0d,\varepsilon^l,\xi\chi(d),X,Y,t)).$$

(2) *Let $\nu(d_0) > 0$. We have*

$$Q^{(1)}_{2r+1}(n; d_0, \varepsilon, \xi, X, Y, t) = \sum_{m=0}^{r} \frac{(-1)^m p^{m-m^2}}{\phi_m(p^{-2})} \widetilde{P}^{(1)}_{2r+1-2m}(n; d_0, \varepsilon, \xi, X, Y, t),$$

and

$$Q^{(0)}_{2r}(n; d_0, \varepsilon, \xi, X, Y, t) = 0.$$

Proof. We prove the assertion (1) by induction on r. Clearly the assertion holds for $r = 0$. Let $r \geq 1$ and suppose that the assertion holds for any $r' < r$. Then by Proposition 5.3.9 we have

$$\tilde{Q}^{(1)}_{2r+1}(n; d_0, \varepsilon^l, \xi, X, Y, t)$$

$$= \tilde{P}^{(1)}_{2r+1}(n; d_0, \varepsilon^l, \xi, X, Y, t) - \sum_{i=1}^{r} \frac{1}{\phi_i(p^{-2})} Q^{(1)}_{2r+1-2i}(n; d_0, \varepsilon^l, \xi, X, Y, t)$$

$$- \sum_{i=0}^{r} \sum_{d \in \mathcal{U}(2r+1, 2i+1; d_0)} \frac{1}{2\phi_i(p^{-2})} Q^{(0)}_{2r-2i}(n; d_0 d, \varepsilon^l, \xi\chi(d), X, Y, t).$$

Then by the induction hypothesis and a direct calculation, we get the desired result for $\tilde{Q}^{(1)}_{2r+1}(n; d_0, \varepsilon^l, \xi, X, Y, t)$. We also get the result for $\tilde{Q}^{(1)}_{2r}(n; d_0, \varepsilon^l, \xi, X, Y, t)$, and this completes the induction. Similarly the assertion (2) can be proved.

□

Theorem 5.3.10. *Let the notation be as in Theorem 5.3.8.*
(1) *Suppose that $\nu(d_0) = 0$ or $\omega = \iota$. Put $\xi_0 = \chi(d_0)$ and $\zeta_d = \chi(d)$. Then*

$$\widetilde{R}_{n-1}(d_0, \omega, X, Y, t) = (1 - p^{-n}t^2)$$

$$\times \{ \sum_{l=0}^{(n-2)/2} \sum_{d \in \mathcal{U}(n-1, n-1-2l, d_0)} \widetilde{P}^{(0)}_{2l}(n; d_0 d, \omega, \zeta_d, X, Y, t) \prod_{i=1}^{(n-2-2l)/2} (1 - p^{-2l-n-2i}t^4) T_{2l}(d_0, d, Y)$$

$$+ \sum_{l=0}^{(n-2)/2} \widetilde{P}^{(1)}_{2l+1}(n; d_0, 1, X, Y, \omega, t) \prod_{i=2}^{(n-2-2l)/2} (1 - p^{-2l-n-2i}t^4) T_{2l+1}(d_0, Y, t) \},$$

where $T_{2r}(d_0, d, Y)$ is a polynomial in Y, and $T_{2r+1}(d_0, Y, t)$ is a polynomial in Y and t and of degree at most 2 with respect to t, and in particular

$$T_{n-2}(d_0, d, Y) = \frac{1}{2}(1 - p^{-1/2}\xi_0 Y) p^{(n-2)/2-1/2} Y \zeta_d (1 + \zeta_d Y p^{(n-2)/2-1/2}) \prod_{i=1}^{(n-4)/2} (1 - p^{2i-1}Y^2),$$

and

$$T_{n-1}(d_0, Y, t) = \prod_{i=1}^{(n-2)/2} (1 - p^{2i-1}Y^2).$$

(2) *Suppose that* $\nu(d_0) > 0$ *and* $\omega = \varepsilon$. *Then*

$$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t) = (1-\xi_0 p^{-1/2}Y)\sum_{l=0}^{(n-2)/2}\widetilde{P}^{(1)}_{2l+1}(n;d_0,\omega,1,X,Y,t)$$
$$\times\frac{(p^{2l+1}Y^2)^{(n-2l-2)/2}\prod_{i=1}^{l}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2-2l)/2}(1-p^{-2l-n-2i-2}t^4)}{\phi_{(n-2-2l)/2}(p^{-2})}.$$

Proof. (1) By Theorem 5.3.8 and Corollary to Proposition 5.3.9, we have

$$\widetilde{R}_{n-1}(d_0,\omega;X,Y,t)$$
$$=\sum_{r=0}^{(n-2)/2}\frac{\prod_{i=1}^{r-1}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2r-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{2\phi_{(n-2r-2)/2}(p^{-2})}$$
$$\times\sum_{d_1\in\mathcal{U}(n-1,n-2r-1,d_0)}D_{2r}(d_0,d_1,Y,t)\{\sum_{m=0}^{r}\sum_{d_2\in\mathcal{U}(2r,2m,d_0d_1)}\frac{(-1)^m(\chi_p(d_2)+p^{-m})p^{-m^2}}{2\phi_m(p^{-2})}$$
$$\times\tilde{P}^{(0)}_{2r-2m}(n;d_0d_1d_2,\omega,\chi(d_1)\chi(d_2),X,Y;t)$$
$$+\sum_{m=0}^{r-1}\frac{(-1)^{m+1}p^{-m-m^2}}{\phi_m(p^{-2})}\tilde{P}^{(1)}_{2r-2m-1}(n;d_0d_1,\omega,\chi(d_1),X,Y;t))\}$$
$$+\sum_{r=0}^{(n-2)/2}\frac{\prod_{i=1}^{r}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2r-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{2\phi_{(n-2r-2)/2}(p^{-2})}$$
$$\times(1-\xi_0p^{-1/2}Y)\{\sum_{m=0}^{r}\frac{(-1)^mp^{-m}p^{-m^2}}{\phi_m(p^{-2})}\tilde{P}^{(1)}_{2r+1-2m}(n;d_0,\omega,1,X,Y,t)$$
$$+\sum_{m=0}^{r-1}\sum_{d_2\in\mathcal{U}(2r+1,2m+1,d_0)}\frac{(-1)^{m+1}p^{-m-m^2}}{2\phi_m(p^{-2})}\tilde{P}^{(0)}_{2r-2m}(n;d_0d_2,\omega,\chi(d_2),X,Y,t)\}.$$

We note that for any $d_1\in\mathcal{U}$ we have

$$\tilde{P}^{(1)}_{2r+1-2m}(n;d_0d_1,\omega,\chi(d_1),X,Y,t)=\tilde{P}^{(1)}_{2r+1-2m}(n;d_0,\omega,1,X,Y,t).$$

Hence

(A) $$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)$$
$$=\sum_{l=0}^{(n-2)/2}\sum_{d\in\mathcal{U}(n-1,n-1-2l,d_0)}\widetilde{P}^{(0)}_{2l}(n;d_0d,\omega,\chi(d),X,Y,t)$$
$$\times\{\sum_{m=0}^{(n-2-2l)/2}(\frac{1}{2}\sum_{d_1\in\mathcal{U}(n-1-2l,2m,d_0d)}D_{2l+2m}(d_0,d_1,Y,t)(\chi(d_1)\chi(d)+p^{-m})(-1)^mp^{-m^2}$$
$$-u_{2l+2m}(Y)(1-\xi_0p^{-1/2}Y)(-1)^mp^{-m-m^2}))$$
$$\times\frac{\prod_{i=1}^{l+m-1}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{2\phi_m(p^{-2})\phi_{(n-2-2l)/2-m}(p^{-2})}\}$$

$$+\sum_{l=0}^{(n-2)/2}\widetilde{P}^{(1)}_{2l+1}(n;d_0,\omega,1,X,Y,t)$$
$$\times\{\sum_{m=0}^{(n-2-2l)/2}((1-\xi_0p^{-1/2}Y)(-1)^mp^{-m-m^2})$$
$$\times\frac{\prod_{i=1}^{l+m}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-2-2l)/2-m}(p^{-2})}$$
$$-\sum_{m=0}^{(n-4-2l)/2}\frac{1}{2}\sum_{d\in\mathcal{U}}D_{2l+2m+2}(d_0,d,Y,t)(-1)^mp^{-m-m^2}$$
$$\times\frac{\prod_{i=1}^{l+m}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-4)/2}(1-p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-4-2l)/2-m}(p^{-2})}\},$$

where $u_{2l+2m}(Y)=1$ or $1-p^{2l+2m-1}Y^2$ according as $l+m=0$ or not.

Now recall that $\mathcal{U}(n-1,n-1,d_0)$ is $\{d_0\}$ or the empty set according as $\nu(d_0)=0$ or not. Hence for $d\in\mathcal{U}(n-1,n-1,d_0)$ we have

$$\frac{1}{2}\sum_{d_1\in\mathcal{U}(n-1,2m,d_0d)}D_{2m}(d_0,d_1,Y,t)(\chi(d_1)\chi(d)+p^{-m})$$
$$-(1-p^{2m-1}Y^2)(1-\xi_0p^{-1/2}Y)p^{-m}$$
$$=\xi_0p^{-1/2}Y(1-p^{-n}t^2)\quad\text{or}\quad\xi_0(1-p^{-1}Y^2)(1-p^{-n}t^2)p^{m-1/2}Y$$

according as $m=0$ or $1\le m\le(n-2)/2$. Furthermore we have

$$\frac{1}{2}\sum_{d_1\in\mathcal{U}(n-1-2l,2m,d_0d)}D_{2l+2m}(d_0,d_1,Y,t)(\chi(d_1)\chi(d)+p^{-m})$$
$$-(1-p^{2l+2m-1}Y^2)(1-\xi_0p^{-1/2}Y)p^{-m}$$
$$=\zeta_d(1-p^{-1/2}\xi_0Y)(1-p^{-n}t^2)p^{l+m-1/2}Y(1+\zeta_dYp^{l-1/2})$$

for any $1\le l\le(n-2)/2, 0\le m\le(n-2l-2)/2$ and $d\in\mathcal{U}(n-1,n-1-2l,d_0)$. We also have

$$(1-\xi_0p^{-1/2}Y)p^{-m}(1-p^{-2n+2l+2m+1}Y^2t^4)$$
$$-\frac{1}{2}\sum_{d_1\in\mathcal{U}}D_{2l+2m+2}(d_0,d_1,Y,t)p^{-m}(1-p^{-n+2m+2l+2})$$
$$=(1-p^{-1/2}\xi_0Y)p^{-n+m+2l+2}(1-p^{-n+m+2l+2}Y^2t^4)$$
$$+(1-p^{-1/2}\xi_0Y)(1-p^{-n+2m+2l+2})p^{2l+m-n+1}Y^2t^2(1-p^{-n}t^2)$$

for any $0\le l\le(n-2)/2$ and $0\le m\le(n-2l-2)/2$. Hence

$$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)=\xi_0p^{-1/2}Y(1-p^{-n}t^2)$$
$$\times\sum_{m=0}^{(n-2)/2}(-1)^mp^{m-m^2}\frac{\prod_{i=0}^{m-1}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2m-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-2)/2-m}(p^{-2})}$$

$$+\sum_{l=1}^{(n-2)/2}\sum_{d\in\mathcal{U}(n-1,n-1-2l,d_0)}\widetilde{P}_{2l}^{(0)}(n;d_0d,\omega,\zeta_d,X,Y,t)$$
$$\times(1-\xi_0p^{-1/2}Y)(1-p^{-n}t^2)p^{l-1/2}\zeta_dY(1+\zeta_dYp^{l-1/2})$$
$$\times\sum_{m=0}^{(n-2-2l)/2}(-1)^mp^{m-m^2}\frac{\prod_{i=1}^{l+m-1}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{2\phi_m(p^{-2})\phi_{(n-2-2l)/2-m}(p^{-2})}$$
$$+\sum_{l=0}^{(n-2)/2}\widetilde{P}_{2l+1}^{(1)}(n;d_0,\omega,1,X,Y,t)$$
$$\times\{(1-\xi_0p^{-1/2}Y)p^{-n+2+2l}$$
$$\times\sum_{m=0}^{(n-2-2l)/2}(-1)^mp^{m-m^2}\frac{\prod_{i=1}^{l+m}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-2-2l)/2-m}(p^{-2})}$$
$$+Y^2t^2p^{2l-n+1}(1-\xi_0p^{-1/2}Y)(1-p^{-n}t^2)$$
$$\times\sum_{m=0}^{(n-4-2l)/2}(-1)^mp^{m-m^2}\frac{\prod_{i=1}^{l+m}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-4)/2}(1-p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-4-2l)/2-m}(p^{-2})}\}.$$

Then the assertion (1) follows from Lemma 5.1.6.

(2) By (1.2) and (2.2) of Theorem 5.3.8 and (2) of Corollary to Proposition 5.3.9, we have

$$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)=\sum_{l=0}^{(n-2)/2}\widetilde{P}_{2l+1}^{(1)}(n;d_0,\omega,1,X,Y,t)$$
$$\times\sum_{m=0}^{(n-2-2l)/2}((-1)^m(1-\xi_0p^{-1/2}Y)p^{m-m^2})$$
$$\times\frac{\prod_{i=1}^{l+m}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2l-2m-2)/2}(1-p^{-2i-n-1}Y^2t^4)}{\phi_m(p^{-2})\phi_{(n-2-2l)/2-m}(p^{-2})}.$$

Thus the assertion follows from Lemma 5.1.6.

□

5.4. **Explicit formulas of formal power series of Koecher-Maass type.**

In this section we give an explicit formula for $P_m^{(l)}(d_0,\omega,\xi,X,t)$ for $\omega=\iota,\varepsilon$. We write $P_m^{(l)}(d_0,\omega,X,t)=P_m^{(l)}(d_0,\omega,1,X,t)$ as stated before.

Theorem 5.4.1. *Let m be even, and $d_0\in\mathcal{F}_p$. Put $\xi_0=\chi(d_0)$. Then we have the following:*

(1)

$$P_m^{(0)}(d_0,\iota,X,t)=\frac{(p^{-1}t)^{\nu(d_0)}}{\phi_{m/2-1}(p^{-2})(1-p^{-m/2}\xi_0)}$$

$$\times\frac{(1+t^2p^{-m/2-3/2})(1+t^2p^{-m/2-5/2}\xi_0^2)-\xi_0t^2p^{-m/2-2}(X+X^{-1}+p^{1/2-m/2}+p^{-1/2+m/2})}{(1-p^{-2}Xt^2)(1-p^{-2}X^{-1}t^2)\prod_{i=1}^{m/2-1}(1-t^2p^{-2i-1}X)(1-t^2p^{-2i-1}X^{-1})}.$$

(2)

$$P_m^{(0)}(d_0,\varepsilon,X,t)=\frac{1}{\phi_{m/2-1}(p^{-2})(1-p^{-m/2}\xi_0)}$$
$$\times\frac{\xi_0^2}{\prod_{i=1}^{m/2}(1-t^2p^{-2i}X)(1-t^2p^{-2i}X^{-1})},$$

where δ_{2p} is Kronecker's delta.

Theorem 5.4.2. *Let m be even, and $d_0\in\mathcal{F}_p$. Put $\xi_0=\chi(d_0)$. Then we have the following:*

(1)

$$P_{m-1}^{(1)}(d_0,\iota,X,t)=\frac{(p^{-1}t)^{\nu(d_0)}(1-\xi_0t^2p^{-5/2})}{\phi_{(m-2)/2}(p^{-2})(1-t^2p^{-2}X)(1-t^2p^{-2}X^{-1})}$$
$$\times\frac{1}{\prod_{i=1}^{(m-2)/2}(1-t^2p^{-2i-1}X)(1-t^2p^{-2i-1}X^{-1})}.$$

(2)

$$P_m^{(1)}(d_0,\varepsilon,X,t)=\frac{(p^{-1}t)^{\nu(d_0)}(1-\xi_0t^2p^{-1/2-m})}{\phi_{(m-2)/2}(p^{-2})}$$
$$\times\frac{1}{\prod_{i=1}^{(m-2)/2}(1-t^2p^{-2i}X)(1-t^2p^{-2i}X^{-1})}.$$

Theorem 5.4.1 follows from [[IK06], Theorem 3.1], and Theorem 5.4.2 can be proved in the same way as in Theorem 5.4.1, but for the convenience of readers we give a proof to them. Let m be an even positive integer. For $l=0,1$ and $j=0,1$ put

$$K_{m-l}^{(l)}(d_0,\varepsilon^j,X,t)$$
$$=\kappa(d_0,m-l,j,t)^{-1}\sum_{B'\in\mathcal{L}_{r,p}^{(l)}(d_0)}\frac{G_p^{(l)}(2^{l-1}B',p^{-(m+1)/2}X)\varepsilon(B')^j}{\alpha_p(B')}X^{-\mathfrak{e}^{(l)}(B')}t^{\nu(\det B')}.$$

Proposition 5.4.3. *Let m and d_0 be as above. Then, for $l=0,1$, we have*

$$P_{m-l}^{(l)}(d_0,\omega,X,t)=\prod_{i=1}^{m-l}(1-t^2Xp^{i-m+l-2})^{-1}K_{m-l}^{(l)}(d_0,\omega,X,t).$$

Proof. We note that B' belongs to $\mathcal{L}_{m-l,p}^{(l)}(d_0)$ if B belongs to $\mathcal{L}_{m-l,p}^{(l)}(d_0)$ and $\alpha_p(B',B)\neq 0$. Hence by Lemma 5.2.2 for $\omega=\varepsilon^j$ with $j=0,1$ we have

$$P_{m-l}^{(l)}(d_0,\omega,X,t)=\kappa(d_0,m-l,j,t)^{-1}$$

$$\times \sum_{B\in\mathcal{L}^{(l)}_{m-l,p}(d_0)} \frac{1}{\alpha_p(B)} \sum_{B'} \frac{G^{(l)}_p(B', p^{-(m+1)/2}X)X^{-\mathfrak{e}^{(l)}(B')}\alpha_p(B', B)\omega(B')}{\alpha_p(B')}$$
$$\times (p^{-1}X)^{(\nu(\det B)-\nu(\det B'))/2} t^{\nu(\det B)}$$
$$= \kappa(d_0, m-l, j, t)^{-1} \sum_{B'\in\mathcal{L}^{(l)}_{m-l,p}(d_0)} \frac{G^{(l)}_p(B', p^{-(m+1)/2}X)\omega(B')}{\alpha_p(B')} X^{-\mathfrak{e}^{(l)}(B')}$$
$$\times \sum_{B\in\mathcal{L}^{(l)}_{m-l,p}(d_0)} \frac{\alpha_p(B', B)}{\alpha_p(B)} (p^{-1}X)^{(\nu(\det B)-\nu(\det B'))/2} t^{\nu(\det B)}.$$

Hence by [[BS87], Theorem 5], and by (1) of Lemma 5.1.1, we have

$$\sum_B \frac{\alpha_p(B', B)}{\alpha_p(B)} (p^{-1}X)^{(\nu(\det B)-\nu(\det B'))/2} t^{\nu(\det B)}$$
$$= \sum_W \alpha_p(B', B'[W])(t^2 X p^{-1} p^{-m+l})^{\nu(\det W)} t^{\nu(\det B')}$$
$$= \prod_{i=1}^{m-l} (1 - t^2 X p^{i-m+l-2})^{-1} t^{\nu(\det B')}.$$

Thus the assertion holds. □

Proof of Theorem 5.4.1. Let $b^*_p(T, s)$ be the primitive Siegel series in (cf. [IK06], Page 176), and

$$\widetilde{D}(t, \sigma, d_0, \omega) = \sum_{B'\in S_{m,p}(d_0)_e} \frac{b^*_p(B'/2, \sigma)\omega(B')}{\alpha_p(B')} t^{\mathfrak{e}^{(0)}(B')}$$

for $\omega = \varepsilon^l$. Then

$$K^{(0)}_m(d_0, \omega, p^{-\sigma}, t)$$
$$= \kappa(d_0, m, l, t)^{-1} t^{\nu(d_0)} \frac{1 - \chi(d_0)p^{m/2-\sigma}}{(1-p^{-\sigma})\prod_{i=1}^{m/2}(1-p^{2i-2\sigma})} \widetilde{D}(tp^\sigma, \sigma+(m+1)/2, d_0, \omega)$$

for any complex number σ. The both-hand sides of the above are polynomials in $p^{-\sigma}$. Now, for a p-adic number d define a formal power series $D(t, \sigma, d, \omega)$ by

$$D(t, \sigma, d, \omega) = \sum_{i=0}^{\infty} \sum_{\substack{B\in S_{m,p;e} \\ \det B = p^i d}} \frac{b^*_p(B'/2, \sigma)\omega(B')}{\alpha_p(B')} t^i$$

as in [IK06]. Let $d_0 \in \mathcal{F}_p$ and $l = \nu(d_0)$. Then

$$t^l \widetilde{D}(t, \sigma, d_0, \omega) = \frac{1}{2}(D(t, \sigma, p^{-l}(-1)^{m/2}d_0, \omega) + (-1)^l D(-t, \sigma, p^{-l}(-1)^{m/2}d_0, \omega)).$$

Thus the assertion follows from Theorem 3.1 in [IK06] and Proposition 5.4.3. □

Remark. We should remark that there is a misprint in [IK06]; the right hand side on page 186, line 3 of it should be

$$\frac{2^{(-2s-2)\delta_{2p}}p^{-1}p^{-s}(1-p^{-k})(1+p^{-1-k-2s})}{(p^{-2})^{n/2-1}}.$$

In order to prove Theorems 5.4.2, we introduce some notation. Let r be an even integer. For $l=0,1$ and $d_0 \in \mathbf{Z}_p^\times$ put

$$\zeta_{r-1}(d_0,\varepsilon^l,u)=\kappa(d_0,r-1,l,u)^{-1}\sum_{T\in S_{r-1,p}(d_0)/\sim}\frac{\varepsilon(T)^l}{\alpha_p(T)}u^{\nu(\det T)},$$

and

$$\zeta_r(d_0,\varepsilon^l,u)=\kappa(d_0,r,l,u)^{-1}\sum_{T\in S_{r,p;e}(d_0)/\sim}\frac{\varepsilon(T)^l}{\alpha_p(T)}u^{\nu(\det T)}.$$

We make the convention that $\zeta_0(d_0,\varepsilon^l,u)=1$ or 0 according as $d_0 \in \mathbf{Z}_p^*$ or not. Now for an integer m, and $d \in \mathbf{Z}_p$, let $Z_m(u,\varepsilon^l,d)$ and $Z_m^*(u,\varepsilon^l,d)$ be the formal power series in Theorems 5.1, 5.2, and 5.3 of [IS95]. Put

$$Z_{m,e}(u,\varepsilon^l,d)=\frac{1}{2}(Z_m(u,\varepsilon^l,d)+Z_m(-u,\varepsilon^l,d)),$$

$$Z_{m,o}(u,\varepsilon^l,d)=\frac{1}{2}(Z_m(u,\varepsilon^l,d)-Z_m(-u,\varepsilon^l,d)).$$

We also define $Z_{m,e}^*(u,\varepsilon^l,d)$ and $Z_{m,o}^*(u,\varepsilon^l,d)$ in the same way. Furthermore put $x(i)=e$ or o according as i is even or odd. Let $p\neq 2$, and $p^{-i}d_0 \in \mathbf{Z}_p^*$ with $i=0$ or 1. Then

$$\zeta_m(d_0,\varepsilon^l,u)=Z_{m,x(i)}(p^{-(m+1)/2}((-1)^{(m+1)/2},p)_p u,\varepsilon^l,p^{-i}(-1)^{(m+1)/2}d_0)$$

or

$$\zeta_m(d_0,\varepsilon^l,u)=Z_{m,x(i)}(p^{-(m+1)/2}u,\varepsilon^l,p^{-i}(-1)^{[(m+1)/2]}d_0)$$

according as m is odd and $l=1$, or not. Let $p=2$ and m is odd. Then

$$\zeta_m(d_0,\varepsilon^l,u)=2^m Z_{m,x(\nu(d_0))}(2^{-(m+1)/2}u,\varepsilon^l,2^{-\nu(d_0)}(-1)^{(m+1)/2}d_0).$$

Let $p=2$ and m be an even integer. First suppose $d_0\equiv 1 \bmod 4$. Then

$$\zeta_m^*(d_0,\varepsilon^l,u)=2^m Z_{m,e}^*(2^{-(m+1)/2}u,\varepsilon^l,(-1)^{m/2}d_0).$$

Next suppose $4^{-1}d_0\equiv -1 \bmod 4$. Then

$$\zeta_m^*(d_0,\varepsilon^l,u)=2^m Z_{m,e}^*(2^{-(m+1)/2}u,\varepsilon^l,4^{-1}(-1)^{m/2}d_0).$$

Finally suppose $8^{-1}d_0 \in \mathbf{Z}_2^*$. Then

$$\zeta_m^*(d_0,\varepsilon^l,u)=2^m Z_{n,o}^*(2^{-(m+1)/2}u,\varepsilon^l,8^{-1}(-1)^{m/2}d_0).$$

Here we recall that the definition of local density in our paper is a little bit different from that in [IS95].

Proposition 5.4.4. *Let m be a positive even integer. Let $d_0 \in \mathcal{F}_p$. For a positive even integer r and $d \in \mathcal{U}$ put*

$$c(r, d_0, d, X) = (1 - \chi(d_0)p^{-1/2}X) \times \prod_{i=1}^{r/2-1} (1 - p^{2i-1}X^2)(1 - \chi(d)p^{r/2-1/2}X).$$

Here we understand that $c(0, d_0, d, X) = 1$. Furthermore, for a positive odd integer r put

$$c(r, d_0, X) = (1 - \chi(d_0)p^{-1/2}X) \prod_{i=1}^{(r-1)/2} (1 - p^{2i-1}X^2).$$

(1) *Let $p \neq 2$.*

(1.1) *Let $l = 0$ or $\nu(d_0) = 0$. Then*

$$K_{m-1}^{(1)}(d_0, \varepsilon^l, X, t) = X^{\nu(d_0)/2}\Big\{ \sum_{r=0}^{(m-2)/2} \sum_{d \in \mathcal{U}(m-1, m-2r-1, d_0)} \frac{p^{-r(2r+1)}(tX^{-1/2})^{2r} c(2r, d_0, d, X)}{2^{1-\delta_{(m-2)/2,r}} \phi_{(m-2r-2)/2}(p^{-2})} \times (p, d_0 d)_p^l \zeta_{2r}(d_0 d, \varepsilon^l, tX^{-1/2}) + \sum_{r=0}^{(m-2)/2} \frac{p^{-(r+1)(2r+1)}(tX^{-1/2})^{2r+1} c(2r+1, d_0, X)}{\phi_{(m-2r-2)/2}(p^{-2})} \Big\} \times \zeta_{2r+1}(p^{-1}d_0, \varepsilon^l, tX^{-1/2}).$$

(1.2) *Let $\nu(d_0) = 1$. Then*

$$K_{m-1}^{(1)}(d_0, \varepsilon, X, t) = (X^{1/2})^{\nu(d_0)} \sum_{r=0}^{(m-2)/2} \frac{p^{-(r+1)(2r+1)}(p, -1)^{r+1}(tX^{-1/2})^{2r+1} c(2r+1, d_0, X)}{\phi_{(m-2r-2)/2}(p^{-2})} \times \zeta_{2r+1}(p^{-1}d_0, \varepsilon, tX^{-1/2}).$$

(2) *Let $p = 2$.*

(2.1) *Let $l = 0$ or $(-1)^{m/2}d_0 \equiv 1 \bmod 4$. Then*

$$K_{m-1}^{(1)}(d_0, \varepsilon^l, X, t) = (X^{1/2})^{\nu(d_0)}\Big\{ \sum_{r=0}^{(m-2)/2} \sum_{d \in \mathcal{U}(m-1, m-2r-1, d_0)} (tX^{-1})^{2r} 2^{-r(2r+1)} \frac{c(2r, d_0, d, X)}{2^{1-\delta_{(m-2)/2,r}} \phi_{(m-2r-2)/2}(2^{-2})} \times \zeta_{2r}^*(d_0 d, \varepsilon, tX^{-1/2}) + \sum_{r=0}^{(m-2)/2} (tX^{-1/2})^{2r+1} 2^{-(r+1)(2r+1)} \frac{2^{-\widetilde{\nu}(d_0)(m-2r-2)/2} c(2r+1, d_0, X)}{\phi_{(m-2r-2)/2}(2^{-2})} \times \zeta_{2r+1}(d_0, \varepsilon, tX^{-1/2}) \Big\},$$

where $\widetilde{\nu}(d_0) = 0$ *or* 1 *according as* $\nu(d_0)$ *is even or odd.*

(2.2) *Suppose that* $(-1)^{m/2}4^{-1}d_0 \equiv 1 \bmod 4$ *or* $8^{-1}d_0 \in \mathbf{Z}_2^*$. *Then*

$$K_{m-1}^{(1)}(d_0,\varepsilon,X,t) = X^{\nu(d_0)/2}\sum_{r=0}^{(m-2)/2}(tX^{-1/2})^{2r+1}2^{-(r+1)(2r+1)}\frac{c(2r+1,d_0,X)}{\phi_{(m-2r-2)/2}(2^{-2})}$$

$$\times\zeta_{2r+1}(d_0,\varepsilon,tX^{-1/2}).$$

Proof. Let $p \neq 2$, and let $l = 0$ or $\nu(d_0) = 0$. Then by Lemma 5.2.1 and Proposition 5.3.5, and by using the same argument as in (1) of Theorem 5.3.8, we have

$$K_{m-1}^{(1)}(d_0,\varepsilon^l,X,t) = \sum_{r=0}^{(m-2)/2}\sum_{d\in\mathcal{U}(m-1,m-2r-1,d_0)}\frac{c(2r,d_0,d,X)}{2^{1-\delta_{(m-2)/2,r}}\phi_{(m-2r-2)/2}(p^{-2})}$$

$$\times\sum_{B\in S_{2r,p}(d_0d)\cap S_{2r,p}}\frac{\varepsilon(pB)^l}{\alpha_p(pB)}X^{-\mathfrak{e}^{(0)}(pB)}t^{\nu(\det(pB)}$$

$$+\sum_{r=0}^{(m-2)/2}\frac{(tX^{-1/2})^{2r+1}c(2r+1,d_0,d,X)}{\phi_{(m-2r-2)/2}(p^{-2})}$$

$$\times\sum_{B\in p^{-1}S_{2r+1,p}(d_0)\cap S_{2r+1,p}}\frac{\varepsilon(pB)^l}{\alpha_p(pB)}X^{-\mathfrak{e}^{(1)}(pB)}t^{\nu(\det(pB)}.$$

Thus the assertion (1.1) follows from Lemma 5.3.3 by remarking that $p^{-1}S_{2r,p}(d_0d)\cap S_{2r,p} = S_{2r,p}(d_0d)$ and $p^{-1}S_{2r+1,p}(d_0)\cap S_{2r+1,p} = S_{2r+1}(p^{1-2[(\nu(d_0)+1)/2]}d_0)$. Similarly the assertion (1.2) can be proved by remarking that $\zeta_{2r}(d_0,\varepsilon,tX^{-1/2}) = 0$. The assertion for $p = 2$ can also be proved by using the same argument as in (2) of Theorem 5.3.8 by remarking that

$$4^{-1}S_{2r,2}(d_0)\cap S_{2r,2} = S_{2r,2}(d_0)$$

and

$$4^{-1}S_{2r+1,2}(d_0)\cap S_{2r+1,2} = S_{2r+1,2}(d_0).$$

□

Proof of Theorem 5.4.2 in case $p \neq 2$**.** (1) First let $d_0 \in \mathbf{Z}_p^*$. Then by (1.1) of Proposition 5.4.4, we have

$$K_{m-1}(d_0,\iota,X,t) = \frac{1}{\phi_{(m-2)/2}(p^{-2})}$$

$$+2^{-1}\sum_{r=1}^{(m-2)/2}\sum_{d\in\mathcal{U}(m-1,m-2r-1,d_0)}p^{-r(2r+1)}(t^2X^{-1})^r\prod_{i=1}^{r-1}(1-p^{2i-1}X^2)\phi_{(m-2r-2)/2}(p^{-2})^{-1}$$

$$\times(1-p^{-1/2}\xi_0X)(1+\xi p^{r-1/2}X)\zeta_{2r}(d_0d,\iota,tX^{-1/2})$$

$$+\sum_{r=0}^{(m-2)/2} p^{-(2r+1)(r+1)}(t^2X^{-1})^{r+1/2}\prod_{i=1}^{r}(1-p^{2i-1}X^2)\phi_{(m-2r-2)/2}(p^{-2})^{-1}$$
$$\times(1-p^{-1/2}\xi_0 X)\zeta_{2r+1}(pd_0,\iota,tX^{-1/2}).$$

Here we put $\xi=\chi(d)$ for $d\in\mathcal{U}$. By Theorem 5.1 of [IS95], we have

$$\zeta_{2r+1}(pd_0,\iota,tX^{-1/2})=\frac{p^{-1}tX^{-1/2}}{\phi_r(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{r}(1-p^{2i-3-2r}t^2X^{-1})},$$

and

$$\zeta_{2r}(d_0d,\iota,tX^{-1/2})=\frac{(1+\xi_0\xi p^{-r})(1-\xi_0\xi p^{-r-2}t^2X^{-1})}{\phi_r(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{r}(1-p^{2i-3-2r}t^2X^{-1})}.$$

Hence $K_{m-1}(d_0,\iota,X,t)$ can be expressed as

$$K_{m-1}(d_0,\iota,X,t)=\frac{S(d_0,\iota,X,t)}{\phi_{(m-2)/2}(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{(m-2)/2}(1-p^{2i-m-1}t^2X^{-1})},$$

where $S(d_0,\iota,X,t)$ is a polynomial in t of degree m. Now we have

$$2^{-1}(1-p^{-1/2}\xi_0X)\sum_{\xi=\pm1}(1+\xi p^{(m-2)/2-1/2}X)(1+\xi_0\xi p^{-(m-2)/2})(1-\xi_0\xi p^{-(m-2)/2-2}t^2X^{-1})$$
$$=(1-p^{-1}X^2)(1-p^{-2}t^2X^{-1})+p^{-2}t^2X^{-1}(1-\xi_0p^{-1/2}X)(1-p^{-m+2}).$$

Hence

$$2^{-1}\sum_{d\in\mathcal{U}(m-1,m-2r-1,d_0)} p^{(m-1)(-m+2)/2}(t^2X^{-1})^{(m-2)/2}\prod_{i=1}^{(m-2)/2-1}(1-p^{2i-1}X^2)$$
$$\times(1-p^{-1/2}\xi_0X)(1+\xi p^{(m-2)/2-1/2}X)\zeta_{m-2}(d_0d,\iota,tX^{-1/2})$$
$$+p^{-(m-1)m/2}(t^2X^{-1})^{(m-2)/2+1/2}\prod_{i=1}^{(m-2)/2}(1-p^{2i-1}X^2)(p^{-2})^{-1}$$
$$\times(1-p^{-1/2}\xi_0X)\zeta_{m-1}(pd_0,\iota,tX^{-1/2})$$
$$=\frac{(1-\xi_0p^{-5/2}t^2)\prod_{i=0}^{(m-2)/2-1}(1-p^{2i-1}X^2)}{\phi_{(m-2)/2}(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{(m-2)/2}(1-p^{2i-m-1}t^2X^{-1})},$$

and therefore $S(d_0,\iota,X,t)$ can be expressed as

(B) $$S(d_0,\iota,X,t)$$
$$=\prod_{i=0}^{(m-2)/2-1}(1-p^{2i-1}X^2)(1-p^{-5/2}\xi_0t^2)+(1-p^{-m+1}t^2X^{-1})U(X,t),$$

where $U(X,t)$ is a polynomial in X,X^{-1} and t. Now by Proposition 5.4.3, we have

$$P^{(1)}_{m-1}(d_0,\iota,X,t)$$
$$=\frac{S(d_0,\iota,X,t)}{\phi_{(m-2)/2}(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^{(m-2)/2}(1-p^{2i-m-1}t^2X^{-1})\prod_{i=1}^{m-1}(1-p^{i-m-1}Xt^2)}.$$

Hence the power series $P^{(1)}_{m-1}(d_0,\iota,X,t)$ is a rational function in t. Since we have $\widetilde{F}^{(1)}_p(T,X^{-1})=\widetilde{F}^{(1)}_p(T,X)$ for any $T\in\mathcal{L}^{(1)}_{m-1,p}$, we have $P^{(1)}_{m-1}(d_0,\iota,X^{-1},t)=P^{(1)}_{m-1}(d_0,\iota,X,t)$. This implies that the reduced denominator of the rational function $P^{(1)}_{m-1}(d_0,\iota,X,t)$ in t is at most

$$(1-p^{-2}t^2X^{-1})(1-p^{-2}t^2X)\prod_{i=1}^{(m-2)/2}\{(1-p^{2i-m-1}t^2X^{-1})(1-p^{2i-m-1}t^2X)\}.$$

Hence we have

$$\text{(C)}\qquad S(\iota,d_0,X,t)=\prod_{i=1}^{(m-2)/2}(1-p^{2i-m-2}t^2X)(a_0(X)+a_1(X)t^2)$$

with $a_0(X),a_1(X)$ are polynomials in $X+X^{-1}$. We easily see $a_0(X)=1$. By substituting $p^{(m-1)/2}X^{1/2}$ for t in (B) and (C), and comparing them we see $a_1(X)=-p^{-5/2}\xi_0$. This proves the assertion.

Next let $d_0\in pd_0'$ with $d_0'\in\mathbf{Z}_p^*$. Put $\xi_0=\chi(d_0')$. Then by (1.1) of Proposition 5.4.4, we have

$$\begin{aligned}&K_{m-1}(d_0,\iota,X,t)\\ &=X^{1/2}\{2^{-1}\sum_{r=1}^{(m-2)/2}\sum_{d\in\mathcal{U}(m-1,m-2r-1,d_0)}p^{-r(2r+1)}(t^2X^{-1})^r\prod_{i=1}^{r-1}(1-p^{2i-1}X^2)\phi_{(m-2r-2)/2}(p^{-2})^{-1}\\ &\qquad\times(1+\xi p^{r-1/2}X)\zeta_{2r}(d_0d,\iota,tX^{-1/2})\\ &+\sum_{r=0}^{(m-2)/2}p^{-(2r+1)(r+1)}(t^2X^{-1})^{r+1/2}\prod_{i=1}^{r}(1-p^{2i-1}X^2)\phi_{(m-2r-2)/2}(p^{-2})^{-1}\\ &\qquad\times(1-p^{-1/2}\xi_0X)\zeta_{2r+1}(p^{-1}d_0,\iota,tX^{-1/2})\}.\end{aligned}$$

By Theorem 5.1 of [IS95], we have

$$\zeta_{2r+1}(p^{-1}d_0,\iota,tX^{-1/2})=\frac{1}{\phi_r(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^r(1-p^{2i-3-2r}t^2X^{-1})},$$

and

$$\zeta_{2r}(d_0d,\iota,tX^{-1/2})=\frac{p^{-1}tX^{-1/2}}{\phi_r(p^{-2})(1-p^{-2}t^2X^{-1})\prod_{i=1}^r(1-p^{2i-3-2r}t^2X^{-1})}.$$

Thus the assertion can be proved in the same manner as above.

(2) First let $d_0\in\mathbf{Z}_p^*$. Then by (1.1) of Proposition 5.4.4, we have

$$\begin{aligned}&K_{m-1}(d_0,\varepsilon,X,t)=\frac{1}{\phi_{(m-2)/2}(p^{-2})}\\ &+2^{-1}\sum_{r=1}^{(m-2)/2}\sum_{d\in\mathcal{U}(m-1,m-2r-1,d_0)}p^{-r(2r+1)}(t^2X^{-1})^r\prod_{i=1}^{r-1}(1-p^{2i-1}X^2)\phi_{(m-2r-2)/2}(p^{-2})^{-1}\\ &\qquad\times(1-p^{-1/2}\xi_0X)(1+\xi p^{r-1/2}X)\xi_0\xi\zeta_{2r}(d_0d,\varepsilon,tX^{-1/2})\end{aligned}$$

$$+\sum_{r=0}^{(m-2)/2} p^{-(2r+1)(r+1)}(t^2X^{-1})^{r+1/2}\prod_{i=1}^{r}(1-p^{2i-1}X^2)\phi_{(m-2r-2)/2}(p^{-2})^{-1}$$
$$\times(1-p^{-1/2}\xi_0 X)\zeta_{2r+1}(pd_0,\varepsilon,tX^{-1/2}).$$

By Theorem 5.2 of [IS95],

$$\zeta_{2r}(d_0d,\varepsilon,tX^{-1/2})=\frac{1+\xi_0\xi p^{-r}}{\phi_r(p^{-2})\prod_{i=1}^{r}(1-p^{-2i}t^2X^{-1})},$$

and

$$\zeta_{2r+1}(pd_0,\varepsilon,tX^{-1/2})=\frac{p^{-r-1}tX^{-1/2}}{\phi_r(p^{-2})\prod_{i=1}^{r+1}(1-p^{-2i}t^2X^{-1})}.$$

Thus the assertion can be proved in the same as in (1).

Next let $d_0\in p\mathbf{Z}_p^*$. Then by (1.2) of Proposition 5.4.4, we have

$$K_{m-1}(d_0,\varepsilon,X,t)$$
$$=X^{1/2}\{\sum_{r=0}^{(m-2)/2} p^{-(2r+1)(r+1)}(t^2X^{-1})^{r+1/2}\prod_{i=1}^{r}(1-p^{2i-1}X^2)\phi_{(m-2r-2)/2}(p^{-2})^{-1}$$
$$\times(1-p^{-1/2}\xi_0X)\zeta_{2r+1}(p^{-1}d_0,\varepsilon,tX^{-1/2}).\}$$

By Theorem 5.2 of [IS95],

$$\zeta_{2r+1}(p^{-1}d_0,\varepsilon,tX^{-1/2})=\frac{1}{\phi_r(p^{-2})\prod_{i=1}^{r}(1-p^{-2i}t^2X^{-1})}.$$

Hence

$$K_{m-1}(d_0,\varepsilon,X,t)$$
$$=p^{-1}t\sum_{r=0}^{(m-2)/2} p^{-(2r+1)r}(p^{-2}t^2X^{-1})^r\prod_{i=1}^{r}(1-p^{2i-1}X^2)\phi_{(m-2r-2)/2}(p^{-2})^{-1}$$
$$\times\frac{1}{\phi_r(p^{-2})\prod_{i=1}^{r}(1-p^{-2i}t^2X^{-1})}.$$

The assertion can be proved in the same as in (1). □

Proof of Theorem 5.4.2 in case $p=2$**.** The assertion can also be proved by using the same argument as above. □

Theorem 5.4.5. *Let* $d_0\in\mathcal{F}_p$ *and* $\xi_0=\chi(d_0)$*. Let* $\xi=\pm1$.
(1) *Let* m *be even. Then*

$$P_m^{(0)}(d_0,\iota,\xi,X,t)=\frac{(p^{-1}t)^{\nu(d_0)}}{\phi_{m/2-1}(p^{-2})(1-p^{-m/2}\xi_0)}$$
$$\times\frac{(1+t^2p^{-m/2-3/2}\xi)(1+t^2p^{-m/2-5/2}\xi\xi_0)-\xi_0t^2p^{-m/2-2}(X+X^{-1}+p^{1/2-m/2}\xi+p^{-1/2+m/2}\xi)}{(1-p^{-2}Xt^2)(1-p^{-2}X^{-1}t^2)\prod_{i=1}^{m/2-1}(1-t^2p^{-2i-1}X)(1-t^2p^{-2i-1}X^{-1})},$$

and

$$P_m^{(0)}(d_0,\varepsilon,\xi,X,t)=\frac{1}{\phi_{m/2-1}(p^{-2})(1-p^{-m/2}\xi_0)}\frac{\xi_0^2}{\prod_{i=1}^{m/2}(1-t^2p^{-2i}X)(1-t^2p^{-2i}X^{-1})}.$$

(2) *Let m be even. Then*

$$P_{m-1}^{(1)}(d_0,\iota,\xi,X,t)$$

$$=\frac{(p^{-1}t)^{\nu(d_0)}(1-\xi_0t^2p^{-5/2}\xi)}{(1-t^2p^{-2}X)(1-t^2p^{-2}X^{-1})\prod_{i=1}^{(m-2)/2}(1-t^2p^{-2i-1}X)(1-t^2p^{-2i-1}X^{-1})\phi_{(m-2)/2}(p^{-2})},$$

and

$$P_{m-1}^{(1)}(d_0,\varepsilon,\xi,X,t)$$

$$=\frac{(p^{-1}t)^{\nu(d_0)}(1-\xi_0t^2p^{(-1/2-m)}\xi)}{\prod_{i=1}^{m/2}(1-t^2p^{-2i}X)(1-t^2p^{-2i}X^{-1})\phi_{(m-2)/2}(p^{-2})}.$$

Proof. We note that there exist polynomials $S_{m-l}^{(l)}(d_0,\omega_p,\xi,X,t)$ and $S_{m-l}^{(l)}(d_0,\omega_p,X,t)$ such that

$$P_{m-l}^{(l)}(d_0,\omega_p,\xi,X,t)/t^{\nu(d_0)}=S_{m-l}^{(l)}(d_0,\omega_p,\xi,X,t^2)$$

and

$$P_{m-l}^{(l)}(d_0,\omega_p,X,t)/t^{\nu(d_0)}=S_{m-l}^{(l)}(d_0,\omega_p,X,t^2)$$

for $l=0,1$. We also note that

$$S_{m-l}^{(l)}(d_0,\omega_p,\xi,X,t^2)=S_{m-l}^{(l)}(d_0,\omega_p,\xi X,\xi t^2).$$

Thus the assertion follows from Theorems 5.4.1 and 5.4.2. □

5.5. **Explicit formulas of formal power series of Rankin-Selberg type.**

We prove our main result in this section.

Theorem 5.5.1. *Let $d_0\in\mathcal{F}_p$ and put $\xi_0=\chi(d_0)$.*
(1) *We have*

$$H_{n-1}(d_0,\iota,X,Y,t)$$

$$=(2^{-(n-1)(n-2)/2}t^{n-2})^{\delta_{2,p}}\phi_{(n-2)/2}(p^{-2})^{-1}(p^{-1}t)^{\nu(d_0)}(1-p^{-n}t^2)\prod_{i=1}^{n/2-1}(1-p^{-2n+2i}t^4)$$

$$\times\frac{(1+p^{-2}t^2)(1+p^{-3}\xi_0^2t^2)-p^{-5/2}t^2\xi_0(X+X^{-1}+Y+Y^{-1})}{(1-p^{-2}XYt^2)(1-p^{-2}XY^{-1}t^2)(1-p^{-2}X^{-1}Yt^2)(1-p^{-2}X^{-1}Y^{-1}t^2)}$$

$$\times\frac{1}{\prod_{i=1}^{n/2-1}(1-p^{-2i-1}XYt^2)(1-p^{-2i-1}XY^{-1}t^2)(1-p^{-2i-1}X^{-1}Yt^2)(1-p^{-2i-1}X^{-1}Y^{-1}t^2)}.$$

(2) *We have*

$$H_{n-1}(d_0,\varepsilon,X,Y,t)=((-1)^{n(n-2)/8}2^{-(n-1)(n-2)/2}t^{n-2})^{\delta_{2,p}}$$

$$\times((-1)^{n/2},(-1)^{n/2}d_0)_p\phi_{(n-2)/2}(p^{-2})^{-1}(1-p^{-n}t^2)\prod_{i=1}^{n/2-1}(1-p^{-2n+2i}t^4)(tp^{-n/2})^{\nu(d_0)}$$
$$\times\frac{(1+p^{-n}t^2)(1+p^{-n-1}\xi_0^2t^2)-p^{-1/2-n}t^2\xi_0(X+X^{-1}+Y+Y^{-1})}{\{(1-p^{-n}XYt^2)(1-p^{-n}XY^{-1}t^2)(1-p^{-n}X^{-1}Yt^2)(1-p^{-n}X^{-1}Y^{-1}t^2)}$$
$$\times\frac{1}{\prod_{i=1}^{n/2-1}(1-p^{-2i}XYt^2)(1-p^{-2i}XY^{-1}t^2)(1-p^{-2i}X^{-1}Yt^2)(1-p^{-2i}X^{-1}Y^{-1}t^2)},$$

where $(a,b)_p$ *denotes the Hilbert symbol of* $a,b\in\mathbf{Q}_p$.

Proof. First suppose that $\nu(d_0)=0$ and $\omega=\iota$. For an integer l put

$$V(l,X,Y,t)$$
$$=(1-t^2p^{-2}XY^{-1})(1-t^2p^{-2}X^{-1}Y^{-1})\prod_{i=1}^{l}(1-t^2p^{-2i-1}XY^{-1})(1-t^2p^{-2i-1}X^{-1}Y^{-1}).$$

Then by Proposition 5.3.1 and Theorems 5.4.1, 5.4.2, and (1) of Theorem 5.3.10, we have

$$\widetilde{R}_{n-1}(d_0,\omega,X,Y,t)=(1-p^{-n}t^2)$$
$$\times[\sum_{l=0}^{(n-4)/2}\sum_{d\in\mathcal{U}(n-1,n-1-2l,d_0)}\prod_{i=1}^{(n-2-2l)/2}(1-p^{-2l-n-2i}t^4)T_{2l}(d_0,d,Y)$$
$$\times\frac{\prod_{i=1}^{2l}(1-t^4p^{-n-2l-2+i})S_{2l}^{(0)}(d_0d,\iota,\zeta_d,X,Y,t)}{V(l,X,Y,t)}$$
$$+\sum_{l=0}^{(n-2)/2}\prod_{i=2}^{(n-2-2l)/2}(1-p^{-2l-n-2i}t^4)T_{2l+1}(d_0,d,Y,t)$$
$$\times\frac{\prod_{i=1}^{2l+1}(1-t^4p^{-n-2l-3+i})S_{2l+1}^{(1)}(d_0,\iota,X,Y,t)}{V(l-1,X,Y,t)}$$
$$+\frac{\prod_{i=1}^{(n-3)/2}(1-p^{2i-1}Y^2)\prod_{i=1}^{n-2}(1-t^4p^{-2n+i})}{\phi_{(n-4)/2}(p^{-2})V((n-2)/2,X,Y,t)}$$
$$\times\frac{1}{2}\sum_{d\in\mathcal{U}}\frac{(1-p^{-1/2}\xi_0Y)p^{(n-2)/2-1/2}\zeta_dY(1+\zeta_dYp^{(n-2)/2-1/2})}{(1-p^{(-n+2)/2}\xi_0\zeta_d)}$$
$$\times\{(1+t^2Y^{-1}p^{-(n-2)/2-3/2}\zeta_d)(1+t^2Y^{-1}p^{-(n-2)/2-5/2}\zeta_d)$$
$$-\xi_0\zeta_dt^2Y^{-1}p^{-(n-2)/2}(X+X^{-1}+p^{1/2-(n-2)/2}\zeta_d+p^{-1/2+(n-2)/2}\zeta_d)\}$$
$$+\frac{\prod_{i=1}^{(n-1)/2}(1-p^{2i-1}Y^2)\prod_{i=1}^{n-1}(1-t^4p^{-2n-1+i})(1-\xi_0t^2p^{-5/2}Y^{-1})}{\phi_{(n-2)/2}(p^{-2})V((n-2)/2,X,Y,t)},$$

where $S_{2r}^{(0)}(d_0d,\iota,\zeta_d,X,Y,t)$ and $S_{2r+1}^{(1)}(d_0,\iota,X,Y,t)$ are polynomials in t of degree at most 4 and 2, respectively. Hence $\widetilde{R}_{n-1}(d_0,\iota,X,Y,t)$ can be expressed as

$$\widetilde{R}_{n-1}(d_0,\iota,X,Y,t)$$

$$= \frac{(1-p^{-n}t^2)\prod_{i=1}^{(n-2)/2}(1-p^{-n-2i}t^4)S(d_0,\iota,X,Y,t)}{\phi_{(n-2)/2}(p^{-2})V((n-2)/2,X,Y,t)}],$$

where $S(d_0,\iota,X,Y,t)$ is a polynomial in t of degree at most $2n$ such that

(D) $S(d_0,\iota,X,Y,t)$

$$= \frac{1}{\phi_{(n-2)/2}(p^{-2})}\{(1-p^{-1/2}\xi_0 Y)p^{(n-2)/2-1/2}Y\sum_{\zeta=\pm1}(1+\zeta p^{(n-3)/2}Y)\zeta$$

$$\times\prod_{i=1}^{(n-2)/2-1}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2)/2}(1-p^{-2i-1-n}t^4)\frac{1-p^{-n+2}}{1-p^{(-n+2)/2}\xi_0\zeta}$$

$$\times((1+t^2Y^{-1}p^{-(n-2)/2-3/2}\zeta)(1+t^2Y^{-1}p^{-(n-2)/2-5/2}\zeta)$$

$$-\xi_0\zeta t^2Y^{-1}p^{-(n-2)/2}(X+X^{-1}+p^{1/2-(n-2)/2}\zeta+p^{-1/2+(n-2)/2}\zeta))$$

$$+(1-p^{-1/2}\xi_0 Y)\prod_{i=1}^{(n-2)/2}(1-p^{2i-1}Y^2)\prod_{i=1}^{(n-2)/2}(1-p^{-2i-1-n}t^4)(1+p^{-n}t^2)(1-p^{-5/2}\xi_0 t^2Y^{-1})\}$$

$$+(1-p^{-n+1}XY^{-1}t^2)(1-p^{-n+1}X^{-1}Y^{-1}t^2)U(d_0,X,Y,\iota,t)$$

with $U(d_0,\iota,X,Y,t)$ a polynomial in t. Hence by Theorem 5.2.6 we have

$$H_{n-1}(d_0,\iota,X,Y,t)=\kappa(d_0,n-1,l,t)(1-p^{-n}t^2)\prod_{i=1}^{n/2-1}(1-p^{-2n+2i}t^4)$$

$$\times\frac{S(d_0,\iota,X,Y,t)}{(1-p^{-2}XYt^2)(1-p^{-2}XY^{-1}t^2)(1-p^{-2}X^{-1}Yt^2)(1-p^{-2}X^{-1}Y^{-1}t^2)}$$

$$\times\frac{1}{\prod_{i=1}^{n/2-1}(1-p^{-2i-1}XYt^2)(1-p^{-2i-1}XY^{-1}t^2)(1-p^{-2i-1}X^{-1}Yt^2)(1-p^{-2i-1}X^{-1}Y^{-1}t^2)}$$

$$\times\frac{1}{\prod_{i=1}^{(n-2)/2}(1-p^{-2i}XYt^2)(1-p^{-2i}X^{-1}Yt^2)}.$$

Hence the power series $\widetilde{R}_{n-1}(d_0,\iota,X,Y,t)$ is a rational function in t, is invariant under the transformation $Y\mapsto Y^{-1}$ (cf. the proof of Theorem 5.4.2). This implies that the reduced denominator of the rational function $H_{n-1}(d_0,\iota,X,Y,t)$ in t is at most

$$(1-p^{-2}XYt^2)(1-p^{-2}XY^{-1}t^2)(1-p^{-2}X^{-1}Yt^2)(1-p^{-2}X^{-1}Y^{-1}t^2)$$

$$\times\prod_{i=1}^{n/2-1}(1-p^{-2i-1}XYt^2)(1-p^{-2i-1}XY^{-1}t^2)(1-p^{-2i-1}X^{-1}Yt^2)(1-p^{-2i-1}X^{-1}Y^{-1}t^2)$$

and therefore we have

(E) $S(d_0,\iota,X,Y,t)$

$$= t^{\nu(d_0)} \frac{a_0(d_0,X,Y) + a_1(d_0,X,Y)t^2 + a_2(d_0,X,Y)t^4}{\phi_{(n-2)/2}(p^{-2})}$$
$$\times \prod_{i=1}^{(n-2)/2} (1-p^{-2i}XYt^2)(1-p^{-2i}X^{-1}Yt^2),$$

where $a_i(d_0,X,Y)$ $(i=0,1,2)$ is a Laurent polynomial in X and Y. We can easily see $a_0(d_0,X,Y)=1$. First let $\nu(d_0)=0$. Then by substituting $p^{(n-1)/2}X^{i/2}Y^{1/2}$ $(i=\pm 1)$ for t in (D) and (E), and comparing them, we obtain

$$1 + a_1(d_0,X,Y)p^{n-1}X^iY + a_2(d_0,X,Y)p^{2(n-1)}X^{2i}Y^2$$

$$= 1+p^{n-1}X^iY(p^{-2}+p^{-3}-p^{-5/2}(X+X^{-1}+Y+Y^{-1})\xi_0)+p^{2n-2}X^{2i}Y^2p^{-5}$$

for $i=\pm 1$. Hence $a_1(d_0,X,Y) = p^{-2}+p^{-3}-p^{-5/2}(X+X^{-1}+Y+Y^{-1})\xi_0$ and $a_2(d_0,X,Y)=p^{-5}$. This proves the assertion in case $\nu(d_0)=0$. In case $\nu(d_0)>0$, in the same manner as above we have

$$1 + a_1(d_0,X,Y)p^{n-1}X^iY + a_2(d_0,X,Y)p^{2(n-1)}X^{2i}Y^2 = 1 + p^{n-3}X^iY$$

for $i=\pm 1$. Hence $a_1(d_0,X,Y)=p^{-2}$ and $a_2(d_0,X,Y)=0$. This proves the assertion in case $\nu(d_0)>0$.

Similarly the assertion for $\nu(d_0)=0$ and $\omega=\varepsilon$ can be proved. Next suppose that $\nu(d_0)>0$ and $\omega=\varepsilon$. Then the assertion can be proved similarly by using Proposition 5.3.1 and Theorems 5.4.1, 5.4.2, and (2) of Theorem 5.3.10. □

6. Proof of Conjecture B

Now we give an explicit form of $R(s,\sigma_{n-1}(\phi_{I_n(f),1}))$ for the first Fourier-Jacobi coefficient $\phi_{I_n(f),1}$ of the Duke-Imamoḡlu-Ikeda lift.

Proposition 6.1. *Let k and n be positive even integers. Given a primitive form f in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$, let $\widetilde{f} \in \mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$ be as in Section 2. Then*

$$R(s,\widetilde{f}) = L(2s-2k+n+1,f,\mathrm{Ad}) \sum_{d_0 \in \mathcal{F}^{(-1)^{n/2}}} |c(|d_0|)|^2 |d_0|^{-s}$$
$$\times \prod_p \{(1+p^{-2s+2k-n-1})(1+p^{-2s+2k-n-2}\chi_p(d_0)^2) - 2p^{-2s+2k-n-3/2}\chi_p(d_0)a(p)\},$$

where $c(|d_0|)$ is the $|d_0|$-th Fourier coefficient of $\widetilde{f}$, and $a(p)$ is the p-th Fourier coefficient of f.

Proof. By using the same argument as in Theorem 4.2, we can show that we have

$$R(s,\widetilde{f}) = \kappa_1 \sum_{d_0\in\mathcal{F}^{(-1)}} |c_{\widetilde{f}}(|d_0|)|^2 |d_0|^{n/2-k+1/2}$$

$$\times\left\{\prod_p H_{1,p}(d_0,\iota_p,\alpha_p,\alpha_p,p^{-s+k-n/2+1/2}) + \prod_p H_{1,p}(d_0,\varepsilon,\alpha_p,\alpha_p,p^{-s+k-n/2+1/2})\right\}.$$

By Theorem 5.5.1, for any d_0 we have

$$\prod_p H_{1,p}(d_0,\iota_p,\alpha_p,\alpha_p,p^{-s+k-n/2+1/2}) = \prod_p H_{1,p}(d_0,\varepsilon_p,\alpha_p,\alpha_p,p^{-s+k-n/2+1/2})$$

$$= |d_0|^{-s+k-n/2-1/2} L(2s-2k+n+1,f,\mathrm{Ad})$$

$$\times\prod_p((1+p^{-2s+2k-n-1})(1+p^{-2s-2k-n-2}\chi_p(d_0)^2) - 2p^{-2s+2k-n-3/2}\chi_p(d_0)a(p)).$$

Thus the assertion holds. □

Theorem 6.2. *Let k and n be positive even integers. Given a primitive form $f\in\mathfrak{S}_{2k-n}(\Gamma^{(1)})$, let $\widetilde{f}\in\mathfrak{S}^+_{k-n/2+1/2}(\Gamma_0(4))$ and $\phi_{I_n(f),1}\in J^{\mathrm{cusp}}_{k,1}(\Gamma^{(n-1),J})$ be as in Section 2 and Section 3, respectively. Put $\lambda_n = \frac{e_{n-1}}{2}\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)$. Then, we have*

$$R(s,\sigma_{n-1}(\phi_{I_n(f),1})) = \lambda_n 2^{(-s-1/2)(n-2)}\zeta(2s+n-2k+1)^{-1}\prod_{i=1}^{\frac{n-2}{2}}\zeta(4s+2n-4k+2-2i)^{-1}$$

$$\times\{R(s-n/2+1,\widetilde{f})\zeta(2s-2k+3)\prod_{i=1}^{\frac{n-2}{2}}L(2s-2k+2i+2,f,\mathrm{Ad})\zeta(2s-2k+2i+2)$$

$$+(-1)^{n(n-2)/8}R(s,\widetilde{f})\zeta(2s-2k+n+1)\prod_{i=1}^{\frac{n-2}{2}}L(2s-2k+2i+1,f,\mathrm{Ad})\zeta(2s-2k+2i+1)\}.$$

Proof. The assertion follows directly from Theorems 4.2 and 5.5.1, and Proposition 6.1. □

Theorem 6.3. *Conjecture B holds true for any positive even integer n.*

Proof. The assertion trivially holds if $n=2$. Suppose that $n\geq 4$. By Theorem 6.2 we have

$$\text{(F)}\qquad \mathcal{R}(s,\sigma_{n-1}(\phi_{I_n(f),1})) = \prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)2^{(-s-1/2)(n-2)}\mathcal{T}(s)$$

$$\times\left\{\mathcal{U}(s)^{-1}\mathcal{R}(s-n/2+1,\widetilde{f})\prod_{i=1}^{\frac{n-2}{2}}\widetilde{\Lambda}(2s-2k+2i+2,f,\mathrm{Ad})\widetilde{\xi}(2s-2k+2i+2)\right.$$

$$+(-1)^{n(n-2)/8}R(s,\widetilde{f})\prod_{i=1}^{\frac{n-2}{2}}L(2s-2k+2i+1,f,\mathrm{Ad})\zeta(2s-2k+2i+1)\Bigg\},$$

where

$$\mathcal{T}(s)=\Gamma_{\mathbf{R}}(2s+n-2k+1)\prod_{i=1}^{(n-2)/2}\Gamma_{\mathbf{R}}(4s+2n-4k+2-2i)\prod_{i=1}^{n-1}\Gamma_{\mathbf{R}}(2s-i+1),$$

and

$$\mathcal{U}(s)=\Gamma_{\mathbf{R}}(2s-2k+3)\Gamma_{\mathbf{R}}(2s-n+2)$$
$$\times\prod_{i=1}^{(n-2)/2}(\Gamma_{\mathbf{C}}(2s-2k+2i+2)\Gamma_{\mathbf{C}}(2s-n+2i+1)\Gamma_{\mathbf{R}}(2s-2k+2i+2)).$$

We note that $\mathcal{R}(s,\widetilde{f})$ is holomorphic at $s=k-1/2$. Thus by taking the residue of the both-sides of (F) at $s=k-1/2$, we get

$$\mathrm{Res}_{s=k-1/2}\mathcal{R}(s,\sigma_{n-1}(\phi_{I_n(f),1}))=2^{-k(n-2)}\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)\frac{\mathcal{T}(k-1/2)}{\mathcal{U}(k-1/2)}$$
$$\times\mathrm{Res}_{s=k-n/2+1/2}\mathcal{R}(s,\widetilde{f})\prod_{i=1}^{\frac{n-2}{2}}\Lambda(2i+1,f,\mathrm{Ad})\xi(2i+1).$$

We easily see that

$$\frac{\mathcal{T}(k-1/2)}{\mathcal{U}(k-1/2)}=2^{(n-1)(n-2)/2}.$$

By Theorem 1 in [KZ81], we have

$$\mathrm{Res}_{s=k-n/2+1/2}\mathcal{R}(s,\widetilde{f})=2^{2k-n}\langle\widetilde{f},\widetilde{f}\rangle.$$

Thus the assertion follows from Corollary to Proposition 3.1. □

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