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CONGRUENCE BETWEEN DUKE-IMAMOGLU-IKEDA LIFTS AND NON-DUKE-IMAMOGLU-IKEDA LIFTS

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1. INTRODUCTION

As is well known, there is a congruence between the Fourier coefficients of Eisenstein series and those of cuspidal Hecke eigenforms for $SL_2(\mathbf{Z})$. This type of congruence is not only interesting in its own right but also has an important application to number theory as shown by Ribet [Ri1]. Since then, there have been so many important results about the congruence of elliptic modular forms (cf. [Hi1],[Hi2],[Hi3].) In the case of Hilbert modular forms or Siegel modular forms, it is sometimes more natural and important to consider the congruence between the Hecke eigenvalues of Hecke eigenforms modulo a prime ideal $\mathfrak{P}$. We call such a $\mathfrak{P}$ a prime ideal giving the congruence or a congruence prime. For a cuspidal Hecke eigenform f for $SL_2(\mathbf{Z})$, let $\hat{f}$ be a lift of f to the space $\mathfrak{M}_l(I')$ of modular forms of weight l for a modular group I' . Here we mean by the lift of f a cuspidal Hecke eigenform whose certain L-function can be expressed in terms of certain L-functions of f . As typical examples of the lift we can consider the Doi-Naganuma lift, the Saito-Kurokawa lift, and the Duke-Imamoglu-Ikeda lift. We then consider the following problem:

Problem. *Characterize the prime ideals giving the congruence between $\hat{f}$ and a cuspidal Hecke eigenform in $\mathfrak{M}_l(I')$ not coming from the lift. In particular characterize them in terms of special values of certain L-functions of f .*

This type of problem was first investigated in the Doi-Naganuma lift case by Doi, Hida, and Ishii [D-H-I]. In our previous paper [Ka2], we considered the relationship between the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbf{Z})$ and the special values of their standard zeta functions. In particular, we proposed a conjecture concerning the congruence between Saito-Kurokawa lifts and non-Saito-Kurokawa

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lifts, and proved it under certain condition. In this paper, we consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, which is a generalization of our previous conjecture.

In Section 3, we review a result concerning the relationship between the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbf{Z})$ and the special values of their standard zeta functions. In Section 4, we propose a conjecture concerning the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, and prove it under a certain condition.

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Notation. For a commutative ring R , we denote by $M_{mn}(R)$ the set of (m, n) -matrices with entries in R . In particular put $M_n(R) = M_{nn}(R)$. Here we understand $M_{mn}(R)$ the set of the *empty matrix* if $m = 0$ or $n = 0$. For an (m, n) -matrix X and an (m, m) -matrix A , we write $A[X] = {}^tXAX$, where tX denotes the transpose of X . Let a be an element of R . Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the coset $X \bmod aM_{mn}(R)$. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix A , and R^* denotes the unit group of R . Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R . Furthermore, for an integral domain R of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R , that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices of degree n whose (i, j) -component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. In particular, if S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. Let R' be a subring of R . Two symmetric matrices A and A' with entries in R are called equivalent over R' with each other and write $A \sim_{R'} A'$ if there is an element X of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2. STANDARD ZETA FUNCTIONS OF SIEGEL MODULAR FORMS

For a complex number x put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$. Furthermore put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where 1_n denotes the unit matrix of degree n . For

a subring K of $\mathbf{R}$ put

$$GSp_n(K)^+ = \{M \in GL_{2n}(K) \mid J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) > 0\},$$

and

$$Sp_n(K) = \{M \in GSp_n(K)^+ \mid J_n[M] = J_n\}.$$

Furthermore, put

$$\Gamma^{(n)} = Sp_n(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \mid J_n[M] = J_n\}.$$

We sometimes write an element M of $GSp_n(K)$ as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

with $A, B, C, D \in M_2(K)$. We define a subgroup $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$ as

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \mid C \equiv O_n \pmod{N} \right\}.$$

Let $\mathbf{H}_n$ be Siegel's upper half-space. For each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n(\mathbf{R})^+$ and $Z \in \mathbf{H}_n$ put

$$M(Z) = (AZ + B)(CZ + D)^{-1}$$

and

$$j(M, Z) = \det(CZ + D).$$

Furthermore, for a function f on $\mathbf{H}_n$ and an integer k we define $f|_k M$ as

$$(f|_k M)(Z) = \det(M)^{k/2} j(M, Z)^{-k} f(M(Z)).$$

For an integer or half integer l and the subgroup $\Gamma_0^{(n)}(N)$ of $\Gamma^{(n)}$, we denote by $\mathfrak{M}_k(\Gamma_0^{(n)}(N))$ (resp. $\mathfrak{M}_k^\infty(\Gamma_0^{(n)}(N))$) the space of holomorphic (resp. C^∞ -) modular forms of weight k with respect to $\Gamma_0^{(n)}(N)$. We denote by $\mathfrak{S}_k(\Gamma_0^{(n)}(N))$ the sub-space of $\mathfrak{M}_k(\Gamma_0^{(n)}(N))$ consisting of cusp forms. Let f be a holomorphic modular form of weight k with respect to $\Gamma_0^{(n)}(N)$. Then f has the following Fourier expansion:

$$f(Z) = \sum_{A \in \mathcal{H}(\mathbf{Z})_{\geq 0}} a_f(A) \mathbf{e}(\text{tr}(AZ)),$$

and in particular if f is a cusp form, f has the following Fourier expansion:

$$f(Z) = \sum_{A \in \mathcal{H}(\mathbf{Z})_{> 0}} a_f(A) \mathbf{e}(\text{tr}(AZ)),$$

where tr denotes the trace of a matrix. Let dv denote the invariant volume element on $\mathbf{H}_n$ defined by $dv = \det(\text{Im}(Z))^{-n-1} \wedge_{1 \leq j \leq l \leq n} (dx_{jl} \wedge dy_{jl})$. Here for $Z \in \mathbf{H}_n$ we write $Z = (x_{jl}) + \sqrt{-1}(y_{jl})$ with real matrices

(x_{jl}) and (y_{jl}) . For two C^∞ -modular forms f and g of weight l with respect to $\Gamma_0^{(n)}(N)$ we define the Petersson scalar product $\langle f, g \rangle$ by

$$\langle f, g \rangle = [\Gamma^{(n)} : \Gamma_0^{(n)}(N)]^{-1} \int_{\Gamma_0^{(n)}(N) \backslash \mathbf{H}_n} f(Z) \overline{g(Z)} \det(\operatorname{Im}(Z))^l dv,$$

provided the integral converges.

Let $\mathbf{L}_n = \mathbf{L}_{\mathbf{Q}}(GSp_n(\mathbf{Q})^+, \Gamma^{(n)})$ denote the Hecke algebra over $\mathbf{Q}$ associated with the Hecke pair $(GSp_n(\mathbf{Q})^+, \Gamma^{(n)})$. Furthermore, let $\mathbf{L}_n^\circ = \mathbf{L}_{\mathbf{Q}}(Sp_n(\mathbf{Q}), \Gamma^{(n)})$ denote the Hecke algebra over $\mathbf{Q}$ associated with the Hecke pair $(Sp_n(\mathbf{Q}), \Gamma^{(n)})$. For each integer m define an element $T(m)$ of $\mathbf{L}_n$ by

$$T(m) = \sum_{d_1, \dots, d_n, e_1, \dots, e_n} \Gamma^{(n)}(d_1 \perp \dots \perp d_n \perp e_1 \perp \dots \perp e_n) \Gamma^{(n)},$$

where $d_1, \dots, d_n, e_1, \dots, e_n$ run over all positive integer satisfying

$$d_i | d_{i+1}, e_{i+1} | e_i \ (i = 1, \dots, n-1), d_n | e_n, d_i e_i = m \ (i = 1, \dots, n).$$

Furthermore, for $i = 1, \dots, n$ and a prime number p put

$$T_i(p^2) = \Gamma^{(n)}(1_{n-i} \perp p 1_i \perp p^2 1_{n-i} \perp p 1_i) \Gamma^{(n)},$$

and $(p^{\pm 1}) = \Gamma^{(n)}(p^{\pm 1} 1_n) \Gamma^{(n)}$. As is well known, $\mathbf{L}_n$ is generated over $\mathbf{Q}$ by all $T(p), T_i(p^2)$ ($i = 1, \dots, n$), and $(p^{\pm 1})$. We denote by $\mathbf{L}_n'$ the subalgebra of $\mathbf{L}_n$ generated by over $\mathbf{Z}$ by all $T(p)$ and $T_i(p^2)$ ($i = 1, \dots, n$). Let $T = \Gamma^{(n)} M \Gamma^{(n)}$ be an element of $\mathbf{L}_n \otimes \mathbf{C}$. Write T as $T = \cup_\gamma \Gamma^{(n)} \gamma$ and for $f \in \mathfrak{M}_k(\Gamma^{(n)})$ define the Hecke operator $|_k T$ associated to T as

$$f|_k T = \det(M)^{k/2-(n+1)/2} \sum_{\gamma} f|_k \gamma.$$

We call this action the Hecke operator as usual (cf. [A].) If f is an eigenfunction of a Hecke operator $T \in \mathbf{L}_n \otimes \mathbf{C}$, we denote by $\lambda_f(T)$ its eigenvalue. Let $\mathbf{L}$ be a subalgebra of $\mathbf{L}_n$. We call $f \in \mathfrak{M}_k(\Gamma^{(n)})$ a Hecke eigenform for $\mathbf{L}$ if it is a common eigenfunction of all Hecke operators in $\mathbf{L}$. In particular if $\mathbf{L} = \mathbf{L}_n$ we simply call f a Hecke eigenform. Furthermore, we denote by $\mathbf{Q}(f)$ the field generated over $\mathbf{Q}$ by eigenvalues of all $T \in \mathbf{L}_n$ as in Section 1. As is well known, $\mathbf{Q}(f)$ is a totally real algebraic number field of finite degree. Now, first we consider the integrality of the eigenvalues of Hecke operators. For an algebraic number field K , let $\mathfrak{O}_K$ denote the ring of integers in K . The following assertion has been proved in [Mi2] (see also [Ka2].)

Theorem 2.1 *Let $k \geq n+1$. Let $f \in \mathfrak{S}_k(\Gamma^{(n)})$ be a common eigenform in $\mathbf{L}_n'$. Then $\lambda_f(T)$ belongs to $\mathfrak{O}_{\mathbf{Q}(f)}$ for any $T \in \mathbf{L}_n'$.*

Let $\mathbf{L}_{np} = \mathbf{L}(GSp_n(\mathbf{Q})^+ \cap GL_{2n}(\mathbf{Z}[p^{-1}]), \Gamma^{(n)})$ be the Hecke algebra associated with the pair $(GSp_n(\mathbf{Q})^+ \cap GL_{2n}(\mathbf{Z}[p^{-1}]), \Gamma^{(n)})$. $\mathbf{L}_{np}$ can be considered as a subalgebra of $\mathbf{L}_n$, and is generated over $\mathbf{Q}$ by $T(p)$ and $T_i(p^2)$ ($i = 1, 2, \dots, n$). We now review the Satake p -parameters of $\mathbf{L}_{np}$; let $\mathbf{P}_n = \mathbf{Q}[X_0^\pm, X_1^\pm, \dots, X_n^\pm]$ be the ring of Laurent polynomials in $X_0, X_1, \dots, X_n$ over $\mathbf{Q}$. Let $\mathbf{W}_n$ be the group of $\mathbf{Q}$ -automorphisms of $\mathbf{P}_n$ generated by all permutations in variables $X_1, \dots, X_n$ and by the automorphisms $\tau_1, \dots, \tau_n$ defined by

$$\tau_i(X_0) = X_0 X_i, \tau_i(X_i) = X_i^{-1}, \tau_i(X_j) = X_j \ (j \neq i).$$

Furthermore, a group $\tilde{\mathbf{W}}_n$ isomorphic to $\mathbf{W}_n$ acts on the set $T_n = (\mathbf{C}^\times)^{n+1}$ in a way similarly to above. Then there exists a $\mathbf{Q}$ -algebra isomorphism Φ_{np} , called the Satake isomorphism, from $\mathbf{L}_{np}$ to the $\mathbf{W}_n$ -invariant subring $\mathbf{P}_n^{\mathbf{W}_n}$ of $\mathbf{P}_n$. Then for a $\mathbf{Q}$ -algebra homomorphism λ from $\mathbf{L}_{np}$ to $\mathbf{C}$, there exists an element $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \dots, \alpha_n(p, \lambda))$ of T_n satisfying

$$\lambda(\Phi_{np}^{-1}(F(X_0, X_1, \dots, X_n))) = F(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \dots, \alpha_n(p, \lambda))$$

for $F \in \mathbf{P}_n^{\mathbf{W}_n}$. The equivalence class of $(\alpha_0(p, \lambda), \alpha_1(p, \lambda), \dots, \alpha_n(p, \lambda))$ under the action of $\tilde{\mathbf{W}}_n$ is uniquely determined by λ . We call this the Satake parameters of $\mathbf{L}_{np}$ determined by λ .

Now assume that an element f of $M_k(Sp_n(\mathbf{Z}))$ is a Hecke eigenform. Then for each prime number p , f defines a $\mathbf{Q}$ -algebra homomorphism $\lambda_{f,p}$ from $\mathbf{L}_{np}$ to $\mathbf{C}$ in a usual way, and we denote by $\alpha_0(p), \alpha_1(p), \dots, \alpha_n(p)$ the Satake parameters of $\mathbf{L}_{np}$ determined by f . We then define the standard zeta function $L(f, s, \underline{\text{St}})$ by

$$L(s, f, \underline{\text{St}}) = \prod_p \prod_{i=1}^n \{(1 - p^{-s})(1 - \alpha_i(p)p^{-s})(1 - \alpha_i(p)^{-1}p^{-s})\}^{-1}.$$

Let $f(z) = \sum_{A \in \mathcal{H}_n(\mathbf{Z})_{>0}} a(A) \mathbf{e}(\text{tr}(Az))$ be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$.

For a positive integer $m \leq k - n$ such that $m \equiv n \pmod{2}$ put

$$\begin{aligned} \Lambda(f, m, \underline{\text{St}}) &= (-1)^{n(m+1)/2+1} 2^{-4kn+3n^2+n+(n-1)m+2} \\ &\times \Gamma(m+1) \prod_{i=1}^n \Gamma(2k-n-i) \frac{L(f, m, \underline{\text{St}})}{\langle f, f \rangle \pi^{-n(n+1)/2+nk+(n+1)m}}. \end{aligned}$$

Then the following theorem is due to Böcherer [B2] and Mizumoto [Mi].

Theorem 2.2. *Let l, k and n be a positive integers such that $\rho(n) \leq l \leq k - n$, where $\rho(n) = 3$, or 1 according as $n \equiv 1 \pmod{4}$ and $n \geq 5$, or not. Let $f \in \mathfrak{S}_k(\Gamma^{(n)})$ be a Hecke eigenform. Then $\Lambda(f, m, \underline{\text{St}})$ belongs to $\mathbf{Q}(f)$.*

For later purpose, we consider a special element in $\mathbf{L}_{np}$; the polynomial $X_0^2 X_1 X_2 \cdots X_n \sum_{i=1}^n (X_i + X_i^{-1})$ is an element of $\mathbf{P}_n^{\mathbf{W}_n}$, and thus we can define an element $\Phi_{np}^{-1}(X_0^2 X_1 X_2 \cdots X_n \sum_{i=1}^n (X_i + X_i^{-1}))$ of $\mathbf{L}_{np}$, which is denoted by $\mathbf{r}_1$.

Proposition 2.3. *Under the above notation the element $\mathbf{r}_1$ belongs to $\mathbf{L}'_n$, and we have*

$$\lambda_f(\mathbf{r}_1) = p^{n(k-(n+1)/2)} \sum_{i=1}^n (\alpha_i(p) + \alpha_i(p)^{-1}).$$

Proof. By a careful analysis of the computation in page 159-160 of [A], we see that $\mathbf{r}_1$ is a $\mathbf{Z}$ -linear combination of $T_i(p^2)$ ($i = 1, \dots, n$), and therefore we can prove the first assertion. Furthermore, by Lemma 3.3.34 of [A], we can prove the second assertion.

3. CONGRUENCE OF MODULAR FORMS AND SPECIAL VALUES OF THE STANDARD ZETA FUNCTIONS

In this section we review a result concerning the congruence between the Hecke eigenvalues of modular forms of the same weight following [Ka2]. Let K be an algebraic number field, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in K . For a prime ideal $\mathfrak{P}$ of $\mathfrak{O}$, we denote by $\mathfrak{O}_{(\mathfrak{P})}$ the localization of $\mathfrak{O}$ at $\mathfrak{P}$ in K . Let $\mathfrak{A}$ be a fractional ideal in K . If $\mathfrak{A} = \mathfrak{P}^e \mathfrak{B}$ with $\mathfrak{B} \mathfrak{O}_{(\mathfrak{P})} = \mathfrak{O}_{(\mathfrak{P})}$ we write $\text{ord}_{\mathfrak{P}} = e$. We simply write $\text{ord}_{\mathfrak{P}}(c) = \text{ord}_{\mathfrak{P}}((c))$ for $c \in K$. Now let f be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$ and M be a subspace of $\mathfrak{S}_k(\Gamma^{(n)})$ stable under Hecke operators $T \in \mathbf{L}_n$. Assume that M is contained in $(\mathbf{C}f)^\perp$, where $(\mathbf{C}f)^\perp$ is the orthogonal complement of $\mathbf{C}f$ in $\mathfrak{S}_k(\Gamma^{(n)})$ with respect to the Petersson product. Let K be an algebraic number field containing $\mathbf{Q}(f)$. A prime ideal $\mathfrak{P}$ of $\mathfrak{O}_K$ is called a congruence prime of f with respect to M if there exists a Hecke eigenform $g \in M$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \pmod{\tilde{\mathfrak{P}}}$$

for any $T \in \mathbf{L}'_n$, where $\tilde{\mathfrak{P}}$ is the prime ideal of $\mathfrak{O}_{K\mathbf{Q}(g)}$ lying above $\mathfrak{P}$. If $M = (\mathbf{C}f)^\perp$, we simply call $\mathfrak{P}$ a congruence prime of f .

Now we consider the relation between the congruence primes and the standard zeta values. To consider this, we have to normalize the standard zeta value $\Lambda(f, l, \underline{\text{St}})$ for a Hecke eigenform f because it is not uniquely determined by the system of Hecke eigenvalues of f . We note that there is no reasonable normalization of cuspidal Hecke eigenform in the higher degree case unlike the elliptic modular case.

Thus we define the following quantities: for a Hecke eigenform $f(z) = \sum_A a_f(A) \mathbf{e}(\text{tr}(Az))$ in $\mathfrak{S}_k(\Gamma^{(n)})$, let $\mathfrak{I}_f$ be the $\mathfrak{D}_{\mathbf{Q}(f)}$ -module generated by all $a_f(A)$'s. Assume that there exists a complex number c such that all the Fourier coefficients cf belongs to $\mathbf{Q}(f)$. Then $\mathfrak{I}_f$ is a fractional ideal in $\mathbf{Q}(f)$, and therefore, so is $\Lambda(f, l, \underline{\text{St}}) \mathfrak{I}_f^2$ if l satisfies the condition in Theorem 2.2. We note that this fractional ideal does not depend on the choice of c . We also note that the value $N_{\mathbf{Q}(f)}(\Lambda(f, l, \underline{\text{St}})) N(\mathfrak{I}_f)^2$ does not depend on the choice of c , where $N(\mathfrak{I}_f)$ is the norm of the ideal $\mathfrak{I}_f$. Then, we have

Theorem 3.1. *Let f be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$. Assume that there exists a complex number c such that all the Fourier coefficients cf belongs to $\mathbf{Q}(f)$. Let l be a positive integer satisfying the condition in Theorem 2.2. Let $\mathfrak{P}$ be a prime ideal of $\mathfrak{D}$. Assume that $\text{ord}_{\mathfrak{P}}(\Lambda(f, l, \underline{\text{St}}) \mathfrak{I}_f^2) < 0$ and that it does not divide $(2l-1)!$. Then $\mathfrak{P}$ is a congruence prime of f . In particular, if a rational prime number p divides the denominator of $N_{\mathbf{Q}(f)}(\Lambda(f, l, \underline{\text{St}})) N(\mathfrak{I}_f)^2$, then p is divisible by some congruence prime of f .*

Now for a Hecke eigenform f in $\mathfrak{S}_k(\Gamma^{(n)})$, let $\mathfrak{Z}_f$ denote the subspace of $\mathfrak{S}_k(\Gamma^{(n)})$ spanned by all Hecke eigenforms with the same system of the Hecke eigenvalues as f .

Corollary. *In addition to the above notation and the assumption, assume that $\mathfrak{S}_k(\Gamma^{(n)})$ has the multiplicity one property. Then $\mathfrak{P}$ is a congruence prime of f with respect to $\mathfrak{Z}_f^\perp$. In particular, if a rational prime number p divides the denominator of $N_{\mathbf{Q}(f)}(\Lambda(f, l, \underline{\text{St}})) N(\mathfrak{I}_f)^2$, then p is divisible by some congruence prime of f with respect to $\mathfrak{Z}_f^\perp$.*

4. CONGRUENCE BETWEEN DUKE-IMAMOGLU-IKEDA LIFTS AND NON-DUKE-IMAMOGLU-IKEDA LIFTS

In this section, we consider the congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts. Throughout this section and the next, we assume that n and k are even positive integers. Let

$$f(z) = \sum_{m=1}^{\infty} a(m) \mathbf{e}(mz)$$

be a normalized Hecke eigenform of weight $2k - n$ with respect to $SL_2(\mathbf{Z})$. For a Dirichlet character χ , we then define the L-function

$L(s, f)$ of f twisted by χ by

$$L(s, f) = \prod_p \{(1 - \chi(p)\beta_p p^{k-n/2-1/2-s})(1 - \chi(p)\beta_p^{-1} p^{k-n/2-1/2-s})\}^{-1},$$

where β_p is a non-zero complex number such that $\beta_p + \beta_p^{-1} = p^{-k+n/2+1/2}a(p)$. We simply write $L(s, f)$ as $L(s, f, \chi)$ if χ is the principal character. Furthermore, let $\tilde{f}$ be the cusp form of weight $k-n/2+1/2$ belonging to the Kohnen plus space corresponding to f via the Shimura correspondence (cf. [Ko1]). Then $\tilde{f}$ has the following Fourier expansion:

$$\tilde{f}(z) = \sum_e c(e) \mathbf{e}(ez),$$

where e runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \pmod{4}$. We then put

$$a_{I_n(f)}(T) = c(|\mathfrak{b}_T|) \prod_p (p^{k-n/2-1/2} \beta_p)^{\nu_p(\mathfrak{f}_T)} F_p(T, p^{-(n+1)/2} \beta_p^{-1}).$$

We note that $a_{I_n(f)}(T)$ does not depend on the choice of β_p . Define a Fourier series $I_n(f)(Z)$ by

$$I_n(f)(Z) = \sum_{T \in \mathcal{H}_n(\mathbf{Z})_{>0}} a_{I_n(f)}(T) \mathbf{e}(\mathrm{tr}(TZ)).$$

In [Ik1] Ikeda showed that $I_n(f)(Z)$ is a cusp form of weight k with respect to $\Gamma^{(n)}$ and a Hecke eigenform for $\mathbf{L}_n^\circ$ such that

$$L(s, I_n(f), \underline{\mathrm{St}}) = \zeta(s) \prod_{i=1}^n L(s + k - i, f).$$

This was first conjecture by Duke and Imamoglu. Thus we call $I_n(f)$ the Duke-Imamoglu-Ikeda lift of f . We note that we have $\mathbf{Q}(\tilde{f}) = \mathbf{Q}(I_n(f)) = \mathbf{Q}(f)$. Furthermore, we have $\mathfrak{I}_{\tilde{f}} = \mathfrak{I}_{I_n(f)}$, where $\mathfrak{I}_{\tilde{f}}$ is the $\mathfrak{O}_{\mathbf{Q}(f)}$ -module generated by all the Fourier coefficients of $\tilde{f}$.

Now to consider a congruence between Duke-Imamoglu-Ikeda lifts and non-Duke-Imamoglu-Ikeda lifts, first we prove the following:

Proposition 4.1 *$I_n(f)$ is a Hecke eigenform.*

We note that Ikeda proved in [Ik1] that $I_n(f)$ is a Hecke eigenform for $\mathbf{L}_n^\circ$ but has not proved that it is a Hecke eigenform for $\mathbf{L}_n$. This was pointed to us by B. Heim (see [He].) We thank him for his comment. We also note that an explicit form of the spinor L-function of $I_n(f)$ was obtained by Murakawa [Mu] and Schmidt [Sch] assuming that $I_n(f)$ is a Hecke eigenform.

Proof of Proposition 4.1. We have only to prove that $I_n(f)$ is an eigenfunction of $T(p)$ for any prime p . The proof may be more or less well known, but for the convenience of the readers we here give the proof. For a modular form

$$F(Z) = \sum_B c_F(B) \mathbf{e}(\mathrm{tr}(BZ)),$$

let $c_F^{(p)}(B)$ be the B -th Fourier coefficient of $F|T(p)$. Then for any positive definite matrix B we have

$$\begin{aligned} c_F^{(p)}(B) &= p^{nk-n(n+1)/2} \sum_{d_1|d_2|\dots|d_n|p} d_1^n d_2^{n-1} \dots d_n \\ &\times \sum_{D \in \Lambda_n(d_1 \perp \dots d_n) \Lambda_n} \det D^{-k} c_F(p^{-1} A[tD]), \end{aligned}$$

where $\Lambda_n = GL_n(\mathbf{Z})$.

Now let $E_{n,k}(Z)$ be the Siegel Eisenstein series of degree n and of weight k defined by

$$E_{n,k}(Z) = \sum_{\gamma \in \Gamma_{n,\infty} \backslash \Gamma_n} j(\gamma, Z)^{-k}.$$

For $k \geq n+1$, the Siegel Eisenstein series $E_{n,k}(Z)$ is a holomorphic modular form of weight k with respect to Γ_n . Furthermore, $E_{n,k}(Z)$ is a Hecke eigenform and in particular we have

$$E_{n,k}|T(p)(Z) = h_{n,p}(p^k) E_{n,k}(Z),$$

where

$$h_{n,p}(X) = 1 + \sum_{r=1}^n \sum_{1 \leq i_1 < \dots < i_r \leq n} p^{-\sum_{j=1}^r i_j} X^r.$$

Let $c_{n,k}(B)$ be the B -th Fourier coefficient of $E_{n,k}(Z)$. Then we have

$$\begin{aligned} h_{n,p}(p^k) c_{n,k}(B) &= p^{nk-n(n+1)/2} \sum_{d_1|d_2|\dots|d_n|p} d_1^n d_2^{n-1} \dots d_n \\ &\times \sum_{D \in \Lambda_n(d_1 \perp \dots d_n) \Lambda_n} \det D^{-k} c_{n,k}(p^{-1} B[tD]). \end{aligned}$$

Let B be positive definite. Then we have

$$c_{n,k}(B) = a_{n,k} (\det 2B)^{k-(n+1)/2} L(k - n/2, \chi_B) \prod_q F_q(B, p^{-k}),$$

where $a_{n,k}$ is a non-zero constant depending only on n and k . We note that we have

$$F_q(p^{-1} B[tD], X) = F_q(B, X)$$

for any $D \in \Lambda_n(d_1 \perp \cdots d_n) \Lambda_n$ with $d_1 | \cdots | d_n | p$ if $q \neq p$. Thus we have

$$\begin{aligned} h_{n,p}(p^k) F_p(B, p^{-k}) &= \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{ne_1 + (n-1)e_2 + \cdots + e_n} p^{(e_1 + \cdots + e_n)(k-n-1)} \\ &\quad \times \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1} \perp \cdots p^{e_n}) \Lambda_n} F_p(p^{-1} B[tD], p^{-k}). \end{aligned}$$

The both-hand sides of the above are polynomials in p^k and the equality holds for infinitely many k . Thus we have

$$\begin{aligned} h_{n,p}(X^{-1}) F_p(B, X) &= \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{ne_1 + (n-1)e_2 + \cdots + e_n} (X^{-1} p^{-n-1})^{(e_1 + \cdots + e_n)} \\ &\quad \times \sum_{D \in \Lambda_n \setminus \Lambda_n(p^{e_1} \perp \cdots p^{e_n}) \Lambda_n} F_p(p^{-1} B[tD], X) \end{aligned}$$

as polynomials in X and X^{-1} . Thus we have

$$\begin{aligned} (p^{k-(n+1)/2} X)^{n/2} h_{n,p}(p^{(n+1)/2} X^{-1}) (p^{k-(n+1)/2} X^{-1})^{\nu_p(\mathfrak{f}_B)} F_p(B, p^{-(n+1)/2} X) \\ = p^{nk-n(n+1)/2} \sum_{e_1 \leq e_2 \leq \cdots \leq e_n \leq 1} p^{ne_1 + (n-1)e_2 + \cdots + e_n} \\ \times \sum_{D \in \Lambda_n(p^{e_1} \perp \cdots p^{e_n}) \Lambda_n} \det D^{-k} (p^{k-(n+1)/2} X^{-1})^{\nu_p(\mathfrak{f}_{p^{-1}B[tD]})} F_p(p^{-1} B[tD], p^{-(n+1)/2} X). \end{aligned}$$

We recall that we have

$$c_{I_n(f)}(B) = c_{\tilde{f}}(|\mathfrak{d}_B|) \mathfrak{f}_B^{k-(n+1)/2} \prod_q (\beta_q)^{\nu_q(\mathfrak{f}_B)} F_q(B, q^{-(n+1)/2} \beta_q^{-1}),$$

where β_q is the Satake q -parameter of f . We also note that $c_{\tilde{f}}(|\mathfrak{d}_{p^{-1}B[tD]}|) = c_{\tilde{f}}(|\mathfrak{d}_B|)$ for any D . Thus we have

$$\begin{aligned} (p^{k-(n+1)/2} \alpha_p^{-1})^{n/2} h_{n,p}(p^{(n+1)/2} \alpha_p) c_{I_n(f)}(B) \\ = p^{nk-n(n+1)/2} \sum_{d_1 | d_2 | \cdots | d_n | p} d_1^n d_2^{n-1} \cdots d_n \sum_{D \in \Lambda_n(d_1 \perp \cdots d_n) \Lambda_n} \det D^{-k} c_{I_n(f)}(p^{-1} B[tD]). \end{aligned}$$

This proves the assertion.

Let f be a primitive form in $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$. Let $\{f_1, \dots, f_d\}$ be a basis of $\mathfrak{S}_{2k-n}(\Gamma^{(1)})$ consisting of primitive forms. Let K be an algebraic number field containing $\mathbf{Q}(f_1) \cdots \mathbf{Q}(f_d)$, and $A = \mathfrak{O}_K$. To formulate our conjecture exactly, we introduce the Eichler-Shimura periods as follows (cf. Hida [Hi3].) Let $\mathfrak{P}$ be a prime ideal in K . Let $A_{\mathfrak{P}}$ be a valuation ring in K corresponding to $\mathfrak{P}$. Assume that the residual characteristic of $A_{\mathfrak{P}}$ is greater than or equal to 5. Let $L(2k-n-2, A_{\mathfrak{P}})$ be the module of homogeneous polynomials of degree $2k-n-2$ in the variables X, Y

with coefficients in $A_{\mathfrak{p}}$. We define the action of $M_2(\mathbf{Z}) \cap GL_2(\mathbf{Q})$ on $L(2k - n - 2, A_{\mathfrak{p}})$ via

$$\gamma \cdot P(X, Y) = P({}^t(X, Y)(\gamma)^\iota),$$

where $\gamma^\iota = (\det \gamma)\gamma^{-1}$. Let $H_P^1(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ be the parabolic cohomology group of $\Gamma^{(1)}$ with values in $L(2k - n - 2, A_{\mathfrak{p}})$. Fix a point $z_0 \in \mathbf{H}_1$. Let $g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$ or $g \in \overline{\mathfrak{S}_{2k-n}(\Gamma^{(1)})}$. We then define the differential $\omega(g)$ as

$$\omega(g)(z) = \begin{cases} 2\pi i g(z)(X - zY)^n dz & \text{if } g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \\ 2\pi \sqrt{-1} g(z)(X - \bar{z}Y)^n dz & \text{if } g \in \overline{\mathfrak{S}_{2k-n}(\Gamma^{(1)})}, \end{cases}$$

and define the cohomology class $\delta(g)$ of the 1-cocycle of $\Gamma^{(1)}$ as

$$\gamma \in \Gamma^{(1)} \longrightarrow \int_{z_0}^{\gamma(z_0)} \omega(g).$$

The mapping $g \longrightarrow \delta(g)$ induces the isomorphism

$$\delta : \mathfrak{S}_{2k-n}(\Gamma^{(1)}) \oplus \overline{\mathfrak{S}_{2k-n}(\Gamma^{(1)})} \longrightarrow H_P^1(\Gamma^{(1)}, L(2k - n - 2, \mathbf{C})),$$

which is called the Eichler-Shimura isomorphism. We can define the action of Hecke algebra $\mathbf{L}'_1$ on $H_P^1(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ in a natural manner. Furthermore, we can define the action F_∞ on $H_P^1(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ as

$$F_\infty(\delta(g)(z)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \delta(g)(-\bar{z}),$$

and this action commutes with the Hecke action. For a primitive form f and $j = \pm 1$, we define the subspace $H_P^1(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))[f, j]$ of $H_P^1(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))$ as

$$H_P^1(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))[f, j]$$

$$= \{x \in H_P^1(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}})) ; x|T = \lambda_f(T)x \text{ for } T \in \mathbf{L}_1, \text{ and } F_\infty(x) = jx\}.$$

Since $A_{\mathfrak{p}}$ is a principal ideal domain, $H_P^1(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))[f, j]$ is a free module of rank one over $A_{\mathfrak{p}}$. For each $j = \pm 1$ take a basis $\eta(f, j, A_{\mathfrak{p}})$ of $H_P^1(\Gamma^{(1)}, L(2k - n - 2, A_{\mathfrak{p}}))[f, j]$ and define a complex number $\Omega(f, j; A_{\mathfrak{p}})$ by

$$(\delta(f) + jF_\infty(\delta(f)))/2 = \Omega(f, j; A_{\mathfrak{p}})\eta(f, j; A_{\mathfrak{p}}).$$

This $\Omega(f, j; A_{\mathfrak{p}})$ is uniquely determined up to constant multiple of units in $A_{\mathfrak{p}}$. We call $\Omega(f, +; A_{\mathfrak{p}})$ and $\Omega(f, -; A_{\mathfrak{p}})$ the Eichler-Shimura periods. For $j = \pm 1$, $1 \leq l \leq 2k - n - 1$, and a Dirichlet character χ such

that $\chi(-1) = j(-1)^{l-1}$, put

$$\mathbf{L}(l, f, \chi) = \mathbf{L}(l, f, \chi; A_{\mathfrak{P}}) = \frac{\Gamma(l)L(l, f, \chi)}{\tau(\chi)(2\pi\sqrt{-1})^l \Omega(f, j; A_{\mathfrak{P}})},$$

where $\tau(\chi)$ is the Gauss sum of χ . In particular, put $\mathbf{L}(l, f; A_{\mathfrak{P}}) = \mathbf{L}(l, f, \chi; \mathfrak{P})$ if χ is the principal character. Furthermore, put

$$\mathbf{L}(s, f, \underline{\text{St}}) = 4(2\pi)^{-2s-2k+n+1} \Gamma(s) \Gamma(s+2k-n-1) L(s, f, \underline{\text{St}}).$$

It is well-known that $\mathbf{L}(l, f, \chi)$ belongs to the field $K(\chi)$ generated over K by all the values of χ , and $\mathbf{L}(l, f, \underline{\text{St}})$ belongs to $\mathbf{Q}(f)$ (cf. [Bo].) Let $I_n(f)$ be the Duke-Imamoglu-Ikeda lift of f . Let $\mathfrak{S}_k(\Gamma^{(n)})^*$ be the subspace of $\mathfrak{S}_k(\Gamma_n)$ generated by all the Duke-Imamoglu-Ikeda lifts $I(g)^n$ of primitive forms $g \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$. We remark that $\mathfrak{S}_k(\Gamma^{(2)})^*$ is the Maass subspace of $\mathfrak{S}_k(\Gamma^{(2)})$.

Conjecture A. *Let K and f be as above. Assume that $k > n$. Let $\mathfrak{P}$ be a prime ideal of K not dividing $(2k-1)!$. Then $\mathfrak{P}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{S}_k(\Gamma^{(n)})^*)^\perp$ if $\mathfrak{P}$ divides $\mathbf{L}(k, f) \prod_{i=1}^{n/2-1} \mathbf{L}(2i+1, f, \text{St})$.*

Remark. This is an analogue of the Doi-Hida-Ishii conjecture concerning the congruence primes of the Doi-Naganuma lifting [D-H-I]. (See also [Ka1].) We also note that this type of conjecture has been proposed by Harder [Ha] in the case of vector valued Siegel modular forms.

Now to explain why our conjecture is reasonable, we refer to Ikeda's conjecture on the Petersson inner product of the Duke-Imamoglu-Ikeda lifting. Let f and $\tilde{f}$ be as above. Put

$$\tilde{\xi}(s) = 2(2\pi)^{-s} \Gamma(s) \zeta(s),$$

and

$$\Lambda(s, f) = 2(2\pi)^{-s} \Gamma(s) L(s, f).$$

Theorem 4.2. (Katsurada and Kawamura [K-K]) *In addition to the above notation and the assumption, assume $k > n$. Then we have*

$$\tilde{\xi}(n) \Lambda(k, f) \prod_{i=1}^{n/2-1} \mathbf{L}(2i-1, f, \underline{\text{St}}) \tilde{\xi}(2i) = 2^\alpha \frac{\langle I_n(f)f, I_n(f) \rangle}{\langle f, f \rangle^{n/2-1} \langle \tilde{f}, \tilde{f} \rangle},$$

where α is an integer depending only on n and k .

We note that the above theorem was conjectured by Ikeda [Ik2] under more general setting. We note that the theorem has been proved by Kohnen and Skoruppa [K-S] in case $n = 2$.

Proposition 4.3 *Under the above notation and the assumption we have for any fundamental discriminant D such that $(-1)^{n/2}D > 0$ and $L(k - n/2, f, \chi_D) \neq 0$ we have*

$$\frac{c(|D|)^2}{\langle I_n(f), I_n(f) \rangle} = a_{n,k} \frac{\langle f, f \rangle^{n/2} |D|^{k-n/2} \mathbf{L}(k - n/2, f, \chi_D)}{\mathbf{L}(k, f) \tilde{\xi}(n) \prod_{i=1}^{n/2-1} \mathbf{L}(2i+1, f, \text{St}) \tilde{\xi}(2i)}$$

with some algebraic number $a_{n,k}$ depending only on n, k .

Proof. By the result in Kohnen-Zagier [K-Z], for any fundamental discriminant D such that $(-1)^{n/2}D > 0$ we have

$$\frac{c(|D|)^2}{\langle \tilde{f}, \tilde{f} \rangle} = \frac{2^{k-n/2-1} |D|^{k-n/2-1/2} \Lambda(k - n/2, f, \chi_D)}{\langle f, f \rangle}.$$

Thus the assertion holds.

Lemma 4.4. *Let f be as above.*

(1) *Let $\mathbf{r}_1$ be an element of $\mathbf{L}'_n$ in Proposition 2.3. Then we have*

$$\lambda_{I_n(f)}(\mathbf{r}_1) = p^{(n-1)k-n(n+1)/2} a_f(p) \sum_{i=1}^n p^i.$$

(2) *Let $n = 2$. Then we have*

$$\lambda_{I_2(f)}(T(p)) = a_f(p) + p^{2k-n-1} + p^{2k-n-2}.$$

Lemma 4.5. *Let d be a fundamental discriminant such that $(-1)^{n/2}d > 0$.*

(1) *Assume that $d \neq 1$. Then there exists a positive definite half integral matrix A of degree n such that $(-1)^{n/2} \det(2A) = d$.*

(2) *Assume $n \equiv 0 \pmod{8}$. Then there exists a positive definite half integral matrix A of degree n such that $\det(2A) = 1$.*

(3) *Assume that $n \equiv 4 \pmod{8}$. Then for any prime number q there exists a positive definite half integral matrix A of degree n such that $\det(2A) = q^2$.*

Proof. (1) For a non-degenerate symmetric matrix A with entries in $\mathbf{Q}_p$ let $h_p(A)$ be the Hasse invariant of A . First let $n \equiv 2 \pmod{4}$ and $d = -4$. Take a family $\{A_p\}_p$ of half integral matrices over $\mathbf{Z}_p$ of

degree n such that $A_p = 1_n$ if $p \neq 2$, and $A_2 = (-1)^{(n-2)/4} 1_2 \perp H_{n/2-1}$, where $H_r = \overbrace{H \perp \dots \perp H}^r$ with $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$. Then we have $\det A = 2^{2-n} \in \mathbf{Q}_p^\times / (\mathbf{Q}_p^\times)^2$ for any p , and $h_p(A) = 1$ for any p . Thus by [I-S, Proposition 2.1], there exists an element A of $\mathcal{L}_{n,2>0}$ such that $A \sim A_p$ for any p . In particular we have $(-1)^{n/2} \det(2A) = -4$. Next let $d = (-1)^{n/2} 8$. We take $A_p = (-1)^{n/2} 2 \perp 1_{n-1}$ if $p \neq 2$. We can take $\xi \in \mathbf{Z}_2^*$ such that $(2, \xi) = (-1)^{(n-2)(n+4)/8}$, and put $A_2 = 2\xi \perp (-\xi) \perp H_{n/2-1}$. Then we have $\det A = (-1)^{n/2} 2^{3-n} \in \mathbf{Q}_p^\times / (\mathbf{Q}_p^\times)^2$ for any p , and $h_p(A) = 1$ for any p . Thus again by [I-S, Proposition 2.1], we prove the assertion for this case. Finally assume that d contains a odd prime factor q . For $p \neq p_0$ we take a matrix A_p so that $\det A_p = 2^{-n} d \in \mathbf{Q}_p^\times / (\mathbf{Q}_p^\times)^2$. Then for almost all p we have $h_p(A_p) = 1$. We take $\xi \in \mathbf{Z}_q^*$ such that $(q, -\xi) = \prod_{p \neq q} h_p(A_p)$, and put $A_q = \xi d \perp \xi \perp 1_{n-2}$. Then we have $\det A_q 2^{-n} d \in \mathbf{Q}_q^\times / (\mathbf{Q}_q^\times)^2$, and $h_p(A_q) \prod_{p \neq q} = 1$. Thus again by [I-S, Proposition 2.1], we prove the assertion for this case.

(2) It is well known that there exists a positive definite half-integral

matrix E_8 of degree 8 such that $\det(2E_8) = 1$. Thus $A = \overbrace{E_8 \perp \dots \perp E_8}^{n/8}$ satisfies the required condition.

(3) Let $q \neq 2$. Then, take a family $\{A_p\}$ of half-integral matrices over $\mathbf{Z}_p$ of degree n such that $A_q \sim_{\mathbf{Z}_q} q \perp (-q\xi) \perp (-\xi) \perp 1_{n-3}$ with $(\frac{\xi}{q}) = -1$, $A_2 = H_{n/2}$, and $A_p = 1_n$ for $p \neq 2$. Then by the same argument as in (1) we can show that there exists a positive definite half integral matrix A of degree n such that $\det(2A) = q^2$ such that $A \sim_{\mathbf{Z}_p} A_p$ for any p . Let $q = 2$. Then the matrix $A' = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1 \end{pmatrix}$ is a positive

definite and $\det(2A') = 4$. Thus the matrix $A' \perp \overbrace{E_8 \perp \dots \perp E_8}^{(n-4)/8}$ satisfies the required condition.

Proposition 4.6. *Let k and n be positive even integer. Let d be a fundamental discriminant. Let f be a primitive form in $S_{2k-n}(\Gamma_1)$. Let $\mathfrak{P}$ be a prime ideal in K . Then there exists a positive definite half integral matrices A of degree n such that $c_{I_n(f)}(A) = c_{\tilde{f}}(|d|)q$ with q not divisible by $\mathfrak{P}$.*

Proof. First assume that $d \neq 1$, or $n \not\equiv 4 \pmod{8}$. (1) By (1) and (2) of Lemma 4.5, there exists a matrix A such that $\mathfrak{d}_A = d$. Thus we have $c_{I_n(f)}(A) = c_{\bar{f}}(|d|)$. This proves the assertion.

Next assume that $n \equiv 4 \pmod{8}$ and that $d = 1$. Assume that $c_f(q) + q^{k-n/2-1}(-q-1)$ is divisible by $\mathfrak{P}$ for any prime number q . Let p be a prime number divisible by $\mathfrak{P}$. Fix an imbedding $\iota_p : \bar{\mathbf{Q}} \longrightarrow \bar{\mathbf{Q}}_p$, and let $\rho_{f,p} : \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \text{GL}_2(\bar{\mathbf{Q}}_p)$ be the Galois representation attached to f . Then by Chebotarev density theorem, the semi-simplification $\bar{\rho}_{f,p}^{ss}$ of $\bar{\rho}_{f,p}$ can be expressed as

$$\bar{\rho}_{f,p}^{ss} = \overline{\chi}_p^{k-n/2} \oplus \overline{\chi}_p^{k-n/2-1}$$

with $\overline{\chi}_p$ the p -adic mod p cyclotomic character. On the other hand, by the Fontaine-Messing [Fo-Me] and Fontaine-Laffaille [Fo-La], $\bar{\rho}_{f,p}^{ss}|_{I_p}$ should be $\overline{\chi}_p^{2k-n-1} \oplus 1$ or $\omega_2^{2k-n-1} \oplus \omega_2^{p(2k-n-1)}$ with ω_2 the fundamental character of level 2, where I_p denotes the inertia group of p in $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. This is impossible because $k > 2$. Thus there exists a prime number q such that $c_f(q) + q^{k-n/2-1}(-q-1)$ is not divisible by $\mathfrak{P}$. For such a q , take a positive definite matrix A in (3) of Lemma 4.5. Then

$$c_{I_n(f)}(A) = c(1)q^{k-(n+1)/2}\beta_q F_q(A, q^{-(n+1)/2}\beta_q^{-1}).$$

By [Ka1], we have

$$F_q(B, X) = 1 - Xq^{(n-2)/2}(q^2 + q) + q^3(Xq^{(n-2)/2})^2.$$

Thus we have

$$c_{I_n(f)}(A) = c(1)(c_f(q) + q^{k-n/2-1}(-q-1)).$$

Thus the assertion holds.

Theorem 4.7. *Let $k \geq 2n+4$. Let K and f be as above. Assume that the Conjecture B holds for f . Let $\mathfrak{P}$ be a prime ideal of K . Furthermore assume that*

- (1) $\mathfrak{P}$ divides $\mathbf{L}(k, f) \prod_{i=1}^{n/2-1} \mathbf{L}(2i+1, f, \text{St})$.
- (2) $\mathfrak{P}$ does not divide

$$\tilde{\xi}(2m) \prod_{i=1}^n \mathbf{L}(2m+k-i, f) \mathbf{L}(k-n/2, f, \chi_D) D(2k-1)!$$

for some integer $n/2+1 \leq m \leq k/2 - n/2 - 1$, and for some fundamental discriminant D such that $(-1)^{n/2}D > 0$.

Then $\mathfrak{P}$ is a congruence prime of $I_n(f)$ with respect to $\mathbf{C}I_n(f)^\perp$. Furthermore assume that the following condition hold:

(3) $\mathfrak{P}$ does not divide

$$C_{k,n} \frac{\langle f, f \rangle}{\Omega(f, +, A_{\mathfrak{P}}) \Omega(f, -, A_{\mathfrak{P}})},$$

where $C_{k,n} = 1$ or $\prod_{q \leq (2k-n)/12} (1 + q + \cdots + q^{n-1})$ according as $n = 2$ or not.

Then $\mathfrak{P}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{S}_k(\Gamma_n)^*)^\perp$.

Proof. Let $\mathfrak{P}$ be a prime ideal satisfying the condition (1) and (2). For the D above, take a matrix $A \in \mathcal{H}_n(\mathbf{Z})_{>0}$ so that $c_{I_n(f)}(A) = c_{\tilde{f}}(|D|)q$ with q not divisible by $\mathfrak{P}$. Then by Proposition 4.3, we have

$$\begin{aligned} \Lambda(2m, I_n(f), \underline{\text{St}}) |c_{I_n(f)}(A)|^2 &= \Lambda(2m, I_n(f), \underline{\text{St}}) |c_{\tilde{f}}(|D|)|^2 q^2 \\ &= \epsilon_{k,m} \frac{\prod_{i=1}^n \mathbf{L}(2m+k-i, f) |D|^{k-n/2} \mathbf{L}(k-n/2, f, \chi_D)}{\mathbf{L}(k, f) \tilde{\xi}(n) \prod_{i=1}^{n/2-1} \mathbf{L}(2i+1, f, \underline{\text{St}}) \tilde{\xi}(2i)} \\ &\quad \times \left(\frac{\Omega(f, +; \mathfrak{P}) \Omega(f, -; A_{\mathfrak{P}})}{\langle f, f \rangle} \right)^{n/2}, \end{aligned}$$

where $\epsilon_{k,m}$ is a rational number whose numerator is not divided by $\mathfrak{P}$.

We note that $\frac{\langle f, f \rangle}{\Omega(f, +; A_{\mathfrak{P}}) \Omega(f, -; A_{\mathfrak{P}})}$ is $\mathfrak{P}$ -integral. Thus by assumptions (1) and (2), $\mathfrak{P}$ divides $(\Lambda(2m, I_n(f), \underline{\text{St}}) c_{I_n(f)}(A)^2)^{-1}$, and thus it divides $(\Lambda(2m, I_n(f), \underline{\text{St}}) \mathfrak{I}_{I_n(f)}^2)^{-1}$. We note that $I_n(f)$ satisfies the assumption in Theorem 3.1. Thus by Theorem 3.1, there exists a Hecke eigenform $G \in \mathbf{C}(I_n(f))^\perp$ such that

$$\lambda_G(T) \equiv \lambda_{I_n(f)}(T) \pmod{\mathfrak{P}}$$

for any $T \in \mathbf{L}'_n$. Assume that we have $G = I_n(g)$ with some primitive form $g(z) = \sum_{m=1}^\infty a_g(m) \mathbf{e}(mz) \in \mathfrak{S}_{2k-n}(\Gamma^{(1)})$. Let $n = 2$. Then by (1) of Proposition 4.2, $\mathfrak{P}$ is also a congruence prime of f . Let $n \geq 4$. Then by (1) of Proposition 4.4, we have

$$(p^{n-1} + \cdots + p + 1) a_f(p) \equiv (p^{n-1} + \cdots + p + 1) a_g(p) \pmod{\mathfrak{P}}$$

for any prime number p not divisible by $\mathfrak{P}$. By assumption (3), in particular, for any $p \leq (2k-n)/12$, we have

$$a_f(p) \equiv a_g(p) \pmod{\mathfrak{P}}.$$

Thus by Sturm [Stur], $\mathfrak{P}$ is also a congruence prime of f . Thus by [Hi2] and [Ri2], $\mathfrak{P}$ divides $\frac{\langle f, f \rangle}{\Omega(f, +; A_{\mathfrak{P}}) \Omega(f, -; A_{\mathfrak{P}})}$, which contradicts the assumption (3). Thus $\mathfrak{P}$ is a congruence prime of $I_n(f)$ with respect to $(\mathfrak{S}_k(\Gamma^{(n)})^*)^\perp$.

□

Example Let $n = 4$ and $k = 18$. Then we have $\dim S_{18}(\Gamma_4) \approx 16$ (cf. Poor and Yuen[P-Y]) and $\dim S_{18}(\Gamma_4)^* = \dim S_{32}(\Gamma_1) = 2$. Take a primitive form $f \in S_{32}(\Gamma_1)$. Then we have $[\mathbf{Q}(f) : \mathbf{Q}] = 2$, and $211 = \mathfrak{P}\mathfrak{P}'$ in $\mathbf{Q}(f)$. Then we have

$$N_{\mathbf{Q}(f)/\mathbf{Q}}(\mathbf{L}(18, f)) = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 211,$$

$$N_{\mathbf{Q}(f)/\mathbf{Q}}\left(\prod_{i=1}^4 \mathbf{L}(24-i, f)\right) = 2^{19} \cdot 3^{13} \cdot 5^5 \cdot 7^8 \cdot 11^2 \cdot 13^5 \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 503 \cdot 1307 \cdot 14243,$$

$$\tilde{\xi}(6) = 2^{-2} \cdot 3^{-2} \cdot 7^{-1}$$

and

$$N_{\mathbf{Q}(f)/\mathbf{Q}}(\mathbf{L}(16, f, \chi_1)) = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13^2.$$

(cf. Stein [Ste].) Furthermore, by a direct computation we see neither $\mathfrak{P}$ nor $\mathfrak{P}'$ is a congruence prime of $\hat{f}$ with respect to $\mathbf{C}\hat{g}$ for another primitive form $g \in S_{32}(\Gamma_1)$. Thus by Theorem 4.7, $\mathfrak{P}$ or $\mathfrak{P}'$ is a congruence prime of $\hat{f}$ with respect to $S_{18}(\Gamma_4)^{\perp}$.

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