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THE WARPING DEGREE OF A NANOWORD

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Abstract. A. Kawauchi has introduced the notion of warping degrees of knot diagrams and A. Shimizu has given an inequality for warping degrees and crossing number of knot diagrams in the paper [5]. In this paper, we extend the notion of warping degrees and Shimizu's inequality to nanowords. Moreover, to describe the condition for the equality, we introduce the new notion on nanowords, "the alternating nanowards", which corresponds to the alternating knot diagrams.

keywords: knot diagrams, nanowords, warping degree

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1. Introduction.

In this paper, a *knot* is the image of smooth embedding of S^1 into $\mathbb{R}^3$. To study knots, we often use knot diagrams. A *knot diagram* is a smooth immersion of S^1 into $\mathbb{R}^2$ with transversal double points such that the two paths at each double point are assigned to be over path and the under path respectively (we call a double point of such immersion a *crossing*). If a knot diagram D is obtained as the image of a knot by a projection of $\mathbb{R}^3$ to $\mathbb{R}^2$, then we call D a diagram of the knot.

Another approach to study knot is introduced by V. Turaev in [6] and [7] (see also [8]). Turaev introduced the notion of nanowords which is extension of the theory of knots from a view point of Gauss codes. Let α be an alphabet endowed with an involution $\tau : \alpha \to \alpha$. Let $\mathcal{A}$ be an alphabet endowed with a mapping $|\cdot| : \mathcal{A} \to \alpha$ which is called a projection. We call this $\mathcal{A}$ an α-alphabet. Then we call a pair an α-alphabet $\mathcal{A}$ and a word on $\mathcal{A}$ an étale word. If all letters in $\mathcal{A}$ appear exactly twice, then we call this étale word a nanoword. Turaev showed a special case of the theory of nanowords corresponds to the theory of stable homeomorphic theory of knot diagrams on surfaces (see Section 2.2 for more details).

In [4], A. Kawauchi introduced the warping degree of a knot diagram. Furthermore, in [5], A. Shimizu studied the warping degree of a knot diagram and showed some inequality on the warping degree and the crossing number of a knot diagram and characterized a condition for equality to hold.

In this paper, we extend Shimizu's inequality to nanowords (which contain the theory of knot diagrams on surfaces). We define the warping degree of a nanoword and show a inequality on the warping degree of a nanoword and the number of letters in a nanoword. Moreover we characterize nanowords which satisfy the equality in the inequality. To do this we introduce a new notion which corresponds to alternating knot diagrams (we call this notion alternating nanowords).

The rest of this paper organized as follows. In Section 2, we review the definition of étale words and nanowords. Moreover we discuss the relation between knot diagrams on surfaces and nanowords. In Section 3, we review the definition of the warping degree of a knot diagrams and introduce Shimizu's inequality. In Section 4, we define the warping degree of a nanoword and show a inequality which is generalization of Shimizu's inequality. In Section 5, we introduce a new notion which is called alternating nanowords and show a necessary and sufficient condition for equality to hold is the nanoword is an alternating nanoword.

2. Étale words and nanowords.

In this section, we introduce the notion of nanowords which was defined by V. Turaev in the paper [6] (See also [7] and [8]). Furthermore, we introduce a relation between nanowords and knot diagrams on surfaces (See [7] for more details).

2.1. **The Definition of Étale Words and Nanowords.** Throughout this paper an *alphabet* is a finite set and *letters* means its element. A *word* of length n on an alphabet $\mathcal{A}$ is a mapping $w : \hat{n} \to \mathcal{A}$ where $\hat{n} := \{1, 2, \cdots n\}$. A multiplicity of a letter $A \in \mathcal{A}$ in a word w on $\mathcal{A}$ is a number of A in the word w. We denote multiplicity of $A \in \mathcal{A}$ by $m_w(A)$. Let α be an alphabet endowed with an involution $\tau : \alpha \to \alpha$. An *α-alphabet* is a pair (An alphabet $\mathcal{A}$, mapping $|\cdot| : \mathcal{A} \to \alpha$). We call the mapping $|\cdot|$ *projection*.

In the paper [6], V. Turaev defined generalized words which is called étale words. An *étale word* over α is a pair (An α-alphabet $\mathcal{A}$, A word on $\mathcal{A}$). A words w on an α gives rise to an étale phrase (α, w) where the projection $\alpha \to \alpha$ is the identity mapping. In this meaning étale words are generalization of usual words.

Next we define nanowords. To define nanowords, we remember the definition of Gauss words. A *Gauss word* on an alphabet $\mathcal{A}$ is a word w on $\mathcal{A}$ which all letters in $\mathcal{A}$ appear exactly twice in w.

A *nanoword* over α is a pair (An α-alphabet $\mathcal{A}$, A Gauss word on $\mathcal{A}$). Instead of writing $(\mathcal{A}, w)$ for a nanoword over α, we often write simply w. The alphabet $\mathcal{A}$ can be uniquely recovered. However the projection should be always specified.

For a nanoword $w = A_1 A_2 \cdots A_n$, we define *inverse word* w^- by $\overline{A_n A_{n-1}} \cdots \overline{A_1}$ with $|\overline{A_i}|$ is equal to $\nu(|A_i|)$ for all $i \in \{1, \cdots n\}$.

2.2. **Knot Diagrams on Surfaces and Nanowords.** In this subsection, we review correspondence of knot diagrams on surfaces and nanowords.

Let α_* be a four element set $\{a_+, a_-, b_+, b_-\}$. We consider an involution ν_* on α_* which is defined by $\nu_*(a_\pm)$ is equal to $b_\pm$.

First, we consider the method of making a nanoword $w(D_a)$ over α_* from a pointed knot diagram D_a on surface. Let us label the double points of D_a by distinct letters $A_1, \cdots, A_n$. Starting at the base point a of D_a and following along D_a in the positive direction, we write down the labels of double points which we passes until the return to the base point. Then we obtain a word w on the alphabet $\mathcal{A} = \{A_1, \cdots, A_n\}$. For a crossing A of a pointed diagram D_a, we denote sign of A by $\epsilon(A)$. Let t_i^1 (respectively, t_i^2) be the tangent vector to D_a at the double point labeled A_i appearing at the first (respectively, second) passage through this point.

Set $|A_i| = a_{\epsilon(A_i)}$, if the pair (t_i^1, t_i^2) is positively oriented, and $|A_i| = b_{\epsilon(A_i)}$ otherwise. Then we obtain a required nanoword $w(D_a) := (\mathcal{A}, w)$.

Next we consider the shift move for nanowords over α with involution ν. For a nanoword w over α, the *ν-shift move* is defined as follows :

$$w = A_1 A_2 A_3 \cdots A_n \longrightarrow A_2 A_3 \cdots A_n \overline{A_1},$$

where $|\overline{A_1}|$ is defined by $\nu(|A_1|)$.

Then we obtain following theorem.

Theorem 2.1 (cf. Turaev [7])**.** *$\{$nanowords over $\alpha_*\}/(\nu_* -$ shift move$)$ is one to one corresponds to $\{$knot diagrams on surfaces$\}/($stably homeomorphism$)$.*

Remark 2.1. The theory of stably equivalence of knot diagrams is equivalent to the theory of virtual knots. See [1] for more details.

3. The warping degree of a knot diagram and Shimizu's inequality.

In this section, we review the warping degree of a knot diagram which was defined in [4] and Shimizu's result on the warping degree of a knot diagram [5].

Let D be a knot diagram. A *pointed knot diagram* is a knot diagram D endowed with a point $a \in D$ which is not double point of D. We denote this pointed knot diagram D_a. A crossing point of D_a is a *warping crossing point* if we meet the point first at the under-crossing when we walk along the orientation of D by starting from a. Now we define the warping degree of a knot diagram. First we define the warping degree of a pointed knot diagram. The *warping degree of a pointed knot diagram* D_a is the number of warping crossing points of D_a (we denote this number $d(D_a)$). Then we define the *warping degree of a knot diagram* D by minimal warping degree for all base points of D (we denote this $d(D)$).

In the paper [5], Shimizu proved a following inequality.

Theorem 3.1 (A.Shimizu [5])**.** *Let D be an oriented knot diagram which has at least one crossing point. Then we have the following inequality:*

$$d(D) + d(-D) + 1 \leq c(D)$$

where $-D$ is an orientation reverse knot diagram of D, and $c(D)$ is a crossing number of D. Further, the equality holds if and only if D is an alternating diagram.

In this paper, we extend this result to nanowords.

4. The warping degree of a nanoword.

In this section we extend the definition of warping degree of a knot diagram to a nanoword.

4.1. **Definition of the Warping Degree of a Nanoword.** Let α be an alphabet endowed with involution ν. we fix a *ν-orientation* (complete representative system of α/ν) $\{a_1, \cdots, a_l, a_{l+1}, \cdots, a_n\}$ such that $\nu(a_i)$ is not equal to a_i for all $i \in \{1, \cdots, l\}$ and $\nu(a_i)$ is equal to a_i for all $i \in \{l+1, \cdots, n\}$. We denote a fixed ν-orientation by $\alpha_{\nu\text{-ori}}$.

Now we define the warping degree of a nanoword. For a letter $A \in \mathcal{A}$, A is a *warping letter* if $|A|$ is an element of $\alpha_{\nu\text{-ori}}$.

Definition 4.1. Let $(\mathcal{A}, w)$ be a nanoword over α and I be a subset of $\{1, \cdots, n\}$. We define the I-warping degree $d(w)_I$ by

$$d(w)_I = \sharp\{A \in \mathcal{A} || A| \in a_i \text{ for some } i \in I\}.$$

Let $\hat{a_j}$ be a set $\{a_j, \nu(a_j)\}$. Consider a nanoword $(\mathcal{A}_w, w)$. For $A \in \mathcal{A}_w$, it is trivial that $|A|_{\mathcal{A}_w}$ is equal to a_i for some $i \in \hat{n}$ or $|A|_{\mathcal{A}_w}$ is equal to $\nu(a_i)$ for some $i \in \hat{l}$ (note that a_i is not equal to $\nu(a_i)$ for all $i \in \hat{l}$). Moreover for a letter $A \in \mathcal{A}_w$ such that $\widehat{|A|_{\mathcal{A}_w}}$ is equal to $\hat{a_i}$ for some $i \in \hat{l}$, if $|A|_{\mathcal{A}_w}$ is not equal to a_i, then $|A|_{\mathcal{A}_{w^-}}$ is equal to a_i. Therefore if $\widehat{|A|_{\mathcal{A}_w}}$ is equal to $\hat{a_i}$ for some $i \in \hat{l}$, then $A \in \mathcal{A}_w$ contributes either $d(w)_I$ or $d(w^-)_I$ for all $I \subset \hat{l}$ (note that $\mathcal{A}_w$ is equal to $\mathcal{A}_{w^-}$ as sets). By the above we obtain a following lemma.

Lemma 4.1. *Let w be a nanoword over α. Let $\mathcal{A}_I$ be a set of letters which satisfy $|A|$ is equal to $\hat{a_i}$ for some $i \in I$. If I is a subset of $\hat{l}$, then*

$$d(w)_I + d(w^-)_I = \sharp\mathcal{A}_I.$$

Let w be a nanoword over α. We denote a ν-shift move equivalent class by $[w]$. Further we denote a nanoword which obtained from w by k times ν-shift move by w_{ν^k} (for negative integer k, notation w_{ν^k} means $-k$ times inverse ν-shift move).

Remark 4.1. Note that by definition of ν-shift move, we obtain $w_{\nu^{2\sharp\mathcal{A}}}$ is equal to w.

Lemma 4.2. *Let $(\mathcal{A}_w, w)$ be a nanoword over α. If there is a letter $A \in \mathcal{A}_w$ with $|A| \in a_i$ for some $i \in I \subset \hat{l}$. Then*

$$\max_n d(w_{\nu^n})_I - \min_n d(w_{\nu^n})_I \geq 1.$$

Proof. Without loss of generality, we can assume $w{=}w(1)w(2)\cdots w(n)$ with $|w(i)|$ is equal to a_i for some $i \in I \subset \hat{l}$. Then $w_\nu{=}w(2)w(3)\cdots w(n)\overline{w(1)}$ with $|\overline{w(1)}| = \nu(|w(1)|) \neq |w(1)|$ and $|w(i)| \notin a_i$. Hence $d(w_\nu)_I$ is equal to $d(w)_I - 1$. □

Now we describe the main theorem. Let $d([w])_I$ be equal to $\min_n\{d(w_{\nu^n})_I\}$. Then a following theorem hold

Theorem 4.1. *Let $(\mathcal{A}, w)$ be a nanoword over α. Suppose I is a subset of $\hat{l}$ Then*

$$d([w])_I + d([w^-])_I + 1 \leq \sharp\mathcal{A}_I$$

where $\sharp\mathcal{A}_I$ is equal to $\sharp\{A \in \mathcal{A} || A| \in \widehat{a_i} \text{ for some } i \in I\}$.

Proof. Let n_I and m_I be positive integers which satisfy $d([w])_I$ is equal to $d(w_{\nu^{n_I}})_I$ and $d([w^-])_I$ is equal to $d(w^-_{\nu_{m_I}})_I$. Note that $\max_n d(w_{\nu^n})_I$ is equal to $d(w_{\nu^{m_I}})_I$. Therefore

$$\max_n d(w_{\nu^n})_I - \max_m d(w_{\nu^m})_I \geq 1$$

and this equivalent to

$$d(w_{\nu^{m_I}})_I - d(w_{\nu^{n_I}})_I \geq 1.$$

Hence

$$d(w_{\nu^{n_I}})_I + 1 \leq d(w_{\nu^{m_I}})_I.$$

By adding $d(w^-_{\nu^{m_I}})_I$ to each side

$$d(w_{\nu^{n_I}})_I + d(w^-_{\nu^{m_I}})_I + 1 \leq d(w_{\nu^{m_I}})_I + d(w^-_{\nu^{m_I}})_I = \sharp\mathcal{A}_i.$$

Then we obtain

$$d([w])_I + d([w^-])_I + 1 \leq \sharp\mathcal{A}_I$$

□

Remark 4.2. A condition for equality to hold is discussed in the next section.

Remark 4.3. If we put α is equal to α_*, ν is equal to ν_* and I is equal to $\{a_-, b_+\}$, then we obtain Theorem 3.1.

5. A condition for equality to hold.

In this section we describe a necessary and sufficient condition for equality in Theorem 4.1 hold. To do this, we introduce a new notion which is correspond to alternating knot diagrams.

5.1. **Definition of Alternating Nanowords.** For a nanoword over α, $(\mathcal{A}, w)$ we define a notation $\overline{w(i)}$ for each $i \in \widehat{\text{length}(w)}$ as follows: Suppose $w(i)$ is equal to $w(j)$ for $i < j$. If $w(i) \in \alpha_{\nu\text{-ori}}$, then we put overline on $w(i)$ and do nothing to $w(j)$. If $w(i) \notin \alpha_{\nu\text{-ori}}$, then we put overline on $w(j)$ and do nothing to $w(i)$. Now we define an alternating nanoword.

Definition 5.1. For nanoword $(\mathcal{A}, w)$ over α, we call $(\mathcal{A}, w)$ *an alternating nanoword* if

$$w = \overline{w(1)}w(2)\overline{w(3)}w(4)\cdots\overline{w(n-1)}w(n)$$

or

$$w = w(1)\overline{w(2)}w(3)\overline{w(4)}\cdots w(n-1)\overline{w(n)}.$$

Moreover $(\mathcal{A}, w)$ is *I-alternating nanoword* if $\mathcal{U}_I(w)$ is alternating nanoword where $\mathcal{U}_I(w)$ is a nanoword which is obtained by deleting all letters $A \in \mathcal{A}$ such that $|A| \notin I \cup \nu(I)$ from both $\mathcal{A}$ and w.

Remark 5.1. Consider a nanoword $w = \overline{w(1)}w(2)\cdots w(l)\cdots w(n)$ such that $w(1)$ is equal to $w(l)$. Then $w_\nu = w(2)\cdots w(l)\cdots w(n)\overline{w(1)}$ since $|w_\nu(l)| = |w_\nu(n)| = \nu(|w(1)|) \notin \alpha_{\nu\text{-ori}}$.

Remark 5.2. Let i_A (respectively j_A) be $\min\{i \in \mathbb{N} | w(i) = A\}$ (respectively $\max\{i \in \mathbb{N} | w(i) = A\}$). Then $|w(i_A)| \in \alpha_{\nu\text{-ori}}$ if and only if $w(i_A)$ has overline. Hence $d(w)_I$ is equal to $\sharp\{A \in \mathcal{A}_I | w(i_A) \text{ has overline}\}$. Therefore, if $w(1)$ has overline, then $d(w)_I$ is equal to $d(w)_I - 1$ by Remark 5.1. Similarly if $w(1)$ does not have overline, then $d(w)_I$ is equal to $d(w)_I + 1$.

Now we describe a necessary and sufficient condition for equality in Theorem 4.1 holds.

Theorem 5.1. *The equality in Theorem 4.1 holds, in other words*

$$d([w])_I + d([w^-])_I = \sharp\mathcal{A}_I,$$

if and only if w is I-alternating nanoword.

Proof. By Remarks 5.1 and 5.2, if w is an I-alternating nanoword, then $d(w)_I$ is equal to $d([w])_I$ if and only if $d(w^-)_I$ is equal to $d([w^-])_I + 1$. Similarly $d(w^-)_I$ is equal to $d([w^-])_I$ if and only if $d(w)_I$ is equal to $d([w])_I + 1$. By the above and equation $d(w)_I + d(w^-)_I$ is equal to $\sharp\mathcal{A}_I$, we obtain

$$d([w])_I + d([w^-])_I = \sharp\mathcal{A}_I.$$

Therefore if w is an I-alternating nanoword, then equality in Theorem 4.1 holds.

Conversely, suppose the equality in Theorem 4.1 holds. Assume w is not I-alternating, then there exist a positive integer n such that $w_{\nu^n} = \overline{w(1)}\cdots\overline{w(l)}\cdots$ where $\widehat{|w(1)|} = \widehat{|w(l)|} = \widehat{a_i}$ for some $i \in I$, $|w(2)|, \cdots |w(l-1)| \notin \widehat{a_i}$ for all $i \in I$. Therefore $d(w_{\nu^n})_I$ is equal to $d(w_{\nu^{n+2}})_I$. Moreover

$$\begin{aligned} \sharp\mathcal{A}_I &= d(w_{\nu^n})_I + d(w^-_{\nu^n})_I, \\ &> d(w_{\nu^{n+2}})_I + 1 + d(w^-_{\nu^n})_I, \\ &\geq d([w])_I + d([w^-])_I + 1. \end{aligned}$$

Note that $(w_{\nu^n})^-$ is equal to $(w^-)_{\nu^{-n}}$. Therefore if w is not an alternating nanoword, the equality in Theorem 4.1 does not hold.

Now we completed the proof. □

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