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AN APPROACH TO ROTATING BOUNDARY LAYERS BASED ON VECTOR RADON MEASURES

YOSHIKAZU GIGA
Graduate School of Mathematical Sciences, University of Tokyo
Komaba 3-8-1 Meguro, Tokyo 153-8914, Japan
labgiga@ms.u-tokyo.ac.jp

JÜRGEN SAAL
(corresponding author)
Center of Smart Interfaces, TU Darmstadt
Petersenstrasse 32, 64287 Darmstadt, Germany
Phone: +49 (0)6151 16-5415
Fax: +49 (0)6151 16-72022
saal@csi.tu-darmstadt.de

ABSTRACT. In this paper we develop a new approach to rotating boundary layers via Fourier transformed finite vector Radon measures. As an application we consider the Ekman boundary layer. By our methods we can derive very explicit bounds for existence intervals and solutions of the linearized and the nonlinear Ekman system. For example, we can prove these bounds to be uniform with respect to the angular velocity of rotation which has proved to be relevant for several aspects (see introduction). Another advantage of our approach is that we obtain well-posedness in classes containing nondecaying vector fields such as almost periodic functions. These outcomes give respect to the nature of boundary layer problems and cannot be obtained by approaches in standard function spaces such as Lebesgue, Bessel-potential, Hölder or Besov spaces.

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[2]*Key words and phrases.* Vector-valued Radon measures, nondecaying functions, operator-valued multipliers, rotating boundary layers, uniform well-posedness, Ekman layer

1. Introduction and main results

The main purpose of this note is the development of a new approach to rotating boundary layer problems. The method is based on the introduction and the establishment of an operator theory on spaces of Fourier transformed finite vector-valued Radon measures. An essential advantage in dealing with Fourier transformed quantities lies in the fact that all performed calculations and estimations become rather explicit. As a consequence we can derive detailed information on how the solution depends on involved parameters such as time, viscosity, layer thickness, and angular velocity of rotation.

It is also typical for geostrophic boundary layer problems that existing stationary solutions are nondecaying and oscillating in tangential direction. To give respect to this fact, it seems natural to consider this type boundary layer problems in classes containing nondecaying functions. Hence, the frequently performed L^p approach for $1 < p < \infty$ to the corresponding mathematical models fails in this situation. The spaces introduced here, however, include nondecaying - in particular almost periodic - functions.

By an application to the Ekman boundary layer problem we demonstrate the strengths of our theory. Mathematically this geophysical problem is modeled by the system

$$\left\{\begin{array}{rcll} \partial_t v - \nu\Delta v + \omega \mathbf{e}_3 \times v + (v\cdot\nabla)v &=& -\nabla q & \text{in } (0,T)\times G,\\ \text{div } v &=& 0 & \text{in } (0,T)\times G,\\ v &=& \mathbf{U}^E|_{\partial G} & \text{on } (0,T)\times \partial G,\\ v|_{t=0} &=& v_0 & \text{in } G, \end{array}\right. \tag{1.1}$$

which represents the 3D Navier-Stokes equations with Coriolis force. Here $\mathbf{e}_3 = (0,0,1)^T$, $\nu > 0$ is the viscosity coefficient, and $\omega \in \mathbb{R}$ is the Coriolis parameter which equals twice the angular velocity of rotation. For G we will consider simultaneously the half-space $\mathbb{R}^3_+$ or a layer, i.e., we have $G = \mathbb{R}^2 \times D$ with $D = (0,d)$ and either fixed $d \in (0,\infty)$ or $d = \infty$. The vector field $\mathbf{U}^E$ is the so-called Ekman spiral (introduced by the geophysicist V.W. Ekman [17]) given as

$$\mathbf{U}^E(x_3) = U_\infty(1 - e^{-x_3/\delta}\cos(x_3/\delta),\ e^{-x_3/\delta}\sin(x_3/\delta),\ 0)^T, \quad x_3 \geq 0. \tag{1.2}$$

System (1.1) is known to be a well-established model for the layer arising in a rotating system (e.g. the earth) between a straight geostrophic flow (e.g. wind) and the surface on which the no slip condition is imposed. Observe that in the above model rotation about the x_3-axis is assumed, whereas U_∞ denotes the total velocity of the flow, blowing in direction of the x_1-axis. The parameter δ denotes the layer thickness given by $\delta = \sqrt{2\nu/|\omega|}$. By geostrophic approximation (see [37]) (1.1) is a reasonable model at least for the upper part of the northern hemisphere. The couple $(\mathbf{U}^E, p^E)$ with pressure

$$p^E(x_2) = -\omega U_\infty x_2$$

represents a stationary solution of system (1.1). Observe that $\mathbf{U}^E(0) = 0$, i.e. sytem (1.1) is subject to Dirichlet conditions at the lower boundary, and that $\mathbf{U}^E$

is oscillating and nondecaying in tangential direction. We note that remarkable persistent stability of $\mathbf{U}^E$ is observed in geophysical literature.

Mathematically, an approach to stability in time is given in [13] and to asymptotic stability in [31]. These two papers consider the problem in the L^2 setting which, of course, does not include nondecaying perturbations of $\mathbf{U}^E$. We refer to [30], [10], [11], and [38] for more mathematical literature on the Ekman problem dealing also with vanishing Rossby and Ekman numbers. For a spectral analysis of the linearized problem we refer to [28], [34]. Local-in-time well-posedness in the homogeneous L^p-valued Besov space $\dot{B}^0_{\infty,1}(\mathbb{R}^2, L^p(\mathbb{R}_+))$ is obtained in [25]. By the fact that almost periodic functions are contained in $\dot{B}^0_{\infty,1}(\mathbb{R}^2)$, this represents the first result in a space including nondecaying functions. On the other hand, the space $\dot{B}^0_{\infty,1}(\mathbb{R}^2, L^p(\mathbb{R}_+))$ turned out to be not very useful concerning stability investigations. In fact, the semigroup corresponding to the linearized equations is expected to be increasing in time and in ω in the space $\dot{B}^0_{\infty,1}(\mathbb{R}^2, L^p(\mathbb{R}_+))$.

This is underlined by the following observation. Roughly speaking, the part of the linear solution operator coming from the Coriolis force is given by a group which is called Poincaré-Riesz group or occasionally also Poincaré-Sobolev group. Its symbol consists essentially of functions as

$$m(\xi) = e^{-it\omega\xi_j/|\xi|}, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.3}$$

In the L^2 setting, by Plancherel's theorem, the uniform boundedness of m guarantees the uniform boundedness in t and ω of the Poincaré-Riesz semigroup and hence of the full semigroup associated to the linearization of (1.1). This fact, which relies on the skew symmetry of the Coriolis force, is used in [13] and [31]. However, the application of typical multiplier conditions involving e.g. derivatives of m, cause a growth in t and ω. Thus, stability in these two parameters is not expected in standard function spaces not isomorphic to a Hilbert space such as L^p for $p \neq 2$ or $\dot{B}^0_{\infty,1}(\mathbb{R}^2, L^p(\mathbb{R}_+))$. In fact, for the case of L^p, $p \neq 2$, polynomial growth in ω of the Poincaré-Sobolev group is explicitly derived in [16].

The uniformness in ω of appearing quantities such as an existence interval or a bound for solutions, however, is interesting for several aspects. For instance, it is important for the investigation of statistical properties of turbulence as it is demonstrated in the textbooks [36] and [44]. It also represents the basis for the examination of rapidly oscillating limits as $\omega \to \infty$. In a series of papers [5], [6], [7], [33] Babin, Mahalov, and Nicolaenko proved the striking result of global-in-time regularization of a flow in periodic domains, if the rotation is sufficiently fast. This also represents the first rigorous mathematical verification of the Taylor-Proudman theorem, the physical principle behind that phenomenon. For an alternative proof in $\mathbb{R}^n$ based on dispersive effects, see also [11]. We also refer to the classical books [29] and [37] for an introduction to rotating fluids in geophysics, and to the nice monograph [11] for an introduction from the mathematical point of view. The results obtained by Babin, Mahalov, and Nicolaenko are not only mathematically of great interest. They could also play a significant role in applied situations. This is justified by the fact that in applications the angular velocity of rotation is often much higher than other appearing parameters. This is true in geophysical situations, e.g.

for the rotating earth, but also in technological applications such as the spin-coating-process, cf. [12]. Furthermore, a rather explicit knowledge on the dependence of the norm of solutions on the appearing parameters time t, viscosity ν, layer thickness δ, and angular velocity $\omega/2$ could also help to improve numerical codes used for simulations of the Ekman layer.

The discussion above demonstrates the importance of two requirements that we want the solution of a rotating boundary layer problem to satisfy in this note:

(1) almost periodic perturbations of stationary solutions should be included, i.e., the ground space should contain a sufficiently large class of nondecaying functions.
(2) rather explicit knowledge on the dependence of appearing quantities on the involved parameters is desired. In particular, an existence time interval or a bound for the solution should be uniform in the angular velocity of rotation ω.

In standard spaces, however, apparently at least one of the requirements cannot be satisfied. For example, in L^2 or $\dot{B}^0_{2,2}$ we can fulfill (2), but (1) fails; in $\dot{B}^0_{\infty,1}$, L^∞, BUC, or C^α we can achieve (1), but (2) seems to fail; in L^p for $p \neq 2$ even both conditions cannot be satisfied. This is the reason for the development of the theory presented in this note. In order to prove that we can indeed satisfy both requirements, next we formulate our main results on the Ekman boundary layer problem (1.1). For a rigorous definition of the appearing spaces we refer to Sections 2 and 3. Let $\mathrm{M}_0(\mathbb{R}^2, L^2(D)^3)$ denote the space of finite $L^2(D)^3$-valued Radon measures with no point mass at the origin. We set

$$\mathrm{FM}_0(\mathbb{R}^2, L^2(D)^3) := \{\mathcal{F}\mu : \ \mu \in \mathrm{M}_0(\mathbb{R}^2, L^2(D)^3)\}$$

and equip it with its canonical norm. It can be shown that $\mathrm{FM}_0(\mathbb{R}^2, L^2(D)^3) \subset \mathrm{BUC}(\mathbb{R}^2, L^2(D)^3)$ (see Lemma 2.12(iii)), hence the Fourier transform $\mathcal{F}\mu$ is well-defined. Moreover, the Helmholtz projection P is bounded on $\mathrm{FM}_0(\mathbb{R}^2, L^2(D)^3)$ (see Lemma 3.4). Thus we may define its solenoidal part as

$$\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(D)^3) := P(\mathrm{FM}_0(\mathbb{R}^2, L^2(D)^3)).$$

Next, we set $u_0 := v_0 - \mathbf{U}^E$, $u = v - \mathbf{U}^E$, and $p := q - p^E$. Then (v, q) solves (1.1) if and only if (u, p) solves the transformed system

$$\begin{aligned} \partial_t u - \nu\Delta u + \omega \mathbf{e}_3 \times u + (\mathbf{U}^E \cdot \nabla)u + u^3 \partial_3 \mathbf{U}^E + (u \cdot \nabla)u &= -\nabla p && \text{in } (0,T) \times G, \\ \operatorname{div} u &= 0 && \text{in } (0,T) \times G, \\ u &= 0 && \text{on } (0,T) \times \partial G, \\ u|_{t=0} &= u_0 && \text{in } G. \end{aligned} \tag{1.4}$$

The Stokes-Coriolis-Ekman operator A_{SCE} is defined as the full linear operator of the linearized Cauchy problem associated to (1.4) (see Section 3 for a rigorous definition). For A_{SCE} we prove the following theorem, which is our main result for the linearized Ekman problem (see also Theorem 3.10).

Theorem 1.1. *Let $T \in (0,\infty)$, $\nu, U_\infty > 0$, $d \in (0,\infty]$, and set $X_2 := L^2(0,d)^3$. Then A_{SCE} is the generator of a holomorphic C_0-semigroup $T_{SCE}(t) = e^{-tA_{SCE}}$ on $\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)$ satisfying the following estimates:*

(i) $\|T_{SCE}(t)\|_{\mathscr{L}(\mathrm{FM}_0(\mathbb{R}^2,X_2))} \le \exp(tU_\infty^2/8\nu) \quad (t \ge 0)$,

(ii) $\|\nabla T_{SCE} u_0\|_{L^2((0,T),\mathrm{FM}_0(\mathbb{R}^2,X_2))} \le \sqrt{2/\nu}\, \exp(TU_\infty^2/8\nu)\|u_0\|_{\mathrm{FM}_0(\mathbb{R}^2,X_2)}$,

for all $u_0 \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)$. In particular, all estimates are uniform in $\omega \in \mathbb{R}$.

Based on Theorem 1.1 and a fixed point argument, in Section 4 we derive the following main result for the full nonlinear Ekman problem (1.1).

Theorem 1.2. *Let the assumptions of Theorem 1.1 be satisfied and let $\mathbf{U}^E$ be the Ekman spiral given in (1.2). Then for every $v_0 \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2) + \mathbf{U}^E$ there is a $T_0 > 0$ and a unique (mild) solution v of (1.1) satisfying*

$$\begin{aligned} v - \mathbf{U}^E &\in \mathrm{BC}\left((0,T_0),\, \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)\right), \\ \nabla(v - \mathbf{U}^E) &\in L^2\left((0,T_0),\, \mathrm{FM}_0(\mathbb{R}^2, X_2)\right). \end{aligned}$$

Additionally, the existence time T_0 can be estimated from below by

$$T_0 > T^* := \min\left\{ \frac{\pi^4 \nu^3}{2 \cdot 48^4 \exp(U_\infty^2/\nu)\|v_0 - \mathbf{U}^E\|^4_{\mathrm{FM}_0(\mathbb{R}^2,X_2)}},\ 1 \right\},$$

and for $T \le T^*$ the solution can be estimated from above as

$$\begin{aligned} \|v - \mathbf{U}^E\|_{L^\infty((0,T),\mathrm{FM}_0(\mathbb{R}^2,X_2))} &\le 4\exp(U_\infty^2/8\nu)\|v_0 - \mathbf{U}^E\|_{\mathrm{FM}_0(\mathbb{R}^2,X_2)}, \\ \|\nabla(v - \mathbf{U}^E)\|_{L^2((0,T),\mathrm{FM}_0(\mathbb{R}^2,X_2))} &\le 4\sqrt{\frac{2}{\nu}}\exp(U_\infty^2/8\nu)\|v_0 - \mathbf{U}^E\|_{\mathrm{FM}_0(\mathbb{R}^2,X_2)}. \end{aligned}$$

In particular, all estimates above are uniform in $\omega \in \mathbb{R}$, i.e., with respect to the angular velocity of rotation.

Remark 1.3. (a) It is not difficult to show that $\mathrm{FM}_0(\mathbb{R}^2, L^2(D)^3)$ contains vector fields which are almost periodic in tangential direction. In particular, it includes functions of the form

$$x \mapsto \sum_{j=1}^{\infty} a_j \mathrm{e}^{-i\lambda_j \cdot x}, \quad x \in \mathbb{R}^2,$$

where $(\lambda_j)_{j\in\mathbb{N}} \subset \mathbb{R}^2 \setminus \{0\}$ denotes the sequence of frequencies and where $(a_j)_{j\in\mathbb{N}} \subset L^2(D)^3$ satisfies $\sum_{j=1}^{\infty} \|a_j\|_{L^2(D)^3} < \infty$. This shows that Theorem 1.1 and Theorem 1.2 satisfy our requirements (1) and (2).

(b) Applying iteratively derivatives to the mild formulation (4.1) of problem (1.1), it can be proved that the solution v given by Theorem 1.2 enjoys higher regularity in $\mathrm{FM}_0(\mathbb{R}^2, X_2)$. By this fact we can recover the pressure via

$$\nabla q = (I - P)(\nu\Delta v - \omega \mathbf{e}_3 \times v - (v \cdot \nabla)v).$$

Then it can be shown that

$$(v, q) \in C^\infty((0,T) \times \mathbb{R}^2 \times (0,d)),$$

i.e., (v, q) is the unique classical solution of problem (1.1). However, we will not carry out this standard procedure here. This result will be included in a forthcoming work.

(c) We note that the applicability of the theory developed here is by far not limited to local-in-time well-posedness. In the same forthcoming work we will prove a global-in-time existence result as well as exponential stability for the Ekman spiral. Also in this context we will derive precise estimates and uniformness in ω.

(d) The power 4 of the norm of $v_0 - \mathbf{U}^E$ in the estimate for T_0 is natural from scaling point of view. For local existence the main term of (1.4) is the Navier-Stokes part. In the classical Navier-Stokes equations (i.e. (1.4) with $\omega = 0$, $\mathbf{U}^E = 0$), if (u, p) is a solution in $\mathbb{R}^n \times (0, \infty)$, so is (u_k, p_k) with $u_k(x, t) = ku(kx, k^2t)$, $p_k(x, t) = k^2p(kx, k^2t)$, $k > 0$. To reflect this invariance one assigns scaling dimensions as given in [9], [21]. For example we assign dimension 2 to time variable and dimension 1 to spatial variable. We assign dimension -1 to the velocity field. Checking the dimension of the norm of $v_0 - \mathbf{U}^E$ in Theorem 1.2, it has the scaling dimension $-1/2$ while T_0 has the scaling dimension 2, so in the estimate both hand sides are balanced.

The idea to use the space of Fourier transformed Radon measures for the treatment of the Navier-Stokes equations with Coriolis force, first appeared in [22]. There the local-in-time well-posedness in the whole space $\mathbb{R}^3$, i.e. in the class $\mathrm{FM}_0(\mathbb{R}^3) := \mathrm{FM}_0(\mathbb{R}^3, \mathbb{R}^3)$, uniformly in ω for the system

$$\begin{aligned} \partial_t v - \nu\Delta v + \omega \mathbf{e}_3 \times v + (v \cdot \nabla)v &= -\nabla q && \text{in } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} v &= 0 && \text{in } (0, T) \times \mathbb{R}^3, \\ v|_{t=0} &= v_0 && \text{in } \mathbb{R}^3 \end{aligned}$$

is proved. Global-in-time well-posedness and stability results in $\mathrm{FM}_0(\mathbb{R}^3)$ are derived in [23] and [24]. Moreover, in [26] it is proved that almost periodicity in space is preserved if it is initially almost periodic.

We note that the situation in $\mathbb{R}^3$ is much easier to handle. Then one can work in the setting of standard finite nonnegative Radon measures. It seems to be difficult to stay completely in the FM setting, when a boundary is present. In order to handle the case of a half-space or a layer, therefore we developed the vector-valued approach given here.

A crucial advantage in dealing with spaces of Fourier transformed quantities lies in the fact that, concerning multiplier results, we can obtain a situation similar to L^2. Indeed, merely boundedness and continuity is required in order to turn a symbol into a multiplier (see Proposition 2.13, or e.g. [22, Lemma 2.2]). As seen from the discussion before and after (1.3) this preserves the important uniform boundedness in ω of related solution operators. In fact, we even obtain (scalar- or operator-valued) a relation as

$$\|op(m)\|_{\mathscr{L}(\mathrm{FM})} = \|m\|_\infty = \|op(m)\|_{\mathscr{L}(L^2)} \tag{1.5}$$

for the associated operator formally given by $op(m) = \mathcal{F}^{-1}m\mathcal{F}$ with $\mathcal{F}$ the Fourier transformation. This shows that the boundedness of an operator in FM and L^2 is equivalent, if it has a bounded and continuous symbol. This is a very remarkable

property of the space FM, in particular by the fact that it contains nondecaying functions, but L^2 does not.

By this fact, the space FM also turns out to be very interesting for investigations on instability. A typical ansatz in order to prove instability for large Reynolds numbers is to decompose the solution into a sum of wave functions. Inserting this ansatz into the equations the problem reduces to an ODE for the corresponding Eigenvalues and Eigenvectors of the waves producing instability. At least numerically these problems in many cases can be solved; we refer to [32] for a numerical proof of linear instability for the Ekman spiral. For an approach to nonlinear instability in L^2 based on [32] we refer to [14]. A problem with the approach in L^2 is that the unstable waves are non-decaying, hence do not belong to L^2 and they have to be realized as approximate Eigenfunctions. However, they belong to FM. This means in contrast to the situation in L^2, in FM the unstable Eigenvalues belong to the point spectrum and the corresponding unstable Eigenfunctions instantly prove instability in FM. But then, due to equality (1.5), we immediately obtain linear instability in L^2 as well. Based on the results in [32] by this method a relatively short proof of linear and nonlinear instability for the Ekman problem is performed in [18]. This exhibits another nice application of the theory developed here and again demonstrates its valuability in the treatment of rotating boundary layer problems.

Apart from the properties mentioned above, the space FM displays a couple of further remarkable mathematical properties, for instance, concerning maximal regularity. Indeed, the negative Laplacian $-\Delta$ can be proved to have L^1 maximal regularity on $\mathrm{FM}(\mathbb{R}^n)$, cf. [27]. This is noteworthy, since for L^1 maximal regularity no higher regularity for the initial value is required (note that $-\Delta$ does not even have L^1 maximal regularity on $L^2(\mathbb{R}^n)$). However, we do not use these further properties in this note.

The paper is organized as follows. After the introduction in the current section, in Section 2 we first recall some basic facts on vector Radon measures. Then we establish systematically a theory for operator-valued symbols on spaces of vector Radon measures. Accordingly, this leads to an operator-valued Fourier multiplier theory on spaces of Fourier transformed finite Banach space valued Radon measures. Furthermore, we prove useful properties and estimates in these spaces. In Section 3 we apply the theory developed in Section 2 to the linearized Ekman boundary layer problem. We establish the Helmholtz decomposition of the space $\mathrm{FM}_0(\mathbb{R}^2, L^p(D)^3)$ and prove the Stokes operator to be a generator of a bounded analytic semigroup. Here we essentially make use of relation (1.5) and the fact that the obtained results are known in $L^p(\mathbb{R}^2 \times D)$. Based on these facts we prove Theorem 1.1. Finally, relying to the uniform estimates obtained for the linearized equations, in Section 4 we prove uniform local-in-time existence for (1.1), that is Theorem 1.2, by applying the contraction mapping principle.

2. Vector measures and abstract setting

We use standard notation throughout this article. The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ denote reals, complex numbers, and integers, respectively. We also write $\mathbb{N} = \{1, 2, 3, \dots\}$

for the naturals and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbols X, Y, Z usually denote Banach spaces, whereas $\mathscr{L}(X,Y)$ stands for the set of bounded linear operators from X to Y. If $X = Y$, we write $\mathscr{L}(X)$. For a measure space $(G, \mathscr{A}, \mu)$ and $1 \leq p \leq \infty$, $L^p(G, X, \mu)$ denotes the X-valued Lebesgue space with respect to $(G, \mathscr{A}, \mu)$. If $G \subset \mathbb{R}^n$ is open and $\mu = \Lambda$, i.e. the Lebesgue measure, we simply write $L^p(G, X)$. For $k \in \mathbb{N}$, $W^{k,p}(G, X)$ denotes the X-valued Sobolev space. For $p = 2$ we also write $H^k(G, X)$. If $X = \mathbb{C}^m$ or $X = \mathbb{R}^m$ we use the common notation $L^p(G)$, $W^{k,p}(G)$, $H^k(G)$ (occasionally also $L^p(G)^m$, $W^{k,p}(G)^m$, $H^k(G)^m$ if confusion seems likely). The space $C_c^\infty(G, X)$ is the set of smooth and compactly supported functions. Its closure in $W^{k,p}$ (H^k) is denoted by $W_0^{k,p}(G, X)$ ($H_0^k(G, X)$). We will also write $\mathrm{BC}(G, X)$ and $\mathrm{BUC}(G, X)$ for the space of bounded and continuous functions and the space of bounded and uniformly continuous functions, respectively. Spaces of continuous and continuously differentiable functions are as usual denoted by $C(G, X)$, $C^k(G, X)$, $\mathrm{BC}^k(G, X)$, and so on. The Fourier transformation on the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ in this note is defined as

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi x} u(x) \mathrm{d}x, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

As usual, its extension by duality to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n, X) := \mathscr{L}(\mathcal{S}(\mathbb{R}^n), X)$ is again denoted by $\mathcal{F}u$ or $\widehat{u}$ for $u \in \mathcal{S}'(\mathbb{R}^n, X)$. For the duality pairing of a topological vector space E with its dual space we use the notation

$$\langle x', x \rangle_{E',E}, \quad x \in E,\ x' \in E'.$$

Next we recall some basic definitions related to *X-valued measures*, cf. [15].

Definition 2.1. Let X be a Banach space, Ω be a set, $\mathscr{A}$ be a σ-algebra over Ω, and $\mu : \mathscr{A} \to X$ be a set function.

(i) The function μ is called σ-additive, if it satisfies

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

for all pairwise disjoint sets $A_j \in \mathscr{A}$, $j = 1, 2, \ldots$. (The convergence of the right hand side is in X.)

(ii) If μ is σ-additive and satisfies $\mu(\emptyset) = 0$, then μ is called X-valued measure or vector measure.

(iii) The variation of an X-valued measure μ is defined as

$$|\mu|(\mathcal{O}) := \sup \left\{ \sum_{A \in \Pi(\mathcal{O})} \|\mu(A)\|_X : \ \Pi(\mathcal{O}) \subset \mathscr{A} \text{ finite decomposition of } \mathcal{O} \right\}.$$

for $\mathcal{O} \in \mathscr{A}$. (Note that $\Pi(\mathcal{O})$ is a decomposition of $\mathcal{O} \in \mathscr{A}$, if $A \cap B = \emptyset$ for all $A, B \in \Pi(\mathcal{O})$ with $A \neq B$ and $\bigcup_{A \in \Pi(\mathcal{O})} A = \mathcal{O}$.)

(iv) The quantity $|\mu|(\Omega)$ is called total variation of μ. If $|\mu|(\Omega) < \infty$, then μ is called finite or of bounded variation.

Remark 2.2. (a) Observe that $|\mu| : \mathscr{A} \to \mathbb{R}$ is a positive measure. In fact, let $(A_j)_{j\in\mathbb{N}}$ be a family of disjoint sets and $(E_k)_{k=1}^N$ be a finite decomposition of $\bigcup_{j=1}^\infty A_j$. Then, for each $j \in \mathbb{N}$ the family $(E_k \cap A_j)_{k=1}^N$ is a finite decomposition of A_j and we obtain by the σ-additivity of μ that

$$\sum_k \|\mu(E_k)\|_X = \sum_k \|\mu\Big(\bigcup_{j=1}^\infty E_k \cap A_j\Big)\|_X \le \sum_j \sum_k \|\mu(E_k \cap A_j)\|_X.$$

Thus, $|\mu|\Big(\bigcup_{j=1}^\infty A_j\Big) \le \sum_{j=1}^\infty |\mu|(A_j)$. On the other hand, if $(E_{j,k})_{k=1}^N$ is a decomposition of A_j, then $(E_{j,k})_{j,k}$ is one of $\bigcup_{j=1}^\infty A_j$. Assuming for each $j \in \mathbb{N}$ that $|\mu|(A_j) \le \sum_k \|\mu(E_{j,k})\|_X + \varepsilon/2^j$ yields

$$\sum_j |\mu|(A_j) \le \sum_j \sum_k \|\mu(E_{j,k})\|_X + \varepsilon,$$

and therefore $\sum_{j=1}^\infty |\mu|(A_j) \le |\mu|\Big(\bigcup_{j=1}^\infty A_j\Big)$. The remaining properties are obvious.

(b) For $|\mu|$-measurable bounded scalar-valued functions the integral with respect to an X-valued measure μ can be defined in a standard way via the approximation by simple functions, see [15, page 5]. Thus, for each $A \in \mathscr{A}$ the map

$$\int_A \cdot \,\mathrm{d}\mu : L^\infty(A,\mu) \to X, \quad f \mapsto \int_A f \mathrm{d}\mu$$

is well-defined.

In this note we mainly deal with Radon measures which will be defined next. For this purpose let $\Omega \subset \mathbb{R}^n$ be open, $\mathscr{A}$ be a σ-algebra over Ω, and denote by $\mathscr{B}(\Omega)$ the Borel σ-algebra over Ω. Recall that $\eta : \mathscr{A} \to [0,\infty)$ is a *Radon measure* if it is Borel regular, that is, if $\mathscr{B}(\Omega) \subseteq \mathscr{A}$ and if for each $A \subseteq \Omega$ there exists a $B \in \mathscr{B}(\Omega)$ such that $A \subseteq B$ and $\eta^*(A) = \eta^*(B)$, where η^* denotes the outer measure associated to η given by

$$\eta^*(A) := \inf\left\{\sum_{j=1}^\infty \eta(E_j) :\ (E_j)_{j\in\mathbb{N}} \subseteq \mathscr{A},\ A \subseteq \bigcup_{j=1}^\infty E_j\right\}. \tag{2.1}$$

Also observe that in the sequel we identify a measure η by its outer measure, so that η is complete in the sense that all subsets B of a set $A \in \mathscr{A}$ satisfying $\eta(A) = 0$ belong to $\mathscr{A}$.

Definition 2.3. Let $\Omega \subset \mathbb{R}^n$ be open, X be a Banach space, and $\mathscr{A}$ be a σ-algebra over Ω. The set function $\mu : \mathscr{A} \to X$ is called a *finite X-valued Radon measure*, if μ is an X-valued measure and if the variation $|\mu|$ is a finite Radon measure. The set of all finite X-valued Radon measures is denoted by $\mathrm{M}(\Omega, X)$.

Let us quickly show

Lemma 2.4. *The set $\mathrm{M}(\Omega, X)$ enhanced with the total variation as a norm, i.e. with $\|\cdot\|_{\mathrm{M}} := |\cdot|(\Omega)$, forms a Banach space.*

Proof. It is easily checked that $\|\cdot\|_{\mathrm{M}}$ is indeed a norm and consequently $\mathrm{M}(\Omega, X)$ is a normed linear space. Let $(\mu_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $(\mathrm{M}(\Omega, X), \|\cdot\|_{\mathrm{M}})$. Obviously for each $A \in \mathscr{A}$ the sequence $(\mu_k(A))_{k\in\mathbb{N}}$ has a limit in X. We define

$$\mu(A) := \lim_{k\to\infty} \mu_k(A)$$

and show that $\mu \in \mathrm{M}(\Omega, X)$. Clearly, $\mu(\emptyset) = 0$ and for disjoint sets $A, B \in \mathscr{A}$ we have that

$$\begin{aligned}\mu(A\cup B) &= \lim_{k\to\infty} \mu_k(A\cup B)\\ &= \lim_{k\to\infty} (\mu_k(A)+\mu_k(B)) = \mu(A)+\mu(B).\end{aligned}$$

In order to see that μ is even σ-additive it suffices to prove that $\|\mu(A_j)\|_X \to 0$ as $j\to\infty$ for $A_j \in \mathscr{A}$, $j = 1, 2, \ldots$, such that $A_j \subset A_{j-1}$ and $\bigcap A_j = \emptyset$. By the triangle inequality we obtain

$$\begin{aligned}\|\mu_k(A_j)\|_X &\le \|(\mu_k-\mu_\ell)(A_j)\|_X + \|\mu_\ell(A_j)\|_X\\ &\le \|\mu_k-\mu_\ell\|_{\mathrm{M}} + \|\mu_\ell(A_j)\|_X\\ &\le \varepsilon/2 + \|\mu_\ell(A_j)\|_X\end{aligned}$$

for all $k, \ell \ge N(\varepsilon)$ and $j = 1, 2, \ldots$. Fixing $\ell \ge N(\varepsilon)$ and letting $k\to\infty$ yields

$$\|\mu(A_j)\|_X \le \varepsilon/2 + \|\mu_\ell(A_j)\|_X \quad (j = 1, 2, \ldots).$$

By the σ-additivity of μ_ℓ we therefore may choose $j(\varepsilon)$ large enough so that

$$\|\mu_\ell(A_j)\|_X \le \varepsilon/2 \quad (j \ge j(\varepsilon)).$$

Hence, $\mu : \mathscr{A} \to X$ is a finite X-valued measure and by the lower semicontinuity of the supremum we obtain

$$\begin{aligned}\|\mu_k-\mu\|_{\mathrm{M}} &= \sup_{\Pi(\Omega)} \sum_{E\in\Pi(\Omega)} \|(\mu_k-\mu)(E)\|_X\\ &= \sup_{\Pi(\Omega)} \sum_{E\in\Pi(\Omega)} \lim_{\ell\to\infty} \|(\mu_k-\mu_\ell)(E)\|_X\\ &\le \liminf_{\ell\to\infty} \sup_{\Pi(\Omega)} \sum_{E\in\Pi(\Omega)} \|(\mu_k-\mu_\ell)(E)\|_X\\ &\le \varepsilon \quad (k \ge N(\varepsilon)).\end{aligned}$$

In order to see that $|\mu|$ is a Radon measure we observe that

$$\big|\, |\mu_k|(\Omega) - |\mu|(\Omega)\,\big| \le \|\mu_k-\mu\|_{\mathrm{M}} \to 0 \quad (k\to\infty).$$

Thus $|\mu_k| \to |\mu|$ in $\mathrm{M}(\Omega, \mathbb{R})$ which is known to be a Banach space. In fact, by the Riesz representation theorem we have $\mathrm{M}(\Omega, \mathbb{R}) = \mathscr{L}(C_\infty(\Omega), \mathbb{R})$, where

$$C_\infty(\Omega) = \{u \in C(\Omega) : \lim_{R\to\infty} \sup_{x\in\Omega\setminus B_R(0)} |u(x)| \to 0\}$$

(see [41]). Thus, $\mu \in \mathrm{M}(\Omega, X)$ which proves the assertion. □

Next we consider vector measures which are (absolutely) continuous with respect to a positive finite measure. For this purpose assume that X is a Banach space and $(\Omega, \mathscr{A}, \eta)$ a finite measure space. Recall that a finite vector measure $\mu : \mathscr{A} \to X$ is called η-continuous, if

$$\eta(\mathcal{O}) = 0 \quad \Rightarrow \quad \mu(\mathcal{O}) = 0, \quad \mathcal{O} \in \mathscr{A}.$$

For instance, it is clear by definition that μ is $|\mu|$-continuous. A fundamental question for an X-valued η-continuous measure μ is, under which circumstances it has a *Radon-Nikodým derivative* with respect to $(\Omega, \mathscr{A}, \eta)$, that is, when there exists a representation by a Bochner integral as

$$\mu(\mathcal{O}) = \int_{\mathcal{O}} g \mathrm{d}\eta \quad (\mathcal{O} \in \mathscr{A}), \tag{2.2}$$

with a $g \in L^1(\Omega, \eta, X)$. It turns out that this is essentially a matter of properties of the range of the measure μ and therefore it can be regarded as a property of the space X. Indeed, a space X is said to have the *Radon-Nikodým property with respect to the finite measure space* $(\Omega, \mathscr{A}, \eta)$, if each η-continuous vector measure $\mu : \mathscr{A} \to X$ admits a representation (2.2). The space X is said to have the *Radon-Nikodým property*, if X has the Radon-Nikodým property with respect to every finite measure space $(\Omega, \mathscr{A}, \eta)$. This question and its answer go back to fundamental works of Dunford and Pettis in the first half of the 20-th century and was continued by a couple of famous mathematicians in the second half. We refer to [15] for a comprehensive approach to this topic and for further references. In this note we mostly deal with reflexive spaces X which are known to have the Radon-Nikodým property, see e.g. [15, page 76, Corollary 13].

So, from now on assume X to have the Radon-Nikodým property and $\Omega \subset \mathbb{R}^n$ to be open. By $\rho_\mu \in L^1(\Omega, X, |\mu|)$ we denote the Radon-Nikodým derivative of a measure $\mu \in \mathrm{M}(\Omega, X)$ with respect to $(\Omega, \mathscr{A}, |\mu|)$, i.e., we have

$$\mu(\mathcal{O}) = \int_{\mathcal{O}} \rho_\mu \mathrm{d}|\mu| \quad (\mathcal{O} \in \mathscr{A}).$$

Next, let $\psi \in L^\infty(\Omega, |\mu|, \mathscr{L}(X, Y))$, where Y is another Banach space. By the definition of measurability, i.e., via the approximation by simple functions, it is easily checked that the $|\mu|$-measurability of ψ as a function from Ω to $\mathscr{L}(X, Y)$ and of ρ_μ as a function from Ω to X implies also $\psi\rho_\mu : \Omega \to Y$ to be $|\mu|$-measurable. Moreover, we have $\psi\rho_\mu \in L^1(\Omega, |\mu|, Y)$. Thus, the following is well-defined.

Definition 2.5. The multiplication of the $|\mu|$-almost everywhere bounded $\mathscr{L}(X, Y)$-valued function ψ with the X-valued measure μ is defined as Y-valued measure $\mu \lfloor \psi$ of the form

$$\mu \lfloor \psi(\mathcal{O}) := \int_{\mathcal{O}} \psi\rho_\mu \mathrm{d}|\mu| \quad (\mathcal{O} \in \mathscr{A}). \tag{2.3}$$

Note that again by means of approximation by a sequence of simple functions and by the definition of the variation it can be shown that

$$|\mu|(\mathcal{O}) = \int_{\mathcal{O}} \|f\|_X \mathrm{d}\eta \quad (\mathcal{O} \in \mathscr{A}), \tag{2.4}$$

for $f \in L^1(\Omega, \eta, X)$ and a positive measure η, and if the X-valued measure μ is represented by $\mu(\mathcal{O}) = \int_{\mathcal{O}} f \mathrm{d}\eta$, $\mathcal{O} \in \mathscr{A}$, see [15, page 46, Theorem 4(iv)] for the details. For the case that $f = \rho_\mu$ and $\eta = |\mu|$, relation (2.4) immediately implies that

$$\|\nu(t)\|_X = 1 \quad \text{for } |\mu|\text{-almost all } t \in \Omega. \tag{2.5}$$

For $\mu \in \mathrm{M}(\Omega, X)$ expression (2.3) defines a new vector Radon measure. This fact and the expected properties of this new measure are proved in

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be open and let X, Y, Z be Banach spaces having the Radon-Nikodým property. Furthermore, let $\mu \in \mathrm{M}(\Omega, X)$ and the functions $\psi \in L^\infty(\Omega, \mathscr{L}(X,Y), |\mu|)$ and $\phi \in L^\infty(\Omega, \mathscr{L}(Y,Z), |\mu|)$ be given. Then we have*

(i) $|\mu\lfloor\psi| = |\mu|\lfloor\|\psi\rho_\mu\|_Y \le |\mu|\lfloor\|\psi\|_{\mathscr{L}(X,Y)}$,
(ii) $\mu\lfloor\psi \in \mathrm{M}(\Omega, Y)$ *and therefore* $\mu\lfloor\psi(\mathcal{O}) = \int_{\mathcal{O}} \rho_{\mu\lfloor\psi} \mathrm{d}|\mu\lfloor\psi|$, $\mathcal{O} \in \mathscr{A}$,
(iii) $(\mu\lfloor\psi)\lfloor\phi = \mu\lfloor(\phi\psi)$.

Proof. (i) Since $\mu\lfloor\psi$ is defined by an integral with respect to a positive measure it clearly defines an Y-valued measure. From (2.4) and a straight forward estimation we obtain that

$$\begin{aligned} |\mu\lfloor\psi|(\mathcal{O}) &= |\mu|\lfloor\|\psi\rho_\mu\|_Y(\mathcal{O}) \\ &\le \int_{\mathcal{O}} \|\psi\|_{\mathscr{L}(X,Y)} \|\rho_\mu\|_X \mathrm{d}|\mu| \\ &= |\mu|\lfloor\|\psi\|_{\mathscr{L}(X,Y)}(\mathcal{O}) \quad (\mathcal{O} \in \mathscr{A}), \end{aligned}$$

where the last equality is a consequence of (2.5).

(ii) The estimate in (i) shows that $\mu\lfloor\psi$ is finite. By the equality part in (i) the assertion therefore follows from the fact, that for $f \in L^\infty(\Omega, \eta)$, $f \ge 0$, the expression $\mu(\cdot) = \int_\cdot f \mathrm{d}\eta$ defines a finite positive Radon measure if η does so. This can be seen as follows. Obviously μ is a finite positive measure. In order to see that μ is Borel-regular, let $A \subseteq \Omega$ be arbitrary and pick a Borel set $B \supseteq A$ such that

$$\eta^*(A) = \eta^*(B) = \eta(B).$$

We will show that the same set B will do for μ, i.e., that $\mu^*(A) = \mu(B)$. By definition (2.1) of the outer measure η^*, for arbitrary $\varepsilon > 0$ we may choose $(E_j)_{j\in\mathbb{N}} \subseteq \mathscr{A}$ in a way such that $A \subset \bigcup_{j=1}^\infty E_j$ and that

$$\begin{aligned} \mu\Big(\bigcup_{j=1}^\infty E_j\Big) &\le \mu^*(A) + \varepsilon \quad \text{and} \\ \eta\Big(\bigcup_{j=1}^\infty E_j\Big) &\le \eta^*(A) + \frac{\varepsilon}{\|f\|_\infty}. \end{aligned}$$

By construction and the properties of B we may assume without loss of generality that $B \subset \bigcup_{j=1}^\infty E_j$. Thus, the second inequality implies that

$$\eta\Big(\bigcup_{j=1}^\infty E_j \setminus B\Big) = \eta\Big(\bigcup_{j=1}^\infty E_j\Big) - \eta^*(A) \le \frac{\varepsilon}{\|f\|_\infty}.$$

In view of $\mu \le \|f\|_\infty \eta$ we therefore deduce

$$\mu\Big(\bigcup_{j=1}^{\infty} E_j \setminus B\Big) \le \varepsilon.$$

Hence, we conclude

$$\begin{aligned} \mu(B) - \varepsilon &\le \mu\Big(\bigcup_{j=1}^{\infty} E_j\Big) - \varepsilon \le \mu^*(A) \\ &\le \mu\Big(\bigcup_{j=1}^{\infty} E_j\Big) \le \mu(B) + \varepsilon. \end{aligned}$$

This implies (ii).

(iii) First we prove the relation for simple functions ϕ. To this end, let

$$\phi : \Omega \to \mathscr{L}(Y,Z), \quad \phi(t) = \sum_{j=1}^{m} a_j \chi_{E_j}(t),$$

with $a_j \in \mathscr{L}(Y,Z)$, $E_j \in \mathscr{A}$ pairwise disjoint, and where χ_{E_j} denotes the characteristic function to the set E_j. Further, let ρ_μ and $\rho_{\mu\lfloor\psi}$ be the Radon-Nikodým derivative of the X-valued Radon measure μ and the Y-valued Radon measure $\mu\lfloor\psi$ with respect to $|\mu|$ and $|\mu\lfloor\psi|$ respectively. Since $\psi\rho_\mu$ is $|\mu|$-measurable [1] and $\rho_{\mu\lfloor\psi}$ is $|\mu\lfloor\psi|$-measurable, obviously also $a_j\psi\rho_\mu$ and $a_j\rho_{\mu\lfloor\psi}$ are $|\mu|$- and $|\mu\lfloor\psi|$-measurable respectively. Thus, finite sums of these functions are measurable as well, and we obtain

$$\begin{aligned} (\mu\lfloor\psi)\lfloor\phi(\mathcal{O}) &= \int_{\mathcal{O}} \phi\rho_{\mu\lfloor\psi}\mathrm{d}|\mu\lfloor\psi| = \sum_{j=1}^{m} \int_{E_j\cap\mathcal{O}} a_j\rho_{\mu\lfloor\psi}\mathrm{d}|\mu\lfloor\psi| \\ &= \sum_{j=1}^{m} a_j\mu\lfloor\psi(E_j\cap\mathcal{O}) = \sum_{j=1}^{m} a_j \int_{E_j\cap\mathcal{O}} \psi\rho_\mu\mathrm{d}|\mu| \\ &= \int_{\mathcal{O}} \sum_{j=1}^{m} a_j\psi\rho_\mu\chi_{E_j}\,\mathrm{d}|\mu| = \int_{\mathcal{O}} \phi\psi\rho_\mu\mathrm{d}|\mu| \\ &= \mu\lfloor(\phi\psi)(\mathcal{O}) \quad (\mathcal{O}\in\mathscr{A}). \end{aligned}$$

Next, let $\phi \in L^\infty(\Omega,|\mu|,\mathscr{L}(Y,Z))$ and (ϕ_k) be an approximating sequence of simple functions. Then, applying twice Lebesgue's dominated convergence theorem for Bochner integrals implies

$$\begin{aligned} (\mu\lfloor\psi)\lfloor\phi(\mathcal{O}) &= \int_{\mathcal{O}} \phi\rho_{\mu\lfloor\psi}\mathrm{d}|\mu\lfloor\psi| = \lim_{k\to\infty}\int_{\mathcal{O}} \phi_k\rho_{\mu\lfloor\psi}\mathrm{d}|\mu\lfloor\psi| \\ &= \lim_{k\to\infty} (\mu\lfloor\psi)\lfloor\phi_k(\mathcal{O}) = \lim_{k\to\infty} \mu\lfloor(\phi_k\psi)(\mathcal{O}) \\ &= \lim_{k\to\infty}\int_{\mathcal{O}} \phi_k\psi\rho_\mu\mathrm{d}|\mu| = \int_{\mathcal{O}} \phi\psi\rho_\mu\mathrm{d}|\mu| \end{aligned}$$

[1] That means it can be approximated by a sequence of simple functions in $L^\infty(\Omega,Y,|\mu|)$.

$$= \quad \mu\lfloor(\phi\psi)(\mathcal{O}) \quad (\mathcal{O} \in \mathscr{A}).$$

This completes the proof. □

Remark 2.7. An important observation for the applications in subsequent sections is the following fact: since every element of $\mathrm{M}(\Omega, X)$ is defined on the Borel σ-algebra $\mathscr{B}(\Omega)$ and since every continuous function is Borel measurable, we obtain that

$$\mathrm{BC}(\Omega, \mathscr{L}(X,Y)) \subseteq L^\infty(\Omega, \mathscr{L}(X,Y), |\mu|) \quad (\mu \in \mathrm{M}(\Omega, X)). \tag{2.6}$$

That means expression $\mu\lfloor\psi$ is well defined and all assertions in Lemma 2.6 hold for every $\mu \in \mathrm{M}(\Omega, X)$ and every $\psi \in \mathrm{BC}(\Omega, \mathscr{L}(X,Y))$.

In view of the above remark it is also quite obvious that

$$T_\mu f := \int_{\mathbb{R}^n} f \rho_\mu \mathrm{d}|\mu| = \mu\lfloor f(\mathbb{R}^n), \quad f \in C_\infty(\mathbb{R}^n), \tag{2.7}$$

defines a bounded linear operator T_μ from $C_\infty(\mathbb{R}^n)$ to X. A standard argument shows that $\mu \mapsto T_\mu$ is injective. Thus we always have a continuous and injective embedding of the form

$$\mathrm{M}(\mathbb{R}^n, X) \hookrightarrow \mathscr{L}(C_\infty(\mathbb{R}^n), X). \tag{2.8}$$

A much more delicate issue is the question for the converse direction, that is, when a bounded operator $T : C_\infty(\mathbb{R}^n) \to X$ has a representation by an X-valued measure as above, or in other words, when does the Riesz representation theorem hold. This question is is closely related to the existence of a Radon-Nikodým derivative, see [15]. As mentioned above, if $X = \mathbb{R}$, the space of finite Radon measures can be identified with the space $\mathscr{L}(C_\infty(\mathbb{R}^n), \mathbb{R})$. Based on results established in [15], it can be shown that this identification generalizes to reflexive Banach spaces X. Since we will not use it in the sequel, we just state this fact as a further remark.

Remark 2.8. For reflexive Banach spaces X we have that

$$\mathrm{M}(\mathbb{R}^n, X) = \mathscr{L}(C_\infty(\mathbb{R}^n), X)$$

in the sense that every bounded linear operator $T : C_\infty(\mathbb{R}^n) \to X$ has a representation by a finite X-valued Radon measure μ given through (2.7). In particular, for all $\mu \in \mathrm{M}(\mathbb{R}^n, X)$ and $\mathcal{O} \in \mathscr{A}$ we have that

$$|\mu|(\mathcal{O}) = \sup\{\|T_\mu f\|_X : \ f \in C_c^\infty(\mathcal{O}), \ \|f\|_\infty \le 1\}.$$

By the intention to introduce the Fourier transform of vector Radon measures, from now on we assume $\Omega = \mathbb{R}^n$. In order to get a better feeling of the space $\mathrm{M}(\mathbb{R}^n, X)$, we next derive some properties and relations to known function spaces. We still assume that X has the Radon-Nikodým property. First, note that we obviously have

$$L^1(\mathbb{R}^n, X) \hookrightarrow \mathrm{M}(\mathbb{R}^n, X) \tag{2.9}$$

in the sense of the identification

$$f \mapsto \Lambda\lfloor f, \quad f \in L^1(\mathbb{R}^n, X),$$

where Λ denotes the Lebesgue measure on $\mathbb{R}^n$. But observe that the space $\mathrm{M}(\mathbb{R}^n, X)$ is strictly larger than $L^1(\mathbb{R}^n, X)$, since each Dirac measure δ_{t_0} with respect to the point $t_0 \in \mathbb{R}^n$ defines for every $x \in X$ via $\delta_{t_0} x$ an element in $\mathrm{M}(\mathbb{R}^n, X) \setminus L^1(\mathbb{R}^n, X)$. On the other hand, the fact that the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions with its canonical topology is continuously and densely embedded in $C_\infty(\mathbb{R}^n)$ gives us

$$\mathscr{L}(C_\infty(\mathbb{R}^n), X) \hookrightarrow \mathscr{L}(\mathcal{S}(\mathbb{R}^n), X) = \mathcal{S}'(\mathbb{R}^n, X).$$

Thus, in the sense of the identification

$$\mu \mapsto T_\mu, \quad T_\mu f := \mu \lfloor f(\mathbb{R}^n),$$

by relation (2.8) we have the embedding

$$\mathrm{M}(\mathbb{R}^n, X) \hookrightarrow \mathcal{S}'(\mathbb{R}^n, X).$$

Consequently, the Fourier transform on $\mathrm{M}(\mathbb{R}^n, X)$ is well-defined. Fubini's theorem implies

$$\widehat{T_\mu}(f) = T_\mu(\widehat{f}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{e}^{-it\cdot\xi} \rho_\mu \mathrm{d}|\mu|(t) f(\xi) \mathrm{d}\xi$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$, and therefore we obtain

$$\widehat{\mu}(\xi) = \mu \lfloor \varphi_\xi(\mathbb{R}^n) \tag{2.10}$$

with $\varphi_\xi(t) := (2\pi)^{-n/2} \mathrm{e}^{-it\cdot\xi}$. This allows for the definition of the space

$$\mathrm{FM}(\mathbb{R}^n, X) := \{\widehat{\mu} : \ \mu \in \mathrm{M}(\mathbb{R}^n, X)\},$$

which we equip with the canonical norm

$$\|u\|_{\mathrm{FM}} := \|\mathcal{F}^{-1} u\|_{\mathrm{M}}.$$

Observe that replacing the Fourier transform by its inverse in the definition does not change the value of the norm, i.e., we have $\|\cdot\|_{\mathrm{FM}} = \|\mathcal{F}\cdot\|_{\mathrm{M}} = \|\mathcal{F}^{-1}\cdot\|_{\mathrm{M}}$. In order to define multipliers with symbols not necessarily continuous at the origin, we also introduce the spaces

$$\mathrm{M}_0(\mathbb{R}^n, X) := \{\mu \in \mathrm{M}(\mathbb{R}^n, X) : \ \mu(\{0\}) = 0\},$$

that is, the subspace of Radon measures with no point mass at the origin and

$$\mathrm{FM}_0(\mathbb{R}^n, X) := \{\widehat{\mu} : \ \mu \in \mathrm{M}_0(\mathbb{R}^n, X)\}.$$

Definition 2.9. For $n \in \mathbb{N}$ and a Banach space X having the Radon-Nikodým property, we call $\mathrm{FM}(\mathbb{R}^n, X)$ and $\mathrm{FM}_0(\mathbb{R}^n, X)$ *(X-valued) spaces of Fourier transformed (finite) Radon measures.*

Remark 2.10. Note that every $\mu \in \mathrm{M}(\mathbb{R}^n, X)$ decomposes uniquely as

$$\mu = (\mu - \delta_0 \mu(\{0\})) + \delta_0 \mu(\{0\}).$$

In other words, we have $\mathrm{M}(\mathbb{R}^n, X) = \mathrm{M}_0(\mathbb{R}^n, X) \oplus \{\delta_0 x; \ x \in X\}$. Since by definition $\mathcal{F} : \mathrm{M}(\mathbb{R}^n, X) \to \mathrm{FM}(\mathbb{R}^n, X)$ is isomorphic, this implies that

$$\mathrm{FM}(\mathbb{R}^n, X) = \mathrm{FM}_0(\mathbb{R}^n, X) \oplus X. \tag{2.11}$$

Next, we list some useful properties of the spaces just introduced. We start with a convolution for vector Radon measures. For this purpose, let X, X_1, X_2 be Banach spaces having the Radon-Nikodým property. Suppose further that a multiplication of elements in X_2 with elements in X_1 is defined in a way that

$$X_2 \cdot X_1 \hookrightarrow X.$$

For example, X_2 can be a suitable subspace of $\mathscr{L}(X_1, X)$ or it can be the multiplication of functions in suitable L^p or Sobolev spaces. By 'suitable' we particularly mean that these spaces have the Radon-Nikodým property, which does in general not hold for the whole of $\mathscr{L}(X_1, X)$, see [15, page 219]. For two nonnegative measures η_1, η_2 over $\mathscr{B}(\mathbb{R}^n)$, we denote the corresponding product measure over $\mathscr{B}(\mathbb{R}^{2n})$ by $\eta_1 \otimes \eta_2$. For $\mu \in \mathrm{M}(\mathbb{R}^n, X_1)$ and $\eta \in \mathrm{M}(\mathbb{R}^n, X_2)$ we define the convolution formally as

$$\eta * \mu(\mathcal{O}) := \int_{\mathbb{R}^{2n}} \chi_{\mathcal{O}}(t+s)\rho_\eta(t)\rho_\mu(s)\,\mathrm{d}(|\mu| \otimes |\eta|)(t,s) \quad (\mathcal{O} \in \mathscr{B}(\mathbb{R}^n)). \tag{2.12}$$

Lemma 2.11. *In the situation above we have*

(i) $\eta * \mu \in \mathrm{M}(\mathbb{R}^n, X)$*, in particular, Young's inequality*

$$\|\eta * \mu\|_{\mathrm{M}(\mathbb{R}^n,X)} \le \|\eta\|_{\mathrm{M}(\mathbb{R}^n,X_2)}\|\mu\|_{\mathrm{M}(\mathbb{R}^n,X_1)}$$

and the (familiar looking) representation

$$\eta * \mu(\mathcal{O}) = \int_{\mathbb{R}^n} \eta(\mathcal{O} - s)\rho_\mu(s)\mathrm{d}|\mu|(s), \quad \mathcal{O} \in \mathscr{B}(\mathbb{R}^n),$$

hold;

(ii) *for all* $f \in L^1(\mathbb{R}^n, X_1)$, $g \in L^1(\mathbb{R}^n, X_2))$ *that* $(\Lambda\lfloor g) * (\Lambda\lfloor f) = \Lambda\lfloor(g * f)$*, i.e., in this case (2.12) coincides with the standard convolution in* L^1*;*

(iii) *for all* $f \in \mathrm{BC}(\mathbb{R}^n, \mathscr{L}(X, Y))$ *that*

$$(\eta*\mu)\lfloor f(\mathcal{O}) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{\mathcal{O}}(t+s)f(t+s)\rho_\eta(t)\rho_\mu(s)\mathrm{d}|\mu|(s)\mathrm{d}|\eta|(s) \quad (\mathcal{O} \in \mathscr{B}(\mathbb{R}^n));$$

(iv) $\mathcal{F}(\eta * \mu) = (2\pi)^{n/2}\widehat{\eta} \cdot \widehat{\mu}$*;*

(v) $\eta * \mu = \mu * \eta$*, if the multiplication* $X_2 \cdot X_1$ *is commutative;*

(vi) *that* $(\mathrm{M}(\mathbb{R}^n, X), *)$ *is an (abelian) algebra (with unit), if* $(X, \cdot)$ *is an (abelian) algebra (with unit).*

Proof. (i) Regarding the $|\eta|$-measurable function ρ_η and the $|\mu|$-measurable function ρ_μ as constant in s and as constant in t respectively, we see that these functions are obviously measurable in $\mathscr{B}(\mathbb{R}^{2n})$, too. Therefore, for each $\mathcal{O} \in \mathscr{B}(\mathbb{R}^n)$, $(t,s) \mapsto \chi_{\mathcal{O}}(t+s)\rho_\eta(t)\rho_\mu(s)$ is $|\eta| \otimes |\mu|$-measurable. Hence the integral in representation (2.12) is well-defined. Thanks to $\chi_{\mathcal{O}}(t+s) = \chi_{\mathcal{O}-s}(t)$ and to Fubini's theorem then we know that

$$\int_{\mathbb{R}^n} \chi_{\mathcal{O}}(t+s)\rho_\eta(t)\mathrm{d}|\eta|(t) = \int_{\mathcal{O}-s} \rho_\eta(t)\mathrm{d}|\eta|(t) = \eta(\mathcal{O}-s), \quad s \in \mathbb{R}^n,$$

is $|\mu|$-measurable. More precisely, we have that

$$\left(s \mapsto \int_{\mathbb{R}^n} \chi_{\mathcal{O}}(t+s)\rho_\eta(t)\mathrm{d}|\eta|(t)\right) \in L^\infty(\mathbb{R}^n, X_2, |\mu|)$$

and that

$$\begin{aligned}(\eta * \mu)(\mathcal{O}) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{\mathcal{O}}(t+s)\rho_\eta(t)\rho_\mu(s)\mathrm{d}|\mu|(s)\mathrm{d}|\eta|(t) \\ &= \int_{\mathbb{R}^n} \eta(\mathcal{O}-s)\rho_\mu(s)\mathrm{d}|\mu|(s) \quad (\mathcal{O}\in\mathscr{B}(\mathbb{R}^n)).\end{aligned} \tag{2.13}$$

From the latter representation it readily follows that $\eta * \mu$ is a finite X-valued measure. In order to see that its variation $|\eta * \mu|$ is a Radon measure, observe that (2.4) implies that

$$\begin{aligned}|\eta * \mu|(\mathcal{O}) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{\mathcal{O}}(t+s)\|\rho_\eta(t)\rho_\mu(s)\|_X\mathrm{d}|\mu|(s)\mathrm{d}|\eta|(t) \\ &= \int_{\mathbb{R}^n} |\mu|\lfloor\|\rho_\eta(t)\rho_\mu(\cdot)\|_X(\mathcal{O}-t)\mathrm{d}|\eta|(t) \quad (\mathcal{O}\in\mathscr{B}(\mathbb{R}^n)).\end{aligned}$$

Obviously $\sigma_t(\mathcal{O}) := |\mu|\lfloor\|\rho_\eta(t)\rho_\mu(\cdot)\|_X(\mathcal{O}-t)$ is a finite Radon measure for each fixed $t\in\mathbb{R}^n$ (see proof of Lemma 2.6(ii)). Furthermore, dominated convergence implies for the corresponding outer measures that

$$|\eta * \mu|^*(\mathcal{O}) = \int_{\mathbb{R}^n} \sigma_t^*(\mathcal{O})\mathrm{d}|\eta|(t),$$

from which we easily see that $|\eta * \mu|$ is Borel regular. Consequently, also $|\eta * \mu|$ is a Radon measure which yields $\eta * \mu \in \mathrm{M}(\mathbb{R}^n, X)$. Young's inequality in (i) is easily obtained from representation (2.13).

(ii) By virtue of translation invariance of the Lebesgue measure and by a repeated application of Fubini's theorem we obtain

$$\begin{aligned}\Lambda\lfloor g * \Lambda\lfloor f(\mathcal{O}) &= \int_{\mathbb{R}^n} \Lambda\lfloor g(\mathcal{O}-s)\rho_{\Lambda\lfloor f}(s)\mathrm{d}|\Lambda\lfloor f|(s) \\ &= \int_{\mathbb{R}^n}\int_{\mathcal{O}-s} g(t)\mathrm{d}t\,\rho_{\Lambda\lfloor f}(s)\mathrm{d}|\Lambda\lfloor f|(s) \\ &= \int_{\mathbb{R}^n} g(t)\int_{\mathbb{R}^n} \chi_{\mathcal{O}}(t+s)\rho_{\Lambda\lfloor f}(s)\mathrm{d}|\Lambda\lfloor f|(s)\mathrm{d}t \\ &= \int_{\mathbb{R}^n} g(t)\Lambda\lfloor f(\mathcal{O}-t)\mathrm{d}t \\ &= \int_{\mathbb{R}^n} g(t)\int_{\mathcal{O}-t} f(s)\mathrm{d}s\mathrm{d}t \\ &= \int_{\mathbb{R}^n} g(t)\int_{\mathcal{O}} f(s-t)\mathrm{d}s\mathrm{d}t \\ &= \int_{\mathcal{O}}\int_{\mathbb{R}^n} g(t)f(s-t)\mathrm{d}t\mathrm{d}s \\ &= \int_{\mathcal{O}}\int_{\mathbb{R}^n} g(t-s)f(s)\mathrm{d}t\mathrm{d}s \\ &= (\Lambda\lfloor(g*f))(\mathcal{O}) \quad (\mathcal{O}\in\mathscr{B}(\mathbb{R}^n)).\end{aligned}$$

(iii) Let $f(t) = \sum_{j=1}^m a_j \chi_{E_j}(t)$ with $a_j \in \mathscr{L}(X,Y)$ and $E_j \in \mathscr{B}(\mathbb{R}^n)$ be a simple function. Then we have

$$\begin{aligned}
(\eta * \mu)\lfloor f(\mathcal{O}) &= \sum_{j=1}^m a_j \int_{\mathcal{O}} \chi_{E_j}(t)\rho_{\eta*\mu}(t)\mathrm{d}|\eta*\mu|(t) \\
&= \sum_{j=1}^m a_j (\eta*\mu)(\mathcal{O}\cap E_j) \\
&= \sum_{j=1}^m a_j \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{\mathcal{O}\cap E_j}(t+s)\rho_\eta(t)\rho_\mu(s)\mathrm{d}|\mu|(s)\mathrm{d}|\eta|(t) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{\mathcal{O}}(t+s)\sum_{j=1}^m a_j\chi_{E_j}(t+s)\rho_\eta(t)\rho_\mu(s)\mathrm{d}|\mu|(s)\mathrm{d}|\eta|(t) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_{\mathcal{O}}(t+s) f(t+s)\rho_\eta(t)\rho_\mu(s)\mathrm{d}|\mu|(s)\mathrm{d}|\eta|(t) \quad (\mathcal{O}\in\mathscr{B}(\mathbb{R}^n)).
\end{aligned}$$

Dominated convergence yields the result for general f.

(iv) Set $\varphi_\xi(t) = (2\pi)^{-n/2}\mathrm{e}^{-i\xi t}$. Representation (2.10), (iii), and Fubini's theorem imply

$$\begin{aligned}
\mathcal{F}(\eta*\mu)(\xi) &= (\eta*\mu)\lfloor\varphi_\xi(\mathbb{R}^n) \\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{-i\xi(t+s)}\rho_\eta(t)\rho_\mu(s)\mathrm{d}|\mu|(s)\mathrm{d}|\eta|(t) \\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\mathrm{e}^{-i\xi s}\int_{\mathbb{R}^n}\mathrm{e}^{-i\xi t}\rho_\eta(t)\mathrm{d}|\eta|(t)\rho_\mu(s)\mathrm{d}|\mu|(s) \\
&= (2\pi)^{n/2}\widehat{\eta}(\xi)\cdot\widehat{\mu}(\xi), \quad \xi\in\mathbb{R}^n.
\end{aligned}$$

Assertions (v) and (vi) are obvious consequences of (i). Note that if e is the unit in $(X,\cdot)$, then $\delta_0 e$ is the unit in $(\mathrm{M}(\mathbb{R}^n,X),*)$, where δ_0 denotes the Dirac measure in $x=0$. □

With Lemma 2.11 at hand we can show the announced properties of the introduced spaces of Fourier transformed Radon measures.

Lemma 2.12. *Suppose X, X_1, X_2 are Banach spaces having the Radon-Nikodým property and that $X_2 \cdot X_1 \hookrightarrow X$. Then the following assertions hold.*

(i) *The spaces $\mathrm{FM}(\mathbb{R}^n,X)$, $\mathrm{M}_0(\mathbb{R}^n,X)$, and $\mathrm{FM}_0(\mathbb{R}^n,X)$ are Banach spaces.*

(ii) *For all $u\in\mathrm{FM}(\mathbb{R}^n,X_2)$ and $v\in\mathrm{FM}(\mathbb{R}^n,X_1)$ we have that*

$$\|u\cdot v\|_{\mathrm{FM}(\mathbb{R}^n,X)} \le (2\pi)^{-n/2}\|u\|_{\mathrm{FM}(\mathbb{R}^n,X_2)}\|v\|_{\mathrm{FM}(\mathbb{R}^n,X_1)},$$

i.e., $\mathrm{FM}(\mathbb{R}^n,X_2)\cdot\mathrm{FM}(\mathbb{R}^n,X_1)\hookrightarrow\mathrm{FM}(\mathbb{R}^n,X)$. In particular, $(\mathrm{FM}(\mathbb{R}^n,X),\cdot)$ is an (abelian) algebra (with unit), if $(X,\cdot)$ is an (abelian) algebra (with unit).

(iii) *We have*

$$\mathrm{FM}(\mathbb{R}^n,X)\hookrightarrow\mathrm{BUC}(\mathbb{R}^n,X) \tag{2.14}$$

and

$$\mathcal{F}L^1(\mathbb{R}^n,X)\hookrightarrow \mathrm{FM}_0(\mathbb{R}^n,X)\hookrightarrow \dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n,X)\hookrightarrow \mathrm{BUC}(\mathbb{R}^n,X), \tag{2.15}$$

where $\mathcal{B}^0_{\infty,1}(\mathbb{R}^n,X)$ denotes the homogeneous Besov space (see proof for a precise Definition).

Proof. (i) This follows immediately by the definition and Lemma 2.4.

(ii) This is an obvious consequence of Lemma 2.11(i),(iv), and (vi).

(iii) By (2.9) the first embedding in (2.15) is obvious. The third embedding in (2.15) is well-known, hence we omit a proof here (see e.g. [39, Example 2.3]). For the second embedding we first prove that

$$\mathrm{FM}(\mathbb{R}^n,X)\hookrightarrow L^\infty(\mathbb{R}^n,X). \tag{2.16}$$

Set $\varphi_\xi(t)=(2\pi)^{-n/2}\mathrm{e}^{-it\xi}$. Indeed, by virtue of Lemma 2.6(i) we obtain

$$\|\widehat{\mu}(\xi)\|_X\le |\mu\lfloor\varphi_\xi|(\mathbb{R}^n)\le \|\mu\|_{\mathrm{M}}\|\varphi_\xi\|_\infty\le C\|\widehat{\mu}\|_{\mathrm{FM}}\quad(\xi\in\mathbb{R}^n),$$

which shows the validity of (2.16). Recall that the Besov space $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n,X)$ is defined by

$$\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n,X)=\left\{u\in\mathcal{S}'(\mathbb{R}^n,X):\ \|u\|_{\dot{\mathcal{B}}^0_{\infty,1}}<\infty,\ u=\sum_{j\in\mathbb{Z}}\widehat{\phi}_j*u \text{ in } \mathcal{S}'(\mathbb{R}^n,X)\right\},$$

where the norm reads as

$$\|u\|_{\dot{\mathcal{B}}^0_{\infty,1}}=\sum_{j\in\mathbb{Z}}\|\widehat{\phi}_j*u\|_{L^\infty(\mathbb{R}^n,X)},$$

cf. [43]. Here $(\phi_j)_{j\in\mathbb{Z}}$ is a standard Littlewood-Paley decomposition given by a family of functions $\phi_j\in\mathcal{S}(\mathbb{R}^n)$ satisfying $\sum_{j\in\mathbb{Z}}\phi_j(\xi)=1$ for $\xi\in\mathbb{R}^n\setminus\{0\}$, where $\phi_j(\xi):=\phi_0(2^{-j}\xi)$ and $0\neq\phi_0\in\mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\phi_0\subseteq\{1/2\le|\xi|\le 2\}$. Since $\mathrm{FM}_0(\mathbb{R}^n,X)$ does not contain constants (i.e. elements in X), the convergence $\sum_{j=-N}^N\widehat{\phi}_j*u\to u$ for $N\to\infty$ is clear for every u in this space. It remains to derive the estimate for the corresponding norms. An application of Lemma 2.6(iii), Lemma 2.11(iv), and (2.10) yields

$$\begin{aligned}\widehat{\phi}_j*\widehat{\mu}(\xi) &= (2\pi)^{n/2}\mathcal{F}(\phi_j\mu)(\xi)\\ &= (2\pi)^{n/2}\mathcal{F}((\mu\lfloor\chi_{E_j})\lfloor\phi_j)(\xi)\\ &= \left[(\mu\lfloor\chi_{E_j})\lfloor\phi_j\varphi_\xi\right](\mathbb{R}^n)\quad(\xi\in\mathbb{R}^n),\end{aligned}$$

where $E_j=\{2^{j-1}\le|\xi|\le 2^{j+1}\}$ and where we made use of the fact that $\operatorname{supp}\phi_j\subseteq E_j$. Relation (2.16) and once more Lemma 2.6(i) then imply

$$\begin{aligned}\|\widehat{\phi}_j*\widehat{\mu}\|_{L^\infty(\mathbb{R}^n,X)} &\le C\sup_{\xi\in\mathbb{R}^n}(|\mu|\lfloor\chi_{E_j})(\mathbb{R}^n)\|\phi_0\varphi_\xi\|_{L^\infty(\mathbb{R}^n)}\\ &\le C(|\mu|\lfloor\chi_{E_j})(\mathbb{R}^n)\|\phi_0\|_{L^\infty(\mathbb{R}^n)}\\ &\le C|\mu|(E_j)\quad(j\in\mathbb{Z}).\end{aligned}$$

Since $E_j \cap E_k = \emptyset$ for $|j-k| \geq 3$ we conclude by the σ-additivity of $|\mu|$ that

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \|\widehat{\phi}_j * \widehat{\mu}\|_{L^\infty(\mathbb{R}^n,X)} &\leq C \sum_{j=-\infty}^{\infty} |\mu|(E_j) \\ &\leq C \sum_{k=0}^{2} \sum_{j=-\infty}^{\infty} |\mu|(E_{3j+k}) \\ &\leq 3C|\mu|(\mathbb{R}^n) \\ &\leq 3C\|\widehat{\mu}\|_{\mathrm{FM}} \quad (\widehat{\mu} \in \mathrm{FM}_0(\mathbb{R}^n,X)).\end{aligned}$$

The fact that $u \in \mathrm{FM}_0(\mathbb{R}^n,X)$ such that $\|u\|_{\dot{\mathcal{B}}^0_{\infty,1}} = 0$ implies $u \in X$. Thanks to (2.11) then $u = 0$. Thus, $\mathrm{FM}_0(\mathbb{R}^n,X)$ is continuously embedded in $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n,X)$ and (2.15) is proved. Relation (2.14) now follows from the fact that $\mathrm{FM}_0(\mathbb{R}^n,X) \hookrightarrow \mathrm{BUC}(\mathbb{R}^n,X)$ and (2.11). □

Observe that the space $\mathrm{FM}_0(\mathbb{R}^n,X)$ is not an algebra, but at least we obtain by Lemma 2.12(ii) that

$$\mathrm{FM}_0(\mathbb{R}^n,X_2) \cdot \mathrm{FM}_0(\mathbb{R}^n,X_1) \hookrightarrow \mathrm{FM}(\mathbb{R}^n,X),$$

which will be important for later purposes.

The fact that $\delta_{t_0} x \in \mathrm{M}_0(\mathbb{R}^n,X)$ for Dirac measures δ_{t_0}, $t_0 \in \mathbb{R}^n \setminus \{0\}$, and $x \in X$, gives rise to another interesting class of functions contained in the space $\mathrm{FM}_0(\mathbb{R}^n,X)$. In fact, every sequence $(a_j)_{j\in\mathbb{N}} \subseteq X$ satisfying $\sum_{j=1}^{\infty} \|a_j\|_X < \infty$ defines for each sequence of frequencies $(\lambda_j)_{j\in\mathbb{N}} \subseteq \mathbb{R}^n \setminus \{0\}$ an element

$$\left(x \mapsto \sum_{j=1}^{\infty} a_j \mathrm{e}^{-i\lambda_j \cdot x} \right) \in \mathrm{FM}_0(\mathbb{R}^n,X),$$

by the fact that $\sum_{j=1}^{\infty} \delta_{\lambda_j} a_j \in \mathrm{M}_0(\mathbb{R}^n,X)$. This class of almost periodic functions is significant for applications to rotating boundary layers as explained in the introduction.

Next, we present an operator valued multiplier result in $\mathrm{FM}_0(\mathbb{R}^n,X)$. It will play a crucial role in deriving stability and uniformness in the appearing parameters for the boundary layer problem treated in Sections 3 and 4.

For $\sigma \in \mathrm{BC}(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(X,Y))$ we define

$$op(\sigma) f := \mathcal{F}^{-1} \widehat{f} \lfloor \sigma, \quad f \in \mathrm{FM}_0(\mathbb{R}^n,X). \tag{2.17}$$

As a consequence of the theory for vector measures developed above we obtain the following multiplier result.

Proposition 2.13. *Let X, Y be Banach spaces having the Radon-Nikodým property and suppose that $\sigma \in \mathrm{BC}(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(X,Y))$. Then $op(\sigma)$ as defined in (2.17) is bounded from $\mathrm{FM}_0(\mathbb{R}^n,X)$ to $\mathrm{FM}_0(\mathbb{R}^n,Y)$ and we have*

$$\|op(\sigma)\|_{\mathscr{L}(\mathrm{FM}_0(\mathbb{R}^n,X),\mathrm{FM}_0(\mathbb{R}^n,Y))} = \|\sigma\|_{L^\infty(\mathbb{R}^n\setminus\{0\},\mathscr{L}(X,Y))}.$$

If σ is also continuous at the origin, then $op(\sigma) \in \mathscr{L}(\mathrm{FM}(\mathbb{R}^n, X),\ \mathrm{FM}(\mathbb{R}^n, Y))$ with the corresponding equality for the operator norm.

Proof. Lemma 2.6(i) implies

$$\begin{aligned} \|op(\sigma)f\|_{\mathrm{FM}} &= \|\widehat{f}\lfloor\sigma\|_{\mathrm{M}} \leq |\widehat{f}|\lfloor\|\sigma\|_{\mathscr{L}(X,Y)} \\ &\leq \|\sigma\|_{L^\infty(\mathbb{R}^n\setminus\{0\},\mathscr{L}(X,Y))}\|f\|_{\mathrm{FM}} \end{aligned}$$

for all $f \in \mathrm{FM}_0(\mathbb{R}^n, X)$, where we made use of the fact that $\|\cdot\|_{\mathrm{FM}} = \|\mathcal{F}\cdot\|_{\mathrm{M}} = \|\mathcal{F}^{-1}\cdot\|_{\mathrm{M}}$. This shows

$$\|op(\sigma)\| := \|op(\sigma)\|_{\mathscr{L}(\mathrm{FM}_0(\mathbb{R}^n,X),\mathrm{FM}_0(\mathbb{R}^n,Y))} \leq \|\sigma\|_{L^\infty(\mathbb{R}^n\setminus\{0\},\mathscr{L}(X,Y))}.$$

To see the converse, suppose that $\|op(\sigma)\| < \|\sigma\|_{L^\infty(\mathbb{R}^n\setminus\{0\},\mathscr{L}(X,Y))}$. In detail this means there is an $x \in X$ with $\|x\|_X = 1$ and an $t_0 \in \mathbb{R}^n \setminus \{0\}$ such that

$$\|\sigma(t_0)x\|_X > \|op(\sigma)\|.$$

We set $u := \mathcal{F}^{-1}\delta_{t_0}$. Then $|\widehat{u}|(\mathbb{R}^n) = 1$. Furthermore, we have $ux \in \mathrm{FM}_0(\mathbb{R}^n, X)$ and that the Radon-Nikodým derivative of $\widehat{ux} = \widehat{u}x$ with respect to $|\widehat{u}x| = |\widehat{u}|$ reads $\rho_{\widehat{u}x} = x$. This results in the contradiction

$$\begin{aligned} \|op(\sigma)\| &= \int_{\mathbb{R}^n} \|op(\sigma)\|\mathrm{d}|\widehat{u}|(\xi) \\ &< \int_{\mathbb{R}^n} \|\sigma(\xi)x\|_X \mathrm{d}|\widehat{u}|(\xi) \\ &= |\widehat{u}|\lfloor\|\sigma(\cdot)x\|_X(\mathbb{R}^n) = |(\widehat{u}x)\lfloor\sigma|(\mathbb{R}^n) \\ &= \|\mathcal{F}^{-1}[(\widehat{u}x)\lfloor\sigma]\|_{\mathrm{FM}} = \|op(\sigma)(ux)\|_{\mathrm{FM}} \\ &\leq \|op(\sigma)\|\|u\|_{\mathrm{FM}(\mathbb{R}^n,\mathbb{C})}\|x\|_X \\ &= \|op(\sigma)\|, \end{aligned}$$

where we applied Lemma 2.6(i) in the third equality. The additional assertion is then obvious. □

Remark 2.14. If H_1 and H_2 are Hilbert spaces, Plancherel's theorem implies that the right hand side of the equality in Proposition 2.13 equals the operator norm of $op(\sigma)$ in $\mathscr{L}(L^2(\mathbb{R}^n, H_1),\ L^2(\mathbb{R}^n, H_2))$. Hence in this case we have

$$\|op(\sigma)\|_{\mathscr{L}(\mathrm{FM}_0(\mathbb{R}^n,H_1),\mathrm{FM}_0(\mathbb{R}^n,H_2))} = \|\sigma\|_\infty = \|op(\sigma)\|_{\mathscr{L}(L^2(\mathbb{R}^n,H_1),\,L^2(\mathbb{R}^n,H_2))}.$$

For Banach spaces X_1 and X_2 and $1 \leq p \leq \infty$ Plancherel's theorem does not hold, but still we have that

$$\|op(\sigma)\|_{\mathscr{L}(\mathrm{FM}_0(\mathbb{R}^n,X_1),\mathrm{FM}_0(\mathbb{R}^n,X_2))} = \|\sigma\|_\infty \leq \|op(\sigma)\|_{\mathscr{L}(L^p(\mathbb{R}^n,X_1),\,L^p(\mathbb{R}^n,X_2))} \tag{2.18}$$

(see Lemma 3.1). By these facts, we see that results which are known in a $L^p(\mathbb{R}^n, X)$ framework and which fit into the multiplier context above immediately transfer to $\mathrm{FM}_0(\mathbb{R}^n, X)$ (for $X = X_1 = X_2$). This observation will be very helpful for our purposes in the next section.

In the last part of this section the domain $\mathbb{R}^n$ is essentially fixed. So we occasionally supress $\mathbb{R}^n$ and simply write $\mathrm{FM}(X)$ instead of $\mathrm{FM}(\mathbb{R}^n, X)$ and so on. In applications we will often use the fact noted in the above remark for the case that H_1 and H_2 are certain L^2-spaces. In the same spirit the following lemma will turn out to be helpful.

Lemma 2.15. *Let $J \subset \mathbb{R}$ be an interval and let H_1 and H_2 be Hilbert spaces. Assume that*

$$L \in \mathscr{L}(L^2(\mathbb{R}^n, H_1),\, L^2(J, L^2(\mathbb{R}^n, H_2))) \quad \text{with} \quad \|L\|_{\mathscr{L}(L^2(H_1),\, L^2(J, L^2(H_2)))} \leq M$$

is an operator with a symbol σ_L satisfying

$$\sigma_L \in C\left(\mathbb{R}^n \setminus \{0\},\, \mathscr{L}(H_1,\, L^2(J, H_2))\right).$$

Then we have

$$L \in \mathscr{L}(\mathrm{FM}_0(\mathbb{R}^n, H_1),\, L^2(J, \mathrm{FM}_0(\mathbb{R}^n, H_2))), \quad \|L\|_{\mathscr{L}(\mathrm{FM}_0(H_1),\, L^2(J,\, \mathrm{FM}_0(H_2)))} \leq M.$$

Proof. We set $E := L^2(J, H_2)$. Note that by Fubini's theorem we have

$$L^2(J, L^2(\mathbb{R}^n, H_2)) \cong L^2(\mathbb{R}^n, E).$$

Thus we may regard L as an operator such that

$$L \in \mathscr{L}(L^2(\mathbb{R}^n, H_1),\, L^2(\mathbb{R}^n, E))$$

with an unchanged operator norm. Since L is assumed to have a symbol σ_L, Plancherel's theorem and the assumption on σ_L imply that

$$\sigma_L \in \mathrm{BC}\left(\mathbb{R}^n \setminus \{0\},\, \mathscr{L}(H_1,\, E)\right), \quad \|\sigma_L\|_{L^\infty(\mathscr{L}(H_1,\, E))} \leq M.$$

Proposition 2.13 then yields

$$L \in \mathscr{L}(\mathrm{FM}_0(\mathbb{R}^n, H_1),\, \mathrm{FM}_0(\mathbb{R}^n, E)), \quad \|L\|_{\mathscr{L}(\mathrm{FM}_0(H_1),\, \mathrm{FM}_0(E))} \leq M. \tag{2.19}$$

By the fact that only the multiplier σ_L depends on $t \in J$ and not the measure to that it is applied, we can estimate as follows:

$$\begin{aligned}
\|Lu\|_{L^2(J, \mathrm{FM}(H_2))} &= \| \, |\widehat{u}\lfloor \sigma_L(\cdot)|_{H_2}(\mathbb{R}^n) \, \|_{L^2(J)} \\
&= \left\| \int_{\mathbb{R}^n} \|\sigma_L(\cdot, \xi)\rho_{\widehat{u}}(\xi)\|_{H_2} \mathrm{d}|\widehat{u}|(\xi) \right\|_{L^2(J)} \\
&\leq \int_{\mathbb{R}^n} \|\sigma_L(\cdot, \xi)\rho_{\widehat{u}}(\xi)\|_E \mathrm{d}|\widehat{u}|(\xi) \\
&= |\widehat{u}\lfloor \sigma_L|_E(\mathbb{R}^n) \\
&= \|Lu\|_{\mathrm{FM}(E)} \quad (u \in \mathrm{FM}_0(\mathbb{R}^n, H_1)).
\end{aligned}$$

Here we employed the notation $|\cdot|_{H_2}$ and $|\cdot|_E$ in order to highlight that $\widehat{u}\lfloor \sigma_L$ is once regarded as an H_2-valued and once as an E-valued measure. Observe that the first equality in Lemma 2.6(i) is crucial in order to obtain the second last equality above. Relation (2.19) now implies the assertion. □

We also prepare the following general result on vector-valued convolution.

Lemma 2.16. *Let X, Y be Banach spaces, $1 \le p \le \infty$, $T \in (0, \infty]$, and set $J = (0, T)$. For $g \in \mathscr{L}(X, L^p(J, Y))$ and $f \in L^1(J, X)$ we have*

$$\left(t \mapsto g \star f := \int_0^t g(t-s) f(s) \mathrm{d}s \right) \in L^p(J, Y)$$

and

$$\|g \star f\|_{L^p(J,Y)} \le \|g\|_{\mathscr{L}(X, L^p(J,Y))} \|f\|_{L^1(J,X)}.$$

Proof. For a function h defined on J we set

$$\tilde{h}(t) := \begin{cases} h(t), & t \in J, \\ 0, & \text{elsewhere.} \end{cases}$$

Pick $g \in \mathscr{L}(X, L^p(J, Y))$ and $f \in L^1(J, X)$. By assumption we have for a.e. $s \in J$ that

$$(t \mapsto \tilde{g}(t-s)\tilde{f}(s)) \in L^p(\mathbb{R}, Y)$$

and that

$$\begin{aligned} \|\tilde{g}(t-s)\tilde{f}(s)\|_{L^p(\mathbb{R},Y)} &= \left(\int_s^{T+s} \|g(t-s)f(s)\|_Y^p \mathrm{d}t \right)^{1/p} \\ &= \left(\int_0^T \|g(r)f(s)\|_Y^p \mathrm{d}r \right)^{1/p} \\ &\le \|g\|_{\mathscr{L}(X,L^p(J,Y))} \|f(s)\|_X \\ &= \|g\|_{\mathscr{L}(X,L^p(J,Y))} \|\tilde{f}(s)\|_X. \end{aligned} \tag{2.20}$$

For $s \in \mathbb{R} \setminus J$ this estimate is trivially true. This yields

$$\left(s \mapsto \|\tilde{g}(t-s)\tilde{f}(s)\|_{L^p(\mathbb{R},Y)} \right) \in L^1(\mathbb{R}).$$

Hence $\int_{\mathbb{R}} \tilde{g}(t-s)\tilde{f}(s)\mathrm{d}s$, and therefore also $g \star f(t)$, exists as a Bochner integral with values in $L^p(J, Y)$. Thanks to (2.20) we also obtain

$$\begin{aligned} \|g \star f\|_{L^p(J,Y)} &= \| \int_{\mathbb{R}} \tilde{g}(t-s)\tilde{f}(s)\mathrm{d}s \|_{L^p(\mathbb{R},Y)} \\ &\le \int_{\mathbb{R}} \|\tilde{g}(t-s)\tilde{f}(s)\|_{L^p(J,Y)} \mathrm{d}s \\ &\le \|g\|_{\mathscr{L}(X,L^p(J,Y))} \int_{\mathbb{R}} \|\tilde{f}(s)\|_X \mathrm{d}s \\ &= \|g\|_{\mathscr{L}(X,L^p(J,Y))} \|f\|_{L^1(J,X)}. \end{aligned}$$

□

Finally, we establish some density properties. For $k \in \mathbb{N}$ we set

$$\begin{aligned} \mathrm{FM}^k(\mathbb{R}^n, X) &:= \{u \in \mathrm{FM}(\mathbb{R}^n, X) : \partial^\alpha u \in \mathrm{FM}(\mathbb{R}^n, X) \ (0 \le |\alpha| \le k)\}, \\ \|u\|_{\mathrm{FM}^k(X)} &:= \sum_{0 \le |\alpha| \le k} \|\partial^\alpha u\|_{\mathrm{FM}(X)}. \end{aligned}$$

Further we define $\mathrm{FM}^\infty(\mathbb{R}^n,X):=\bigcap_{k=0}^\infty \mathrm{FM}^k(\mathbb{R}^n,X)$ equipped with the canonical topology. The spaces $\mathrm{FM}_0^k(\mathbb{R}^n,X)$, $k\in\mathbb{N}_0\cup\{\infty\}$, are defined accordingly.

Lemma 2.17. *Let a Banach space X having the Radon-Nikodým property and a domain $\Omega\subseteq\mathbb{R}^m$ be given. Then we have:*

(i) *For every $k\in\mathbb{N}_0\cup\{\infty\}$ the space $\mathrm{FM}^k(\mathbb{R}^n,X)$ lies dense in $\mathrm{FM}(\mathbb{R}^n,X)$.*
(ii) *For every $k\in\mathbb{N}_0\cup\{\infty\}$ and $p\in[1,\infty)$ the space $\mathrm{FM}^k(\mathbb{R}^n,C_c^\infty(\Omega)^n)$ lies dense in $\mathrm{FM}(\mathbb{R}^n,L^p(\Omega)^n)$.*

Assertions (i) and (ii) remain true, if FM *is replaced by* FM_0.

Remark 2.18. (a) Although $C_c^\infty(\Omega)$ is not a Banach space it is clear how to understand $\mathrm{FM}^k(\mathbb{R}^n,C_c^\infty(\Omega)^n)$, by the fact that $C_c^\infty(\Omega)\hookrightarrow L^p(\Omega)$, for example.

(b) We note that $\mathrm{FM}^k(\mathbb{R}^n,X)\hookrightarrow \mathrm{BC}^k(\mathbb{R}^n,X)$ for $k\in\mathbb{N}\cup\{\infty\}$ (see Lemma 2.12(iii)).

(c) Also observe that there is no dense set of decaying functions in $\mathrm{FM}(\mathbb{R}^n,X)$. This follows by the fact that the norm in $\mathrm{FM}(\mathbb{R}^n,X)$ is stronger than the norm in $L^\infty(\mathbb{R}^n,X)$.

Proof. (i) Choose a mollifier φ_ε, i.e. $\varphi_\varepsilon(x)=\frac{1}{\varepsilon^n}\varphi_0(x/\varepsilon)$ with $0\neq\varphi_0\in C_c^\infty(\mathbb{R}^n)$, $\varphi_0\geq 0$, and $\int_{\mathbb{R}^n}\varphi_0(x)\mathrm{d}x=1$. For $u\in\mathrm{FM}(\mathbb{R}^n,X)$ we set

$$u_\varepsilon:=\varphi_\varepsilon * u=\int_{\mathbb{R}^n}\varphi_\varepsilon(x-y)u(y)\mathrm{d}y.$$

Since $\mathrm{FM}(\mathbb{R}^n,X)\hookrightarrow\mathrm{BUC}(\mathbb{R}^n,X)$ we readily have $u_\varepsilon\in C^\infty(\mathbb{R}^n,X)$. Furthermore, we obtain

$$\begin{aligned}
\partial^\alpha(\varphi_\varepsilon * u)(x) &= \int_{\mathbb{R}^n} u(x-y)\partial^\alpha\varphi_\varepsilon(y)\mathrm{d}y\\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\xi}\rho_{\widehat{u}}(\xi)\mathrm{d}|\widehat{u}|(\xi)\partial^\alpha\varphi_\varepsilon(y)\mathrm{d}y\\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\xi}\rho_{\widehat{u}}(\xi)\int_{\mathbb{R}^n} e^{-iy\xi}\partial^\alpha\varphi_\varepsilon(y)\mathrm{d}y\mathrm{d}|\widehat{u}|(\xi)\\
&= \int_{\mathbb{R}^n} e^{ix\xi}\rho_{\widehat{u}}(\xi)\mathcal{F}\partial^\alpha\varphi_\varepsilon(\xi)\mathrm{d}|\widehat{u}|(\xi)\\
&= (2\pi)^{n/2}\mathcal{F}^{-1}(\widehat{u}\lfloor\mathcal{F}\partial^\alpha\varphi_\varepsilon)(x)\quad(x\in\mathbb{R}^n,\ \alpha\in\mathbb{N}_0^n),
\end{aligned}$$

where we applied Lemma 2.6(iii) in the last step. This implies that $\mathcal{F}\partial^\alpha u_\varepsilon=(2\pi)^{n/2}\widehat{u}\lfloor\mathcal{F}\partial^\alpha\varphi_\varepsilon\in\mathrm{M}(X)$. Consequently $\partial^\alpha u_\varepsilon\in\mathrm{FM}(X)$ for $\varepsilon>0$ and $\alpha\in\mathbb{N}_0^n$. Next, observe that

$$(\widehat{u}_\varepsilon-\widehat{u})(\mathcal{O})=\int_{\mathcal{O}}(\hat{\varphi}_\varepsilon\rho_{\widehat{u}}-\rho_{\widehat{u}})\mathrm{d}|\widehat{u}|\quad(\mathcal{O}\in\mathscr{B}(\mathbb{R}^n)).$$

Thus,

$$\|u_\varepsilon-u\|_{\mathrm{FM}}=\|\widehat{u}_\varepsilon-\widehat{u}\|_{\mathrm{M}}\leq\int_{\mathbb{R}^n}|\hat{\varphi}_\varepsilon-1|\|\rho_{\widehat{u}}\|_X\mathrm{d}|\widehat{u}|\to 0\quad(\varepsilon\to 0)$$

by dominated convergence, and (i) follows.

(ii) Pick $u \in \mathrm{FM}(\mathbb{R}^n, L^p(\Omega)^n)$. By (i) we may even assume that

$$u \in \mathrm{FM}^\infty(\mathbb{R}^n, L^p(\Omega)^n).$$

In the present situation we choose a mollifier φ_ε satisfying the properties in (i) in the last variable, that is, $\varphi_\varepsilon : \mathbb{R}^m \to [0,\infty)$. Furthermore, let $(K_j)_{j\in\mathbb{N}}$ be an exhausting sequence of Ω, i.e., each $K_j \subset \Omega$ is compact and we have

$$K_j \subset \mathring{K}_{j+1} \quad (j \in \mathbb{N}), \qquad \Omega = \bigcup_{j=1}^{\infty} K_j,$$

where $\mathring{K}_{j+1}$ denotes the interior of K_{j+1}. We also choose $\psi_j \in C^\infty(\Omega)$ such that $\psi_j \equiv 1$ on K_j, $\psi_j \equiv 0$ outside K_{j+1}, and $0 \le \psi_j \le 1$ on Ω. Then $(\psi_j)_{j\in\mathbb{N}} \subset C_c^\infty(\Omega)$ and $\psi_j \to 1$ pointwisely in Ω. We consider the sequence

$$u_{\varepsilon,j}(t,x) := \psi_j(x)(\varphi_\varepsilon *_x u)(t,x) \quad (\varepsilon > 0,\ j \in \mathbb{N},\ t \in \mathbb{R}^n,\ x \in \Omega).$$

Here '$*_x$' denotes the convolution with respect to $x \in \mathbb{R}^m$, i.e., we set

$$(\varphi_\varepsilon *_x u)(t,x) = \int_\Omega \varphi_\varepsilon(x-y)u(t,y)\mathrm{d}y, \quad x \in \Omega,\ t \in \mathbb{R}^n.$$

For functions $a \in \mathrm{BC}(\mathbb{R}^m)$ and $b \in L^1(\mathbb{R}^m)$ it is easily checked that

$$[(t,x) \mapsto a(x)(b *_x u)(t,x)] \in \mathrm{FM}(\mathbb{R}^n, L^p(\Omega)^n). \tag{2.21}$$

From this we already obtain

$$\partial^\alpha u_{\varepsilon,j} \in \mathrm{FM}(\mathbb{R}^n, L^p(\Omega)^n) \quad (\varepsilon > 0,\ j \in \mathbb{N},\ \alpha \in \mathbb{N}_0^{n+m}). \tag{2.22}$$

In fact, writing $\partial^\alpha = \partial_t^{\alpha_1}\partial_x^{\alpha_2}$, the Leibniz rule gives us

$$\partial^\alpha u_{\varepsilon,j}(t,x) = \sum_{\beta\le\alpha_2} c(\alpha_2,\beta)(\partial^{\alpha_2-\beta}\psi_j)(x)([\partial^\beta\varphi_\varepsilon] *_x \partial_t^{\alpha_1}u)(t,x)$$

with certain constants $c(\alpha_2,\beta) > 0$ and where $\beta \le \alpha$ is understood componentwise. From this representation we see that the single summands possess the structure given in (2.21). Thus (2.22) follows. So, we have proved $u_{\varepsilon,j} \in \mathrm{FM}^\infty(\mathbb{R}^n, W^{k,p}(\Omega))$ for all $k \in \mathbb{N}$, hence that $u_{\varepsilon,j} \in \mathrm{FM}^\infty(\mathbb{R}^n, C_c^\infty(\Omega))$ by the Sobolev embedding and since ψ_j has compact support.

In view of

$$(\widehat{u}_{\varepsilon,j} - \widehat{u})(\mathcal{O},x) = \int_{\mathcal{O}} \big[\psi_j(x)(\varphi_\varepsilon *_x \rho_{\widehat{u}})(t,x) - \rho_{\widehat{u}}(t,x)\big]\mathrm{d}|\widehat{u}|(t),$$

we can calculate

$$\begin{aligned}
\|u_{\varepsilon,j} - u\|_{\mathrm{FM}(L^p)} &= \|\widehat{u}_{\varepsilon,j} - \widehat{u}\|_{\mathrm{M}(L^p)} \\
&\le \sup_{\Pi(\mathbb{R}^n)} \sum_{E_\ell\in\Pi(\mathbb{R}^n)} \|(\widehat{u}_{\varepsilon,j} - \widehat{u})(E_\ell)\|_p \\
&\le \int_{\mathbb{R}^n} \|\psi_j(\varphi_\varepsilon *_x \rho_{\widehat{u}})(t) - \rho_{\widehat{u}}(t)\|_p \,\mathrm{d}|\widehat{u}|(t).
\end{aligned}$$

Clearly, we have

$$\|\psi_j(\varphi_\varepsilon *_x \rho_{\widehat{u}})(t) - \rho_{\widehat{u}}(t)\|_p \to 0 \quad (\varepsilon \to 0,\ j \to \infty)$$

for a.e. $t \in \mathbb{R}^n$. Moreover,

$$\begin{aligned}\|\psi_j(\varphi_\varepsilon *_x \rho_{\widehat{u}})(t) - \rho_{\widehat{u}}(t)\|_p &\leq \|\varphi_\varepsilon *_x \rho_{\widehat{u}}(t)\|_p + \|\rho_{\widehat{u}}(t)\|_p \\ &\leq (\|\varphi_\varepsilon\|_1 + 1)\|\rho_{\widehat{u}}(t)\|_p \\ &\leq 2\|\rho_{\widehat{u}}(t)\|_p \quad (t \in \mathbb{R}^n).\end{aligned}$$

The dominated convergence theorem therefore yields

$$\|u_{\varepsilon,j} - u\|_{\mathrm{FM}(L^p)} \to 0 \quad (\varepsilon \to 0,\ j \to \infty).$$

This implies (ii). The additional assertion is obvious. □

3. The linearized Ekman problem

From now on let $X = X_p := L^p(D)^n$, where D denotes either one of the intervals $(0, \infty)$ or $(0, d)$ for some fixed $d > 0$. We start with deriving the Helmholtz decomposition of the space $\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$ and the required generator result for the Stokes operator on its solenoidal subspace. These facts will be proved in the spirit of Remark 2.14, in particular of relation (2.18). Since this is not so standard, let us briefly explain the strategy we pursue here.

The main point is to derive suitable operator-valued representations for the symbols of Helmholtz projection and Stokes resolvent (see (3.7) and (3.16)). This can be obtained by taking advantage of the well-known L^p-versions of the desired results (see Lemma 3.2). Thanks to the L^p-boundedness of the operators under discussion and due to Lemma 3.1 below, we can prove their symbols to be bounded functions. Continuity of the symbols can be read off directly from their representations. Proposition 2.13 then yields boundedness of the corresponding operators in $\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$. In other words, we rigorously verified relation (2.18) for the symbol of the Helmholtz projection and of the Stokes resolvent. A priori this method merely yields boundedness of operators associated to certain symbols. Thus, finally it has to be shown that these operators indeed possess the desired properties of the Helmholtz projection or the Stokes resolvent also in $\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$. This again reduces to the validity of these properties in L^p.

Of course, Helmholtz decomposition and sectoriality of the Stokes operator in $\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$ could also be obtained without utilizing the counterparts of these results in L^p. Based on the multiplier result Proposition 2.13, these facts could be obtained by directly estimating corresponding explicit solution formulas in $\mathbb{R}^{n-1} \times D$. Such formulas, however, are somewhat lengthy and of intricate structure, especially corresponding representations for the Stokes resolvent in a layer, cf. [1], [2]. As a consequence, this more direct way would enlarge the proofs enourmously. It is much more convenient to take advantage of knowledge in L^p and, based on this, to deal with the operator-valued formulas (3.7) and (3.16). Those have a nicer and much more compact structure than explicit formulas in $\mathbb{R}^{n-1} \times D$, especially in the case $d < \infty$.

In order to follow the strategy just explained, next we recall three known results from the L^p-setting. Note that also here and in Section 4 we will frequently make

use of the short hand notation $\mathrm{M}(X)$, $\mathrm{FM}(X)$ and so on, since the domain $\mathbb{R}^{n-1}$ essentially will be fixed.

Lemma 3.1. *Let E, F be Banach spaces and $1 < p < \infty$. We denote by the set $\mathcal{M}_p(\mathbb{R}^n, \mathscr{L}(E,F))$ the class of all $\mathscr{L}(E,F)$-valued Fourier multipliers on $L^p(\mathbb{R}^n, E)$, i.e.,*

$$\mathcal{M}_p(\mathbb{R}^n, \mathscr{L}(E,F)) := \big\{ m : \mathbb{R}^n \setminus \{0\} \to \mathscr{L}(E,F); \\ \mathcal{F}^{-1} m \mathcal{F} \in \mathscr{L}\big(L^p(\mathbb{R}^n, E),\, L^p(\mathbb{R}^n, F)\big) \big\}$$

endowed with the norm $\|m\|_{\mathcal{M}_p(\mathbb{R}^n, \mathscr{L}(E,F))} = \|\mathcal{F}^{-1} m \mathcal{F}\|_{\mathscr{L}(L^p(\mathbb{R}^n,E),\, L^p(\mathbb{R}^n,F))}$. Then, $\mathcal{M}_p(\mathbb{R}^n, \mathscr{L}(E,F)) \hookrightarrow L^\infty(\mathbb{R}^n, \mathscr{L}(E,F))$. More precisely, we have

$$\|m\|_{L^\infty(\mathbb{R}^n, \mathscr{L}(E,F))} \le \|m\|_{\mathcal{M}_p(\mathbb{R}^n, \mathscr{L}(E,F))} \quad (m \in \mathcal{M}_p(\mathbb{R}^n, \mathscr{L}(E,F))).$$

Proof. Pick $m \in \mathcal{M}_p(\mathbb{R}^n, \mathscr{L}(E,F))$, $x \in E$, and $y' \in F'$. Then $\langle m(\cdot)x, y' \rangle$ is a scalar-valued multiplier. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing of F with F'. For such multipliers it is well-known that

$$\mathcal{M}_p(\mathbb{R}^n) \hookrightarrow \mathcal{M}_2(\mathbb{R}^n) = L^\infty(\mathbb{R}^n).$$

In particular, we have $\|m\|_{L^\infty(\mathbb{R}^n)} \le \|m\|_{\mathcal{M}_p(\mathbb{R}^n)}$ for all $m \in \mathcal{M}_p(\mathbb{R}^n)$. Note that these classes were introduced by Mikhlin, cf. [42], [43]. By the commutativity of suprema this yields

$$\begin{aligned}
\|m\|_{L^\infty(\mathbb{R}^n, \mathscr{L}(E,F))} &= \sup_{\xi \in \mathbb{R}^n} \sup_{\|x\|_E = 1} \sup_{\|y'\|_{F'} = 1} |\langle m(\xi)x, y' \rangle| \\
&= \sup_{\|x\|_E = 1} \sup_{\|y'\|_{F'} = 1} \|\langle mx, y' \rangle\|_{L^\infty(\mathbb{R}^n)} \\
&\le \sup_{\|x\|_E = 1} \sup_{\|y'\|_{F'} = 1} \|\mathcal{F}^{-1} \langle mx, y' \rangle \mathcal{F}\|_{\mathscr{L}(L^p(\mathbb{R}^n))} \\
&\le \sup_{\|x\|_E = 1} \sup_{\|y'\|_{F'} = 1} \sup_{\|f\|_{L^p(\mathbb{R}^n)} = 1} \|\langle \mathcal{F}^{-1} m \mathcal{F} f x, y' \rangle\|_{L^p(\mathbb{R}^n)} \\
&\le \sup_{\|x\|_E = 1} \sup_{\|f\|_{L^p(\mathbb{R}^n)} = 1} \|\mathcal{F}^{-1} m \mathcal{F} f x\|_{L^p(\mathbb{R}^n, F)} \\
&\le \|\mathcal{F}^{-1} m \mathcal{F}\|_{\mathscr{L}(L^p(\mathbb{R}^n, E),\, L^p(\mathbb{R}^n, F))}.
\end{aligned}$$

□

In what follows, we denote by $\mathcal{N}$ the outer normal vector at the boundary ∂G of a domain $G \subset \mathbb{R}^n$ and by Σ_θ the complex sector

$$\Sigma_\theta := \{ z \in \mathbb{C} \setminus \{0\} : \ |\arg z| < \theta \}$$

for $\theta \in (0, \pi)$.

Lemma 3.2. *Let $1 < p < \infty$, $\nu > 0$, $d \in (0, \infty]$, and set $G = \mathbb{R}^{n-1} \times (0, d)$.*

(i) *On G we have the Helmholtz decomposition*

$$L^p(G) = L^p_\sigma(G) \oplus G_p(G),$$

where

$$\begin{aligned} L^p_\sigma(G) &= \{u \in L^p(G) :\ \operatorname{div} u = 0,\ \mathcal{N}\cdot u = 0 \text{ on } \partial G\} \\ &= \overline{\{u \in C^\infty_c(G)^n :\ \operatorname{div} u = 0\}}^{\|\cdot\|_p}, \\ G_p(G) &= \{\nabla p :\ p \in L^1_{loc}(G),\ \nabla p \in L^p(G)\}. \end{aligned}$$

In particular, the associated Helmholtz projection $P : L^p(G) \to L^p(G)$ *is bounded and we have* $\|P\|_{\mathscr{L}(L^2(G))} = 1$.

(ii) *The Stokes operator*

$$A_\nu := -\nu P\Delta, \quad \mathscr{D}(A_\nu) := W^{2,p}(G) \cap W^{1,p}_0(G) \cap L^p_\sigma(G)$$

is the generator of a bounded holomorphic C_0*-semigroup* $(e^{-tA_\nu})_{t\geq 0}$ *on* $L^p_\sigma(G)$ *and for each* $\varphi_0 \in (0,\pi)$ *there exists a* C_{φ_0} *such that*

$$\|\lambda^{k/2}\partial^\alpha(\lambda + A_\nu)^{-1}\|_{\mathscr{L}(L^p_\sigma(G),\, L^p(G))} \leq C_{\varphi_0} \quad (\lambda \in \Sigma_{\varphi_0},\ k + |\alpha| = 2).$$

Here $k \in \mathbb{N}_0$ *and* $\alpha \in \mathbb{N}^n_0$. *In particular, for* $p = 2$ A_ν *is positive selfadjoint and the family* $(e^{-tA_\nu})_{t\geq 0}$ *forms a semigroup of contractions.*

Proof. For the case $d = \infty$ the existence of the Helmholtz projection in (i) for instance is proved in [40]; assertion (ii) for the case of $d = \infty$ is obtained in [35]. We also refer to [8] for (i) and (ii) and the case $d = \infty$. For the case of a layer, that is $d < \infty$, statement (ii) is proved in [1] and [2]; for both (i) and (ii) and the case of $d < \infty$ we refer to [3], [4]. □

Now we are in position to establish the Helmholtz decomposition of the space $\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$. To be precise, we will show that

$$\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p) = \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p) \oplus G_{\mathrm{FM}}, \tag{3.1}$$

with

$$\begin{aligned} &\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p) \\ &:= \overline{\left\{u \in \mathrm{FM}^\infty_0\left(\mathbb{R}^{n-1}, \bigcap_{k=0}^\infty W^{k,p}(D)\right);\ \operatorname{div} u = 0,\ \mathcal{N}\cdot u|_{\partial(\mathbb{R}^{n-1}\times D)} = 0\right\}}^{\|\cdot\|_{\mathrm{FM}}} \end{aligned} \tag{3.2}$$

and

$$G_{\mathrm{FM}} = \{\nabla p;\ p \in L^1_{loc}(\mathbb{R}^{n-1}\times D),\ \nabla p \in \mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)\}. \tag{3.3}$$

Recall that the existence of (3.1) is (at first formally) equivalent to the unique solvability (modulo constants) of the Neumann problem

$$\left\{\begin{array}{rcl} \Delta p &=& \operatorname{div} u \quad \text{in } \mathbb{R}^{n-1}\times D, \\ \partial_{\mathcal{N}} p &=& \mathcal{N}\cdot u \quad \text{on } \partial(\mathbb{R}^{n-1}\times D), \end{array}\right. \tag{3.4}$$

in the weak sense. The corresponding Helmholtz projection associated to (3.1) then is given as

$$Pu = u - \nabla p$$

with p the solution of (3.4).

Definition 3.3. In what follows we will make use of the notation $x = (x', x_n)$, i.e., $x' \in \mathbb{R}^{n-1}$ denotes the tangential part of $x \in \mathbb{R}^n$. The same notation we will use for vector fields u, that is, we write $u = (u', u^n)$, $\rho = (\rho', \rho^n)$, etc.

As explained in the beginning of this section, the boundedness of P is proved in the spirit of Remark 2.14. Thus, we start with deriving a suitable operator-valued symbol representation. For this purpose, assume that

$$u \in L^p(\mathbb{R}^{n-1}, C_c^\infty(D)^n) \overset{d}{\hookrightarrow} L^p(\mathbb{R}^{n-1}, X_p) \cong L^p(\mathbb{R}^{n-1} \times D)^n.$$

Dealing with those u has the advantage that the trace condition in (3.4) is homogeneous. Applying Fourier transformation in tangential direction to (3.4), we are left with the ODE

$$\left\{ \begin{array}{rcll} (|\xi'|^2 - \partial_n^2)\widehat{p} & = & -i\xi' \cdot \widehat{u}' - \partial_n \widehat{u}^n & \text{in } D, \\ \partial_{\mathcal{N}} \widehat{p} & = & 0 & \text{on } \partial D. \end{array} \right. \tag{3.5}$$

Let $\Delta_{N,n}$ denote the Neumann Laplacian in the normal variable x_n. By well-known results for this operator the solution of (3.5) is represented by

$$\widehat{p}(\xi', \cdot) = -i\xi' \cdot (|\xi'|^2 - \Delta_{N,n})^{-1} \widehat{u}'(\xi', \cdot) - (|\xi'|^2 - \Delta_{N,n})^{-1} \partial_n \widehat{u}^n(\xi', \cdot), \quad \xi' \neq 0. \tag{3.6}$$

We define the operator-valued symbol σ_P by

$$\begin{aligned} \sigma_P(\xi')v &= v + \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} \left(i\xi' \cdot (|\xi'|^2 - \Delta_{N,n})^{-1} v' + (|\xi'|^2 - \Delta_{N,n})^{-1} \partial_n v^n \right) \\ &= v + \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} (|\xi'|^2 - \Delta_{N,n})^{-1} \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix}^T v \end{aligned} \tag{3.7}$$

for $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ and $v \in C_c^\infty(D)^n$. Obviously this is exactly the symbol of the Helmholtz projection and we have

$$Pu = u - \nabla p = \mathcal{F}^{-1} \sigma_P \mathcal{F} u \quad (u \in L^p(\mathbb{R}^{n-1}, C_c^\infty(D))). \tag{3.8}$$

By virtue of Lemma 3.2(i) and Lemma 3.1 it is clear that σ_P extends to X_p. However, in the sequel it will be advantageous to know that also representation (3.7) is still valid for $v \in X_p$ (see the latter part of the proof of Lemma 3.4). This can be seen via abstract arguments for sectorial operators. In fact, by utilizing these (well-known) arguments, the resolvent of $\Delta_{N,n}$ has a natural extension on $W_0^{-1,p}(D) := (W^{1,p'}(D))'$, which for simplicity again is denoted by $(\lambda - \Delta_{N,n})^{-1}$. Indeed, we have $(\lambda - \Delta_{N,n})^{-1} \in \mathscr{L}(W_0^{-1,p}(D), W^{1,p}(D))$. Since $\partial_n(L^p(D)) \hookrightarrow W_0^{-1,p}(D)$, we see that $(|\xi'|^2 - \Delta_{N,n})^{-1} \partial_n v^n$, $\xi \neq 0$, is a well-defined element in $W^{1,p}(D)$ for every $v^n \in L^p(D)$. Therefore representations (3.7) and (3.8) hold for all $v \in X_p$ and all $u \in L^p(\mathbb{R}^{n-1} \times D)^n$, respectively.

We also remark that then $p = \mathcal{F}^{-1}\widehat{p}$ with $\widehat{p}$ given through (3.6) with the generalized resolvent of $\Delta_{N,n}$ represents the solution of (3.4) for general data $u \in L^p(\mathbb{R}^{n-1} \times D)^n$. This fact, however, will not be used in the sequel. In the following proof we just concentrate on the symbol σ_P defined by representation (3.7) on X_p, with $(|\xi'|^2 - \Delta_{N,n})^{-1}$ being interpreted as the above constructed extension on $W_0^{-1,p}(D)$.

Lemma 3.4. *The Helmholtz projection P is bounded on $\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$ and decomposition (3.1) holds.*

Proof. Lemma 3.2(i) and Lemma 3.1 imply

$$\|\sigma_P(\xi')v\|_{X_p} \le C\|v\|_{X_p} \quad (\xi' \in \mathbb{R}^{n-1} \setminus \{0\},\ v \in X_p), \tag{3.9}$$

even with $C = 1$ if $p = 2$. With the help of representation (3.7) and estimate (3.9) it is not difficult to verify that σ_P satisfies

$$\sigma_P \in \mathrm{BC}(\mathbb{R}^{n-1} \setminus \{0\}, \mathscr{L}(X_p)). \tag{3.10}$$

Proposition 2.13 therefore implies that

$$P = op(\sigma_P) \in \mathscr{L}(\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)). \tag{3.11}$$

Moreover, the representation

$$Pu = \mathcal{F}^{-1}(\widehat{u}\lfloor\sigma_P)$$

with σ_P given through (3.7) holds for all $u \in \mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$.

Note that by the fact that $L^p(\mathbb{R}^{n-1} \times D)^n \cap \mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$ is not dense in $\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$ we still have to prove that P as an operator on $\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$ admits the usual properties of the Helmholtz projection. To this end, we take advantage of the function $\psi(x') := e^{-|x'|^2/2}$, which is known to represent a fixed point for the Fourier transformation. First we show that P is indeed a projection on $\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$. By $P^2u = Pu$ for all $u \in L^p(\mathbb{R}^{n-1} \times D)^n$ we deduce

$$(\sigma_P^2(\xi') - \sigma_P(\xi'))\widehat{u}(\xi') = 0 \quad (u \in L^p(\mathbb{R}^{n-1} \times D)^n,\ \xi' \in \mathbb{R}^{n-1} \setminus \{0\}).$$

Let $v \in X_p$ and set $u := \psi v$. Then we have $u \in L^p(\mathbb{R}^{n-1} \times D)^n$ and

$$\psi(\xi')(\sigma_P^2(\xi') - \sigma_P(\xi'))v = 0 \quad (\xi' \in \mathbb{R}^{n-1} \setminus \{0\}).$$

This implies $\sigma_P^2 v = \sigma_P v$ for all $v \in X_p$, from which we conclude $\widehat{u}\lfloor\sigma_P^2 = \widehat{u}\lfloor\sigma_P$ for all $u \in \mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$. So, we have proved

$$P^2u = Pu \quad (u \in \mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)).$$

We set

$$\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p) = P(\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)).$$

It remains to prove characterizations (3.2) and (3.3).

The fact that $\operatorname{div} Pu = 0$ for $u \in \mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$ follows from the validity of this equation in L^p in the same manner as we proved $P^2u = Pu$ (or by a direct calculation). Next, again by known facts for $\Delta_{N,n}$ we have

$$(|\xi'|^2 - \Delta_{N,n})^{-1}(W^{k,p}(D)) \hookrightarrow W^{k,p}(D) \quad (k \in \mathbb{N}_0,\ 1 < p < \infty).$$

Utilizing this relation, from the formula

$$\partial^\alpha Pu = \mathcal{F}^{-1}((\mathcal{F}\partial_{x'}^{\alpha_1}u)\lfloor\partial_{x_n}^{\alpha_2}\sigma_P) \quad (\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^n)$$

and representation (3.7) we can derive

$$P(\mathrm{FM}^\infty(\mathbb{R}^{n-1}, C_c^\infty(D))) \hookrightarrow \mathrm{FM}^\infty(\mathbb{R}^{n-1}, W^{k,p}(D)) \quad (k \in \mathbb{N}_0).$$

Noting that $\mathrm{FM}^\infty(\mathbb{R}^{n-1}, W^{k,p}(D)) \hookrightarrow \mathrm{BUC}(\mathbb{R}^{n-1}, \mathrm{BUC}(\overline{D}))$ for $k \geq 1$, we see that the trace $\mathcal{N}\cdot u|_{\partial(\mathbb{R}^{n-1}\times D)}$ makes sense for functions $u \in P(\mathrm{FM}^\infty(\mathbb{R}^{n-1}, C_c^\infty(D)))$. We consider the case $D = (0,\infty)$. Then $\mathcal{N}\cdot(Pu)|_{\partial(\mathbb{R}^{n-1}\times D)} = -(Pu)^n|_{x_n=0}$. Since the trace operator $\gamma : v \mapsto v|_{x_n=0}$ acts as a continuous linear operator from $\mathrm{BUC}(\mathbb{R}_+)$ to $\mathbb{R}$, we have $\gamma(Pu)^n = \mathcal{F}^{-1}\int_{\mathbb{R}^{n-1}} \gamma(\sigma_P(\xi')\rho_{\widehat{u}}(\xi'))^n \mathrm{d}|\widehat{u}|(\xi')$.

Thus, it remains to show that

$$(\sigma_P\rho_{\widehat{u}})^n(\xi', 0) = 0 \tag{3.12}$$

for $|\widehat{u}|$-a.e. $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$. On the other hand, from (3.7) we conclude

$$\begin{aligned}(\sigma_P\rho_{\widehat{u}})^n(\xi', \cdot) \ &= \rho^n_{\widehat{u}}(\xi', \cdot) + i\xi'\cdot\partial_n(|\xi'|^2 - \Delta_{N,n})^{-1}\rho'_{\widehat{u}}(\xi', \cdot) \\ &\quad + \partial_n(|\xi'|^2 - \Delta_{N,n})^{-1}\partial_n\rho^n_{\widehat{u}}(\xi', \cdot).\end{aligned} \tag{3.13}$$

Also observe that

$$0 = \gamma u(x') = \frac{1}{(2\pi)^{(n-1)/2}}\int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'}\gamma\rho_{\widehat{u}}(\xi')\mathrm{d}|\widehat{u}|(\xi') \quad (x' \in \mathbb{R}^{n-1}).$$

Hence we have $\rho_{\widehat{u}}(\xi', 0) = 0$ $|\widehat{u}|$-a.e. The fact that $u \in \mathrm{FM}^\infty(\mathbb{R}^{n-1}, W^{k,p}(D))$ also implies that $\partial_n\rho^n_{\widehat{u}}(\xi', \cdot) \in L^p(D)$. Taking trace in (3.13) therefore yields (3.12). The case of a layer, i.e. $D = (0, d)$, follows completely analogous. A density argument then results in characterization (3.2).

To see (3.3), let $w := (I - P)u$ for $u \in \mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)$. Then $w \in L^1_{loc}(\mathbb{R}^{n-1}\times D)$ and $Pw = 0$. This yields

$$0 = \hat{w}\lfloor\sigma_P(\mathcal{O}) = \int_{\mathcal{O}} \sigma_P(\xi')\rho_{\hat{w}}(\xi')\mathrm{d}|\hat{w}|(\xi') \quad (\mathcal{O} \in \mathscr{B}(\mathbb{R}^{n-1})),$$

from which we conclude that $\sigma_P(\xi')\rho_{\hat{w}}(\xi') = 0$ for $|\hat{w}|$-a.e. $\xi' \in \mathbb{R}^{n-1}$. Furthermore, with the help of representation (3.7) we easily find that

$$\begin{aligned}&\int_D \overline{\sigma_P(\xi')\varphi(x_n)}^T\psi(x_n)\mathrm{d}x_n = \int_D \overline{\varphi(x_n)}^T\sigma_P(\xi')\psi(x_n)\,\mathrm{d}x_n \\ &\quad (\xi' \in \mathbb{R}^{n-1}\setminus\{0\},\ \varphi \in X_p,\ \psi \in X_{p'}).\end{aligned}$$

This gives us

$$\begin{aligned}\langle w,\, v\rangle_{L^\infty,L^1} &= \langle w,\, Pv\rangle_{L^\infty,L^1} \\ &= \int_D \langle w,\, Pv(x_n)\rangle_{\mathcal{S}'(\mathbb{R}^{n-1},X_p),\mathcal{S}(\mathbb{R}^{n-1})}(x_n)\mathrm{d}x_n \\ &= \int_D \left(\hat{w}\lfloor[\overline{(\sigma_P\hat{v})(x_n)}^T]\right)(x_n)\mathrm{d}x_n \\ &= \int_D\int_{\mathbb{R}^{n-1}} \overline{(\sigma_P\hat{v})(\xi', x_n)}^T\rho_{\hat{w}}(\xi', x_n)\mathrm{d}|\hat{w}|(\xi')\mathrm{d}x_n \\ &= \int_{\mathbb{R}^{n-1}}\int_D \overline{(\sigma_P\hat{v})(\xi', x_n)}^T\rho_{\hat{w}}(\xi', x_n)\mathrm{d}x_n\mathrm{d}|\hat{w}|(\xi') \\ &= \int_{\mathbb{R}^{n-1}}\int_D \overline{\hat{v}(\xi', x_n)}^T\sigma_P\rho_{\hat{w}}(\xi', x_n)\mathrm{d}x_n\mathrm{d}|\hat{w}|(\xi')\end{aligned}$$

$$\begin{aligned} &= \int_D \int_{\mathbb{R}^{n-1}} \overline{\hat{v}(\xi', x_n)}^T \sigma_P \rho_{\hat{w}}(\xi', x_n) \mathrm{d}|\hat{w}|(\xi') \mathrm{d}x_n \\ &= 0 \qquad (v \in C^\infty_{c,\sigma}(\mathbb{R}^{n-1} \times D)). \end{aligned}$$

In view of [19, Lemma III 1.1] then there exists a $p \in W^{1,1}_{loc}(\mathbb{R}^{n-1} \times D)$ such that $w = \nabla p$. This implies that $(I-P)(\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)) \subset G_{\mathrm{FM}}$.

Conversely, let $\nabla p \in G_{\mathrm{FM}}$. We intend to prove

$$\sigma_P(\xi) \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} \rho_{\widehat{p}}(\xi') = 0 \quad (\xi' \in \mathbb{R}^n \setminus \{0\})$$

which will yield the result. A crucial point here is to give a sense to $\widehat{p}$ and consequently to $\rho_{\widehat{p}}$, since a priori we only know $\nabla p \in \mathcal{S}'(\mathbb{R}^{n-1}, X_p)$. For every $x' \in \mathbb{R}^{n-1}$ we have

$$p(x') \in \widehat{W}^{1,p}(D) := \{v \in L^1_{\mathrm{loc}}(D) : \ \partial_n v \in L^p(D)\}/\mathbb{C}.$$

The fact that $\partial_n p \in \mathrm{FM}_0(\mathbb{R}^{n-1}, L^p(D))$ then implies that $p \in \mathrm{FM}_0(\mathbb{R}^{n-1}, \widehat{W}^{1,p}(D))$. Thus, the Fourier transform $\widehat{p}$ is well-defined in $\mathcal{S}'(\mathbb{R}^{n-1}, \widehat{W}^{1,p}(D))$. Therefore, also $\widehat{p}\lfloor i\xi_1(\mathcal{O})$ is well-defined for every compact $\mathcal{O} \in \mathscr{B}(\mathbb{R}^{n-1})$. By virtue of $\partial_1 p \in \mathrm{FM}_0(\mathbb{R}^{n-1}, L^p(D))$ we deduce $\widehat{p}\lfloor i\xi_1(\mathcal{O}) \in L^p(D)$. Due to Lemma 2.6, this, in turn, gives us that

$$\widehat{p}(\mathcal{O}) = (\widehat{p}\lfloor i\xi_1)\lfloor(1/i\xi_1)(\mathcal{O}) \in L^p(D)$$

for every compact $\mathcal{O} \in \mathscr{B}(\mathbb{R}^{n-1} \setminus \{0\})$. Together with $\widehat{p}(\mathcal{O}) \in \widehat{W}^{1,p}(D)$ we can conclude that $\rho_{\widehat{p}}(\xi') \in W^{1,p}(D)$ for $|\widehat{p}|$-a.e. $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$. On the other hand, for each $q \in W^{1,p}(D)$ the validity of formula (3.7) on X_p yields

$$\begin{aligned} \sigma_P(\xi') \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} q &= \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} \left(q + (|\xi'|^2 - \Delta_{N,n})^{-1} \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix}^T \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} q \right) \\ &= \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} (q - q) = 0 \quad (\xi \in \mathbb{R}^{n-1} \setminus \{0\}). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathcal{F}P\nabla p(\mathcal{O}) &= \widehat{p}\lfloor \left[\sigma_P \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} \right] (\mathcal{O}) \\ &= \int_{\mathcal{O}} \sigma_P(\xi') \begin{pmatrix} i\xi' \\ \partial_n \end{pmatrix} \rho_{\widehat{p}}(\xi') \mathrm{d}|\widehat{p}|(\xi') \\ &= 0 \qquad (\mathcal{O} \in \mathscr{B}(\mathbb{R}^{n-1} \setminus \{0\}) \text{ compact}). \end{aligned}$$

Taking into account that $\widehat{p}$ has no point mass at the origin we obtain $\mathcal{F}P\nabla p(\mathcal{O}) = 0$ for any $\mathcal{O} \in \mathscr{B}(\mathbb{R}^{n-1})$. This shows that $G_{\mathrm{FM}} \subset (I-P)(\mathrm{FM}_0(\mathbb{R}^{n-1}, X_p))$. Thus (3.3) follows and the proof is complete. □

Next, we define as usual the Stokes operator by

$$\begin{aligned} A_\nu &= -\nu P\Delta \\ \mathscr{D}(A_\nu) &= \big\{ u \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p) : \ \partial^\alpha u \in \mathrm{FM}(\mathbb{R}^{n-1}, X_p), \\ &\qquad \alpha \in \mathbb{N}^n_0, \ |\alpha| \le 2, \ u|_{\partial(\mathbb{R}^{n-1} \times D)} = 0 \big\}. \end{aligned}$$

The same strategy performed in Lemma 3.4 will result in sectoriality of A_ν. For this purpose, an operator-valued representation for the symbol of the resolvent of A_ν is in order.

Recall that the Stokes resolvent problem is given by

$$\left\{\begin{array}{rcll} (\lambda-\nu\Delta)u+\nabla p &=& f & \text{in } \mathbb{R}^{n-1}\times D,\\ \operatorname{div} u &=& 0 & \text{in } \mathbb{R}^{n-1}\times D,\\ u &=& 0 & \text{on } \partial(\mathbb{R}^{n-1}\times D). \end{array}\right. \tag{3.14}$$

Assume that $f\in L^p_\sigma(\mathbb{R}^{n-1}\times D)$, i.e., again first we consider the situation in L^p. Applying the Helmholtz projection P to the first line of (3.14) and then Fourier transform in tangential direction x', (3.14) turns into the problem

$$\left\{\begin{array}{rcll} (\lambda+\nu|\xi'|^2)\widehat{u}-\sigma_P(\xi')\nu\partial_n^2\widehat{u} &=& \widehat{f} & \text{in } D,\\ \widehat{u} &=& 0 & \text{on } \partial D, \end{array}\right. \tag{3.15}$$

where σ_P as before denotes the symbol of the Helmholtz projection defined in (3.7). Note that, if (3.15) is solved, the pressure p can be recovered by $\nabla p=-(I-P)\nu\Delta u$. In this sense, problems (3.14) and (3.15) are equivalent.

By Lemma 3.2(ii), (3.14) is uniquely solvable in $L^p(\mathbb{R}^{n-1},X_p)\hookrightarrow\mathcal{S}'(\mathbb{R}^{n-1},X_p)$. Since $\mathcal{F}:\mathcal{S}'(\mathbb{R}^{n-1},X_p)\to\mathcal{S}'(\mathbb{R}^{n-1},X_p)$ is isomorphic, also the solution $\widehat{u}$ of (3.15) is unique. Let $S:\mathbb{R}^n\setminus\{0\}\to\mathscr{L}(X_p)$ denote the solution operator of (3.15), i.e., we set

$$\widehat{u}=S(\xi')\widehat{f}:=(\lambda+\nu|\xi'|^2-\sigma(\xi')\nu\partial_n^2)^{-1}\widehat{f}. \tag{3.16}$$

By the uniqueness of the solution, S represents exactly the symbol of the resolvent of the Stokes operator. To be precise, we have

$$(\lambda+A_\nu)^{-1}f=\mathcal{F}^{-1}S\mathcal{F}f\quad(f\in L^p_\sigma(\mathbb{R}^{n-1}\times D)). \tag{3.17}$$

Based on the operator-valued representation (3.16) for the Stokes resolvent we can prove the following result.

Theorem 3.5. *Let $1<p<\infty$, $\nu>0$, $k\in\mathbb{N}$, and $\alpha\in\mathbb{N}_0^n$. The Stokes operator A_ν is the generator of a bounded holomorphic C_0-semigroup on $\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1},X_p)$ and for each $\varphi_0\in(0,\pi)$ there is a $C=C(\varphi_0,\nu)>0$ such that*

$$\|\lambda^{k/2}\partial^\alpha(\lambda+A_\nu)^{-1}\|_{\mathscr{L}(\mathrm{FM}_0(\mathbb{R}^{n-1},X_p))}\le C\quad(\lambda\in\Sigma_{\pi-\varphi_0},\ k+|\alpha|=2).$$

In particular, for $p=2$ A_ν is the generator of a semigroup of contractions.

Proof. Let $\varphi_0\in(0,\pi)$, $k,\ell\in\mathbb{N}_0$, and $\alpha\in\mathbb{N}_0^n$. In order to obtain a symbol in L^p, rather than L^p_σ, we consider $S\sigma_P$ instead of just S in what follows. Identifying $L^p(\mathbb{R}^{n-1}\times D)^n$ with $L^p(\mathbb{R}^{n-1},X_p)$, Lemma 3.2(ii) and Lemma 3.1 immediately imply that

$$\begin{aligned}&\|\lambda^{k/2}\partial_n^\ell(i\xi')^\alpha S(\xi')\sigma_P(\xi')\|_{\mathscr{L}(L^p(D)^n)}\le C(\varphi_0,\nu)\\ &(\lambda\in\Sigma_{\pi-\varphi_0},\ \xi'\in\mathbb{R}^{n-1}\setminus\{0\},\ k+\ell+|\alpha|=2),\end{aligned} \tag{3.18}$$

and, if $p=2$, also that

$$\|\lambda S(\xi')\sigma_P(\xi')\|_{\mathscr{L}(L^2(D)^n)}\le 1\quad(\lambda>0,\ \xi'\in\mathbb{R}^{n-1}\setminus\{0\}). \tag{3.19}$$

Furthermore, since $(\xi' \mapsto \sigma_P(\xi')) \in C(\mathbb{R}^{n-1} \setminus \{0\}, \mathscr{L}(L^p(D)^n))$, by representation (3.16) and estimate (3.18) it is straight forward to show that

$$\Big(\xi' \mapsto \lambda^{k/2}\partial_n^\ell (i\xi')^\alpha S(\xi')\sigma_P(\xi')\Big) \in C(\mathbb{R}^{n-1} \setminus \{0\}, \mathscr{L}(L^p(D)^n)) \tag{3.20}$$

$$(\lambda \in \Sigma_{\pi-\varphi_0},\ k+\ell+|\alpha| = 2). \tag{3.21}$$

(We will perform such a calculation in the proof of Lemma 3.9 for the more complicated resolvent of the Stokes-Coriolis-Ekman operator A_{SCE}. The proof given there can be copied in order to obtain (3.20).) Combining (3.18) and (3.20) we deduce

$$\Big(\xi' \mapsto \lambda^{k/2}\partial_n^\ell (i\xi')^\alpha S(\xi')\sigma_P(\xi')\Big) \in \mathrm{BC}(\mathbb{R}^{n-1} \setminus \{0\}, \mathscr{L}(L^p(D)^n))$$
$$(\lambda \in \Sigma_{\pi-\varphi_0},\ k+\ell+|\alpha| = 2).$$

By Proposition 2.13 we therefore conclude that

$$op(S) = \mathcal{F}^{-1}S\mathcal{F} \in \mathscr{L}(\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p),\, \mathrm{FM}_0(\mathbb{R}^{n-1}, X_p)).$$

More precisely, in combination with (3.18) and (3.19) Proposition 2.13 yields

$$\|\lambda^{k/2}\partial^\alpha op(S)\|_{\mathscr{L}(\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1},X_p),\,\mathrm{FM}_0(\mathbb{R}^{n-1},X_p))} \le C(\varphi_0,\nu)$$
$$(\lambda \in \Sigma_{\pi-\varphi_0},\ k+|\alpha| = 2)$$

and

$$\|\lambda op(S)\|_{\mathscr{L}(\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1},X_2))} \le 1 \quad (\lambda > 0), \tag{3.22}$$

respectively.

It remains to show that the operator $op(S)$ indeed represents the Stokes resolvent in $\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p)$. For, we next show that $op(S)$ maps $\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p)$ into the domain of A_ν. Note that it suffices to verify the trace condition $op(S)f|_{\partial(\mathbb{R}^{n-1}\times D)} = 0$. The fact that $\operatorname{div} op(S)f = 0$ for $f \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p)$ can be proved similarly.

Pick $f \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p)$. Recall that we have

$$op(S)f(x') = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^{n-1}} e^{i\xi'\cdot x'} S(\xi')\rho_{\widehat{f}}(\xi')\mathrm{d}|\widehat{f}|(\xi') \in W^{2,p}(D), \quad x' \in \mathbb{R}^{n-1}.$$

Since the trace operator $\gamma : v \mapsto v|_{x_n=0}$ is bounded from $W^{2,p}(D)$ to $\mathbb{R}$, from this representation we see that $\gamma op(S)f(x') = 0$ for all $x' \in \mathbb{R}^{n-1}$ if $\gamma S(\xi')\rho_{\widehat{f}}(\xi') = 0$ for all $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$. For $h \in L^p(\mathbb{R}^{n-1} \times D)$ we know $\gamma S(\xi')\sigma_p(\xi')\widehat{h}(\xi') = 0$, thanks to Lemma 3.2(ii). Now let $v \in X_p$ and set $\psi(x') := e^{-|x'|^2/2}$ and $h := \psi v$. Then we have $h \in L^p(\mathbb{R}^{n-1} \times D)$ and

$$0 = \gamma S(\xi')\sigma_P(\xi')\widehat{h}(\xi') = \psi(\xi')\gamma S(\xi')\sigma_P(\xi')v \quad (\xi' \in \mathbb{R}^{n-1} \setminus \{0\}).$$

This implies $\gamma S(\xi')\sigma_P(\xi')v = 0$ for all $v \in X_p$ and $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$. Consequently, we also have $\gamma S(\xi')\sigma_P(\xi')\widehat{f}(\xi') = 0$ for all $f \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p)$ and $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$. So, we have proved $op(S)(\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p)) \subset \mathscr{D}(A_\nu)$. Hence we may apply A_ν to $op(S)f$ and by using the symbol representations we easily can show that

$$(\lambda + A_\nu)op(S)f = f \qquad (f \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p)),$$
$$op(S)(\lambda + A_\nu)u = u \qquad (u \in \mathscr{D}(A_\nu)).$$

Consequently,

$$op(S) = (\lambda + A_\nu)^{-1}.$$

Finally, we show that A_ν is densely defined. To this end, it is sufficient to prove that

$$(1 + \frac{1}{j}A_\nu)^{-1} f \to f \quad \text{in } \mathrm{FM}_0(\mathbb{R}^{n-1}, X_p) \quad (j \to \infty,\ f \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p)). \tag{3.23}$$

By virtue of (3.16) and (3.17) we obtain

$$\begin{aligned} &\|(1 + \frac{1}{j}A_\nu)^{-1} f - f\|_{\mathrm{FM}(L^p)} \\ &= \|S\widehat{f} - \widehat{f}\|_{\mathrm{M}(L^p)} \\ &\leq \int_{\mathbb{R}^n} \left\| \left(1 + \frac{1}{j}(\nu|\xi'|^2 - \sigma(\xi')\partial_n^2)\right)^{-1} \rho_{\widehat{f}}(\xi') - \rho_{\widehat{f}}(\xi') \right\|_p \mathrm{d}|\widehat{f}|(\xi'). \end{aligned}$$

The sectoriality of A_ν on $L^p_\sigma(\mathbb{R}^n \times D)$ given by Lemma 3.2(ii) implies the pointwise convergence to 0 of the integrand in case that $f \in L^p_\sigma(\mathbb{R}^{n-1} \times D)$. In a similar way as we reduced the verification of Dirichlet boundary conditions to the situation in $L^p_\sigma(\mathbb{R}^{n-1} \times D)$, we can prove that pointwise convergence remains true for $f \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, X_p)$. By estimate (3.18) it is easily proved that there is an integrable majorant as well. Hence the dominated convergence theorem yields (3.23). The Hille-Yosida characterization for generators of holomorphic C_0-semigroups yields the remaining assertion. The fact that for $p = 2$ A_ν generates a semigroup of contractions follows by (3.22) and the corresponding characterization for contraction semigroups. □

Next, we consider the problem

$$\begin{cases} \partial_t u - \nu\Delta u + \omega \mathbf{e}_3 \times u + (\mathbf{U}^E \cdot \nabla)u + u^3 \partial_3 \mathbf{U}^E &= -\nabla p & \text{in } (0,\infty) \times G, \\ \qquad\qquad \text{div } u &= 0 & \text{in } (0,\infty) \times G, \\ \qquad\qquad u &= 0 & \text{on } (0,\infty) \times \partial G, \\ \qquad\qquad u|_{t=0} &= u_0 & \text{in } G, \end{cases} \tag{3.24}$$

which is the linearized version of system (1.4). From here on we restrict ourselves to the physically relevant case of three space dimensions. That is, $G = \mathbb{R}^2 \times D$ denotes a layer or a half-space in $\mathbb{R}^3$ and $\mathbf{U}^E$ denotes the Ekman spiral given in (1.2). Applying the Helmholtz projection P to the first line and employing the Stokes operator A_ν, these equations reduce to the Cauchy problem

$$\begin{cases} u' + A_{SCE} u &= 0 & \text{in } (0,\infty), \\ u(0) &= u_0. \end{cases}$$

Here $A_{SCE} = A_\nu + B_\omega + B_E$ denotes the *Stokes-Coriolis-Ekman* operator with the Stokes operator A_ν, the operator

$$B_\omega u := \omega P e_3 \times u$$

arising from the Coriolis force and the contribution from the Ekamn spiral $B_E = B_E^1 + B_E^2$, where

$$B_E^1 u = P(\mathbf{U}^E \cdot \nabla)u, \qquad B_E^2 u = Pu^3 \partial_3 \mathbf{U}^E.$$

Obviously B_ω and B_E are of lower order. Thus, in view of Theorem 3.5, they are relatively bounded by A_ν. Since the property of generating an analytic semigroup is stable under this type perturbations, Theorem 3.5 or Lemma 3.2(ii), respectively, immediately imply the following result.

Theorem 3.6. *Let $1 < p < \infty$ and let $F \in \{L^p_\sigma(\mathbb{R}^2 \times (0,d)), \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_p)\}$ Then the Stokes-Coriolis-Ekman operator $A_{SCE} : \mathscr{D}(A_{SCE}) \to F$ with domain $\mathscr{D}(A_{SCE}) = \mathscr{D}(A_\nu)$ is the generator of a holomorphic C_0-semigroup on F. Furthermore, for each $\varphi_0 \in (0,\pi)$ there exists a $\lambda_0 = \lambda_0(\varphi_0) > 0$ and a $C_{\varphi_0} > 0$ such that*

$$\|\lambda^{k/2}\partial^\alpha(\lambda + (A_{SCE} + \lambda_0))^{-1}\|_{\mathscr{L}(F)} \le C_{\varphi_0} \quad (\lambda \in \Sigma_{\pi-\varphi_0},\ k + |\alpha| = 2).$$

However, this result gives no information on the dependence of the semigroup on the parameters ω, ν, δ. Therefore, we restrict our considerations from now on to the case $p = 2$, for which much more can be said. We will see that then the dependence of the norm of $e^{-A_{SCE}}$ on ω, ν, and δ can be determined rather explicitly. In particular, it is uniformly bounded in $\omega \in \mathbb{R}$. This will be the issue of the remaining part of the present section. First we provide

Lemma 3.7. *We have*

$$\begin{aligned} \|e^{-(\cdot)/\alpha} v\|_{L^2((0,d))} &\le \alpha \left(\int_0^{d/\alpha} e^{-2x} x \mathrm{d}x \right)^{1/2} \|v'\|_{L^2((0,d))} \\ &\le \frac{\alpha}{2} \|v'\|_{L^2((0,d))} \quad (\alpha, d > 0,\ v \in H^1(\mathbb{R}_+)). \end{aligned}$$

Proof. We may assume that $v \in C_c^\infty((0,d])$. The fundamental theorem of calculus and the Cauchy-Schwarz inequality yield

$$\begin{aligned} |e^{-x/\alpha} v(x)| &\le e^{-x/\alpha} \int_0^x |v'(s)| \mathrm{d}s \\ &\le e^{-x/\alpha} x^{1/2} \|v'\|_{L^2((0,d))} \qquad (x \in (0,d)). \end{aligned}$$

Integrating with respect to x and the substitution $y = x/\alpha$ imply the assertion. □

With the help of Lemma 3.7 we can obtain the following result for the linear operator A_{SCE} in L^2. It already exhibits the mentioned explicit dependence on the parameters ω, ν, and δ for norm estimates of the solution.

Proposition 3.8. *Let $d \in (0,\infty]$ and $G = \mathbb{R}^2 \times (0,d)$. The analytic C_0-semigroup, generated by the Stokes-Coriolis-Ekman operator A_{SCE} on $L^2_\sigma(G)$ according to Proposition 3.6, satisfies the following uniform estimates: for $\eta = U_\infty^2/8\nu$ we have*

(i) $\|e^{-tA_{SCE}}\|_{\mathscr{L}(L^2(G))} \le e^{\eta t}$,

(ii) $\|\nabla e^{-(\cdot)A_{SCE}} u_0\|_{L^2((0,t),L^2(G))} \le \sqrt{2/\nu}\, e^{\eta t} \|u_0\|_{L^2(G)}$

for all $u_0 \in L^2_\sigma(G)$ and all $t \geq 0$. In particular, all estimates are uniform in $\omega \in \mathbb{R}$.

Proof. For $u_0 \in L^2_\sigma(G)$ we set $u(t) := e^{-tA_{SCE}}u_0$. Then u solves

$$\left\{ \begin{array}{rcll} u' + A_{SCE}u & = & 0 & \text{in } (0,\infty), \\ u(0) & = & u_0. & \end{array} \right.$$

Multiplying the above equation with u, integrating w.r.t. x, and taking into account the skew-symmetry of B_ω and B^1_E we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 + \nu\|\nabla u(t)\|_2^2 + (u^3(t)\partial_3 U^E, u(t)) = 0 \quad (t \geq 0). \tag{3.25}$$

Note that by (1.2) for the derivative of the Ekman spiral we obtain

$$\partial_3 \mathbf{U}^E(x_3) = \frac{U_\infty}{\delta} e^{-x_3/\delta} \begin{pmatrix} \cos(x_3/\delta) + \sin(x_3/\delta) \\ \cos(x_3/\delta) - \sin(x_3/\delta) \\ 0 \end{pmatrix}.$$

The third term in (3.25) we estimate as follows:

$$\begin{aligned} |(u^3(t)\partial_3 U^E, u(t))| &\leq \sum_{j=1}^{2} \|u^3(t)(\partial_3 U^E)^j\|_2 \|u(t)\|_2 \\ &\leq \frac{\sqrt{2}U_\infty}{\delta}\|e^{-(\cdot)/\delta}u^3(t)\|_2\|u(t)\|_2 \\ &\leq \frac{U_\infty}{\sqrt{2}}\left(\varepsilon\|\nabla u^3(t)\|_2^2 + \frac{1}{4\varepsilon}\|u(t)\|_2^2\right) \quad (\varepsilon > 0), \end{aligned}$$

where we applied Lemma 3.7 with $\alpha = \delta$ and Young's inequality in the last line. Inserting this into (3.25) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 + 2\left(\nu - \frac{U_\infty \varepsilon}{\sqrt{2}}\right)\|\nabla u(t)\|_2^2 \leq \frac{U_\infty}{2\sqrt{2}\,\varepsilon}\|u(t)\|_2^2 \quad (t \geq 0). \tag{3.26}$$

Choosing $\varepsilon = \sqrt{2}\,\nu/U_\infty$ and applying Gronwall's lemma we arrive at

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 e^{2\eta t} \quad (t \geq 0). \tag{3.27}$$

This proves (i).

To see (ii), we choose $\varepsilon < \sqrt{2}\,\nu/U_\infty$, say $\varepsilon = \nu/\sqrt{2}\,U_\infty$. Integrating (3.26) with respect to t then implies with the help of (3.27) that

$$\begin{aligned} \nu\int_0^t \|\nabla u(s)\|_2^2 \mathrm{d}s &\leq \frac{U_\infty^2}{2\nu}\int_0^t e^{2\eta s}\mathrm{d}s\|u_0\|_2^2 + \|u_0\|_2^2 \\ &\leq 2(e^{2\eta t} - 1)\|u_0\|_2^2 + \|u_0\|_2^2 \\ &\leq 2e^{2\eta t}\|u_0\|_2^2 \qquad (t \geq 0). \end{aligned}$$

Thus the proposition is proved. □

Next, we transfer the results obtained in Proposition 3.8 in $L^2_\sigma(G)$ to the space $\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)$. Again we intend to exploit the idea pointed out in Remark 2.14. For this purpose we need

Lemma 3.9. *Let $T \in (0,\infty)$ and let T_{SCE} be the holomorphic C_0-semigroup generated by A_{SCE} on $\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)$. Then T_{SCE} has a symbol $\sigma_{T_{SCE}}$ that satisfies*

(i) $(\xi' \mapsto \sigma_{T_{SCE}}(t,\xi')) \in C\big(\mathbb{R}^2 \setminus \{0\},\, \mathscr{L}(X_2)\big)$, $t \geq 0$, *and*

(ii) $(\xi' \mapsto \sigma_{T_{SCE}}(\cdot,\xi')) \in C\big(\mathbb{R}^2 \setminus \{0\},\, \mathscr{L}(X_2,\, L^2((0,T),X_2))\big)$.

Proof. Thanks to the fact that the Ekman spiral $\mathbf{U}^E$ does not depend on $x' \in \mathbb{R}^2$ the Fourier transform of the Stokes-Coriolis-Ekman operator reads as

$$\sigma_{A_{SCE}}(\xi')\widehat{u}(\xi') = \mathcal{F}A_{SCE}\,u(\xi') = \nu|\xi'|^2\widehat{u}(\xi') - \sigma_P(\xi')K(\xi')\widehat{u}(\xi'),$$

where

$$K(\xi')\widehat{u}(\xi') = \nu\partial_n^2\widehat{u}(\xi') - \omega e_3 \times \widehat{u}(\xi') - \sum_{j=1}^{2}(\mathbf{U}^E)^j\xi_j\widehat{u}(\xi') - \widehat{u}^3(\xi')\partial_3\mathbf{U}^E$$

and where σ_P denotes the symbol of the Helmholtz projection defined in (3.7). We fix $\varphi_0 \in (0,\pi/2)$. By Theorem 3.6 we may choose $\lambda_0 > 0$ such that $A_{SCE} + \lambda_0$ generates a bounded holomorphic C_0-semigroup on $L^2_\sigma(G)$. Due to this Theorem and Lemma 3.1 we therefore have that the symbol of the resolvent satisfies

$$\|\lambda^{k/2}(i\xi')^\alpha\partial_n^\ell(\lambda + \sigma_{A_{SCE}}(\xi') + \lambda_0)^{-1}\|_{\mathscr{L}(X_2)} \leq C_{\varphi_0} \tag{3.28}$$

for all $\xi' \in \mathbb{R}^2\setminus\{0\}$, all $\lambda \in \Sigma_{\pi-\varphi_0}$, and all $\alpha \in \mathbb{N}_0^2$, $k,\ell \in \mathbb{N}_0$ such that $k+|\alpha|+\ell = 2$.

Now pick $\xi_0' \in \mathbb{R}^2 \setminus \{0\}$ and $\lambda \in \Sigma_{\pi-\varphi_0}$. By the resolvent identity we obtain

$$\begin{aligned}
&\|(\lambda + \sigma_{A_{SCE}}(\xi') + \lambda_0)^{-1} - (\lambda + \sigma_{A_{SCE}}(\xi_0') + \lambda_0)^{-1}\|_{\mathscr{L}(X_2)}\\
&= \|(\lambda + \sigma_{A_{SCE}}(\xi') + \lambda_0)^{-1}(\sigma_{A_{SCE}}(\xi_0') - \sigma_{A_{SCE}}(\xi'))(\lambda + \sigma_{A_{SCE}}(\xi_0') + \lambda_0)^{-1}\|_{\mathscr{L}(X_2)}\\
&\leq C(\lambda)\|(\sigma_{A_{SCE}}(\xi_0') - \sigma_{A_{SCE}}(\xi'))(\lambda + \sigma_{A_{SCE}}(\xi_0') + \lambda_0)^{-1}\|_{\mathscr{L}(X_2)}.
\end{aligned}$$

We have

$$\begin{aligned}
&\sigma_{A_{SCE}}(\xi_0') - \sigma_{A_{SCE}}(\xi')\\
&\quad= \nu(|\xi_0'|^2 - |\xi'|^2) + \sigma_P(\xi')(K(\xi') - K(\xi_0')) + (\sigma_P(\xi') - \sigma_P(\xi_0'))K(\xi_0').
\end{aligned} \tag{3.29}$$

For the first term in (3.29) we deduce

$$\begin{aligned}
\|\nu(|\xi_0'|^2 - |\xi'|^2)(\lambda + \sigma_{A_{SCE}}(\xi_0') + \lambda_0)^{-1}\|_{\mathscr{L}(X_2)} &\leq C(\varphi_0,\lambda)(|\xi_0'|^2 - |\xi'|^2)\\
&\to 0 \quad (\xi' \to \xi_0').
\end{aligned}$$

In view of (3.10) and since

$$K(\xi') - K(\xi_0') = \sum_{j=1}^{2}(\mathbf{U}^E)^j((\xi_0')_j - \xi_j'),$$

we obtain for the second term in (3.29) that

$$\begin{aligned}
&\|\sigma_P(\xi')(K(\xi') - K(\xi_0'))(\lambda + \sigma_{A_{SCE}}(\xi_0') + \lambda_0)^{-1}\|_{\mathscr{L}(X_2)}\\
&\leq\ C(\varphi_0,\lambda)\|\mathbf{U}^E\|_\infty|\xi_0' - \xi'|\\
&\to\ 0 \quad (\xi' \to \xi_0').
\end{aligned}$$

By (3.28) we also have

$$\|K(\xi_0')(\lambda+\sigma_{A_{SCE}}(\xi_0')+\lambda_0)^{-1}\|_{\mathscr{L}(X_2)}\leq C(\varphi_0,\lambda)\|\mathbf{U}^E\|_\infty.$$

Hence, by the continuity of σ_P, i.e. by virtue of (3.10), we conclude for the third term in (3.29) that

$$\begin{aligned}&\|(\sigma_P(\xi')-\sigma_P(\xi_0'))K(\xi_0')(\lambda+\sigma_{A_{SCE}}(\xi_0')+\lambda_0)^{-1}\|_{\mathscr{L}(X_2)}\\ &\leq\ C(\varphi_0,\lambda)\|\mathbf{U}^E\|_\infty\|(\sigma_P(\xi')-\sigma_P(\xi_0'))\|_{\mathscr{L}(X_2)}\\ &\to\ 0\quad(\xi'\to\xi_0').\end{aligned}$$

So, we have proved that

$$(\lambda+\sigma_{A_{SCE}}(\cdot)+\lambda_0)^{-1}\in C\left(\mathbb{R}^2\setminus\{0\},\,\mathscr{L}(X_2)\right)\tag{3.30}$$

for every $\lambda\in\Sigma_{\pi-\varphi_0}$.

In order to establish this also for the symbol of the holomorphic semigroup $(\exp(-t(A_{SCE}+\lambda_0)))_{t\geq0}$ we employ the Dunford integral representation

$$\exp(-t(A_{SCE}+\lambda_0))=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda+A_{SCE}+\lambda_0)^{-1}\mathrm{d}\lambda,\quad t\geq0.\tag{3.31}$$

Here Γ is the usual path $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$ passed through in counterclockwise direction, where

$$\begin{aligned}\Gamma_1&=\{re^{i\theta}:\ \infty>r<\delta\},\\ \Gamma_2&=\{\delta e^{is}:\ \theta\geq s\geq-\theta\},\\ \Gamma_3&=\{re^{-i\theta}:\ \delta<r<\infty\}\end{aligned}$$

for some $\delta>0$ and $\theta\in(\pi/2,\pi-\varphi_0)$. The unitarity of the Fourier transformation on $L^2(\mathbb{R}^2,X_2)$ implies that

$$\begin{aligned}\mathcal{F}\exp(-t(A_{SCE}+\lambda_0))\mathcal{F}^{-1}(\xi')&=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda+\sigma_{A_{SCE}}(\xi')+\lambda_0)^{-1}\mathrm{d}\lambda\\ &=\exp(-t(\sigma_{A_{SCE}}(\xi')+\lambda_0)).\end{aligned}\tag{3.32}$$

Again we fix $\xi_0'\in\mathbb{R}^2\setminus\{0\}$. Thanks to (3.32) we have that

$$\begin{aligned}&\exp(-t(\sigma_{A_{SCE}}(\xi')+\lambda_0))-\exp(-t(\sigma_{A_{SCE}}(\xi_0')+\lambda_0))\\ &=\ \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\left[(\lambda+\sigma_{A_{SCE}}(\xi')+\lambda_0)^{-1}-(\lambda+\sigma_{A_{SCE}}(\xi_0')+\lambda_0)^{-1}\right]\mathrm{d}\lambda.\end{aligned}\tag{3.33}$$

In view of

$$\|e^{\lambda t}\left[(\lambda+\sigma_{A_{SCE}}(\xi')+\lambda_0)^{-1}-(\lambda+\sigma_{A_{SCE}}(\xi_0')+\lambda_0)^{-1}\right]\|_{\mathscr{L}(X_2)}\leq2e^{t\mathrm{Re}\lambda}\frac{C_{\varphi_0}}{|\lambda|}\tag{3.34}$$

for $\lambda\in\Sigma_{\pi-\varphi_0}$ and $\xi'\in\mathbb{R}^2\setminus\{0\}$), we see that on Γ the integrand above has an integrable majorant. Lebegue's dominated convergence theorem then yields by virtue of (3.30) that

$$\|\exp(-t(\sigma_{A_{SCE}}(\xi')+\lambda_0))-\exp(-t(\sigma_{A_{SCE}}(\xi_0')+\lambda_0))\|_{\mathscr{L}(X_2)}\to0\quad(\xi'\to\xi_0')$$

for every $t \geq 0$. By virtue of

$$\begin{aligned}&\sigma_{T_{SCE}}(t,\xi') - \sigma_{T_{SCE}}(t,\xi_0')\\ &= e^{\lambda_0 t}\left(\exp(-t(\sigma_{A_{SCE}}(\xi')+\lambda_0)) - \exp(-t(\sigma_{A_{SCE}}(\xi_0')+\lambda_0))\right)\end{aligned}$$

this gives us

$$\sigma_{T_{SCE}}(t,\cdot) \in C(\mathbb{R}^2 \setminus \{0\},\, \mathscr{L}(X_2))$$

for every $t \geq 0$. Thus (i) is proved.

In order to see (ii) we again employ representation (3.33). First note that

$$\begin{aligned}\|e^{\mathrm{Re}\lambda t}\|_{L^2((0,T))} &= \left(\int_0^T e^{2t\mathrm{Re}\lambda}\mathrm{d}t\right)^{1/2}\\ &\leq \frac{C(T)}{\sqrt{\mathrm{Re}\lambda}} \qquad (\lambda \in \Sigma_{\pi-\varphi_0}).\end{aligned}$$

In combination with (3.34) this gives us

$$\begin{aligned}&\|e^{\lambda(\cdot)}\left[(\lambda+\sigma_{A_{SCE}}(\xi')+\lambda_0)^{-1} - (\lambda+\sigma_{A_{SCE}}(\xi_0')+\lambda_0)^{-1}\right]\|_{\mathscr{L}(X_2,\, L^2((0,T),X_2))}\\ &\leq \frac{C(\varphi_0,T)}{|\lambda|\sqrt{\mathrm{Re}\lambda}} \quad (\lambda \in \Sigma_{\pi-\varphi_0},\ \xi' \in \mathbb{R}^2 \setminus \{0\}).\end{aligned} \tag{3.35}$$

Also observe that on Γ_1 and Γ_3 it holds an estimate as

$$|\lambda| \leq c(\varphi_0,\delta)\mathrm{Re}\lambda \quad (\lambda \in \Gamma_1 \cup \Gamma_3).$$

By this fact the right hand side of (3.35) is integrable on $\Gamma_1 \cup \Gamma_3$. On the other hand, $1/|\lambda|\sqrt{\mathrm{Re}\lambda}$ is obviously integrable on Γ_2. Hence we conclude that the right hand side of (3.35) defines an integrable majorant for the integrand in (3.33), now in the sense of an $\mathscr{L}\left(X_2,\, L^2((0,T),X_2)\right)$-valued Bochner integral. Dominated convergence and relation (3.30) then imply

$$\|\exp(-(\cdot)(\sigma_{A_{SCE}}(\xi')+\lambda_0)) - \exp(-(\cdot)(\sigma_{A_{SCE}}(\xi_0')+\lambda_0))\| \to 0 \quad (\xi' \to \xi_0'),$$

where $\|\cdot\|$ here denotes the operator norm in $\mathscr{L}\left(X_2,\, L^2((0,T),X_2)\right)$. By the fact that

$$\begin{aligned}&\|\sigma_{T_{SCE}}(\cdot,\xi') - \sigma_{T_{SCE}}(\cdot,\xi_0')\|\\ &= \|e^{\lambda_0(\cdot)}\left(\exp(-(\cdot)(\sigma_{A_{SCE}}(\xi')+\lambda_0)) - \exp(-(\cdot)(\sigma_{A_{SCE}}(\xi_0')+\lambda_0))\right)\|\\ &\leq e^{\lambda_0 T}\|\exp(-(\cdot)(\sigma_{A_{SCE}}(\xi')+\lambda_0)) - \exp(-(\cdot)(\sigma_{A_{SCE}}(\xi_0')+\lambda_0))\|\\ &\to 0 \quad (\xi' \to \xi_0'),\end{aligned}$$

we finally arrive at (ii). Hence the proof is complete. □

The previous two results serve as the preparation for the following Theorem. It includes our main result Theorem 1.1 on the linearized Ekman boundary layer problem (3.24).

Theorem 3.10. *Let $T \in (0,\infty)$, $d \in (0,\infty]$, and as before we set $X_2 = L^2(0,d)^3$. Furthermore, let the convolution $f \star g$ be defined as in Lemma 2.16. Then the holomorphic C_0-semigroup $T_{SCE}(t) = e^{-tA_{SCE}}$, generated by the Stokes-Coriolis-Ekman operator A_{SCE} on $\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)$ according to Theorem 3.6, satisfies the following uniform estimates: For $\eta = U_\infty^2/8\nu$ we have*

(i) $\|T_{SCE}(t)\|_{\mathscr{L}(\mathrm{FM}(X_2))} \le e^{\eta t} \quad (t \ge 0)$,
(ii) $\|\nabla T_{SCE} u_0\|_{L^2((0,T),\mathrm{FM}(X_2))} \le \sqrt{2/\nu}\, e^{\eta T} \|u_0\|_{\mathrm{FM}(X_2)}$,
(iii) $\|\nabla T_{SCE} \star f\|_{L^2((0,T),\mathrm{FM}(X_2))} \le \sqrt{2/\nu}\, e^{\eta T} \|f\|_{L^1((0,T),\mathrm{FM}(X_2))}$

for all $u_0 \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)$ and all $f \in L^1((0,T), \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2))$. In particular, all estimates are uniform in $\omega \in \mathbb{R}$.

Proof. Proposition 3.8(i) in combination with Lemma 3.1, Lemma 3.9(i), and (3.10) implies

$$\sigma_{T_{SCE}}(t)\sigma_P \in \mathrm{BC}\left(\mathbb{R}^2 \setminus \{0\},\, \mathscr{L}(X_2)\right)$$

and (with (3.9) for $p = 2$) that

$$\|\sigma_{T_{SCE}}(t)\sigma_P\|_{L^\infty(\mathbb{R}^2\setminus\{0\},\, \mathscr{L}(X_2))} \le \mathrm{e}^{\eta t} \quad (t \ge 0).$$

Proposition 2.13 then yields (i).

From Proposition 3.8(ii) we infer that

$$\nabla T_{SCE} P \in \mathscr{L}\left(L^2(\mathbb{R}^2, X_2),\ L^2((0,T),\, L^2(\mathbb{R}^2, X_2))\right)$$

such that

$$\|\nabla T_{SCE} P\|_{\mathscr{L}(L^2(\mathbb{R}^2,X_2),\ L^2((0,T),\, L^2(\mathbb{R}^2,X_2)))} \le \sqrt{2/\nu}\, e^{\eta T}.$$

With $H_1 = H_2 = X_2$, $J = (0,T)$, $L = T_{SCE}P$, and $M = \sqrt{2/\nu}\, e^{\eta T}$ we therefore see that relation (ii) is obtained as a consequence of Lemma 2.15 if we can show that

$$\sigma_{T_{SCE}}\sigma_P \in C\left(\mathbb{R}^2 \setminus \{0\},\, \mathscr{L}(X_2,\, L^2((0,T), X_2))\right).$$

But this follows from Lemma 3.9(ii) and (3.10).

Assertion (iii) is now obtained as a consequence of (ii) and Lemma 2.16. Indeed, relation (ii) gives us

$$\nabla T_{SCE} \in \mathscr{L}\left(\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2),\ L^2((0,T),\, \mathrm{FM}_0(\mathbb{R}^2, X_2))\right),$$

and

$$\|\nabla T_{SCE}\| = \|\nabla T_{SCE}\|_{\mathscr{L}(\mathrm{FM}_{0,\sigma}(\mathbb{R}^2,X_2),\ L^2((0,T),\, \mathrm{FM}_0(\mathbb{R}^2,X_2)))} \le \sqrt{2/\nu}\, e^{\eta T}.$$

Thus, with $p = 2$, $X = \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)$, $Y = \mathrm{FM}_0(\mathbb{R}^2, X_2)$, and $g = \nabla T_{SCE}$, Lemma 2.16 yields

$$\begin{aligned} \|\nabla T_{SCE} \star f\|_{L^2((0,T),\mathrm{FM}(X_2))} &\le \|\nabla T_{SCE}\| \|f\|_{L^1((0,T),\mathrm{FM}(X_2))} \\ &\le \sqrt{2/\nu}\, e^{\eta T} \|f\|_{L^1((0,T),\mathrm{FM}(X_2))} \end{aligned}$$

for all $f \in L^1((0,T), \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2))$. Hence the theorem is proved. □

4. The Ekman layer - uniform local nonlinear existence

Utilizing the results obtained in the previous section, particularly Theorem 3.10, here we turn to the construction of a local-in-time solution to the full nonlinear problem (1.1). In other words, we prove Theorem 1.2. We again emphasize the explicitness of the dependence on the appearing parameters in the norm estimates, in particular, the uniformness in ω.

Note that we will prove the corresponding result for the transformed system (1.4). This will yield the assertion due to the equivalence of (1.1) and (1.4). To this end, we fix a $u_0 \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)$. Formally the solution u is given by the variation of constant formula

$$u(t) = \exp(-tA_{SCE})u_0 - \int_0^t \exp(-(t-s)A_{SCE})P(u(s)\cdot\nabla)u(s)\mathrm{d}s. \tag{4.1}$$

Let $Hu(t)$ denote the right hand side of (4.1). We expect the solution to belong to the class

$$\begin{aligned}\mathbb{E}_T \ := \ & \mathrm{BC}((0,T), \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, X_2)) \\ & \cap L^2((0,T),\, \mathrm{FM}^1_0(\mathbb{R}^2, X_2)) \\ & \cap L^2((0,T),\, \mathrm{FM}_0(\mathbb{R}^2, H^1_0(0,d)^3)).\end{aligned}$$

For suitable M and T we shall now show that the nonlinear operator H is a contraction on the closed set

$$B_{T,M} := \left\{ u \in \mathbb{E}_T : \ \|u\|_T \le M\|u_0\|_{\mathrm{FM}(X_2)} \right\},$$

where

$$\|u\|_T := (2/\nu)^{1/2}\|u\|_{L^\infty((0,T),\mathrm{FM}(X_2))} + \|\nabla u\|_{L^2((0,T),\mathrm{FM}(X_2))}.$$

In this connection the crucial point is a suitable estimate for the nonlinear term $(u\cdot\nabla)u$. This will be proved in

Lemma 4.1. *For given functions $u, v \in \mathbb{E}_T$ we have that*

$$\|(u\cdot\nabla)v\|_{L^1((0,T),\mathrm{FM}(X_2))} \le \frac{3(2\nu)^{1/4}}{\pi}\, T^{1/4}\|u\|_T\|v\|_T.$$

Proof. In the first step we are tempted to use

$$L^\infty(0,d)\cdot X_2 \hookrightarrow X_2 \tag{4.2}$$

in order to obtain by Lemma 2.12(ii) that

$$\mathrm{FM}(\mathbb{R}^2, L^\infty(0,d))\cdot \mathrm{FM}(\mathbb{R}^2, X_2) \hookrightarrow \mathrm{FM}(\mathbb{R}^2, X_2).$$

However, this argumentation is not possible by the simple fact that $L^\infty(0,d)$ does not enjoy the Radon-Nikodým property. So, we have to argue somewhat more carefully. Applying Young's inequality to the well-known interpolation inequality

$$\|f\|_\infty \le \sqrt{2}\,\|f\|_2^{1/2}\|f'\|_2^{1/2} \quad (f \in H^1_0(0,d)),$$

(for a proof apply the Hölder inequality to $f^2(x) = \int_0^x (f^2)'(t)\mathrm{d}t$, see e.g. [20, Section 6.1]) we can achieve that

$$\|f\|_\infty \le \sqrt{2}\left(\lambda\|f\|_2 + \frac{1}{\lambda}\|f'\|_2\right) =: \|f\|_\lambda \quad (f \in H_0^1(0,d),\ \lambda > 0). \tag{4.3}$$

For each $\lambda > 0$ the norm $\|\cdot\|_\lambda$ is equivalent to the standard norm of $H_0^1(0,d)$. Thus $H^1_{0,\lambda}(0,d) := (H_0^1(0,d), \|\cdot\|_\lambda)$ is a Hilbert space and therefore it enjoys the Radon-Nikodým property. Moreover, relations (4.2) and (4.3) show that

$$H^1_{0,\lambda}(0,d) \cdot X_2 \hookrightarrow X_2.$$

Now, we can apply Lemma 2.12(ii) to the result

$$\begin{aligned}
&\|(u(t)\cdot\nabla)v(t)\|_{\mathrm{FM}(X_2)}\\
&\le \frac{1}{2\pi}\sum_{j=1}^3 \|u^j(t)\|_{\mathrm{FM}(H^1_{0,\lambda}(0,d))}\|\partial_j v(t)\|_{\mathrm{FM}(X_2)}\\
&\le \frac{3}{\pi\sqrt{2}}\left(\lambda\|u(t)\|_{\mathrm{FM}(X_2)} + \frac{1}{\lambda}\|\nabla u(t)\|_{\mathrm{FM}(X_2)}\right)\|\nabla v(t)\|_{\mathrm{FM}(X_2)}
\end{aligned}$$

for all $u, v \in \mathbb{E}_T$, $\lambda > 0$, and $t \in (0,T)$. Integrating with respect to t and applying the Hölder inequality implies

$$\begin{aligned}
&\|(u\cdot\nabla)v\|_{L^1((0,T),\mathrm{FM}(X_2))}\\
&\le \frac{3}{\pi\sqrt{2}}\left(\lambda\|u\|_{L^\infty((0,T),\mathrm{FM}(X_2))}\int_0^T \|\nabla v\|_{\mathrm{FM}(X_2)}\mathrm{d}t\right.\\
&\quad\left. + \frac{1}{\lambda}\int_0^T \|\nabla u(t)\|_{\mathrm{FM}(X_2)}\|\nabla v(t)\|_{\mathrm{FM}(X_2)}\,\mathrm{d}t\right)\\
&\le \frac{3}{\pi\sqrt{2}}\left(\lambda T^{1/2}(\nu/2)^{1/2}\|u\|_T\|v\|_T + \frac{1}{\lambda}\|u\|_T\|v\|_T\right)
\end{aligned}$$

for all $\lambda > 0$. Choosing $\lambda = (T\nu/2)^{-1/4}$ yields the assertion. □

Next, we show that $H(B_{T,M}) \subset B_{T,M}$ for suitable $M, T > 0$. In fact, by virtue of Theorem 3.10(i), taking the $\mathrm{FM}(X_2)$-norm of Hu leads to

$$\begin{aligned}
\|Hu(t)\|_{\mathrm{FM}(X_2)} &\le e^{\eta t}\|u_0\|_{\mathrm{FM}(X_2)} + \int_0^t e^{\eta(t-s)}\|(u(s)\cdot\nabla)u(s)\|_{\mathrm{FM}(X_2)}\mathrm{d}s\\
&\le e^{\eta T}\left(\|u_0\|_{\mathrm{FM}(X_2)} + \|(u\cdot\nabla)u\|_{L^1((0,T),\mathrm{FM}(X_2))}\right).
\end{aligned}$$

Taking the $L^2((0,T),\mathrm{FM}(X_2))$-norm of ∇Hu, we infer from Theorem 3.10(ii) and (iii) that

$$\begin{aligned}
&\|\nabla Hu(t)\|_{L^2((0,T),\mathrm{FM}(X_2))}\\
&\le \|\nabla T_{SCE}u_0\|_{L^2((0,T),\mathrm{FM}(X_2))} + \|\nabla T_{SCE}\star(u\cdot\nabla)u\|_{L^2((0,T),\mathrm{FM}(X_2))}\\
&\le \sqrt{2/\nu}\,e^{\eta T}\left(\|u_0\|_{\mathrm{FM}(X_2)} + \|(u\cdot\nabla)u\|_{L^1((0,T),\mathrm{FM}(X_2))}\right).
\end{aligned}$$

Summing up the above two inequalities yields

$$\|Hu\|_T \le 2\sqrt{2/\nu}\, e^{\eta T}\left(\|u_0\|_{\mathrm{FM}(X_2)} + \|(u\cdot\nabla)u\|_{L^1((0,T),\mathrm{FM}(X_2))}\right).$$

Thanks to Lemma 4.1 we can estimate the latter term to the result

$$\begin{aligned}\|Hu\|_T &\le 2\sqrt{2/\nu}\, e^{\eta T}\left(\|u_0\|_{\mathrm{FM}(X_2)} + \frac{3(2\nu)^{1/4}}{\pi}\, T^{1/4}\|u\|_T^2\right)\\ &\le 2\sqrt{2/\nu}\, e^{\eta T}\|u_0\|_{\mathrm{FM}(X_2)}\left(1 + \frac{3(2\nu)^{1/4}}{\pi}\, T^{1/4}M^2\|u_0\|_{\mathrm{FM}(X_2)}\right).\end{aligned}$$

Thus, choosing $M = 4\sqrt{2/\nu}\, e^{\eta}$ and $T \le \min\left\{\dfrac{\pi^4\nu^3}{2\cdot 48^4 e^{8\eta}\|u_0\|^4_{\mathrm{FM}(X_2)}},\ 1\right\}$ we achieve that

$$\|Hu\|_T \le M\|u_0\|_{\mathrm{FM}(X_2)}. \tag{4.4}$$

The boundary condition $Hu|_{\partial\mathbb{R}^3_+} = 0$ now follows from representation (4.1). In order to see that H is also contractive we observe that

$$Hu - Hv = \int_0^t \exp(-(t-s)A_{SCE})P\left[(u\cdot\nabla)(u-v) + ((u-v)\cdot\nabla)v\right](s)\mathrm{d}s.$$

Again by employing Theorem 3.10 and Lemma 4.1, completely analogous to the calculation above we can obtain

$$\begin{aligned}\|Hu - Hv\|_T \le\ & 2\sqrt{2/\nu}\, e^{\eta T}\Big(\|(u\cdot\nabla)(u-v)\|_{L^1((0,T),\mathrm{FM}(X_2))}\\ &\qquad + \|((u-v)\cdot\nabla)u\|_{L^1((0,T),\mathrm{FM}(X_2))}\Big)\\ \le\ & 2\sqrt{2/\nu}\frac{3(2\nu)^{1/4}}{\pi}\, T^{1/4}\, e^{\eta T}\Big(\|u\|_T\|u-v\|_T + \|u-v\|_T\|u\|_T\Big)\\ \le\ & 4\sqrt{2/\nu}\frac{3(2\nu)^{1/4}}{\pi}\, T^{1/4}\, e^{\eta T}M\|u_0\|_{\mathrm{FM}(X_2)}\|u-v\|_T.\end{aligned}$$

Thus, choosing

$$T < T^* := \min\left\{\frac{\pi^4\nu^3}{2\cdot 48^4 e^{8\eta}\|u_0\|^4_{\mathrm{FM}(X_2)}},\ 1\right\},$$

we see that H is a contraction. The contraction mapping principle then yields the existence of a unique fixed point of H in $B_{T,M}$. Since according to estimate (4.4) $\|Hu\|_T$ is bounded up to $T = T^*$, the interval of existence extends to some $T_0 > T^*$. This implies all the assertions of Theorem 1.2.

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