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Author(s)	Takasao, Keisuke
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GRADIENT ESTIMATES AND EXISTENCE OF MEAN CURVATURE FLOW WITH TRANSPORT TERM

KEISUKE TAKASAO

Abstract. In this paper we consider a hypersurface of the graph of the mean curvature flow with transport term. The existence of the mean curvature flow with transport term was proved by Liu, Sato and Tonegawa by using geometric measure theory techniques. We give a proof of the gradient estimates and the short time existence for the mean curvature flow with transport term by applying the backward heat kernel.

1. Introduction

A family of hypersurfaces $\{\Gamma(t)\}_{0\leq t<\infty}$ in $\mathbb{R}^n$ moves by mean curvature if the velocity of $\{\Gamma(t)\}_{0\leq t<\infty}$ is

$$V_\Gamma = H\nu \qquad \text{on} \qquad \Gamma(t), \qquad t \geq 0.$$

Here ν is the unit normal vector, H is the mean curvature of $\Gamma(t)$.

Brakke proved the existence of the generalized evolution $\{\Gamma(t)\}_{0<t<\infty}$ by using varifold methods from geometric measure theory [1]. Ecker and Huisken studied the interior estimates for the mean curvature flow [4, 5, 6]. In [2] and [8], they proved the existence of the viscosity solutions of mean curvature flow by using the level set method. Colding and Minicozzi proved the sharp estimates of the interior gradient and the area for the graph of the mean curvature flow [3].

In this paper we consider the family of hypersurfaces $\{\Gamma(t)\}_{0\leq t<\infty}$ in $\mathbb{R}^n$ whose velocity is

$$V_\Gamma = (F\cdot\nu)\nu + H\nu \qquad \text{on} \qquad \Gamma(t), \qquad t \geq 0. \tag{1.1}$$

Here F is the transport term. In this paper we assume that $\nu \cdot e_n > 0$ on $\Gamma(t)$ for $t \geq 0$. From the assumption there exists $u = u(x,t)$ such that $\Gamma(t) = \{(x, u(x,t)) \,|\, x \in \mathbb{R}^{n-1}\}$ for $t \geq 0$.

Let $\Omega = (\mathbb{R}/\mathbb{Z})^{n-1}$. The first main result of the present paper is that there exists $C, T > 0$ depending only on $n, \|F\|_{L^\infty}, \|DF\|_{L^\infty}$ and $\|Du(\cdot,0)\|_{L^\infty}$ such that $|Du(t)| \leq C$, for any $(x,t) \in \Omega \times (0,T)$ and the second main result is that there exist the family of hypersurfaces $\{\Gamma(t)\}_{0\leq t\leq T}$ (see Theorem 2.1 and Theorem 2.2 for the precise statement).

The main results are related to the pioneering work by Liu, Sato and Tonegawa [14]. They proved the existence of the generalized evolution $\{\Gamma(t)\}_{0<t<\infty}$ in dimension $n = 2, 3$ by using geometric measure theory techniques by Brakke [1].

The purpose of this paper is to give a new, simple proof of the gradient estimate of u to prove the short time existence of the graph $\Gamma(t)$ for any dimension. We remark that by using the level set method we may prove the existence of (1.1) but we can not prove the regularity of the surface on the condition of this paper.

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The mean curvature flow is related to the prescribed mean curvature equation:

$$\text{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = H, \tag{1.2}$$

here H is given. Trudinger [16, 17] and Korevaar [12] proved the interior gradient estimate for (1.2). They used the height of the graph to estimate the interior gradient.

In [10] to estimate the gradient of u they used the following:

$$\begin{aligned} w(y) \leq & \frac{1}{\omega_{n-1}R^{n-1}}\int_{\mathfrak{S}_R} w\, d\mathcal{H}^{n-1} + \frac{1}{4\omega_{n-1}}\int_{\mathfrak{S}_R} wH^2r^2(r^{-n+1}-R^{-n+1})\, d\mathcal{H}^{n-1} \\ & - \frac{1}{(n-1)\omega_{n-1}}\int_{\mathfrak{S}_R}\psi(r)\Delta w\, d\mathcal{H}^{n-1}, \end{aligned}$$

here $R>0$, $w=\log\sqrt{1+|Du|^2}$, $r=r(x)=|x|$, $\psi(r)=\int_r^R \tau(\tau^{-n+1}-R^{-n+1})d\tau \geq 0$, ω_{n-1} is the volume of unit ball in $\mathbb{R}^{n-1}$, Δ is Laplace-Beltrami operator of the surface $\mathfrak{S}=\{(x,u(x))\}$ and $\mathfrak{S}_R$ is the intersection of $\mathfrak{S}$ and the ball of radius R centered at $(y,u(y))$. In this paper we adapt the inequality for (1.1).

The organization of the paper is as follows. In Section 2 of this paper we set out the basic definitions and explain the main theorems. In Section 3 we prove the main theorems. The proof of the first main result is based on the techniques of the backward heat kernel [11] and by a standard argument we obtain the short time existence of $\Gamma(t)$.

2. Preliminaries and Main results

Let $n\geq 2$, $\Omega=(\mathbb{R}/\mathbb{Z})^{n-1}\simeq[0,1)^{n-1}$ and $F:\Omega\times\mathbb{R}\times[0,\infty)\to\mathbb{R}^n$ be a C^1 vector valued function. We consider the mean curvature flow with transport term:

$$\begin{cases} \dfrac{\partial_t u}{v} = H + F(x,u,t)\cdot\nu, & (x,t)\in\Omega\times(0,\infty), \\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

here $H=\text{div}\left(\dfrac{du}{v}\right)$, $du=(\partial_{x_1}u,\partial_{x_2}u,\dots,\partial_{x_{n-1}}u)$, $v=(1+|du|^2)^{\frac{1}{2}}$ and $\nu=(\nu^1,\nu^2,\dots,\nu^n)=\dfrac{(-du,1)}{v}$. We remark that we may obtain this PDE from (1.1). Let $G=\sup_{(x,y)\in\Omega\times\mathbb{R},t\in[0,1]}(|F|^2+|DF|)$ and $v_0=\max_{x\in\Omega}v(x,0)$, here $DF=(dF,\partial_{x_n}F)$. We denote $Q_T=\Omega\times(0,T)$ and $Q_T^\varepsilon=\Omega\times(\varepsilon,T)$. The first main result is the following:

Theorem 2.1. *We assume that $u\in C([0,1];C^2(\Omega))\cap C^1((0,1);C(\Omega))$ is a solution of (2.1), $F\in C^1(\Omega\times\mathbb{R}\times[0,1];\mathbb{R}^n)$ and $G<\infty$. Then there exists $T>0$ such that*

$$v(x,t)\leq 2v_0^2, \qquad (x,t)\in Q_T, \tag{2.2}$$

here $T=\min\left\{\dfrac{C}{Gv_0^6},1\right\}$ and $C>0$ is a constant depending only on n.

By Theorem 2.1 we obtain the second main result:

Theorem 2.2. *Fix $\alpha\in(0,1)$. We assume that*

$$K:=\max\{\|DF\|_{L^\infty(Q_T)},\|\partial_t F\|_{L^\infty(Q_T)},\sup_{c\in\mathbb{R}}\|F(\cdot,c,\cdot)\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}\}<+\infty$$

and u_0 is a Lipschitz function, namely there exists $L > 0$ such that $|u_0(x) - u_0(y)| < L|x-y|$ for any $x, y \in \Omega$. Then there exists a unique solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T) \cap C(\overline{Q_T})$ of (2.1). Furthermore there exists $C > 0$ depending only on n, α, L, K and $\varepsilon > 0$ such that

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T^\varepsilon)} < C. \tag{2.3}$$

Throughout this paper, δ and $\Delta_{\Gamma(t)}$ are the induced covariant derivative and laplacian of $\Gamma(t)$. Namely, for $g = g(x_1, x_2, \dots, x_n)$,

$$\delta g = (\delta_1 g, \delta_2 g, \dots, \delta_n g) = Dg - (\nu \cdot Dg)\nu, \qquad \Delta_{\Gamma(t)} g = \sum_{i=1}^{n} \delta_i \delta_i g,$$

here $Dg = (dg, \partial_{x_n} g)$. Next we define the backward heat kernel.

Definition 2.3. *For $s, t > 0$ $(s > t)$ and $X, Y \in \mathbb{R}^n$ we define $\rho = \rho_{(Y,s)}(X,t)$ by*

$$\rho_{(Y,s)}(X,t) = \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} e^{-\frac{|X-Y|^2}{4(s-t)}}.$$

We remark that for continuous function g and $x, y \in \mathbb{R}^{n-1}$ we have

$$\lim_{t \nearrow s} \int_{\Gamma(t)} g(\cdot, u(\cdot,t), t) \rho_{(y,u(y,s),s)}(\cdot, u(\cdot,t), t)\, d\mathcal{H}^{n-1} = g(y, u(y,s), s). \tag{2.4}$$

We provide the lemmas for the proof of Theorem 2.1 and Theorem 2.2. The next lemma is a derivation of Huisken's monotonicity formula [11].

Lemma 2.4. *Assume that u satisfies (2.1) and $\Gamma(t)$ is the surface of (2.1) extended periodically to all of $x \in \mathbb{R}^{n-1}$. Let $g = g(x,t), (x,t) \in \mathbb{R}^{n-1} \times [0, \infty)$ be a non-negative $C^{2,1}$ function. Then we have*

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma(t)} g\rho\, d\mathcal{H}^{n-1} \le \int_{\Gamma(t)} \rho \partial_t g - \rho \Delta_{\Gamma(t)} g + \rho (dg \cdot \overline{\nu}) \frac{\partial_t u}{v} \\ + \frac{1}{4} g \rho f^2(u)\, d\mathcal{H}^{n-1}, \end{aligned} \tag{2.5}$$

here $\overline{\nu} = \dfrac{-du}{v}$ and $f(u) = F(x, u(x,t), t) \cdot \nu$.

Proof. First we will show the following:

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma(t)} g\rho\, d\mathcal{H}^{n-1} = \int_{\Gamma(t)} \partial_t g \rho + g \partial_t \rho + \rho (dg \cdot \overline{\nu}) \frac{\partial_t u}{v} \\ + g(D\rho \cdot \nu)(H + f(u)) - g\rho H (H + f(u))\, d\mathcal{H}^{n-1}. \end{aligned} \tag{2.6}$$

We compute that

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma(t)} g\rho\, d\mathcal{H}^{n-1} &= \frac{d}{dt} \int_{\mathbb{R}^{n-1}} g(x,t) \rho(x, u(x,t), t) v(x,t)\, dx \\ &= \int_{\mathbb{R}^{n-1}} \partial_t g \rho v + g \partial_t \rho v + g \partial_{x_n} \rho \left(\frac{\partial_t u}{v} \right) v^2 + g\rho \frac{du \cdot d\partial_t u}{v}\, dx, \end{aligned} \tag{2.7}$$

here $\partial_t v = \dfrac{du \cdot d\partial_t u}{v}$ is used. From integration by parts we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} g\rho \frac{du \cdot d\partial_t u}{v}\, dx = -\int_{\mathbb{R}^{n-1}} \mathrm{div}\, \left(g\rho \frac{du}{v}\right) \partial_t u\, dx \\
&= -\int_{\mathbb{R}^{n-1}} \left\{ \left(dg \cdot \frac{du}{v} \right) \rho + g(d\rho + \partial_{x_n}\rho du) \cdot \frac{du}{v} + g\rho \mathrm{div}\, \left(\frac{du}{v}\right) \right\} \partial_t u\, dx \\
&= \int_{\mathbb{R}^{n-1}} \left\{ \rho(dg \cdot \overline{\nu}) \frac{\partial_t u}{v} + g\frac{\partial_t u}{v}(d\rho + \partial_{x_n}\rho du) \cdot \overline{\nu} - g\rho H \frac{\partial_t u}{v} \right\} v\, dx,
\end{aligned} \tag{2.8}$$

here $H = \mathrm{div}\left(\dfrac{du}{v}\right)$ and $\overline{\nu} = \dfrac{-du}{v}$. We also compute that

$$\begin{aligned}
&g\partial_{x_n}\rho \left(\frac{\partial_t u}{v}\right) v + g\frac{\partial_t u}{v}(d\rho + \partial_{x_n}\rho du) \cdot \overline{\nu} \\
=& g\frac{\partial_t u}{v}\{d\rho \cdot \overline{\nu} + \partial_{x_n}\rho(v + du \cdot \overline{\nu})\} = g\frac{\partial_t u}{v}(d\rho \cdot \overline{\nu} + \partial_{x_n}\rho \nu^n) \\
=& g\frac{\partial_t u}{v}(D\rho \cdot \nu) = g(D\rho \cdot \nu)(H + f(u)),
\end{aligned} \tag{2.9}$$

here $\nu^n = v^{-1}$ and $\dfrac{\partial_t u}{v} = H + f(u)$ are used. From (2.7), (2.8), (2.9) we obtain (2.6). Next we prove (2.5). We compute that

$$g(D\rho \cdot \nu)H - g\rho H^2 = -g\left(\frac{D\rho \cdot \nu}{\rho^{\frac{1}{2}}} - \rho^{\frac{1}{2}} H\right)^2 + g\frac{(D\rho \cdot \nu)^2}{\rho} - g(D\rho \cdot \nu)H.$$

Hence by Young's inequality we have

$$\begin{aligned}
&g(D\rho \cdot \nu)(H + f(u)) - g\rho H(H + f(u)) \\
=& -g\left(\frac{D\rho \cdot \nu}{\rho^{\frac{1}{2}}} - \rho^{\frac{1}{2}} H\right)^2 + g\frac{(D\rho \cdot \nu)^2}{\rho} - g(D\rho \cdot \nu)H + gf(u)(D\rho \cdot \nu - \rho H) \\
\leq& \frac{1}{4} g\rho f^2(u) + g\frac{(D\rho \cdot \nu)^2}{\rho} - g(D\rho \cdot \nu)H.
\end{aligned} \tag{2.10}$$

We denote $S = (\delta_{ij} - \nu^i \nu^j)$ and $\mathrm{div}_{\,\Gamma(t)} h = \mathrm{tr}(SDh)$ for $h \in C^1(\mathbb{R}^n; \mathbb{R}^n)$. By the divergence theorem [7] we have

$$\begin{aligned}
\int_{\Gamma(t)} -gH(D\rho \cdot \nu)\, d\mathcal{H}^{n-1} &= \int_{\Gamma(t)} \mathrm{div}_{\,\Gamma(t)}(gD\rho)\, d\mathcal{H}^{n-1} \\
&= \int_{\Gamma(t)} \mathrm{tr}(SD^2\rho + S(\partial_{x_i} g \partial_{x_j}\rho))\, d\mathcal{H}^{n-1} \\
&= \int_{\Gamma(t)} g\{\Delta\rho - \nu(D^2\rho)\nu^T\} + \delta g \cdot \delta\rho\, d\mathcal{H}^{n-1} \\
&= \int_{\Gamma(t)} g\{\Delta\rho - \nu(D^2\rho)\nu^T\} - \Delta_{\Gamma(t)} g\rho\, d\mathcal{H}^{n-1},
\end{aligned} \tag{2.11}$$

here $\Delta\rho = \sum_{i=1}^{n} \partial_{x_i}\partial_{x_i}\rho$ and $D^2\rho = (\partial_{ij}\rho)$.

By using (2.6), (2.10), (2.11) and

$$\partial_t \rho + \frac{(D\rho \cdot \nu)^2}{\rho} + \Delta \rho - \nu (D^2 \rho) \nu^T = 0,$$

we obtain (2.5) (See [11]). □

We use (2.5) with $g = v$ in this paper. To estimate the right side of (2.5) we use the following:

Lemma 2.5. *Assume that $u \in C^{3,1}(Q_T)$ satisfies (2.1). Then we have*

$$\partial_t v - \Delta_{\Gamma(t)} v + (\overline{\nu} \cdot dv) \frac{\partial_t u}{v} = -|A|^2 v - 2v^{-1} |\delta v|^2 + du \cdot d(f(u)), \tag{2.12}$$

here A is the second fundamental form of $\Gamma(t)$.

Proof. We remark that by the assumption we have $v \in C^{2,1}(Q_T)$. From [5] we obtain

$$-\Delta_{\Gamma(t)} v + |A|^2 v + 2v^{-1} |\delta v|^2 + v^2 (\delta H \cdot e_n) = 0. \tag{2.13}$$

By $H = \dfrac{\partial_t u}{v} - f(u)$ we have

$$\begin{aligned} v^2 (\delta H \cdot e_n) &= v^2 \delta_n H = v^2 (\partial_{x_n} H - (DH \cdot \nu) \nu^n) \\ &= v^2 (0 - (dH \cdot \overline{\nu}) v^{-1}) = du \cdot d \left(\frac{\partial_t u}{v} - f(u) \right) \\ &= \frac{du \cdot d \partial_t u}{v} - \frac{\partial_t u (du \cdot dv)}{v^2} - du \cdot d(f(u)) \\ &= \partial_t v + (\overline{\nu} \cdot dv) \frac{\partial_t u}{v} - du \cdot d(f(u)). \end{aligned} \tag{2.14}$$

From (2.13) and (2.14) we obtain (2.12). □

To use Schauder estimates we provide the following:

Lemma 2.6. *We assume that $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$ is a solution of (2.1). Then*

$$\|u\|_{L^\infty(Q_T)} \leq \sup_{c \in \mathbb{R}} \|F(\cdot, c, \cdot)\|_{L^\infty(Q_T)} T + \|u_0\|_{L^\infty(\Omega)}. \tag{2.15}$$

Proof. We denote $w(x,t) = \sup_{c \in \mathbb{R}} \|F(\cdot, c, \cdot)\|_{L^\infty(Q_T)} t + \|u_0\|_{L^\infty(\Omega)}$. We remark that

$$\partial_t w \geq \sqrt{1 + |dw|^2} \mathrm{div} \left(\frac{dw}{\sqrt{1 + |dw|^2}} \right) + F(x, w, t) \cdot (-dw, 1).$$

By the maximum principle [9] we obtain that

$$w \geq u, \qquad (x,t) \in Q_T.$$

Similarly to the above argument, we have

$$u \geq -w, \qquad (x,t) \in Q_T.$$

Hence we obtain (2.15). □

3. Proof of main results

1. We assume that $u \in C^{3,1}(Q_T) \cap C(\overline{Q_T})$ for the time being. First we estimate $\dfrac{d}{dt}\displaystyle\int_{\Gamma(t)} v\rho \, d\mathcal{H}^{n-1}$. By the Lemma 2.4 and Lemma 2.5 we obtain that

$$\frac{d}{dt}\int_{\Gamma(t)} v\rho \, d\mathcal{H}^{n-1} \leq \int_{\Gamma(t)} \frac{1}{4} v\rho f^2(u) - |A|^2 v\rho - 2v^{-1}|\delta v|^2\rho + du \cdot d(f(u))\rho \, d\mathcal{H}^{n-1}. \tag{3.1}$$

We denote $v_\infty = \sup_{x\in\Omega, 0\leq t\leq T} v(x,t)$ and $T > 0$ is a arbitrary number to be selected later. We remark that there exists $C = C(n) > 0$ such that

$$H^2 \leq C|A|^2. \tag{3.2}$$

The direct computations show that

$$du \cdot d(f(u)) = du \cdot d(F(u)\cdot\nu) = du \cdot d(F(u)) \cdot \nu + du \cdot d\nu \cdot F(u). \tag{3.3}$$

By using $(1 + |du|) \leq 2v$, we have

$$|du \cdot d(F(u)) \cdot \nu| \leq |du| \ |dF(u) + (\partial_{x_n} F)(u) du| \leq C v_\infty^2 |DF|, \tag{3.4}$$

here $DF = (dF, \partial_{x_n} F)$ and $C = C(n) > 0$. We compute that

$$\begin{aligned}\delta_n \nu^i =& \partial_{x_n}\nu^i - (D\nu^i \cdot \nu)\nu^n = 0 - (d\nu^i \cdot \overline{\nu})v^{-1} = -d\nu^i \cdot (-du)v^{-2} \\ =& v^{-2} du \cdot d\nu^i, \qquad i = 1, 2, \ldots, n.\end{aligned}$$

Therefore

$$du \cdot d\nu \cdot F(u) = v^2 \delta_n \nu \cdot F(u).$$

Hence

$$|du \cdot d\nu \cdot F(u)| \leq \sum_{i=1}^n v^2 |\delta_n \nu^i| \ |F(u)| \leq \frac{1}{2}|A|^2 v + C v_\infty^3 |F(u)|^2, \tag{3.5}$$

here (3.2) and $|A|^2 = \displaystyle\sum_{i,j=1}^n (\delta_i \nu^j)^2$ are used. Therefore by (3.3), (3.4) and (3.5) we obtain

$$|du \cdot d(f(u))\rho| \leq C v_\infty^2 |DF|\rho + |A|^2 v\rho + C v_\infty^3 |F(u)|^2 \rho. \tag{3.6}$$

We choose $(y, s) \in \overline{\Omega} \times [0, T]$ such that $v_\infty = v(y, s)$. From (3.1) and (3.6) we obtain

$$\frac{d}{dt}\int_{\Gamma(t)} v\rho_\infty \, d\mathcal{H}^{n-1} \leq C\int_{\Gamma(t)} v_\infty^3 |F|^2 \rho_\infty + v_\infty^2 |DF| \rho_\infty \, d\mathcal{H}^{n-1}, \tag{3.7}$$

here $\rho_\infty = \rho_{(y,u(y,s),s)}$. We remark that we may obtain (3.7) for $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$.

2. Next we prove Theorem 2.1. We denote $X = (x, u(x,t))$ and $Y = (y, u(y,s))$. We have that

$$\begin{aligned}\int_{\Gamma(t)} \rho_\infty \, d\mathcal{H}^{n-1} =& \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|X-Y|^2}{4(s-t)}} v \, dx \\ \leq& \frac{v_\infty}{(4\pi(s-t))^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x-y|^2}{4(s-t)}} \, dx = v_\infty.\end{aligned} \tag{3.8}$$

Therefore from (3.7) and (3.8) we obtain

$$\frac{d}{dt}\int_{\Gamma(t)} v\rho_\infty \, d\mathcal{H}^{n-1} \leq CGv_\infty^4,$$

here $C = C(n) > 0$. We have

$$\int_{\Gamma(t)} v\rho_\infty \, d\mathcal{H}^{n-1}\Big|_{t=0} \leq \int_{\mathbb{R}^{n-1}} \rho_\infty(x,0) v_0^2 \, dx \leq v_0^2, \tag{3.9}$$

here $v_0 = \max_{x\in\Omega} v(x,0)$ and $d\mathcal{H}^{n-1} = v\, dx$. Hence by (2.4) and (3.9) we obtain

$$v_\infty - v_0^2 \leq \int_0^s \frac{d}{dt}\left(\int_{\Gamma(t)} v\rho_\infty \, d\mathcal{H}^{n-1}\right) dt.$$

Therefore

$$C_0 G s v_\infty^4 - v_\infty + v_0^2 \geq 0, \tag{3.10}$$

here $C_0 = C_0(n) > 0$. We consider the function $f(r) = C_0 G s r^4 - r + v_0^2$. We remark that $v_0 \geq 1$, $f(0) = v_0^2$ and $f(v_0) > 0$. Furthermore $f'(r) < -\dfrac{1}{2}$ for any $r \in \left(0, \dfrac{1}{2(C_0 G s)^{\frac{1}{3}}}\right)$. Hence if $2v_0^2 < \dfrac{1}{2(C_0 G s)^{\frac{1}{3}}}$ then

$$f'(r) < -\frac{1}{2},$$

for any $r \in (0, 2v_0^2)$. Therefore there exists $\alpha \in (v_0, 2v_0^2)$ such that

$$f(\alpha) < 0. \tag{3.11}$$

Let $T := \dfrac{1}{2^6 C_0 G v_0^6}$. We have $s \leq T$. We assume

$$v_\infty > 2v_0^2.$$

Then there exist $s' \in (0, s)$ and $y' \in \Omega$ such that

$$v_1 := \max_{x\in\Omega,\ t\in[0,s']} v(x,t) = v(y', s') = \alpha.$$

By (3.11) we have

$$C_0 G s' v_1^4 - v_1 + v_0^2 < 0.$$

But by (3.10) we have

$$C_0 G s' v_1^4 - v_1 + v_0^2 \geq 0.$$

This is a contradiction. Hence

$$v_\infty \leq 2v_0^2.$$

Thus Theorem 2.1 is proved.

Remark 3.1. *If u is a global solution of* (2.1) *and $F \equiv 0$ then we obtain $T = \infty$.*

3. Finally we prove Theorem 2.2. We assume that $u_0 \in C^{2+\alpha}(\Omega)$ for the time being. We denote

$$X = \{u \in C^{\alpha, \frac{\alpha}{2}}(Q_T) |\ \|u\|_X = \|u\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} + \|du\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} < \infty\}.$$

We consider the following linear parabolic type equation:

$$\begin{cases} \partial_t u = \displaystyle\sum_{i,j=1}^{n-1} a_{ij}(w)\partial_{x_i}\partial_{x_j} u + F(x,w,t)\cdot(du,1), & \text{in } Q_T, \\ u = u_0, & \text{on } \Omega, \end{cases} \tag{3.12}$$

here $w \in X$ and $a_{ij}(w) = \left(\delta_{ij} - \dfrac{\partial_{x_i} w \partial_{x_j} w}{1+|dw|^2}\right)$.

Remark 3.2.

(i) *If $u = w$ then (2.1) and (3.12) are the same PDE.*
(ii) *The least eigenvalue of $(a_{ij}(w))$ is w^{-2}. Hence if $\|w\|_{L^\infty(Q_T)} < \infty$ then $(a_{ij}(w))$ is uniformly elliptic in Q_T.*

We have

$$\|a_{ij}(w)\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C, \qquad w \in X, \tag{3.13}$$

here $C = C(n, \|w\|_X) > 0$. By the assumption, there exists $C = C(K, \|w\|_X) > 0$ (K as in Theorem 2.2) such that

$$\|F(\cdot, w, \cdot)\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C, \tag{3.14}$$

for any $w \in X$. Hence, there exists unique solution $u_w \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T) \subset X$ of (3.12) such that

$$\|u_w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} \le C, \tag{3.15}$$

here $C = C(n, \alpha, K, \|w\|_X, \|u_0\|_{C^{2+\alpha}(Q_T)}) > 0$ (See Theorem 4.5.1 of [13]). We define $A : X \to X$ by $Aw = u_w$. We remark that A is compact.

We will show that

$$S = \{u \mid u = \sigma A u, \text{ for some } \sigma \in [0,1]\}$$

is bounded in X. If $u \in S$ then

$$\begin{cases} \partial_t u = \displaystyle\sum_{i,j=1}^{n-1} a_{ij}(u)\partial_{x_i}\partial_{x_j} u + F(x,u,t)\cdot(du,\sigma), & \text{in } Q_T, \\ u = \sigma u_0, \qquad \text{on } \Omega. \end{cases} \tag{3.16}$$

By the Theorem 2.1 we have

$$\|Du\|_{L^\infty(Q_T)} \le C_1, \tag{3.17}$$

here $C_1 = C_1(\|Du_0\|_{L^\infty(\Omega)}) > 0$. Then we obtain that

$$\|Du\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \le C_2, \tag{3.18}$$

here $C_2 = C_2(M, C_1, \|u_0\|_{C^2(\Omega)}) > 0$ and

$$M = \max_{i,j,k=1,2,\dots,n-1} \max_{Q_T} |a_{ij}, F, \partial_{x_k} a_{ij}, \partial_t F, \partial_u a_{ij}, \partial_t a_{ij}, \partial_{x_k} a_{ij}, \partial_{(\partial_{x_k} u)} a_{ij}, \partial_u F, \partial_{x_k} F|,$$

(See Theorem 6.2.3 of [13]). We remark that M depends only on C_1 and K. Hence $C_2 = C_2(K, \|u_0\|_{C^2(\Omega)}) > 0$. By the similar argument we obtain

$$\|u\|_X \le C_3, \tag{3.19}$$

here $C_3 = C_3(K, \|u_0\|_{C^2(\Omega)}) > 0$. Thus S is bounded in X. By Schaefer's Fixed point theorem there exists the solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T) \cap C(\overline{Q_T})$ of (2.1) (See [15]).

We recur to the assumption that u_0 is a Lipschitz function. We choose smooth functions u_0^k converging uniformly to u_0 on Ω. Then there exists $C = C(K, L, \varepsilon, \|u\|_{L^\infty(Q_T)}) > 0$ such that

$$\sup_k \|du^k\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T^\varepsilon)} < C,$$

here u^k is the solution of (2.1) with $u^k(x,0) = u_0^k(x)$ in Ω (See Theorem 3.11.1 [13]). We remark that by (2.15), $\|u\|_{L^\infty(Q_T)}$ is estimated from above by K, T and $\|u_0\|_{L^\infty(\Omega)}$. Therefore Schauder estimates imply that

$$\sup_k \|u^k\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T^\varepsilon)} < \infty,$$

for any $\varepsilon > 0$. Hence there exists a solution $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T) \cap C(\overline{Q_T})$ and we obtain (2.3). The maximum principle implies the uniqueness of u [9]. Thus Theorem 2.2 is proved.

References

[1] Brakke, K. A., *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, N.J., (1978).

[2] Chen, Y.-G., Giga, Y., Goto, S., *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom. **33** (1991), no.3, 749-786.

[3] Colding, T. H. and Minicozzi, II, W. P., *Sharp estimates for mean curvature flow of graphs*, J. Reine Angew. Math., **574** (2004), 187-195.

[4] Ecker, K., *Estimates for evolutionary surfaces of prescribed mean curvature*, Math. Z. **180** (1982), no. 2, 179-192.

[5] Ecker, K. and Huisken, G., *Interior curvature estimates for hypersurfaces of prescribed mean curvature*, Ann. Inst. Henri Poincaré, **6** (1989), no. 4, 251-260.

[6] Ecker, K. and Huisken, G., *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no.3, 547-569.

[7] Ecker, K., *Regularity theory for mean curvature flow*, Birkhäuser Verlag, Basel, (2004).

[8] Evans, L. C. and Spruck, J., *Motion of level sets by mean curvature I*, J. Differential Geom. **33** (1991), no. 3, 635-681.

[9] Friedman, A., *Partial differential equations of parabolic type*, Prentice-Hall Inc., Englewood Cliffs, N.J., (1964).

[10] Gilbarg, D. and Trudinger, N. S., *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, (1977).

[11] Huisken, G., *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom., **31** (1990), no.1, 285-299.

[12] Korevaar, N., *An easy proof of the interior gradient bound for solutions to the prescribed mean curvature equation*, Proc. Sympos. Pure Math. **45** Part 2, Amer. Math. Soc. (1986), 81-89.

[13] Ladyženskaja, O. A. and Solonnikov, V. A. and Ural'ceva, N. N., *Linear and quasi-linear equations of parabolic type*, Amer. Math. Soc. (1968) (Translated from Russian).

[14] Liu, C. and Sato, N. and Tonegawa, Y., *On the existence of mean curvature flow with transport term*, Interfaces Free Bound., **12** (2010), no.2, 251-277.

[15] Trudinger, N. S., *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math., **21** (1968), 205-226.

[16] Trudinger, N. S., *A new proof of the interior gradient bound for the minimal surface equation in n dimensions*, Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 821-823.

[17] Trudinger, N. S., *Gradient estimates and mean curvature*, Math. Z. **131** (1973), 165-175.

Department of Mathematics, Hokkaido University, Sapporo 060-0810 Japan

E-mail address: takasao@math.sci.hokudai.ac.jp