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Title	A remark on a Liouville problem with boundary for the Stokes and the Navier-Stokes equations
Author(s)	GIGA, YOSHIKAZU
Citation	Hokkaido University Preprint Series in Mathematics, 989, 1-16
Issue Date	2011-11-18
DOI	https://doi.org/10.14943/84136
Doc URL	https://hdl.handle.net/2115/69795
Type	departmental bulletin paper
File Information	re989.pdf

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A remark on a Liouville problem with boundary for the Stokes and the Navier-Stokes equations

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Abstract

We construct a Poiseuille type flow which is a bounded entire solution of the nonstationary Navier-Stokes and the Stokes equations in a half space with non-slip boundary condition. Our result in particular implies that there is a nontrivial solution for the Liouville problem under the non-slip boundary condition. A review for cases of the whole space and a slip boundary condition is included.

2000 *Mathematics Subject Classification.* Primary: 35Q30; Secondary: 35B40, 76D05, 76D07.

Key words and phrases. Liouville problem, Navier-Stokes equations, non-slip boundary condition, Poiseuille type flow.

*Partly supported by the Japan Society for the Promotion of Science through grant Kiban (S) 21224001.

1 Introduction

A classical Liouville theorem says that there is no bounded harmonic function in $\mathbf{R}^n$ other than a constant function. Non-existence of nontrivial entire solutions for a partial differential equation is often called a Liouville problem. Such a problem is important not only for classification of entire solutions but also for applications to a blow-up argument.

A blow-up argument is a powerful tool to obtain a bound for solutions and their regularity. It was first introduced by De Giorgi [8] (see also [14, Theorem 8.1]) for the study of minimal surfaces. To obtain an a priori bound for a semilinear elliptic equation Gidas and Spruck [9] first introduced a blow-up argument. Among other results they obtained a bound for solutions of

$$\Delta u + u^p = f,\ u \geq 0,\ 1 < p < (n+2)/(n-2) \tag{1.1}$$

in a smoothly bounded domain in $\mathbf{R}^n$ ($n \geq 2$) with homogeneous Dirichlet condition, i.e. zero boundary condition, where f is a given data. The author [11] adjusted a blow-up argument for semilinear parabolic equations typically of the form

$$u_t = \Delta u + u^p,\ u \geq 0,\ 1 < p < (n+2)/(n-2)$$

to establish a global uniform bound for global solutions depending only on sup norm of the initial data. See also [12] for derivation of blow-up rate of a blowing up solutions for the above semilinear parabolic equation. Such a blow-up argument has been more sophisticated to derive several a priori bounds for semilinear elliptic and parabolic equations [22] (see also [24]).

To understand the basic idea we recall a blow-up argument to obtain an a priori bound for a bounded positive solution u of (1.1) in $\mathbf{R}^n$. We shall prove that

$$\|u\|_\infty \leq C \tag{1.2}$$

for a bounded (classical) positive solution of u of (1.1) in $\mathbf{R}^n$ with C depending only on the sup norm $\|f\|_\infty$ of f. Suppose that such estimates were false. Then there would exist a sequence $\{u_k\}_{k=1}^\infty$ of solutions of (1.1) (with $f = f_k$ satisfying $\sup_k \|f_k\|_\infty < \infty$) such that $\|u_k\|_\infty \geq k$. By definition there exists $\{x_k\}_{k=1}^\infty$ such that

$$u_k(x_k) \geq \|u_k\|_\infty - 1.$$

We rescale u_k with respect to x_k to define

$$U_k(x) = k^{-1} u_k \left(x_k + x k^{-(p-1)/2} \right)$$

to observe that

$$\|U_k\|_\infty \leq 1, \tag{1.3}$$

$$U_k(0) \geq 1 - 1/k \tag{1.4}$$

and U_k solves

$$\Delta U_k + U_k^p = F_k/k^p \tag{1.5}$$

for F_k defined by

$$F_k(x) = f_k\left(x_k + xk^{-(p-1)/2}\right).$$

Here we invoked the scaling property of (1.1). There are two crucial steps. One is a *compactness* argument to guarantee that there is a limit U of U_k which solves the limit equation in some sense and U_k converges to U locally uniformly near zero. In our particular problem this step is carried out since p is subcritical in the sense $p < (n+2)/(n-2)$. Since $\sup_k \|f_k\|_\infty < \infty$, the limit equation of (1.5) is

$$\Delta U + U^p = 0 \quad \text{in} \quad \mathbf{R}^n \tag{1.6}$$

with $U(0) = 1$ and

$$\|U\|_\infty \le 1. \tag{1.7}$$

Here we invoke (1.3) and (1.4).

The second step is the *Liouville type uniqueness* for the limit equation (1.6). Fortunately, in [9] it is shown that a nonnegative solution to (1.6) must be $U \equiv 0$. (We need not invoke (1.7).) This leads a contradiction to $U(0) = 1$ so we get a bound (1.2). This is a simple variant of the proof of [9], where they discussed a priori bound when Ω is bounded with the Dirichlet condition. (In their case we also need to invoke the Liouville problem in a half space since x_k may tend to the boundary $\partial\Omega$ of Ω.)

The blow-up argument also plays an important role for the analysis of the singularities for geometric flows like the harmonic map heat flow [27] and the Ricci flow [15]. It is quite recent that the blow-up argument applies to the Navier-Stokes equations or the Stokes equations. Koch, Nadirashvilli, Seregin and Svěrák [18] applied it to show that a type I axisymmetric three dimensional Navier-Stokes flow must be regular; see [25] for a local version and also [5], [6] for different proofs. The blow-up argument is also applied to show that a type I Navier-Stokes flow with continuous alignment of vorticity direction must be regular [13]. The latter is very closely related to a result of Constantin and Fefferman [7], where they prove a similar assertion with no type I assumption but for finite kinetic energy solutions with Lipschitz continuous alignment. (The result in [13] applies to a general mild solution not necessarily of finite kinetic energy.)

It should be noted that a blow-up argument applies to obtain the analyticity of the Stokes semigroup in L^∞ type spaces when the domain has a curved boundary [1]. This problem was a long standing open problem in the theory of the Stokes equations.

In this paper we consider a Liouville type problem for the Navier-Stokes equations in $\mathbf{R}^n \times (-\infty, 0)$. The question is whether there is a nontrivial classical backward global solution whose velocity is bounded in $\mathbf{R}^n \times (-\infty, 0)$. For two-dimensional

problem ($n = 2$) this is essentially known. There is no nontrivial bounded solution [18], [13]. This result is very useful to complete a blow-up argument in [18] and [13]. In fact, their limit equation after blow-up (rescaling) has only trivial solution which completes the proof. However, if one considers a half-space $\mathbf{R}^n_+$ instead of $\mathbf{R}^n$, the situation is different. If one considers a slip boundary condition, the problem is reduced to a whole space problem. However, if one considers the Dirichlet boundary condition (non-slip condition), then the situation is quite different. It turns out that there is a nontrivial Poiseuille type flow which is a bounded entire solution. This is a new discovery of the present paper. Such a phenomenon seems to be related to production of vorticity near the boundary which becomes more explicit by a recent work of Maekawa [20].

In our Liouville type results we only assume boundedness of the velocity for a behavior near spatial infinity. No assumptions on the pressure are required. This point was not emphasized in the literature although it is essentially known in [18] and [13]. For the two-dimensional half space problem with non-slip condition if we further impose nonnegativity of the vorticity and some decay of the tangential component u^1 of the velocity as well as boundedness of the velocity and its derivatives, then the vorticity must be zero. The decay condition we need is that $u^1(x_1, x_2) \to 0$ as $x_2 \to \infty$ uniformly in x_1, where $\mathbf{R}^2_+ = \big\{(x_1, x_2) \,|\, x_2 > 0\big\}$. This result is due to Y. Maekawa. We shall mention it at the end of this paper. Note that our Poiseuille type flow can be arranged so that the vorticity is positive. However, u^1 cannot decay as $x_2 \to \infty$ so it does not violate Maekawa's criterion.

There are several articles on a Liouville type problem for the Navier-Stokes equations [2], [3], [4]. However, their velocity and pressure are assumed to decay at spatial infinity and their solutions are defined in $\mathbf{R}^n \times (0, T)$ so the situation is quite different from ours.

There are many studies on a backward global solution of the Navier-Stokes equations, i.e. a solution in $\mathbf{R}^n \times (-\infty, 0)$ with some spatial decay. We just point out a recent result of Hsu and Maekawa [16], where they derive several non-existence results for a special type of backward and forward self-similar solutions with a linear background flow called a linear strain. The reader is referred to [16] and references theirin and [10] for background of the problem.

The author is grateful to Professor Yasunori Maekawa who allows the author to include his result in the present paper with the proof. The author is also grateful to Mr. Pen-Yuan Hsu for careful reading of this manuscript with useful comments.

2 Liouville type problem

We study a backward global solution called an *ancient solution* for the Navier-Stokes equations in a whole space, i.e.,

$$u_t - \Delta u + (u \cdot \nabla)\, u + \nabla p = 0,\ \text{div}\ u = 0 \quad \text{in} \quad \mathbf{R}^n \times (-\infty, 0). \tag{2.1}$$

If one considers spatially nondecaying solutions, there always exists a trivial solution of the form

$$u\,(x,t) = g\,(t),\ p\,(x,t) = -g'(t) \cdot x + h(t) \tag{2.2}$$

for any spatially constant functions $g\,(t)$ (and $h(t)$). This solution is called a *parastic solution* in [18]. Here is a version of Liouville problem for the nonstationary Navier-Stokes equations.

By a classical solution we mean that all derivatives appeared in the equation (2.1) are continuous.

Problem. *Let $(u, \nabla p)$ be a classical solution of (2.1). Assume that $|u|$ is bounded in $\mathbf{R}^n \times (-\infty, 0)$. Then is u always a parastic solution (2.2)?*

As far as the author knows, the answer is not known for $n \geq 3$. If this problem is solved affirmatively, then one can conclude that there is no type I blow-up solution i.e. of the Navier-Stokes initial value problem; see [13, Proposition 2.1], [18]. By type I we mean that

$$\left|u\,(x,t)\right| \leq \frac{C}{\sqrt{T-t}}\,,$$

where T is a blow-up time and C is independent of time t and x.

For $n = 2$ this problem is solved affirmatively.

Theorem 2.1. *Assume that $(u, \nabla p)$ be a classical solution of (2.1). Assume that $n = 2$. If $|u|$ is bounded in $\mathbf{R}^n \times (-\infty, 0)$, then u is a parastic solution.*

This is essentially proved in [18] and [13] provided that we admit that $|\nabla u|$ is also bounded in $\mathbf{R}^n \times (-\infty, 0)$. Such a bound is well-known and it goes back to [26], [21]. Once $|\nabla u|$ is bounded the problem is reduced to the Liouville problem for the vorticity equations. Note that no assumption at spatial infinity for the pressure p is imposed.

Lemma 2.2. *Under the assumption of Theorem 2.1 if $\omega = \text{curl}\ u$ $(= \partial u^2/\partial x_1 - \partial u^1/\partial x_2)$ is bounded in $\mathbf{R}^2 \times (-\infty, 0)$, then $\omega \equiv 0$.*

Proof of Theorem 2.1. By the interior regularity theory [26], [21] we know that $|\nabla u|$ is also bounded so that $|\omega|$ is bounded. By Lemma 2.2 the vorticity $\omega \equiv 0$. Since

$$-\Delta u = \text{curl}\ \omega,$$

the classical Liouville theorem implies that u is spatially constant. Thus u must be a parastic solution. □

The proof of Lemma 2.2 is essentially known in [18] by using stability of the strong maximum principle. A shorter proof is given in [13, Lemma 2.3]. For the reader's convenience we reproduce it here.

Proof of Lemma 2.2. We may assume that u and ω are smooth in $\mathbf{R}^2 \times (-\infty, 0]$ by shifting time. Suppose that $L = \|\omega\|_{L^\infty\left(\mathbf{R}^2\times(-\infty,0]\right)} > 0$. Then there would exist a sequence of points $(x_k, t_k) \in \mathbf{R}^2 \times (-\infty, 0]$ satisfying $\omega(x_k, t_k) \to L$ (or $-L$). We may assume that the limit is L since tha case of $-L$ can be treated in the same way. We shift (u, ω) as

$$u^k(x,t) := u(x + x_k, t + t_k),\ \omega^k(x,t) := \omega(x + x_k, t + t_k).$$

Then there exists (U, Ω) such that (u^k, ω^k) subsequently converges to (U, Ω) respectively in locally uniformly (with its derivatives appeared in the vorticity equations (2.3)) as $k \to \infty$ and (U, Ω) solves the two dimensional vorticity equations

$$\Omega_t - \Delta\Omega + (U \cdot \nabla)\,\Omega = 0,\ -\Delta U = \text{curl}\ \Omega \tag{2.3}$$

in $\mathbf{R}^2 \times (-\infty, 0]$. (Here we have invoked L^∞ theory of the vorticity equations (cf. [10]).) By the choice of (x_k, t_k) we see that $\Omega(0, 0) = L$ and $|\Omega| \le L$. By the strong maximum principle [23] we have $\Omega \equiv L$. Since $-\Delta U = \text{curl}\ \Omega$, the Liouville theorem implies that U is spatially constant which implies $\Omega \equiv 0$. This contradicts to the assumption $\Omega \equiv L$ so we conclude that $L = 0$. Thus $\omega \equiv 0$ in $\mathbf{R}^n \times (-\infty, 0]$. □

Remark 1. The crucial part is that Ω solves the two-dimensional vorticity equations (2.3). The first equation of (2.3) enjoys a strong maximum principle which is the key of the proof. The first equation of the vorticity equations is obtained by taking curl of (2.1). For three dimensional case ($n = 3$) it is

$$\omega_t - \Delta\omega + (u \cdot \nabla)\,\omega - (\omega \cdot \nabla)\,u = 0, \tag{2.4}$$

where the vorticity stretching term $(\omega \cdot \nabla)\,u$ appears. The vorticity ω cannot be regarded as a scalar function. Moreover, there is no property like the strong maximum principle for (2.4). So up to now Theorem 2.1 is only known for $n = 2$.

We now consider a boundary value problem for $n = 2$. We begin with a slip boundary condition. Instead of (2.1) we consider

$$u_t - \Delta u + (u \cdot \nabla)\,u + \nabla p = 0,\ \text{div}\ u = 0 \quad \text{in} \quad \mathbf{R}^2_+ \times (-\infty, 0) \tag{2.5}$$

with

$$\omega = 0,\ u^2 = 0 \quad \text{on} \quad \partial\mathbf{R}^2_+ \times (-\infty, 0), \tag{2.6}$$

where $u = (u^1, u^2)$, $\mathbf{R}^2_+ = \left\{ (x_1, x_2) \,|\, x_2 > 0 \right\}$ so that u^2 is the normal trace of the velocity on the boundary.

Theorem 2.3. *Assume that $(u, \nabla p)$ be a classical solution of (2.5)–(2.6). If $|u|$ is bounded in $\mathbf{R}^2_+ \times (-\infty, 0)$, then u is a parastic solution with $u^2 = 0$.*

This is easily reduced to the whole space problem. In fact, for $x_2 < 0$ we extend

$$u^1(x_1, x_2, t) = u^1(x_1, -x_2, t),\ u^2(x_1, x_2, t) = -u^2(x_1, -x_2, t)$$
$$p\,(x_1, x_2, t) = p\,(x_1, -x_2, t)$$

so that $\omega\,(x_1, x_2, t) = -\omega\,(x_1, -x_2, t)$. Then it turns out that this extended function formally solves (2.1) with $n = 2$ including the line $x_2 = 0$. In fact, note that all tangential derivatives and time derivatives of u and p are continuous across $x_2 = 0$. Since u^2 is an odd extension, we see that ω is continuous across $x_2 = 0$. Since div $u = 0$ up to $x_2 = 0$, one observes that $\partial^2 u^2/\partial x_2^2 = -\partial^2 u^1/\partial x_1 \partial x_2 = \partial\omega/\partial x_1 - \partial^2 u^2/\partial x_1^2$. This implies that $\partial^2 u/\partial x_2^2$ is continuous across $\{x_2 = 0\}$. Arguing in such a way we observe that all quantities appeared in (2.5) are continuous across $\{x_2 = 0\}$ so one is able to reduce the problem for the whole space problem.

We apply Theorem 2.1 to conclude that u is a parastic solution. Of course, since we have assume $u^2 = 0$ on $\{x_2 = 0\}$, the second component of the resulting parastic solution is identically zero.

If one imposes the Dirichlet condition, the situation is quite different.

3 An entire solution for the Dirichlet problem

We shall construct a nontrivial (non-zero) entire solution for the Navier-Stokes equations in a half space with the non-slip boundary condition. We consider the Navier-Stokes equations

$$u_t - \Delta u + (u \cdot \nabla)\, u + \nabla p = 0,\ \text{div}\ u = 0 \quad \text{in} \quad \mathbf{R}^n_+ \times (-\infty, \infty) \tag{3.1}$$

with the Dirichlet boundary condition

$$u\ (x', 0, t) = 0 \quad \text{on} \quad \partial\mathbf{R}^n_+ \times (-\infty, \infty), \tag{3.2}$$

where $\mathbf{R}^n_+ = \left\{\, (x', x_n) \,|\, x_n > 0,\ x' \in \mathbf{R}^{n-1} \,\right\}$.

Theorem 3.1. *There is a non-zero solution $(u, \nabla p)$ for (3.1)–(3.2) which is smooth in $\bar{\mathbf{R}}^n_+ \times (-\infty, \infty)$ satisfying the following properties.*

(i) $|u|$ and $|\nabla p|$ is bounded in $\mathbf{R}^n_+ \times (-\infty, \infty)$.

(ii) $|\nabla u|$ is bounded in $\mathbf{R}^n_+ \times (-\infty, \infty)$ and $\nabla u \not\equiv 0$.

(iii) u depends only on x_n and t while p depends only on t and one tangential variable say x_1. Moreover, u is a vector field parallel to the boundary.

(iv) $\sup \left\{\, |\nabla u|\,(x_n, t) \;\middle|\; t \in \mathbf{R},\ x_n \geq L \,\right\} \leq C/L$ *with some constant $C > 0$ independent of L.*

Proof. We shall construct a Poiseuille type flow. We consider an inhomogeneous heat equation

$$\frac{\partial u_T^1}{\partial t} - \Delta u_T^1 = f \quad \text{in} \quad \mathbf{R}_+^n \times (-T, \infty), \tag{3.3}$$

$$u_T^1(x', 0, t) = 0 \quad \text{on} \quad \partial\mathbf{R}_+^n \times (-T, \infty), \tag{3.4}$$

$$u_T^1(x, -T) = 0 \quad \text{on} \quad \mathbf{R}_+^n. \tag{3.5}$$

Here f is a spatially constant function depending only on t. It is easy to solve this equation explicitly. Let $\tilde{f}$ be an odd extension of f, i.e.

$$\tilde{f}(t) = \begin{cases} f(t), & x_n > 0, \\ -f(t), & x_n < 0. \end{cases}$$

Then

$$u_T^1\ (x_n, t) = \int_{-T}^{t} e^{(t-s)\Delta_1} \tilde{f}(s)\, \mathrm{d}s \tag{3.6}$$

is a smooth solution of (3.3)–(3.5) provided that f is smooth in $t \in \mathbf{R}$. (Note that u_T^1 is independent of x' since $\tilde{f}$ is spatially constant.) Here $e^{t\Delta_1}$ denotes the one-dimensional heat semigroup, i.e.,

$$\left(e^{t\Delta_1} v\right)(x_n) = \int_{-\infty}^{\infty} g_t\left(x_n - y_n\right) v\left(y_n\right) dy_n,\ g_t\left(x_n\right) = \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{|x_n|^2}{4t}\right).$$

We extend u_T^1 as zero for $t < T$.

If

$$\int_{-\infty}^{\infty} \left| f(t) \right| \mathrm{d}t < \infty \tag{3.7}$$

then u_T^1 converges to a unique limit u^1 uniformly in $\mathbf{R}_+ \times (-\infty, \infty)$ as $T \to \infty$. Indeed, u_T^1 is a Cauchy sequence in $L^\infty\big(\mathbf{R}_+ \times (-\infty, \infty)\big)$ under (3.7) since we observe from (3.6) that

$$\begin{aligned} \left\| u_T^1 - u_{T'}^1 \right\|_\infty &= \left\| \int_{-T'}^{-T} e^{(t-s)\Delta_1} \tilde{f}(s)\, \mathrm{d}s \right\|_\infty \\ &\le \int_{-T'}^{-T} \left| \tilde{f}(s) \right| \mathrm{d}s \quad \text{for} \quad T' > T, \end{aligned}$$

where $\|\cdot\|$ is the norm in $L^\infty\big(\mathbf{R}_+ \times (-\infty, \infty)\big)$. Thus, u^1 is bounded in $\mathbf{R}_+ \times (-\infty, \infty)$. Moreover, the function u^1 solves (3.3)–(3.4) for all $T > 0$ at least in some weak sense.

However, the standard regularity theory [19] implies that u is a smooth solution of (3.3)–(3.4) for all $T > 0$ provided that f is smooth in time. (Of course, for this particular solution it is not difficult to prove directly that all derivatives of u_T^1 converge locally uniformly to those of u^1 in $\bar{\mathbf{R}}_+ \times (-\infty, \infty)$.)

We next estimate the spatial derivative of u^1 i.e. $\partial_{x_n} u^1$ $(= \partial u^1/\partial x_n)$. We observe by (3.6) that

$$\partial_{x_n} u_T^1(x_n, t) = \int_{-T}^{t} e^{(t-s)\Delta_1} 2f(s)\, \delta_{x_n} \mathrm{d}s \tag{3.8}$$

$$= \int_{-T}^{t} 2g_{t-s}(x_n) f(s)\, \mathrm{d}s. \tag{3.9}$$

We invoke an elementary estimate

$$z^\alpha e^{-Mz} \le (\alpha/eM)^\alpha \quad \text{for} \quad \alpha > 0,\ M > 0,\ z > 0$$

to estimate

$$0 \le g_{t-s}(x_n) \le \frac{1}{(2\pi e)^{1/2}} \frac{1}{L} \quad \text{for} \quad x_n \ge L$$

by setting $z^{-1} = 4(t-s)$, $M = L^2$, $\alpha = 1/2$. Thus

$$\left| \partial_{x_n} u_T^1(x_n, t) \right| \le \frac{2}{L} \int_{-\infty}^{\infty} \left| f(s) \right| \mathrm{d}s \, \frac{1}{(2\pi e)^{1/2}} \quad \text{for} \quad x_n \ge L,\ t \in \mathbf{R}.$$

By (3.7) this implies the decay of gradient for u^1 of the form

$$\sup \left\{ \left| \partial_{x_n} u^1(x_n, t) \right| \ \middle|\ x_n \ge L,\ t \in \mathbf{R} \right\} \le C/L \tag{3.10}$$

with $C = 2\int_{-\infty}^{\infty} \left| f(s) \right| \mathrm{d}s \cdot (2\pi e)^{-1/2}$. To get a uniform bound we estimate (3.9) in a different way.

We further assume that

$$\sup_{s \in \mathbf{R}} \left| f(s) \right| < \infty. \tag{3.11}$$

We divide the modulus of the integral (3.9)

$$\left| \int_{-T}^{t} g_{t-s}(x_n) f(s)\, \mathrm{d}s \right| \le \int_{-\infty}^{\infty} g_{t-s}(x_n) \left| \tilde{f}(s) \right| \mathrm{d}s$$

into two regions $|t-s| > 1$ and $|t-s| \le 1$. Since $g_t(x) \le 1/(4\pi t)^{1/2}$, we estimate

$$\left| \int_{-T}^{t} g_{t-s}(x_n) f(s)\, \mathrm{d}s \right| \le \frac{1}{(4\pi)^{1/2}} \left\{ \int_{-\infty}^{\infty} \left| f(s) \right| \mathrm{d}s + \int_0^1 \frac{\mathrm{d}s}{s^{1/2}} \sup_{s \in \mathbf{R}} \left| f(s) \right| \right\}.$$

So if we assume (3.11), then $\big\{\,|\,\partial_{x_n} u^1_T\,|\big\}_{T>0}$ is bounded in $L^\infty\big(\mathbf{R}_+ \times (-\infty,\infty)\big)$ so that $\partial_{x_n} u^1 \in L^\infty\big(\mathbf{R}_+ \times (-\infty,\infty)\big)$.

We are now ready to construct a desired solution $(u, \nabla p)$ for (3.1) and (3.2). We set

$$u = \big(u^1(x_n,t), 0, \dots, 0\big),\ p\,(x,t) = -f(t)\,x_1 \tag{3.12}$$

with $f \in C^\infty(-\infty,\infty)$ satisfying (3.7) and (3.11); i.e. $f \in L^1 \cap L^\infty(-\infty,\infty)$. It is clear that div $u = 0$ in $\mathbf{R}^n_+ \times (-\infty,\infty)$. Moreover, $(u\cdot\nabla)\,u^1 = u^1 \partial_{x_1} u^1 = 0$ since u^1 is independent of x_1 and $u^2 = \cdots = u^n = 0$. Thus, our $(u, \nabla p)$ solves (3.1) as well as the Stokes equations (i.e. (3.1) with no convective term $(u\cdot\nabla)\,u$). The boundary condition (3.2) is trivially fulfilled. The property (iii) is trivially satisfied by the forms (3.12) of our u and p. The desired bound for u in (i), (ii), (iv) has been established for u^1 under (3.7) and (3.11). The bound for ∇p in (i) follows from (3.11). If $f \not\equiv 0$, the function u^1 is non zero as well as its derivative $\partial_{x_n} u^1$. We have thus proved that our solution $(u, \nabla p)$ satisfies all desired properties so the proof is now complete. □

Remark 2. In two dimensional case ($n = 2$) the vorticity curl $u = \partial u^2/\partial x_1 - \partial u^1/\partial x_2$ equals $-\partial u^1/\partial x_2$. If f is taken negative, then $-\partial u^1/\partial x_2$ of our solution is always positive. Thus different from the whole plane there is a nontrivial global solution having positive vorticity for a half space with the non-slip boundary condition. In particular, Problem 3.4 of [13] has a nontrivial bounded ancient solution (with positive vorticity) even if there is no convective term. For the reader's convenience we state it below.

Theorem 3.2. *There is a smooth solution* (u, ω)

$$\omega_t - \Delta\omega + (u\cdot\nabla)\,\omega = 0,\ -\Delta u = \text{curl}\ \omega \quad \text{in} \quad \mathbf{R}^2_+ \times (-\infty,\infty)$$

with

$$u = 0 \quad \text{on} \quad \partial\mathbf{R}^2_+ \times (-\infty,\infty)$$

such that $|u|$ *and* $|\nabla u|$ *are bounded in* $\mathbf{R}^2_+ \times (-\infty,\infty)$ *and* $\omega > 0$ *in* $\mathbf{R}^2_+ \times (-\infty,\infty)$ *with*

$$\sup\big\{\,\omega\,(x_1,x_2,t)\,|\,x_2 \ge L, x_1, t \in \mathbf{R}\,\big\} \to 0 \quad \text{as} \quad L \to \infty.$$

The example we constructed has a linear pressure like the Poiseuille flow so we call the solution (3.12) a *Poiseuille type* solution. One may ask a question whether or not there exists an ancient solution other than a Poiseuille type solution.

Problem. *Let* $(u, \nabla p)$ *be a classical solution of (3.1) in* $\mathbf{R}^n_+ \times (-\infty, 0)$ *with (3.2). Assume that* $|u|$ *and* $|\nabla u|$ *is bounded in* $\mathbf{R}^n_+ \times (-\infty, 0)$. *Then is* u *always a Poiseuille type solution?*

This problem is left open even if $n = 2$, for which we conjecture that the answer is yes. The reason we assume a bound for $|\nabla u|$ is that normal derivative cannot be controlled by a bound for $|u|$ near the boundary even if one considers the Stokes problem [17].

Remark 3. If one considers a time independent solution for (3.1)–(3.2), the Poiseuille type flow we constructed has a velocity u quadratic in x_n so it violates the growth condition for u. So it is not clear whether there is a nontrivial stationary solution to (3.1)–(3.2) with bounded velocity and vorticity.

Remark 4. One cannot impose the decay condition

$$\lim_{R\to\infty} \sup \left\{ \left|u^1(x_1, x_2)\right| \;\middle|\; x_1 \in \mathbf{R}, x_2 \geq R \right\} = 0 \tag{3.13}$$

to have a nontrivial solution in Theorem 3.2. This is because of the following non-existence results due to Y. Maekawa.

Theorem 3.3. *Let $u = (u^1, u^2)$ be a C^1-vector field in $\mathbf{R}^2_+$ satisfying*

$$\operatorname{div} u = 0 \quad \text{in} \quad \mathbf{R}^2_+. \tag{3.14}$$

Assume that the vorticity ω is nonnegative in $\mathbf{R}^2_+$. Assume furthermore that u and ∇u is bounded in $\mathbf{R}^2_+$ and that u is continuous up to the boundary of $\mathbf{R}^2_+$ with

$$u = 0 \quad \text{on} \quad \partial\mathbf{R}^2_+. \tag{3.15}$$

If u^1 fulfills (3.13), then $\omega \equiv 0$.

Corollary 1. *Let $(u, \nabla p)$ be a classical solution of (3.1) in $\mathbf{R}^2_+ \times (-\infty, 0)$ with (3.2). Assume that $|u|$ and $|\nabla u|$ is bounded in $\mathbf{R}^2_+ \times (-\infty, 0)$ and that $\omega \geq 0$ in $\mathbf{R}^2_+ \times (-\infty, t_0]$ for some time $t_0 < \infty$. If $u^1(t_0) = u^1(x_1, x_2, t_0)$ fulfills the decay condition (3.13) as $x_2 \to \infty$, then $\omega(t) = 0$ for $t \leq t_0$.*

Proof of Corollary 1. By Theorem 3.3 it is clear that $\omega(t_0)\big(= \omega(x_1, x_2, t_0)\big) \equiv 0$ in $\mathbf{R}^2_+$. Since $\omega \geq 0$ in $\mathbf{R}^2_+ \times (-\infty, t_0]$ by the strong maximum principle [23] we conclude that $\omega \equiv 0$ in $\mathbf{R}^2_+ \times (-\infty, t_0]$. □

We shall prove Theorem 3.3. Let E be the fundamental solution of $-\Delta$ in $\mathbf{R}^2$, i.e. $E(x) = -(2\pi)^{-1} \log|x|$. We define

$$K(x, y) = -(\partial_2 E)(x - y^*)\,\omega(y),\ y^* = (y_1, -y_2),\ \partial_2 = \partial/\partial x_2.$$

The next Lemma is a key observation of Y. Maekawa.

Lemma 3.4. *Assume that $u \in C^2(\mathbf{R}^2_+) \cap C(\bar{\mathbf{R}}^2_+)$ fulfills (3.14) and (3.15). Assume that $|u|$ and $|\nabla u|$ is bounded in $\mathbf{R}^2_+$. If $\omega = \partial_1 u^2 - \partial_2 u^1 \geq 0$ in $\mathbf{R}^2_+$, then $K \geq 0$ in $\mathbf{R}^2_+ \times \mathbf{R}^2_+$. Moreover,*

(i) $K(x,y)$ is integrable in y for each $x \in \mathbf{R}^2_+$.

(ii) $M(x) := \int_{\mathbf{R}^2_+} K(x,y)\,\mathrm{d}y$ is a constant independent of $x \in \mathbf{R}^2_+$.

(iii) $\big|M(x)\big| \leq C \liminf\limits_{R\to\infty} \sup\limits_{R\leq y_2\leq 2R} \frac{1}{y_2} \sup\limits_{x_1\in\mathbf{R}} \big|\psi(x_1,y_2)\big|$

with $\psi(y_1,y_2) = \int_0^{y_2} u^1(y_1,z_2)\mathrm{d}z_2$, where C is a constant depending only on the bound for $\|u^1\|_\infty$.

Proof of Theorem 3.3. If we assume (3.13), then

$$\begin{aligned}\sup_{R\leq y_2\leq 2R} \frac{1}{y_2} \sup_{x_1\in\mathbf{R}} \big|\psi(x_1,y_2)\big| &\leq \sup_{1\leq z\leq 2} \int_0^1 \sup_{x_1} \big|u^1(x_2,Rzs)\big|\,\mathrm{d}s \\ &\leq \int_0^1 \sup\Big\{ \big|u^1(x_1,Rzs)\big| \;\Big|\; 1\leq z\leq 2, x_1\in\mathbf{R}\Big\}\,\mathrm{d}s \to 0 \quad \text{as} \quad R\to\infty\end{aligned}$$

by Lebesgue's bounded convergence theorem. Thus $M \equiv 0$ so that $K \equiv 0$ in $\mathbf{R}^2_+ \times \mathbf{R}^2_+$. This implies that $\omega \equiv 0$. □

It remains to prove Lemma 3.4. One should note that the integrability (i) is not automatic since $(\partial_2 E)(x-y^*)$ is not integrable in $\mathbf{R}^2_+$ as a function of y for $x \in \mathbf{R}^2_+$. Since $\operatorname{div} u = 0$ in $\mathbf{R}^2_+$ and $u = 0$ on $\partial\mathbf{R}^2_+$, we are tempted to calculate by integration by parts and observing that $(\partial_2 E)(x-y^*) = \partial_{y_2}\big(E(x-y^*)\big)$ to get

$$\begin{aligned}\int_{\mathbf{R}^2_+} K(x,y)\,\mathrm{d}y &= \int_{\mathbf{R}^2_+} -(\partial_2 E)(x-y^*)(\partial_1 u^2 - \partial_2 u^1)(y)\,\mathrm{d}y \\ &= \int_{\mathbf{R}^2_+} \big\{-(\partial_2\partial_1 E)(x-y^*)u^2(y) - (\partial_2^2 E)(x-y^*)\,u^1(y)\big\}\,\mathrm{d}y \\ &= \int_{\mathbf{R}^2_+} \big\{(\partial_1 E)(x-y^*)\partial_2 u^2(y) - (\partial_2^2 E)(x-y^*)\,u^1(y)\big\}\,\mathrm{d}y \\ &= -\int_{\mathbf{R}^2_+} \big((\partial_1^2+\partial_2^2)\,E\big)(x-y^*)\,u^1(y)\,\mathrm{d}y = 0.\end{aligned}$$

However, this calculation is wrong because $(\partial_2 E)(x-y^*)$ is not integrable though the corresponding calculation for $\int \partial_j K(x,y)\,\mathrm{d}y$ can be justified to conclude that $\nabla M(x) = 0$.

Proof of Lemma 3.4. Since $\omega \geq 0$ and $(\partial_2 E)(x-y^*) \leq 0$, it is clear that $K \geq 0$ in $\mathbf{R}^2_+ \times \mathbf{R}^2_+$.

We shall prove (i)–(iii). Let ϕ be a nonnegative cut-off function with compact support, which will be specified later. Set

$$K_\phi(x,y) = K(x,y)\phi(y). \tag{3.16}$$

We will freely use $\operatorname{div} u = 0$ in $\mathbf{R}^2_+$, $u = 0$ on $x_2 = 0$, $\partial_{x_2}\big(E(x-y^*)\big) = \partial_{y_2}\big(E(x-y^*)\big)$, and $\Delta_x E(x-y^*) = 0$ in $x, y \in \mathbf{R}^2_+$. Then by the integration by parts we have

$$\begin{aligned}
M_\phi(x) &:= \int_{\mathbf{R}^2_+} K_\phi(x,y)\,\mathrm{d}y \\
&= -\int_{\mathbf{R}^2_+} \partial_{x_2} E(x-y^*)\big(\partial_1 u^2(y) - \partial_2 u^1(y)\big)\phi(y)\,\mathrm{d}y \\
&= \int_{\mathbf{R}^2_+} \partial_{x_1} E(x-y^*)\big(\phi\partial_2 u^2 + u^2\partial_2\phi\big)\,\mathrm{d}y \\
&\quad - \int_{\mathbf{R}^2_+} \partial^2_{x_2} E(x-y^*)u^1\phi\,\mathrm{d}y + \int_{\mathbf{R}^2_+} \partial_{x_2} E(x-y^*)\big(u^2\partial_1\phi - u^1\partial_2\phi\big)\,\mathrm{d}y \\
&= \int_{\mathbf{R}^2_+} \partial_{x_1} E(x-y^*)\big(-\phi\partial_1 u^1 + u^2\partial_2\phi\big)\,\mathrm{d}y \\
&\quad - \int_{\mathbf{R}^2_+} \partial^2_{x_2} E(x-y^*)u^1\phi\,\mathrm{d}y + \int_{\mathbf{R}^2_+} \partial_{x_2} E(x-y^*)\big(u^2\partial_1\phi - u^1\partial_2\phi\big)\,\mathrm{d}y \\
&= \int_{\mathbf{R}^2_+} \partial_{x_1} E(x-y^*)\big(u^1\partial_1\phi + u^2\partial_2\phi\big)\mathrm{d}y \\
&\quad + \int_{\mathbf{R}^2_+} \partial_{x_2} E(x-y^*)\big(u^2\partial_1\phi - u^1\partial_2\phi\big)\mathrm{d}y.
\end{aligned}$$

Note that the function ψ satisfies

$$\psi = \partial_2\psi = 0 \quad \text{on } \partial\mathbf{R}^2_+, \qquad \partial_1\psi = -u^2, \quad \partial_2\psi = u^1, \qquad \big|\psi(x)\big| \le C|x_2|. \tag{3.17}$$

(Actually, ψ is a stream function.) Thus from the above equality we may write $M_\phi(x)$ as

$$\begin{aligned}
M_\phi(x) &= 2\int_{\mathbf{R}^2_+} \partial^2_{x_2} E(x-y^*)\psi(y)\partial_2\phi(y)\mathrm{d}y - 2\int_{\mathbf{R}^2_+} \partial_{x_1}\partial_{x_2} E(x-y^*)\psi(y)\partial_1\phi(y)\mathrm{d}y \\
&\quad + \int_{\mathbf{R}^2_+} \partial_{x_2} E(x-y^*)\psi(y)\partial_1^2\phi(y)\mathrm{d}y + \int_{\mathbf{R}^2_+} \partial_{x_2} E(x-y^*)\psi(y)\partial_2^2\phi(y)\mathrm{d}y \\
&= \sum_{j=1}^4 I_j.
\end{aligned}$$

We take $\phi(x) = \phi_{R_1}(x_1)\phi_{R_2}(x_2)$ with $\phi_R(x) = \varphi\left(|x|/R\right)$ and $\varphi \in C^\infty[0,\infty)$ such that $\varphi(s) = 1$ for $s \le 1$ and $\varphi(s) = 0$ for $s \ge 2$ and $0 \le \varphi \le 1$. Then it is easy to verify

$$I_1 + I_4 \le C \sup_{R_2 \le y_2 \le 2R_2} \frac{1}{y_2} \big\| \psi(\cdot, y_2) \big\|_{L^\infty_{x_1}(\mathbf{R})}, \tag{3.18}$$

$$I_2 + I_3 \le C R_1^{-1} \int_0^{2R_2} \left(|x_2 + y_2|^{-1} + R_1^{-1} \right) \big\| \psi(\cdot, y_2) \big\|_{L^\infty_{x_1}(\mathbf{R})} \mathrm{d}y_2. \tag{3.19}$$

We first fix $R_2 > 0$ and let $R_1 \to \infty$. Then by the monotone convergence theorem we have

$$\begin{aligned} \int_{\mathbf{R}^2_+} K(x,y)\phi_{R_2}(y_2)\mathrm{d}y &= \lim_{R_1 \to \infty} \int_{\mathbf{R}^2_+} K(x,y)\phi_{R_1}(y_1)\phi_{R_2}(y_2)\mathrm{d}y \\ &\le C \sup_{R_2 \le y_2 \le 2R_2} \frac{1}{y_2} \big\| \psi(\cdot, y_2) \big\|_{L^\infty_{x_1}(\mathbf{R})}. \end{aligned}$$

Then by letting $R_2 \to \infty$ and by the monotone convergence theorem we have

$$M(x) = \lim_{R_2 \to \infty} \int_{\mathbf{R}^2_+} K(x,y)\phi_{R_2}(y_2)\mathrm{d}y \le C \liminf_{R_2 \to \infty} \sup_{R_2 \le y_2 \le 2R_2} \frac{1}{y_2} \big\| \psi(\cdot, y_2) \big\|_{L^\infty_{x_1}(\mathbf{R})}.$$

This proves (i) and (iii). Similar calculation yields $\left|\nabla M(x)\right| = 0$ in $\mathbf{R}^2_+$. So the proof is now complete. □

Remark 5. We note that a stream function Ψ solves $-\Delta\Psi = \omega$. If the non-slip boundary condition is imposed, then Ψ fulfills the Neumann and Dirichlet boundary condition on $x_2 = 0$ so that the Biot-Savart law deduces

$$\begin{aligned} \Psi(x) &= \int_{\mathbf{R}^2_+} \left\{ E(x-y) - E(x-y^*) \right\} \omega(y)\, \mathrm{d}y \\ &= \int_{\mathbf{R}^2_+} \left\{ E(x-y) + E(x-y^*) \right\} \omega(y)\, \mathrm{d}y \end{aligned}$$

when ω sufficiently decay near spatial infinity. Thus this implies that

$$\int_{\mathbf{R}^2_+} E(x-y^*)\, \omega(y)\, \mathrm{d}y \equiv 0.$$

Our M is the minus of the derivative of this quantity in x_2. (Note that $-\partial_2\Psi = u^1$ and $\partial_1\Psi = u^2$.)

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