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Multi-specialization and multi-asymptotic expansions

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Abstract

In this paper we extend the notion of specialization functor to the case of several closed submanifolds satisfying some suitable conditions. Applying this functor to the sheaf of Whitney holomorphic functions we construct different kinds of sheaves of multi-asymptotically developable functions, whose definitions are natural extensions of the definition of strongly asymptotically developable functions introduced by Majima.

Contents

1 Multi-normal deformation **4**

2 Multi-actions **12**

3 Multi-cones **13**

4 Multi-normal cones **14**

5 Morphisms between multi-normal deformations **22**

6 Multi-specialization **31**

7 Multi-asymptotic expansions **38**

8 Multi-specialization and asymptotic expansions **56**

A Conic sheaves **64**

B Multi-conic sheaves **67**

References **75**

Introduction

Asymptotically developable expansions of holomorphic functions on a sector are an important tool to study ordinary differential equations with irregular singularities. When we study, in higher dimension, a completely integrable connection with irregular singularities along a normal crossing divisor $H = H_1 \cup \cdots \cup H_\ell \subset X$, it is known that these asymptotic expansions are too weak for this purpose. Hence H. Majima, in [15], introduced the notion of *strongly asymptotically developable expansion* along H for a holomorphic function defined on a poly-sector S, and the one of *consistent family of coefficients* to which f is strongly asymptotically developable.

We can understand these notions from a view point of a locally defined multi-action. For each smooth submanifold H_k $(k = 1, 2, \ldots, \ell)$, we can locally identify X with the normal bundle $T_{H_k}X$ of H_k near H_k. A conic action on $T_{H_k}X$ by $\mathbb{R}^+$ induces a local action μ_k on X near H_k. Then a poly-sector S on which f is defined can be regarded as *a multi-cone* with respect to a multi-action μ_1, $\ldots$, μ_ℓ in the sense that it is an intersection of open sets V_k $(k = 1, 2, \ldots, \ell)$ where each V_k is a (locally) conic subset with respect to the action μ_k and its edge is contained in H_k. A strongly asymptotically developable expansion of f is, roughly speaking, a formal Taylor expansion with respect to an orbit generated by these actions μ_1, $\ldots$, μ_ℓ.

Hence, from this point of view, one can expect that strongly asymptotically developability extends to that along a more general H. As a matter of fact, we have succeeded to construct the sheaves of *multi-asymptotically developable functions* along several kinds of H by the aid of a multi-action, whose definitions are natural extensions of the one introduced by H. Majima in [15]. These sheaves contain, as important cases, not only that of strongly asymptotically developable functions but also that associated with a multi-cone which appears in *a bi-normal deformation* introduced by P. Schapira and K. Takeuchi in [21].

Now an important problem is to establish relations between these sheaves on different spaces along different kinds of H, to be more precise, we need to construct operations such as inverse images including restrictions and direct images for these sheaves. For that purpose, we need a uniform machinery allowing us to treat geometries associated with these multi-actions. Namely, we need the notion of *multi-normal deformation* and the one of *multi-specialization* introduced in this paper, which are our main subjects.

Let us briefly explain these new notions.

Let X be a n-dimensional real analytic manifold with $\dim X = n$, and let

$\chi = \{M_1, \ldots, M_\ell\}$ be a family of connected closed submanifolds satisfying some suitable conditions. The multi-normal deformation of X with respect to χ is constructed as follows. We first construct the normal deformation $\widetilde{X}_{M_1}$ of X along M_1 defined by M. Kashiwara and P. Schapira in [10]. Then, taking the pull-back of M_2 in $\widetilde{X}_{M_1}$, we can obtain the normal deformation $\widetilde{X}_{M_1,M_2}$ of $\widetilde{X}_{M_1}$ along the pull-back of M_2. Then we can define recursively the normal deformation along χ as $\widetilde{X} = \widetilde{X}_{M_1,\ldots,M_\ell} := (\widetilde{X}_{M_1,\ldots,M_{\ell-1}})^{\sim}_{\widetilde{M}_\ell}$. This manifold is of dimension $n+\ell$, it is locally isomorphic to $X \times \mathbb{R}^\ell$ and in the zero section $X \times \{0\}$ it is isomorphic to

$$\underset{X,1\leq j\leq \ell}{\times} T_{M_j}\iota(M_j) := T_{M_1}\iota(M_1) \underset{X}{\times} T_{M_2}\iota(M_2) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_\ell}\iota(M_\ell),$$

where $\iota(M_j)$ denotes the intersection of the M_k's strictly containing M_j (or X if M_j is maximal). There is also an action of $(\mathbb{R}^+)^\ell$ on $\widetilde{X}$, which is obtained as a natural extension of the $\mathbb{R}^+$-action of the normal deformation with respect to one submanifold. Its restriction to the zero section will be crucial for the definition of multi-asymptotic functions. There are also natural notions of multi-cone and multi-normal cone extending the one of P. Schapira and K. Takeuchi, which will be the key to understand the geometry of the sections of the multi-specialization functor. Given a morphism of real analytic manifolds $f : X \to Y$, we are also able to extend f to a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$. This is done by repeatedly employing the the usual construction of a morphism between normal deformations, i.e. we extend f to $\widetilde{f}_1 : \widetilde{X}_{M_1} \to \widetilde{Y}_{N_1}$, then we extend $\widetilde{f}_1$ to $\widetilde{f}_{1,2} : \widetilde{X}_{M_1,M_2} \to \widetilde{Y}_{N_1,N_2}$. Then we can define recursively $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ by extending the morphism $\widetilde{f}_{1,\ldots,\ell-1} : \widetilde{X}_{M_1,\ldots,M_{\ell-1}} \to \widetilde{Y}_{N_1,\ldots,N_{\ell-1}}$ to the normal deformations with respect to M_ℓ and N_ℓ respectively. This morphism enable us to make a link between different kinds of multi-normal deformations. As a kind of example, desingularization map makes a link between the normal deformation with respect to a normal crossing divisor and the binormal deformation of P. Schapira and K. Takeuchi.

Once we have constructed the multi-normal deformation, we are able to extend the definition of the specialization functor to the case of several submanifolds. Roughly speaking, it is a functor associating to a subanalytic sheaf on X a subanalytic sheaf on $\underset{X,1\leq j\leq \ell}{\times} T_{M_j}\iota(M_j)$. We perform all this constructions in the subanalytic setting in order to treat sheaves of functions with growth conditions. Given a morphism of real analytic manifolds $f : X \to Y$ we give conditions for the commutation of the direct and inverse images with respect to the multi-specialization functor.

When we apply the multi-specialization functor to the subanalytic sheaf of Whitney holomorphic functions we obtain the sheaf of multi-asymptotically developable functions along χ and, outside the zero section in the normal crossing case, the sheaf of strongly asymptotically developable functions of H. Majima. When we apply the multi-specialization functor to the subanalytic sheaf of Whitney holomorphic functions vanishing up to infinity on $M_1 \cup \cdots \cup M_\ell$ (resp. Whitney holomorphic functions on $M_1 \cup \cdots \cup M_\ell$) we obtain the sheaf of flat multi-asymptotically developable functions (resp. consistent families of coefficients) along χ. The vanishing of the H^1 of flat multi-asymptotically developable functions allows us to prove a Borel-Ritt exact sequence for multi-asymptotic functions.

The paper is organized in the following way. In Section 1 we introduce the notion of multi-normal deformation. In Section 2 we define the multi-action of $(\mathbb{R}^+)^\ell$ on the zero section of $\widetilde{X}$. In Sections 3 and 4 we give the definitions of multi-cone and multi-normal cone which are essential to understand the sections of the multi-specialization. Morphisms between multi-normal deformations and their restriction to the zero sections are studied in Section 5. The functorial construction is performed in Section 6 with the definition of the multi-specialization functor and its relations with the functors of direct and inverse image. In Section 7 we define the sheaves of multi-asymptotically developable functions along χ whose functorial nature is proved in Section 8, where we apply the multi-specialization functor to the subanalytic sheaf of holomorphic functions with Whitney growth conditions.

We end this work with an Appendix in which we introduce the category of multi-conic sheaves. Using o-minimal geometry we construct suitable coverings of subanalytic open subsets which are helpful in order to study the sections of multi-conic sheaves.

1 Multi-normal deformation

Let X be a real analytic manifold with $\dim X = n$, and let $\chi = \{M_1, \ldots, M_\ell\}$ be a family of closed submanifolds in X ($\ell \geq 1$). We set, for $N \in \chi$ and $p \in N$,

$$\mathrm{NR}_p(N) := \{M_j \in \chi;\, p \in M_j,\, N \not\subseteq M_j \text{ and } M_j \not\subseteq N\}.$$

Let us consider the following conditions for χ.

H1 Each $M_j \in \chi$ is connected and the submanifolds are mutually distinct, i.e. $M_j \neq M_{j'}$ for $j \neq j'$.

H2 For any $N \in \chi$ and $p \in N$ with $\mathrm{NR}_p(N) \neq \emptyset$, we have

$$\left(\bigcap_{M_j \in \mathrm{NR}_p(N)} T_p M_j \right) + T_p N = T_p X. \tag{1.1}$$

Note that, if χ satisfies the condition H2, the configuration of two submanifolds must be either 1. or 2. below.

1. Both submanifolds intersect transversely.
2. One of them contains the other.

We set, for $N \in \chi$,

$$\iota_\chi(N) := \begin{cases} X & \text{There exists no } M_j \in \chi \text{ with } N \subsetneqq M_j, \\ \displaystyle\bigcap_{N \subsetneqq M_j} M_j & \text{Otherwise.} \end{cases}$$

When there is no risk of confusion, we write for short $\iota(N)$ instead of $\iota_\chi(N)$.

Since $T_{M_j}\iota(M_j)$ is contained in the zero section $T_X X$ if M_j satisfies $M_j = \iota(M_j)$, we assume the condition H3 below for simplicity.

H3 $M_j \neq \iota(M_j)$ for any $j \in \{1, 2, \ldots, \ell\}$.

Example 1.1 *Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3).*

(i) Let $\chi = \{M_1, M_2, M_3\}$ with $M_i = \{x_i = 0\}$, $i = 1, 2, 3$. Then clearly χ satisfies H1. We have $\iota(M_i) = X$, $i = 1, 2, 3$ and $T_0 M_i + T_0 M_j = T_0 X$, $i, j \in \{1, 2, 3\}$, $i \neq j$. Hence χ satisfies H2,H3.

(ii) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$, $M_3 = \{x_3 = 0\}$. Then clearly χ satisfies H1. We have $\iota(M_1) = M_2$, $\iota(M_2) = M_3$, $\iota(M_3) = X$ and $NR_0(M_i) = \emptyset$ for $i = 1, 2, 3$. Hence χ satisfies H2,H3.

(iii) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}$, $M_2 = \{x_1 = 0\}$, $M_3 = \{x_2 = 0\}$. Then clearly χ satisfies H1. We have $NR_0(M_1) = \emptyset$ and $T_0 M_2 + T_0 M_3 = T_0 X$. We have $\iota(M_1) = M_2 \cap M_3 \supsetneqq M_1$ and $\iota(M_2) = \iota(M_3) = X$. Hence χ satisfies H2,H3.

(iv) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}$, $M_2 = \{x_1 = x_2 = 0\}$, $M_3 = \{x_3 = 0\}$. Then clearly χ satisfies H1. We have $NR_0(M_1) = \emptyset$ and $T_0 M_2 + T_0 M_3 = T_0 X$. We have $\iota(M_2) = \iota(M_3) = X$ and $\iota(M_1) = M_2 \cap M_3 = M_1$. Hence χ satisfies H2 but not H3.

(v) *Let* $\chi = \{M_1, M_2\}$ *with* $M_1 = \{x_1 = x_2 = 0\}$, $M_2 = \{x_2 = x_3 = 0\}$. *Then clearly* χ *satisfies H1. We have* $T_0M_1 + T_0M_2 \subsetneqq T_0X$. *Then* χ *does not satisfy H2.*

(vi) *Let* $\chi = \{M_1, M_2, M_3\}$ *with* $M_1 = \{x_1 = x_2\}$, $M_2 = \{x_1 = 0\}$, $M_3 = \{x_2 = 0\}$. *Then clearly* χ *satisfies H1. We have* $T_0M_i + \bigcap_{i\neq j} T_0M_j = T_0M_i \subsetneqq T_0X$ *for* $i = 1, 2, 3$. *Then* χ *does not satisfy H2.*

(vii) *Let* $\chi = \{M_1, M_2, M_3\}$ *with* $M_1 = \{x_1 = x_2^2\}$, $M_2 = \{x_1 = 0\}$, $M_3 = \{x_2 = 0\}$. *Then clearly* χ *satisfies H1. We have* $T_0M_1 + \bigcap_{1\neq j} T_0M_j = T_0M_1 \subsetneqq T_0X$. *Then* χ *does not satisfy H2 (even if* $\bigcap_{3\neq j} T_0M_j + T_0M_3 = T_0X$*).*

Proposition 1.2 *The following conditions are equivalent.*

1. *The family* χ *satisfies the condition H2.*
2. *For any* $p \in X$, *there exist a neighborhood* V *of* p *in* X, *a system of local coordinates* $(x_1, \ldots, x_n)$ *in* V *and a family of subsets* $\{I_j\}_{j=1}^{\ell}$ *of the set* $\{1, 2, \ldots, n\}$ *for which the following conditions hold.*
   (a) *Either* $I_k \subset I_j$, $I_j \subset I_k$ *or* $I_k \cap I_j = \emptyset$ *holds* $(k, j \in \{1, 2, \ldots, \ell\})$.
   (b) *A submanifold* $M_j \in \chi$ *with* $p \in M_j$ $(j = 1, 2, \ldots, \ell)$ *is defined by* $\{x_i = 0; i \in I_j\}$ *in* V.

Proof. Clearly 2. implies 1. We will show the converse implication. We may assume, by taking V sufficiently small, $p \in M_j$ for any j. We set, for $N \in \chi$,

$$d(N) := \max\{k;\ N = M_{j_1} \subsetneqq M_{j_2} \subsetneqq \cdots \subsetneqq M_{j_k},\ M_{j_1}, \ldots, M_{j_k} \in \chi\},$$

and

$$d(\chi) := \max\{d(N);\ N \in \chi\}.$$

We show the proposition by induction with respect to $d(\chi)$. Clearly the equivalence holds for a family χ with $d(\chi) = 1$. Let us consider χ with $d(\chi) = \kappa > 1$, and suppose that 1. $\Rightarrow$ 2. were true for any χ with $d(\chi) < \kappa$. Let $L_1, \ldots, L_m$ denote the least elements in χ with respect to the partial order $\subset$ of sets.

For each $k = 1, 2, \ldots, m$, we will determine defining functions $f_{k,1}, \ldots, f_{k,i_k}$ of the submanifold L_k in the following way. Set, for any $N \in \chi$,

$$\chi_N = \{L \in \chi;\ N \subsetneqq L\}.$$

Since we have $d(\chi_{L_k}) < \kappa$ $(k = 1, 2, \ldots, m)$, and since the family χ_{L_k} also satisfies the condition 1. of the proposition, by the induction hypothesis, there exist local coordinate functions $(\varphi_{k,1}, \ldots, \varphi_{k,n})$ that satisfies the condition 2. for the family χ_{L_k}. Then, for $k = 1, 2, \ldots, m$, we take defining functions $\{f_{k,1}, \ldots, f_{k,i_k}\}$ of L_k so that they contain all the coordinate functions $\varphi_{k,i}$ which vanish on L_k.

As $L_{j'} \in \mathrm{NR}_p(L_j)$ holds for $1 \leq j \neq j' \leq m$, it follows from the condition (1.1) that we have

$$\bigwedge_{1 \leq k \leq m,\, 1 \leq i \leq i_k} df_{k,i} \neq 0 \qquad \text{near } p.$$

Therefore the family of these functions $\{f_{k,i}\}$ can be extended to a system of local coordinates near p, for which we can easily verify 2. of the proposition. This completes the proof. □

Let X be a n-dimensional real analytic manifold and let $\chi = \{M_1, \ldots, M_\ell\}$ be a family of closed smooth submanifolds of X satisfying H2. First recall the construction of the normal deformation of X along M_1. We denote it by $\widetilde{X}_{M_1}$ and we denote by $t_1 \in \mathbb{R}$ the deformation parameter. Let $\Omega_{M_1} = \{t_1 > 0\}$ and let us identify $s^{-1}(0)$ with $T_{M_1}X$. We have the commutative diagram

$$\begin{array}{ccccc} T_{M_1}X & \xrightarrow{s_{M_1}} & \widetilde{X}_{M_1} & \xleftarrow{i_{\Omega_{M_1}}} & \Omega_{M_1} \\ \downarrow{\scriptstyle \tau_{M_1}} & & \downarrow{\scriptstyle p_{M_1}} & \swarrow{\scriptstyle \widetilde{p}_{M_1}} & \\ M & \xrightarrow{i_{M_1}} & X. & & \end{array} \tag{1.2}$$

Set $\widetilde{\Omega}_{M_1} = \{(x, t_1)\ ;\ t_1 \neq 0\}$ and define

$$\widetilde{M_2} := \overline{(p_{M_1}|_{\widetilde{\Omega}_{M_1}})^{-1} M_2}.$$

Then $\widetilde{M_2}$ is a closed smooth submanifold of $\widetilde{X}_{M_1}$. Now we can define the normal deformation along M_1, M_2 as

$$\widetilde{X}_{M_1,M_2} := (\widetilde{X}_{M_1})^{\sim}_{\widetilde{M_2}}.$$

Then we can define recursively the normal deformation along χ as

$$\widetilde{X} = \widetilde{X}_{M_1,\ldots,M_\ell} := (\widetilde{X}_{M_1,\ldots,M_{\ell-1}})^{\sim}_{\widetilde{M_\ell}}.$$

Set $S = \{t_1, \dots, t_\ell = 0\}$, $M = \bigcap_{i=1}^{\ell} M_i$ and $\Omega = \{t_1, \dots, t_\ell > 0\}$. Then we have the commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{s} & \widetilde{X} & \xleftarrow{i_\Omega} & \Omega \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \widetilde{p}} & \\ M & \xrightarrow{i_M} & X. & & \end{array} \tag{1.3}$$

In local coordinate let $I_1, \dots I_\ell \subseteq \{1, \dots, n\}$ such that $M_i = \{x_k = 0\ ;\ k \in I_i\}$. Set

$$J_i = \{k \in \{1, \dots, \ell\}\ ;\ i \in I_k\}, \qquad t_{J_i} = \prod_{k \in J_i} t_k,$$

where $t_1, \dots, t_\ell \in \mathbb{R}$ and $t_{J_i} = 1$ if $J_i = \emptyset$. Then $p : \widetilde{X} \to X$ is defined by

$$(x_1, \dots, x_n, t_1, \dots, t_\ell) \mapsto (t_{J_1} x_1, \dots, t_{J_n} x_n).$$

We are interested in the bundle structure of the zero section

$$S := \{t_1 = \dots = t_\ell = 0\} \subset \widetilde{X}.$$

Let us consider the canonical map $T_{M_j}\iota(M_j) \to M_j \hookrightarrow X$, $j = 1, \dots, \ell$, we write for short

$$\underset{X, 1 \le j \le \ell}{\times} T_{M_j}\iota(M_j)$$

instead of

$$T_{M_1}\iota(M_1) \underset{X}{\times} T_{M_2}\iota(M_2) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_\ell}\iota(M_\ell).$$

Proposition 1.3 *Assume that χ satisfies the conditions H1, H2 and H3. Then we have*

$$S \simeq \underset{X, 1 \le j \le \ell}{\times} T_{M_j}\iota(M_j). \tag{1.4}$$

Proof. Let $\hat{X}$ be a copy of X, and $\varphi : X \to \hat{X}$ a local coordinate transformation near $p \in X$. We set $\hat{M}_j = \varphi(M_j)$. We may assume that $X = \mathbb{R}^n$ (resp. $\hat{X} = \mathbb{R}^n$) with coordinates $(x_1, \dots, x_n)$ (resp. $(\hat{x}_1, \dots, \hat{x}_n)$), and φ is given by $\hat{x}_i = \varphi_i(x_1, \dots, x_n)$ $(i = 1, 2, \dots, n)$. Moreover, by Proposition 1.2, we may also suppose that there exist a family of subsets $\{I_j\}_{j=1}^{\ell}$ of $\{1, 2, \dots, n\}$ that satisfies 2. (a) and 2. (b) of Proposition 1.2 for the both coordinate systems.

Let $J_k \subset \{1, 2, \dots, \ell\}$ $(k = 1, 2, \dots, n)$ denote the set

$$\{j \in \{1, 2, \dots, \ell\};\ k \in I_j\}. \tag{1.5}$$

For a subset J of $\{1, 2, \ldots, \ell\}$, we set $t_J := \prod_{j \in J} t_j$ if J is non-empty and $t_J := 1$ if $J = \emptyset$.

Then, outside of $t_1 t_2 \ldots t_\ell = 0$, the coordinate transformation between multi-normal deformations

$$(x_1, \ldots, x_n; t_1, \ldots, t_\ell) \to (\hat{x}_1, \ldots, \hat{x}_n; \hat{t}_1, \ldots, \hat{t}_\ell)$$

is given by

$$\begin{cases} \hat{t}_{J_k} \hat{x}_k = \varphi_k(t_{J_1} x_1, t_{J_2} x_2, \ldots, t_{J_n} x_n) & (k = 1, 2, \ldots, n), \\ \hat{t}_j = t_j & (j = 1, 2, \ldots, \ell). \end{cases}$$

For any $k \in \bigcup_{1 \le j \le \ell} I_j$, we set

$$I(k) := \bigcap_{k \in I_j} I_j = \bigcap_{j \in J_k} I_j.$$

As the condition 2. (a) of Proposition 1.2 is equivalently saying that

$$I_j \cap I_{j'} \neq \emptyset \Rightarrow I_j \subset I_{j'} \text{ or } I_{j'} \subset I_j,$$

the set $\{I_j;\, k \in I_j\}$ is totally ordered with respect to "$\subset$", and $I(k)$ is its minimal element. Hence, for any $k \in \bigcup_{1 \le j \le \ell} I_j$, there exists $j(k) \in \{1, 2, \ldots, \ell\}$ such that $I(k) = I_{j(k)}$. By expanding $\varphi_k(x_1, \ldots, x_n)$ along the submanifold $M_{j(k)}$, we obtain

$$\varphi_k(x_1, \ldots, x_n) = \sum_{i \in I_{j(k)}} \left.\frac{\partial \varphi_k}{\partial x_i}\right|_{M_{j(k)}} x_i + \frac{1}{2} \sum_{i_1, i_2 \in I_{j(k)}} \left.\frac{\partial^2 \varphi_k}{\partial x_{i_1} \partial x_{i_2}}\right|_{M_{j(k)}} x_{i_1} x_{i_2} + \ldots,$$

as $\varphi_k|_{M_{j(k)}} = 0$ holds. Then we get

$$t_{J_k} \hat{x}_k = \sum_{i \in I_{j(k)}} \left.\frac{\partial \varphi_k}{\partial x_i}\right|_{M_{j(k)}} t_{J_i} x_i + \frac{1}{2} \sum_{i_1, i_2 \in I_{j(k)}} \left.\frac{\partial^2 \varphi_k}{\partial x_{i_1} \partial x_{i_2}}\right|_{M_{j(k)}} t_{J_{i_1}} t_{J_{i_2}} x_{i_1} x_{i_2} + \ldots.$$

We can easily see $J_k \subseteq J_i$ for any $i \in I_{j(k)}$. Indeed, this follows from the facts

$$j \in J_k \text{ and } i \in I_{j(k)} \Longrightarrow k \in I_j \text{ and } i \in I(k) = \bigcap_{k \in I_\beta} I_\beta$$

$$\Longrightarrow i \in I_j \Longrightarrow j \in J_i.$$

Therefore, by letting $t \to 0$, we have

$$\hat{x}_k = \sum_{i \in I_{j(k)},\, J_k = J_i} \left. \frac{\partial \varphi_k}{\partial x_i} \right|_M x_i.$$

where $M := \bigcap_{1 \le j \le l} M_j$. Now we claim, for $k \in \bigcup_{1 \le j \le \ell} I_j$,

$$\left\{ i \in I_{j(k)};\, J_k = J_i \right\} = \left\{ i \in \{1, 2, \ldots, n\};\, j(k) = j(i) \right\},$$

which is proved in the following way: We first prove the implication ($\Leftarrow$). Assume that i satisfies $j(k) = j(i)$, which implies $i \in I_{j(i)} = I_{j(k)}$. We have already proved the fact $J_k \subseteq J_i$ for $i \in I_{j(k)}$. Therefore it suffices to show $J_i \subseteq J_k$. Let $\beta \in J_i$. Then, as $i \in I_\beta$, we have $k \in I_{j(k)} = I_{j(i)} \subset I_\beta$, from which $\beta \in J_k$ follows.

The converse implication comes from

$$I_{j(k)} = \bigcap_{\beta \in J_k} I_\beta = \bigcap_{\beta \in J_i} I_\beta = I_{j(i)}.$$

We divide the set $\bigcup_{1 \le j \le \ell} I_j \subset \{1, 2, \ldots, n\}$ into equivalent classes $\{B_\alpha\}$ by the equivalence relation "$i \sim k \iff j(i) = j(k)$". Then, for an equivalent class B, we obtain

$$\hat{x}_i = \sum_{k \in B} \left. \frac{\partial \varphi_i}{\partial x_k} \right|_M x_k \qquad \text{for } i \in B.$$

We denote by E_B the vector bundle over M defined by the above equations where its fiber coordinates are given by x_k's ($k \in B$). Then, by the above observation, S is a direct sum of bundles E_B's over M. Note that, since each equivalent class can be written in the form

$$I_{j(k)} \setminus \left(\bigcup_{I_{j(i)} \subsetneqq I_{j(k)}} I_{j(i)} \right) = I_{j(k)} \setminus \left(\bigcup_{I_j \subsetneqq I_{j(k)}} I_j \right)$$

for some $j(k) \in \{1, 2, \ldots, l\}$ with $k \in \bigcup_{1 \le j \le \ell} I_j$, the bundle E_B is isomorphic to $T_{M_{j(k)}} \iota(M_{j(k)}) \underset{X}{\times} M$.

For any $j \in \{1, 2, \ldots, \ell\}$, the set

$$\hat{I}_j := I_j \setminus \left(\bigcup_{I_k \subsetneqq I_j} I_k \right)$$

is not empty by the condition H3, and it gives an equivalent class of $\bigcup_{1\leq j\leq \ell} I_j/\sim$. Further it follows from the condition H1 and H2 that we get

$$\bigcup_{1\leq j\leq \ell} I_j = \hat{I}_1 \sqcup \cdots \sqcup \hat{I}_l.$$

Hence we have obtained that S is a direct sum of bundles $T_{M_j}\iota(M_j) \underset{X}{\times} M$ $(j = 1, 2, \ldots.\ell)$. This completes the proof. □

Example 1.4 *Let us see three typical examples of multi-normal deformations.*

1. *(Majima) Let $X = \mathbb{C}^2$ ($\simeq \mathbb{R}^4$ as a real manifold) with coordinates (z_1, z_2) and let $\chi = \{M_1, M_2\}$ with $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1\}$, $I_2 = \{2\}$, $J_1 = \{1\}$, $J_2 = \{2\}$ (in $\mathbb{R}^4$, if $z_1 = (x_1, x_2)$ and $z_2 = (x_3, x_4)$ we have $I_1 = \{1, 2\}$, $I_2 = \{3, 4\}$, $J_1 = J_2 = \{1\}$, $J_3 = J_4 = \{2\}$). The map $p : \widetilde{X} \to X$ is defined by*

   $$(z_1, z_2, t_1, t_2) \mapsto (t_1 z_1, t_2 z_2).$$

   We have $\iota(M_1) = \iota(M_2) = X$ and then the zero section S of $\widetilde{X}$ is isomorphic to $T_{M_1}X \underset{X}{\times} T_{M_2}X$.

2. *(Takeuchi) Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1, 2, 3\}$, $I_2 = \{2, 3\}$, $I_3 = \{3\}$, $J_1 = \{1\}$, $J_2 = \{1, 2\}$, $J_3 = \{1, 2, 3\}$. The map $p : \widetilde{X} \to X$ is defined by*

   $$(x_1, x_2, x_3, t_1, t_2, t_3) \mapsto (t_1 x_1, t_1 t_2 x_2, t_1 t_2 t_3 x_3).$$

   We have $\iota(M_1) = M_2$, $\iota(M_2) = M_3$, $\iota(M_3) = X$ and then the zero section S of $\widetilde{X}$ is isomorphic to $T_{M_1}M_2 \underset{X}{\times} T_{M_2}M_3 \underset{X}{\times} T_{M_3}X$.

3. *(Mixed) Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1, 2, 3\}$, $I_2 = \{2\}$, $I_3 = \{3\}$, $J_1 = \{1\}$, $J_2 = \{1, 2\}$, $J_3 = \{1, 3\}$. The map $p : \widetilde{X} \to X$ is defined by*

   $$(x_1, x_2, x_3, t_1, t_2, t_3) \mapsto (t_1 x_1, t_1 t_2 x_2, t_1 t_3 x_3).$$

We have $\iota(M_1) = M_2 \cap M_3$, $\iota(M_2) = \iota(M_3) = X$ and then the zero section S is isomorphic to $T_{M_1}(M_2 \cap M_3) \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_3}X$.

2 Multi-actions

Let X be a real analytic manifold and let $\chi = \{M_1, M_2, \dots, M_\ell\}$ be closed submanifolds. In what follows, we always assume that χ satisfies the conditions H1, H2 and H3. Consider the diagram (1.3). There is a $(\mathbb{R}^+)^\ell$ action

$$\mu : \widetilde{X} \times (\mathbb{R}^+)^\ell \to \widetilde{X}$$

which is described in local coordinate system by

$$((x_1, \dots, x_n, t_1, \dots, t_\ell), (c_1, \dots, c_\ell)) \mapsto \left(c_{J_1}x_1, \dots, c_{J_n}x_n, \frac{t_1}{c_1}, \dots, \frac{t_\ell}{c_\ell}\right).$$

More precisely, the j-th component of the action is given by

$$\mu_j : ((x_1, \dots, x_n, t_1, \dots, t_\ell), c_j) \mapsto \left(c_{1j}x_1, \dots, c_{nj}x_n, t_1, \dots, \frac{t_j}{c_j}, \dots, t_\ell\right),$$

where $c_{ij} = c_j$ if $i \in I_j$ and $c_{ij} = 1$ otherwise.

First we consider the actions on $\widetilde{X}$ which are compatible with those on $\underset{X,1\le\alpha\le\ell}{\times} T_{M_\alpha}\iota(M_\alpha)$. Set

$$J_{\supsetneq M_j} := \{\alpha \in \{1, 2, \dots, \ell\};\ M_\alpha \supsetneq M_j,\ \text{ there is no } \beta \text{ with } M_\alpha \supsetneq M_\beta \supsetneq M_j\ \}$$
$$J_{\subsetneq M_j} := \{\alpha \in \{1, 2, \dots, \ell\};\ M_\alpha \subsetneq M_j,\ \text{ there is no } \beta \text{ with } M_\alpha \subsetneq M_\beta \subsetneq M_j\ \}$$

Note that, by the conditions, the set $J_{\subsetneq M_j}$ either is empty or consists of only 1 index.

Using the actions μ_j for $j = 1, 2, \dots, \ell$, we define the action $\tau_j(\lambda) : \widetilde{X} \to \widetilde{X}$ by

$$\tau_j(\lambda) := \mu_j(\lambda) \prod_{\beta \in J_{\supsetneq M_j}} \mu_\beta(\lambda^{-1}) \qquad (j = 1, 2, \dots, \ell).$$

On the zero section $\underset{X,1\le\alpha\le\ell}{\times} T_{M_\alpha}\iota(M_\alpha)$ of $\widetilde{X}$, the action $\tau_j(\lambda)$ coincides with

$$\tau_j(\lambda)|_{T_{M_\alpha}\iota(M_\alpha)} = \begin{cases} T_{M_\alpha}\iota(M_\alpha) \xrightarrow{\lambda\cdot} T_{M_\alpha}\iota(M_\alpha) & (\alpha = j) \\ \mathrm{id}_{T_{M_\alpha}\iota(M_\alpha)} & (\alpha \neq j). \end{cases}$$

Conversely, we can recover the original actions $\{\mu_j\}$ by $\{\tau_j\}$ in the following way.

$$\mu_j(\lambda) := \prod_{M_j \subseteq M_\alpha} \tau_\alpha(\lambda) \qquad (j = 1, 2, \ldots, l).$$

Hence both multi-actions on $\widetilde{X}$ associated with $\{\mu_j\}$ and $\{\tau_j\}$ are equivalent.

Example 2.1 *Let us see some example of multi-actions.*

1. *(Majima) Let $X = \mathbb{C}^2$ with coordinates (z_1, z_2) and let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then*

$$\begin{aligned}
\mu : (z_1, z_2, t_1, t_2) &\mapsto \left(c_1 z_1, c_2 z_2, \frac{t_1}{c_1}, \frac{t_2}{c_2}\right) \\
\tau_1 : (z_1, z_2, t_1, t_2) &\mapsto (\lambda z_1, z_2, \lambda^{-1} t_1, t_2) \\
\tau_2 : (z_1, z_2, t_1, t_2) &\mapsto (z_1, \lambda z_2, t_1, \lambda^{-1} t_2).
\end{aligned}$$

2. *(Takeuchi) Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then*

$$\begin{aligned}
\mu : (x_1, x_2, x_3, t_1, t_2, t_3) &\mapsto \left(c_1 x_1, c_1 c_2 x_2, c_1 c_2 c_3 x_3, \frac{t_1}{c_1}, \frac{t_2}{c_2}, \frac{t_3}{c_3}\right) \\
\tau_1 : (x_1, x_2, x_3, t_1, t_2, t_3) &\mapsto (\lambda x_1, x_2, x_3, \lambda^{-1} t_1, \lambda t_2, t_3) \\
\tau_2 : (x_1, x_2, x_3, t_1, t_2, t_3) &\mapsto (x_1, \lambda x_2, x_3, t_1, \lambda^{-1} t_2, \lambda t_3) \\
\tau_3 : (x_1, x_2, x_3, t_1, t_2, t_3) &\mapsto (x_1, x_2, \lambda x_3, t_1, t_2, \lambda^{-1} t_3).
\end{aligned}$$

3. *(Mixed) Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then*

$$\begin{aligned}
\mu : (x_1, x_2, x_3, t_1, t_2, t_3) &\mapsto \left(c_1 x_1, c_1 c_2 x_2, c_1 c_3 x_3, \frac{t_1}{c_1}, \frac{t_2}{c_2}, \frac{t_3}{c_3}\right) \\
\tau_1 : (x_1, x_2, x_3, t_1, t_2, t_3) &\mapsto (\lambda x_1, x_2, x_3, \lambda^{-1} t_1, \lambda t_2, \lambda t_3) \\
\tau_2 : (x_1, x_2, x_3, t_1, t_2, t_3) &\mapsto (x_1, \lambda x_2, x_3, t_1, \lambda^{-1} t_2, t_3) \\
\tau_3 : (x_1, x_2, x_3, t_1, t_2, t_3) &\mapsto (x_1, x_2, \lambda x_3, t_1, t_2, \lambda^{-1} t_3).
\end{aligned}$$

3 Multi-cones

Let $q \in \bigcap_{1 \le j \le \ell} M_j$ and $p_j = (q; \xi_j)$ be a point in $T_{M_j}\iota(M_j)$ $(j = 1, 2, \ldots, \ell)$. We set

$$p = p_1 \underset{X}{\times} \ldots \underset{X}{\times} p_\ell \in \underset{X, 1 \le j \le \ell}{\times} T_{M_j}\iota(M_j),$$

and $\tilde{p}_j = (q; \tilde{\xi}_j) \in T_{M_j}X$ designates the image of the point p_j by the canonical embedding $T_{M_j}\iota(M_j) \hookrightarrow T_{M_j}X$. We denote by $\mathrm{Cone}_{\chi,j}(p)$ $(j = 1, 2, \ldots, \ell)$ the set of open conic cones in $(T_{M_j}X)_q \simeq \mathbb{R}^{n-\dim M_j}$ that contain the point $\tilde{\xi}_j \in (T_{M_j}X)_q \simeq \mathbb{R}^{n-\dim M_j}$.

Definition 3.1 *We say that an open set $G \subset (TX)_q$ is a multi-cone along χ with direction to $p \in \left(\underset{X,1\leq j\leq \ell}{\times} T_{M_j}\iota(M_j)\right)_q$ if G is written in the form*

$$G = \bigcap_{1\leq j\leq \ell} \pi_{j,q}^{-1}(G_j) \qquad G_j \in \mathrm{Cone}_{\chi,j}(p)$$

where $\pi_{j,q} : (TX)_q \to (T_{M_j}X)_q$ is the canonical projection. We denote by $\mathrm{Cone}_\chi(p)$ the set of multi-cones along χ with direction to p.

Example 3.2 *We now give three typical examples of multi-cones.*

1. *(Majima) Let $X = \mathbb{C}^2$ with coordinates (z_1, z_2) and let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then $\mathrm{Cone}_\chi(p)$ for $p = (0, 0; 1, 1)$ is nothing but the set of multi sectors along $Z_1 \cup Z_2$ with their direction to $(1, 1)$.*

2. *(Takeuchi) Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0, 0, 0; 1, 1, 1) \in T_{M_1}M_2 \underset{X}{\times} T_{M_2}M_3 \underset{X}{\times} T_{M_3}X$, it is easy to see that a cofinal set of $\mathrm{Cone}_\chi(p)$ is, for example, given by the family of the sets*

$$\{(\xi_1, \xi_2, \xi_3);\ |\xi_2| + |\xi_3| < \epsilon\xi_1,\ |\xi_3| < \epsilon\xi_2,\ \xi_3 > 0\}_{\epsilon>0}.$$

3. *(Mixed) Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0, 0, 0; 1, 1, 1) \in T_{M_1}(M_2 \cap M_3) \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_3}X$, a cofinal set of $\mathrm{Cone}_\chi(p)$ is, for example, given by the family of the sets*

$$\{(\xi_1, \xi_2, \xi_3);\ |\xi_2| + |\xi_3| < \epsilon\xi_1,\ \xi_2 > 0,\ \xi_3 > 0\}_{\epsilon>0}.$$

4 Multi-normal cones

Definition 4.1 *Let Z be a subset of X. The multi-normal cone to Z along χ is the set*

$$C_\chi(Z) = \overline{\widetilde{p}^{-1}(Z)} \cap S.$$

For any $q \in X$, there exists an isomorphism

$$\psi : X \xrightarrow{\sim} (TX)_q$$

near q which satisfies $\psi(q) = (q; 0)$ and $\psi(M_j) = (TM_j)_q$ for any $j = 1, \ldots, \ell$. The existence of such a ψ is guaranteed by Proposition 1.2.

Lemma 4.2 *Let Z be a subset of X. We have the following equivalence: $p \notin C_\chi(Z)$ if and only if there exist an open subset $\psi(q) \in U \subset (TX)_q$ and a multi-cone $G \in \mathrm{Cone}_\chi(p)$ such that*

$$\psi(Z) \cap G \cap U = \emptyset$$

holds.

Proof. The problem is local. Hence we identify $X \simeq (TX)_q \simeq \mathbb{R}^n$ by ψ, and we consider, in what follows, $\mathrm{Cone}_\chi(p)$ as the set of cones defined in X. We may assume $q = 0$. Recall that $\hat{I}_j \subset \{1, 2, \ldots, n\}$ is defined as

$$\hat{I}_j := I_j \setminus \left(\bigcup_{I_k \subsetneq I_j} I_k \right).$$

The equivalent class $\hat{I}_j$ corresponds to $T_{M_j}\iota(M_j)$ for $j \in \{1, \ldots, \ell\}$, and set $B := \bigcup_{1 \le j \le \ell} I_j = \bigcup_{1 \le j \le \ell} \hat{I}_j \subset \{1, 2, \ldots, n\}$. It suffices to show the following claim

$$p \in C_\chi(Z) \iff \text{For any } p \in U \text{ and } G \in \mathrm{Cone}_\chi(p),\ \psi(Z) \cap G \cap U \neq \emptyset.$$

First we will show ($\Rightarrow$): Assume that

$$p = (q; \{\xi_i\}_{i \in B}) = \underset{X, 1 \le j \le \ell}{\times} p_j \in C_\chi(Z)$$

where p_j is a point in $T_{M_j}\iota(M_j)$. By the definition, we have a sequence

$$p^{(m)} = (x_1^{(m)}, \ldots, x_n^{(m)}; t_1^{(m)}, \ldots, t_\ell^{(m)}) \in \widetilde{p}^{-1}(Z) \subset \widetilde{X} \qquad m = 1, 2, \ldots$$

satisfying that $x_i^{(m)} \to \xi_i$ for $i \in B$, $x_i^{(m)} \to 0$ for $j \notin B$, and $t_j^{(m)} \to 0$ for any j. For $j \in \{1, 2, \ldots, \ell\}$, let us consider the commutative diagram

$$\begin{array}{ccc} \underset{X, 1 \le \alpha \le \ell}{\times} T_{M_\alpha}\iota(M_\alpha) & \to & T_{M_j}X \\ \downarrow & & \downarrow \\ \varphi : \widetilde{X} & \to & \widetilde{X}_{M_j} \\ & \searrow & \downarrow \\ & & X \end{array}$$

where the top horizontal arrow is given by the composition of morphisms

$$\underset{X,1\leq\alpha\leq\ell}{\times} T_{M_\alpha}\iota(M_\alpha) \to T_{M_j}\iota(M_j) \underset{X}{\times} M \hookrightarrow T_{M_j}X$$

and $\varphi : (x_1,\ldots,x_n;\, t_1,\ldots,t_\ell) \to (\hat{x}_1,\ldots,\hat{x}_n;\, \hat{t})$ is defined by

$$\begin{aligned}
\hat{t} &= \prod_{M_\beta\subset M_j} t_\beta, \\
\hat{x}_i &= t_{J_i} x_i && (i\notin I_j), \\
\hat{x}_i &= \left(\prod_{\{\beta\in J_i; M_j\subsetneq M_\beta\}} t_\beta\right) x_i && (i\in I_j).
\end{aligned}$$

Here the definitions of J_i, etc. were given in the proof of Proposition 1.2. Then $\varphi(p^{(m)})$ converges to $\overline{p}_j \in T_{M_j}X$ where $\overline{p}_j$ is the image of p_j by the canonical embedding $T_{M_j}\iota(M_j) \hookrightarrow T_{M_j}X$. This implies $\overline{p}_j \in C_{M_j}(Z)$. Therefore it follows from the definition of a usual normal cone that, for any cone $G_j \in \mathrm{Cone}_{\chi,j}(p)$, we have $\widetilde{p}_{M_j}(\varphi(p^{(m)})) \in \pi_{j,q}^{-1}(G_j)$ for any sufficiently large m. This entails that, for any multi-cone $G \in \mathrm{Cone}_\chi(p)$, we have $\widetilde{p}(p^{(m)}) \in G$ for any sufficiently large m, in particular, we have $Z\cap G\cap U \neq \emptyset$ for any U.

Let us show converse ($\Leftarrow$): Set, for a subset $I \subset \{1,2,\ldots,n\}$,

$$|x|_I := \sum_{i\in I} |x_i|.$$

Note that, if I is empty, we set $|x|_I = 1$. Let

$$p = (q;\, \xi) = \underset{X,1\leq j\leq\ell}{\times} p_j \in \underset{X,1\leq j\leq\ell}{\times} T_{M_j}\iota(M_j)$$

where $p_j = (q;\, \xi_j) \in T_{M_j}\iota(M_j)$. By the conic actions $\tau_j(\cdot)$, we may assume either $|\xi_j| = |\xi|_{\hat{I}_j} = 1$ or $|\xi_j| = |\xi|_{\hat{I}_j} = 0$ for $1 \leq j \leq \ell$.

Let $\{G_j^{(m)}\}_{m=1,2,\ldots}$ be a cofinal set of $\mathrm{Cone}_{\chi,j}(p)$ and $\{U^{(m)}\}_{m=1,2,\ldots}$ a set of fundamental neighborhoods of q. We set

$$G^{(m)} := \bigcap_{1\leq j\leq\ell} \pi_{j,q}^{-1}(G_j^{(m)}) \qquad m = 1,2,\ldots.$$

Choose points in X as

$$q^{(m)} \in Z \cap G^{(m)} \cap U^{(m)} \subset X \qquad m = 1,2,\ldots,$$

and define a sequence in $\widetilde{X}$ by $p^{(m)} := (q^{(m)}; 1, \dots, 1)$. Clearly we have $\widetilde{p}(p^{(m)}) \in Z \cap G^{(m)} \cap U^{(m)}$. For each $1 \leq j \leq \ell$, by taking a subsequence of $\{q^{(m)}\}$, we may assume either $|q^{(m)}|_{\hat{I}_j} \neq 0$ for every m or $|q^{(m)}|_{\hat{I}_j} = 0$ for all m. We divide the set $\{1, 2, \dots, \ell\}$ into two sets J' and J'' as follows:

$$J' = \left\{ j \in \{1, 2, \dots, l\};\, |\xi_j| = |\xi|_{\hat{I}_j} \neq 0 \right\},$$
$$J'' = \{1, 2, \dots, \ell\} \setminus J'.$$

Note that $|q^{(m)}|_{\hat{I}_j} \neq 0$ $(m = 1, 2, \dots)$ holds for $j \in J'$. Let us determine a sequence $\kappa^{(m)} = (\kappa_1^{(m)}, \dots, \kappa_\ell^{(m)})$ of positive real numbers that satisfies the following conditions.

1. $\kappa^{(m)} \to 0$, $\kappa_j^{(m)} = |q^{(m)}|_{\hat{I}_j}$ for $j \in J'$ and $\dfrac{|q^{(m)}|_{\hat{I}_j}}{\kappa_j^{(m)}} \to 0$ for $j \in J''$.

2. For any pair $\alpha, \beta \in \{1, 2, \dots, \ell\}$ with $M_\alpha \subsetneq M_\beta$, we have $\dfrac{\kappa_\beta^{(m)}}{\kappa_\alpha^{(m)}} \to 0$.

Set, for $j \in \{1, 2, \dots, \ell\}$,

$$d_j := \max\{k; M_j = M_{j_0} \subsetneq M_{j_1} \subsetneq \cdots \subsetneq M_{j_k},\, M_{j_1}, \dots, M_{j_k} \in \chi\}$$

and

$$J'_k := J' \cup \{j \in J''; d_j \leq k\}.$$

Now we construct a sequence $\kappa^{(m)}$ by induction with respect to k. For this purpose, we introduce the following condition Hypo(j, k).

1. $\kappa_j^{(m)} \to 0$.

2. For $\beta' \in J'_k$ with with $M_{\beta'} \subsetneq M_j$, we have $\dfrac{\kappa_j^{(m)}}{\kappa_{\beta'}^{(m)}} \to 0$

3. For $\beta \in \{1, 2, \dots, \ell\}$ with $M_j \subsetneq M_\beta$, we have $\dfrac{|q^{(m)}|_{\hat{I}_\beta}}{\kappa_j^{(m)}} \to 0$. Further we also have $\dfrac{|q^{(m)}|_{\hat{I}_j}}{\kappa_j^{(m)}} \to 0$ if $j \in J''$.

Assume $k=-1$, i.e., $J'_{-1}=J'$. We set $\kappa_j^{(m)}=|q^{(m)}|_{\hat{I}_j}$ for $j\in J'$. It is easy to see that the hypothesis Hypo(j,k) is satisfied for any $j\in J'_{-1}$. Indeed, as $\xi_j\neq 0$ and $q^{(m)}\in\pi_{j,q}^{-1}(G_j^{(m)})$ hold, there exists a sequence $\epsilon^{(m)}\to 0$ such that $|q^{(m)}|_{\hat{I}_\beta}\leq\epsilon^{(m)}|q^{(m)}|_{\hat{I}_j}$ for β with $M_j\subsetneq M_\beta$. This implies 3. of Hypo(j,k). By the same reason, as $\beta'\in J'$, 2. of Hypo(j,k) also holds.

Assume that we have constructed $\kappa_j^{(m)}$ for $j\in J'_k$ and Hypo(j,k) holds for any $j\in J'_k$. Let $j\in J''$ with $d_j=k+1$. Then we can determine $\kappa_j^{(m)}$ so that the hypothesis Hypo(j,k) is satisfied in the following way. First note that $\chi^*=\{M_{\beta'}\in\chi;\,\beta'\in J'_k,\,M_{\beta'}\subsetneq M_j\}$ is a totally ordered set with respect to "$\subset$", in particular, we have the maximal submanifold $M_{j^*}\in\chi^*$. If we can determine $\kappa_j^{(m)}$ satisfying $\dfrac{\kappa_j^{(m)}}{\kappa_{j^*}^{(m)}}\to 0$, then $\dfrac{\kappa_j^{(m)}}{\kappa_{\beta'}^{(m)}}\to 0$ also holds for any $M_{\beta'}\in\chi^*$ by induction hypothesis. It follows from induction hypothesis again that for any $\beta\in\{1,2,\dots,\ell\}$ with $M_{j^*}\subsetneq M_\beta$, we have $\dfrac{|q^{(m)}|_{\hat{I}_\beta}}{\kappa_{j^*}^{(m)}}\to 0$. Therefore we can find $\kappa_j^{(m)}$ such that $\dfrac{\kappa_j^{(m)}}{\kappa_{j^*}^{(m)}}\to 0$ and $\dfrac{|q^{(m)}|_{\hat{I}_\beta}}{\kappa_j^{(m)}}\to 0$ for $\beta\in\{1,2,\dots,\ell\}$ with $M_j\subset M_\beta$ (note that this contains the case $\beta=j$).

By repeating the same procedure, we obtain $\kappa_j^{(m)}$ for any $j\in J''$ with $d_j=k+1$. As $I_j\cap I_{j'}=\emptyset$ for j and j' with $d_j=d_{j'}$, we can conclude that Hypo$(j,k+1)$ holds for any $j\in J'_{k+1}$. Hence we have obtained $\kappa_j^{(m)}$ for all $j\in\{1,2,\dots,\ell\}$.

Let us define points in $\widetilde{X}$ by

$$\overline{p}^{(m)}:=\left(\prod_{j\in\{1,2,\dots,\ell\}}\tau_j\left(\frac{1}{\kappa_j^{(m)}}\right)\right)p^{(m)}.$$

Note that $\widetilde{p}(\overline{p}^{(m)})\in Z$ still holds. Then the value of j-th coordinate t_j of $\overline{p}^{(m)}$ is given by that of

$$\prod_{j\in\{1,2\dots,\ell\}}\mu_j\left(\frac{1}{\kappa_j^{(m)}}\right)\prod_{\beta\in J_{\supsetneq M_j}}\mu_\beta\left(\kappa_j^{(m)}\right)p^{(m)},$$

which is equal to that of

$$\mu_j\left(\frac{1}{\kappa_j^{(m)}}\right)\prod_{j\in J_{\supsetneq M_\beta}}\mu_j\left(\kappa_\beta^{(m)}\right)p^{(m)}=\mu_j\left(\frac{1}{\kappa_j^{(m)}}\right)\prod_{\beta\in J_{\subsetneq M_j}}\mu_j\left(\kappa_\beta^{(m)}\right)p^{(m)}.$$

Note that $J_{\subsetneq M_j}$ consists of at most 1 element. If $J_{\subsetneq M_j}$ is empty, then the j-th component is $\kappa_j^{(m)}$ which clearly tends to 0 when $m \to \infty$. If $J_{\subsetneq M_j} = \{\beta\}$ for some β, then it is given by $\dfrac{\kappa_j^{(m)}}{\kappa_\beta^{(m)}}$ which also tends to 0 by the construction of $\kappa^{(m)}$. As a result, we have $\overline{p}^{(m)} \to p$. This completes the proof. □

Given the family $\chi = \{M_1, \ldots, M_\ell\}$ and a sub-family $\chi_k := \{M_{j_1}, \ldots, M_{j_k}\}$ of χ, we have the natural maps

$$\underset{X,1\le j\le \ell}{\times} T_{M_j}\iota_\chi(M_j) \hookleftarrow \left(\underset{X,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i})\right) \underset{X}{\times} M \hookrightarrow \underset{X,1\le i\le k}{\times} T_{M_{j_i}}\iota_{\chi_k}(M_{j_i}),$$

where $M = \bigcap_{j=1}^{\ell} M_j$. We set for short

$$\underset{M,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i}) := \left(\underset{X,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i})\right) \underset{X}{\times} M.$$

Corollary 4.3 *Let $k \le \ell$ and $\{j_1, \ldots, j_k\}$ be a subset of $\{1, 2, \ldots, \ell\}$. Set $\chi_k = \{M_{j_1}, \ldots, M_{j_k}\}$. Let Z be a subset of X. Then we have*

$$C_\chi(Z) \cap \underset{M,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i}) = C_{\chi_k}(Z) \cap \underset{M,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i}).$$

Proof. Let us prove $\subseteq$. Suppose that $p \in \underset{M,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i})$ does not belong to $C_{\chi_k}(Z)$. By Lemma 4.2 there exists an open subset $\psi(q) \in U \subset (TM)_q$ and a multi-cone $G' \in \mathrm{Cone}_{\chi_k}(p)$ such that $\psi(Z) \cap G' \cap U = \emptyset$ holds. For $\alpha \notin \{j_1, \ldots, j_k\}$, set $G_\alpha = (T_{M_\alpha}X)_q$. Hence $G := \left(\bigcap_{\alpha\notin\{j_1,\ldots,j_k\}} \pi_\alpha^{-1}(G_\alpha)\right) \cap G' = G' \in \mathrm{Cone}_\chi(p)$ and $\psi(Z) \cap G \cap U = \emptyset$. By Lemma 4.2 we obtain $p \notin C_\chi(Z)$.

Let us prove $\supseteq$. Suppose that $p \in \underset{M,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i})$ does not belong to $C_\chi(Z)$. By Lemma 4.2 there exists an open subset $\psi(q) \in U \subset (TM)_q$ and a multi-cone $G \in \mathrm{Cone}_\chi(p)$ such that $\psi(Z) \cap G \cap U = \emptyset$ holds. For $\alpha \notin \{1, \ldots, k\}$, $p_{j_\alpha} = (q; 0)$ and we have $\mathrm{Cone}_{\chi,j_\alpha}(p) = \{(T_{M_{j_\alpha}}X)_q\}$. Hence $G' := G$ can be regarded as an element of $\mathrm{Cone}_{\chi_k}(p)$ and $\psi(Z) \cap G' \cap U = \emptyset$. By Lemma 4.2 we obtain $p \notin C_{\chi_k}(Z)$. □

Definition 4.4 *Denote by* $\mathrm{Op}(X)$ *the category of open subsets of* X*, and let* Z *be a subset of* X*.*

(i) We set $\mathbb{R}^+_j Z = \mu_j(Z, \mathbb{R}^+)$*. If* $U \in \mathrm{Op}(X)$*, then* $\mathbb{R}^+_j U \in \mathrm{Op}(X)$ *since* μ_j *is open for each* $j = 1, \dots, \ell$*.*

(ii) Let $J = \{j_1, \dots, j_k\} \subset \{1, \dots, \ell\}$*. We set*

$$\mathbb{R}^+_J Z = \mathbb{R}^+_{j_1} \cdots \mathbb{R}^+_{j_k} Z = \mu_{j_1}(\cdots \mu_{j_k}(Z, \mathbb{R}^+), \dots, \mathbb{R}^+).$$

We set $(\mathbb{R}^+)^\ell Z = \mathbb{R}^+_{\{1,\dots,\ell\}} Z = \mu(Z, (\mathbb{R}^+)^\ell)$*. If* $U \in \mathrm{Op}(X)$*, then* $\mathbb{R}^+_J U \in \mathrm{Op}(X)$ *since* μ_j *is open for each* $j \in \{1, \dots, \ell\}$*.*

(iii) We say that Z *is* $(\mathbb{R}^+)^\ell$*-conic (*ℓ*-conic for short) if* $Z = (\mathbb{R}^+)^\ell Z$*. In other words,* Z *is invariant by the action of* μ_j*,* $j = 1, \dots, \ell$*.*

Definition 4.5 *(i) We say that a subset* Z *of* X *is* $\mathbb{R}^+_j$*-connected if* $Z \cap \mathbb{R}^+_j x$ *is connected for each* $x \in Z$*.*

(ii) We say that a subset Z *of* X *is* $(\mathbb{R}^+)^\ell$*-connected if there exists a permutation* $\sigma : \{1, \dots, \ell\} \to \{1, \dots, \ell\}$ *such that*

$$\begin{cases} Z \text{ is } \mathbb{R}^+_{\sigma(1)}\text{-connected,} \\ \mathbb{R}^+_{\sigma(1)} Z \text{ is } \mathbb{R}^+_{\sigma(2)}\text{-connected,} \\ \quad \vdots \\ \mathbb{R}^+_{\sigma(1)} \cdots \mathbb{R}^+_{\sigma(\ell-1)} Z \text{ is } \mathbb{R}^+_{\sigma(\ell)}\text{-connected.} \end{cases}$$

The proof of the following is almost the same of that of Proposition 4.1.3 of [10], and we shall not repeat it.

Proposition 4.6 *Let* V *be a* $(\mathbb{R}^+)^\ell$*-conic open subset of the zero section* S*.*

(i) Let W *be an open neighborhood of* V *in* $\widetilde{X}$*, and let* $U = \widetilde{p}(W \cap \Omega)$*. Then* $V \cap C_\chi(X \setminus U) = \emptyset$*.*

(ii) Let U *be an open subset of* X *such that* $V \cap C_\chi(X \setminus U) = \emptyset$*. Then* $\widetilde{p}^{-1}(U) \cup V$ *is an open neighborhood of* V *in* $\overline{\Omega}$*.*

Proposition 4.7 *Let* V *be a* $(\mathbb{R}^+)^\ell$*-conic subanalytic open subset of* S*. Then any subanalytic neighborhood* W *of* V *in* $\widetilde{X}$ *contains* $\widetilde{W}$ *open and subanalytic in* $\widetilde{X}_{sa}$ *such that:*

$$(4.1) \quad \begin{cases} \text{(i) } \widetilde{W} \cap \Omega \text{ is } (\mathbb{R}^+)^\ell\text{-connected,} \\ \text{(ii) } \mathbb{R}^+_1 \cdots \mathbb{R}^+_\ell(\widetilde{W} \cap \Omega) = \widetilde{p}^{-1}(\widetilde{p}(W \cap \Omega)) \text{ is subanalytic in } \widetilde{X}. \end{cases}$$

Proof. Let $\chi_i = \{M_1, \dots, M_{i-1}, M_{i+1}, \dots, M_\ell\}$ and let $\widetilde{X}_i$ be the normal deformation of X with respect to χ_i. Define $p_i : \widetilde{X} \to \widetilde{X}_i$ and $\widetilde{p}_i : \widetilde{X} \cap \{t_i > 0\} \to \widetilde{X}_i$ as in (1.2). Let V be a conic subanalytic open subset of S.

(i) We first prove that any subanalytic neighborhood W of V in $\widetilde{X}$ contains $\widetilde{W} \in \mathrm{Op}_{sa}(\widetilde{X})$ such that:

$$\begin{cases} \text{(i) the fibers of } \widetilde{p}_i : \widetilde{W} \cap \{t_i > 0\} \to \widetilde{X}_i \text{ are connected,} \\ \text{(ii) } \widetilde{p}_i(\widetilde{W} \cap \{t_i > 0\}) \text{ is subanalytic in } \widetilde{X}_i. \end{cases} \tag{4.2}$$

Let W be an open subanalytic neighborhood of V in $\widetilde{X}_{sa}$. Up to shrink W we may suppose $V = W \cap \{t_i = 0\}$. Set $X' = \widetilde{X} \setminus (M_i \times \mathbb{R})$, $Z = X'/\mathbb{R}^+$. Then $\alpha : X' \to Z$ is an $\mathbb{R}^+$-bundle and $\varphi : Z \to \widetilde{X}_i$ is proper. Consider a continuous subanalytic section of $\{t_i = 0\} \to \{t_i = 0\}/\mathbb{R}^+$ (the i-th component of the action), extend it to a continuous subanalytic section σ of $X' \to Z$ and set

$$W' = \bigcup_{x \in \alpha(W)} W'_x \cap \{t_i > 0\},$$

where W'_x denotes the connected component of $\alpha^{-1}(x) \cap W$ containing $\sigma(x)$. By construction, the fibers of $W' \to Z$ are connected and W' is an open neighborhood of $V \setminus M_i$. Let

$$W'' = \bigcup_{x \in M_i \cap W} W''_x \cap \{t_i > 0\},$$

where W''_x denotes the connected component of $(\{x\} \times \mathbb{R}) \cap W$ intersecting M. Up to shrink W'', $\widetilde{W} = W' \cup W'' \cup (W \setminus \{t_i > 0\})$ is an open neighborhood of V and satisfies (4.2)(i).

Let us see that $\widetilde{W}$ is subanalytic and satisfies (4.2) (ii). We may reduce to the case $X = \mathbb{R}^n$, $M_i = \{0\} \times \mathbb{R}^{n-m} \subset X$, $\widetilde{X} = \mathbb{R}^{n+1}$, $\{t_i = 0\} = \mathbb{R}^n \times \{0\} \subset \widetilde{X}$, $\{t_i > 0\} = \mathbb{R}^n \times \mathbb{R}^+ \subset \widetilde{X}$. So that $(x', x'', t) \cdot c \mapsto (cx', x'', c^{-1}t)$ is the action of $\mathbb{R}^+$ on $\widetilde{X}$ and $\widetilde{p} : \{t_i > 0\} \to X$ is the map $(x', x'', t) \mapsto (tx', x'')$. In this situation $X' = \mathbb{R}^{n+1} \setminus (M_i \times \mathbb{R}) \simeq \mathbb{S}^{m-1} \times \mathbb{R}^+ \times \mathbb{R}^{n-m} \times \mathbb{R}$. Moreover $X'/\mathbb{R}^+ \simeq \mathbb{S}^{m-1} \times \mathbb{R}^{n-m} \times \mathbb{R}$, indeed

$$X' = S \times \mathbb{R}^+ = \{(ci(\vartheta), x'', sc^{-1}),\ (\vartheta, x'', s) \in S,\ c \in \mathbb{R}^+\},$$

where $i : \mathbb{S}^{m-1} \hookrightarrow \mathbb{R}^m$ denotes the embedding. The section $\sigma : Z \xrightarrow{\sim} Z \times \{1\} \subset X'$ is a globally subanalytic subset of X'. Let us consider the globally

subanalytic (even semialgebraic) homeomorphism $\psi : \{t_i > 0\} \to \{t_i > 0\}$ defined by $\psi(x', x'', t) = (tx', x'', t^{-1})$. Then $\pi \circ \psi = \widetilde{p}_i$, where $\pi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is the projection. The set $\psi(W \cap \{t_i > 0\})$ is still subanalytic. Let $p \in \mathbb{R}^n$. Then $\pi^{-1}(p) \cap \psi(W \cap X' \cap \{t_i > 0\})$ is a disjoint union of intervals. Let us consider the interval $(m(p), M(p))$, $m(p) < M(p) \in \mathbb{R} \cup \{\pm\infty\}$ intersecting $\psi(\sigma \cap \Omega)$. Then $\psi(W') = \{(p, r) \in \psi(W \cap X' \cap \Omega);\ m(p) < r < M(p)\}$. The set $\psi(W')$ is open subanalytic (it is a consequence of Proposition 1.2, Chapter 6 of [23]). Moreover, up to shrink W we may suppose that $\widetilde{p}_i(W') = \widetilde{p}_i(\sigma \cap W \cap \{t_i > 0\})$ which is subanalytic. Indeed, since we are working in a local chart, we may assume that $W \cap \sigma$ is globally subanalytic. Then $\mathbb{R}^+(W') = p_i^{-1}(\widetilde{p}_i(W'))$ is subanalytic. Let $x'' \in \mathbb{R}^{n-m}$. Then $(\{0\} \times \{x''\} \times \mathbb{R}) \cap W''$ is a disjoint union of (bounded, up to shrink W) intervals. Let us consider the interval $(m(x''), M(x''))$, $m(x'') < M(x'') \in \mathbb{R}$ containing 0. Then $W'' = \{(0, r) \in W \cap (M_i \times \mathbb{R});\ m(x'') < r < M(x'')\}$. The set W'' is subanalytic (it is a consequence of Proposition 1.2, Chapter 6 of [23]). Moreover (up to shrink W'') $\widetilde{p}_i(W'' \cap \Omega) = W'' \cap (M_i \times \{0\})$, which is subanalytic.

(ii) Let us find $\widetilde{W}$ satisfying (4.1). Let us argue by induction on ℓ. The case $\ell = 1$ follows from (i). Let us treat the general case. Set $S_\ell = \{t_\ell = 0\}$, $\Omega_\ell = \{t_1, \dots, t_{\ell-1} > 0\}$. By the induction hypothesis we can find a subanalytic neighborhood W' of V which is $(\mathbb{R}^+)^{\ell-1}$-connected and such that $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{\ell-1}(W' \cap \Omega_\ell)$ is subanalytic. Let us apply (i) with $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{\ell-1}(W' \cap S_\ell \cap \Omega_\ell)$ instead of V and $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{\ell-1}(W' \cap \Omega_\ell)$ instead of W. Then we find W'' subanalytic $(\mathbb{R}^+)^{\ell-1}$-conic and $\mathbb{R}^+_\ell$-connected satisfying (4.2). The set $\widetilde{W} = (W' \cap W'') \cup (W \setminus \Omega)$ is an open neighborhood of V and satisfies (4.1). □

5 Morphisms between multi-normal deformations

Let X and Y be real analytic manifolds of dimension n and m respectively and let $\chi^M = \{M_j\}_{j=1}^\ell$, $\chi^N = \{N_j\}_{j=1}^\ell$ be families of smooth closed submanifolds of X and Y respectively which satisfy H1, H2 and H3. Let $f : X \to Y$ be a morphism of real analytic manifolds such that $f(M_j) \subseteq N_j$, $j = 1, \dots, \ell$.

We want to extend f to a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$. This is done by repeatedly employing the the usual construction of a morphism between normal deformations, i.e. we extend f to $\widetilde{f}_1 : \widetilde{X}_{M_1} \to \widetilde{Y}_{N_1}$, then we extend $\widetilde{f}_1$ to $\widetilde{f}_{1,2} : \widetilde{X}_{M_1,M_2} \to \widetilde{Y}_{N_1,N_2}$. Then we can define recursively $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ by

extending the morphism $\widetilde{f}_{1,\dots,\ell-1} : \widetilde{X}_{M_1,\dots,M_{\ell-1}} \to \widetilde{Y}_{N_1,\dots,N_{\ell-1}}$ to the normal deformations with respect to M_ℓ and N_ℓ respectively.

In a local coordinate system set

$$f(x_1,\dots,x_n) = (f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n)).$$

We define the I_j^M as in 2. of Proposition 1.2 and J_i^M as in (1.5) (and similarly for I_j^N and J_i^N). Then, outside of $t_1t_2\dots t_\ell = 0$, we can define a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$

$$(x_1,\dots,x_n;\, t_1,\dots,t_\ell) \to (y_1,\dots,y_m;\, \hat{t}_1,\dots,\hat{t}_\ell)$$

by setting

$$\begin{cases} \hat{t}_{J_k^N} y_k = f_k(t_{J_1^M} x_1,\dots,t_{J_n^M} x_n) \qquad (k = 1,2,\dots,m), \\ \hat{t}_j = t_j \qquad (j = 1,2,\dots,\ell). \end{cases}$$

In order to understand the restriction of $\widetilde{f}$ to $t_1t_2\cdots t_\ell = 0$, we follow the notations of the proof of Proposition 1.3. For any $k = 1,2,\dots,m$, we set

$$I^N(k) := \bigcap_{k\in I_j^N} I_j^N = \bigcap_{j\in J_k^N} I_j^N.$$

Let $k \in \bigcup_{1\le j\le \ell} I_j^N$, and let $j(k) \in \{1,2,\dots,\ell\}$ such that $I^N(k) = I_{j(k)}^N$. Then, for any $j \in J_k^N$, we get $f(M_j) \subseteq N_{j(k)}$ because of $f(M_j) \subset N_j \subset N_{j(k)}$. We set

$$I = \bigcup_{j\in J_k^N} I_j^M,\ M_I = \bigcap_{j\in J_k^N} M_j.$$

By expanding $f_k(x_1,\dots,x_n)$ along the submanifold M_I, we obtain

$$f_k(x_1,\dots,x_n) = \sum_{i\in I} \frac{\partial f_k}{\partial x_i}\Big|_{M_I} x_i + \frac{1}{2}\sum_{i_1,i_2\in I} \frac{\partial^2 f_k}{\partial x_{i_1}\partial x_{i_2}}\Big|_{M_I} x_{i_1}x_{i_2} + \dots,$$

as $f_k|_{M_I} = 0$ holds due to $f(M_I) \subset N_{j(k)}$. Then we get, on $t_1\dots t_\ell \neq 0$,

$$(5.1)\quad y_k = \sum_{i\in I} \frac{\partial f_k}{\partial x_i}\Big|_{M_I} \frac{t_{J_i^M}}{t_{J_k^N}} x_i + \frac{1}{2}\sum_{i_1,i_2\in I} \frac{\partial^2 f_k}{\partial x_{i_1}\partial x_{i_2}}\Big|_{M_I} \frac{t_{J_{i_1}^M} t_{J_{i_2}^M}}{t_{J_k^N}} x_{i_1}x_{i_2} + \dots.$$

Let $i_1,\dots,i_p$ be a sequence $(i_1,\dots,i_p \in I)$.

(i) If there exists $j \in J_k^N$ such that $\{i_1, \ldots, i_p\} \cap I_j^M = \emptyset$, as $f_k|_{M_j} = 0$ and derivatives $\dfrac{\partial}{\partial x_{i_1}}$, …, $\dfrac{\partial}{\partial x_{i_p}}$ are tangent to M_j, we have

$$\left.\frac{\partial^p f_k}{\partial x_{i_1} \ldots \partial x_{i_p}}\right|_{M_I} = 0.$$

(ii) Suppose that $\{i_1, \ldots, i_p\} \bigcap I_j^M \neq \emptyset$ holds for each $j \in J_k^N$. This implies that I_j^M contains some i_q in $\{i_1, \ldots, i_p\}$, which is equivalent to saying that $j \in J_{i_q}^M$. Therefore any $j \in J_k^N$ belongs to $J_{i_q}^M$ with some $q \in \{1, \ldots, p\}$, from which we obtain

$$J_k^N \subset J_{i_1}^M \cup \cdots \cup J_{i_p}^M.$$

Now we have two cases.

(a) If some pair of $J_{i_1}^M$, …, $J_{i_p}^M$ is not disjoint or if $J_k^N \subsetneqq J_{i_1}^M \sqcup \cdots \sqcup J_{i_p}^M$, then $t_{J_{i_1}^M} \cdots t_{J_{i_p}^M} t_{J_k^N}^{-1} \to 0$ when $t \to 0$.

(b) If $J_k^N = J_{i_1}^M \sqcup \cdots \sqcup J_{i_p}^M$ holds, then the term with its indices $i_1, \ldots, i_p$ in (5.1) becomes, by letting t to 0,

$$\frac{1}{p!}\left.\frac{\partial^p f_k}{\partial x_{i_1} \cdots \partial x_{i_p}}\right|_{M_I} x_{i_1} \cdots x_{i_p}.$$

From these observations, the morphism $\widetilde{f}$ is described by, on $t_1 = \cdots = t_\ell = 0$,

$$y_k = \sum_{J_{i_1}^M \sqcup \cdots \sqcup J_{i_p}^M = J_k^N} \frac{1}{p!}\left.\frac{\partial^p f_k}{\partial x_{i_1} \cdots \partial x_{i_p}}\right|_M x_{i_1} \cdots x_{i_p} \qquad (k \in \bigcup_{1 \leq j \leq \ell} I_j^N).$$

Here $M := M_1 \cap \cdots \cap M_\ell$. Note that if there is no $\{i_1, \ldots, i_p\}$ with $J_k^N = J_{i_1}^M \sqcup \cdots \sqcup J_{i_p}^M$, then we set $y_k := 0$.

Let us study the condition $J_k^N = J_{i_1}^M \sqcup \cdots \sqcup J_{i_p}^M$. For this purpose, we introduce two definitions. Let $k \in \bigcup_{1 \leq j \leq \ell} I_j^N$ and set

$$\sup{}_{\subset} J_k^N := \left\{ j \in J_k^N;\ M_j \text{ is a maximal submanifold in } \{M_\beta\}_{\beta \in J_k^N} \right\}$$

and

$$\inf{}_{\subset} J_k^N := \left\{ j \in J_k^N;\ M_j \text{ is a minimal submanifold in } \{M_\beta\}_{\beta \in J_k^N} \right\}.$$

Note that $\# \inf_{\subset} J_k^N \leq \# \sup_{\subset} J_k^N$ holds.

Example 5.1 *Let us consider closed submanifolds* $\{M_1, M_2, M_3\}$ *in* $X = \mathbb{R}^n$ *and* $\{N_1, N_2, N_3\}$ *in* $Y = \mathbb{R}^m$. *Let* $f : X \to Y$ *be a morphism satisfying* $f(M_j) \subset N_j$ $(j = 1, 2, 3)$. *We assume that* $N_3 \subset N_2 \subset N_1$, $M_3 \subset M_1$, $M_3 \subset M_2$, M_1 *and* M_2 *intersect transversely. For* $k \in I_1^N$, $J_k^N = \{1, 2, 3\}$, $\sup_{\subset} J_k^N = \{1, 2\}$ *and* $\inf_{\subset} J_k^N = \{3\}$. *For* $k \in \hat{I}_2^N = I_2^N \setminus I_1^N$, $J_k^N = \{2, 3\}$, $\sup_{\subset} J_k^N = \{2\}$ *and* $\inf_{\subset} J_k^N = \{3\}$. *For* $k \in \hat{I}_3^N = I_3^N \setminus (I_1^N \cup I_2^N)$, $J_k^N = \{3\}$, $\sup_{\subset} J_k^N = \inf_{\subset} J_k^N = \{3\}$.

Lemma 5.2 *Let* $k \in \bigcup_{1 \le j \le \ell} I_j^N$ *and* $\{i_1, \ldots, i_p\}$ *a subset of* $\bigcup_{1 \le j \le \ell} I_j^M$. *Then* $J_k^N = J_{i_1}^M \sqcup \cdots \sqcup J_{i_p}^M$ *holds if and only if the conditions (a) and (b) below are satisfied.*

(a) k *satisfies the following condition* $(\dagger)_k$

$$\#\inf_{\subset} J_k^N = \#\sup_{\subset} J_k^N, \tag{5.2}$$

$$M_\beta \subset M_j \ (j \in J_k^N,\ \beta \in \{1, 2, \ldots, \ell\}) \Longrightarrow \beta \in J_k^N. \tag{5.3}$$

(b) $p = \#\sup_{\subset} J_k^N$ *and the indices* $i_1, \ldots, i_p$ *satisfy*

$$i_\alpha \in \hat{I}_{\sigma(\alpha)}^M = I_{\sigma(\alpha)}^M \setminus \left(\bigcup_{I_j^M \subsetneq I_{\sigma(\alpha)}^M} I_j^M \right) \qquad (\alpha \in \{1, 2, \ldots, p\}) \tag{5.4}$$

for some bijection $\sigma : \{1, 2 \ldots, p\} \to \sup_{\subset} J_k^N$.

Proof. We first show the claim that if $\#\inf_{\subset} J_k^N < \#\sup_{\subset} J_k^N$, then there exists no $\{i_1, \ldots, i_p\}$ with $J_k^N = J_{i_1}^M \sqcup \cdots \sqcup J_{i_p}^M$. Assume that there exists $\{i_1, \ldots, i_p\}$ with $J_k^N = J_{i_1}^M \sqcup \cdots \sqcup J_{i_p}^M$. Then it follows from $\#\inf_{\subset} J_k^N < \#\sup_{\subset} J_k^N$ that we can find indices j, j', j'' in J_k^N satisfying $I_{j'}^M \subset I_j^M$, $I_{j''}^M \subset I_j^M$ and $I_{j'}^M \cap I_{j''}^M = \emptyset$. By the assumption, there exist α' and α'' in $\{1, 2, \ldots, p\}$ such that $j' \in J_{i_{\alpha'}}^M$ and $j'' \in J_{i_{\alpha''}}^M$. As $I_{j'}^M \cap I_{j''}^M = \emptyset$, we have $\alpha' \neq \alpha''$. On the other hand, $I_{j'}^M \subset I_j^M$ and $I_{j''}^M \subset I_j^M$ implies $j \in J_{i_{\alpha'}}^M$ and $j \in J_{i_{\alpha''}}^M$, which contradicts that $J_{i_{\alpha'}}^M$ and $J_{i_{\alpha''}}^M$ are disjoint. Hence we have obtained the claim.

In what follows, we assume

$$\#\inf_{\subset} J_k^N = \#\sup_{\subset} J_k^N.$$

Suppose that $\{i_1, \ldots, i_p\}$ satisfies the condition $J_k^N = J_{i_1}^M \sqcup \cdots \sqcup J_{i_p}^M$.

Let $j \in J_k^N$ and $\beta \in \{1, 2, \ldots, \ell\}$ with $M_\beta \subset M_j$. Then we can find i_α such that $j \in J_{i_\alpha}^M$. $M_\beta \subset M_j$ implies $\beta \in J_{i_\alpha}^M \subset J_k^N$. Therefore we have (5.3).

Let $j \in \sup_{\subset} J_k^N$. Then some $J_{i_q}^M$ ($q \in \{1, 2, \ldots, p\}$) contains j, which implies $i_q \in I_j^M$. Further we can show that i_q belongs to $\hat{I}_j^M$. If $i_q \in I_j^M \setminus \hat{I}_j^M$, then there exists j' with $i_q \in I_{j'}^M$ and $M_j \subsetneqq M_{j'}$ by the definition of $\hat{I}_j^M$, from which we have $j' \in J_{i_q}^M \subset J_k^N$. This contradicts $j \in \sup_{\subset} J_k^N$. Hence $i_q \in \hat{I}_j^M$ and we have obtained that the set $\{i_1, \ldots, i_p\}$ contains at least one index that belongs to $\hat{I}_j^M$ for any $j \in \sup_{\subset} J_k^N$. Further two or more indices in $\hat{I}_j^M$ cannot belong to $\{i_1, \ldots, i_p\}$ at the same time because any pair of $J_{i_1}^M, \cdots, J_{i_p}^M$ is disjoint.

Now we show that $\{i_1, \ldots, i_p\}$ consists of only these indices. Let i be an element in $\{i_1, \ldots, i_p\}$. Choosing $j' \in J_i^M$, we can find $j \in \sup_{\subset} J_k^N$ with $M_{j'} \subset M_j$. Then, by the above argument, there exists $i_q \in \hat{I}_j^M$ which belongs to $\{i_1, \ldots, i_p\}$. As $i_q \in I_j^M \subset I_{j'}^M$, we have $j' \in J_{i_q}^M$, which implies $J_{i_q}^M \cap J_i^M \neq \emptyset$. Since each pair is disjoint, we have $i = i_q$.

Therefore we can find a bijection $\sigma : \{1, 2, \ldots, p\} \to \sup_{\subset} J_k^N$ such that $i_\alpha \in \hat{I}_{\sigma(\alpha)}^M$ and $J_{i_\alpha}^M \subset J_k^N (\alpha \in \{1, 2, \ldots, p\})$.

Conversely if such a σ exists, then $J_k^N = J_{i_1}^M \sqcup \cdots \sqcup J_{i_p}^M$ easily follows from $\# \inf_{\subset} J_k^N = \# \sup_{\subset} J_k^N$ and $J_{i_\alpha}^M \subset J_k^N$. The last inclusion can be obtained by the following argument. Let $\beta \in J_{i_\alpha}^M$. Then, as $i_\alpha \in \hat{I}_{\sigma(\alpha)}$ and $i_\alpha \in I_\beta$ hold, we have $I_{\sigma(\alpha)} \subset I_\beta$, from which we have $M_\beta \subset M_{\sigma(\alpha)}$ (the inclusion $\supset$ cannot hold because of the maximality of $M_{\sigma(\alpha)}$). Hence we obtain $\beta \in J_k^N$ by the condition $(\dagger)_k$. □

Example 5.3 *Let us consider closed submanifolds $\{M_1, M_2, M_3\}$ in $X = \mathbb{R}^3$ and $\{N_1, N_2, N_3\}$ in $Y = \mathbb{R}^m$. Let $f : X \to Y$ be a morphism satisfying $f(M_j) \subset N_j$ ($j = 1, 2, 3$). We consider these three cases:*

1. *$M_i = \{x_i = 0\}$, $i = 1, 2, 3$ (Majima),*
2. *$M_1 = \{0\}$, $M_2 = \{x_1 = x_2 = 0\}$, $M_3 = \{x_1 = 0\}$ (Takeuchi),*
3. *$M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$, $M_3 = \{0\}$ (Mixed).*

Let us see some examples of conditions (a), (b) of Lemma 5.2.

(i) *Suppose that* $J_k^N = \{1,2,3\}$.
In 1. $\sup_{\subset} J_k^N = \inf_{\subset} J_k^N = \{1,2,3\}$ *and condition* $(\dagger)_k$ *is clearly satisfied. Moreover* $i_1 = \sigma(1), i_2 = \sigma(2), i_3 = \sigma(3)$ *satisfy* (5.4) *for any permutation* σ *of* $\{1,2,3\}$.
In 2. $\sup_{\subset} J_k^N = \{3\}$, $\inf_{\subset} J_k^N = \{1\}$ *and condition* $(\dagger)_k$ *is clearly satisfied. Moreover* $i_1 = 1$ *satisfies* (5.4).
In 3. $\sup_{\subset} J_k^N = \{1,2\}$ *and* $\inf_{\subset} J_k^N = \{3\}$. *Then condition* (5.2) *is not satisfied.*

(ii) *Suppose that* $J_k^N = \{2,3\}$.
In 1. $\sup_{\subset} J_k^N = \inf_{\subset} J_k^N = \{2,3\}$ *and* $(\dagger)_k$ *is satisfied. It is easy to check that the indices* $i_1 = 2$, $i_2 = 3$ *(or* $i_1 = 3$, $i_2 = 2$*) satisfy* (5.4).
In 2. $\sup_{\subset} J_k^N = \{3\}$ *and* $\inf_{\subset} J_k^N = \{2\}$. *Then condition* (5.2) *is satisfied but* $M_2, M_3 \supset M_1$ *and* $1 \notin \sup_{\subset} J_k^N$, *hence* (5.3) *does not hold.*
In 3. $\sup_{\subset} J_k^N = \{2\}$, $\inf_{\subset} J_k^N = \{3\}$ *and* $(\dagger)_k$ *is satisfied. Moreover the index* $i_1 = 2$ *satisfies* (5.4).

(iii) *Suppose that* $J_k^N = \{1,2\}$.
In 1. $\sup_{\subset} J_k^N = \inf_{\subset} J_k^N = \{1,2\}$ *and* $(\dagger)_k$ *is satisfied. It is easy to check that the indices* $i_1 = 1$, $i_2 = 2$ *(or* $i_1 = 2$, $i_2 = 1$*) satisfy* (5.4).
In 2. $\sup_{\subset} J_k^N = \{2\}$ *and* $\inf_{\subset} J_k^N = \{1\}$. *In this case condition* (5.2) *is satisfied and* (5.3) *holds too. Moreover the index* $i_1 = 2$ *satisfies* (5.4).
In 3. $\sup_{\subset} J_k^N = \inf_{\subset} J_k^N = \{1,2\}$. *Then condition* (5.2) *is satisfied but* $M_1, M_2 \supset M_3$ *and* $3 \notin \sup_{\subset} J_k^N$, *hence* (5.3) *does not hold.*

As an immediate consequence of Lemma 5.2, we have the following.

Corollary 5.4 *For* $k \in \bigcup_{1 \le j \le \ell} I_j^N$, y_k *of the morphism* $\widetilde{f}$ *on* $t_1 = \cdots = t_\ell = 0$ *is given by*

$$(5.5)\quad y_k = \begin{cases} \displaystyle\sum_{i_1 \in \hat{I}_{j_1}^M, \ldots, i_p \in \hat{I}_{j_p}^M} \left. \frac{\partial^p f_k}{\partial x_{i_1} \cdots \partial x_{i_p}} \right|_M x_{i_1} \cdots x_{i_p} & ((\dagger)_k \text{ holds}), \\ 0 & (\text{otherwise}). \end{cases}$$

Here $p = \#\sup_{\subset} J_k^N$, $\{j_1, \ldots, j_p\} = \sup_{\subset} J_k^N$ *and the condition* $(\dagger)_k$ *for* k *was given in Lemma 5.2.*

Let $j \in \{1, 2, \ldots, l\}$. Then, for any k and k' in $\hat{I}_j^N$, as $J_k^N = J_{k'}^N$ holds, we have $\inf_{\subset} J_k^N = \inf_{\subset} J_{k'}^N$ and $\sup_{\subset} J_k^N = \sup_{\subset} J_{k'}^N$. This implies that the sets $\inf_{\subset} J_k^N$ and $\sup_{\subset} J_k^N$ do not depend on a choice of $k \in \hat{I}_j^N$. We denote them by $\underline{\mathcal{J}}(j)$ and $\overline{\mathcal{J}}(j)$ respectively for short. As the submanifolds in χ^M and χ^N are connected, the sets $\underline{\mathcal{J}}(j)$ and $\overline{\mathcal{J}}(j)$ are independent of the choice of a local coordinates system. We also say that j satsifies the condtion (†) if the condtion $(\dagger)_k$ holds for some $k \in \hat{I}_j^N$. Note that this definition is independent of a choice of $k \in \hat{I}_j^N$ by the above observation.

For $j \in \{1, 2, \ldots, l\}$ that satisfies (†), by taking Corollary 5.4 into account, we have the map

$$\hat{\varphi}_j : \left(\underset{X,\, \beta \in \overline{\mathcal{J}}(j)}{\times} T_{M_\beta} \iota(M_\beta) \right) \underset{X}{\times} M \to \left(T_{N_j} \iota(N_j) \underset{Y}{\times} N \right) \underset{Y}{\times} M$$

where $N = N_1 \cap \cdots \cap N_\ell$. Although $\hat{\varphi}_j$ is not a morphism of vector bundles over M, we still have

$$\hat{\varphi}_j \left(\tau_{j_1}^X(\lambda_{j_1}) \ldots \tau_{j_p}^X(\lambda_{j_p}) p \right) = \tau_j^Y(\lambda_{j_1} \ldots \lambda_{j_p}) \hat{\varphi}_j(p)$$

where $\{j_1, \ldots, j_p\} = \overline{\mathcal{J}}(j)$ and τ_β^X and τ_β^Y denote the action τ_β on each spaces X and Y respectively. Hence $\hat{\varphi}_j$ gives a multi-linear map between the fibers of vector bundles. This implies, in particular, that the image of a multi-conic set is conic and the inverse image of a conic set is multi-conic.

Summing up, we have

Corollary 5.5 *The restriction of $\widetilde{f}$ to $\{t_1 = \cdots = t_\ell = 0\}$ is equal to the map*

$$\varphi_1 \underset{Y}{\times} \cdots \underset{Y}{\times} \varphi_\ell : \underset{X,\, 1 \le \beta \le \ell}{\times} T_{M_\beta} \iota(M_\beta) \to \underset{Y,\, 1 \le \beta \le \ell}{\times} T_{N_\beta} \iota(N_\beta).$$

Here each map

$$\varphi_j : \underset{X,\, 1 \le \beta \le \ell}{\times} T_{M_\beta} \iota(M_\beta) \to T_{N_j} \iota(N_j) \underset{Y}{\times} N \qquad (j = 1, 2, \ldots, \ell)$$

is defined by the composition

$$\underset{X,\, 1 \le \beta \le \ell}{\times} T_{M_\beta} \iota(M_\beta) \to \left(\underset{X,\, \beta \in \overline{\mathcal{J}}(j)}{\times} T_{M_\beta} \iota(M_\beta) \right) \underset{X}{\times} M \overset{\hat{\varphi}_j}{\to} T_{N_j} \iota(N_j) \underset{Y}{\times} N$$

if j satisfies the condition (†), and by the map which sends a point to the zero section of $T_{N_j} \iota(N_j) \underset{Y}{\times} N$ for other j.

Example 5.6 *Set $Y = \mathbb{C}^2$, let $X = \{(z_1, z_2, \xi_1, \xi_2) \in \mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}},\ \xi_2 z_1 = \xi_1 z_2\}$ and let*

$$\begin{array}{rcl} \pi : X & \to & \mathbb{C}^2 \\ (z_1, z_2, \xi_1, \xi_2) & \mapsto & (z_1, z_2) \end{array}$$

be the desingularization map. Let $N_1 = \{0\}$, $N_2 = \{z_2 = 0\}$, $M_1 = \pi^{-1}(0)$, $M_2 = \{\xi_2 = 0\}$. Locally on X, for example on $U_1 := \{\xi_1 \neq 0\}$, set $\lambda = \dfrac{\xi_2}{\xi_1}$. Then $z_2 = \dfrac{\xi_2}{\xi_1} z_1$ and we have an homeomorphism

$$\begin{array}{rcl} \psi : \mathbb{C}^2 & \xrightarrow{\sim} & U_1 \\ (\lambda, z_1) & \mapsto & (z_1, \lambda z_1, 1, \lambda). \end{array}$$

We have $\psi^{-1}(M_1) = \{z_1 = 0\}$ and $\psi^{-1}(M_2) = \{\lambda = 0\}$. We still denote them by M_1, M_2. The map $f := \pi|_{U_1} \circ \psi$ is given by $(\lambda, z_1) \mapsto (\lambda z_1, z_1)$. Let us consider $\widetilde{f}$. On the zero section, let (w_1, w_2) be the coordinates of $\mathbb{C}^2 \simeq S'$, the zero section of $\widetilde{Y}$. We have $J_1^N = J_1^M = \{1\}$, $J_2^N = \{1, 2\} = J_1^M \sqcup J_2^M$, hence

$$\begin{array}{rcl} w_1 & = & \dfrac{\partial f}{\partial z_1}(0,0) z_1 = z_1 \\ w_2 & = & \dfrac{\partial^2 f}{\partial \lambda \partial z_1}(0,0) \lambda z_1 = \lambda z_1 \end{array}$$

which is a conic map with respect to the $(\mathbb{R}^+)^2$-actions on S and S'.

One can ask when the morphism thus obtained becomes that of vector bundles.

Proposition 5.7 *Suppose the conditions 1. and 2. below.*

1. *$f(M_j) \subset N_j$ for any $1 \leq j \leq \ell$.*
2. *$M_{j'} \subset M_j$ if and only if $N_{j'} \subset N_j$ for $1 \leq j, j' \leq \ell$.*

Then the map

$$\varphi_1 \underset{Y}{\times} \ldots \underset{Y}{\times} \varphi_\ell : \underset{X,\, 1 \leq \beta \leq \ell}{\times} T_{M_\beta} \iota(M_\beta) \to \left(\underset{Y,\, 1 \leq \beta \leq \ell}{\times} T_{N_\beta} \iota(N_\beta) \right) \underset{Y}{\times} M$$

is a morphism of vector bundles over M.

Proof. For any $k \in \{1, 2, \ldots, m\}$, we have $\#\inf_{\subset} J_k^N = \#\sup_{\subset} J_k^N = 1$. Therefore, by (5.5), the map φ_j becomes a bundle map and the result follows. □

Remark 5.8 *More generally, the morphism exists for the case* $\#\chi^N \leq \#\chi^M$.

Let $f : X \to Y$ *and* $\chi^M = \{M_1, \ldots, M_\ell\}$ *in* X *and* $\chi^N = \{N_1, \ldots, N_{\ell'}\}$ *in* Y *for* $\ell' \leq \ell$.

Then the situation decomposes into the following:

$$(X;\, M_1, \ldots, M_\ell) \xrightarrow{\mathrm{Id}} (X;\, M_1, \ldots, M_{\ell'}) \xrightarrow{f} (Y;\, N_1, \ldots, N_{\ell'})$$

For the second arrow, we have already constructed the morphism. We will construct the morphism for the first arrow. In what follows, we assume that $Y = X$, $f = \mathrm{Id}$, $\ell > \ell'$ *and* $N_j = M_j$.

Locally the morphism between multi-normal deformations

$$(x_1, \ldots, x_n;\, t_1, \ldots, t_\ell) \to (x'_1, \ldots, x'_n;\, t'_1, \ldots, t'_{\ell'})$$

is given by:

$$\begin{aligned} t'_j &= t_{\kappa_{\chi^M,\chi^N}(j)} && (1 \leq j \leq \ell'), \\ x'_i &= t_{J_{\chi^M,\chi^N,i}} x_i && (1 \leq i \leq n). \end{aligned}$$

Here κ_{χ^M,χ^N} *is the map from* $\{1, 2, \ldots, \ell'\}$ *to subsets of* $\{1, 2, \ldots, \ell\}$ *defined by*

$$\kappa_{\chi^M,\chi^N}(j) := \left\{ \beta \in \{1, 2, \ldots, \ell\};\, \left(\bigcup_{N_k \subsetneq N_j} N_k \right) \subsetneq f(M_\beta) \subset N_j \right\}$$

and, for $1 \leq i \leq n$,

$$J_{\chi^M,\chi^N,i} := J_i^M \setminus \left(\bigcup_{j \in J_i^N} \kappa_{\chi^M,\chi^N}(j) \right)$$

with $J_i^M := \{j \in \{1, 2, \ldots, \ell\};\, i \in I_j^M\}$ *and* $J_i^N := \{j \in \{1, 2, \ldots, \ell'\};\, i \in I_j^N\}$.

For example, if $\ell' = 2$ *and* $M_\ell \subsetneq M_{\ell-1} \subsetneq \cdots \subsetneq M_1$ *are satisfied, as* $\kappa_{\chi^M,\chi^N}(1) = \{1\}$, $\kappa_{\chi^M,\chi^N}(2) = \{2,\ldots,\ell\}$, $J_{\chi^M,\chi^N,i} = J_i^M$ $(i \notin I_2^M)$ *and* $J_{\chi^M,\chi^N,i} = \emptyset$ $(i \in I_2^M)$, *the local morphism is given by*

$$\begin{array}{ll} t'_1 = t_1, & t'_2 = t_2 \ldots t_\ell, \\ x'_i = t_{J_i^M} x_i \, (i \notin I_2^M), & x'_i = x_i \, (i \in I_2^M). \end{array}$$

We can certainly glue these locally defined morphisms (by the definition of a multi-normal deformation) and obtain the morphism between multi-normal deformed manifolds (say $\widetilde{X}$ *and* $\widetilde{Y}$*) globally. Moreover, as* $J_i^M = \bigsqcup_{j \in J_i^N} \kappa_{\chi^M,\chi^N}(j)$ *is satisfied by definition, the following diagram commutes.*

$$\begin{array}{ccc} \widetilde{X} & \to & \widetilde{Y} \\ & \searrow & \downarrow \\ & & X = Y \end{array}$$

Since $\iota(M_j) \subset \iota(N_j)$ *holds* $(j = 1, \ldots, \ell')$, *we have the canonical injection*

$$T_{M_j}\iota(M_j) \hookrightarrow T_{N_j}\iota(N_j) \qquad (j \leq \ell').$$

These injections and the zero map on $T_{M_j}\iota(M_j)$ *for* $j > \ell'$ *induce the bundle map over* $M := \bigcap_{1 \leq j \leq \ell} M_j$

$$\varphi : \underset{X, 1 \leq j \leq \ell}{\times} T_{M_j}\iota(M_j) \to M \underset{Y}{\times} \left(\underset{Y, 1 \leq j \leq \ell'}{\times} T_{N_j}\iota(N_j) \right)$$

In this case, on the zero section $\{t = 0\}$, *the morphism from* $\widetilde{X}$ *to* $\widetilde{Y}$ *coincides with* φ.

6 Multi-specialization

Let X be a real analytic manifold with $\dim X = n$, and let $\chi = \{M_1, \ldots, M_\ell\}$ be a family of closed submanifolds satisfying H1, H2 and H3. Let $\widetilde{X}$ be the multi-normal deformation of X with respect to the family χ and consider the diagram (1.3).

Denote by $\mathrm{Op}(X_{sa})$ (resp. $\mathrm{Op}^c(X_{sa})$) the category of open (resp. open relatively compact) subanalytic subsets of X. One endows $\mathrm{Op}(X_{sa})$ with the following topology: $S \subset \mathrm{Op}(X_{sa})$ is a covering of $U \in \mathrm{Op}(X_{sa})$ if for any compact K of X there exists a finite subset $S_0 \subset S$ such that

$K \cap \bigcup_{V \in S_0} V = K \cap U$. We will call X_{sa} the subanalytic site and denote by $\rho : X \to X_{sa}$ the natural morphism of sites associated to the inclusion $\mathrm{Op}(X_{sa}) \hookrightarrow \mathrm{Op}(X)$. Let $\mathrm{Mod}(k_{X_{sa}})$ (resp. $D^b(k_{X_{sa}})$ denote the category of sheaves on X_{sa} (resp. bounded derived category of sheaves on X_{sa}). Reference for classical sheaf theory are made to [10], for subanalytic sheaves we refer to [14] and [19]. For an exposition on $(\mathbb{R}^+)^\ell$-conic sheaves see the Appendix.

Lemma 6.1 *Let $F \in D^b(k_{X_{sa}})$. There is a natural isomorphism*

$$s^{-1}R\Gamma_\Omega p^{-1}F \simeq s^!(p^{-1}F)_\Omega.$$

Proof. We prove the assertion in several step. For $I, J \subseteq \{1, \dots, \ell\}$, $I \cap J = \emptyset$, set $S_I = \{t_i = 0,\, i \in I\}$ and $\Omega_J = \{t_j > 0,\, j \in J\}$.

(i) We show that if $\sharp J < \ell$, then $(R\Gamma_{\Omega_J}(p^{-1}F)_\Omega)|_S = 0$. We may reduce to $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. It follows because for any $(\mathbb{R}^+)^\ell$-connected neighborhood V (for example a ball of radius $\varepsilon > 0$) of $p \in S$ and for any $W \subset V$ $(\mathbb{R}^+)^\ell$-connected containing $V \setminus \Omega$

$$R\Gamma(V \cap \Omega_J; p^{-1}F) \simeq R\Gamma(W \cap \Omega_J; p^{-1}F)$$

since $p^{-1}F$ is $(\mathbb{R}^+)^\ell$-conic. This implies

$$(R\Gamma_{\Omega_J}(p^{-1}F)_{\Omega_J \setminus \Omega})|_S \simeq (R\Gamma_{\Omega_J}p^{-1}F)|_S.$$

(ii) We show that if $\sharp(I \cup J) < \ell$, then $(R\Gamma_{S_I \cap \Omega_J}(p^{-1}F)_\Omega)|_S = 0$. We argue by induction on $\sharp I$. If $\sharp I = 0$ then it follows by (i) that $(R\Gamma_{\Omega_J}(p^{-1}F)_\Omega)|_S = 0$. Suppose that it is true for $\sharp I \leq \ell - 2$. Let $i_0 \in I$. We have the distinguished triangle

$$(R\Gamma_{S_I \cap \Omega_J}(p^{-1}F)_\Omega)|_S \to (R\Gamma_{S_{I \setminus \{i_0\}} \cap \Omega_J}(p^{-1}F)_\Omega)|_S \to (R\Gamma_{S_{I \setminus \{i_0\}} \cap \Omega_{J \cup \{i_0\}}}(p^{-1}F)_\Omega)|_S \overset{+}{\to}.$$

The second and the third terms of the triangle are zero since $\sharp(I \setminus \{i_0\}) \leq \ell - 2$. Then the first one is zero.

(iii) We show that if $I \cup J = \ell$ then $(R\Gamma_{S_I \cap \Omega_J}(p^{-1}F)_\Omega)|_S \simeq (R\Gamma_S(p^{-1}F)_\Omega)|_S[\sharp J]$. We argue by induction on $\sharp J$. Let $j_0 \in J$. We have the distinguished triangle

$$(R\Gamma_{S_{I \cup \{j_0\}} \cap \Omega_{J \setminus \{j_0\}}}(p^{-1}F)_\Omega)|_S \to (R\Gamma_{S_I \cap \Omega_{J \setminus \{j_0\}}}(p^{-1}F)_\Omega)|_S \to (R\Gamma_{S_I \cap \Omega_J}(p^{-1}F)_\Omega)|_S \overset{+}{\to}$$

where the second term is zero by (ii). If $\sharp J = 1$ the result follows immediately. Suppose that it is true for $\sharp J \leq \ell - 1$. Then $(R\Gamma_{S_{I \cup \{j_0\}} \cap \Omega_{J \setminus \{j_0\}}}(p^{-1}F)_\Omega)|_S[1] \simeq (R\Gamma_{S_I \cap \Omega_J}(p^{-1}F)_\Omega)|_S$ by (ii) and $(R\Gamma_{S_{I \cup \{j_0\}} \cap \Omega_{J \setminus \{j_0\}}}(p^{-1}F)_\Omega)|_S \simeq (R\Gamma_S(p^{-1}F)_\Omega)|_S[\sharp J - 1]$ by the induction hypothesis.

(iv) By (iii) we obtain $s^!(p^{-1}F)_\Omega[\ell] \simeq s^{-1}R\Gamma_\Omega p^{-1}F$. It remains to show that $(p^{-1}F)_\Omega[\ell] \simeq (p^!F)_\Omega$. We may reduce to the case $F \simeq \varinjlim_i \rho_* F_i$ with $F_i \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_X)$. In this case we have

$$\begin{aligned} H^{k+\ell}(p^{-1}F)_\Omega &\simeq \varinjlim_i \rho_* H^{k+\ell}(p^{-1}F_i)_\Omega \\ &\simeq \varinjlim_i \rho_* H^k(p^!F_i)_\Omega \\ &\simeq H^k(p^!F)_\Omega \end{aligned}$$

where the second isomorphism follows since $\widetilde{p}^{-1}[\ell] \simeq \widetilde{p}^!$ in $\mathrm{Mod}(k_X)$. □

Definition 6.2 *The (multi-)specialization along χ is the functor*

$$\begin{aligned} \nu_\chi^{sa} : D^b(k_{X_{sa}}) &\to D^b(k_{S_{sa}}) \\ F &\mapsto s^{-1}\mathrm{R}\Gamma_\Omega p^{-1}F. \end{aligned}$$

Theorem 6.3 *Let $F \in D^b(k_{X_{sa}})$.*

(i) $\nu_\chi^{sa}F \in D^b_{(\mathbb{R}^+)^\ell}(k_{S_{sa}})$ (see Definition B.7).

(ii) Let V be a $(\mathbb{R}^+)^\ell$-conic subanalytic open subset of S. Then:

$$H^j(V; \nu_\chi^{sa}F) \simeq \varinjlim_U H^j(U; F),$$

where U ranges through the family of $\mathrm{Op}(X_{sa})$ such that $C_\chi(X \setminus U) \cap V = \emptyset$.

Proof. (i) We may reduce to the case $F \in \mathrm{Mod}(k_{X_{sa}})$. Hence $F = \varinjlim_i \rho_* F_i$ with $F_i \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(k_{X_{sa}})$ for each i. We have $p^{-1}\varinjlim_i \rho_* F_i \simeq \varinjlim_i \rho_* p^{-1}F_i$ and $p^{-1}F_i$ is $\mathbb{R}$-constructible and $(\mathbb{R}^+)^\ell$-conic for each i. Hence $p^{-1}F$ is $(\mathbb{R}^+)^\ell$-conic. Since the functors $\mathrm{R}\Gamma_\Omega$ and s^{-1} send $(\mathbb{R}^+)^\ell$-conic sheaves to $(\mathbb{R}^+)^\ell$-conic sheaves we obtain $s^{-1}\mathrm{R}\Gamma_\Omega p^{-1}F = \nu_\chi^{sa}F \in D^b_{(\mathbb{R}^+)^\ell}(k_{S_{sa}})$.

(ii) Let $U \in \mathrm{Op}(X_{sa})$ such that $V \cap C_\chi(X \setminus U) = \emptyset$. We have the chain of morphisms

$$\begin{aligned} \mathrm{R}\Gamma(U; F) &\to \mathrm{R}\Gamma(p^{-1}(U); p^{-1}F) \\ &\to \mathrm{R}\Gamma(p^{-1}(U) \cap \Omega; p^{-1}F) \\ &\to \mathrm{R}\Gamma(\widetilde{p}^{-1}(U) \cup V; \mathrm{R}\Gamma_\Omega p^{-1}F) \\ &\to \mathrm{R}\Gamma(V; \nu_\chi^{sa}F) \end{aligned}$$

where the third arrow exists since $\widetilde{p}^{-1}(U) \cup V$ is a neighborhood of V in $\overline{\Omega}$ by Lemma 4.6 (ii). Let us show that it is an isomorphism. Let V be a conic open subanalytic subset of S. We have

$$\begin{aligned} H^k(V; \nu^{sa}_\chi F) &\simeq \varinjlim_W H^k(W; \mathrm{R}\Gamma_\Omega p^{-1}F) \\ &\simeq \varinjlim_W H^k(W \cap \Omega; p^{-1}F), \end{aligned}$$

where W ranges through the family of subanalytic open neighborhoods of V in $\widetilde{X}$. By Lemma 4.7 we may assume that W satisfies (4.1). Since $p^{-1}F$ is $(\mathbb{R}^+)^\ell$-conic, we have

$$\begin{aligned} H^k(W \cap \Omega; p^{-1}F) &\simeq H^k(p^{-1}(p(W \cap \Omega)); p^{-1}F) \\ &\simeq H^k(p(W \cap \Omega) \times \{(1)_\ell\}; p^{-1}F) \\ &\simeq H^k(p(W \cap \Omega); F), \end{aligned}$$

where $(1)_\ell = (1, \dots, 1) \in \mathbb{R}^\ell$. The second isomorphism follows since every subanalytic neighborhood of $p(W \cap \Omega) \times \{(1)_\ell\}$ contains an $(\mathbb{R}^+)^\ell$-connected subanalytic neighborhood (the proof is similar to that of Lemma 4.7). By Lemma 4.6 (i) we have that $p(W \cap \Omega)$ ranges through the family of subanalytic open subsets U of X such that $V \cap C_\chi(X \setminus U) = \emptyset$ and we obtain the result. □

Remember that a sheaf $F \in \mathrm{Mod}(k_{X_{sa}})$ is said to be quasi-injective if the restriction morphism $\Gamma(U; F) \to \Gamma(V; F)$ is surjective for each $U, V \in \mathrm{Op}^c(X_{sa})$ with $U \supset V$.

Corollary 6.4 *Let $F \in \mathrm{Mod}(k_{X_{sa}})$ be quasi-injective. Then F is ν^{sa}_χ-acyclic.*

Proof. The result follows from Theorem 6.3 and the fact that quasi-injective sheaves are $\Gamma(U; \cdot)$-acyclic for each $U \in \mathrm{Op}(X_{sa})$. □

Corollary 6.5 *Let $F \in D^b(k_{X_{sa}})$ and let $p = (q; \xi) \in S$. Then*

$$(\rho^{-1} H^j \nu^{sa}_\chi F)_p \simeq \varinjlim_{W, \epsilon} H^j(W \cap B_\epsilon; F),$$

where W ranges through the family $\mathrm{Cone}\,\chi(p)$ and B_ϵ ranges through the family of open balls of radius $\epsilon > 0$ containing q. Here we locally identify X with a vector space as in Section 4.

Proof. The result follows since for any subanalytic conic neighborhood V of p, any $U \in \mathrm{Op}(X_{sa})$ such that $C_\chi(X \setminus U) \cap V = \emptyset$ contains $W \cap B_\epsilon$, $q \in B_\epsilon$, $\epsilon > 0$, $W \in \mathrm{Cone}_\chi(p)$. □

Remember that

$$\underset{M,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i}) := \left(\underset{X,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i})\right) \underset{X}{\times} M.$$

Corollary 6.6 *Let $F \in D^b(k_{X_{sa}})$. Let $k \le \ell$ and $\{j_1, \dots, j_k\}$ be a subset of $\{1, 2, \dots, \ell\}$. Set $\chi_k = \{M_{j_1}, \dots, M_{j_k}\}$. Then we have*

$$(\nu^{sa}_\chi F)|_{\underset{M,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i})} \simeq (\nu^{sa}_{\chi_k} F)|_{\underset{M,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i})}.$$

Proof. Set $Z = \underset{M,1\le i\le k}{\times} T_{M_{j_i}}\iota_\chi(M_{j_i})$. On Z the multi-actions induced by $\{\mu_1, \dots, \mu_\ell\}$ and $\{\mu_{j_1}, \dots, \mu_{j_k}\}$ coincide. Let V be a $(\mathbb{R}^+)^\ell$-conic globally subanalytic open subset of Z. Let $U \in \mathrm{Op}(X_{sa})$ such that $V \cap C_\chi(X \setminus U) = \emptyset$. Let W be the complement of $C_\chi(X \setminus U)$. Then W is a $(\mathbb{R}^+)^\ell$-conic open subanalytic neighborhood of V such that $C_\chi(X \setminus U) \cap W = \emptyset$. Then for $j \in \mathbb{Z}$

$$\begin{aligned} H^j(V; \nu^{sa}_\chi F) &\simeq \varinjlim_W H^j(W; \nu^{sa}_\chi F) \\ &\simeq \varinjlim_U H^k(U; F) \\ &\simeq \varinjlim_{U'} H^k(U'; F), \end{aligned}$$

where W ranges through the family of $(\mathbb{R}^+)^\ell$-conic open subanalytic neighborhoods of V, $U \in \mathrm{Op}(X_{sa})$ is such that $C_\chi(X \setminus U) \cap W = \emptyset$ and $U' \in \mathrm{Op}(X_{sa})$ is such that $C_\chi(X \setminus U') \cap V = \emptyset$. Remark that the first isomorphism is a consequence of the fact that $\nu^{sa}_\chi F$ is conic and the fact that by Lemma B.11 every open subanalytic neighborhood of V contains a $(\mathbb{R}^+)^\ell$-connected one. In the same way we see that

$$H^j(V; \nu^{sa}_{\chi_k} F) \simeq \varinjlim_{U'} H^k(U'; F),$$

where $U' \in \mathrm{Op}(X_{sa})$ is such that $C_{\chi_k}(X \setminus U') \cap V = \emptyset$. To show the result it is enough to see that

$$C_\chi(X \setminus U) \cap V = \emptyset \iff C_{\chi_k}(X \setminus U) \cap V = \emptyset.$$

This follows from Corollary 4.3, which says that

$$C_\chi(X \setminus U) \cap Z = C_{\chi_k}(X \setminus U) \cap Z.$$

Hence we have the isomorphism

$$H^j(V; \nu^{sa}_\chi F) \simeq H^j(V; \nu^{sa}_{\chi_k} F)$$

for each $(\mathbb{R}^+)^\ell$-conic globally subanalytic open subset of Z. By Lemma B.10 and the fact that both $\nu^{sa}_\chi F$ and $\nu^{sa}_{\chi_k} F$ are multi-conic we obtain $(\nu^{sa}_\chi F)|_Z \simeq (\nu^{sa}_{\chi_k} F)|_Z$. □

Let $f : X \to Y$ be a morphism of real analytic manifolds, $\chi^M = \{M_1, \dots, M_\ell\}$, $\chi^N = \{N_1, \dots, N_\ell\}$ two families of closed analytic submanifolds of X and Y respectively satisfying the hypothesis H1, H2 and H3. Suppose that $f(M_i) \subseteq N_i$, $i = 1, \dots, \ell$. We call $T_\chi f$ the induced map on the zero section. Let $J \subseteq \{1, \dots, \ell\}$ and set $\chi_J = \{M_{j_1}, \dots, M_{j_k}\}$, $j_k \in J$. The map $T_{\chi_J} f$ denotes the restriction of $\widetilde{f}$ to $\{t_{j_k} = 0,\, j_k \in J\}$. In the following we will denote with the same symbol $C_{\chi_J}(Z)$ the normal cone with respect to χ_J and its inverse image via the map $\widetilde{X} \to \widetilde{X}_{M_{j_1}, \dots, M_{j_k}}$.

Proposition 6.7 *Let $F \in D^b(k_{X_{sa}})$.*

(i) There exists a commutative diagram of canonical morphisms

$$\begin{array}{ccc} R(T_\chi f)_{!!} \nu^{sa}_{\chi^M} F & \longrightarrow & \nu^{sa}_{\chi^N} Rf_{!!} F \\ \downarrow & & \downarrow \\ R(T_\chi f)_* \nu^{sa}_{\chi^M} F & \longleftarrow & \nu^{sa}_{\chi^N} Rf_* F. \end{array}$$

(ii) Moreover if $f : \mathrm{supp} F \to Y$ and $T_{\chi_J} f : C_{\chi_J}(\mathrm{supp} F) \to \{t_{j_1} = \cdots = t_{j_k} = 0\}$ for each $J = \{j_1, \dots, j_k\}$ are proper, and if $\mathrm{supp} F \cap f^{-1}(N_j) \subseteq M_j$, $j \in \{1, \dots, \ell\}$, then the above morphisms are isomorphisms.

Proof. (i) The existence of the arrows is done as in [10] Proposition 4.2.4.

(ii) If $\widetilde{p}^{-1}(\mathrm{supp} F)$ is proper over $\widetilde{X}$, then all the morphisms are isomorphisms. We have to prove that for a closed subset Z of X, the restriction of $\widetilde{f}$ to $\widetilde{p^{-1}(Z)}$ is proper, if $Z \to Y$ and $C_{\chi_J}(\mathrm{supp} F) \to \{t_{j_1} = \cdots = t_{j_k} = 0\}$ for $k \leq \ell$ are proper, and if $Z \cap f^{-1}(N_j) \subseteq M_j$, $j \in \{1, \dots, \ell\}$. We argue by induction on $\sharp\chi$. If $\sharp\chi = 1$ this is Proposition 4.2.4 of Sheaves on Manifolds.

Suppose it is true for $\sharp\chi \leq \ell - 1$. It follows from the hypothesis that the fibers of $\widetilde{f}$ restricted to $\overline{\widetilde{p}^{-1}(Z)}$ are compact (if $t_{j_1} = \cdots = t_{j_k} = 0$ this is a consequence of the fact that $C_{M_{j_1}\ldots M_{j_k}}(\mathrm{supp}F) \to \{t_{j_1} = \cdots = t_{j_k} = 0\}$ is proper). Then it remains to prove that it is a closed map. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $\overline{\widetilde{p}^{-1}(Z)}$ and suppose that $\{\widetilde{f}(u_n)\}_{n\in\mathbb{N}}$ converges. We shall find a convergent subsequence of $\{u_n\}_{n\in\mathbb{N}}$. We may also assume that $\{\widetilde{p}(u_n)\}_{n\in\mathbb{N}}$ converges. The map $\widetilde{p}^{-1}(Z) \setminus \{t_1 = \cdots = t_\ell = 0\} \to \widetilde{Y} \setminus \{t_1 = \cdots = t_\ell = 0\}$ is proper. Indeed, let K be a compact subset of $\widetilde{Y} \setminus \{t_1 = \cdots = t_\ell = 0\}$, and reduce to the case that K is contained in $\{c \leq t_j \leq d\} \subset \widetilde{Y}$, $c, d > 0$, $j \in \{1, \ldots, \ell\}$. Suppose without loss of generality that $j = 1$. Then $K_1 := \widetilde{p}_{N_1}(K)$ is a compact subset of $\widetilde{Y}_{N_2\ldots N_\ell}$. Let us identify K_1 with $\widetilde{p}_{N_1}^{-1}(K_1) \cap \{t_1 = 1\}$. Then $\widetilde{f}^{-1}(K_1)$ is compact by the induction hypothesis. Hence $\widetilde{f}^{-1}(K) \subseteq \mu_1(\widetilde{f}^{-1}(K_1), [c, d])$ is compact since it is closed and contained in a compact subset. We may assume that $\{\widetilde{f}(u_n)\}_{n\in\mathbb{N}}$ converges to a point of $\{t_1 = \cdots = t_\ell = 0\}$. Then $\{\widetilde{p}(u_n)\}_{n\in\mathbb{N}}$ converges to a point of $Z \cap f^{-1}(N_1 \cap \cdots \cap N_\ell) \subseteq M_1 \cap \cdots \cap M_\ell$.

Taking local coordinates systems of X and Y, let $u_n = (x_{1n}, \ldots, x_{mn}, t_{1n}, \ldots, t_{\ell n})$, $t_{jn} > 0$, $j = 1, \ldots, \ell$. Then $t_{jn} \to 0$, $j = 1, \ldots, \ell$ and $t_{J_i^M n} x_{in} \to 0$, $i = 1, \ldots, m$. It is enough to show that $\{|x_{in}|\}_{n\in\mathbb{N}}$ is bounded for each $i = 1, \ldots, m$. We argue by contradiction. Suppose without loss of generality that $|x_{1n}| \to +\infty$ and that $\{x_{1n}/|x_n|\}_{n\in\mathbb{N}}$ is not infinitesimal. Set $u_n^{(\beta)} = (x_n^{(\beta)}, t_n^{(\beta)})$, where $x_{in}^{(\beta)} = t_{in\beta} x_{in}$ with $t_{in\beta} = t_{\beta n}$ if $i \in I_\beta$ and $t_{in\beta} = 1$ if $i \notin I_\beta$, and where $t_{jn}^{(\beta)} = t_{jn}$ if $j \neq \beta$ and $t_{\beta n}^{(\beta)} = 1$. Then u_n belongs to $\widetilde{p}^{-1}(Z) \setminus \{t_1 = \cdots = t_\ell = 0\}$ and $\{\widetilde{f}(u_n^{(\beta)})\}_{n\in\mathbb{N}}$ converges. Since $\widetilde{f}$ is proper on $\widetilde{p}^{-1}(Z) \setminus \{t_1 = \cdots = t_\ell = 0\}$, then $\{|x_n^{(\beta)}|\}_{n\in\mathbb{N}}$ is bounded.

(a) Suppose that there exists $\beta \in \{1, \ldots, \ell\}$ such that $1 \notin I_\beta^M$. The sequence $\{|x_n^{(\beta)}|\}_{n\in\mathbb{N}}$ is bounded. In particular $\{|x_{1n}|\}_{n\in\mathbb{N}}$ is bounded which is a contradiction.

(b) Suppose that $1 \in I_j^M$ for $j = 1, \ldots, \ell$. Suppose without loss of generality that I_1^M is the biggest I_j^M containing 1. Set $\widetilde{u}_n = (\widetilde{x}_n, \widetilde{t}_n)$ where $\widetilde{x}_{in} = x_{in}/|x_n|$ if $i \in I_1^M$ and $\widetilde{x}_{in} = x_{in}$ otherwise and where $\widetilde{t}_{1n} = t_{1n}|x_n|$, $\widetilde{t}_{jn} = t_{jn}$ if $j \neq 1$. Remark that $|t_{1n}x_n| = |t_{1n}x_{1n}||x_n|/|x_{1n}|$ is bounded since the sequences $\{|x_{1n}^{(1)}|\}_{n\in\mathbb{N}}$ and $\{|x_n|/x_{1n}\}_{n\in\mathbb{N}}$ are bounded. By extracting a subsequence $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ converges to a non-zero vector v and $\widetilde{u}_n$ belongs to $\widetilde{p}^{-1}(Z)$. On the other hand $\widetilde{f}(u_n)$ converges and hence $f_k(t_{J_1^M n}x_{1n}, \ldots, t_{J_m^M n}x_{mn})/t_{J_k^N n}|x_n|$ converges to 0 for each k such that $k \in$

I_1^N. Moreover for each $k \in I_2^N \backslash I_1^N$ by (5.1) the sequence $f_k(t_{J_1^m n} x_{1n}, \dots, t_{J_m^M n} x_{mn})/t_{J_k^N n}$ converges to 0. Hence $\widetilde{f}(\mu_{\underline{2}}(v,\lambda)) = \widetilde{f}(v)$ for each $\lambda \in \mathbb{R}^+$ which contradicts the fact that the fibers of $\widetilde{p}^{-1}(Z) \to \widetilde{Y}$ are compact. □

Proposition 6.8 *Let $F \in D^b(k_{Y_{sa}})$.*

(i) There exists a commutative diagram of canonical morphisms

$$\begin{array}{ccc} \omega_{S_X/S_Y} \otimes (T_\chi f)^{-1} \nu^{sa}_{\chi^N} F & \longrightarrow & \nu^{sa}_{\chi^M}(\omega_{S_X/S_Y} \otimes f^{-1}F) \\ \downarrow & & \downarrow \\ T_\chi f^! \nu^{sa}_{\chi^N} F & \longleftarrow & \nu^{sa}_{\chi^M} f^! F. \end{array}$$

(ii) The above morphisms are isomorphisms on the open sets where $T_{\chi_J} f$ is smooth for each $J \subseteq \{1, \dots, \ell\}$.

Proof. (i) The existence of the arrows is done as in [10] Proposition 4.2.5. When (ii) is satisfied the function $\widetilde{f}$ is smooth at any point of the boundary of Ω and all the above morphisms become isomorphisms. □

7 Multi-asymptotic expansions

Let X be the n-dimensional complex vector space $\mathbb{C}^n$ with coordinates $(z_1, z_2, \dots, z_n)$ and $\chi = \{Z_1, \dots, Z_\ell\}$ a family of ℓ-complex submanifolds defined by

$$Z_j := \{z \in X;\ z_{i_{j,1}} = \cdots = z_{i_{j,m_j}} = 0\}$$

where $I_j = \{i_{j,1}, \dots, i_{j,m_j}\}$ is a subset of $\{1, 2, \dots, n\}$ and $\{I_j\}_{j=1,2,\dots,\ell}$ satisfies conditions given in Proposition 1.2 (the conditions H1 and H2), that is,

$$I_j \cap I_{j'} \neq \emptyset \Longrightarrow I_j \subset I_{j'} \text{ or } I_{j'} \subset I_j \qquad (j, j' \in \{1, 2, \dots, \ell\}).$$

We set, for $1 \leq j \leq \ell$,

$$\hat{I}_j := I_j \setminus \left(\bigcup_{I_k \subsetneq I_j} I_k \right)$$

We also assume $\hat{I}_j \neq \emptyset$ for $j = 1, 2, \ldots, \ell$ (the condition H3). Then we have $T_{Z_j}\iota(Z_j) \simeq Z_j \times \mathbb{C}^{\#\hat{I}_j}$ and the equality

$$\bigcup_{1\leq j\leq \ell} I_j = \hat{I}_1 \sqcup \hat{I}_2 \sqcup \cdots \sqcup \hat{I}_\ell,$$

which follows from conditions H1, H2 and H3. For a subset $I = \{i_1, i_2, \ldots, i_m\}$ of $\{1, 2, \ldots, n\}$, we denote by z_I the coordinates $(z_{i_1}, z_{i_2}, \ldots, z_{i_m})$ and by $\mathbb{C}^I$ the complex space $\mathbb{C}^{\#I}$ with coordinates z_I. The map π_I designates the projection from X to $\mathbb{C}^I$. Let $\mu_j(z, \lambda) : X \times \mathbb{R} \to X$ $(j = 1, 2, \ldots, \ell)$ be an action defined by

$$\mu_j(z_1, \ldots, z_n, \lambda) = (\lambda_{j1} z_1, \ldots, \lambda_{jn} z_n)$$

where $\lambda_{ji} = \lambda$ if $i \in I_j$ and $\lambda_{ji} = 1$ otherwise. Let U be an open subset in X. We say that U is a *μ-star shape* if U is non-empty and $\mu_j(x_0, [0, 1]) \subset U$ holds for any $x_0 \in U$ and $1 \leq j \leq \ell$. An open ball with its center being the origin is, for example, a μ-star shape.

Let U be an open convex set in X which is a μ-star shape, and let G_j $(j = 1, 2, \ldots, \ell)$ be an open proper convex cone in $\mathbb{C}^{\hat{I}_j}$. In what follows, we only consider a proper cone, which means that we work outside the zero section of the tangent bundle.

We define an open proper convex multi-cone $S(U, \{G_j\}, \epsilon)$ $(\epsilon > 0)$ in X as follows. We first set

$$V_j := \left\{ z \in U;\, z_{\hat{I}_j} \in G_j,\, |z_{I_j \setminus \hat{I}_j}| < \epsilon |z_{\hat{I}_j}| \right\} \subset X \qquad (j = 1, 2, \ldots, \ell).$$

Here if $I_j \setminus \hat{I}_j = \emptyset$, then we set $|z_{I_j \setminus \hat{I}_j}| = 0$, i.e., no conditions. Then we define

$$S(U, \{G_j\}, \epsilon) := V_1 \cap V_2 \cap \cdots \cap V_\ell \subset X.$$

By convention, the case with no submanifold Z_j is allowed. In this case, we set $S(U, \{G_j\}, \epsilon) := U$. Note that $S := S(U, \{G_j\}, \epsilon)$ is non-empty and

$$S \cap \left(\bigcup_{1\leq j\leq \ell} Z_j \right) = \emptyset$$

follows from the conditions H1, H2 and H3. For example, S gives just a poly-sector when $\bigcup_{1\leq j\leq \ell} Z_j$ forms a normal-crossing divisor. The multi-cone S plays the same geometrical role as that of a sector in an asymptotic expansion.

Let $\mathbb{Z}_\ell$ denote the set of integers $\{1, 2, \ldots, \ell\}$ and $\hat{\mathcal{P}}(\mathbb{Z}_\ell)$ be the set of all the subsets of $\mathbb{Z}_\ell$ except for the empty set. For a $J = \{j_1, j_2, \ldots, j_m\} \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$ and a proper convex multi-cone $S := S(U, \{G_j\}, \epsilon)$, we define

$$\begin{aligned} Z_J &:= Z_{j_1} \cap Z_{j_2} \cap \cdots \cap Z_{j_m}, \\ I_J &:= I_{j_1} \cup I_{j_2} \cup \cdots \cup I_{j_m}, \\ S_J &:= \mathrm{Int}_{Z_J}\left(\overline{S} \cap Z_J\right) \end{aligned}$$

where $\overline{A}$ is the closure of a set A in X and $\mathrm{Int}_{Z_J}(B)$ denotes the interior of a set B in Z_J. We often write S_J by S_j if $J = \{j\}$ from now on. We give some fundamental properties of a multi-cone which are needed later.

Lemma 7.1 *For $S := S(U, \{G_j\}, \epsilon)$ and $J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$, we have the followings.*

1. *S_J is also a non-empty open proper convex multi-cone in Z_J for submanifolds $\{Z_j \cap Z_J\}_{j \in J^*}$. To be more precise, we have in Z_J*

$$S_J = S(U \cap Z_J, \{G_j\}_{j \in J^*}, \epsilon) \tag{7.1}$$

   where $J^* = \{j \in \mathbb{Z}_\ell;\ Z_J \nsubseteq Z_j\}$ and $I_j^* = I_j \setminus I_J$ $(j \in J^*)$ which defines submanifold $Z_j \cap Z_J$. Note that $\{Z_j \cap Z_J\}_{j \in J^*}$ also satisfies the conditions H1, H2 and H3 in Z_J.

2. *$\overline{S_J} = \overline{S} \cap Z_J$ and $Z_{J'} \cap S_J = \emptyset$ for $J' \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$ with $Z_J \nsubseteq Z_{J'}$.*

3. *$\{S_J\}_{J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)}$ covers the edge $\overline{S} \cap \left(\bigcup_{1 \le j \le \ell} Z_j\right) \cap U$ of S, i.e.,*

$$\overline{S} \cap \left(\bigcup_{1 \le j \le \ell} Z_j\right) \cap U = \bigcup_{J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)} S_J. \tag{7.2}$$

Proof. Note that $Z_J \nsubseteq Z_j$ is equivalent to saying that $\hat{I}_j \cap I_J = \emptyset$ under the conditions H1 and H2. We also note an elementary fact that $\overline{A \cap B} = \overline{A} \cap \overline{B}$ if A and B are open convex sets with $A \cap B \neq \emptyset$.

We first show 1. of the lemma. As each V_j is open convex and $V_1 \cap \cdots \cap V_\ell$ is not empty, we have $\overline{S} = \overline{V}_1 \cap \cdots \cap \overline{V}_\ell$. Therefore we have

$$S_J = \mathrm{Int}_{Z_J}(\overline{V}_1 \cap \cdots \cap \overline{V}_\ell \cap Z_J) = \mathrm{Int}_{Z_J}(\overline{V}_1 \cap Z_J) \cap \cdots \cap \mathrm{Int}_{Z_J}(\overline{V}_\ell \cap Z_J)$$

and

$$\mathrm{Int}_{Z_J}(\overline{V}_j \cap Z_J) = \begin{cases} V_j \cap Z_J & (\hat{I}_j \cap I_J = \emptyset), \\ U \cap Z_J & (\text{otherwise}), \end{cases}$$

from which we have

$$S_J = \bigcap_{\hat{I}_j \cap I_J = \emptyset} (V_j \cap Z_J).$$

Hence S_J is the multi-cone $S(U \cap Z_J, \{G_j\}_{j \in J^*}, \epsilon)$ in Z_J.

Next we show 2. of the lemma. As $\overline{V}_j \cap Z_J = \overline{U} \cap Z_J$ if $\hat{I}_j \cap I_J \neq \emptyset$, we have

$$\overline{S_J} = \overline{\bigcap_{\hat{I}_j \cap I_J = \emptyset} V_j \cap Z_J} = \bigcap_{\hat{I}_j \cap I_J = \emptyset} \overline{V_j} \cap Z_J = \bigcap_{1 \leq j \leq \ell} \overline{V_j} \cap Z_J = \overline{S} \cap Z_J.$$

For $Z_J \not\subseteq Z_{J'}$, we can find $j \in J'$ satisfying $Z_J \not\subseteq Z_j$. Then j belongs to J^* by definition, from which $(Z_j \cap Z_J) \cap S_J = \emptyset$ because S_J is an open proper convex multi-cone in Z_J by 1. of the lemma.

Noticing that $\overline{S_J} = \overline{S} \cap Z_J$ and S_J is still a multi-cone, 3. of the lemma can be shown by induction. □

The following properties of $S := S(U, \{G_j\}, \epsilon)$ are easily verified.

1. For any j and $z \in S$, we have $\mu_j(z, (0, 1]) \subset S$.
2. For any $\tilde{z} \in S_j$ $(1 \leq j \leq \ell)$, there exists a point $z \in S$ satisfying $\mu_j(z, 0) = \tilde{z}$.

For a $J = \{j_1, j_2, \ldots, j_m\} \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$, we set

$$\mu_J(z, \lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_m}) = \mu_{j_1}(\mu_{j_2}(\ldots \mu_{j_m}(z, \lambda_{j_m}), \ldots, \lambda_{j_2}), \lambda_{j_1}).$$

Then the above properties (1) and (2) also hold for these multi-actions.

Lemma 7.2 *Let $J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$ and $S := S(U, \{G_j\}, \epsilon)$.*

1. *For any $z \in S$, we have $\mu_J(z, (0, 1], \ldots, (0, 1]) \subset S$.*
2. *For any $\tilde{z} \in S_J$, there exists $z \in S$ with $\mu_J(z, 0, \ldots, 0) = \tilde{z}$.*

Proof. The claim 1. is easy. We prove 2. by induction with respect to the number of elements of J. Assume $\#J > 1$ and fix an element $k \in J$. The set S_k is the multi-cone $S(U \cap Z_k, \{G_j\}_{j \in J^*}, \epsilon)$ in Z_k. Here J^* was given in (7.1). Noticing that $Z_k \subset Z_j$ if $\hat{I}_j \cap I_k \neq \emptyset$, we have

$$S_J = (S_k)_{J^* \cap J}.$$

Then, by induction hypothesis, we can find $z_0 \in S_k$ satisfying

$$\tilde{z} = \mu'_{J^* \cap J}(z_0, 0, \ldots, 0)$$

where $\mu'_{J^*\cap J}$ is the multi action in Z_k defined by $\{I^*_j\}_{j\in J^*}$. We can also find $z_1 \in S$ with $\mu_k(z_1, 0) = z_0$. As the restriction of $\mu_{J^*\cap J}$ to Z_k coincides with $\mu'_{J^*\cap J}$, we have

$$\tilde{z} = \mu_{(J^*\cap J)\cup\{k\}}(z_1, 0, \ldots, 0).$$

Since $\mu_j(z, 0)$ $(j \in J)$ is the identity map on Z_J, we finally obtained $\tilde{z} = \mu_J(z_1, 0, \ldots, 0)$, which completes the proof. □

Thanks to the lemma, we have

Corollary 7.3 *Let $J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$ and $S := S(U, \{G_j\}, \epsilon)$. We have $\pi_J(S) = S_J$ where $\pi_J : X \to Z_J$ is the canonical projection defined by $(z) \mapsto (z_{I_J^{\mathrm{C}}})$ (I_J^{C} is the complement set of I_J).*

Proof. As $\pi_J(z) = \mu_J(z, 0, \ldots, 0)$ holds, the inclusion $S_J \subset \pi_J(S)$ comes from 2. of Lemma 7.2. Let us show a converse inclusion. Let $\tilde{z} \in \pi_J(S)$ with $\tilde{z} = \pi_J(z')$ $(z' \in S)$. Then $z'_\lambda := \mu_J(z', \lambda, \lambda, \ldots, \lambda)$ $(0 < \lambda \leq 1)$ is contained in S, and

$$\lim_{\lambda\to 0^+} z'_\lambda = \mu_J(z', 0, \ldots, 0) = \pi_J(z') = \tilde{z}.$$

Moreover, as $\mu_J(z, 0, \ldots, 0)$ is an open map, an open neighborhood of z' contained in S is mapped to an open neighborhood of $\tilde{z}$ in Z_j by $\mu_J(z, 0, \ldots, 0)$. This implies $\tilde{z} \in S_J$. □

Let $J = \{j_1, \ldots, j_m\} \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$, and let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be variables of parameters for the multi-action $\mu := \mu_{\{1,2,\ldots,\ell\}}$ in X. We denote by $\mathbb{Z}^J_{\geq 0}$ the subset of $\mathbb{Z}^\ell_{\geq 0}$ consisting of $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell) \in \mathbb{Z}^\ell_{\geq 0}$ with $\beta_j = 0$ if $j \notin J$. Note that $\mathbb{Z}^J_{\geq 0}$ is isomorphic to $\mathbb{Z}^{\#J}_{\geq 0}$. We set $\partial^\beta_\lambda := \dfrac{\partial^\beta}{\partial\lambda^\beta}$ for $\beta \in \mathbb{Z}^\ell_{\geq 0}$. When β belongs to $\mathbb{Z}^J_{\geq 0}$, we sometimes denote it by $\partial^\beta_{\lambda,J}$ to emphasize the fact that β is an element of $Z^J_{\geq 0}$.

For $\beta \in \mathbb{Z}^J_{\geq 0}$, we introduce the polynomial of the variables z_{I_J} with constant coefficients by

$$T^\beta_J(z_{I_J}) := \frac{1}{\beta!} \exp(-\mu(z, \lambda))\partial^\beta_{\lambda,J} \exp(\mu(z, \lambda))\Big|_{\lambda=e_{\lambda,J}}. \tag{7.3}$$

Here $e_{\lambda,J} \in \mathbb{C}^\ell$ is the point $(\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_j = 0$ $(j \in J)$ and $\lambda_j = 1$ $(j \notin J)$ and the exponential function exp on $\mathbb{C}^n$ is defined by $\exp(z_1, \ldots, z_n) := e^{z_1} \ldots e^{z_n}$ as usual.

Then, for $N = (n_1, n_2, \ldots, n_\ell) \in \mathbb{Z}^\ell_{\geq 0}$, we define a polynomial of z_{I_J} with constant coefficients by

$$T_J^{<N}(z_{I_J}) := \sum_{\beta <_J N,\, \beta \in \mathbb{Z}^J_{\geq 0}} T_J^\beta(z_{I_J}) \tag{7.4}$$

where $\beta <_J N$ if and only if $\beta_k < n_k$ for any $k \in J$. Note that if there exists no index $\beta <_J N$, then we set $T_J^{<N}(z_{I_J}) := 0$ as usual convention.

We now give a concrete form of the polynomial $T_J^{<N}(z_{I_J})$. Let $I = \{i_1, \ldots, i_m\}$ be a subset of $\{1, 2, \ldots, n\}$. We denote by $\mathbb{Z}^I_{\geq 0}$ the subset of $\mathbb{Z}^n_{\geq 0}$ whose element is $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i = 0$ if $i \notin I$. For $\alpha \in \mathbb{Z}^l_{\geq 0}$, a monomial z^α designates $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. If α belongs to $\mathbb{Z}^I_{\geq 0}$, we also write it by z_I^α. We set

$$|\alpha|_I = \alpha_{i_1} + \cdots + \alpha_{i_m}.$$

Lemma 7.4 *We have*

$$T_J^{<N}(z_{I_J}) = \sum_{\alpha \in A(N)} \frac{1}{\alpha!} z_{I_J}^\alpha \tag{7.5}$$

where the indices set $A(N)$ is given by

$$A(N) := \{\alpha \in \mathbb{Z}^{I_J}_{\geq 0};\ |\alpha|_{I_j} < n_j \text{ for any } j \in J\}.$$

Proof. Let $i \in I_J$ and $\beta = (\beta_1, \ldots, \beta_n) \in Z^J_{\geq 0}$ ($\beta_j = 0$ if $j \notin J$). We first assume $\beta_j \geq 1$ for any $j \in J$. Then we have

$$\begin{aligned}
\frac{\partial}{\partial z_i} T_J^\beta &= \frac{\partial}{\partial z_i} \left(\frac{1}{\beta!} \exp(-\mu(z,\lambda)) \partial_{\lambda,J}^\beta \exp(\mu(z,\lambda)) \Big|_{\lambda = e_{\lambda,J}} \right) \\
&= \frac{1}{\beta!} \exp(-\mu(z,\lambda)) \partial_{\lambda,J}^\beta \left(\prod_{j \in J_i} \lambda_j \exp(\mu(z,\lambda)) \right) \Big|_{\lambda = e_{\lambda,J}} \\
&= \frac{1}{(\beta - e_{J_i})!} \exp(-\mu(z,\lambda)) \partial_{\lambda,J}^{\beta - e_{J_i}} \exp(\mu(z,\lambda)) \Big|_{\lambda = e_{\lambda,J}} = T_J^{\beta - e_{J_i}}
\end{aligned}$$

where $e_{J_i} = (e_{J_i,1}, \ldots, e_{J_i,\ell}) \in \mathbb{Z}^l_{\geq 0}$ is determined by $e_{J_i,j} = 1$ for $j \in J_i$ and $e_{J_i,j} = 0$ otherwise. This formula also holds for any $\beta \in \mathbb{Z}^l_{\geq 0}$ if we set $T_J^\beta(z_{I_J}) := 0$ for $\beta = (\beta_1, \ldots, \beta_l) \in \mathbb{Z}^J$ with some $\beta_j < 0$ ($j \in J$). Hence we obtained, for $N = (n_1, \ldots, n_\ell) \in \mathbb{Z}^l_{\geq 0}$,

$$\frac{\partial}{\partial z_i} T_J^{<N} = T_J^{<N - e_{J_i}}.$$

We prove the lemma by induction with $n = n_1 + \cdots + n_\ell$. If $n = 0$, as both sides of the equation (7.5) are zero by definition, the lemma is true.

Now we prove the lemma for a general $n > 0$. Let us consider the system of partial differential equations of an unknown function $u(z_{I_J})$ defined by

$$\frac{\partial}{\partial z_i} u = T_J^{<N-e_{J_i}} \quad (i \in I_J) \tag{7.6}$$

Then, by induction hypothesis, the right hand side of the above equation is given by $\displaystyle\sum_{\alpha \in A(N-e_{J_i})} \frac{1}{\alpha!} z_{I_J}^{\alpha}$. Clearly both $u = T_J^{<N}(z_{I_J})$ and $u = \displaystyle\sum_{\alpha \in A(N)} \frac{1}{\alpha!} z_{I_J}^{\alpha}$ satisfy the same equation (7.6). The solution of (7.6) is uniquely determined if the initial values at $z_{I_J} = 0$ is given. It is easy to see that $T_J^{<N}(0) = 1$ if $A(N) \neq \emptyset$ and $T_J^{<N}(0) = 0$ if $A(N) = \emptyset$. Hence we have obtained (7.5) for N. This completes the proof. □

Definition 7.5 *Let $S := (U, \{G_i\}, \epsilon)$ be a multi-cone in X. We say that $F := \{F_J\}_{J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)}$ is a total family of coefficients of multi-asymptotic expansion along $\displaystyle\bigcup_{1 \le j \le \ell} Z_j$ on S if each F_J consists of a family of holomorphic functions $\{f_{J,\alpha}(z_{I_J^{\mathrm{C}}})\}_{\alpha \in \mathbb{Z}_{\ge 0}^{I_J}}$ defined on S_J. Here I_J^{C} is the complement set of I_J (note that $z_{I_J^{\mathrm{C}}}$ are the coordinates of the submanifold Z_J).*

Let F be a total family of coefficients defined in the above definition. We introduce a map $\tau_{F,J}$ from polynomials of the variables z_{I_J} to those with coefficients in holomorphic functions on S_J in the following way. Let $p(z_{I_J}) = \displaystyle\sum_{\alpha} c_\alpha z_{I_J}^{\alpha}$ be a polynomial of the variable z_{I_J} with constant coefficients. Then we define $\tau_{F,J}(p)(z)$ by replacing a monomial $z_{I_J}^{\alpha}$ in $p(z_{I_J})$ with $f_{J,\alpha}(z_{I_J^{\mathrm{C}}}) z_{I_J}^{\alpha}$ where $f_{J,\alpha}$ is given in $F_J = \{f_{J,\alpha}\}$, that is,

$$\tau_{F,J}(p)(z) := \sum_{\alpha} c_\alpha f_{J,\alpha}(z_{I_J^{\mathrm{C}}}) z_{I_J}^{\alpha}. \tag{7.7}$$

Note that $\tau_{F,J}(p)(z)$ is a holomorphic function on $\pi_J^{-1}(S_J)$, in particular, it is defined on $S \subset \pi_J^{-1}(S_J)$ by Corollary 7.3. We set

$$\begin{aligned} T_J^{\beta}(F;\, z) &= \tau_{F,J}(T_J^{\beta}), \\ T_J^{<N}(F;\, z) &:= \sum_{\beta <_J N} T_J^{\beta}(F;\, z), \end{aligned} \tag{7.8}$$

and

$$\mathrm{App}^{<N}(F;z) := \sum_{J\in\hat{\mathcal{P}}(\mathbb{Z}_\ell)} (-1)^{(\#J+1)} T_J^{<N}(F;\, z). \tag{7.9}$$

It follows from Lemma 7.4 that, if $\bigcup_{1\le j\le \ell} Z_j$ forms a normal crossing divisor, $\mathrm{App}^{<N}(F;z)$ coincides with one defined by Majima in [15].

Let us recall the definitions of the sets $J_{\supsetneq Z_j}$ and $J_{\subsetneq Z_j}$ for $j \in \mathbb{Z}_\ell$.

$$J_{\supsetneq Z_j} := \{k \in \mathbb{Z}_\ell;\, Z_k \supsetneq Z_j,\ \text{ there is no } m \text{ with } Z_k \supsetneq Z_m \supsetneq Z_j\ \}$$
$$J_{\subsetneq Z_j} := \{k \in \mathbb{Z}_\ell;\, Z_k \subsetneq Z_j,\ \text{ there is no } m \text{ with } Z_k \subsetneq Z_m \subsetneq Z_j\ \}.$$

Then the function $w_j : \mathbb{Z}_{\ge 0}^\ell \to \mathbb{Z}$ $(j = 1, 2, \ldots, \ell)$ is defined by

$$w_j(N) = n_j - \sum_{k\in J_{\supsetneq Z_j}} n_k \tag{7.10}$$

for $N = (n_1, n_2, \ldots, n_\ell) \in \mathbb{Z}_{\ge 0}^\ell$. Note that w_j takes not only positive values but also negative ones.

Let S' be another open proper convex multi-cone $S(U', \{G'_j\}, \epsilon')$. We say that S' is properly contained in S if $\epsilon' < \epsilon$, U' is relatively compact in U and G'_j $(j = 1, 2, \ldots, \ell)$ is properly contained in G_j as a conic cone.

Definition 7.6 *Let f be a holomorphic function on $S = S(U, \{G_j\}, \epsilon)$. We say that f is multi-asymptotically developable along $\bigcup_{1\le j\le \ell} Z_j$ on S if there exists a total family F of coefficients of multi-asymptotic expansion along $\bigcup_{1\le j\le \ell} Z_j$ on S that satisfies the following condition.*

For any open proper convex multi-cone $S' = S(U', \{G'_j\}, \epsilon')$ properly contained in S and for any $N = (n_1, \ldots, n_\ell) \in \mathbb{Z}_{\ge 0}^\ell$, there exists a constant $C_{S',N} > 0$ for which we have an estimate

$$\left|f(z) - \mathrm{App}^{<N}(F;z)\right| \le C_{S',N} \prod_{1\le j\le \ell} \mathrm{dist}(z,\, Z_j)^{w_j(N)} \qquad (z \in S'). \tag{7.11}$$

Let us see some typical examples.

Example 7.7 *Let $X = \mathbb{C}^n$ with the coordinates $(z_1, z_2, \ldots, z_n) = (z_1, z_2, z')$ and Z_j $(j = 1, 2)$ submanifolds defined by $\{z_j = 0\}$ (i.e. $I_j = \{j\}$). Let G_j $(j = 1, 2)$ be a proper open sector in $\mathbb{C}$ and $U_R := B_R^1 \times B_R^1 \times B_R^{n-2}$ where*

B_R^k designates an open ball in $\mathbb{C}^k$ with its radius $R > 0$ and center 0. Note that U is a μ-star shape. In this case, the multi-cone $S = S(U_R, \{G_1, G_2\}, \epsilon)$ is nothing but a poly-sector $(G_1 \times G_2 \times \mathbb{C}^{n-2}) \cap U_R$ and we have

$$S_1 = (G_2 \cap B_R^1) \times B_R^{n-2},\ S_2 = (G_1 \cap B_R^1) \times B_R^{n-2},\ S_{\{1,2\}} = B_R^{n-2}.$$

Let F be a total family of coefficients of multi-asymptotic expansion, that is,

$$\begin{aligned} F &= (F_{\{1\}}, F_{\{2\}}, F_{\{1,2\}}) \\ &= \left(\left\{ f_{\{1\},k}(z_2, z') \right\}_{k \geq 0}, \left\{ f_{\{2\},k}(z_1, z') \right\}_{k \geq 0}, \left\{ f_{\{1,2\},\alpha}(z') \right\}_{\alpha \in \mathbb{Z}_{\geq 0}^2} \right) \end{aligned}$$

where $f_{\{1\},k}$ (resp. $f_{\{2\},k}$ and $f_{\{1,2,\}\alpha}$) is a holomorphic function on S_1 (resp. S_2 and $S_{\{1,2\}}$). For $N = (n_1, n_2) \in \mathbb{Z}_{\geq 0}^2$, an asymptotic expansion $\mathrm{App}^{<N}(F; z)$ is given by

$$\begin{aligned} T_{\{1\}}^{<N}(F;\, z) &= \sum_{k<n_1} f_{\{1\},k}(z_2, z') \frac{z_1^k}{k!}, \\ T_{\{2\}}^{<N}(F;\, z) &= \sum_{k<n_2} f_{\{2\},k}(z_1, z') \frac{z_2^k}{k!}, \\ T_{\{1,2\}}^{<N}(F;\, z) &= \sum_{\alpha_1<n_1, \alpha_2<n_2} f_{\{1,2\},\alpha}(z') \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1! \alpha_2!} \\ \mathrm{App}^{<N}(F;\, z) &= T_{\{1\}}^{<N}(F;\, z) + T_{\{2\}}^{<N}(F;\, z) - T_{\{1,2\}}^{<N}(F;\, z). \end{aligned}$$

As $w_j(N) = n_j$ and $\mathrm{dist}(z, Z_j) = |z_j|$ $(j = 1, 2)$, a holomorphic function f is multi-asymptotically developable to F if, for any poly-sector S' properly contained in S and for any $N = (n_1, n_2) \in \mathbb{Z}_{\geq 0}^2$, there exists a positive constant $C_{S',N}$ such that

$$\left| f(z) - \mathrm{App}^{<N}(F;\, z) \right| \leq C_{S',N} |z_1|^{n_1} |z_2|^{n_2} \qquad (z \in S').$$

Hence our definition coincides with that of strongly asymptotic developability established by Majima in [15]. We give some examples of asymptotics:

$$\begin{aligned} &N = (0,0) && |f(z)| \leq C_{S',N}, \\ &N = (1,0) && |f(z) - f_{\{1\},0}(z_2, z')| \leq C_{S',N} |z_1|, \\ &N = (0,1) && |f(z) - f_{\{2\},0}(z_1, z')| \leq C_{S',N} |z_2|, \\ &N = (1,1) && |f(z) - f_{\{1\},0}(z_2, z') - f_{\{2\},0}(z_1, z') + f_{\{1,2\},(0,0)}(z')| \leq C_{S',N} |z_1||z_2|. \end{aligned}$$

Example 7.8 *Let $X = \mathbb{C}^n$ with the coordinates $(z_1, z_2, \dots, z_n) = (z_1, z_2, z')$, and let $Z_1 = \{z_1 = 0\}$ and $Z_2 = \{z_1 = z_2 = 0\}$. In this case, I_1 and I_2 are given by $\{1\}$ and $\{1, 2\}$ respectively, and we have $\hat{I}_1 = \{1\}$, $\hat{I}_2 = \{2\}$. Let G_1 and G_2 be proper open sectors in $\mathbb{C}$. We set $U_R := B_R^2 \times B_R^{n-2}$, which is a μ-star shape. The multi-cone $S = S(U_R, \{G_1, G_2\}, \epsilon)$ is defined by*

$$\{(z_1, z_2, z') \in (G_1 \times G_2 \times \mathbb{C}_{z'}^{n-2}) \cap U;\ |z_1| < \epsilon |z_2|\}.$$

Then we have

$$S_1 = (G_2 \cap B_R^1) \times B_R^{n-2},\ S_2 = S_{\{1,2\}} = B_R^{n-2}.$$

Let F be a total family of coefficients of multi-asymptotic expansion, which consists of

$$\begin{aligned} F &= (F_{\{1\}}, F_{\{2\}}, F_{\{1,2\}}) \\ &= \left(\left\{ f_{\{1\},k}(z_2, z') \right\}_{k \geq 0},\ \left\{ f_{\{2\},\alpha}(z') \right\}_{\alpha \in \mathbb{Z}_{\geq 0}^2},\ \left\{ f_{\{1,2\},\alpha}(z') \right\}_{\alpha \in \mathbb{Z}_{\geq 0}^2} \right) \end{aligned}$$

where $f_{\{1\},k}$ (resp. $f_{\{2\},\alpha}$ and $f_{\{1,2\},\alpha}$) is a holomorphic function on S_1 (resp. $S_2 = S_{\{1,2\}}$). Then we have, for $N = (n_1, n_2) \in \mathbb{Z}_{\geq 0}^2$,

$$\begin{aligned} T_{\{1\}}^{<N}(F;\ z) &= \sum_{k < n_1} f_{\{1\},k}(z_2, z') \frac{z_1^k}{k!}, \\ T_{\{2\}}^{<N}(F;\ z) &= \sum_{\alpha_1 + \alpha_2 < n_2} f_{\{2\},\alpha}(z') \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1! \alpha_2!}, \\ T_{\{1,2\}}^{<N}(F;\ z) &= \sum_{\alpha_1 < n_1,\ \alpha_1 + \alpha_2 < n_2} f_{\{1,2\},\alpha}(z') \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{\alpha_1! \alpha_2!}, \\ \mathrm{App}^{<N}(F;\ z) &= T_{\{1\}}^{<N}(F;\ z) + T_{\{2\}}^{<N}(F;\ z) - T_{\{1,2\}}^{<N}(F;\ z). \end{aligned}$$

It follows from the definition that we have

$$w_1(N) = n_1 \text{ and } w_2(N) = n_2 - n_1$$

and

$$\mathrm{dist}(z, Z_1) = |z_1| \text{ and } \mathrm{dist}(z, Z_2) \simeq |z_1| + |z_2|.$$

Hence a holomorphic function f is multi-asymptotically developable to F if, for any proper convex multi-cone S' properly contained in S and for any $N = (n_1, n_2) \in \mathbb{Z}_{\geq 0}^2$, there exists a positive constant $C_{S',N}$ such that

$$\left| f(z) - \mathrm{App}^{<N}(F;\ z) \right| \leq C_{S',N} |z_1|^{n_1} (|z_1| + |z_2|)^{n_2 - n_1} \qquad (z \in S').$$

We give some examples of asymptotics:

$$
\begin{aligned}
&N=(0,0) \qquad |f(z)| \le C_{S',N},\\
&N=(1,0) \qquad |f(z)-f_{\{1\},0}(z_2,z')| \le C_{S',N}\frac{|z_1|}{|z_1|+|z_2|},\\
&N=(0,1) \qquad |f(z)-f_{\{2\},0}(z')| \le C_{S',N}(|z_1|+|z_2|),\\
&N=(1,1) \qquad |f(z)-f_{\{1\},0}(z_2,z')-f_{\{2\},0}(z')+f_{\{1,2\},(0,0)}(z')| \le C_{S',N}|z_1|.
\end{aligned}
$$

One of important features of multi-asymptoticity is stability for differentiations.

Proposition 7.9 *Let $S := (U, \{G_i\}, \epsilon)$ be a proper convex multi-cone in X and f a holomorphic function on S. If $f(z)$ is multi-asymptotically developable along $\bigcup_{1\le j\le \ell} Z_j$ on S, then any derivative of f is also multi-asymptotically developable along $\bigcup_{1\le j\le \ell} Z_j$ on S.*

Proof. It suffices to show that, for an $i \in \{1, 2, \ldots, n\}$, $\dfrac{\partial f}{\partial z_i}$ is also multi-asymptotically developable. We need the lemma below. Let $F := \{F_J\}_{J\in\hat{\mathcal{P}}(\mathbb{Z}_\ell)}$ be a total family of coefficients of multi-asymptotic expansion with $F_J = \{f_{J,\alpha}\}_{\alpha\in\mathbb{Z}^{I_J}_{\ge 0}}$.

Lemma 7.10 *Let $N = (n_1, n_2, \ldots, n_\ell) \in \mathbb{Z}^\ell_{\ge 0}$. Then we have*

$$\frac{\partial}{\partial z_i}\mathrm{App}^{<N_+}(F;\, z) = \mathrm{App}^{<N}(F';\, z).$$

Here $N_+ = (n'_1, \ldots, n'_\ell)$ is determined by $n'_j = n_j + 1$ for j with $i \in I_j$ and $n'_j = n_j$ otherwise. The total family $F' = \{F'_J\}$ is given by

$$F'_J := \{f'_{J,\alpha}\} = \left\{\frac{\partial f_{J,\alpha}}{\partial z_i}\right\} \qquad (\textit{for } J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell) \textit{ with } i \notin I_J)$$

and

$$F'_J := \{f'_{J,\alpha}\} := \{f_{J,\alpha+e_i}\} \qquad (\textit{for } J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell) \textit{ with } i \in I_J)$$

where $e_i \in \mathbb{Z}^n_{\ge 0}$ is the unit vector whose i-th element is equal to 1.

Proof. Let $J = (j_1, j_2, \ldots, j_m) \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$. For J with $i \notin I_J$, as the polynomial $T^{<N}_J$ does not contain the variable z_i and $n'_j = n_j$ for $j \in J$, we have

$$\frac{\partial}{\partial z_i}T^{<N_+}_J(F;\, z) = \frac{\partial}{\partial z_i}\tau_{F,J}\left(T^{<N_+}_J\right) = \tau_{F',J}\left(T^{<N_+}_J\right) = T^{<N}_J(F';\, z).$$

Assume that J satisfies $i \in I_J$. Then, by the proof of Lemma 7.4, we have

$$\frac{\partial}{\partial z_i} T_J^{<N_+} = T_J^{<N}.$$

As

$$\frac{\partial}{\partial z_i} \tau_{F,J}(p) = \tau_{F',J}\left(\frac{\partial p}{\partial z_i}\right)$$

holds for a polynomial $p(z_{I_J})$ of the variables z_{I_J}, we have obtained

$$\begin{aligned}\frac{\partial}{\partial z_i} T_J^{<N_+}(F;\, z) &= \frac{\partial}{\partial z_i} \tau_{F,J}(T_J^{<N_+}) = \tau_{F',J}\left(\frac{\partial}{\partial z_i} T_J^{<N_+}\right) \\ &= \tau_{F',J}\left(T_J^{<N}\right) = T_J^{<N}(F';\, z).\end{aligned}$$

□

We continue the proof of the proposition. It suffices to consider the case for $i \in I_{\{1,2,\ldots,\ell\}}$. By the lemma, we have

$$\left|\frac{\partial f}{\partial z_i} - \mathrm{App}^{<N}(F';\, z)\right| = \left|\frac{\partial}{\partial z_i}\left(f(z) - \mathrm{App}^{<N_+}(F;\, z)\right)\right|.$$

Let j_0 be a unique integer with $i \in \hat{I}_{j_0}$ and $S' := S(U', \{G'_j\}, \epsilon')$ (resp. $S'' := S(U'', \{G''_j\}, \epsilon'')$) a proper convex multi-cone properly contained in S (resp. S'). Then, as $\mathrm{dist}(z, Z_{j_0}) \leq \epsilon'' \,\mathrm{dist}(z, Z_j)$ $(z \in S'')$ holds for $Z_j \subset Z_{j_0}$, there exists a positive constant $\kappa > 0$ such that for any point $z^* \in S''$

$$z^* + \{z \in \mathbb{C}^n;\, z_k = 0\,(k \neq i),\, |z_i| \leq \kappa \min\{1,\, \mathrm{dist}(z^*, Z_{j_0})\}\} \subset S'. \tag{7.12}$$

Assume that $z \in S''$. Set

$$D = \left\{(\zeta_1, \ldots, \zeta_n);\, \zeta_\alpha = z_\alpha\,(\alpha \neq i),\, |\zeta_i - z_i| = \frac{\kappa}{2}\,\mathrm{dist}(z, Z_{j_0})\right\}.$$

(Here we may assume $\kappa < 1$). D is a circle in the z_i-plane and the other coordinates are fixed. By (7.12), D is contained in S'.

Now remark the following fact. Let $j \in \{1, 2, \ldots, \ell\}$. Then

$$\mathrm{dist}(z, Z_j) \simeq \sum_{i \in \hat{I}_j} |z_i| \tag{7.13}$$

(note that sum is taken over indices in $\hat{I}_j$ and not in I_j) as long as $z \in S'$, which comes from the fact that each cone G_j is proper. Hence, for $j \neq j_0$, we may assume

$$\mathrm{dist}(z, Z_j) = \mathrm{dist}(\zeta, Z_j) \qquad (\zeta \in D)$$

because $\hat{I}_j$ does not contain the index i and D is a circle in the z_i-plane (the other coordinates are fixed). Further, for $j = j_0$, we have

$$\left(1 - \frac{\kappa}{2}\right) \mathrm{dist}(z, Z_j) \leq \mathrm{dist}(\zeta, Z_j) \leq \left(1 + \frac{\kappa}{2}\right) \mathrm{dist}(z, Z_j) \qquad (\zeta \in D)$$

(note that both inequalities are needed in the estimation (7.14) below, which depends on the sign of $w_{j_0}(N)$).

Since $f(z)$ is multi-asymptotically developable, there exists a constant C_{S',N_+} satisfying

$$\left|h(z) := f(z) - \mathrm{App}^{<N_+}(F;\, z)\right| \leq C_{S',N_+} \prod_{1 \leq j \leq \ell} \mathrm{dist}(z,\, Z_j)^{w_j(N_+)} \qquad (z \in S').$$

Therefore we obtain

$$|h(\zeta_1, \ldots, \zeta_i, \ldots, \zeta_n)| \leq C_\kappa \prod_{1 \leq j \leq \ell} \mathrm{dist}(z, Z_j)^{w_j(N+)} \qquad (\zeta \in D) \tag{7.14}$$

for some positive constant C_κ. By the Cauchy integral formula, we have

$$\frac{\partial h}{\partial z_i}(z) = \int_{\partial D} \frac{h(z_1, \ldots, \zeta_i, \ldots, z_n)}{(\zeta_i - z_i)^2} d\zeta_i,$$

where the path of the integration is the projection of D to the z_i-plane. Putting the estimation of $h(z)$ and the above estimations into the formula, we get

$$\left|\frac{\partial h(z)}{\partial z_i}\right| \leq C \frac{\prod_{1 \leq j \leq \ell} \mathrm{dist}(z,\, Z_j)^{w_j(N_+)}}{\mathrm{dist}(z, Z_{j_0})} \qquad (z \in S'')$$

for a constant $C > 0$. As $w_{j_0}(N) = w_{j_0}(N_+) - 1$ and $w_j(N) = w_j(N_+)$ for $j \neq j_0$, we finally obtain

$$\left|\frac{\partial f}{\partial z_i} - \mathrm{App}^{<N}(F';\, z)\right| \leq C \prod_{1 \leq j \leq \ell} \mathrm{dist}(z,\, Z_j)^{w_j(N)} \qquad (z \in S'').$$

This completes the proof. □

By this proposition and integration by parts, we can obtain the following theorem.

Theorem 7.11 *Let $S := S(U, \{G_i\}, \epsilon)$ be a proper convex multi-cone and f a holomorphic function on S. Then the following conditions are equivalent.*

1. *f is multi-asymptotically developable along $\bigcup_{1\leq j\leq \ell} Z_j$ on S.*

2. *For any open proper convex multi-cone $S' := S(U', \{G'_i\}, \epsilon')$ properly contained in S and for any $\alpha \in \mathbb{Z}^n_{\geq 0}$, $\left|\dfrac{\partial^\alpha f}{\partial z_\alpha}\right|$ is bounded on S'.*

3. *For any open proper convex multi-cone $S' := S(U', \{G'_i\}, \epsilon')$ properly contained in S, the holomorphic function $f|_{S'}$ on S' can be extended to a $\mathcal{C}^\infty$-function on $X_{\mathbb{R}}$ ($X_{\mathbb{R}}$ denotes the underlying real analytic manifold of X).*

Proof. We first show 1. implies 2. By (7.11) with $N = (0, \ldots, 0)$, we obtain that f is bounded on S'. Since each higher derivative of f is still multi-asymptotically developable thanks to Proposition 7.9, $\dfrac{\partial^\alpha f}{\partial z^\alpha}$ is also bounded for any $\alpha \in \mathbb{Z}^n_{\geq 0}$ on S'.

As S' is convex and as f is holomorphic (i.e. $\dfrac{\partial^\alpha f}{\partial \bar{z}^\alpha} = 0$), the claim 3. follows from 2. by the result of Whitney in [25]. Clearly 3. implies 2. Hence the claim 2. and 3. are equivalent.

Now we will show 3. implies 1. Assume that f satisfies 3. In particular, any derivative of f extends to $\overline{S'}$ and is bounded on $\overline{S'}$. It follows from Lemma 7.2 that for $z \in S'$ and $0 \leq \lambda_j \leq 1$ $(j = 1, 2, \ldots, \ell)$, we get $\mu(z, \lambda) \in \overline{S'}$. Therefore, for $N = (n_1, \ldots, n_\ell) \in \mathbb{Z}^\ell_{\geq 0}$,

$$\varphi_N(f; z) := \int_0^\infty \cdots \int_0^\infty \frac{\partial^N f(\mu(z, \lambda))}{\partial \lambda^N} \prod_{1\leq j\leq \ell} K_{n_j-1}(1 - \lambda_j) d\lambda_1 \ldots d\lambda_\ell$$

where

$$K_n(t) := \begin{cases} \delta(t) & (n = -1), \\ \dfrac{t_+^n}{n!} & (n \geq 0) \end{cases}$$

is well-defined on S'. Here δ denotes the Dirac delta function and $t_+ = t$ if $t \geq 0$, $t_+ = 0$ if $t < 0$. We define $e_J = (e_{J,1}, \ldots, e_{J,\ell}) \in \mathbb{Z}^\ell$ by $e_{J,j} = 0$ if $j \in J$ and $e_{J,j} = 1$ otherwise. Then, by integration by parts, we have

$$\varphi_N(f; z) = f(z) - \sum_{J\in \hat{\mathcal{P}}(\mathbb{Z}_\ell)} (-1)^{\#J+1} \sum_{\beta\in\mathbb{Z}^J, \beta<_J N} \frac{1}{\beta!} \partial^\beta_{\lambda,J} f(\mu(z, \lambda))\Big|_{\lambda=e_J}.$$

Hence it suffices to show that $\varphi_N(f;z)$ has an estimation which appears in a multi-asymptotic expansion. Let us consider coordinates transformation of λ as $\lambda_j = \nu_j(z)\tilde{\lambda}_j$ $(j = 1, 2, \ldots, \ell)$ where $\nu_j(z)$ is given by

$$\nu_j(z) := \begin{cases} \dfrac{1}{|z_{I_j}|} & (J_{\subsetneq Z_j} = \emptyset), \\ \dfrac{|z_{I_k}|}{|z_{I_j}|} & (k \in J_{\subsetneq Z_j}). \end{cases}$$

Note that $J_{\subsetneq Z_j}$ consists of at most one element. Then we have

$$\frac{\partial^N f(\mu(z,\lambda))}{\partial \lambda^N} = \frac{1}{\prod\limits_{1\le j\le \ell} \nu_j^{n_j}(z)} \frac{\partial^N}{\partial \tilde{\lambda}^N} f\left(\frac{z_1}{|z_{I_{j(1)}}|}\tilde{\lambda}_{J_1}, \ldots, \frac{z_n}{|z_{I_{j(n)}}|}\tilde{\lambda}_{J_n} \right).$$

Here, for $k \in \{1, 2, \ldots, n\}$, we denote by $j(k)$ the integer j that satisfies $k \in \hat{I}_j$ and we set $J_k = \{j \in \mathbb{Z}_\ell;\, k \in I_j\}$. Since $\dfrac{1}{\nu_j(z)}$ is bounded on S', $\tilde{\lambda}_j$ is also bounded when $0 \le \lambda_j \le 1$. Hence we get

$$\sup_{\lambda\in[0,1]^\ell} \left| \frac{\partial^N f(\mu(z,\lambda))}{\partial \lambda^N} \right| \le \frac{C}{\prod\limits_{1\le j\le \ell} \nu_j^{n_j}(z)} \qquad (z \in S')$$

for some constant $C > 0$. By noticing

$$\frac{1}{\prod\limits_{1\le j\le \ell} \nu_j^{n_j}(z)} = \prod_{1\le j\le \ell} \mathrm{dist}(z, Z_j)^{w_j(N)},$$

we have obtained a multi-asymptotic expansion of f with desired estimation. The proof has been completed. □

Let $S := S(U, \{G_i\}, \epsilon)$ be a proper convex multi-cone and f a holomorphic function that is multi-asymptotically developable along $\bigcup\limits_{1\le j\le \ell} Z_j$ on S. Let $S^{(k)} := S(U^{(k)}, \{G_i^{(k)}\}, \epsilon^{(k)})$ $(k = 1, 2, 3, \ldots)$ be a family of proper convex multi-cones properly contained in S, which satisfies

$$S^{(1)} \subset S^{(2)} \subset S^{(3)} \subset \ldots \text{ and } \bigcup_{k\ge 1} S^{(k)} = S.$$

Let $J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$. Then $\left.\dfrac{\partial^\alpha f}{\partial z^\alpha}\right|_{S^{(k)}}$ has a unique Whitney extension $\varphi^{(k)}$ on $\overline{S^{(k)}}$ by the above theorem. As $\bigcup\limits_{k\ge 1} S_J^{(k)} = S_J$, a family $\left\{ \varphi^{(k)}|_{S_J^{(k)}} \right\}$ determines

a function on S_J, which is holomorphic. We denote it by $\left.\dfrac{\partial^\alpha f}{\partial z^\alpha}\right|_{S_J}$. Note that, as S_J is a multi-cone in Z_J for a family of submanifolds $\chi_J := \{Z_j \cap Z_J\}_{j\in J^*}$ where J^* is given by $\{j \in \mathbb{Z}_\ell; Z_J \not\subseteq Z_j\}$, it follows again from the theorem that the holomorphic function $\left.\dfrac{\partial^\alpha f}{\partial z^\alpha}\right|_{S_J}$ is multi-asymptotically developable along $\bigcup_{j\in J^*}(Z_j \cap Z_J)$ on S_J.

Proposition 7.12 *Let $S := S(U, \{G_i\}, \epsilon)$ be a proper convex multi-cone and f a holomorphic function on S. Assume that f is multi-asymptotically developable to a family $F := \{F_J\}_{J\in\hat{\mathcal{P}}(\mathbb{Z}_\ell)}$ of coefficients of multi-asymptotic expansion with $F_J = \{f_{J,\alpha}\}_{\alpha\in\mathbb{Z}^{I_J}_{\geq 0}}$. Then we have*

$$\left.\frac{\partial^\alpha f}{\partial z^\alpha}\right|_{S_J} = f_{J,\alpha} \qquad (J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell),\ \alpha \in \mathbb{Z}^{I_J}_{\geq 0}). \tag{7.15}$$

In particular, coefficients of multi-asymptotic expansion are unique and they are multi-asymptotically developable functions themselves.

Proof. It suffices to show the proposition for a proper convex multi-cone $S' := S(U', \{G'_i\}, \epsilon')$ that is properly contained in S. Moreover, by Proposition 7.9 and Lemma 7.10, it enough to show (7.15) with $\alpha = 0$. We prove the lemma by induction with respect to the number of elements in J. We use the same symbol f for a unique continuous extension of f on $\overline{S'}$ in what follows.

Let $J = \{j\}$. By taking $N = (n_1, n_2, \ldots, n_\ell)$ with $n_j = 1$ and $n_k = 0$ $(k \neq j)$, we have

$$|f(z) - f_{J,0}(z_{I_J})| \leq C\frac{\mathrm{dist}(z, Z_j)}{\delta_j(z)} \tag{7.16}$$

where $\delta_j(z) = 1$ if $J_{\subsetneq Z_j}$ is an empty set and $\delta_j(z) = \mathrm{dist}(z, Z_k)$ if $J_{\subsetneq Z_j} = \{k\}$. Note that the set $J_{\subsetneq Z_j}$ consists of at most one element. For $\tilde{z}^* \in S'_J$, by Lemma 7.2, we can find a point $z^* \in S'$ with $\mu_j(z^*, 0) = \tilde{z}^*$. Then there exists $\delta > 0$ such that $\mathrm{dist}(\mu_j(z^*, [0, 1]), Z_k) > \delta$ for k with $Z_k \subsetneq Z_j$ because of $\tilde{z}^* \notin Z_k$ (2. of Lemma 7.1). Hence, by putting $z = \mu(z^*, \lambda)$ into (7.16) and letting $\lambda \to 0^+$, we get $f(\tilde{z}^*) = f_{J,0}(\tilde{z}^*)$. This shows (7.15) with $\alpha = 0$.

Let $J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$ with $\#J > 1$, and let J^* be a subset of J consisting of $j \in J$ such that Z_j is a minimal submanifold in $\{Z_k\}_{k\in J}$ with respect to the

order $\subset$. Set $N = (n_1, n_2, \ldots, n_\ell)$ with $n_j = 1$ if $j \in J$ and $n_j = 0$ otherwise. Then, by noticing $\mathrm{dist}(z, Z_k) \leq \epsilon' \mathrm{dist}(z, Z_j)$ if $Z_j \subsetneq Z_k$, we have

$$\begin{aligned} &\left| f(z) + \sum_{J' \subsetneq J, J' \neq \emptyset} (-1)^{\#J'} f_{J',0}(z_{I_{J'}}) + (-1)^{\#J} f_{J,0}(z_{I_J}) \right| \\ &\qquad \leq C \prod_{j \in J^*} \frac{\mathrm{dist}(z, Z_j)}{\delta_j(z)} \end{aligned} \tag{7.17}$$

By induction hypothesis, for $J' \subsetneq J$, the coefficient $f_{J',0}$ also has a unique continuous extension on $\overline{S'_{J'}}$ satisfying $f_{J',0} = f|_{\overline{S'_{J'}}}$. Hence, noticing $S'_J \subset \overline{S'_{J'}} \cap Z_J$ due to 2. of Lemma 7.1, we have

$$f(z) + \sum_{J' \subsetneq J, J' \neq \emptyset} (-1)^{\#J'} f_{J',0}(z_{I_{J'}}) = (-1)^{\#J+1} f(z) \qquad z \in S'_J.$$

Let $\tilde{z} \in S'_J$ and z^* a point in S with $\mu_J(0, \ldots, 0; z^*) = \tilde{z}$. Then, by putting $z = \mu_J(z^*, \lambda, \ldots, \lambda)$ into (7.17) and letting $\lambda \to 0^+$, as the right hand side of (7.17) tends to 0 by the same reason as that for $\#J = 1$, we have $f_{J,0}(\tilde{z}) = f(\tilde{z})$. This completes the proof. □

Let us extend the notion of *a consistent family of coefficients of strongly asymptotic expansion* defined by Majima in [15] to our case. Let $S := S(U, \{G_i\}, \epsilon)$ be a proper convex multi-cone and $F := \{F_J\}_{J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)}$ a family of coefficients of multi-asymptotic expansion with $F_J = \{f_{J,\alpha}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{I_J}}$. For a proper convex cone $S' := S(U', \{G'_i\}, \epsilon')$ properly contained in S, we can consider the natural restriction $F|_{S'}$ of F to S'. Moreover we can also define the restriction of F to a submanifold. Let $J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$ and $\alpha \in \mathbb{Z}_{\geq 0}^{I_J}$. Then the restriction $F|_{J,\alpha}$ of F to Z_J is defined as follows.

Let $J^* := \{j \in \mathbb{Z}_\ell; Z_J \nsubseteq Z_j\}$ and $I_j^* = I_j \setminus I_J$ $(j \in J^*)$. We assume $J^* \neq \emptyset$ and denote by S^* the proper convex multi-cone S_J in Z_J. Then for $K \in \mathcal{P}(J^*) \setminus \emptyset$, a family $F_{K,\alpha}^* := \{f_{K,\alpha,\gamma}^*\}_{\gamma \in \mathbb{Z}_{\geq 0}^{I_K^*}}$ of holomorphic functions on $S_K^* \subset Z_J$ is defined by

$$f_{K,\alpha,\gamma}^* = f_{J \cup K,(\alpha,\gamma)}|_{Z_J}$$

where $(\alpha, \gamma) \in \mathbb{Z}_{\geq 0}^{I_J} \times \mathbb{Z}_{\geq 0}^{I_K^*}$ can be considered as an element in $\mathbb{Z}_{\geq 0}^{I_{J \cup K}}$ by the identification $\mathbb{Z}^{I_J} \times \mathbb{Z}^{I_K^*} = \mathbb{Z}^{I_{J \cup K}}$. Then the restriction $F|_{J,\alpha}$ of F is defined by a family $\{F_{K,\alpha}^*\}_{K \in \mathcal{P}(J^*) \setminus \emptyset}$.

The restriction $F|_{J,\alpha}$ gives a family of coefficients of multi-asymptotic expansion along submanifolds $\{Z_j \cap Z_J\}_{j\in J^*}$ in Z_J on the proper convex multi-cone $S^* = S(U \cap Z_J, \{G_j\}_{j\in J^*}, \epsilon) \subset Z_J$.

Definition 7.13 *Let $S := S(U, \{G_i\}, \epsilon)$ be a proper convex multi-cone and $F := \{F_J\}_{J\in\hat{\mathcal{P}}(\mathbb{Z}_\ell)}$ a family of coefficients of multi-asymptotic expansion with $F_J = \{f_{J,\alpha}\}_{\alpha\in\mathbb{Z}_{\geq 0}^{I_J}}$. We say that F is a consistent family of coefficients of multi-asymptotic expansion along $\bigcup_{1\leq j\leq \ell} Z_j$ on S if the following conditions are satisfied.*

1. *For any $J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$ with $J^* := \{j \in \mathbb{Z}_\ell; Z_J \not\subseteq Z_j\} \neq \emptyset$ and for any $\alpha \in \mathbb{Z}_{\geq 0}^{I_J}$, a holomorphic function $f_{J,\alpha}$ on S_J is multi-asymptotically developable to the family $F|_{J,\alpha}$ along $\{Z_j \cap Z_J\}_{j\in J^*}$ on S_J.*

2. *For any J and $J' \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$ with $S_J = S_{J'}$, we have $F_J = F_{J'}$.*

Example 7.14 *Let us see some typical examples of consistent families.*

1. *(Majima) Let us consider the Example 7.7. The consistent families $F = (F_{\{1\}}, F_{\{2\}}, F_{\{1,2\}})$ are those satisfying*

   - $F_{\{1\}}$ is asymptotic to $F_{\{1,2\}}$ when $z_2 \to 0$,

   - $F_{\{2\}}$ is asymptotic to $F_{\{1,2\}}$ when $z_1 \to 0$.

2. *(Takeuchi) Let us consider the Example 7.8. The consistent families $F = (F_{\{1\}}, F_{\{2\}}, F_{\{1,2\}})$ are those satisfying*

   - $F_{\{1\}}$ is asymptotic to $F_{\{1,2\}}$ when $z_2 \to 0$,

   - $F_{\{2\}} = F_{\{1,2\}}$.

   Therefore the case $N = (1, 1)$ in Example 7.8 is equivalent to

$$|f(z) - f_{\{1\},0}(z_2, z')| \leq C_{S',N}|z_1|.$$

The following corollary immediately follows from Theorem 7.11 and Proposition 7.12.

Corollary 7.15 *Let $S := S(U, \{G_i\}, \epsilon)$ be a proper convex multi-cone and F a family of coefficients of multi-asymptotic expansion along $\bigcup_{1\leq j\leq \ell} Z_j$ on S. If some holomorphic function on S is multi-asymptotically developable to F, then F is a consistent family of multi-asymptotic expansion.*

Let $S' := S(U', \{G'_i\}, \epsilon')$ be a proper convex multi-cone properly contained in S, and let $F := \{F_J\}$ be a consistent family on S with $F_J = \{f_{J,\alpha}\}_{\alpha \in \mathbb{Z}^{I_J}_{\geq 0}}$. By Theorem 7.11, each F_J can be regarded as a $\mathcal{C}^\infty$-Whitney jet on $\overline{S'_J}$. (see [16] for Malgrange's definition of a $\mathcal{C}^\infty$-Whitney jet). As $\overline{S'_J} \cap \overline{S'_{J'}} = \overline{S'_{J \cup J'}}$ holds (J, $J' \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$), Whitney jets defined by F_J and $F_{J'}$ coincide on the set $\overline{S'_{J \cup J'}}$ by Proposition 7.12.

Therefore, as $\bigcup_{J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)} \overline{S'_J} = \overline{S'} \cap \left(\bigcup_{1 \leq j \leq \ell} Z_j \right)$ holds by Lemma 7.1, it follows from Theorem 5.5 of [16] that we obtain the Whitney jet defined on $\overline{S'} \cap \left(\bigcup_{1 \leq j \leq \ell} Z_j \right)$ whose restriction to $\overline{S'_J}$ is equal to the one defined by F_J. Hence we have obtained the following proposition.

Proposition 7.16 *Let $S := S(U, \{G_i\}, \epsilon)$ be a proper convex multi-cone and $F := \{F_J\}$ a family of coefficients of multi-asymptotic expansion along $\bigcup_{1 \leq j \leq \ell} Z_j$ on S with $F_J = \{f_{J,\alpha}\}_{\alpha \in \mathbb{Z}^{I_J}_{\geq 0}}$. Then the following conditions are equivalent.*

1. *F is consistent.*
2. *For any proper convex cone $S' := S(U', \{G_i\}, \epsilon')$ properly contained in S, the restriction $F|_{S'}$ of F to S' can be extended to a $\mathcal{C}^\infty$-function on $X_\mathbb{R}$, that is, there exists a $\mathcal{C}^\infty$-function $\varphi(z)$ on $X_\mathbb{R}$ such that for any $J \in \hat{\mathcal{P}}(\mathbb{Z}_\ell)$ and $\alpha \in \mathbb{Z}^{I_J}_{\geq 0}$ we have $f_{J,\alpha} = \left. \dfrac{\partial^\alpha \varphi}{\partial z^\alpha} \right|_{S'_J}$ for $z \in S'_J$.*

8 Multi-specialization and asymptotic expansions

Let X be a real analytic manifold. We consider a slight generalization of the sheaf of Whitney $\mathcal{C}^\infty$-functions of [14]. As usual, given $F \in D^b(\mathbb{C}_X)$ we set $D'F = R\mathcal{H}om(F, \mathbb{C}_X)$. Remember that an open subset U of X is locally cohomologically trivial (l.c.t. for short) if $D'\mathbb{C}_U \simeq \mathbb{C}_{\overline{U}}$.

Definition 8.1 *Let $F \in \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and let $U \in \mathrm{Op}(X_{sa})$. We define the presheaf $\mathcal{C}^{\infty,\mathrm{w}}_{X|F}$ as follows:*

$$U \mapsto \Gamma(X; H^0 D'\mathbb{C}_U \otimes F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X).$$

Let $U, V \in \mathrm{Op}(X_{sa})$, and consider the exact sequence

$$0 \to \mathbb{C}_{U\cap V} \to \mathbb{C}_U \oplus \mathbb{C}_V \to \mathbb{C}_{U\cup V} \to 0,$$

applying the functor $\mathcal{H}om(\cdot, \mathbb{C}_X) = H^0 D'(\cdot)$ we obtain

$$0 \to H^0 D'\mathbb{C}_{U\cup V} \to H^0 D'\mathbb{C}_U \oplus H^0 D'\mathbb{C}_V \to H^0 D'\mathbb{C}_{U\cap V},$$

applying the exact functors $\cdot\otimes F$, $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X$ and taking global sections we obtain

$$0 \to \mathcal{C}^{\infty,\mathrm{w}}_{X|F}(U \cup V) \to \mathcal{C}^{\infty,\mathrm{w}}_{X|F}(U) \oplus \mathcal{C}^{\infty,\mathrm{w}}_{X|F}(V) \to \mathcal{C}^{\infty,\mathrm{w}}_{X|F}(U \cap V).$$

This implies that $\mathcal{C}^{\infty,\mathrm{w}}_{X|F}$ is a sheaf on X_{sa}. Moreover if $U \in \mathrm{Op}(X_{sa})$ is l.c.t., the morphism $\Gamma(X; \mathcal{C}^{\infty,\mathrm{w}}_{X|F}) \to \Gamma(U; \mathcal{C}^{\infty,\mathrm{w}}_{X|F})$ is surjective and $R\Gamma(U; \mathcal{C}^{\infty,\mathrm{w}}_{X|F})$ is concentrated in degree zero. Let $0 \to F \to G \to H \to 0$ be an exact sequence in $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, we obtain an exact sequence in $\mathrm{Mod}(\mathbb{C}_{X_{sa}})$

$$0 \to \mathcal{C}^{\infty,\mathrm{w}}_{X|F} \to \mathcal{C}^{\infty,\mathrm{w}}_{X|G} \to \mathcal{C}^{\infty,\mathrm{w}}_{X|H} \to 0. \tag{8.1}$$

We can easily extend the sheaf $\mathcal{C}^{\infty,\mathrm{w}}_{X|F}$ to the case of $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, taking a finite resolution of F consisting of locally finite sums $\oplus\mathbb{C}_V$ with V l.c.t. in $\mathrm{Op}^c(X_{sa})$. In fact, the sheaves $\mathcal{C}^{\infty,\mathrm{w}}_{X|\oplus\mathbb{C}_V}$ form a complex quasi-isomorphic to $\mathcal{C}^{\infty,\mathrm{w}}_{X|F}$ consisting of acyclic objects with respect to $\Gamma(U;\cdot)$, where U is l.c.t. in $\mathrm{Op}^c(X_{sa})$.

As in the case of Whitney $\mathcal{C}^\infty$-functions one can prove that, if $G \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ one has

$$\rho^{-1} R\mathcal{H}om(G, \mathcal{C}^{\infty,\mathrm{w}}_{F|X}) \simeq D'G \otimes F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X.$$

Example 8.2 *Setting $F = \mathbb{C}_X$ we obtain the sheaf of Whitney $\mathcal{C}^\infty$-functions. Let Z be a closed subanalytic subset of X. Then $\mathcal{C}^{\infty,\mathrm{w}}_{X|\mathbb{C}_{X\setminus Z}}$ is the sheaf of Whitney $\mathcal{C}^\infty$-functions vanishing on Z with all their derivatives.*

Notations 8.3 *Let S be a locally closed subanalytic subset of X. We set for short $\mathcal{C}^{\infty,\mathrm{w}}_{X|S}$ instead of $\mathcal{C}^{\infty,\mathrm{w}}_{X|\mathbb{C}_S}$.*

Let $\chi = \{M_1, \dots, M_\ell\}$ be a family of closed analytic submanifolds of X satisfying H1, H2 and H3, set $M = \bigcup_{i=1}^\ell M_i$ and consider $\widetilde{X}$. Consider the diagram (1.3).

Set $F = \mathbb{C}_{X\setminus M}$, $G = \mathbb{C}_X$, $H = \mathbb{C}_M$ in (8.1). The exact sequence

$$0 \to \mathcal{C}^{\infty,\mathrm{w}}_{X|X\setminus M} \to \mathcal{C}^{\infty,\mathrm{w}}_X \to \mathcal{C}^{\infty,\mathrm{w}}_{X|M} \to 0$$

induces an exact sequence

$$0 \to \nu^{sa}_{\chi}\mathcal{C}^{\infty,\mathrm{w}}_{X|X\setminus M} \to \nu^{sa}_{\chi}\mathcal{C}^{\infty,\mathrm{w}}_{X} \to \nu^{sa}_{\chi}\mathcal{C}^{\infty,\mathrm{w}}_{X|M} \to 0, \tag{8.2}$$

in fact let V be a l.c.t. conic subanalytic subset of the zero section of $\widetilde{X}$ and $U \in \mathrm{Op}(X_{sa})$ such that $C_{\chi}(X \setminus U) \cap V = \emptyset$, then we can find a l.c.t. $U' \subset U$ satisfying the same property. Applying the functor ρ^{-1} to the exact sequence (8.2) we obtain the exact sequence

$$0 \to \rho^{-1}\nu^{sa}_{\chi}\mathcal{C}^{\infty,\mathrm{w}}_{X|X\setminus M} \to \rho^{-1}\nu^{sa}_{\chi}\mathcal{C}^{\infty,\mathrm{w}}_{X} \to \rho^{-1}\nu^{sa}_{\chi}\mathcal{C}^{\infty,\mathrm{w}}_{X|M} \to 0,$$

where the surjective arrow is the map which associates to a function its asymptotic expansion.

Let X be a complex manifold and let $X_{\mathbb{R}}$ denote the underlying real analytic manifold of X. Let $\chi = \{Z_1, \dots, Z_\ell\}$ be a family of complex submanifolds of X satisfying H1, H2 and H3 and set $Z = \bigcup_{i=1}^{\ell} Z_i$. Let $F \in D^b_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. We denote by $\mathcal{O}^{\mathrm{w}}_{X|F}$ the sheaf defined as follows:

$$\mathcal{O}^{\mathrm{w}}_{X|F} := R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\mathrm{w}}_{X_{\mathbb{R}}|F}).$$

Let $0 \to F \to G \to H \to 0$ be an exact sequence in $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. Then the exact sequence (8.1) gives rise to the distiguished triangle

$$\mathcal{O}^{\mathrm{w}}_{X|F} \to \mathcal{O}^{\mathrm{w}}_{X|G} \to \mathcal{O}^{\mathrm{w}}_{X|H} \overset{+}{\to}. \tag{8.3}$$

Setting $F = \mathbb{C}_{X\setminus Z}$, $G = \mathbb{C}_X$, $H = \mathbb{C}_Z$ in (8.3) and applying the functor of specialization, we have the distinguished triangle

$$\rho^{-1}\nu^{sa}_{\chi}\mathcal{O}^{\mathrm{w}}_{X|X\setminus Z} \to \rho^{-1}\nu^{sa}_{\chi}\mathcal{O}^{\mathrm{w}}_{X} \to \rho^{-1}\nu^{sa}_{\chi}\mathcal{O}^{\mathrm{w}}_{X|Z} \overset{+}{\to}. \tag{8.4}$$

The sheaf $\rho^{-1}\nu^{sa}_{\chi}\mathcal{O}^{\mathrm{w}}_{X}$ is concentrated in degree zero. This follows from the following result of [4, 5]: if $U \in \mathrm{Op}(X_{sa})$ is convex, then $R\Gamma(X; \mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)$ is concentrated in degree zero.

Proposition 8.4 *Outside the zero section of* $\underset{X,1\le k\le \ell}{\times} T_{Z_k}\iota(Z_k)$*, the flat specialization along* $Z := Z_1 \cup \cdots \cup Z_\ell$

$$\rho^{-1}\nu^{sa}_{\chi}\mathcal{O}^{\mathrm{w}}_{X|X\setminus Z}$$

is concentrated in degree zero.

Proof. Set

$$I_j = \{i_{j,1}, \ldots, i_{j,m_j}\} \quad (j = 1, 2, \ldots, \ell).$$

Moreover, by a rotation of the coordinates, we may assume that

$$\hat{I}_j = \{i_{j,1}, \ldots, i_{j,m'_j}\} \quad (m'_j \leq m_j).$$

We can identify $\mathbb{C}^{\hat{I}_j}$ with $(T_{Z_j}\iota(Z_j))_0$. Let $p_j = (1, 0, \ldots, 0) \in (T_{Z_j}\iota(Z_j))_0$. Under the above identification, p_j is the point in $\mathbb{C}^{\hat{I}_j}$ with $z_{i_{j,1}} = 1$ and $z_{i_{j,k}} = 0$ $(1 < k \leq m'_j)$. Let G_j be a closed proper convex cone in $\mathbb{C}^{\hat{I}_j}$ with direction p_j, which is defined by

$$\{(z_{j,1}, \ldots, z_{j,m'_j}) \in \mathbb{C}^{\hat{I}_j};\ |z_{j,1}| \leq \epsilon,\ |\arg(z_{j,1})| \leq \epsilon,\ |z_{j,k}| \leq \epsilon |z_{j,1}|,\ k \geq 2\}$$

for $0 < \epsilon < \frac{\pi}{2}$. Let us define G as the product

$$G := G_1 \times G_2 \times \cdots \times G_\ell \times B_\epsilon,$$

where B_ϵ is a closed ball of radius ϵ and center at the origin in $X' := \mathbb{C}^{n-\sum_{j=1}^{\ell} m'_j}$. Set

$$\begin{aligned} T &:= \{z \in \mathbb{C}^n;\ z_{i_{1,1}} = 0\} \cup \cdots \cup \{z \in \mathbb{C}^n;\ z_{i_{\ell,1}} = 0\} \\ &= \{z \in \mathbb{C}^n;\ z_{i_{1,1}} \cdots z_{i_{\ell,1}} = 0\}. \end{aligned}$$

As

$$G_j \cap \{(z_{i_{j,1}}, \ldots, z_{i_{j,m'_j}}) \in \mathbb{C}^{\hat{I}_j};\ z_{i_{j,1}} = 0\} = \{0\}$$

holds, we have

$$G \setminus T = (G_1 \setminus \{0\}) \times \cdots \times (G_\ell \setminus \{0\}) \times B_\epsilon.$$

Therefore, by Proposition 5.4 of [12] $R\Gamma(X, \mathbb{C}_{G\setminus T} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)$ is quasi-isomorphic to

$$R\Gamma(\mathbb{C}^{\hat{I}_1}; \mathbb{C}_{G_1\setminus\{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}) \widehat{\boxtimes} \cdots \widehat{\boxtimes} R\Gamma(\mathbb{C}^{\hat{I}_\ell}; \mathbb{C}_{G_\ell\setminus\{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}) \widehat{\boxtimes} R\Gamma(X'; \mathbb{C}_{B_\epsilon} \overset{\mathrm{w}}{\otimes} \mathcal{O}),$$

where $\widehat{\boxtimes}$ denotes the topological tensor product of [7].

Each $R\Gamma(\mathbb{C}^{\hat{I}_j}; \mathbb{C}_{G_j\setminus\{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O})$, $j = 1, \ldots, \ell$, is concentrated in degree zero by the following lemma whose proof will be given later.

Lemma 8.5 (Propostion 6.1.1 [13]) *For the closed set*

$$G := \{(z_1, \ldots, z_n) \in \mathbb{C}^n;\ |z_1| \leq \epsilon,\ |\arg(z_1)| \leq \epsilon,\ |z_k| \leq \epsilon |z_1|,\ k \geq 2\},$$

we have $H^k(\mathbb{C}^n; \mathbb{C}_{G\setminus\{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{C}^n}) = 0$ *for* $k \neq 0$.

Hence we have obtained

$$H^k(X, \mathbb{C}_{G\setminus T} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) = 0, \qquad (k \neq 0).$$

Let us consider the the holomorphic map $f : X \to X$

$$f(z_1, \ldots, z_n) = (f_1(z), \ldots, f_n(z)) := \left(\left(\prod_{j \in \hat{J}_1} z_{i_{j,1}} \right) z_1, \ldots, \left(\prod_{j \in \hat{J}_n} z_{i_{j,1}} \right) z_n \right),$$

where, for $k \in \{1, 2, \ldots, n\}$,

$$\hat{J}_k := \{j \in \{1, 2, \ldots, \ell\};\ k \in I_j,\ k \notin \hat{I}_j\}.$$

Clearly $f|_G$ is a proper map (G is compact). We can also prove that f induces an isomorphism on $X \setminus T$. Moreover we have

$$p \in G \cap T \Longleftrightarrow f(p) \in f(G) \cap (Z_1 \cup \cdots \cup Z_\ell).$$

Therefore we get

$$Rf_! \mathbb{C}_{G\setminus T} = \mathbb{C}_{f(G)\setminus (Z_1 \cup \cdots \cup Z_\ell)}.$$

It follows from Theorem 5.7 of [12] that we have

$$R\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X \overset{f}{\to} X}, \mathbb{C}_{G\setminus T} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \simeq R\Gamma(X; \mathbb{C}_{f(G)\setminus (Z_1 \cup \cdots \cup Z_\ell)} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X).$$

Set $\varphi := z_{i_{1,1}} z_{i_{2,1}} \cdots z_{i_{\ell,1}}$. Let $\mathcal{O}_{X,\varphi}$ denote the sheaf of meromorphic functions whose poles are contained in $\{\varphi = 0\}$ (i.e. $\mathcal{O}_{X,\varphi} = \mathcal{O}_X[z_{i_{1,1}}^{-1}, \ldots, z_{i_{\ell,1}}^{-1}]$) and we set $\mathcal{D}_{X,\varphi} := \mathcal{O}_{X,\varphi} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$. As $\mathbb{C}_{G\setminus T} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ is a $\mathcal{D}_{X,\varphi}$-module and $\mathcal{D}_{X \overset{f}{\to} X}$ is left quasi-coherent over $\mathcal{D}_X$, we have

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_{X \overset{f}{\to} X}, \mathbb{C}_{G\setminus T} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_{X,\varphi}}(\mathcal{D}_{X,\varphi} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X \overset{f}{\to} X}, \mathbb{C}_{G\setminus T} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X).$$

We also have

$$\mathcal{D}_{X,\varphi} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X \overset{f}{\to} X} \simeq \mathcal{D}_{X,\varphi}.$$

As a matter of fact, $\mathcal{D}_{X \overset{f}{\to} X}$ is given by

$$\mathcal{D}_{X \overset{f}{\to} X} = \frac{\mathcal{D}_{X\times X}}{\mathcal{D}_{X\times X}(w_1 - f_1(z), \ldots, w_n - f_n(z), \theta_1, \ldots, \theta_n)}$$

where $(w_1, \dots, w_n)$ is a system of coordinates of the second X and the vector field θ_k $(k = 1, 2, \dots, n)$ is defined by

$$\theta_k := \frac{\partial}{\partial z_k} + \sum_{i=1}^{n} \frac{\partial f_i}{\partial z_k} \frac{\partial}{\partial w_i}.$$

We denote by J_f the Jacobian of f. Since f gives an isomorphism outside T, we have $J_f \neq 0$ outside T, which implies $J_f = z_{i_{1,1}}^{\beta_1} \dots z_{i_{\ell,1}}^{\beta_l} h$ for some $\beta_j \geq 0$ $j = 1, \dots, \ell$ and for some h with $h(0) \neq 0$. Hence, as J_f has an inverse in $\mathcal{O}_{X,\varphi}$, we conclude that $\mathcal{D}_{X,\varphi} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X \xrightarrow{f} X}$ is a free $\mathcal{D}_{X,\varphi}$ module of rank 1.

Hence we have obtained

$$R\Gamma(X; \mathbb{C}_{G \setminus T} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X) \simeq R\Gamma(X; \mathbb{C}_{f(G) \setminus (Z_1 \cup \dots \cup Z_\ell)} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X).$$

This implies that $R\Gamma(X; \mathbb{C}_{f(G) \setminus (Z_1 \cup \dots \cup Z_\ell)} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)$ is also concentrated in degree zero. Since $f(G)$ contains $\overline{S}$ for some multi-cone S, and since , for a given multi-cone S, there exists a G such that $\overline{S}$ contains $f(G)$, we have the required result. □

To complete the proof of Proposition 8.4, we give the proof of Lemma 8.5.

Proof of Lemma 8.5. Let us consider the holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$f(z_1, z_2, \dots, z_n) = (z_1, z_1 z_2, \dots, z_1 z_n).$$

Then, by Proposition 5.4 and Theorem 5.7 of [12] and applying the same argument as that in the proof of Propostion 8.4 to f, we have

$$R\Gamma(\mathbb{C}^n; \mathbb{C}_{G \setminus \{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{C}^n}) \simeq R\Gamma(\mathbb{C}; \mathbb{C}_{(K \cap B_\epsilon^1) \setminus \{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{C}}) \widehat{\boxtimes} R\Gamma(\mathbb{C}^{n-1}; \mathbb{C}_{B_\epsilon^{n-1}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{C}^{n-1}})$$

where K is a closed conic cone defined by $\{z \in \mathbb{C}; |\arg(z)| \leq \epsilon\}$ and B_ϵ^k is a closed ball in $\mathbb{C}^k$ of radius ϵ and center at the origin. Therefore it suffices to show

$$H^1(\mathbb{C}; \mathbb{C}_{(K \cap B_\epsilon^1) \setminus \{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{C}}) = 0.$$

We have the commutative diagram

$$\begin{array}{ccccc} \Gamma(\mathbb{P}^1; \mathbb{C}_{\{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^1}) & \to & H^1(\mathbb{P}^1; \mathbb{C}_{\overline{K} \setminus \{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^1}) & & \\ \| & & \downarrow & & \\ \Gamma(\mathbb{C}; \mathbb{C}_{\{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{C}}) & \to & H^1(\mathbb{C}; \mathbb{C}_{(K \cap B_\epsilon^1) \setminus \{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{C}}) & \to & 0 \end{array}$$

Here the second row is exact because of $H^1(\mathbb{C};\, \mathbb{C}_{(K\cap B^1_\epsilon)} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{C}}) = 0$ since $K \cap B^1_\epsilon$ is convex (see [4, 5]) and $\overline{K}$ denotes the closure of K in $\mathbb{P}^1$. By applying Propostion 6.1.1 of [13] to the case $V = \mathbb{C}$ and employing the coordinate transformation which exchanges the origin and the infinity, we get $H^1(\mathbb{P}^1;\, \mathbb{C}_{\overline{K}\setminus\{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^1}) = 0$. Then $H^1(\mathbb{C};\, \mathbb{C}_{(K\cap B^1_\epsilon)\setminus\{0\}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{C}}) = 0$ follows from the exactness of the second row. □

Proposition 8.6 *The distinguished triangle* (8.4) *induces an exact sequence outside the zero section of* $\underset{X,1\le k\le \ell}{\times} T_{Z_k}\iota(Z_k)$

$$0 \to \rho^{-1}H^0\nu^{sa}_\chi \mathcal{O}^{\mathrm{w}}_{X|X\setminus Z} \to \rho^{-1}H^0\nu^{sa}_\chi \mathcal{O}^{\mathrm{w}}_X \to \rho^{-1}H^0\nu^{sa}_\chi \mathcal{O}^{\mathrm{w}}_{X|Z} \to 0. \tag{8.5}$$

All the complexes $\rho^{-1}\nu^{sa}_\chi \mathcal{O}^{\mathrm{w}}_{X|X\setminus Z}$, $\rho^{-1}\nu^{sa}_\chi \mathcal{O}^{\mathrm{w}}_X$ *and* $\rho^{-1}\nu^{sa}_\chi \mathcal{O}^{\mathrm{w}}_{X|Z}$ *are concentrated in degree zero outside the zero section.*

Proof. The exactness of the sequence (8.5) follows from Proposition 8.4 and the fact that outside the zero section $\rho^{-1}\nu^{sa}_\chi \mathcal{O}^{\mathrm{w}}_X$ is concentrated in degree zero. This also proves the vanishing of the cohomology of degree ≥ 1 of the three terms. □

We give a functorial construction of multi-asymptotically developable functions using Whitney holomorphic functions. Let U be an open l.c.t. subanalytic subset in X. Then $\Gamma(U; \mathcal{C}^{\infty,\mathrm{w}}_{X_\mathbb{R}}) \simeq \Gamma(X; \mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} C^\infty_{X_\mathbb{R}})$ is nothing but the set of $\mathcal{C}^\infty$-Whitney jets on $\overline{U}$. Further $H^0(U; \mathcal{O}^{\mathrm{w}}_X) \simeq H^0(X; \mathbb{C}_{\overline{U}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)$ consists of $\mathcal{C}^\infty$-Whitney jets on $\overline{U}$ that satisfy the Cauchy-Riemann system. Therefore, if a proper convex multi-cone $S := S(U, \{G_j\}, \epsilon)$ is subanalytic, the set of holomorphic functions on S that are multi-asymptotically developable along Z is equal to

$$\varinjlim_{S'} \Gamma(X; \mathbb{C}_{\overline{S'}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)$$

where S' runs through the family of open subanalytic proper convex multi-cones $S(U', \{G'_j\}, \epsilon')$ properly contained in S. Moreover, by Proposition 7.16, the set of consistent families of coefficients of multi-asymptotic expansion is given by

$$\varinjlim_{S'} \Gamma(X; \mathbb{C}_{\overline{S'}\cap Z} \overset{\mathrm{w}}{\otimes} \mathcal{O}_X).$$

Let W be an open convex subset in $Z_1 \cap \cdots \cap Z_\ell$ and G_j an open proper convex cone in $\mathbb{C}^{\hat{I}_j}$ $(j = 1, 2, \ldots, \ell)$. Using these W and $\{G_j\}$, we define an $(\mathbb{R}^+)^\ell$-conic open subset $V(W, \{G_j\})$ in $\underset{X, 1 \leq j \leq \ell}{\times} \overset{\circ}{T}_{Z_j} \iota_\chi(Z_j)$ by

$$W \times G_1 \times \cdots \times G_\ell \subset \left(\bigcap Z_j\right) \times \mathbb{C}^{\hat{I}_1} \times \ldots \mathbb{C}^{\hat{I}_\ell} \simeq \underset{X, 1 \leq j \leq \ell}{\times} T_{Z_j} \iota_\chi(Z_j)$$

Note that a family of open sets of type $V(W, \{G_j\})$ forms a basis of topology for $(\mathbb{R}^+)^\ell$-conic sheaves on $\underset{X, 1 \leq j \leq \ell}{\times} \overset{\circ}{T}_{Z_j} \iota(Z_j)$. We define the presheaves $\widetilde{\mathcal{A}}_\chi^{<0}$, $\widetilde{\mathcal{A}}_\chi$ and $\widetilde{\mathcal{A}}_\chi^{CF}$ on $\underset{X, 1 \leq j \leq \ell}{\times} \overset{\circ}{T}_{Z_j} \iota(Z_j)$. For an $(\mathbb{R}^+)^\ell$-conic open set $V := V(W, \{G_j\})$, we set

$$\begin{aligned}
\Gamma(V; \widetilde{\mathcal{A}}_\chi^{<0}) = \varinjlim_{U,\, \epsilon > 0} \{ & f \in \mathcal{O}(S(U, \{G_j\}, \epsilon)); \ f \text{ is multi-asymptotically} \\
& \text{developable to zero along } Z \text{ on } S(U, \{G_j\}, \epsilon)\},
\end{aligned}$$

$$\begin{aligned}
\Gamma(V; \widetilde{\mathcal{A}}_\chi) = \varinjlim_{U,\, \epsilon > 0} \{ & f \in \mathcal{O}(S(U, \{G_j\}, \epsilon)); \ f \text{ is multi-asymptotically} \\
& \text{developable along } Z \text{ on } S(U, \{G_j\}, \epsilon)\},
\end{aligned}$$

$$\begin{aligned}
\Gamma(V; \widetilde{\mathcal{A}}_\chi^{CF}) = \varinjlim_{U,\, \epsilon > 0} \{ & F; \ F \text{ is a consistent family of coefficients of} \\
& \text{multi-asymptotic expansion along } Z \text{ on } S(U, \{G_j\}, \epsilon)\},
\end{aligned}$$

where U ranges through the family open convex neighborhoods of W which are a μ-star shape. Let us consider the multi-specialization of Whitney holomorphic functions. We have

$$\Gamma(V; \rho^{-1} \nu_\chi^{sa} \mathcal{O}^{\mathrm{w}}_{X|X\setminus Z}) = \varprojlim_{V'} \varinjlim_{U'} \Gamma(U'; \mathcal{O}^{\mathrm{w}}_{X|X\setminus Z}),$$

$$\Gamma(V; \rho^{-1} \nu_\chi^{sa} \mathcal{O}^{\mathrm{w}}_{X}) = \varprojlim_{V'} \varinjlim_{U'} \Gamma(U'; \mathcal{O}^{\mathrm{w}}_{X}),$$

$$\Gamma(V; \rho^{-1} \nu_\chi^{sa} \mathcal{O}^{\mathrm{w}}_{X|Z}) = \varprojlim_{V'} \varinjlim_{U'} \Gamma(U'; \mathcal{O}^{\mathrm{w}}_{X|Z}),$$

where V' ranges through the family of subanalytic open cones such that $\overline{V'} \setminus Z \subset V$, U' ranges through the family of $\mathrm{Op}(X_{sa})$ such that $C_\chi(X \setminus U') \cap V' = \emptyset$.

The identity morphism induces morphisms of presheaves

$$\begin{aligned}\widetilde{\mathcal{A}}_\chi^{<0} &\to \rho^{-1}\nu_\chi^{sa}\mathcal{O}^{\mathrm{w}}_{X|X\setminus Z},\\ \widetilde{\mathcal{A}}_\chi &\to \rho^{-1}\nu_\chi^{sa}\mathcal{O}^{\mathrm{w}}_{X},\\ \widetilde{\mathcal{A}}_\chi^{CF} &\to \rho^{-1}\nu_\chi^{sa}\mathcal{O}^{\mathrm{w}}_{X|Z}.\end{aligned}$$

The above morphisms are isomorphisms in the stalks (i.e. in the limit of multi-cones containing a given direction $p \in \underset{X,1\le j\le \ell}{\times} \overset{\circ}{T}_{Z_j}\iota_\chi(Z_j)$). Let $\mathcal{A}_\chi^{<0}$ (resp. $\mathcal{A}_\chi$, resp. $\mathcal{A}_\chi^{CF}$) be the sheaves associated to $\widetilde{\mathcal{A}}_\chi^{<0}$ (resp. $\widetilde{\mathcal{A}}_\chi$, resp. $\widetilde{\mathcal{A}}_\chi^{CF}$). We get

$$\begin{aligned}\mathcal{A}_\chi^{<0} &\xrightarrow{\sim} \rho^{-1}\nu_\chi^{sa}\mathcal{O}^{\mathrm{w}}_{X|X\setminus Z},\\ \mathcal{A}_\chi &\xrightarrow{\sim} \rho^{-1}\nu_\chi^{sa}\mathcal{O}^{\mathrm{w}}_{X},\\ \mathcal{A}_\chi^{CF} &\xrightarrow{\sim} \rho^{-1}\nu_\chi^{sa}\mathcal{O}^{\mathrm{w}}_{X|Z}.\end{aligned}$$

By Proposition 8.6 outside the zero section we have the exact sequence of sheaves

$$0 \to \mathcal{A}_\chi^{<0} \to \mathcal{A}_\chi \to \mathcal{A}_\chi^{CF} \to 0. \tag{8.6}$$

When Z forms a normal crossing divisor, these sheaves are nothing but sheaves of strongly asymptotically developable functions defined by Majima in [15].

Remark 8.7 *The exact sequence* (8.6) *is nothing but a Borel-Ritt exact sequence for multi-asymptotically developable functions. This was already proven for formal specialization in the single divisor case in [2] and for Majima's asymptotic in [6, 8]. Both results are based on the Borel-Ritt theorem in dimension one (for a proof, see [24]). Thanks to Proposition 8.4 we obtained a purely cohomological proof of the Borel-Ritt exact sequence.*

Appendix

A Conic sheaves

Let k be a field. Let X be a real analytic manifold endowed with a subanalytic action μ of $\mathbb{R}^+$. In other words we have a subanalytic map

$$\mu : X \times \mathbb{R}^+ \to X,$$

which satisfies, for each $t_1, t_2 \in \mathbb{R}^+$:

$$\begin{cases} \mu(x, t_1 t_2) = \mu(\mu(x, t_1), t_2), \\ \mu(x, 1) = x. \end{cases}$$

Note that μ is open, in fact let $U \in \mathrm{Op}(X)$ and $(t_1, t_2) \in \mathrm{Op}(\mathbb{R}^+)$. Then $\mu(U, (t_1, t_2)) = \bigcup_{t \in (t_1, t_2)} \mu(U, t)$, and $\mu(\cdot, t) : X \to X$ is a homeomorphism (with inverse $\mu(\cdot, t^{-1})$). We have a diagram

$$X \xrightarrow{j} X \times \mathbb{R}^+ \underset{p}{\overset{\mu}{\rightrightarrows}} X,$$

where $j(x) = (x, 1)$ and p denotes the projection. We have $\mu \circ j = p \circ j = \mathrm{id}$.

Definition A.1 *(i) Let S be a subset of X. We set $\mathbb{R}^+ S = \mu(S, \mathbb{R}^+)$. If $U \in \mathrm{Op}(X)$, then $\mathbb{R}^+ U \in \mathrm{Op}(X)$ since μ is open.*

(ii) Let S be a subset of X. We say that S is conic if $S = \mathbb{R}^+ S$. In other words, S is invariant by the action of μ.

(iii) An orbit of μ is the set $\mathbb{R}^+ x$ with $x \in X$.

We assume that the orbits of μ are contractible. For each $x \in X$ there are two possibilities: either $\mathbb{R}^+ x = x$ or $\mathbb{R}^+ x \simeq \mathbb{R}$.

Definition A.2 *We say that a subset S of X is $\mathbb{R}^+$-connected if $S \cap \mathbb{R}^+ x$ is connected for each $x \in S$.*

Lemma A.3 *Let $S_1, S_2 \subset X$ and suppose that S_2 is conic. Then $\mathbb{R}^+(S_1 \cap S_2) = \mathbb{R}^+ S_1 \cap S_2$. (ii) If S_1 and S_2 are $\mathbb{R}^+$-connected, then $S_1 \cap S_2$ is $\mathbb{R}^+$-connected.*

Proof. (i) The inclusion $\subseteq$ is true since $\mathbb{R}^+(S_1 \cap S_2) \subset \mathbb{R}^+ S_i$, $i = 1, 2$. Let us prove $\supseteq$. Let $x \in \mathbb{R}^+ S_1 \cap S_2$. Then there exists $a \in \mathbb{R}^+$ such that $\mu(x, a) \in S_1$. Since S_2 is conic $\mu(x, a) \in S_2$ and the result follows.

(ii) Let $x_1, x_2 \in S_1 \cap S_2$. Then $x_i = \mu(x, a_i)$ for some $x \in X$ and some $a_i \in \mathbb{R}^+$, $i = 1, 2$. Suppose $a_1 \leq a_2$. Since S_1, S_2 are $\mathbb{R}^+$-connected, $i = 1, 2$ and the orbits of μ are either points or isomorphic to $\mathbb{R}$, we have $\mu(x, [a_1, a_2]) \subseteq S_i$, $i = 1, 2$ and the result follows. □

Let X, Y topological spaces endowed with an action (μ_X and μ_Y respectively) of $\mathbb{R}^+$.

Definition A.4 *A continuous function $f : X \to Y$ is said to be conic if for each $x \in X$, $a \in \mathbb{R}^+$ we have $f(\mu_X(x, a)) = \mu_Y(f(x), a)$.*

Lemma A.5 *Let $f : X \to Y$ be a conic map. (i) Suppose that $S \subset Y$ is $\mathbb{R}^+$-connected (resp. conic). Then $f^{-1}(S)$ is $\mathbb{R}^+$-connected (resp. conic). (ii) Suppose that $Z \subset X$ is conic. Then $f(Z)$ is conic.*

Proof. (i) Let $x_1, x_2 \in f^{-1}(S)$ and suppose that there exists $x \in X$ such that $x_1, x_2 \in \mu_X(x, \mathbb{R}^+)$, i.e. $x_i = \mu_X(x, a_i)$, $a_i \in \mathbb{R}^+$, $i = 1, 2$. Since f is conic we have $f(x_i) = \mu_Y(f(x), a_i) \in \mu_Y(f(x), \mathbb{R}^+) \cap S = f(\mu(x, \mathbb{R}^+) \cap f^{-1}(S))$. Suppose $a_1 \leq a_2$. Since S is $\mathbb{R}^+$-connected $\mu(f(x), [a_1, a_2]) \subseteq S$, hence $\mu(x, [a_1, a_2]) \subseteq f^{-1}(f(\mu(x, [a_1, a_2]))) = f^{-1}(\mu(f(x), [a_1, a_2])) \subseteq f^{-1}(S)$.

If S is conic then $f(\mu_X(f^{-1}(S), \mathbb{R}^+)) = \mu_Y(f(f^{-1}(S)), \mathbb{R}^+) = \mu_Y(S, \mathbb{R}^+) = S$, hence $\mu(f^{-1}(S), \mathbb{R}^+) = f^{-1}(S)$.

(ii) We have $\mu_Y(f(Z), \mathbb{R}^+) = f(\mu(Z, \mathbb{R}^+)) = f(Z)$. □

Definition A.6 *A sheaf of k-modules F on X_{sa} is conic if the restriction morphism $\Gamma(\mathbb{R}^+U; F) \to \Gamma(U; F)$ is an isomorphism for each $\mathbb{R}^+$-connected $U \in \mathrm{Op}^c(X_{sa})$ with $\mathbb{R}^+U \in \mathrm{Op}(X_{sa})$.*

(i) We denote by $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{sa}})$ the subcategory of $\mathrm{Mod}(k_{X_{sa}})$ consisting of conic sheaves.

(ii) We denote by $D^b_{\mathbb{R}^+}(k_{X_{sa}})$, the subcategory of $D^b(k_{X_{sa}})$ consisting of objects F such that $H^j(F)$ belongs to $\mathrm{Mod}_{\mathbb{R}^+}(k_{X_{sa}})$ for all $j \in \mathbb{Z}$.

Assume the hypothesis below:

$$\text{(A.1)} \qquad \begin{cases} \text{(i) for any } U \in \mathrm{Op}^c_{sa}(X) \text{ we have } \mathbb{R}^+U \in \mathrm{Op}(X_{sa}), \\ \text{(ii) for any } x \in X \text{ the set } \mathbb{R}^+x \text{ is contractible,} \\ \text{(iii) there exists a covering } \{V_n\}_{n\in\mathbb{N}} \text{ of } X_{sa} \text{ such that} \\ \quad V_n \text{ is } \mathbb{R}^+\text{-connected and } V_n \subset\subset V_{n+1} \text{ for each } n. \end{cases}$$

The following result was proven in [20].

Proposition A.7 *Assume* (A.1). *Let $U \in \mathrm{Op}(X_{sa})$ be $\mathbb{R}^+$-connected and such that $\mathbb{R}^+U \in \mathrm{Op}(X_{sa})$. Let $F \in D^b_{\mathbb{R}^+}(k_{X_{sa}})$. Then*

$$R\Gamma(\mathbb{R}^+U; F) \xrightarrow{\sim} R\Gamma(U; F).$$

B Multi-conic sheaves

Let X be a topological space with ℓ actions $\{\mu_i\}_{i=1}^{\ell}$ of $\mathbb{R}^+$ such that $\mu_i(\mu_j(x,t_j),t_i) = \mu_j(\mu_i(x,t_i),t_j)$. We have a map

$$\begin{array}{rcl} \mu : X \times (\mathbb{R}^+)^\ell & \to & X \\ (x,t_1,\dots,t_\ell) & \mapsto & \mu_1(\cdots \mu_\ell(x,t_\ell),\dots,t_1). \end{array}$$

Definition B.1 *(i) Let S be a subset of X. We set $\mathbb{R}^+_i S = \mu_i(S,\mathbb{R}^+)$. If $U \in \mathrm{Op}(X)$, then $\mathbb{R}^+_i U \in \mathrm{Op}(X)$ since μ_i is open for each $i = 1,\dots,\ell$.*

(ii) Let S be a subset of X. Let $J = \{i_1,\dots,i_k\} \subseteq \{1,\dots,\ell\}$. We set

$$\mathbb{R}^+_J S = \mathbb{R}^+_{i_1} \cdots \mathbb{R}^+_{i_k} S = \mu_{i_1}(\cdots \mu_{i_k}(S,\mathbb{R}^+),\dots,\mathbb{R}^+), \quad i_1,\dots i_k \in J.$$

We set $(\mathbb{R}^+)^\ell S = \mathbb{R}^+_{\{1,\dots,\ell\}} S = \mu(S,(\mathbb{R}^+)^\ell)$. If $U \in \mathrm{Op}(X)$, then $\mathbb{R}^+_J U \in \mathrm{Op}(X)$ since μ_i is open for each $i \in \{1,\dots,\ell\}$.

(iii) Let S be a subset of X. We say that S is $(\mathbb{R}^+)^\ell$-conic if $S = (\mathbb{R}^+)^\ell S$. In other words, S is invariant by the action of μ_i, $i = 1,\dots,\ell$.

Definition B.2 *(i) We say that a subset S of X is $\mathbb{R}^+_i$-connected if $S \cap \mathbb{R}^+_i x$ is connected for each $x \in S$.*

(ii) We say that a subset S of X is $(\mathbb{R}^+)^\ell$-connected if there exists a permutation $\sigma : \{1,\dots,\ell\} \to \{1,\dots,\ell\}$ such that

$$\text{(B.1)} \qquad \begin{cases} S \text{ is } \mathbb{R}^+_{\sigma(1)}\text{-connected,} \\ \mathbb{R}^+_{\sigma(1)} S \text{ is } \mathbb{R}^+_{\sigma(2)}\text{-connected,} \\ \quad \vdots \\ \mathbb{R}^+_{\sigma(1)} \cdots \mathbb{R}^+_{\sigma(\ell-1)} S \text{ is } \mathbb{R}^+_{\sigma(\ell)}\text{-connected.} \end{cases}$$

The following results follow from the case $\ell = 1$.

Lemma B.3 *(i) $S_1, S_2 \subset X$ and suppose that S_2 is $(\mathbb{R}^+)^\ell$-conic. Then $(\mathbb{R}^+)^\ell(S_1 \cap S_2) = (\mathbb{R}^+)^\ell S_1 \cap S_2$. (ii) If moreover S_1 is $(\mathbb{R}^+)^\ell$-connected then $S_1 \cap S_2$ is $(\mathbb{R}^+)^\ell$-connected.*

Remark B.4 *In (ii) of Lemma B.3 we have to assume that S_2 is $(\mathbb{R}^+)^\ell$-conic. Indeed it is not true that the intersection of two $(\mathbb{R}^+)^\ell$-connected is $(\mathbb{R}^+)^\ell$-connected in general.*

Let X, Y topological spaces endowed with ℓ actions $\{\mu_{Xi}\}_{i=1}^{\ell}$, $\{\mu_{Yi}\}_{i=1}^{\ell}$ of $\mathbb{R}^+$.

Definition B.5 *A continuous function $f : X \to Y$ is said to be $(\mathbb{R}^+)^\ell$-conic if for each $x \in X$, $a \in \mathbb{R}^+$ we have $f(\mu_{Xi}(x,a)) = \mu_{Yi}(f(x),a)$, $i = 1,\dots,\ell$.*

Lemma B.6 *Let $f : X \to Y$ be a $(\mathbb{R}^+)^\ell$-conic map. (i) Suppose that $S \subset Y$ is $(\mathbb{R}^+)^\ell$-connected (resp. $(\mathbb{R}^+)^\ell$-conic). Then $f^{-1}(S)$ is $(\mathbb{R}^+)^\ell$-connected (resp. $(\mathbb{R}^+)^\ell$-conic. (ii) Suppose that $Z \subset X$ is $(\mathbb{R}^+)^\ell$-conic. Then $f(Z)$ is $(\mathbb{R}^+)^\ell$-conic.*

Definition B.7 *A sheaf of k-modules F on X_{sa} is $(\mathbb{R}^+)^\ell$-conic if it is conic with respect to each μ_i.*

(i) We denote by $\mathrm{Mod}_{(\mathbb{R}^+)^\ell}(k_{X_{sa}})$ the subcategory of $\mathrm{Mod}(k_{X_{sa}})$ consisting of $(\mathbb{R}^+)^\ell$-conic sheaves.

(ii) We denote by $D^b_{(\mathbb{R}^+)^\ell}(k_{X_{sa}})$, the subcategory of $D^b(k_{X_{sa}})$ consisting of objects F such that $H^j(F)$ belongs to $\mathrm{Mod}_{(\mathbb{R}^+)^\ell}(k_{X_{sa}})$ for all $j \in \mathbb{Z}$.

Let us assume the following hypothesis

$$\text{(B.2)}\quad \begin{cases} \text{(i) every } U \in \mathrm{Op}^c_{sa}(X) \text{ has a finite covering consisting} \\ \quad \text{of } (\mathbb{R}^+)^\ell\text{-connected subanalytic open subsets,} \\ \text{(ii) we have } \mathbb{R}^+_J U \in \mathrm{Op}(X_{sa}) \text{ for any } U \in \mathrm{Op}^c_{sa}(X) \\ \quad \text{and any } J \subset \{1,\dots,\ell\}, \\ \text{(iii) } \mathbb{R}^+_i x \text{ is contractible for any } x \in X \text{ and any } i = 1,\dots,\ell, \\ \text{(iv) there exists a covering } \{V_n\}_{n\in\mathbb{N}} \text{ of } X_{sa} \text{ such that} \\ \quad V_n \text{ is } (\mathbb{R}^+)^\ell\text{-connected and } V_n \subset\subset V_{n+1} \text{ for each } n. \end{cases}$$

In this situation the orbits of μ_i, $i = 1,\dots,\ell$ are either $\mathbb{R}^+x \simeq \mathbb{R}$ or $\mathbb{R}^+x = x$.

Proposition B.8 *Let $U \in \mathrm{Op}(X_{sa})$ be $(\mathbb{R}^+)^\ell$-connected. Let $F \in D^b_{(\mathbb{R}^+)^\ell}(k_{X_{sa}})$. Then*

$$R\Gamma((\mathbb{R}^+)^\ell U; F) \xrightarrow{\sim} R\Gamma(U; F).$$

Proof. Suppose that U satisfies (B.1). By Proposition A.7 we have

$$R\Gamma(U;F) \simeq R\Gamma(\mathbb{R}^+_{\sigma(1)}U;F) \simeq \cdots \simeq R\Gamma(\mathbb{R}^+_{\sigma(1)}\cdots\mathbb{R}^+_{\sigma(\ell)}U;F)$$

and $\mathbb{R}^+_{\sigma(1)}\cdots\mathbb{R}^+_{\sigma(\ell)}U = (\mathbb{R}^+)^\ell U$. □

If X satisfies (B.2) (i)-(iv), then it follows from Proposition B.8 that for $(\mathbb{R}^+)^\ell$-conic subanalytic sheaves it is enough to study the cohomology of the sections on $(\mathbb{R}^+)^\ell$-conic open subsets.

Let $\{\mu_i\}_{i=1}^\ell$, $\ell \leq n$ be actions of $\mathbb{R}^+$ on $\mathbb{R}^n$ defined by

$$\mu_j : (x_1, \ldots, x_n, c_j) \mapsto (c_{1j}x_1, \ldots, c_{nj}x_n),$$

where $c_{ij} = 1$ if $i \notin I_j$, $c_{ij} = c_j \in \mathbb{R}^+$ if $i \in I_j$. We assume the following hypothesis

(B.3) Either $I_i \subset I_j$, $I_j \subset I_i$ or $I_i \cap I_j = \emptyset$ holds $(i, j = 1, 2, \ldots, \ell)$.

$$I_i \supsetneqq \bigcup_{I_j \subsetneqq I_i} I_j \quad \text{for } i = 1, 2, \ldots, \ell. \tag{B.4}$$

Definition B.9 *Let $i, j = 1, \ldots, \ell$. When $I_i \cap I_j = \emptyset$ the actions μ_i and μ_j are said to be orthogonal and we write $\mu_i \perp \mu_j$.*

Lemma B.10 *Each globally subanalytic $U \in \mathrm{Op}(\mathbb{R}^n_{sa})$ has a finite covering consisting of globally subanalytic $(\mathbb{R}^+)^\ell$-connected open subsets.*

Proof. We may assume that U is connected. We will prove the assertion in several steps. In the rest of the proof we write for short subanalytic instead of globally subanalytic.

(i) First of all, as in [20], let us prove the case $\ell = 1$, i.e. there exists a finite open covering $\{U_\alpha\}$ of U with U_α subanalytic and $\mathbb{R}^+$-connected. Let us consider an action

$$\begin{array}{rcl} \mu : \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^+ & \to & \mathbb{R}^n \\ (x_1, \ldots, x_m, x_{m+1}, \ldots, x_n, \lambda) & \mapsto & (\lambda x_1, \ldots, \lambda x_m, x_{m+1}, \ldots, x_n). \end{array}$$

Let us consider the morphism of manifolds

$$\begin{array}{rcl} \varphi : \mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^{n-m} & \to & \mathbb{R}^n \\ (\vartheta, r, z) & \mapsto & (ri(\vartheta), z), \end{array}$$

where $i : \mathbb{S}^{m-1} \hookrightarrow \mathbb{R}^m$ denotes the embedding. The map φ is proper and subanalytic. The subset $\varphi^{-1}(U)$ is subanalytic in $\mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^{n-m}$. Set $Z = \{0\} \times \mathbb{R}^{n-m}$.

($\mathrm{i_a}$) Let us consider a cylindrical cell decomposition of $\varphi^{-1}(U \setminus Z)$ (for the definition see [3]) with respect to the fibers of the projection $\pi : \mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^{m-n} \to \mathbb{S}^{m-1} \times \mathbb{R}^{m-n}$. Each cell D is defined by $(f, g) := \{(x, y);\ f(x) < y < g(y)\}$, where $f, g : D' \to \mathbb{R}$ are subanalytic functions and D' is a cell of $\pi(\varphi^{-1}(U \setminus Z))$. Consider $h : D' \to \mathbb{R}$, $f < h < g$, for example $h = \frac{f+g}{2}$. Extend the graph of h to $\widetilde{h}$ on an open subanalytic neighborhood U' of D' such that the graph of $\widetilde{h}$ is contained in $\varphi^{-1}(U \setminus Z)$. For $p \in U'$ let $(m(p), M(p))$ be the connected component of $\pi^{-1}(p) \cap \varphi^{-1}(U \setminus Z)$ containing $\widetilde{h}(p)$. The functions $m, M : U' \to \mathbb{R}$ are subanalytic and the open set $U_D = \{(p, y);\ p \in U',\ m(p) < y < M(p)\}$ has the required properties. In this way we obtain a finite covering $\{W_j\}_{j \in J}$ of $\varphi^{-1}(U \setminus Z)$ consisting of $\mathbb{R}^+$-connected subanalytic open subsets.

($\mathrm{i_b}$) Let $p \in \pi(\varphi^{-1}(U \cap Z))$. Then $\pi^{-1}(p) \cap \varphi^{-1}(U)$ is a disjoint union of intervals. Let us consider the interval $(m(p), M(p))$, $m(p) < M(p) \in \mathbb{R}$ containing 0. Set $W_Z = \{(p, r) \in U;\ m(p) < r < M(p)\}$. The set W_Z is open subanalytic, contains $\varphi^{-1}(U \cap Z)$ and its intersections with the fibers of π are connected. Then $\varphi(W_Z)$ is an open $\mathbb{R}^+$-connected subanalytic neighborhood of $U \cap Z$ and it is contained in U.

By ($\mathrm{i_a}$) there exists a finite covering $\{\varphi(W_j)\}_{j \in J}$ of $U \setminus Z$ consisting of $\mathbb{R}^+$-connected subanalytic open subsets, and $\varphi(W_Z) \cup \bigcup_{j \in J} \varphi(W_j) = U$.

(ii) Suppose that $I_i \cap I_j = \emptyset$ for each $i, j \in \{1, \dots, \ell\}$, i.e. all the actions are orthogonal. Let U be $\mathbb{R}^+_1$-connected and such that $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{j-1} U$ is $\mathbb{R}^+_j$-connected for each $j = 2, \dots, \ell - 1$. We are going to find a finite cover $\{U_\alpha\}$ with U_α $\mathbb{R}^+_1$-connected and such that $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{j-1} U$ is $\mathbb{R}^+_j$-connected for each $j = 2, \dots, \ell$.

For any subset $J = \{i_1, \dots, i_k\}$ of $\{1, \dots, \ell - 1\}$ let us consider

$$S_J := \left\{ x \in \mathbb{R}^n;\ \left\| \sum_{i \in I_k} x_i^2 = 1 \right\|\ (k \in J),\ x_j = 0\ (j \in I_k,\ k \notin J) \right\}.$$

By construction S_J is invariant under μ_ℓ. Let us consider $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{\ell-1} U \cap S_J$. By (i) it admits a finite covering $\{V_{\beta_J}\}$ consisting of $\mathbb{R}^+_\ell$-connected subanalytic open subsets of S_J. For each β_J consider $U_{\beta_J} := \mathbb{R}^+_J (V_{\beta_J} \times \mathbb{R}^{n_J})$, where $n_J = n - \sharp \bigcup_{i \in J} I_i$ (hence $\mathbb{R}^{n_J}$ is represented by the coordinates x_k, $k \notin \bigcup_{i \in J} I_i$). We want to show that $U \cap U_{\beta_J}$ has the desired properties. First remark that by Lemma B.3 (i) $\mathbb{R}^+_1 \cdots \mathbb{R}^+_j (U \cap U_{\beta_J}) = \mathbb{R}^+_1 \cdots \mathbb{R}^+_j U \cap U_{\beta_J}$

for each $j = 1, \dots, \ell - 1$ since U_{β_J} is conic with respect to $\mu_1, \dots, \mu_{\ell-1}$ by construction. In particular by Lemma B.3 it is $\mathbb{R}^+_j$-connected for each $j = 1, \dots, \ell - 1$. Moreover $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{\ell-1} U \cap U_{\beta_J} = U_{\beta_J}$ which is $\mathbb{R}^+_\ell$-connected by construction. Then $U \cap U_{\beta_J}$ has the desired properties and the result follows. Remark that this proof does not depend on the choice of the permutation σ in (B.1).

(iii) Let us consider the general case. We argue by decreasing induction on $\sharp J$, where

$$J = \left\{ i \in \{1, \dots, \ell\},\ I_i \cap \bigcup_{i \neq j} I_j = \emptyset \right\},$$

i.e. $\sharp J$ denotes the number of μ_i such that $\mu_i \perp \mu_j$ for each $j \neq i$. If $\sharp J = \ell$ the result follows by (ii). Suppose that the result is true for $\sharp J = 1, \dots, \ell$, we shall prove it for $\sharp J = 0$, i.e. there are no actions which are orthogonal to all the other ones. Thanks to (B.3) and (B.4), there exists $i \in \{1, \dots, \ell\}$ such that $I_i \setminus \bigcup_{j \neq i} I_j \neq \emptyset$. Up to take a permutation of $\{1, \dots, \ell\}$ we may assume $i = \ell$. Let U be $\mathbb{R}^+_1$-connected and such that $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{j-1} U$ is $\mathbb{R}^+_j$-connected for each $j = 1, \dots, \ell - 1$. We are going to find a finite cover $\{U_\alpha\}$ with U_α $\mathbb{R}^+_1$-connected and such that $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{j-1} U_\alpha$ is $\mathbb{R}^+_j$-connected for each $j = 1, \dots, \ell$.

(iii_a) Given $i_0 \in I_\ell \setminus \bigcup_{j \neq \ell} I_j$ let us consider $U' := U \cap \{x_{i_0} \neq 0\}$ and consider the subanalytic homeomorphism ψ of $\{x_{i_0} \neq 0\}$ given by

$$\begin{cases} \psi(x)_k = \dfrac{x_k}{x_{i_0}} & \text{if } k \in I_j \subset I_\ell,\ j \neq \ell, \\ \psi(x)_k = x_k & \text{otherwise.} \end{cases}$$

Set $\mu'_k = \mu_k \circ (\psi \times \mathrm{id}_{(\mathbb{R}^+)^\ell})$. By construction we have $\mu'_\ell \perp \mu'_j$ for each $j \neq \ell$, hence by the induction hypothesis we may find a finite cover $\{V_\alpha\}$ of $\psi(U')$ with the required properties. Then $\{\psi^{-1}(V_\alpha)\}$ is a finite cover of U' consisting of $(\mathbb{R}^+)^\ell$-connected open subanalytic subsets.

(iii_b) By (B.3) $\bigcup_{I_j \subsetneq I_\ell} I_j$ is a disjoint union $I_{i_1} \sqcup \cdots \sqcup I_{i_k}$, $\{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell - 1\}$. Set $x = (x_i)$, $i \in I_\ell \setminus \bigcup_{j \neq \ell} I_j$, $y = (x_j)$, $j \in I_{i_1} \sqcup \cdots \sqcup I_{i_k}$ $z = (x_k)$, $k \notin I_\ell$ Assume that $I_\ell \setminus \bigcup_{j \neq \ell} I_j = \{1, \dots, m\}$. Set $S_0 := \{x = 0\}$, and consider $V_0 := U \cap S_0$. It is an open subanalytic subset of S_0, it is $\mathbb{R}^+_1$-connected and such that $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{j-1} V_0$ is $\mathbb{R}^+_j$-connected for each $j = 1, \dots, \ell$. Indeed on S_0 μ_ℓ is generated by $\mu_1, \dots, \mu_{\ell-1}$ and

hence $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{\ell-1} V_0$ is conic with respect to μ_ℓ. Let us consider the morphism of manifolds

$$\begin{array}{rcl} \varphi : \mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^{n-m} & \to & \mathbb{R}^n \\ (\vartheta, r, y, z) & \mapsto & (ri(\vartheta), y, z), \end{array}$$

where $i : \mathbb{S}^{m-1} \hookrightarrow \mathbb{R}^m$ denotes the embedding. Endow $\mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^{n-m}$ with the actions $\mu'_j = \mu_j$, $j = 1, \ldots, \ell-1$, $\mu'_\ell(\vartheta, r, y, z, \lambda) = (\vartheta, \lambda r, \lambda y, z)$. Then φ is a conic map.

Denote by $\pi : \mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^{n-m} \to \mathbb{S}^{m-1} \times \mathbb{R}^{n-m}$ the projection. Up to shrink U, we may suppose that $\varphi^{-1}(U) = \varphi^{-1}(U) \cap \pi^{-1}(\pi(\varphi^{-1}(V_0)))$. Let $p \in \varphi^{-1}(\mathbb{R}^+_1 \cdots \mathbb{R}^+_{\ell-1} V_0)$. Then $\pi^{-1}(p) \cap \varphi^{-1}(\mathbb{R}^+_1 \cdots \mathbb{R}^+_{\ell-1} U)$ is a disjoint union of intervals. Let us consider the interval $(m(p), M(p))$, $m(p) < M(p) \in \mathbb{R}$ containing 0. Set

$$U_0 = \varphi\left(\{(p, r);\ p \in \varphi^{-1}(\mathbb{R}^+_1 \cdots \mathbb{R}^+_{\ell-1} V_0),\ m(p) < r < M(p)\}\right).$$

The set U_0 is open subanalytic, contains $U \cap S_0$ and if $(x, y, z) \in U_0$, then $(0, y, z) \in U_0$. Moreover by Lemma B.6 (ii) U_0 is conic with respect to $\mu_1, \ldots, \mu_{\ell-1}$. We want to show that $U \cap U_0$ has the desired properties. First remark that by Lemma B.3 (i) $\mathbb{R}^+_1 \cdots \mathbb{R}^+_j (U \cap U_0) = \mathbb{R}^+_1 \cdots \mathbb{R}^+_j U \cap U_0$ for each $j = 1, \ldots, \ell-1$ since U_0 is conic with respect to $\mu_1, \ldots, \mu_{\ell-1}$. In particular by Lemma B.3 (ii) it is $\mathbb{R}^+_j$-connected for each $j = 1, \ldots, \ell-1$. We prove that $\mathbb{R}^+_1 \cdots \mathbb{R}^+_{\ell-1} U \cap U_0 = U_0$ is $\mathbb{R}^+_\ell$-connected. Suppose that $(x, y, z), (\lambda x, \lambda y, z) \in U_0$ for some $\lambda \in \mathbb{R}^+$. By construction $(0, y, z), (0, \lambda y, z) \in U_0$. We may assume without loss of generality $\lambda > 1$. We argue by contradiction. Suppose that there exists $1 < \eta < \lambda$ such that $(\eta x, \eta y, z) \notin U_0$. Since U_0 is conic with respect to $\mu_{i_1}, \ldots, \mu_{i_k}$, for each $t \in \mathbb{R}^+$ we have $(\eta x, t\eta y, z) \notin U_0$. Set $t = \lambda/\eta$. Then $(\eta x, \lambda y, z) \notin U_0$ which leads to a contradiction since by construction $(\eta x, \lambda y, z) \in U_0$ for each $\eta \in (0, \lambda)$.

Hence we have found coverings of $U \cap \{x_{i_k} \neq 0\}$ for each $i_k \in I_i \setminus \bigcup_{j \neq i} I_j$ and a neighborhood of $U \cap S_0$ with the required properties. Then the result follows. □

Lemma B.11 *Let Z be a globally subanalytic $(\mathbb{R}^+)^\ell$-conic subset, and let V be a globally subanalytic $(\mathbb{R}^+)^\ell$-conic open subset of Z. Then each subanalytic neighborhood W of V in $\mathbb{R}^n$ contains $\widetilde{W} \in \mathrm{Op}(\mathbb{R}^n_{sa})$ such that:*

$$\text{(B.5)} \qquad \begin{cases} \textit{(i) } \widetilde{W} \textit{ is } (\mathbb{R}^+)^\ell\textit{-connected,} \\ \textit{(ii) } \mathbb{R}^+_1 \cdots \mathbb{R}^+_\ell \widetilde{W} \textit{ is subanalytic.} \end{cases}$$

Proof. We argue by induction on the number ℓ of actions. Up to shrink W we may suppose that $W \cap Z = V$.

(i) Let us consider the case $\ell = 1$. Let us consider an action

$$\begin{array}{rcl} \mu : \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^+ & \to & \mathbb{R}^n \\ (x_1, \ldots, x_m, x_{m+1}, \ldots, x_n, \lambda) & \mapsto & (\lambda x_1, \ldots, \lambda x_m, x_{m+1}, \ldots, x_n). \end{array}$$

There are two possibilities:

$$\begin{cases} (\mathrm{i_a})\ V \not\subseteq \{0\} \times \mathbb{R}^{n-m}, \\ (\mathrm{i_b})\ V \subseteq \{0\} \times \mathbb{R}^{n-m}. \end{cases}$$

Let us consider the morphism of manifolds

$$\begin{array}{rcl} \varphi : \mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^{n-m} & \to & \mathbb{R}^n \\ (\vartheta, r, z) & \mapsto & (ri(\vartheta), z), \end{array}$$

where $i : \mathbb{S}^{m-1} \hookrightarrow \mathbb{R}^m$ denotes the embedding. Denote by $\pi : \mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^{n-m} \to \mathbb{S}^{m-1} \times \mathbb{R}^{n-m}$ the projection.

($\mathrm{i_a}$) Let $p \in \varphi^{-1}(W) \cap \{r = 1\}$. Up to shrink W, we may suppose that $\varphi^{-1}(W) \cap \{r = 1\}$ is globally subanalytic. Then $\pi^{-1}(p) \cap \varphi^{-1}(W)$ is a disjoint union of intervals. Let us consider the interval $(m(p), M(p))$, $m(p) < M(p) \in \mathbb{R}$ containing 1. Set $W' = \{(p, r) \in \varphi^{-1}(W);\ m(p) < r < M(p)\}$. Remark that in the case $p \in \varphi^{-1}(V)$, if $0 \in (m(p), M(p))$, then $(m(p), M(p)) = \mathbb{R}$ since V is conic. The set W' is open subanalytic, contains $\varphi^{-1}(V)$ and its intersections with the fibers of π are connected. Then $\varphi(W')$ is an open $\mathbb{R}^+$-connected subanalytic neighborhood of V and it is contained in W. Moreover, since $\varphi^{-1}(W) \cap \{r = 1\}$ is globally subanalytic, then $\mathbb{R}^+\varphi(W') = \varphi(\mathbb{R}^+W') = \varphi(\mathbb{R}^+(W' \cap \{r = 1\}))$ is globally subanalytic.

($\mathrm{i_b}$) Let $p \in \varphi^{-1}(V) \subseteq \{r = 0\}$. Then $\pi^{-1}(p) \cap \varphi^{-1}(W)$ is a disjoint union of intervals. Let us consider the interval $(m(p), M(p))$, $m(p) < M(p) \in \mathbb{R}$ containing 0. Set $W' = \{(p, r) \in U;\ m(p) < r < M(p)\}$. The set W' is open subanalytic, contains $\varphi^{-1}(V)$ and its intersections with the fibers of π are connected. Then $\varphi(W')$ is an open $\mathbb{R}^+$-connected subanalytic neighborhood of V and it is contained in W. Moreover, $\mathbb{R}^+\varphi(W') = \varphi(\mathbb{R}^+W') = \varphi((W' \cap \{r = 0\}) \times \mathbb{R}) = \varphi(\pi^{-1}(\pi(\varphi^{-1}(V))))$ is globally subanalytic.

(ii) Suppose that $I_i \cap I_j = \emptyset$ (i.e. $\mu_i \perp \mu_j$) for each $i, j \in \{1, \dots, \ell\}$. Let W be a subanalytic neighborhood of V which is $\mathbb{R}_1^+$-connected and such that $\mathbb{R}_1^+ \cdots \mathbb{R}_{j-1}^+ W$ is $\mathbb{R}_j^+$-connected for $j = 2, \dots, \ell - 1$. Suppose that

$$\begin{aligned} \mu_\ell : \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^+ &\to \mathbb{R}^n \\ (x_1, \dots, x_m, x_{m+1}, \dots, x_n, \lambda) &\mapsto (\lambda x_1, \dots, \lambda x_m, x_{m+1}, \dots, x_n), \end{aligned}$$

and consider the morphism of manifolds

$$\begin{aligned} \varphi : \mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^{n-m} &\to \mathbb{R}^n \\ (\vartheta, r, z) &\mapsto (ri(\vartheta), z), \end{aligned}$$

where $i : \mathbb{S}^{m-1} \hookrightarrow \mathbb{R}^m$ denotes the embedding. Following the construction of (i), $\mathbb{R}_1^+ \cdots \mathbb{R}_{\ell-1}^+ W$ contains a neighborhood W' of V which is $\mathbb{R}_\ell^+$-connected and conic with respect to $\mu_1, \dots, \mu_{\ell-1}$. By Lemma B.3 (i) we have $\mathbb{R}_1^+ \cdots \mathbb{R}_j^+ (W \cap W') = \mathbb{R}_1^+ \cdots \mathbb{R}_j^+ W \cap W'$, $j = 1, \dots, \ell - 1$, hence by Lemma B.3 (ii) the set $W \cap W'$ has the desired properties.

(iii) Let us consider the general case. We argue by decreasing induction on $\sharp J$, where

$$J = \left\{ i \in \{1, \dots, \ell\},\ I_i \cap \bigcup_{i \neq j} I_j = \emptyset \right\},$$

i.e. J denotes the number of μ_i such that $\mu_i \perp \mu_j$ for each $j \neq i$. If $\sharp J = \ell$ the result follows by (ii). Suppose that the result is true for $\sharp J = 1, \dots, \ell$, we shall prove it for $\sharp J = 0$, i.e. there are no actions which are orthogonal to all the other ones. Thanks to (B.3) and (B.4), there exists $i \in \{1, \dots, \ell\}$ such that $I_i \setminus \bigcup_{j \neq i} I_j \neq \emptyset$. Up to take a permutation of $\{1, \dots, \ell\}$ we may assume $i = \ell$. By (B.3) $\bigcup_{I_j \subsetneq I_\ell} I_j$ is a disjoint union $I_{i_1} \sqcup \cdots \sqcup I_{i_k}$, $\{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell - 1\}$. Set $x = (x_i)$, $i \in I_\ell \setminus \bigcup_{j \neq \ell} I_j$, $y = (x_j)$, $j \in I_{i_1} \sqcup \cdots \sqcup I_{i_k}$ $z = (x_k)$, $k \notin \bigcup_{i=1}^{\ell} I_i$. Assume that $I_\ell \setminus \bigcup_{j \neq \ell} I_j = \{1, \dots, m\}$. Let us consider the morphism of manifolds

$$\begin{aligned} \varphi : \mathbb{S}^{m-1} \times \mathbb{R} \times \mathbb{R}^{n-m} &\to \mathbb{R}^n \\ (\vartheta, r, y, z) &\mapsto (ri(\vartheta), y, z), \end{aligned}$$

where $i : \mathbb{S}^{m-1} \hookrightarrow \mathbb{R}^m$ denotes the embedding. Let W be a subanalytic neighborhood of V which is $\mathbb{R}_1^+$-connected and such that $\mathbb{R}_1^+ \cdots \mathbb{R}_{j-1}^+ W$ is $\mathbb{R}_j^+$-connected for $j = 2, \dots, \ell - 1$. Following the construction of (i), $\mathbb{R}_1^+ \cdots \mathbb{R}_{\ell-1}^+ W$ contains a neighborhood W' of V which is $\mathbb{R}_\ell^+$-connected and

conic with respect to $\mu_1, \ldots, \mu_{\ell-1}$. Then, arguing as in Lemma B.10 $(\text{iii})_\text{b}$ $W \cap W'$ has the desired properties. □

Theorem B.12 *Let $\{\mu_i\}_{i=1}^{\ell}$, $\ell \leq n$ be actions of $\mathbb{R}^+$ on $\mathbb{R}^n$ defined by*

$$\mu_j : (x_1, \ldots, x_n, c_j) \mapsto (c_{1j}x_1, \ldots, c_{nj}x_n),$$

where $c_{ij} = 1$ or $c_{ij} = c_j \in \mathbb{R}^+$ satisfying (B.3) *and* (B.4)*. Then $\mathbb{R}^n$ satisfies* (B.2) *(i)-(iv).*

Proof. (B.2) (i) follows from Lemma B.10, (B.2) (ii) follows from the fact that μ_i is a globally subanalytic map for $i = 1, \ldots, \ell$, hence if U is globally subanalytic $\mu_i(U, \mathbb{R}^+)$ is still globally subanalytic, (iii) and (iv) are trivial. □

References

[1] E. Bierstorne, D. Milmann, Semianalytic and subanalytic sets, Publ. I.H.E.S. **67**, pp. 5-42 (1988).

[2] V. Colin, Specialization of formal cohomology and asymptotic expansions, Publ. RIMS, Kyoto Univ. **37** pp. 37-69 (2001).

[3] M. Coste, An introduction to o-minimal geometry, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa (2000).

[4] A. D'Agnolo, P. Schapira, Leray's quantization of projective duality, Duke Math. Journal **84** pp. 453-496 (1996).

[5] A. Dufresnoy, Sur l'opérateur d" et les fonctions différentiables au sens de Whitney, Ann. Inst. Fourier, Grenoble, **29** N. **1** pp. 229-238 (1979).

[6] F. Galindo, J. Sanz, On strongly asymptotically developable functions and the Borel-Ritt theorem, Studia Math. **133** N. **3** pp. 231-248 (1999).

[7] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc. **16** (1955).

[8] J. Hernández, J. Sanz, Constructive Borel-Ritt interpolation results for functions of several variables, Asymptot. Anal. **24** N. **2** pp. 167-182 (2000).

[9] J. Hernández, J. Sanz, Gérard-Sibuya's versus Majima's concept of asymptotic expansion in several variables, J. Aust. Math. Soc. **71** N. **1** pp. 21-35 (2001).

[10] M. Kashiwara, P. Schapira, Sheaves on manifolds, Grundlehren der Math. **292**, Springer-Verlag, Berlin (1990).

[11] M. Kashiwara, P. Schapira, Categories and sheaves, Grundlehren der Math. **332**, Springer-Verlag, Berlin (2006).

[12] M. Kashiwara, P. Schapira, Moderate and formal cohomology associated with constructible sheaves, Mémoires Soc. Math. France **64** (1996).

[13] M. Kashiwara, P. Schapira, Integral transforms with exponential kernels and Laplace transform, Journal of the AMS **4** vol. **10** (1997).

[14] M. Kashiwara, P. Schapira, Ind-sheaves, Astérisque **271** (2001).

[15] H. Majima, Asymptotic Analysis for Integrable Connections with Irregular Singular Points, Lecture Notes in Math. **1075**, Springer-Verlag, Berlin (1984).

[16] B. Malgrange, Ideals of differentiable functions, Tata Institute, Oxford University Press (1966).

[17] J. Mozo Fernández, Topological tensor products and asymptotic developments. Ann. Fac. Sci. Toulouse Math. Vol. **8**, N. **2**, pp. 281-295 (1999).

[18] J. Mozo Fernández, Cohomology theorems for asymptotic sheaves, Tohoku Math. J. Volume **51**, N. **4**, pp. 447-460 (1999).

[19] L. Prelli, Sheaves on subanalytic sites, Rend. Sem. Mat. Univ. Padova Vol. **120**, pp. 167-216 (2008).

[20] L. Prelli, Microlocalization of subanalytic sheaves, C. R. Acad. Sci. Paris Math. **345**, pp. 127-132 (2007), arXiv:math.AG/0702459v3.

[21] P. Schapira, K. Takeuchi, Déformation normale et bispécialisation, C. R. Acad. Sci. Paris Math. **319**, pp. 707-712 (1994).

[22] K. Takeuchi, Binormal deformation and bimicrolocalization, Publ. RIMS, Kyoto Univ. **32** pp. 277-322 (1996).

[23] L. Van Der Dries, Tame topology and o-minimal structures, London Math. Society Lecture Notes Series **248**, Cambridge University Press, Cambridge (1998).

[24] W. Wasow, Asymptotic expansions for ordinary differential equations, Interscience Publishers John Wiley & Sons Inc., New York (1965).

[25] H. Whitney, Functions differentiable on the boundaries of regions, Ann. Math. **35** pp. 482-485 (1934).

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