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Pseudo-spherical evolutes of curves on a spacelike surface in three dimensional Lorentz-Minkowski space

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Abstract

In this paper we introduce the notion of pseudo-spherical evolutes of curves on a spacelike surface in three dimensional Lorentz-Minkowski space which is analogous to the notion of evolutes of curves on the hyperbolic plane. We investigate the the singularities and geometric properties of pseudo-spherical evolutes of curves on a spacelike surface.

1 Introduction

In this paper we study the the singularities of pseudo-spherical evolutes of curves on a spacelike surface in three dimensional Lorentz-Minkowski space. The study of the extrinsic differential geometry of submanifolds in Minkowski space is of special interest in relativity theory. There are some papers that investigate codimension two spacelike submanifolds in Lorentz-Minkowski space, for instance, in [3]. Inspired by these papers, we are particularly interested in spacelike curves in three dimensional Lorentz-Minkowski space as a special case, that is, submanifolds of codimension two in the space. As an application of the idea in [3], we considered curves on a spacelike surface in three dimensional Lorentz-Minkowski space in the last paper [6]. Moreover in [2], it was investigated the evolutes of hyperbolic plane curves and those geometric properties. Inspired by the above results, we introduce the notion of pseudo-spherical evolutes of curves on a spacelike surface in three dimensional Lorentz-Minkowski space and investigate the geometric propoerties. The principal tools for the study of evolutes of hyperbolic plane are the hyperbolic Frenet-Serret formula and hyperbolic height functions on hyperbolic plane curves (cf., [2]). In our case, we adopt a special pseudo-orthonormal frame in $\mathbb{R}^3_1$ which is called the *Lorentzian Darboux frame* and have the Frenet-Serret type formulae (cf., §2). Since we consider a spacelike surface M, we can choose a future directed unit timelike normal vector field $\boldsymbol{n}$ along the surface. For a curve $\boldsymbol{\gamma}$ on the surface, we restrict the normal vector field $\boldsymbol{n}$ along $\boldsymbol{\gamma}$, so that we have a unit timelike normal vector field $\boldsymbol{n}_{\boldsymbol{\gamma}}$ along $\boldsymbol{\gamma}$. Moreover, we choose the unit tangent vector field $\boldsymbol{t}$ and another normal vector field $\boldsymbol{b}$ along $\boldsymbol{\gamma}$. As a result, we construct a pseudo-orthonormal frame $\{\boldsymbol{t}, \boldsymbol{n}_{\boldsymbol{\gamma}}, \boldsymbol{b}\}$ along the curve $\boldsymbol{\gamma}$. We define two families of functions on a curve, timelike height function H^T and spacelike height function H^S. By differentiating these functions, we

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obtain new invariants σ_H and σ_D which characterize the each conditions of derivation of H^T and H^S. For instance, $\frac{\partial H^T}{\partial t} = \frac{\partial^2 H^T}{\partial t^2} = \frac{\partial^3 H^T}{\partial t^3} = 0$ if and only if $\sigma_H = 0$ and some other conditions. In §3, we also define two important curves $\boldsymbol{h}_{\boldsymbol{\gamma}}$ in the hyperbolic space and $\boldsymbol{d}_{\boldsymbol{\gamma}}$ in de Sitter space by observing the conditions of first and second derivation of H^T and H^S respectively. We call $\boldsymbol{h}_{\boldsymbol{\gamma}}$ a *hyperbolic evolute of* $\boldsymbol{\gamma}$ *relative to* M and $\boldsymbol{d}_{\boldsymbol{\gamma}}$ a *de Sitter evolute of* $\boldsymbol{\gamma}$ *relative to* M. We show that the hyperbolic evolute $\boldsymbol{h}_{\boldsymbol{\gamma}}$ is constant if and only if $\sigma_H \equiv 0$. In this case the curve $\boldsymbol{\gamma}$ is a special curve on the surface M, which is called a *hyperbolic-slice* (or an *H-slice*) of M. We also show that the de Sitter evolute $\boldsymbol{d}_{\boldsymbol{\gamma}}$ is constant if and only if $\sigma_D \equiv 0$ and define a special curve on the surface M called a *de Sitter-slice*(or an *D-slice*) of M. We consider H-slice and D-slice of M as the model curves on the surface M. Moreover where $\sigma_H = 0$ and $\sigma_H' \neq 0$, it is a point where $\boldsymbol{\gamma}$ has third order contact with one of the tangent H-slices of M, and we also the same argument about $\boldsymbol{\gamma}$ and tangent D-slice where $\sigma_D = 0$ and $\sigma_D' \neq 0$. Here we remark that tangent D-slice is always non-singular, but we have the case that H-slice has a singular point (cf., §3). As an application of the theory of unfoldings of functions in [1], we give a classification of singularities of both the hyperbolic evolute and the de Sitter evolute in Theorem 5.4, which is one of the main results in this paper.

We explain in §2 the basic notions of Lorentz-Minkowski space and introduce Lorentzian Darboux frame that will be used throughout the paper. In §3 is devoted to the study of two important height functions, the hyperbolic height function and the de Sitter height function by considering the relationship with invariants σ_H and σ_D. Moreover, in §4, we have the main results in this paper that local properties of the curve and the hyperbolic evolute and the de Sitter evolute of the curve provided by the invariant σ_H and σ_D. Finally in §5 we consider Euclidean plane curves and the hyperbolic plane curves and curves on a surface of the graph of a function as a special case.

2 Definitions and Basic Facts

In this section we prepare some definitions and basic facts which we will use in this paper. For basic concepts and details of properties, see [5, 6]. Let $\mathbb{R}^3$ be a three-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, x_2), \boldsymbol{y} = (y_0, y_1, y_2) \in \mathbb{R}^3$, the pseudo-scalar product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2$. We call $(\mathbb{R}^3, \langle , \rangle)$ *Minkowski* 3*-space*. We write $\mathbb{R}^3_1$ instead of $(\mathbb{R}^3, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is *spacelike, lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\| \boldsymbol{x} \| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Here we define the notion of *planes*. For a non-zero vector $\boldsymbol{v} \in \mathbb{R}^3_1$ and a real number c, we define a *plane* with *pseudo-normal* $\boldsymbol{v}$ by

$$P(\boldsymbol{v}, c) = \{ \boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c \}.$$

We call $P(\boldsymbol{v}, c)$ a *spacelike plane*, a *timelike plane* or a *lightlike plane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. We now define *Hyperbolic plane* by

$$H^2_+(-1) = \{ \boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0 \}$$

and *de Sitter* 2*-space* by

$$S^2_1 = \{ \boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \}.$$

Here we define

$$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix},$$

where $\boldsymbol{a} = (a_0, a_1, a_2), \boldsymbol{b} = (b_0, b_1, b_2)$ and $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ is the canonical basis of $\mathbb{R}^3$.

We now prepare some basic facts of curves on a spacelike surface. We consider a spacelilke embedding $\boldsymbol{X} : U \longrightarrow \mathbb{R}^3_1$ from an open subset $U \subset \mathbb{R}^2$. We write $M = \boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. Here, we say that $\boldsymbol{X}$ is a *spacelike embedding* if the tangent space T_pM consists of spacelike vectors at any $p = \boldsymbol{X}(u)$. Let $\bar{\boldsymbol{\gamma}} : I \longrightarrow U$ be a regular curve and we have a curve $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3_1$ defined by $\boldsymbol{\gamma}(s) = \boldsymbol{X}(\bar{\boldsymbol{\gamma}}(s))$. We say that $\boldsymbol{\gamma}$ is a *curve on the spacelike surface M.* Since $\boldsymbol{\gamma}$ is a spacelike curve, we can reparameterize it by the arc-length s. So we have the unit tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ of $\boldsymbol{\gamma}(s)$. Since $\boldsymbol{X}$ is a spacelike embedding, we have a unit timelike normal vector field $\boldsymbol{n}$ along $M = \boldsymbol{X}(U)$ defined by

$$\boldsymbol{n}(p) = \frac{\boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)}{\|\boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)\|},$$

for $p = \boldsymbol{X}(u)$. We say that $\boldsymbol{n}$ is *future directed* if $\langle \boldsymbol{n}, \boldsymbol{e}_0 \rangle < 0$. We choose the orientation of M such that $\boldsymbol{n}$ is future directed. We define $\boldsymbol{n}_{\boldsymbol{\gamma}}(s) = \boldsymbol{n} \circ \boldsymbol{\gamma}(s)$, so that we have a unit timelike normal vector field $\boldsymbol{n}_{\boldsymbol{\gamma}}$ along $\boldsymbol{\gamma}$. Therefore we can construct a spacelike unit normal section $\boldsymbol{b}(s) \in N_p(M)$ by $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}_\gamma(s)$. It follows that we have $\langle \boldsymbol{n}_\gamma, \boldsymbol{n}_\gamma \rangle = -1$, $\langle \boldsymbol{n}_\gamma, \boldsymbol{b} \rangle = 0$, $\langle \boldsymbol{b}, \boldsymbol{b} \rangle = 1$. Then we have a pseudo-orthonormal frame $\{\boldsymbol{t}(s), \boldsymbol{n}_\gamma(s), \boldsymbol{b}(s)\}$, which is called the *Lorentzian Darboux frame* along $\boldsymbol{\gamma}$. By standard arguments, we have the following *Frenet-Serret type formulae*:

$$\begin{cases} \boldsymbol{t}'(s) = \kappa_n(s)\boldsymbol{n}_\gamma(s) + \kappa_g(s)\boldsymbol{b}(s), \\ \boldsymbol{n}_\gamma'(s) = \kappa_n(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) = -\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_\gamma(s), \end{cases}$$

where $\kappa_n(s) = -\langle \boldsymbol{t}'(s), \boldsymbol{n}_\gamma(s) \rangle$, $\kappa_g(s) = \langle \boldsymbol{t}'(s), \boldsymbol{b}(s) \rangle$ and $\tau_g(s) = -\langle \boldsymbol{b}'(s), \boldsymbol{n}_\gamma(s) \rangle$. Here, we have the following properties of $\boldsymbol{\gamma}$ characterized by the conditions of $\kappa_g, \kappa_n, \tau_g$.

$$\boldsymbol{\gamma} \text{ is } \begin{cases} \text{a geodesic curve if and only if } \kappa_g \equiv 0 \\ \text{an acymptotic curve if and only if } \kappa_n \equiv 0 \\ \text{a principal curve if and only if } \tau_g \equiv 0 \end{cases}$$

3 Height Functions

In this section, we introduce two families of functions on a curve on a spacelike surface M $\boldsymbol{\gamma} : I \longrightarrow M \subset \mathbb{R}^3_1$. We now define two families of functions as follow:

$$H^T : I \times H^2_+(-1) \longrightarrow \mathbb{R}; \quad (s, \boldsymbol{v}) \longmapsto \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle,$$

$$H^S : I \times S^2_1 \longrightarrow \mathbb{R}; \quad (s, \boldsymbol{v}) \longmapsto \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle,$$

We call H^T the *timelike height function* of $\boldsymbol{\gamma}$ on M and H^S the *spacelike height function* of $\boldsymbol{\gamma}$ on M. We denote $h^T_{\boldsymbol{v}}(s) = H^T(s, \boldsymbol{v})$ for any fixed $\boldsymbol{v} \in H^2_+(-1)$ and $h^S_{\boldsymbol{v}}(s) = H^S(s, \boldsymbol{v})$ for any fixed $\boldsymbol{v} \in S^2_1$.

Here we calculate the derivative of H^T. By *Frenet-Serret type formulae*, we have

$$\frac{\partial H^T}{\partial s} = \langle \boldsymbol{\gamma}'(s), \boldsymbol{v} \rangle = \langle \boldsymbol{t}, \boldsymbol{v} \rangle = 0$$

Since $\boldsymbol{v} \in H^2_+(-1)$, there are $\lambda, \mu \in \mathbb{R}$ such that $\boldsymbol{v} = \lambda \boldsymbol{n}_\gamma(s) + \mu \boldsymbol{b}(s)$ and $-\lambda^2 + \mu^2 = -1$ if $\frac{\partial H^T}{\partial s} = 0$. Here we have

$$\frac{\partial^2 H^T}{\partial s^2} = \langle \boldsymbol{t}'(s), \boldsymbol{v} \rangle = \langle \kappa_n \boldsymbol{n}_{\boldsymbol{\gamma}} + \kappa_g \boldsymbol{b}, \lambda \boldsymbol{n}_{\boldsymbol{\gamma}} + \mu \boldsymbol{b} \rangle = -\lambda \kappa_n + \mu \kappa_g = 0.$$

Therefore, $\frac{\partial H^T}{\partial s} = \frac{\partial^2 H^T}{\partial s^2} = 0$ if and only if $\boldsymbol{v} = \lambda \boldsymbol{n}_\gamma(s) + \mu \boldsymbol{b}(s)$, $-\lambda^2 + \mu^2 = -1$ and $\lambda \kappa_n = \mu \kappa_g$. It means that

$$\lambda^2(\kappa_g^2 - \kappa_n^2) = \kappa_g^2$$

Since $\lambda, \mu \in \mathbb{R}$, we can consider under the condition $\kappa_g^2 > \kappa_n^2$, then we have

$$\boldsymbol{v} = \pm \frac{1}{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}} (\kappa_g(s) \boldsymbol{n}_{\boldsymbol{\gamma}}(s) + \kappa_n(s) \boldsymbol{b}(s)).$$

Moreover we have that $\frac{\partial H^T}{\partial s} = \frac{\partial^2 H^T}{\partial s^2} = \frac{\partial^3 H^T}{\partial s^3} = 0$ if and only if

$$\kappa_g {\kappa_n}' + \kappa_g^2 \tau_g - \kappa_n^2 \tau_g - \kappa_n {\kappa_g}' = 0. \qquad (*)$$

Under the condition $\kappa_g^2 > \kappa_n^2$, (which implies $\kappa_g^2 > 0$,) the equation $(^*)$ is divided by κ_g^2, so that we have

$$\left(\frac{\kappa_n}{\kappa_g}\right)' + \tau_g \left(1 - \left(\frac{\kappa_n}{\kappa_g}\right)^2\right) = 0.$$

We now define σ_H by $\sigma_H = \left(\frac{\kappa_n}{\kappa_g}\right)' + \tau_g \left(1 - \left(\frac{\kappa_n}{\kappa_g}\right)^2\right)$. If we calculate the 4th derivative of H^S, we can show that the above conditions with the extra condition

$$\frac{\partial^4 H^T}{\partial s^4} = -\kappa_g {\kappa_n}'' - 2\kappa_g {\kappa_g}' \tau_h - {\kappa_g}^2 {\tau_g}' + 2\kappa_n {\kappa_n}' \tau_g + {\kappa_n}^2 {\tau_g}^2 + \kappa_n {\kappa_g}'' = 0$$

is equivalent to the conditions $\sigma_H(s) = 0$ and $(\sigma_H)'(s) = 0$.
Moreover, we calculate the derivatives of H^S under the condition $\kappa_n^2 > \kappa_g^2$, (which implies $\kappa_n^2 > 0$,) as the above, and we also define $\sigma_D = \left(\frac{\kappa_g}{\kappa_n}\right)' + \tau_g \left(1 - \left(\frac{\kappa_g}{\kappa_n}\right)^2\right)$.
As a consequence, we have the following proposition:

Proposition 3.1 (A) *For any $(s, \boldsymbol{v}) \in I \times H^2(-1)$, we have the followings:*
(1) *$(h^T_{\boldsymbol{v}})'(s) = 0$ if and only if $\boldsymbol{v} = \lambda \boldsymbol{n}_\gamma(s) + \mu \boldsymbol{b}(s)$ where $\lambda, \mu \in \mathbb{R}$ such that $-\lambda^2 + \mu^2 = -1$.*
(2) *$(h^T_{\boldsymbol{v}})'(s) = (h^T_{\boldsymbol{v}})''(s) = 0$ if and only if*

$$\boldsymbol{v} = \pm \frac{1}{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}} (\kappa_g(s) \boldsymbol{n}_{\boldsymbol{\gamma}}(s) + \kappa_n(s) \boldsymbol{b}(s))$$

and $\kappa_g^2(s) > \kappa_n^2(s)$.
(3) *$(h^T_{\boldsymbol{v}})'(s) = (h^T_{\boldsymbol{v}})''(s) = (h^T_{\boldsymbol{v}})'''(s) = 0$ if and only if*

$$\boldsymbol{v} = \pm \frac{1}{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}} (\kappa_g(s) \boldsymbol{n}_{\boldsymbol{\gamma}}(s) + \kappa_n(s) \boldsymbol{b}(s)),$$

$\kappa_g^2(s) > \kappa_n^2(s)$, *and* $\sigma_H = 0$.
(4) $(h_{\boldsymbol{v}}^T)'(s) = (h_{\boldsymbol{v}}^T)''(s) = (h_{\boldsymbol{v}}^T)'''(s) = (h_{\boldsymbol{v}}^T)^{(4)}(s) = 0$ *if and only if*

$$\boldsymbol{v} = \pm\frac{1}{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}}(\kappa_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s) + \kappa_n(s)\boldsymbol{b}(s)),$$

$\kappa_g^2(s) > \kappa_n^2(s)$, $\sigma_H = 0$ *and* $(\sigma_H)' = 0$.

(B) *For any* $(s, \boldsymbol{v}) \in I \times S_1^2$, *we have the followings:*
(1) $(h_{\boldsymbol{v}}^S)'(s) = 0$ *if and only if* $\boldsymbol{v} = \lambda\boldsymbol{n}_\gamma(s) + \mu\boldsymbol{b}(s)$ *where* $\lambda, \mu \in \mathbb{R}$ *such that* $-\lambda^2 + \mu^2 = -1$.
(2) $(h_{\boldsymbol{v}}^S)'(s) = (h_{\boldsymbol{v}}^S)''(s) = 0$ *if and only if*

$$\boldsymbol{v} = \pm\frac{1}{\sqrt{\kappa_n^2(s) - \kappa_g^2(s)}}(\kappa_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s) - \kappa_n(s)\boldsymbol{b}(s))$$

and $\kappa_n^2(s) > \kappa_g^2(s)$.
(3) $(h_{\boldsymbol{v}}^S)'(s) = (h_{\boldsymbol{v}}^S)''(s) = (h_{\boldsymbol{v}}^S)'''(s) = 0$ *if and only if*

$$\boldsymbol{v} = \pm\frac{1}{\sqrt{\kappa_n^2(s) - \kappa_g^2(s)}}(\kappa_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s) - \kappa_n(s)\boldsymbol{b}(s)),$$

$\kappa_n^2(s) > \kappa_g^2(s)$, *and* $\sigma_D = 0$.
(4) $(h_{\boldsymbol{v}}^S)'(s) = (h_{\boldsymbol{v}}^S)''(s) = (h_{\boldsymbol{v}}^S)'''(s) = (h_{\boldsymbol{v}}^S)^{(4)}(s) = 0$ *if and only if*

$$\boldsymbol{v} = \pm\frac{1}{\sqrt{\kappa_n^2(s) - \kappa_g^2(s)}}(\kappa_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s) - \kappa_n(s)\boldsymbol{b}(s)),$$

$\kappa_n^2(s) > \kappa_g^2(s)$, $\sigma_D = 0$ *and* $(\sigma_D)' = 0$.

By the above proposition, we have two invariants σ_H and σ_D.
We now respectively define curves $\boldsymbol{h}_{\boldsymbol{\gamma}} : I \longrightarrow H_+^2(-1)$ and $\boldsymbol{d}_{\boldsymbol{\gamma}} : I \longrightarrow S_1^2$ by

$$\begin{aligned}
\boldsymbol{h}_{\boldsymbol{\gamma}}(s) &= \frac{\kappa_g(s)}{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}}\boldsymbol{n}_{\boldsymbol{\gamma}}(s) + \frac{\kappa_n(s)}{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}}\boldsymbol{b}(s) \\
\boldsymbol{d}_{\boldsymbol{\gamma}}(s) &= \frac{\kappa_g(s)}{\sqrt{\kappa_n^2(s) - \kappa_g^2(s)}}\boldsymbol{n}_{\boldsymbol{\gamma}}(s) - \frac{\kappa_n(s)}{\sqrt{\kappa_n^2(s) - \kappa_g^2(s)}}\boldsymbol{b}(s)
\end{aligned}$$

We call $\boldsymbol{h}_{\boldsymbol{\gamma}}$ a *hyperbolic evolute* of γ relative to M and $\boldsymbol{d}_{\boldsymbol{\gamma}}$ a *de Sitter evolute* of $\boldsymbol{\gamma}$ relative to M respectively. We also call one of $\boldsymbol{h}_{\boldsymbol{\gamma}}$ and $\boldsymbol{d}_{\boldsymbol{\gamma}}$ a *pseudo-spherical evolute* of $\boldsymbol{\gamma}$ relative to M. By straightforward calculations, we have $\boldsymbol{h}'_{\boldsymbol{\gamma}}(s) = 0$ if and only if $\sigma_H(s) = 0$. Moreover, we have $\boldsymbol{d}'_{\boldsymbol{\gamma}}(s) = 0$ if and only if $\sigma_D(s) = 0$. Therefore, $\boldsymbol{h}_{\boldsymbol{\gamma}}(s) = \boldsymbol{v}_0$ is constant if and only if $\sigma_H(s) \equiv 0$. In this case, by Proposition 3.1, $h_{\boldsymbol{v}_0}^T(s)$ is constant, that is, there is a real number $c \in \mathbb{R}$ such that $\langle \boldsymbol{\gamma}(s), \boldsymbol{v}_0 \rangle = c$. It means that $\operatorname{Im}\boldsymbol{\gamma} = P(\boldsymbol{v}_0, c) \cap M$. It suggests that curves of the form $P(\boldsymbol{v}, c) \cap M$ for $\boldsymbol{v} \in H_+^2(-1)$ are the candidates of model curves on M. These might play a similar role to lines in Euclidean plane. We call it a *hyperbolic-slice* (or, an *H-slice*) of M. By the same way, we can also define *de Sitter-slice* (or, a *D-slice*) of M by $P(\boldsymbol{v}, c) \cap M$ for $\boldsymbol{v} \in S_1^2$.

Here we remark that we can consider the above H-slice under the condition $\kappa_n \neq 0$. If $\kappa_n = 0$, we have $\boldsymbol{h}_{\boldsymbol{\gamma}}(s) = \boldsymbol{n}_{\boldsymbol{\gamma}}$. It means that $P(\boldsymbol{v}, c)$ is tangent space at $p = \boldsymbol{X}(u)$ for $\boldsymbol{v} \in H^2_+(-1)$ and $P(\boldsymbol{v}, c) \cap M$ has a singular point.

We consider the geometric meanings of the invariants σ_H and σ_D. For the purpose, we introduce the following another family of function:

$$\mathcal{H}^T : \mathbb{R}^3_1 \times H^2_+(-1) \longrightarrow \mathbb{R}; \quad (\boldsymbol{x}, \boldsymbol{v}) \longmapsto \langle \boldsymbol{x}, \boldsymbol{v} \rangle.$$

We denote $\mathfrak{h}^T_{\boldsymbol{v}_0}(\boldsymbol{x}) = \mathcal{H}^T(\boldsymbol{x}, \boldsymbol{v}_0)$ for any fixed $\boldsymbol{v}_0 \in H^2_+(-1)$, then, we have

$$h^T_{\boldsymbol{v}_0}(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v}_0 \rangle = \mathcal{H}(\boldsymbol{\gamma}(s), \boldsymbol{v}_0) = \mathfrak{h}^T_{\boldsymbol{v}_0}(\boldsymbol{\gamma}(s))$$

Moreover, for any $s_0 \in \mathbb{R}$ and $\boldsymbol{v}_0 = \boldsymbol{h}_{\boldsymbol{\gamma}}(s_0)$, $(\mathfrak{h}^T_{\boldsymbol{v}_0}|_M)^{-1}(c)$ is an H-slice of M. By Proposition 3.1, $(\mathfrak{h}^T_{\boldsymbol{v}_0})^{-1}(c_0) = P(\boldsymbol{v}_0, c_0)$ is tangent to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$, so that $(\mathfrak{h}^T_{\boldsymbol{v}_0}|_M)^{-1}(c_0)$ is an H-slice of M tangent to $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$, where $c_0 = h^T_{\boldsymbol{v}_0}(s_0)$. We denote that $TP^S_{\boldsymbol{v}_0, \boldsymbol{\gamma}(s_0)} = P(\boldsymbol{v}_0, c_0)$ which is called a *spacelike tangent plane of* $\boldsymbol{\gamma}$ *at* $\boldsymbol{\gamma}(s_0)$ *with respect to* $\boldsymbol{v}_0 = \boldsymbol{h}_{\boldsymbol{\gamma}}(s_0)$. Moreover, if $\kappa_n(s_0) \neq 0$, then the H-slice $(\mathfrak{h}^T_{\boldsymbol{v}_0}|_M)^{-1}(c_0)$ is non-singular at $\boldsymbol{\gamma}(s_0)$. Then we call it a *tangent H-slice of* $\boldsymbol{\gamma}$ *at* $\boldsymbol{\gamma}(s_0)$ *relative to* M. It is denoted by $T^H_{M, \boldsymbol{\gamma}(s_0)}$. Now let $F : \mathbb{R}^3_1 \longrightarrow \mathbb{R}$ (respectively, $F|_M : M \longrightarrow \mathbb{R}$) be a submersion and $\boldsymbol{\gamma}(s_0) \subset F^{-1}(0) \cap M$. We say that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ (respectively, $F^{-1}(0) \cap M$) have *contact of order* k if the function $f = F \circ \boldsymbol{\gamma}(s)$ satisfies $f(s_0) = f'(s_0) = \cdots = f^{(k)}(s_0) = 0$ and $f^{(k+1)}(s_0) \neq 0$. Under the above notations and by Proposition 3.1, we can conclude $\boldsymbol{\gamma}$ and $TP^S_{\boldsymbol{v}_0, \boldsymbol{\gamma}(s_0)}$ have contact of order three at $\boldsymbol{\gamma}(s_0)$ if and only if $\sigma_H(s_0) = 0$ and $\sigma_H'(s_0) \neq 0$. Under the assumption that $\kappa_n(s_0) \neq 0$ the above conditions are equivalent to the condition that $\boldsymbol{\gamma}$ and $T^H_{M, \boldsymbol{\gamma}(s_0)}$ have contact of order three at $\boldsymbol{\gamma}(s_0)$. Therefore, we have the following proposition:

Proposition 3.2 *Let* $\boldsymbol{\gamma} : I \longrightarrow M$ *be a regular curve on* M. *Then the following conditions are equivalent:*

(1) $\boldsymbol{\gamma}$ *and the spacelike tangent plane* $TP^S_{\boldsymbol{v}_0, \boldsymbol{\gamma}(s_0)}$ *have contact order three,*

(2) $\sigma_H(s_0) = 0$ *and* $\sigma_H'(s_0) \neq 0$.

If $\kappa_n(s_0) \neq 0$, *then the tangent H-slice* $T^H_{M, \boldsymbol{\gamma}(s_0)}$ *of* $\boldsymbol{\gamma}$ *at* $\boldsymbol{\gamma}(s_0)$ *is non-singular and the above two conditions are equivalent to the following condition:*

(3) $\boldsymbol{\gamma}$ *and the the tangent H-slice* $T^H_{M, \boldsymbol{\gamma}(s_0)}$ *have contact order three.*

Moreover, if $\kappa_n(s_0) = 0$ *then the spacelike tangent plane* $TP^S_{\boldsymbol{v}_0, \boldsymbol{\gamma}(s_0)}$ *is the tangent plane* $TM_{\boldsymbol{\gamma}(s_0)}$ *of* M *and* $\boldsymbol{t}(s_0)$ *is an asymptotic direction of* M *at* $\boldsymbol{\gamma}(s_0)$.

On the other hand, $P(\boldsymbol{v}, c)$ for $\boldsymbol{v} \in S^2_1$ is a timelike plane. Therefore a D-slice is always a regular curve. We also define

$$\mathcal{H}^S : \mathbb{R}^3_1 \times S^2_1 \longrightarrow \mathbb{R}; \quad (\boldsymbol{x}, \boldsymbol{v}) \longmapsto \langle \boldsymbol{x}, \boldsymbol{v} \rangle.$$

By exactly the same arguments as the above case, we have the notions of *timelike tangent plane of* $\boldsymbol{\gamma}$ *at* $\boldsymbol{\gamma}(s_0)$ *with respect to* $\boldsymbol{v}_0 = \boldsymbol{d}_{\boldsymbol{\gamma}}(s_0)$ and *tangent D-slice of* $\boldsymbol{\gamma}$ *at* $\boldsymbol{\gamma}(s_0)$ *relative to* M, *respectively. We respectively denote these as* $TP^T_{\boldsymbol{v}_0, \boldsymbol{\gamma}(s_0)}$ *and* $T^D_{M, \boldsymbol{\gamma}(s_0)}$. *In this case the tangent D-slice is always non-singular at* $\boldsymbol{\gamma}(s_0)$. It also follows from Proposition 3.1 that $\boldsymbol{\gamma}$ and $T^D_{M, \boldsymbol{\gamma}(s_0)}$ have contact of order three at $\boldsymbol{\gamma}(s_0)$ if and only if $\sigma_D(s_0) = 0$ and $\sigma_D'(s_0) \neq 0$.

Proposition 3.3 *The following conditions are equivalent:*

(1) $\boldsymbol{\gamma}$ *and the timelikelike tangent plane* $TP^T_{\boldsymbol{v}_0, \boldsymbol{\gamma}(s_0)}$ *have contact order three,*

(2) $\sigma_D(s_0) = 0$ *and* $\sigma_D'(s_0) \neq 0$,

(3) $\boldsymbol{\gamma}$ *and the the tangent D-slice* $T^D_{M, \boldsymbol{\gamma}(s_0)}$ *have contact order three.*

As a consequence, we have the model curves on M if pseudo-spherical evolutes are constant, that is, $\boldsymbol{h}'_{\boldsymbol{\gamma}} \equiv 0$ or $\boldsymbol{d}'_{\boldsymbol{\gamma}} \equiv 0$. In the next section, we investigate the singularities of pseudo-spherical evolute.

4 Unfoldings of Functions

In order to investigate the singularities of pseudo-spherical evolutes, we apply the theory of unfoldings of functions.

First, we give a quick review on the theory of unfoldings of functions of one variable. Detailed descriptions are found in the book[1]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F_{x_0}(s)$. We say that f has *type* A_k at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leqq p \leqq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has *type* $A_{\geq k}$ at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leqq p \leqq k$. Let F be an unfolding of f and $f(s)$ has type A_k $(k \geqq 1)$ at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji} s^j$ for $i = 1, \ldots, r$. Then F is called an $\mathcal{R}^+$*-versal unfolding* if the $(k-1) \times r$ matrix of coefficients (α_{ji}) has rank $k-1$ $(k-1 \leqq r)$. Under the same condition as the above, F is called an $\mathcal{R}$*-versal unfolding* if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank k $(k \leqq r)$, where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, x_0)$.

We now introduce important sets concerning the unfoldings relative to the above notions. The *bifurcation set* B_F of F is

$$B_F = \{x \in \mathbb{R}^r | \exists s;\ \text{with}\ \frac{\partial F}{\partial s}(s, x) = \frac{\partial^2 F}{\partial s^2}(s, x) = 0\ \}.$$

Then we have the following fundamental result of the unfolding theory (cf., [1]):

Theorem 4.1 *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of f which has the type A_k at s_0. If F is a $\mathcal{R}^+$-versal unfolding and k=3, then the germ of B_F at x_0 is diffeomorphic to $(C \times \mathbb{R}^{r-2}, \mathbf{0})$ as set germs. Here, $C = \{(x_1, x_2) |\ {x_1}^2 = {x_2}^3\}$ is the ordinary cusp.*

Here we have the following proposition:

Proposition 4.2 *Let $\boldsymbol{\gamma} : I \longrightarrow M$ be a unit speed curve with $\kappa_n(s_0) \neq 0$.*
(1) If $h^T_{\boldsymbol{v}_0}(s)$ has type A_3 at s_0, then H^T is an $\mathcal{R}^+$-versal unfolding of $h^T_{\boldsymbol{v}_0}(s)$.
(2) If $h^S_{\boldsymbol{v}_0}(s)$ has type A_3 at s_0, then H^S is an $\mathcal{R}^+$-versal unfolding of $h^S_{\boldsymbol{v}_0}(s)$.

Proof. We denote $\boldsymbol{\gamma}(s) = (x_0(s), x_1(s), x_2(s))$, $\boldsymbol{v} = (v_0, v_1, \sqrt{-1 + v_0^2 - v_1^2}) \in H^2(-1)$. Therefore we have

$$H^T(s, \boldsymbol{v}) = -x_0(s)v_0 + x_1 v_1 + \sqrt{-1 + v_0^2 - v_1^2}$$

and

$$\frac{\partial H^T}{\partial v_0} = -x_0(s) + \frac{v_0}{\sqrt{-1 + v_0^2 - v_1^2}} x_2(s)\ \ , \quad \frac{\partial H^T}{\partial v_1} = x_1(s) - \frac{v_1}{\sqrt{-1 + v_0^2 - v_1^2}} x_2(s)$$

$$\frac{\partial^2 H^T}{\partial s \partial v_0} = -x_0'(s) + \frac{v_0}{\sqrt{-1 + v_0^2 - v_1^2}} x_2'(s)\ \ , \quad \frac{\partial^2 H^T}{\partial s \partial v_1} = x_1'(s) - \frac{v_1}{\sqrt{-1 + v_0^2 - v_1^2}} x_2'(s)$$

$$\frac{\partial^3 H^T}{\partial s^2 \partial v_0} = -x_0''(s) + \frac{v_0}{\sqrt{-1+v_0^2-v_1^2}} x_2''(s) \quad , \quad \frac{\partial^3 H^T}{\partial s^2 \partial v_1} = x_1''(s) - \frac{v_1}{\sqrt{-1+v_0^2-v_1^2}} x_2''(s)$$

(1) By Proposition 3.1, $h_{\boldsymbol{v}}^T(s_0)$ has type A_3 at s_0 if and only if

$$\boldsymbol{v} = \pm \frac{1}{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}} (\kappa_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s) + \kappa_n(s)\boldsymbol{b}(s)),$$

$\kappa_g^2(s) > \kappa_n^2(s)$, $\sigma_H = 0$, and $\sigma_H' \neq 0$.

For the purpose, we require the

$$A = \begin{pmatrix} -x_0'(s) + \frac{v_0}{\sqrt{-1+v_0^2-v_1^2}} x_2'(s) & x_1'(s) - \frac{v_1}{\sqrt{-1+v_0^2-v_1^2}} x_2'(s) \\ -x_0''(s) + \frac{v_0}{\sqrt{-1+v_0^2-v_1^2}} x_2''(s) & x_1''(s) - \frac{v_1}{\sqrt{-1+v_0^2-v_1^2}} x_2''(s) \end{pmatrix}$$

to be non-singular. Therefore we calculate determinant of this matrix.

$$\begin{aligned}
\det A &= (-(x_1'x_2'' - x_2'x_1''), x_2'x_0'' - x_0'x_2'', x_0'x_1'' - x_1'x_0') \begin{pmatrix} \frac{v_0}{\sqrt{-1+v_0^2-v_1^2}} \\ -\frac{v_1}{\sqrt{-1+v_0^2-v_1^2}} \\ -1 \end{pmatrix} \\
&= ((x_0', x_1', x_2') \wedge (x_0'', x_1'', x_2'')) \begin{pmatrix} \frac{v_0}{\sqrt{-1+v_0^2-v_1^2}} \\ -\frac{v_1}{\sqrt{-1+v_0^2-v_1^2}} \\ -1 \end{pmatrix} \\
&= -\frac{1}{\sqrt{-1+v_0^2-v_1^2}} (\boldsymbol{t} \wedge (\kappa_n \boldsymbol{n}_{\boldsymbol{\gamma}} + \kappa_g \boldsymbol{b})) \begin{pmatrix} -v_0 \\ v_1 \\ \sqrt{-1+v_0^2-v_1^2} \end{pmatrix} \\
&= -\frac{1}{\sqrt{-1+v_0^2-v_1^2}} \langle \kappa_n \boldsymbol{b} + \kappa_g \boldsymbol{n}_{\boldsymbol{\gamma}}, \pm \frac{1}{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}} (\kappa_g \boldsymbol{n}_{\boldsymbol{\gamma}}(s) + \kappa_n(s)\boldsymbol{b}(s)) \rangle \\
&= \mp \frac{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}}{\sqrt{-1+v_0^2-v_1^2}} \neq 0
\end{aligned}$$

By the same way, if we consider the spacelike height function H^S, we can prove (2). □

As a consequence, we have the following theorem:

Theorem 4.3 *Let $\boldsymbol{\gamma} : I \longrightarrow M$ be a regular curve. Then we have the following assertions:*
(A1) *The hyperbolic evolute at $\boldsymbol{h}_{\boldsymbol{\gamma}}(s_0)$ is regular if $\sigma_H(s_0) \neq 0$.*
(A2) *The following conditions are equivalent:*
(i) *The germ of the hyperbolic evolute at $\boldsymbol{h}_{\boldsymbol{\gamma}}(s_0)$ is diffeomorphic to the ordinary cusp,*
(ii) *$\sigma_H(s_0) = 0$ and $\sigma_H'(s_0) \neq 0$,*
(iii) *$\boldsymbol{\gamma}$ and the spacelike tangent plane $TP^S_{\boldsymbol{v}_0, \boldsymbol{\gamma}(s_0)}$ have contact order three where $\boldsymbol{v}_0 = \boldsymbol{h}_{\boldsymbol{\gamma}}(s_0)$,*
(iv) *If $\kappa_n(s_0) \neq 0$, then the tangent H-slice $T^H_{M, \boldsymbol{\gamma}(s_0)}$ of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$ is non-singular, and $\boldsymbol{\gamma}$ and the the tangent H-slice $T^H_{M, \boldsymbol{\gamma}(s_0)}$ have contact order three.*

(B1) *The de Sitter evolute at* $\boldsymbol{d}_{\boldsymbol{\gamma}}(s_0)$ *is regular if* $\sigma_D(s_0) \neq 0$.

(B2) *The following conditions are equivalent:*

(i) *The germ of the de Sitter evolute at* $\boldsymbol{d}_{\boldsymbol{\gamma}}(s_0)$ *is diffeomorphic to the ordinary cusp,*

(ii) $\boldsymbol{\gamma}$ *and the timelikelike tangent plane* $TP^T_{\boldsymbol{v}_0,\boldsymbol{\gamma}(s_0)}$ *have contact order three,*

(iii) $\sigma_D(s_0) = 0$ *and* $\sigma_D'(s_0) \neq 0$.

(iv) $\boldsymbol{\gamma}$ *and the the tangent D-slice* $T^D_{M,\boldsymbol{\gamma}(s_0)}$ *have contact order three.*

Proof. (A1) By the assertion of Proposition 3.1, we have $\boldsymbol{h}'_{\boldsymbol{\gamma}}(s) = 0$ if $\sigma_H(s) = 0$. It means that the hyperbolic evolute at $\boldsymbol{h}_{\boldsymbol{\gamma}}(s_0)$ is regular if $\sigma_H(s_0) \neq 0$.

(A2) By Proposition 3.1, the bifurcation set of H^T is

$$B_F = \left\{ \boldsymbol{h}_{\boldsymbol{\gamma}}(s) = \frac{\kappa_g(s)}{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}}\boldsymbol{n}_{\boldsymbol{\gamma}}(s) + \frac{\kappa_n(s)}{\sqrt{\kappa_g^2(s) - \kappa_n^2(s)}}\boldsymbol{b}(s) \;\middle|\; \kappa_g^2 > \kappa_n^2 \right\}.$$

By Theorem 4.1 and Proposition 4.2, the germ of the bifurcation set is diffeomorphic to the ordinary cusp if $\sigma_H = 0$ and $\sigma_H' \neq 0$. Moreover we have other equivalences by Proposition 3.2. This complete the proof for (A1) and (A2).

If we consider the spacelike height function H^S, we can prove the remaining assertions of the theorem. □

5 Examples

In this section we consider some examples which explain the difference from the Euclidean plane curves.

5.1 Curves on a spacelike plane

Suppose that $M = \mathbb{R}^2_0 = \{\boldsymbol{x} = (x_0, x_1, x_2) \in \mathbb{R}^3_1 \mid x_0 = 0\}$. We consider a plane curve $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^2_0$. In this case we have $\boldsymbol{n}_{\boldsymbol{\gamma}} = \boldsymbol{e}_0$, $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ and $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{e}_0$. It follows that $\kappa_n(s) \equiv \tau_g(s) \equiv 0$ and $\kappa_g(s) = \langle \boldsymbol{t}'(s), \boldsymbol{b}(s) \rangle = \kappa(s)$. Then we have the following classical Frenet-Serret formulae on Euclidean plane:

$$\begin{cases} \boldsymbol{t}'(s) = \kappa(s)\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) = -\kappa(s)\boldsymbol{t}(s). \end{cases}$$

Here we have $\sigma_H \equiv 0$ and the constant hyperbolic evolute $\boldsymbol{h}_{\boldsymbol{\gamma}}(s) = \boldsymbol{n}_{\boldsymbol{\gamma}} = \boldsymbol{e}_0$ of $\boldsymbol{\gamma}$ relative to M It means that H-slice is $M = P(\boldsymbol{v}, c) \cap M$. Moreover, we do not have de Sitter evolutes. It is known [1, page 37, 2.30] that the evolute of $\boldsymbol{\gamma}$ is defined by

$$\boldsymbol{e}_{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(s) + \frac{1}{\kappa(s)}\boldsymbol{b}(s).$$

Therefore, we have a completely different situation from those of the Euclidean plane curves.

5.2 Curves on the hyperbolic plane $H^2_+(-1)$

Suppose that $M = H^2_+(-1)$. In this case, we have $\boldsymbol{n}_{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(s)$, $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ with $\|\boldsymbol{t}(s)\| = 1$ and $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}_{\boldsymbol{\gamma}}(s)$. We call $\{\boldsymbol{t}, \boldsymbol{\gamma}, \boldsymbol{b}\}$ the *Lorentzian Saban frame*. Therefore, we have $\kappa_n(s) \equiv 1, \tau_g(s) \equiv 0$. By the Frenet-Serret type formulae in §3, we have the following[2]:

$$\begin{cases} \boldsymbol{t}'(s) = \boldsymbol{\gamma} + \kappa_g(s)\boldsymbol{b}(s), \\ \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{b}'(s) = -\kappa_g(s)\boldsymbol{t}(s), \end{cases}$$

Here, we have $\sigma_H = (\frac{1}{\kappa_g})'$, and $\sigma_D = {\kappa_g}'$. Moreover we have the hyperbolic evolute of $\boldsymbol{\gamma}$ relative to M, $\boldsymbol{h}_{\boldsymbol{\gamma}}(s) = \left(\frac{\kappa_g}{\sqrt{\kappa_g^2-1}}\boldsymbol{n}_{\boldsymbol{\gamma}} + \frac{1}{\sqrt{\kappa_g^2-1}}\boldsymbol{b}\right)(s)$ for $\kappa_g^2(s) > 1$ and the de Sitter evolute of $\boldsymbol{\gamma}$ relative to M, $\boldsymbol{d}_{\boldsymbol{\gamma}}(s) = \left(\frac{\kappa_g}{\sqrt{1-\kappa_g^2}}\boldsymbol{n}_{\boldsymbol{\gamma}} - \frac{1}{\sqrt{1-\kappa_g^2}}\boldsymbol{b}\right)(s)$ for $\kappa_g^2(s) < 1$. The singularities of the hyperbolic evolute $\boldsymbol{h}_{\boldsymbol{\gamma}}(s)$ was investigated in [2]. By Theorem 4.3, the evolute at $\boldsymbol{h}_{\boldsymbol{\gamma}}(s_0)$ is the regular if $\sigma_H = (\frac{1}{\kappa_g})' \neq 0$, and is the ordinary cusp locally if $\sigma_H = (\frac{1}{\kappa_g})' = 0$, and ${\sigma_H}' = (\frac{1}{\kappa_g})'' \neq 0$. Since $\sigma_H = \frac{-\kappa_g'}{\kappa_g^2}$ and $\sigma'_H = \frac{-\kappa_g''\kappa_g^2 - 2(\kappa_g')^2}{\kappa_g^3}$, $\sigma_H(s) = \sigma'_H(s) = 0$ if and only if $\kappa_g'(s) = \kappa_g''(s) = 0$. Moreover the de Sitter evolute at $\boldsymbol{d}_{\boldsymbol{\gamma}}(s_0)$ is the regular if $\sigma_D = {\kappa_g}' \neq 0$, and is the ordinary cusp locally if $\sigma_D = {\kappa_g}' = 0$, and ${\sigma_D}' = {\kappa_g}'' \neq 0$. Therefore the cusp of the pseudo-spherical evolute corresponding to the point $\boldsymbol{\gamma}(s)$ with $\kappa_g'(s) = 0$ and $\kappa_g''(s) \neq 0$. We call it the *ordinary vertex of* $\boldsymbol{\gamma}$.

Here we consider H-slices under the condition $\kappa_g^2 > 1$ and consider D-slices under the condition $\kappa_g^2 < 1$. Since we have a well-known classification theorem of totally umbilical surfaces (cf.,[4]), we can conclude that an H-slice is a (hyperbolic) circle if $\kappa_g \neq 0$ and $\kappa_g^2 > 1$, and a D-slice is an equidistant curve if $\kappa_g \neq 0$ and $\kappa_g^2 < 1$.

5.3 Curves on a surface of graphical representation

Suppose that $\boldsymbol{X}(x, y) = (f(x, y), x, y)$ with $f(0, 0) = 0$ and $\partial f/\partial x(0, 0) = \partial f/\partial y(0, 0) = 0$, which is called the *Monge form*. Here we denote $f_x = \partial f/\partial x$, $f_y = \partial f/\partial y$, $\boldsymbol{X}_x = \partial \boldsymbol{X}/\partial x$ and $\boldsymbol{X}_y = \partial \boldsymbol{X}/\partial y$. Since $\boldsymbol{X}$ is a spacelike embedding, we have $\|\boldsymbol{X}_x\| = -f_x^2 + 1 > 0$, $\|\boldsymbol{X}_y\| = -f_y^2 + 1 > 0$, and a unit timelike normal vector field $\boldsymbol{n}(p) = \frac{\boldsymbol{X}_x(u) \wedge \boldsymbol{X}_y(u)}{\|\boldsymbol{X}_x(u) \wedge \boldsymbol{X}_y(u)\|} = -\frac{1}{\sqrt{1-f_x^2-f_y^2}}(1, f_x, f_y)$ with $-1 + f_x^2 + f_y^2 < 0$ where $p = \boldsymbol{X}(u) = \boldsymbol{X}(x, y)$.

(1) De Sitter evolutes

We now consider the curve $\boldsymbol{\gamma}(x) = \boldsymbol{X}(x, 0) = (f(x, 0), x, 0)$, where $f(x, y)$ is a smooth function. Here we denote $\frac{d\gamma}{dx} = \dot{\boldsymbol{\gamma}}$, $f_x = f_x(x, 0)$, and $f_y(x, 0)$. Since $\dot{\boldsymbol{\gamma}}(x) = (f_x, 1, 0)$, we have the unit tangent vector field $\boldsymbol{t}(x) = \frac{1}{\sqrt{1-f_x^2}}(f_x, 1, 0)$, and the two unit normal vector fields $\boldsymbol{n}_{\boldsymbol{\gamma}}(x) = \frac{-1}{\sqrt{1-f_x^2-f_y^2}}(1, f_x, f_y)$, $\boldsymbol{b}(x) = \frac{1}{\sqrt{1-f_x^2-f_y^2}\sqrt{1-f_x^2}}(f_y, f_x f_y, 1 - f_x^2)$. By straightforward calculations, we have $\kappa_g(x) = \left\langle \frac{d\boldsymbol{t}}{ds}(x), \boldsymbol{b}(x) \right\rangle = \frac{-f_y f_{xx}}{(1-f_x^2))^{\frac{3}{2}}(1-f_x^2-f_y^2)^{\frac{1}{2}}}$, and $\kappa_n(x) = \left\langle \frac{d\boldsymbol{t}}{ds}(x), \boldsymbol{n}_{\boldsymbol{\gamma}}(x) \right\rangle = \frac{f_{xx}}{(1-f_x^2)(1-f_x^2-f_y^2)^{\frac{1}{2}}}$, where s is the arc-length. Therefore we have $\frac{\kappa_g}{\kappa_n}(x) = \frac{-f_y}{(1-f_x^2)^{\frac{1}{2}}}$, that is, $\left(\frac{\kappa_g}{\kappa_n}\right)(x)(1-f_x^2)^{\frac{1}{2}} = -f_y$. By differentiation of the both sides, we have $\left(\frac{\kappa_g}{\kappa_n}\right)'(1-f_x^2)^{\frac{1}{2}} + \left(\frac{\kappa_g}{\kappa_n}\right)\frac{-f_x f_{xx}}{(1-f_x^2)^{\frac{1}{2}}} = -f_{yx}$.

Moreover, we have $\left(\frac{\kappa_g}{\kappa_n}\right)''(1-f_x^2)^{\frac{1}{2}} - 2\left(\frac{\kappa_g}{\kappa_n}\right)'\frac{f_xf_{xx}}{(1-f_x^2)^{\frac{1}{2}}} + \left(\frac{\kappa_g}{\kappa_n}\right)\left\{\frac{-f_xf_{xx}}{(1-f_x^2)^{\frac{1}{2}}}\right\}' = -f_{yxx}$. Therefore we have

$$\frac{\kappa_g}{\kappa_n}(0) = -f_y(0,0) = 0, \left(\frac{\kappa_g}{\kappa_n}\right)'(0) = -f_{yx}(0,0), \left(\frac{\kappa_g}{\kappa_n}\right)''(0) = -f_{yxx}(0,0).$$

On the other hand, we have

$$\frac{d\boldsymbol{n}_{\boldsymbol{\gamma}}}{ds}(x) = \frac{1}{(1-f_x^2)^{\frac{1}{2}}(1-f_x^2-f_y^2)^{\frac{3}{2}}}\left(\begin{vmatrix} -f_y & f_{xx} \\ f_x & f_{yx}\end{vmatrix}, \begin{vmatrix} -f_xf_y & f_{xx} \\ 1-f_y^2 & f_{yx}\end{vmatrix}, \begin{vmatrix} f_x^2-1 & f_{xx} \\ -f_xf_y & f_{yx}\end{vmatrix}\right).$$

Then we have

$$\begin{aligned}\tau_g(x) &= \left\langle \boldsymbol{b}(x), \frac{d\boldsymbol{n}_{\boldsymbol{\gamma}}}{ds}(x)\right\rangle \\ &= \frac{1}{(1-f_x^2)(1-f_x^2-f_y^2)^2}\{f_y^2f_{yx} - f_x^2f_y^2f_{yx} + f_xf_yf_{xx} \\ &\quad + f_{xx}f_xf_y^3 - f_x^3f_yf_{xx} - f_{yx} + 2f_{yx}f_x^2 - f_{yx}f_x^4\}.\end{aligned}$$

By the same calculations as those for $\frac{\kappa_g}{\kappa_n}$, we have

$$\tau_g(0) = -f_{yx}(0,0), \text{ and } \tau_g'(0) = -f_{yxx}(0,0).$$

We now consider the special case

$$f(x,y) = a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3.$$

Then we have $f_{yx}(0,0) = a_{11},\ f_{yxx}(0,0) = a_{21},\ f_{xx}(0,0) = a_{20},\ f_{xxx}(0,0) = a_{30}$. Since $\kappa_g(0) = 0$, we can define $\boldsymbol{d}_{\boldsymbol{\gamma}}$ where $\kappa_n(0) = a_{20} \neq 0$. Moreover we have

$$\sigma_D(0) = -2a_{11},\ \sigma_D'(0) = -2a_{21}$$

It means that we have $\sigma_D(0) = 0, \sigma_D'(0) \neq 0$ if $a_{11} = 0$ and $a_{21} \neq 0$. Therefore $f(x,y)$, which is $a_{20} \neq 0$, $a_{11} = 0$ and $a_{21} \neq 0$, is an example of de Sitter evolute which has a cusp at the origin. For instance, we have$f(x,y) = x^2y - \frac{1}{2}x^2$.

(2) Hyperbolic evolutes

We now consider $\boldsymbol{X}(x,y) = (f(x,y), x, y)$ where $f(x,y) = \frac{1}{2}(y - g(x))$ and $g(x)$ is a smooth function, and we investigate the curve $\boldsymbol{\gamma}(x) = \boldsymbol{X}(x, g(x)) = (0, x, g(x))$. Here we denote $\frac{d\boldsymbol{\gamma}}{dx} = \dot{\boldsymbol{\gamma}}$, $\frac{dg}{dx} = \dot{g}$, $f_x = f_x(x,0)$, and $f_y(x,0)$. Since $\dot{\boldsymbol{\gamma}}(x) = (0,1,\dot{g}(x))$, we have the unit tangent vector field $\boldsymbol{t}(x) = \frac{1}{\sqrt{1+\dot{g}^2(x)}}(0,1,\dot{g}(x))$, and the two normal vector fields

$$\boldsymbol{n}_{\boldsymbol{\gamma}}(x) = \frac{-1}{\sqrt{1-f_x^2(x,0)-f_y^2(x,0)}}(1, f_x(x,0), f_y(x,0)),$$

$$\boldsymbol{b}(x) = \frac{1}{\sqrt{1-f_x^2(x,0)-f_y^2(x,0)}\sqrt{1+\dot{g}^2(x)}}(\dot{g}(x)f_x(x,0) - f_y(x,0), -\dot{g}(x), 1).$$

By the same arguments as those in the case for de Sitter evolutes, we have

$$\frac{\kappa_g}{\kappa_n}(x) = \frac{-(1+\dot{g}^2(x))^{\frac{1}{2}}}{-\dot{g}(x)f_x(x,0) + f_y(x,0)},\ \tau_g(x) = \frac{-1}{(1+\dot{g}^2(x))(1-f_x^2-f_y^2)^2}(-\dot{g}(x)f_{xx}(x,0) + f_{yx}(x,0))$$

Moreover we have

$$\frac{\kappa_g}{\kappa_n}(0) = -2,\ \left(\frac{\kappa_g}{\kappa_n}\right)'(0) = 0,\ \left(\frac{\kappa_g}{\kappa_n}\right)''(0) = 8\ddot{g}(0),\ \tau_g(0) = 0,\ \tau_g'(0) = -\frac{8}{3}\ddot{g}^2(0).$$

We now consider the special case $g(x) = a_2x^2 + a_3x^3$. Then we have $\ddot{g}_{xx}(0) = a_2,\ \dddot{g}(0) = a_3$. Since $\kappa_n(0) = -\frac{4}{3}a_2$, we can define $\boldsymbol{h_\gamma}$ where $a_2 \neq 0$. Moreover we have

$$\sigma_H(0) = 0,\ \sigma_H'(0) \neq 0$$

if $a_{20} \neq 0$. Therefore $g(x)$, which is $a_2 \neq 0$, is an example of hyperbolic evolute which has a cusp at the origin.

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