HOKKAIDO UNIVERSITY

Title	Construction of dynamics and time-ordered exponential for unbounded non-symmetric Hamiltonians
Author(s)	Futakuchi, Shinichiro; Usui, Kouta
Citation	Hokkaido University Preprint Series in Mathematics, 1041, 1-41
Issue Date	2013-10-15
DOI	https://doi.org/10.14943/84185
Doc URL	https://hdl.handle.net/2115/69845
Type	departmental bulletin paper
File Information	re1041.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

Construction of dynamics and time-ordered exponential for unbounded non-symmetric Hamiltonians

October 11, 2013

Shinichiro Futakuchi and Kouta Usui

Department of Mathematics, Hokkaido University
060-0810, Sapporo, Japan.

Abstract

We prove under certain assumptions that there exists a solution of the Schrödinger or the Heisenberg equation of motion generated by a linear operator H acting in some complex Hilbert space $\mathcal{H}$, which may be unbounded, *not symmetric*, or *not normal*. We also prove that, under the same assumptions, there exists a time evolution operator in the interaction picture and that the evolution operator enjoys a useful series expansion formula. This expansion is considered to be one of the mathematically rigorous realizations of so called "time-ordered exponential", which is familiar in the physics literature. We apply the general theory to prove the existence of dynamics for the mathematical model of Quantum Electrodynamics (QED) quantized in the Lorenz gauge, the interaction Hamiltonian of which is not even symmetric or normal.

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space and H be a linear operator on $\mathcal{H}$. We consider the initial value problem for the Schrödinger equation

$$\frac{\partial \xi(t)}{\partial t} = -iH\xi(t), \quad \xi(0) = \xi, \tag{1.1}$$

or for the Heisenberg equation

$$\frac{dB(t)}{dt} = [iH, B(t)], \quad B(0) = B, \tag{1.2}$$

where B is a possibly unbounded linear operator on $\mathcal{H}$, and $[X, Y] := XY - YX$. In the context of quantum mechanics, the parameter $t \in \mathbb{R}$ represents time, and H is regarded as a Hamiltonian of the quantum system under consideration. At time $t \in \mathbb{R}$, $\xi(t)$ or $B(t)$ describes a time developed state vector or a time developed observable, respectively. Then, the general mathematical study of the initial value problems (1.1) or (1.2) is of great interest since it will reveal dynamics of a certain class of quantum systems.

In the ordinary formulation of quantum mechanics, a Hamiltonian H is assumed to be a self-adjoint operator. In this case, the solutions of these equations are given by

$$\xi(t) = e^{-itH}\xi, \tag{1.3}$$

$$B(t) = e^{itH}Be^{-itH}, \tag{1.4}$$

with some suitable conditions for operator domains in the Heisenberg case (See [2], in detail). However, in some models, the Hamiltonian H may not be self-adjoint or not even normal. When H is unbounded and not normal, the above time evolution operator e^{-itH} does not immediately make sense since for unbounded H it is usually defined through operational calculus. In such cases, it is not obvious at all that there exist solutions of these equations.

The most important realistic examples that can cause this difficulty contain the mathematical model of Quantum Electrodymamics (QED) when it is quantized in a Lorentz covariant gauge such as Lorenz gauge [13, 7]. In the Lorenz-gauge QED, we have to adopt a vector space with an indefinite metric as a vector space of quantum mechanical state vectors, in order to realize the canonical commutation relations. Indefinite metric results in a non-symmetric Hamiltonian which is not even normal, and thus it is far from trivial that dynamics of the Lorenz-gauge QED really exists. To obtain dynamics for such models, one may apply the general theory of evolution equations or Cauchy problems by estimating the resolvent operators [3, 6], but we will take another way to avoid hard resolvent estimates. The first motivation of the present study is to establish a general theory as to the existence of dynamics with Hamiltonians which is not symmetric and not even normal.

Another motivation of the present work also comes from quantum theory. We consider a system with a Hamiltonian of the type

$$H = H_0 + H_1, \tag{1.5}$$

where H_0 is a *solvable* Hamiltonian (the dynamics of which we already know) and H_1 is an interaction Hamiltonian which causes unknown dynamics. To study a quantum mechanical scattering problem with Hamiltonians of this form, it is often useful to employ the so called interaction picture, in which both state vectors and observables evolve in time. The evolution operator in the interaction picture from time t' to time t — which is usually denoted by $U(t, t')$ — is a solution of the differential equations

$$\frac{\partial}{\partial t} U(t, t') = -iH_1(t)U(t, t'), \tag{1.6}$$

$$\frac{\partial}{\partial t'} U(t, t') = iU(t, t')H_1(t'), \tag{1.7}$$

with

$$H_1(t) := e^{itH_0} H_1 e^{-itH_0}. \tag{1.8}$$

It is easy to *heuristically* derive the series expansion of $U(t, t')$

$$U(t, t') = 1 + (-i) \int_{t'}^{t} d\tau_1 \, H_1(\tau_1) + (-i)^2 \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2 \, H_1(\tau_1)H_1(\tau_2) + \dots . \tag{1.9}$$

This expansion formula (1.9) is well known to be quite useful in computing scattering amplitudes of elementary particles such as electrons or photons, and the results dramatically agree with the high energy experiments, even though these computations contain a lot of mathematically unrigorous steps [15, 16, 10].

The series expansion (1.9) has already been rigorously analyzed, in the case where H_1 is bounded (See, e.g., Refs. [5], [11] Section X.12, [4], [9], [8]). However, in the case where H_1 is not bounded, it seems that there have been few mathematically rigorous studies of the series expansion (1.9) in an abstract or a general form. The second motivation of the present work is to prove in mathematically rigorous manner with certain assumptions that there exists a time evolution operator $U(t, t')$ satisfying (1.6) and (1.7) which possesses the series representation (1.9) on certain dense subspace, including the case where H_1 is neither bounded nor normal. The solutions of Schrödinger or Heisenberg equation will be constructed by using the operator $U(t, t')$. Here, we stress that our proof does not only state the

existence of the solutions abstractly but also derive an explicit series expansion (1.9) for them, which would be useful for the practical applications in mathematical physics.

This paper is organized as follows. In Section 2, we will summarize our results. In Section 3, a solution $U(t,t')$ of the differential equations (1.6), (1.7) is explicitly constructed. In Section 4, we will derive several properties of the solution $U(t,t')$. In Section 5, we will construct solutions of Schrödinger and Heisenberg equations of motion. In Section 6, QED quantized in the Lorenz gauge will be discussed and it will be proved that there exists a solution of the Heisenberg equations of motion for quantized fields. In Appendix, some further mathematical properties of $U(t,t')$ will be studied.

2 Main Results

The inner product and the norm of $\mathcal{H}$ are denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ (anti-linear in the first variable) and $\|\cdot\|_{\mathcal{H}}$ respectively. When there can be no danger of confusion, then the subscript $\mathcal{H}$ in $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ is omitted. For a linear operator T in $\mathcal{H}$, we denote its domain (resp. range) by $D(T)$ (resp. $R(T)$). We also denote the adjoint of T by T^* and the closure by $\bar{T}$ if these exist. For a self-adjoint operator T, $E_T(\cdot)$ denotes the spectral measure of T. The symbol $T \restriction D$ denotes the restriction of a linear operator T to the subspace D.

Let H_0 be a self-adjoint operator on $\mathcal{H}$ and H_1 be a densely defined closed operator on $\mathcal{H}$. Set

$$H := H_0 + H_1, \tag{2.1}$$

with the domain $D(H_0) \cap D(H_1)$.

First, we assume that there exists an operator A in $\mathcal{H}$ satisfying the following conditions:

Assumption 2.1. *(I) A is self-adjoint and non-negative.*

(II) A and H_0 are strongly commuting.

(III) H_1 is $A^{1/2}$- bounded, where $A^{1/2}$ defined through operational calculus.

(IV) There exists a constant $b > 0$ such that, for all $L \geq 0$, $\xi \in R(E_A([0,L]))$ implies $H_1\xi \in R(E_A([0,L+b]))$.

We remark that the above condition (IV) comes from the following physical consideration. Suppose that H is the Hamiltonian of a certain quantum system. The above self-adjoint operator A is expected to be an observable quantity of the quantum system under consideration, typically a particle number in application to quantum field theories (see application in Section 6.). Roughly speaking, the condition (IV) says that the value of the observable A increase at most b by one interaction.

Hereafter, we use the following notations:

$$V_L := R(E_A([0,L])), \quad L \geq 0, \tag{2.2}$$

$$D := \bigcup_{L\geq 0} V_L. \tag{2.3}$$

Since A is assumed to be self-adjoint, it follows that D is a dense subspace in $\mathcal{H}$.

Our first result is:

Theorem 2.1. *Under Assumption 2.1, for each $t,t' \in \mathbb{R}$, $\xi \in D$, the series:*

$$U(t,t')\xi := \xi + (-i)\int_{t'}^{t} d\tau_1\, H_1(\tau_1)\xi + (-i)^2 \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2\, H_1(\tau_1)H_1(\tau_2)\xi + \cdots \tag{2.4}$$

converges absolutely, where each of integrals are strong integrals. Furthermore, $U(t,t')$ forms an evolution operator on D, that is, the following (i), (ii) hold.

(i) For fixed $t' \in \mathbb{R}$ and $\xi \in D$, the vector valued function $\mathbb{R} \ni t \mapsto U(t,t')\xi$ is strongly continuously differentiable, and $U(t,t')\xi \in D(H_1(t))$. Moreover, $U(t,t')\xi$ satisfies

$$\frac{\partial}{\partial t}U(t,t')\xi = -iH_1(t)U(t,t')\xi. \tag{2.5}$$

(ii) For fixed $t \in \mathbb{R}$ and $\xi \in D$, the vector valued function $\mathbb{R} \ni t' \mapsto U(t,t')\xi$ is strongly continuously differentiable, and satisfies

$$\frac{\partial}{\partial t'}U(t,t')\xi = iU(t,t')H_1(t')\xi. \tag{2.6}$$

Next, we assume the following properties in addition.

Assumption 2.2. *(I) H_1^* is $A^{1/2}$- bounded.*

(II) There exists a constant $b' > 0$ such that, for all $L \geq 0$, $\xi \in V_L$ implies $H_1^*\xi \in V_{L+b'}$.

Then, we have

Theorem 2.2. *Under Assumptions 2.1-2.2, it follows that $D \subset D(U(t,t')^*)$, and for all $\xi \in D$, $U(t,t')^*\xi$ is strongly continuously differentiable with respect to t and t', and satisfies*

$$\frac{\partial}{\partial t}U(t,t')^*\xi = iU(t,t')^*H_1(t)^*\xi, \tag{2.7}$$

$$\frac{\partial}{\partial t'}U(t,t')^*\xi = -iH_1(t')^*U(t,t')^*\xi. \tag{2.8}$$

In particular, $U(t,t')$ is closable.

The time evolution operator $U(t,t')$ has the following properties.

Theorem 2.3. *Under Assumptions 2.1-2.2, the following (i) and (ii) hold.*

(i) For all $\xi \in D$, $t, t', t'' \in \mathbb{R}$, $U(t,t)\xi = \xi$ and the operator equality

$$\overline{U(t,t')}U(t',t'') = U(t,t''). \tag{2.9}$$

holds.

(ii) For any $s, t, t' \in \mathbb{R}$, the operator equality

$$e^{isH_0}\overline{U(t,t')}e^{-isH_0} = \overline{U(t+s,t'+s)} \tag{2.10}$$

holds.

If we assume in addition that H_1 is symmetric, then Assumption 2.1 implies Assumption 2.2 and stronger results follow:

Theorem 2.4. *Suppose that Assumption 2.1 holds, and let H_1 be a closed symmetric operator. Then, $\overline{U(t,t')}$ is unitary and the following properties hold.*

(i) The operator $U(t,t')$ satisfies the following operator equalities:

$$\overline{U(t,t)} = I, \quad \overline{U(t,t')}\,\overline{U(t',t'')} = \overline{U(t,t'')}, \tag{2.11}$$

where I denotes the identity operator.

(ii) $U(t,t')$ *is unique in the following sense. If there exist a dense subspace* $\widetilde{D}$ *in* $\mathcal{H}$ *and an operator valued function* $V(t,t')$ $(t,t' \in \mathbb{R})$ *such that* $\widetilde{D} \subset D(V(t,t'))$ *for all* $t,t' \in \mathbb{R}$ *and for* $\xi \in \widetilde{D}$, $V(t,t')\xi$ *is strongly differentiable with respect to t, and* $V(t,t')\xi \in D(H_1(t))$, *which satisfies*

$$V(t,t)\xi = \xi, \quad \frac{\partial}{\partial t}V(t,t')\xi = -iH_1(t)V(t,t')\xi, \quad \xi \in \widetilde{D}, \quad t,t' \in \mathbb{R}, \tag{2.12}$$

then $V(t,t') \restriction \widetilde{D}$ *is closable and* $\overline{V(t,t') \restriction \widetilde{D}} = \overline{U(t,t')}$. *In particular, if* $D(V(t,t')) = \mathcal{H}$ *and* $V(t,t')$ *is bounded for all* $t,t' \in \mathbb{R}$, *then* $V(t,t') = \overline{U(t,t')}$.

We discuss the existence of the dynamics generated by H. Let

$$W(t) := e^{-itH_0}\overline{U(t,0)}, \ t \in \mathbb{R}. \tag{2.13}$$

Theorem 2.5. *Suppose that Assumptions 2.1-2.2 hold. Then, for each* $\xi \in D(H_0) \cap D$, *the vector valued function* $t \mapsto \xi(t) := W(t)\xi$ *is a solution of the initial value problem for the Schrödinger equation:*

$$\frac{d}{dt}\xi(t) = -iH\xi(t), \quad \xi(0) = \xi. \tag{2.14}$$

If H is symmetric, we obtain the following result:

Theorem 2.6. *Suppose that Assumption 2.1 holds and let* H_1 *be a closed symmetric operator. Then there exists a unique self-adjoint operator* $\widetilde{H}$ *such that*

$$W(t) = e^{-it\widetilde{H}}, \quad t \in \mathbb{R}. \tag{2.15}$$

Moreover

$$\overline{U(t,t')} = e^{itH_0}e^{-i(t-t')\widetilde{H}}e^{-it'H_0}, \quad t,t' \in \mathbb{R} \tag{2.16}$$

and

$$\overline{H \restriction D \cap D(H_0)} \subset \widetilde{H}, \tag{2.17}$$

In particular, if H *is essentially self-adjoint on* $D \cap D(H_0)$, *then we have*

$$\overline{H} = \widetilde{H}. \tag{2.18}$$

The existence of a solution of the Heisenberg equation (1.2) is ensured under the following assumptions. Let B be a linear operator in $\mathcal{H}$. We assume

Assumption 2.3. *(I)* B *and* B^* *are* $A^{1/2}$*-bounded and closed.*

(II) There exists a constant $b_0 > 0$ *such that, for all* $L \geq 0$, $\xi \in V_L$ *implies* $B\xi, B^*\xi \in V_{L+b_0}$.

Then, we have

Theorem 2.7. *Under Assumptions 2.1, 2.2 and 2.3, it follows that* $D \subset D(W(-t)BW(t))$ *and the operator valued function* $B(t)$ *defined as*

$$D(B(t)) := D, \quad B(t)\xi := W(-t)BW(t)\xi, \quad \xi \in D, \quad t \in \mathbb{R}, \tag{2.19}$$

is a solution of weak Heisenberg equation:

$$\text{w-}\frac{d}{dt}B(t) = \text{w-}[iH, B(t)] \quad on \quad D(H_0) \cap D, \tag{2.20}$$

where (2.20) *is the abbreviated notation for*

$$\frac{d}{dt}\langle \eta, B(t)\xi \rangle = \langle (iH)^*\eta, B(t)\xi \rangle - \langle B(t)^*\eta, iH\xi \rangle, \quad \xi, \eta \in D(H_0) \cap D. \tag{2.21}$$

Finally, we discuss the existence of a strong solution of the Heisenberg equation (1.2). Let

$$B_0(t) := e^{itH_0} B e^{-itH_0}, \quad t \in \mathbb{R}. \tag{2.22}$$

If the operator valued function $\mathbb{R} \ni t \mapsto B_0(t)$ satisfies the following conditions, then the stronger result holds.

Assumption 2.4. *(I) For each $\xi \in D(A^{1/2})$, $B_0(t)\xi$ is strongly continuously differentiable.*

(II) The strong derivative of $B_0(t)$ on $D(A^{1/2})$,

$$B_0'(t)\xi := \frac{d}{dt} B_0(t)\xi, \quad \xi \in D(A^{1/2})$$

is closable for all $t \in \mathbb{R}$.

(III) $B_0'(t)$ ($t \in \mathbb{R}$) are uniformly $A^{1/2}$-bounded. That is, there exist constants $c_0, c_1 \geq 0$ such that for all $t \in \mathbb{R}$ and $\xi \in D(A^{1/2})$,

$$\|B_0'(t)\xi\| \leq c_0 \|A^{1/2}\xi\| + c_1 \|\xi\|.$$

The next theorem is concerned with the existence of a strong solution of the Heisenberg equation of motion (1.2).

Theorem 2.8. *Under Assumptions 2.1, 2.2, 2.3 and 2.4, it follows that for each $\xi \in D$, the function $\mathbb{R} \ni t \mapsto B(t)\xi$ is strongly continuously differentiable, and satisfies*

$$\frac{d}{dt} B(t)\xi = W(-t)[iH_1, B]W(t)\xi + U(0,t)\overline{B_0'(t)}U(t,0)\xi. \tag{2.23}$$

Moreover, for each $\xi \in D(H_0) \cap D$, the equality

$$\frac{d}{dt} B(t)\xi = [iH, B(t)]\xi. \tag{2.24}$$

holds.

3 Iterative construction of an evolution operator $U(t, t')$

In this section, we prove Theorem 2.1 and Theorem 2.2.

Lemma 3.1. *Under Assumption 2.1, $H_1(t)(A+1)^{-1/2}$ is bounded and there exists a constant $C \geq 0$ independent of $t \in \mathbb{R}$ such that*

$$\|H_1(t)(A+1)^{-1/2}\| \leq C, \quad t \in \mathbb{R}. \tag{3.1}$$

Proof. Since H_1 is $A^{1/2}$-bounded, there exist constants $c_0, c_1 \geq 0$ satisfying

$$\|H_1\xi\| \leq c_0 \|A^{1/2}\xi\| + c_1 \|\xi\|, \quad \xi \in D. \tag{3.2}$$

Hence, for each $\xi \in D$, we obtain by operational calculus

$$\begin{aligned} \|H_1(t)\xi\| &= \|H_1 e^{-itH_0}\xi\| \\ &\leq c_0 \|A^{1/2} e^{-itH_0}\xi\| + c_1 \|\xi\| \\ &= c_0 \|A^{1/2}\xi\| + c_1 \|\xi\|. \end{aligned}$$

On the other hand, we have

$$\|(A+1)^{1/2}\xi\|^2 = \|A^{1/2}\xi\|^2 + \|\xi\|^2.$$

Thus, from the elementary inequality

$$(a+b)^2 \le 2a^2 + 2b^2, \quad a, b \ge 0,$$

we obtain

$$\|H_1(t)\xi\| \le C\|(A+1)^{1/2}\xi\|, \tag{3.3}$$

for some constant $C \ge 0$. This implies that $H_1(t)(A+1)^{-1/2}$ is bounded and

$$\|H_1(t)(A+1)^{-1/2}\| \le C.$$

□

Define

$$L_\xi := \inf\{L \ge 0 \,|\, \xi \in V_L\}, \quad \xi \in D.$$

For $t, t' \in \mathbb{R}$, we define a sequence of operators $U_n(t,t')$ for $n = 0, 1, 2, \dots$ in the following way: For $n = 0$, put

$$D(U_0(t,t')) = D, \quad U_0(t,t')\xi = \xi, \quad \xi \in D. \tag{3.4}$$

For $n \ge 1$, we inductively define

$$D(U_n(t,t')) = D, \quad U_n(t,t')\xi = -i\int_{t'}^{t} d\tau H_1(\tau)U_{n-1}(\tau,t')\xi, \quad \xi \in D, \tag{3.5}$$

where the integration is understood as a strong Riemann integral. It should be confirmed that $U_n(t,t')$ is certainly well defined. This follows from the following lemma:

Lemma 3.2. *Suppose that Assumption 2.1 holds. Let $t, t' \in \mathbb{R}$. Then, for $n = 0, 1, 2, \dots$, there uniquely exists an operator $U_n(t,t')$ such that*

(i) $D(U_n(t,t')) = D$,

(ii) $U_n(t,t')\xi$ *is strongly continuous in* t,

(iii) For all $t, t' \in \mathbb{R}$,

$$U_n(t,t')\xi \in V_{L_\xi + nb},$$

(iv) $H_1(t)U_n(t,t')\xi$ *is strongly continuous in* t,

and satisfies the recursion relations

$$U_{n+1}(t,t')\xi = -i\int_{t'}^{t} d\tau H_1(\tau)U_n(\tau,t')\xi, \quad \xi \in D, \tag{3.6}$$

for $n = 0, 1, \dots$, where the integration is a strong Riemann integral.

Proof. If they exist, the uniqueness is obvious by (3.6).

We prove the existence by induction. Let $n = 0$. If we define $U_0(t,t')$ as above

$$D(U_0(t,t')) = D, \quad U_0(t,t')\xi = \xi, \quad \xi \in D, \tag{3.7}$$

then (i), (ii), and (iii) clearly hold. To prove (iv), we note that

$$H_1(t)\xi = e^{itH_0}H_1(A+1)^{-1/2}e^{-itH_0}(A+1)^{1/2}\xi.$$

The right-hand side is strongly continuous since $H_1(A+1)^{-1/2}$ is bounded. This proves (iv) in the case where $n = 0$.

Suppose that the lemma is true for $n = 0, 1, 2, \ldots, k$ for some $k \geq 0$. Then, we can define $U_{k+1}(t, t')$ via strong Riemann integral as

$$D(U_{k+1}(t,t')) = D, \quad U_{k+1}(t,t')\xi = -i\int_{t'}^{t} d\tau H_1(\tau)U_k(\tau,t')\xi, \quad \xi \in D, \tag{3.8}$$

due to (iv). The operator $U_{k+1}(t, t')$ clearly satisfies (i) and (ii). Let us prove $U_{k+1}(t, t')$ satisfies (iii). From the induction hypothesis (iii), and Assumption 2.1 (II), (IV), we find

$$H_1(t)U_k(t,t')\xi \in V_{L_\xi+(k+1)b}.$$

Since the subspace $V_{L_\xi+(k+1)b}$ is closed, the strong Riemann integral

$$\int_{t'}^{t} d\tau H_1(\tau)U_k(\tau,t')\xi$$

also belongs to $V_{L_\xi+(k+1)b}$. This proves (iii) for $n = k + 1$. The condition (iv) is proved as follows. Since $U_{k+1}(t, t')\xi$ is strongly differentiable with respect to t, it is strongly continuous. On the other hand, the map

$$t \mapsto e^{itH_0}H_1(A+1)^{-1/2}e^{-itH_0}(A+1)^{1/2}$$

is continuous in the strong operator topology. Thus, we have for $h \in \mathbb{R}$

$$\begin{aligned}
&\|H_1(t+h)U_{k+1}(t+h,t')\xi - H_1(t)U_{k+1}(t,t')\xi\| \\
&\quad \leq \|(e^{i(t+h)H_0}H_1(A+1)^{-1/2}e^{-i(t+h)H_0} - e^{itH_0}H_1(A+1)^{-1/2}e^{-itH_0})(A+1)^{1/2}U_{k+1}(t,t')\xi\| \\
&\qquad + \|H_1(A+1)^{-1/2}\|\,\|(A+1)^{1/2}(U_{k+1}(t+h,t') - U_{k+1}(t,t'))\xi\|,
\end{aligned} \tag{3.9}$$

which shows (iv). The formula (3.6) holds by the construction. Therefore, the lemma remains true for $n = k + 1$ and this completes the proof. □

Lemma 3.3. *Suppose that Assumption 2.1 holds. Let $t, t' \in \mathbb{R}$ and $\xi \in D$. Then, we can estimate*

$$\|U_n(t,t')\xi\| \leq \frac{|t-t'|^n}{n!}C^n(L_\xi + (n-1)b + 1)^{1/2}\cdots(L_\xi+1)^{1/2}\|\xi\|, \tag{3.10}$$

for $n = 1, 2, \ldots$.

Proof. We prove by induction. Let $n = 1$. Then we have

$$\begin{aligned}
\|U_1(t,t')\xi\| &\leq \int_{t'}^{t} d\tau\,\|H_1(\tau)\xi\| \\
&\leq |t-t'|C(L_\xi+1)^{1/2}\|\xi\|.
\end{aligned} \tag{3.11}$$

Thus, (3.10) holds for $n = 1$. Assume that (3.10) holds for some $n \geq 0$. Then, if $t' \leq t$, we have

$$\begin{aligned}
\|U_{n+1}(t,t')\xi\| &\leq \int_{t'}^{t} d\tau\|H_1(\tau)U_n(\tau,t')\xi\| \\
&\leq \int_{t'}^{t} d\tau\|H_1(\tau)(A+1)^{-1/2}\|\,\|(A+1)^{1/2}U_n(\tau,t')\xi\| \\
&\leq \int_{t'}^{t} d\tau\, C\cdot(L_\xi+nb+1)^{1/2}\frac{|\tau-t'|^n}{n!}C^n(L_\xi+(n-1)b+1)^{1/2}\cdots(L_\xi+1)^{1/2}\|\xi\| \\
&\leq \frac{|t-t'|^{n+1}}{(n+1)!}C^{n+1}(L_\xi+nb+1)^{1/2}\cdots(L_\xi+1)^{1/2}\|\xi\|.
\end{aligned} \tag{3.12}$$

By a similar computation, one finds that this estimate is also true in the case where $t < t'$. Thus the induction completes and we obtain (3.10). □

Lemma 3.4. *Under Assumption 2.1, for all $t, t' \in \mathbb{R}$ and $\xi \in D$, the followings hold.*

$$\sum_{n=0}^{\infty} \|U_n(t,t')\xi\| < \infty, \tag{3.13}$$

$$\sum_{n=0}^{\infty} \|H_1(t)U_n(t,t')\xi\| < \infty, \tag{3.14}$$

$$\sum_{n=0}^{\infty} \|U_n(t,t')H_1(t')\xi\| < \infty, \tag{3.15}$$

Furthermore, these convergences are uniform in (t,t') on any compact subset in $\mathbb{R}^2$.

Proof. From Lemma 3.3, we know

$$\sum_{n=0}^{\infty} \|U_n(t,t')\xi\| \le \sum_{n=0}^{\infty} \frac{|t-t'|^n}{n!} C^n (L_\xi + (n-1)b + 1)^{1/2} \cdots (L_\xi + 1)^{1/2} \|\xi\|. \tag{3.16}$$

Let $a_n(t,t')$ be the n-th term of the summation in the right-hand side of (3.16). One can see that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(t,t')}{a_n(t,t')} \right| = 0,$$

uniformly in (t,t') on any compact subset in the plane. By using d'Alembert's ratio test, the right hand side converges uniformly in (t,t') on any compact subset, and obtain (3.13).

The convergence of the other two series' (3.14) and (3.15) are also proved in a similar way, and we omit the proof. □

Lemma 3.5. *Suppose that Assumption 2.1 holds. Let $\xi \in D$ and $n = 1, 2, \ldots$. Then, the n-variable function from $\mathbb{R}^n$ into $\mathcal{H}$*

$$\mathbb{R}^n \ni (t_1, \ldots, t_n) \mapsto H_1(t_1) \ldots H_1(t_n)\xi \in D$$

is continuous on $\mathbb{R}^n$ with respect to the usual topology in $\mathbb{R}^n$ and the strong topology in $\mathcal{H}$.

Proof. Fix $\xi \in D$. We prove by induction with respect to n.

Set $n = 1$. We will prove

$$t \mapsto H_1(t)\xi$$

is strongly continuous. But, this has already been proved in the proof of Lemma 3.2.

Suppose that the assertion is valid for some $n \ge 1$. We prove the $(n+1)$-variable function

$$(t_1, \ldots, t_{n+1}) \mapsto H_1(t_1) \ldots H_1(t_{n+1})\xi$$

is strongly continuous at any $(t_1, \ldots, t_{n+1}) \in \mathbb{R}^{n+1}$. We use the abbreviated notations such as

$$\boldsymbol{t} = (t_2, \ldots, t_{n+1}) \in \mathbb{R}^n, \quad (t_1, \boldsymbol{t}) = (t_1, \ldots, t_{n+1}) \in \mathbb{R}^{n+1},$$

and $|\boldsymbol{t} - \boldsymbol{s}|$ denotes the standard Euclidean distance. Choose arbitrary $\epsilon > 0$. By the induction hypothesis, there is a $\delta(\boldsymbol{t}, \epsilon) > 0$ such that for all $\boldsymbol{s} = (s_2, \ldots, s_{n+1}) \in \mathbb{R}^n$ with $|\boldsymbol{t} - \boldsymbol{s}| < \delta(\boldsymbol{t}, \epsilon)$,

$$\|(H_1(t_2) \ldots H_1(t_{n+1}) - H_1(s_2) \ldots H_1(s_{n+1}))\xi\| < \frac{\epsilon}{2C(L_\xi + nb + 1)^{1/2}}.$$

On the other hand, since the mapping from $\mathbb{R}$ to the set of bounded linear operators in $\mathcal{H}$

$$t \mapsto e^{itH_0} H_1 (A+1)^{-1/2} e^{-itH_0}$$

is strongly continuous, there is a $\delta'(t_1, \boldsymbol{t}, \epsilon) > 0$ such that for all $s_1 \in \mathbb{R}$ with $|t_1 - s_1| < \delta'(t_1, \boldsymbol{t}, \epsilon)$

$$\|(H_1(t_1) - H_1(s_1))H_1(t_2)\dots H_1(t_{n+1})\xi\| < \frac{\epsilon}{2}, \tag{3.17}$$

because for any $\psi \in D(A^{1/2})$, we have

$$(H_1(t_1) - H_1(s_1))\psi = (e^{it_1H_0}H_1(A+1)^{-1/2}e^{-it_1H_0} - e^{is_1H_0}H_1(A+1)^{-1/2}e^{-is_1H_0}) \cdot (A+1)^{1/2}\psi.$$

From these estimates, one finds that for all $(s_1, \boldsymbol{s}) \in \mathbb{R}^{n+1}$ with $|(t_1, \boldsymbol{t}) - (s_1, \boldsymbol{s})| < \min(\delta(\boldsymbol{t}, \epsilon), \delta'(t_1, \boldsymbol{t}, \epsilon))$

$$\begin{aligned}
&\|H_1(t_1)\dots H_1(t_{n+1})\xi - H_1(s_1)\dots H_1(s_{n+1})\xi\| \\
&\le \|(H_1(t_1) - H_1(s_1))H_1(t_2)\dots H_1(t_{n+1})\xi\| + \\
&\quad + \|H_1(s_1)(H_1(t_2)\dots H_1(t_{n+1}) - H_1(s_2)\dots H_1(s_{n+1}))\xi\| \\
&\le \|(H_1(t_1) - H_1(s_1))H_1(t_2)\dots H_1(t_{n+1})\xi\| + \\
&\quad + C(L_\xi + nb + 1)^{1/2}\|(H_1(t_2)\dots H_1(t_{n+1}) - H_1(s_2)\dots H_1(s_{n+1}))\xi\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2C(L_\xi + nb + 1)^{1/2}} \cdot C(L_\xi + nb + 1)^{1/2} = \epsilon,
\end{aligned} \tag{3.18}$$

where we have used the fact that the vector $(H_1(t_2)\dots H_1(t_{n+1}) - H_1(s_2)\dots H_1(s_{n+1}))\xi$ belongs to $V_{L_\xi + nb}$. This proves the lemma. □

From Lemma 3.5, we can define a strong Bochner integral in $\mathbb{R}^n$

$$\int_A d^n\tau\, H_1(\tau_1)\dots H_1(\tau_n)\xi, \quad \xi \in D, \tag{3.19}$$

for any Borel measurable bounded subset $A \subset \mathbb{R}^n$, where $d^n\tau$ denotes n-dimensional Lebesgue measure. In particular, since H_1 is closed, we obtain for $t' \le t$ and $\xi \in D$,

$$\begin{aligned}
\int_{t \ge \tau_1 \ge \dots \ge \tau_n \ge t'} d^n\tau\, H_1(\tau_1)\dots H_1(\tau_n)\xi &= \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2 \dots \int_{t'}^{\tau_{n-1}} d\tau_n\, H_1(\tau_1)H_1(\tau_2)\dots H_1(\tau_n)\xi \\
&= \int_{t'}^{t} d\tau_1\, H_1(\tau_1) \int_{t'}^{\tau_1} d\tau_2\, H_1(\tau_2)\dots \int_{t'}^{\tau_{n-1}} d\tau_n\, H_1(\tau_n)\xi \\
&= \int_{t'}^{t} d\tau_1\, H_1(\tau_1)\dots \int_{t'}^{\tau_{n-2}} iH_1(\tau_{n-1})U_1(\tau_{n-1}, t')\xi \\
&= \dots = i^n U_n(t, t')\xi,
\end{aligned} \tag{3.20}$$

where (3.6) was used in the third equality.

Proof of Theorem 2.1. Let $\xi \in D$ and define

$$S_n(t, t')\xi := \sum_{j=0}^{n} U_j(t, t')\xi, \quad n \ge 0, \quad t, t' \in \mathbb{R}.$$

It is clear from Lemma 3.4 (3.13) that $\{S_n(t, t')\xi\}_n$ is Cauchy in $\mathcal{H}$. Thus, we can define

$$D(U(t, t')) = D, \quad U(t, t')\xi = \lim_{n\to\infty} S_n(t, t')\xi = \sum_{n=0}^{\infty} U_n(t, t')\xi, \quad \xi \in D, \quad t, t' \in \mathbb{R}. \tag{3.21}$$

This infinite summation converges absolutely and uniformly in (t, t') on any compact set $K \subset \mathbb{R}^2$. Note that $U_k(t, t')\xi$ $(k = 0, 1, \ldots, n)$ are strongly differentiable with respect to t, so is $S_n(t, t')\xi$. The derivative of $S_n(t, t')$ with respect to t becomes

$$\begin{aligned}\frac{\partial S_n(t,t')\xi}{\partial t} &= \sum_{j=0}^{n} \frac{\partial U_j(t,t')\xi}{\partial t} \\ &= -i \sum_{j=1}^{n} H_1(t) U_{j-1}(t,t')\xi \\ &= -iH_1(t) \sum_{j=0}^{n-1} U_j(t,t')\xi \\ &= -iH_1(t) S_{n-1}(t,t')\xi. \end{aligned} \tag{3.22}$$

By (3.14), one finds that $\{H_1(t)S_n(t, t')\xi\}_n$ is — and therefore $\{(\partial/\partial t)S_n(t, t')\xi\}_n$ is — Cauchy. Hence, the limit

$$\lim_{n\to\infty} \frac{\partial S_n(t,t')\xi}{\partial t} = -i \lim_{n\to\infty} H_1(t) S_{n-1}(t,t')\xi$$

exists. Due to the fact that $H_1(t)$ is closed, this implies $U(t, t')\xi \in D(H_1(t))$ and

$$\frac{\partial}{\partial t} S_n(t,t')\xi \to -iH_1(t)U(t,t')\xi, \quad (n \to \infty), \tag{3.23}$$

uniformly in (t, t') on any compact set $K \subset \mathbb{R}^2$. Since the function $t \mapsto (\partial/\partial t)\, S_n(t, t')\xi$ is strongly continuous, so is its uniform limit $-iH_1(t)U(t, t')\xi$. Then, by exchanging limit and integration, we have

$$\lim_{n\to\infty} \int_{t'}^{t} d\tau \frac{\partial}{\partial \tau} S_n(\tau, t')\xi = -i \int_{t'}^{t} d\tau H_1(\tau) U(\tau, t')\xi,$$

where the convergence is uniform on K. Since

$$\int_{t'}^{t} d\tau \frac{\partial}{\partial \tau} S_n(\tau, t')\xi = S_n(t, t')\xi - \xi \to U(t, t')\xi - \xi, \quad (n \to \infty), \tag{3.24}$$

we have

$$U(t, t')\xi = \xi - i \int_{t'}^{t} d\tau H_1(\tau) U(\tau, t')\xi, \tag{3.25}$$

which implies that $U(t, t')\xi$ is strongly continuously differentiable with respect to t at all $(t, t') \in K$. Since K is arbitrary, one concludes that (2.5) holds.

Next, we prove (2.6). Let $t' \le t$. By interchanging the order of integrations, we have from (3.20)

$$\begin{aligned} U_n(t,t')\xi &= (-i)^n \int_{t' \le \tau_n \le \cdots \le \tau_2 \le \tau_1 \le t} d^n\tau\, H_1(\tau_1) H_1(\tau_2) \cdots H_1(\tau_n)\xi \\ &= (-i)^n \int_{t'}^{t} d\tau_n \int_{\tau_n}^{t} d\tau_{n-1} \cdots \int_{\tau_2}^{t} d\tau_1\, H_1(\tau_1) \cdots H_1(\tau_{n-1}) H_1(\tau_n)\xi \\ &= i^n \int_{t}^{t'} d\tau_n \int_{t}^{\tau_n} d\tau_{n-1} \cdots \int_{t}^{\tau_2} d\tau_1 H_1(\tau_1) \cdots H_1(\tau_{n-1}) H_1(\tau_n)\xi, \end{aligned} \tag{3.26}$$

and this implies

$$U_{n+1}(t,t')\xi = i \int_{t}^{t'} d\tau U_n(t,\tau) H_1(\tau)\xi. \tag{3.27}$$

One can check in the same manner that (3.27) remains valid even if $t < t'$. Hence we find that $U_n(t,t')\xi$ is differentiable with respect to t', and

$$\frac{\partial}{\partial t'}U_{n+1}(t,t')\xi = iU_n(t,t')H_1(t')\xi. \tag{3.28}$$

By using (3.15), we can repeat a discussion similar to the one in the previous paragraph to obtain (2.5), to learn that $U(t,t')\xi$ is strongly continuously differentiable with respect to t' at all $(t,t') \in \mathbb{R}^2$, and satisfies (2.6). □

In the rest of the present section, we also use Assumption 2.2 in order to obtain more detailed results. We can derive the following lemmas in the same manner as before.

Lemma 3.6. *Under Assumption 2.2, $H_1(t)^*(A+1)^{-1/2}$ is bounded and there exists a constant $C' \geq 0$ independent of $t \in \mathbb{R}$ such that*

$$\|H_1(t)^*(A+1)^{-1/2}\| \leq C', \quad t \in \mathbb{R}. \tag{3.29}$$

Lemma 3.7. *Under Assumption 2.2, for all $\xi \in D$, the n-variable function*

$$\mathbb{R}^n \ni (t_1, \ldots, t_n) \mapsto H_1(t_1)^* \ldots H_1(t_n)^*\xi \in D$$

is strongly continuous on $\mathbb{R}^n$.

Lemma 3.7 ensures the existence of a strong Bochner integral

$$\int_A d^n\tau\, H_1(\tau_1)^* \ldots H_1(\tau_n)^*\xi, \quad \xi \in D, \tag{3.30}$$

for any bounded Borel set $A \subset \mathbb{R}^n$ and allows us to perform computations such as (3.20) with H_1 replaced by H_1^*.

Lemma 3.8. *Let Assumption 2.1 and Assumption 2.2 hold. Then, $D \subset D(U_n(t,t')^*)$ and for all $\xi \in D$,*

$$\begin{aligned} U_n(t,t')^*\xi &= i^n \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2 \cdots \int_{t'}^{\tau_{n-1}} d\tau_n\, H_1(\tau_n)^* \cdots H_1(\tau_2)^* H_1(\tau_1)^*\xi \\ &= (-i)^n \int_{t}^{t'} d\tau_n \int_{t}^{\tau_n} d\tau_{n-1} \cdots \int_{t}^{\tau_2} d\tau_1\, H_1(\tau_n)^* H_1(\tau_{n-1})^* \cdots H_1(\tau_1)^*\xi. \end{aligned} \tag{3.31}$$

In particular, $U_n(t,t')^*\xi$ is strongly continuously differentiable with respect to t and t'.

Proof. Choose arbitrary $\xi, \eta \in D$. Then

$$\begin{aligned} \langle U_n(t,t')\eta, \xi\rangle &= i^n \int_{t'}^{t} d\tau_1 \ldots \int_{t'}^{\tau_{n-1}} d\tau_n \,\langle H_1(\tau_1)\ldots H_1(\tau_n)\eta, \xi\rangle \\ &= i^n \int_{t'}^{t} d\tau_1 \ldots \int_{t'}^{\tau_{n-1}} d\tau_n \,\langle \eta, H_1(\tau_n)^* \ldots H_1(\tau_1)^*\xi\rangle \\ &= \left\langle \eta, i^n \int_{t'}^{t} d\tau_1 \ldots \int_{t'}^{\tau_{n-1}} d\tau_n\, H_1(\tau_n)^* \ldots H_1(\tau_1)^*\xi \right\rangle \\ &= \left\langle \eta, i^n \int_{t'}^{t} d\tau_n \int_{\tau_n}^{t} d\tau_{n-1} \ldots \int_{\tau_2}^{t} d\tau_1\, H_1(\tau_n)^* \ldots H_1(\tau_1)^*\xi \right\rangle \\ &= \left\langle \eta, (-i)^n \int_{t}^{t'} d\tau_n \int_{t}^{\tau_n} d\tau_{n-1} \ldots \int_{t}^{\tau_2} d\tau_1\, H_1(\tau_n)^* \ldots H_1(\tau_1)^*\xi \right\rangle. \end{aligned} \tag{3.32}$$

This implies $\xi \in D(U_n(t,t')^*)$ and (3.31). □

From Lemma 3.6, and Lemma 3.8, we can derive the following estimation for $\|U_n(t,t')^*\xi\|$, whose proof will be omitted since it is very similar to that of Lemma 3.3.

Lemma 3.9. *Let Assumption 2.1 and Assumption 2.2 hold. Let $t, t' \in \mathbb{R}$ and $\xi \in D$. Then, the estimate*

$$\|U_n(t,t')^*\xi\| \leq \frac{|t-t'|^n}{n!} C'^n (L_\xi + (n-1)b + 1)^{1/2} \cdots (L_\xi + 1)^{1/2} \|\xi\|, \tag{3.33}$$

holds for $n = 1, 2, \ldots$.

Once this estimate is obtained, the corresponding statement to Lemma 3.4 is proved:

Lemma 3.10. *Under Assumptions 2.1-2.2, the following summations converge uniformly in (t,t') on any compact set in the plane.*

$$\sum_{n=0}^{\infty} \|U_n(t,t')^*\xi\| < \infty, \tag{3.34}$$

$$\sum_{n=0}^{\infty} \|U_n(t,t')^* H_1(t)^*\xi\| < \infty, \tag{3.35}$$

$$\sum_{n=0}^{\infty} \|H_1(t')^* U_n(t,t')^*\xi\| < \infty. \tag{3.36}$$

Proof of Theorem 2.2. From Lemma 3.10 (3.34), one finds

$$\sum_{n=0}^{N} U_n(t,t')^*\xi, \quad \xi \in D \tag{3.37}$$

absolutely converges uniformly in (t,t') on any compact set. For all $\xi, \eta \in D$, we obtain

$$\begin{aligned} \langle \eta, U(t,t')\xi \rangle &= \sum_{n=0}^{\infty} \langle \eta, U_n(t,t')\xi \rangle \\ &= \sum_{n=0}^{\infty} \langle U_n(t,t')^*\eta, \xi \rangle \\ &= \left\langle \sum_{n=0}^{\infty} U_n(t,t')^*\eta, \xi \right\rangle, \end{aligned} \tag{3.38}$$

since $\eta \in D(U_n(t,t')^*)$ for all n. Thus, we obtain $D \subset D(U(t,t')^*)$ and

$$U(t,t')^*\xi = \sum_{n=0}^{\infty} U_n(t,t')^*\xi, \quad \xi \in D. \tag{3.39}$$

From (3.39), we can mimic the proof of Theorem 2.1 by using (3.35) and (3.36), to obtain (2.7) and (2.8). □

4 Properties of time evolution operator

In the present section, we prove several properties of the time evolution operator $U(t,t')$, and prove Theorems 2.3 and 2.4.

Proposition 4.1. *Suppose that Assumptions 2.1 and 2.2 hold. If $\xi \in D$, then for each $t, t', s, s' \in \mathbb{R}$, $U(s, s')\xi \in D(\overline{U(t, t')})$ and*

$$\overline{U(t, t')}U(s, s')\xi = \sum_{m,n=0}^{\infty} U_m(t, t')U_n(s, s')\xi, \tag{4.1}$$

where the right hand side converges absolutely, and does not depend upon the summation order.

Proof. For all $\xi \in D$ and all $(t, t'), (s, s') \in \mathbb{R}^2$, it is clear that $S_n(s, s')\xi \in D(U(t, t'))$. Since $S_n(s, s')\xi$ converges to $U(s, s')\xi$ as n tends to infinity, it suffices to prove that $U(t, t')S_n(s, s')\xi$ converges as $n \to \infty$. We have already know that

$$U(t, t')S_n(s, s')\xi = \sum_{m=0}^{\infty} \sum_{j=0}^{n} U_m(t, t')U_j(s, s')\xi,$$

therefore, it is sufficient to derive

$$\sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \|U_m(t, t')U_j(s, s')\xi\| < \infty.$$

By using (3.10),

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \|U_m(t, t')U_j(s, s')\xi\| \\
&\quad \le \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{|t - t'|^m |s - s'|^j}{m!\, j!} C^{m+j} (L_\xi + (m + j - 1)b + 1)^{1/2} \cdots (L_\xi + 1)^{1/2} \|\xi\| \\
&\quad = \sum_{N=0}^{\infty} \sum_{m=0}^{N} \frac{|t - t'|^m |s - s'|^{N-m}}{m!(N - m)!} C^N (L_\xi + (N - 1)b + 1)^{1/2} \cdots (L_\xi + 1)^{1/2} \|\xi\| \\
&\quad = \sum_{N=0}^{\infty} \frac{1}{N!} \left(C(|t - t'| + |s - s'|)\right)^N (L_\xi + (N - 1)b + 1)^{1/2} \cdots (L_\xi + 1)^{1/2} \|\xi\|
\end{aligned} \tag{4.2}$$

From the d'Alembert's ratio test, this is finite, which proves (4.1). □

Proof of Theorem 2.3. We first prove (i). Take arbitrary $\xi, \eta \in D$. $U(t, t)\xi = \xi$ is obvious. By Theorems 2.1 and 2.2, the function $t' \mapsto \left\langle \eta, \overline{U(t, t')}U(t', t'')\xi \right\rangle = \langle U(t, t')^* \eta, U(t', t'')\xi \rangle$ is differentiable and

$$\begin{aligned}
&\frac{\partial}{\partial t'} \left\langle \eta, \overline{U(t, t')}U(t', t'')\xi \right\rangle \\
&\quad = \langle -iH_1(t')^* U(t, t')^* \eta, U(t', t'')\xi \rangle + \langle U(t, t')^* \eta, -iH_1(t')U(t', t'')\xi \rangle \\
&\quad = 0.
\end{aligned} \tag{4.3}$$

Thus $\left\langle \eta, \overline{U(t, t')}U(t', t'')\xi \right\rangle$ is independent of t', which implies

$$\begin{aligned}
\left\langle \eta, \overline{U(t, t')}U(t', t'')\xi \right\rangle &= \left\langle \eta, \overline{U(t, t'')}U(t'', t'')\xi \right\rangle \\
&= \langle \eta, U(t, t'')\xi \rangle .
\end{aligned} \tag{4.4}$$

Since $\eta \in D$ is arbitrary, it follows that

$$\overline{U(t, t')}U(t', t'')\xi = U(t, t'')\xi. \tag{4.5}$$

Hence, we obtain (2.9) because $\xi \in D$ is arbitrary and $D = D(\overline{U(t,t')}U(t',t''))$.

Next, we prove (ii). Observe $e^{isH_0}H_1(t)e^{-isH_0} = H_1(t+s)$ by definition. Suppose $t \geq t'$. Then, for each $n \in \mathbb{N}$, $\xi \in D$, we obtain

$$\begin{aligned} & e^{isH_0}U_n(t,t')e^{-isH_0}\xi \\ =& \int_{t\geq\tau_1\geq\cdots\geq\tau_n\geq t'} d^n\tau\, H_1(\tau_1+s)\dots H_1(\tau_n+s)\xi \\ =& \int_{t+s\geq\tau_1\geq\cdots\geq\tau_n\geq t'+s} d^n\tau\, H_1(\tau_1)\dots H_1(\tau_n)\xi \\ =& U_n(t+s,t'+s)\xi. \end{aligned} \tag{4.6}$$

The relation (4.6) remains valid in the case where $t < t'$. Thus we have for all $(t,t') \in \mathbb{R}^2$,

$$\begin{aligned} e^{isH_0}U(t,t')e^{-isH_0}\xi &= \sum_{n=0}^{\infty} e^{isH_0}U_n(t,t')e^{-isH_0}\xi \\ &= \sum_{n=0}^{\infty} U_n(t+s,t'+s)\xi \\ &= U(t+s,t'+s)\xi. \end{aligned}$$

Since D is common core of $\overline{U(t,t')}$ and $\overline{U(t+s,t'+s)}$, we obtain the desired result. □

Proof of Theorem 2.4. We first prove that under the present situation, Assumption 2.1 implies Assumption 2.2. Let H_1 be symmetric and Assumption 2.1 hold. Then, by the fact that H_1 is symmetric and $A^{1/2}$-bounded, we find

$$D(A^{1/2}) \subset D(H_1) \subset D(H_1^*), \tag{4.7}$$

which means that H_1^* is also $A^{1/2}$- bounded. Moreover, for each $\xi \in V_L$, one finds

$$H_1^*\xi = H_1\xi \in V_{L+b}. \tag{4.8}$$

Thus, Assumption 2.2 is satisfied.

Next, we prove the unitarity. Since H_1 is symmetric, one obtains for all $\xi \in D$

$$\begin{aligned} \frac{\partial}{\partial t}\|U(t,t')\xi\|^2 =& \langle -iH_1(t)U(t,t')\xi, U(t,t')\xi\rangle + \langle U(t,t')\xi, -iH_1(t)U(t,t')\xi\rangle \\ =& 0. \end{aligned} \tag{4.9}$$

Therefore, $\overline{U(t,t')}$ is isometry, in particular, bounded. By using Theorem 2.3, one finds the operator equality

$$\overline{U(t,t')}\,\overline{U(t',t)} = I, \quad t,t' \in \mathbb{R}, \tag{4.10}$$

which implies that $U(t,t')$ is surjective. Hence, it is unitary.

The statement (i) is directly follows from Theorem 2.3.

We prove (ii). For each $\xi \in D$, $\eta \in \widetilde{D}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} \frac{\partial}{\partial t}\langle V(t,t')\eta, U(t,t')\xi\rangle &= \langle -iH_1(t)V(t,t')\eta, U(t,t')\xi\rangle + \langle V(t,t')\eta, -iH_1(t)U(t,t')\xi\rangle \\ &= 0. \end{aligned} \tag{4.11}$$

Thus, we obtain

$$\langle V(t,t')\eta, U(t,t')\xi\rangle = \langle \eta, \xi\rangle\,, \quad \xi \in D,\ \eta \in \widetilde{D}. \tag{4.12}$$

Since D is dense in $\mathcal{H}$ and since $\overline{U(t,t')}$ is unitary and satisfies $\overline{U(t,t')}\,\overline{U(t',t)} = I$, (4.12) yields for all $\eta \in \widetilde{D}$,

$$V(t,t')\eta = \overline{U(t,t')}\eta, \quad \eta \in \widetilde{D}. \tag{4.13}$$

Suppose that a sequence $\{\eta_n\}_n \subset \widetilde{D}$ satisfies that $\eta_n \to 0$ as n tends to infinity. Then, (4.13) shows that $V(t,t')\eta_n$ converges to 0, which means that $V(t,t')$ is closable. Take arbitrary $\psi \in \mathcal{H}$. Then, there is a sequence $\{\eta_n\}_n$ which converges to ψ as $n \to \infty$, since $\widetilde{D}$ is dense in $\mathcal{H}$. Then, (4.13) implies $\psi \in D(\overline{V(t,t') \restriction \widetilde{D}})$ and $\overline{V(t,t') \restriction \widetilde{D}}\psi = \overline{U(t,t')}\psi$ for all $\psi \in \mathcal{H}$. □

5 Schrödinger and Heisenberg equations of motion

In this section, we construct solutions of the Schrödinger and Heisenberg equations of motion via the time evolution operator $U(t,t')$, and prove Theorems 2.5 and 2.7. Throughout this section, we use Assumptions 2.1 and 2.2. Hereafter, we denote the closure of $U(t,t')$ by the same symbol. Recall that $W(t) = e^{-itH_0}U(t,0)$, $t \in \mathbb{R}$. Put

$$D' := D \cap D(H_0).$$

We remark that D' is dense in $\mathcal{H}$ under Assumption 2.1 (I) and (II). This can be seen as follows. Let $\psi \in D(H_0)$ and, for $n \in \mathbb{N}$,

$$\psi_n := E_A([0,n])\psi.$$

Then $\psi_n \in D$. By Assumption 2.1 (II), $\psi_n \in D(H_0)$. Hence, $\psi_n \in D'$. It is clear that $\psi_n \to \psi$ as n tends to infinity. Thus D' is dense in $D(H_0)$ and then also in $\mathcal{H}$.

Proof of Theorem 2.5. For all $\eta \in D(H_0)$,

$$\begin{aligned}\frac{d}{dt}\langle \eta, W(t)\xi\rangle &= \frac{d}{dt}\left\langle e^{itH_0}\eta, U(t,0)\xi\right\rangle \\ &= \langle iH_0\eta, W(t)\xi\rangle + \langle \eta, -iH_1W(t)\xi\rangle\,.\end{aligned} \tag{5.1}$$

By Theorem 2.3 (ii), $W(t)$ can be rewritten as $U(0,-t)e^{-itH_0}$. Since, for all $\xi \in D'$, the functions $e^{-itH_0}\xi$ and $U(0,-t)\xi$ are strongly differentiable and since $H_0D' \subset D$, it follows that the function $W(t)\xi$ is also strongly differentiable and the derivative becomes

$$\begin{aligned}\frac{d}{dt}W(t)\xi &= U(0,-t)(-iH_1(-t) - iH_0)e^{-itH_0}\xi \\ &= W(t)(-iH)\xi.\end{aligned} \tag{5.2}$$

Hence, by (5.1) and (5.2), we have

$$\langle iH_0\eta, W(t)\xi\rangle + \langle \eta, -iH_1W(t)\xi\rangle = \langle \eta, W(t)(-iH)\xi\rangle\,, \quad \eta \in D(H_0), \tag{5.3}$$

which implies that $W(t)\xi \in D(H_0)$

$$-iW(t)H\xi = -iHW(t)\xi \tag{5.4}$$

Therefore, we obtain (2.14). □

Proof of Theorem 2.6. . By Theorems 2.1 and 2.4, for all $t \in \mathbb{R}$, $W(t)$ is unitary with $W(0) = I$ and strongly continuous in $t \in \mathbb{R}$. In the present case, Assumption 2.2 holds too. Hence, by Theorem 2.3 (ii), we have

$$W(t)W(s) = W(t+s), \quad s,t \in \mathbb{R}.$$

Thus $\{W(t)\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group. Hence, by Stone's theorem, the first statement of the theorem holds.

By (2.15), we have for all $t \in \mathbb{R}$

$$\overline{U(t,0)} = e^{itH_0}e^{-it\widetilde{H}}.$$

By this equation and Theorem 2.4 (i), we obtain (2.16).

It follows from Theorem 2.5 that $D' = D \cap D(H_0) \subset D(\widetilde{H})$ and $H\xi = \widetilde{H}\xi$, $\xi \in D'$. Hence (2.17) follows.

If H is essentially self-adjoint on the subspace D', then one finds

$$\overline{H} \subset \widetilde{H}.$$

But since both $\overline{H}$ and $\widetilde{H}$ are self-adjoint, we have the equality. □

Next, we prepare some lemmas to prove Theorem 2.7.

Lemma 5.1. *Under Assumption 2.3, $B(A+1)^{-1/2}$ and $B^*(A+1)^{-1/2}$ are bounded and there exists a constant $C_0 \geq 0$ such that*

$$\|B(A+1)^{-1/2}\|, \|B^*(A+1)^{-1/2}\| \leq C_0. \tag{5.5}$$

Proof. This can be proved in the same way as Lemma 3.1 □

Lemma 5.2. *Under Assumptions 2.1, 2.2 and 2.3, the followings hold.*

(i) For all $\xi \in D \cap D(H_0)$, the function $W(t)^*\xi$ is strongly differentiable and satisfies

$$\frac{d}{dt}W(t)^*\xi = iH^*W(t)^*\xi = iW(t)^*H^*\xi. \tag{5.6}$$

(ii) $D \subset D(W(-t)BW(t))$.

(iii) $D \subset D(W(t)^*B^*W(-t)^*)$ and

$$B(t)^*\xi = W(t)^*B^*W(-t)^*\xi, \quad \xi \in D, \tag{5.7}$$

hold.

(iv) For all $\xi \in D'$, the function $BW(t)\xi$ is strongly differentiable and satisfies

$$\frac{d}{dt}BW(t)\xi = -iBW(t)H\xi. \tag{5.8}$$

Proof. (i) Note that $W(t)^*$ can be rewritten as

$$W(t)^* = U(t,0)^*e^{itH_0} = e^{itH_0}U(0,-t)^* \tag{5.9}$$

since e^{itH_0} is unitary. By using Theorem 2.2, for all $\eta \in D(H)$ and $\xi \in D'$, we have

$$\frac{d}{dt}\langle \eta, W(t)^*\xi\rangle = \langle -iH\eta, W(t)^*\xi\rangle. \tag{5.10}$$

On the other hand, we can see that $W(t)^*\xi$ is strongly differentiable and the derivative becomes

$$\frac{d}{dt}W(t)^*\xi = U(t,0)^*(iH_1(t)^* + iH_0)e^{itH_0}\xi = iW(t)^*H^*\xi, \tag{5.11}$$

in the same way as (5.2). Hence, by (5.10) and (5.11), we have

$$\langle -iH\eta, W(t)^*\xi\rangle = \langle \eta, iW(t)^*H^*\xi\rangle, \tag{5.12}$$

which implies that $W(t)^*\xi \in D(H^*)$ and $iW(t)^*H^*\xi = iH^*W(t)^*\xi$. Therefore, we obtain (5.6).

(ii) Firstly, we show that $W(t)\xi \in D(B)$ for each $\xi \in D$. By using (5.5) and Lemma 3.3, one finds

$$\begin{aligned}
\|Be^{-itH_0}S_n(t,0)\xi\| &\le \sum_{j=0}^{n}\|Be^{-itH_0}U_j(t,0)\xi\| \\
&\le \sum_{j=0}^{\infty} C_0(L_\xi + jb_0 + 1)^{1/2}\frac{|t|^j}{j!}C^j(L_\xi + (j-1)b + 1)^{1/2}\cdots(L_\xi+1)^{1/2}\|\xi\| \\
&< \infty,
\end{aligned} \tag{5.13}$$

where the convergence is uniform in t on any compact set $K \subset \mathbb{R}$. Since $W(t)\xi = \sum_{n=0}^{\infty} e^{-itH_0}U_n(t,0)\xi$, it follows that $W(t)\xi \in D(B)$ and that

$$BW(t)\xi = \sum_{n=0}^{\infty} Be^{-itH_0}U_n(t,0)\xi = \lim_{n\to\infty} Be^{-itH_0}S_n(t,0)\xi$$

from the closedness of B. Next, we show that $BW(t)\xi \in D(W(-t))$. Since $W(-t)$ is closed, it is sufficient to prove that the sequence

$$W(-t)Be^{-itH_0}S_n(t,0)\xi$$

converges. But, this follows because

$$\begin{aligned}
&\|W(-t)Be^{-itH_0}S_n(t,0)\xi\| \\
&\quad= \|e^{itH_0}U(-t,0)Be^{-itH_0}S_n(t,0)\xi\| \\
&\quad\le \sum_{m,j=0}^{\infty}\|e^{itH_0}U_m(-t,0)Be^{-itH_0}U_j(t,0)\xi\| \\
&\quad\le \sum_{m,j=0}^{\infty}\frac{|t|^m}{m!}C^m(L_\xi + (m+j-1)b + b_0 + 1)^{1/2}\cdots(L_\xi + jb + b_0 + 1)^{1/2} \\
&\qquad\times C_0(L_\xi + jb + 1)^{1/2}\frac{|t|^j}{j!}C^j(L_\xi + (j-1)b + 1)^{1/2}\cdots(L_\xi+1)^{1/2}\|\xi\| \\
&\quad\le \sum_{N=0}^{\infty}\frac{(2|t|)^N}{N!}C^N(L_\xi + (N-1)b + b_0 + 1)^{1/2}\cdots(L_\xi + b_0 + 1)^{1/2}C_0(L_\xi + Nb + 1)^{1/2}\|\xi\| \\
&\quad< \infty.
\end{aligned} \tag{5.14}$$

Hence, it follows that $BW(t)\xi \in D(W(-t))$ and

$$W(-t)BW(t)\xi = \sum_{m,j=0}^{\infty} e^{itH_0}U_m(-t,0)Be^{-itH_0}U_j(t,0)\xi, \tag{5.15}$$

where the right hand side converges absolutely. This means that $D \subset D(W(-t)BW(t))$.

(iii) By using (5.5) and Lemma 3.10, we get the desired conclusion in the same way as (ii), since $B(t)^* \supset W(-t)^* B^* W(t)^*$ in general.

(iv) By Assumption 2.3 (II) and (3.28), the derivative of $BS_n(0,-t)e^{-itH_0}\xi$ becomes

$$\frac{d}{dt}BS_n(0,-t)e^{-itH_0}\xi = -iBS_{n-1}(0,-t)e^{-itH_0}H\xi. \tag{5.16}$$

From the estimation (5.13), one finds that $\{(d/dt)BS_n(0,-t)e^{-itH_0}\xi\}_n$ is Cauchy for all $\xi \in D'$. Hence, the limit $\lim_{n\to\infty}(d/dt)BS_n(0,-t)e^{-itH_0}\xi = -i\lim_{n\to\infty} BS_{n-1}(0,-t)e^{-itH_0}H\xi$ exists. Due to the fact that B is closed, this implies $W(t)H\xi \in D(B)$ and

$$\frac{d}{dt}BS_n(0,-t)e^{-itH_0}\xi \to -iBW(t)H\xi, \quad (n\to\infty), \tag{5.17}$$

uniformly on any finite interval K. Since the function $\frac{d}{dt}BS_n(0,-t)e^{-itH_0}\xi$ is strongly continuous, so is its uniform limit $-iBW(t)H\xi$. Then, by exchanging limit and integration, we have

$$\lim_{n\to\infty}\int_0^t d\tau \frac{d}{d\tau}BS_n(0,-\tau)e^{-i\tau H_0}\xi = -i\int_0^t d\tau BW(\tau)H\xi, \tag{5.18}$$

where the convergence is uniform on K. Since the left hand side is equal to $BW(t)\xi - B\xi$, we obtain

$$BW(t)\xi - B\xi = -i\int_0^t d\tau BW(\tau)H\xi, \tag{5.19}$$

which implies that $BW(t)\xi$ is strongly continuously differentiable in $t \in K$. Since K is arbitrary, one concludes that (5.8) holds.

□

Proof of Theorem 2.7. By Lemma 5.2, for each $\xi, \eta \in D'$, we know that the functions $W(-t)^*\eta$ and $BW(t)\xi$ are strongly differentiable. Hence we have

$$\begin{aligned}\frac{d}{dt}\langle \eta, B(t)\xi\rangle &= \frac{d}{dt}\langle W(-t)^*\eta, BW(t)\xi\rangle \\ &= \langle -iW(-t)^*H^*\eta, BW(t)\xi\rangle + \langle W(-t)^*\eta, -iBW(t)H\xi\rangle \\ &= \langle (iH)^*\eta, W(-t)BW(t)\xi\rangle - \langle W(t)^*B^*W(-t)^*\eta, iH\xi\rangle\,. \end{aligned} \tag{5.20}$$

Therefore, by (5.7), we obtain (2.21). □

In the rest of this section, we give a proof of Theorem 2.8. We denote the closure of $B_0'(t)$ by the same symbol.

Lemma 5.3. *Under Assumptions 2.1, 2.2, 2.3 and 2.4, the following (i)-(iv) hold.*

(i) $B_0'(t)(A+1)^{-1/2}$ *is bounded and there exists a constant* $C_1 \geq 0$ *independent of* $t \in \mathbb{R}$ *such that*

$$\|B_0'(t)(A+1)^{-1/2}\| \leq C_1.$$

(ii) For all $t \in \mathbb{R}$ *and* $L \geq 0$, $\xi \in V_L$ *implies* $B_0'(t)\xi \in V_{L+b_0}$, *where* b_0 *is given in Assumption 2.3 (II).*

(iii) For all $t \in \mathbb{R}$, $D \subset D(U(0,t)H_1(t)B(t)U(t,0)) \cap D(U(0,t)B(t)H_1(t)U(t,0)) \cap D(U(0,t)B_0'(t)U(t,0))$, and for each $\xi \in D$,

$$U(0,t)H_1(t)B(t)U(t,0)\xi = \sum_{m,n=0}^{\infty} U_m(0,t)H_1(t)B(t)U_n(t,0)\xi, \tag{5.21}$$

$$U(0,t)B(t)H_1(t)U(t,0)\xi = \sum_{m,n=0}^{\infty} U_m(0,t)B(t)H_1(t)U_n(t,0)\xi, \tag{5.22}$$

$$U(0,t)B_0'(t)U(t,0)\xi = \sum_{m,n=0}^{\infty} U_m(0,t)B_0'(t)U_n(t,0)\xi, \tag{5.23}$$

where the right hand side in each equations above converges absolutely and uniformly in t on any compact interval, and does not depend upon the summation order.

(iv) For all $\xi \in D(A^{1/2}) \cap D(H_0)$ and $t \in \mathbb{R}$,

$$B_0'(t)\xi = e^{itH_0}[iH_0, B]e^{-itH_0}\xi. \tag{5.24}$$

Proof. The statement (i) follows from Assumption 2.4 (III).

We prove (ii). Let $L \geq 0$ and $\xi \in V_L$. By the definition of $B_0'(t)$,

$$B_0'(t)\xi = \lim_{\delta \to 0} \frac{B_0(t+\delta)\xi - B_0(t)\xi}{\delta}, \quad t \in \mathbb{R}. \tag{5.25}$$

Since H_0 and A are strongly commuting, it follows that $B_0(t)\xi \in V_{L+b_0}$ for all t. Hence, $B_0'(t)\xi \in V_{L+b_0}$ by the closedness of V_{L+b_0}.

By using (i)-(ii), Lemma 3.1 and 5.1, we can check (iii) in the same manner as in the proof of Lemma 5.2 (ii).

We prove (iv). For each $\eta \in D(H_0)$ and $\xi \in D(A^{1/2}) \cap D(H_0)$, we have

$$\frac{d}{dt}\langle \eta, B_0(t)\xi\rangle = \langle -iH_0\eta, B_0(t)\xi\rangle + \langle \eta, B_0(t)(-iH_0)\xi\rangle. \tag{5.26}$$

On the other hand, By Assumption 2.4 and the definition of $B_0'(t)$, we have

$$\frac{d}{dt}\langle \eta, B_0(t)\xi\rangle = \left\langle \eta, B_0'(t)\xi\right\rangle. \tag{5.27}$$

Comparing (5.26) and (5.27), we obtain $B_0(t)\xi \in D(H_0)$ and

$$B_0'(t)\xi = [iH_0, B_0(t)]\xi = e^{itH_0}[iH_0, B]e^{-itH_0}\xi. \tag{5.28}$$

□

Proof of Theorem 2.8. By using Lemma 5.3 (i) and (ii), it follows that for each $\xi \in D$ and $m, n = 0, 1, 2, \ldots$, the function $\mathbb{R} \ni t \mapsto S_m(0,t)B_0(t)S_n(t,0)\xi$ is strongly differentiable and the derivative becomes

$$\begin{aligned}\frac{d}{dt}S_m(0,t)B_0(t)S_n(t,0)\xi =& S_{m-1}(0,t)iH_1(t)B_0(t)S_n(t,0)\xi + S_m(0,t)B_0(t)(-iH_1(t))S_{n-1}(t,0)\xi \\ &+ S_m(0,t)B_0'(t)S_n(t,0)\xi,\end{aligned} \tag{5.29}$$

where $S_{-1}(\cdot,\cdot) := 0$. From Lemma 5.3 (iii), one finds that

$$\lim_{m,n\to\infty} \frac{d}{dt}S_m(0,t)B_0(t)S_n(t,0)\xi = U(0,t)[iH_1(t), B_0(t)]U(t,0)\xi + U(0,t)B_0'(t)U(t,0)\xi, \tag{5.30}$$

uniformly in t on any compact set $K \subset \mathbb{R}$. Since the right hand side of (5.29) is strongly continuous by Assumption 2.4 (I), so is the left hand side of (5.30). Hence, by exchanging limit and integration, we get

$$\lim_{m,n\to\infty} \int_0^t d\tau \, \frac{d}{d\tau} S_m(0,\tau) B_0(\tau) S_n(\tau,0)\xi = \int_0^t d\tau \left(U(0,\tau)[iH_1(\tau), B_0(\tau)]U(\tau,0)\xi + U(0,\tau)B_0'(\tau)U(\tau,0)\xi \right), \tag{5.31}$$

where the convergence is uniform on the compact set K. Since the left hand side of (5.31) is equal to $B(t)\xi - B\xi$ due to (5.15), one concludes that $B(t)\xi$ is strongly continuously differentiable in $t \in K$, and the derivative becomes

$$\begin{aligned} \frac{d}{dt} B(t)\xi &= U(0,t)[iH_1(t), B_0(t)]U(t,0)\xi + U(0,t)B_0'(t)U(t,0)\xi \\ &= W(-t)[iH_1, B]W(t)\xi + U(0,t)B_0'(t)U(t,0)\xi. \end{aligned} \tag{5.32}$$

Therefore, (2.23) follows from the arbitrariness of K.

It remains to prove (2.24). Let $\xi \in D'$. Note that for all $t \in \mathbb{R}$, we have $U(t,0)\xi \in D(A^{1/2})$ from Theorem A.1, and $U(t,0)\xi \in D(H_0)$ from Theorem 2.5. From these facts and Lemma 5.3 (iv) and (2.23), we obtain

$$\frac{d}{dt} B(t)\xi = W(-t)[iH, B]W(t)\xi. \tag{5.33}$$

From (5.4) in the proof of Theorem 2.5, we have

$$W(-t)BHW(t)\xi = W(-t)BW(t)H\xi = B(t)H\xi. \tag{5.34}$$

By using Lemma 5.2 (i), we have for all $\eta \in D'$

$$\begin{aligned} \langle \eta, W(-t)HBW(t)\xi \rangle &= \langle H^* W(-t)^* \eta, BW(t)\xi \rangle \\ &= \langle W(-t)^* H^* \eta, BW(t)\xi \rangle \\ &= \langle \eta, HW(-t)BW(t)\xi \rangle . \end{aligned}$$

Thus

$$W(-t)HBW(t)\xi = HW(-t)BW(t)\xi = HB(t)\xi, \tag{5.35}$$

because D' is dense. Hence, (2.24) follows from (5.33), (5.34) and (5.35). □

6 Application to QED in Lorenz gauge

In this section, we apply the general theory obtained in the preceding sections to a mathematical model of QED, quantized in the Lorenz gauge. As we emphasized in Introduction, our construction of $U(t,t')$ does not require that H be self-adjoint, and Theorems 2.5, and 2.7 are independent of the self-adjointness of H. This method is particularly valid for analyzing Lorenz-gauge QED, whose Hamiltonian is not self-adjoint and not even normal. We expect that our theory would be applicable to a wider class of mathematical models of quantum systems. Other possible applications, including to a model with ordinary self-adjoint Hamiltonians and a more detailed analysis of Lorenz-gauge QED, are in progress and will be presented in separated papers.

QED describes a system in which the quantum radiation field and the quantum Dirac field are minimally interacting. It is well known that, in the Coulomb gauge, one can employ a state space constructed by usual Fock spaces, which equip a positive definite metric, at the cost of the Lorentz covariance. In this

formulation, the Hamiltonian H is self-adjoint [14], hence, there clearly exists the time evolution operator e^{-itH} such that $\xi(t) = e^{-itH}\xi$ or $B(t) = e^{itH}Be^{-itH}$ is the unique solutions of the initial value problems (1.1) or (1.2), respectively. In contrast to the case of Coulomb gauge, in the Lorenz gauge, the Hamiltonian is neither self-adjoint nor normal in consequence of the inevitability of an indefinite metric [13], and hence the time evolution operator e^{-itH} does not necessarily exist. As a result, even the existence of solutions of (1.1) and (1.2) becomes a highly nontrivial problem. It does not seem to be easy to apply the general theory of evolution operators through hard analyses of the resolvent of the Hamiltonian of Lorenz-gauge QED. But our general theory works well to construct an appropriate time evolution as we will see in the present section.

6.1 Radiation fields

We introduce the photon field quantized in the Lorenz gauge.

We adopt as the one-photon Hilbert space

$$\mathcal{H}_{\rm ph} := L^2(\mathbb{R}^3_{\mathbf{k}}; \mathbb{C}^4). \tag{6.1}$$

The above $\mathbb{R}^3_{\mathbf{k}} := \{\mathbf{k} = (k^1, k^2, k^3) \,|\, k^j \in \mathbb{R},\ j = 1, 2, 3\}$ physically represents the momentum space of photons. If there is no danger of confusion, we omit the subscript $\mathbf{k}$ in $\mathbb{R}^3_{\mathbf{k}}$. $\mathcal{H}_{\rm ph}$ can be identified as $\oplus^4 L^2(\mathbb{R}^3_{\mathbf{k}})$. We freely use this identification. The Hilbert space for the quantized radiation field in the Lorenz gauge is given by

$$\mathcal{F}_{\rm ph} := \bigoplus_{n=0}^{\infty} \bigotimes_{\rm s}^{n} \mathcal{H}_{\rm ph} = \Big\{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \,\Big|\, \Psi^{(n)} \in \bigotimes_{\rm s}^{n} \mathcal{H}_{\rm ph},\ \|\Psi\|^2_{\mathcal{F}_{\rm ph}} = \sum_{n=0}^{\infty} \big\|\Psi^{(n)}\big\|^2_{\otimes_{\rm s}^n \mathcal{H}_{\rm ph}} < \infty\Big\}, \tag{6.2}$$

the Boson Fock space over $\mathcal{H}_{\rm ph}$, where $\otimes_{\rm s}^n$ denotes the n-fold symmetric tensor product with the convention $\otimes_{\rm s}^0 \mathcal{H}_{\rm ph} := \mathbb{C}$.

One-photon Hamiltonian in $\mathcal{H}_{\rm ph}$ is the multiplication operator by the function $\omega(\mathbf{k}) := |\mathbf{k}|\,(\mathbf{k} \in \mathbb{R}^3)$. We also denote by ω the matrix valued function

$$\mathbf{k} \mapsto \begin{pmatrix} \omega(\mathbf{k}) & 0 & 0 & 0 \\ 0 & \omega(\mathbf{k}) & 0 & 0 \\ 0 & 0 & \omega(\mathbf{k}) & 0 \\ 0 & 0 & 0 & \omega(\mathbf{k}) \end{pmatrix} \tag{6.3}$$

and the multiplication operator by it. Then, the free Hamiltonian of the quantum radiation field is given by

$$H_{\rm ph} := {\rm d}\Gamma_{\rm b}(\omega) := \bigoplus_{n=0}^{\infty} \omega^{(n)}, \tag{6.4}$$

the second quantization of ω, where $\omega^{(0)} := 0$ and $\omega^{(n)}$ $(n \geq 1)$ is defined by

$$\omega^{(n)} := \overline{\Big(\sum_{j=1}^{n} I \otimes \cdots \otimes I \overset{j\text{-th}}{\overset{\vee}{\omega}} \otimes I \otimes \cdots \otimes I\Big) \restriction \widehat{\bigotimes}^{n} D(\omega)} : \bigotimes_{\rm s}^{n} \mathcal{H}_{\rm ph} \to \bigotimes_{\rm s}^{n} \mathcal{H}_{\rm ph}, \tag{6.5}$$

and $\widehat{\otimes}$ denotes the algebraic tensor product.

Let $g = (g_{\mu\nu})_{\mu,\nu=0,1,2,3}$ be the 4×4 matrix given by

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6.6}$$

Now we introduce the indefinite metric on $\mathcal{F}_{\rm ph}$. The matrix g naturally defines the unitary operator acting on $L^2(\mathbb{R}^3_{\mathbf{k}};\mathbb{C}^4)$. We denote it by the same symbol g. We define η by the second quantization of $-g$, i.e.,

$$\eta := \Gamma_{\rm b}(-g) := \bigoplus_{n=0}^{\infty} \otimes^n (-g) : \mathcal{F}_{\rm ph} \to \mathcal{F}_{\rm ph}. \tag{6.7}$$

Then η is unitary and satisfies $\eta^* = \eta$, $\eta^2 = I$. By using η we introduce an indefinite metric on $\mathcal{F}_{\rm ph}$ by

$$\langle \Psi | \Phi \rangle := \langle \Psi, \eta \Phi \rangle_{\mathcal{F}_{\rm ph}}, \quad \Psi, \Phi \in \mathcal{F}_{\rm ph}. \tag{6.8}$$

In order to define the adjoint with respect to indefinite metric (6.8), we introduce the η-adjoint. For a densely defined linear operator T on $\mathcal{F}_{\rm ph}$, the adjoint operator $T^\dagger$ with respect to the metric $\langle \cdot | \cdot \rangle$ is defined by

$$T^\dagger := \eta T^* \eta. \tag{6.9}$$

It follows that

$$\langle \Psi | T\Phi \rangle = \left\langle T^\dagger \Psi | \Phi \right\rangle, \quad \Psi \in D(T^\dagger), \quad \Phi \in D(T). \tag{6.10}$$

We introduce notions of η-symmetry, η-self-adjointness and η-unitarity [7, 13] below.

Definition 6.1. *(i) A densely defined linear operator T is η-symmetric if $T \subset T^\dagger$.*

(ii) A densely defined linear operator T is η-self-adjoint if $T^\dagger = T$.

(iii) A densely defined linear operator T is essentially η-self-adjoint if $\overline{T}$ is η-self-adjoint.

(iv) A densely defined linear operator T is η-unitary if T is injective and $T^\dagger = T^{-1}$.

Lemma 6.1. *(i) T is η-symmetric if and only if ηT is symmetric.*

(ii) T is η-self-adjoint if and only if ηT is self-adjoint.

(iii) T is essentially η-self-adjoint if and only if ηT is essentially self-adjoint.

(iv) If T is η-symmetric then T is closable.

(v) Let T be η-self-adjoint and ηT is essentially self-adjoint on a subspace D, Then D is a core of T.

Proof. See [7]. □

Note that the free Hamiltonian $H_{\rm ph}$ is self-adjoint and η-self-adjoint.

The annihilation operator $a(F)$ with $F \in \mathcal{H}_{\rm ph}$ is defined to be a densely defined closed linear operator on $\mathcal{F}_{\rm ph}$ whose adjoint is given by

$$(a(F)^*\Psi)^{(0)} = 0, \quad (a(F)^*\Psi)^{(n)} = \sqrt{n} S_n (F \otimes \Psi^{(n-1)}), \quad n \geq 1, \quad \Psi \in D(a(F)^*), \tag{6.11}$$

where S_n denotes the symmetrization operator on $\otimes^n \mathcal{H}_{\rm ph}$, i.e. $S_n(\otimes^n \mathcal{H}_{\rm ph}) = \otimes^n_{\rm s} \mathcal{H}_{\rm ph}$. We note that $a(F)$ is anti-linear in F and $a(F')^*$ linear in F'. As is well known, the creation and annihilation operators leave the finite particle subspace

$$\mathcal{F}_{\rm b,0}(\mathcal{H}_{\rm ph}) := \left\{ \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\rm ph} \,\middle|\, \Psi^{(n)} = 0 \text{ for all } n \geq n_0 \text{ with some } n_0 \right\} \tag{6.12}$$

invariant and satisfy the canonical commutation relations:

$$[a(F), a(F')^*] = \langle F, F' \rangle_{\mathcal{H}_{\rm ph}}, \quad [a(F), a(F')] = [a(F)^*, a(F')^*] = 0, \tag{6.13}$$

on $\mathcal{F}_{\mathrm{b},0}(\mathcal{H}_{\mathrm{ph}})$. For each $f \in L^2(\mathbb{R}^3_{\mathbf{k}})$, we use the notation:

$$a^{(0)}(f) := a(f,0,0,0), \quad a^{(1)}(f) := a(0,f,0,0), \tag{6.14}$$

$$a^{(2)}(f) := a(0,0,f,0), \quad a^{(3)}(f) := a(0,0,0,f). \tag{6.15}$$

Then, the operator equalities

$$a^{(0)}(f)^\dagger = -a^{(0)}(f)^*, \tag{6.16}$$

$$a^{(j)}(f)^\dagger = a^{(j)}(f)^*, \quad j = 1,2,3, \tag{6.17}$$

hold. For each $f \in L^2(\mathbb{R}^3_{\mathbf{k}})$ and $\mu = 0,1,2,3$, we define

$$a_\mu(f) := a(fe^{(0)}_\mu,\, fe^{(1)}_\mu,\, fe^{(2)}_\mu,\, fe^{(3)}_\mu), \tag{6.18}$$

where $e^{(\lambda)}(\mathbf{k}) = (e^{(\lambda)}_\mu(\mathbf{k}))^3_{\mu=0} \in \mathbb{C}^4$ $(\lambda = 0,1,2,3)$ are the polarization vectors satisfying

$$\sum_{\lambda,\lambda'=0}^{3} e^{(\lambda)}_\mu(\mathbf{k}) g_{\lambda\lambda'} e^{(\lambda')}_\nu(\mathbf{k})^* = g_{\mu\nu}, \quad \text{a.e. } \mathbf{k} \in \mathbb{R}^3, \quad \mu,\nu = 0,1,2,3. \tag{6.19}$$

In this paper, for simplicity, we choose the special Lorentz frame, that is,

$$e^{(0)}(\mathbf{k}) := (1,0,0,0), \quad e^{(3)} := (0, \frac{\mathbf{k}}{|\mathbf{k}|}), \tag{6.20}$$

$$e^{(r)}(\mathbf{k}) := (0, \mathbf{e}^{(r)}(\mathbf{k})), \quad r = 1,2, \tag{6.21}$$

where $\mathbf{e}^{(r)}$ $(r = 1,2)$ are $\mathbb{R}^3$-valued continuous functions on the nonsimply connected space $K_0 := \mathbb{R}^3 \backslash \{(0,0,k^3) \,|\, k^3 \in \mathbb{R}\}$ such that, for all $\mathbf{k} \in K_0$,

$$\mathbf{e}^{(r)}(\mathbf{k}) \cdot \mathbf{e}^{(r')}(\mathbf{k}) = \delta_{rr'}, \quad \mathbf{e}^{(r)}(\mathbf{k}) \cdot \mathbf{k} = 0, \quad r,r' = 1,2. \tag{6.22}$$

We often use the notation $a^\dagger_\mu(f) = a_\mu(f)^\dagger$. Then, $a_\mu(f)$ and $a^\dagger_\mu(f)$ are closed, and satisfy the commutation relations:

$$[a_\mu(f), a^\dagger_\nu(g)] = -g_{\mu\nu} \langle f, g \rangle_{L^2(\mathbb{R}^3)},$$

$$[a_\mu(f), a_\nu(g)] = [a^\dagger_\mu(f), a^\dagger_\nu(g)] = 0,$$

on $\mathcal{F}_{\mathrm{b},0}(\mathcal{H}_{\mathrm{ph}})$.

For all $f \in L^2(\mathbb{R}^3_{\mathbf{x}})$ satisfying $\widehat{f}/\sqrt{\omega} \in L^2(\mathbb{R}^3_{\mathbf{k}})$, we set

$$A_\mu(0,f) := a_\mu\Big(\frac{\widehat{f^*}}{\sqrt{2\omega}}\Big) + a^\dagger_\mu\Big(\frac{\widehat{f}}{\sqrt{2\omega}}\Big), \tag{6.23}$$

where $\widehat{f}$ denotes the Fourier transform of f, and f^* denotes the complex conjugate of f. The functional $\mathcal{S}(\mathbb{R}^3_{\mathbf{x}}) \ni f \mapsto A_\mu(0,f)$ gives an operator-valued distribution (Cf. [1] Definition 7-1) acting on $(\mathcal{F}_{\mathrm{ph}}, \mathcal{F}_{\mathrm{b},0}(\mathcal{H}_{\mathrm{ph}}))$ and it is called the quantized radiation field at time $t = 0$. Now, fix $\chi_{\mathrm{ph}} \in L^2(\mathbb{R}^3_{\mathbf{x}})$ such that it is real and satisfies $\widehat{\chi_{\mathrm{ph}}}/\sqrt{\omega} \in L^2(\mathbb{R}^3_{\mathbf{k}})$. We set

$$A_\mu(\mathbf{x}) := A_\mu(0, \chi^{\mathbf{x}}_{\mathrm{ph}}), \tag{6.24}$$

$$\chi^{\mathbf{x}}_{\mathrm{ph}}(\mathbf{y}) := \chi_{\mathrm{ph}}(\mathbf{y} - \mathbf{x}), \quad \mathbf{y} \in \mathbb{R}^3. \tag{6.25}$$

$A_\mu(\mathbf{x})$ is called the point-like quantized radiation field with momentum cutoff $\widehat{\chi_{\mathrm{ph}}}$. As will be seen later, for real-valued f, the closures of $A_\mu(f)$, $\mu = 0,1,2,3$, are η-self-adjoint but not even normal.

6.2 Dirac fields

Next, we define the quantized Dirac field. We adopt as the one-electron Hilbert space

$$\mathcal{H}_{\rm el} := L^2(\mathbb{R}^3_{\mathbf{p}}; \mathbb{C}^4), \tag{6.26}$$

where $\mathbb{R}^3_{\mathbf{p}} := \{\mathbf{p} = (p^1, p^2, p^3) \,|\, p^j \in \mathbb{R},\ j = 1, 2, 3\}$ physically represents the momentum space of electrons. The Hilbert space for the quantized Dirac field is given by

$$\mathcal{F}_{\rm el} := \bigoplus_{n=0}^{\infty} \overset{n}{\wedge} \mathcal{H}_{\rm el} = \Big\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \,\Big|\, \Psi^{(n)} \in \overset{n}{\wedge} \mathcal{H}_{\rm el},\ \|\Psi\|^2_{\mathcal{F}_{\rm el}} = \sum_{n=0}^{\infty} \big\|\Psi^{(n)}\big\|^2_{\wedge^n \mathcal{H}_{\rm el}} < \infty \Big\}, \tag{6.27}$$

the Fermion Fock space over $\mathcal{H}_{\rm el}$, where $\wedge^n$ denotes the n-fold anti-symmetric tensor product with the convention $\wedge^0 \mathcal{H}_{\rm el} := \mathbb{C}$.

We denote the mass of the Dirac particle by $M > 0$. One-electron Hamiltonian in $\mathcal{H}_{\rm el}$ is the multiplication operator by the function $E_M(\mathbf{p}) := \sqrt{\mathbf{p}^2 + M^2}$ ($\mathbf{p} \in \mathbb{R}^3$). The Hamiltonian of the free quantum Dirac field is given by

$$H_{\rm el} := {\rm d}\Gamma_{\rm f}(E_M) := \bigoplus_{n=0}^{\infty} E_M^{(n)}, \tag{6.28}$$

where

$$E_M^{(n)} := \overline{\Big(\sum_{j=1}^{n} I \otimes \cdots \otimes I \overset{j\text{-th}}{\check{E}_M} \otimes I \otimes \cdots \otimes I \Big) \restriction \overset{n}{\widehat{\otimes}} D(E_M)} : \overset{n}{\wedge} \mathcal{H}_{\rm el} \to \overset{n}{\wedge} \mathcal{H}_{\rm el}. \tag{6.29}$$

The operator $H_{\rm el}$ is self-adjoint and non-negative.

Let γ^μ ($\mu = 0, 1, 2, 3$) be 4×4 gamma matrices, i.e., γ^0 is hermitian and γ^j ($j = 1, 2, 3$) are anti-hermitian, satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3, \tag{6.30}$$

where $\{X, Y\} := XY + YX$. Let $\alpha^\mu := \gamma^0\gamma^\mu$, $\beta := \gamma^0$, and let $s_1 := \frac{i}{2}\gamma^2\gamma^3$, $s_2 := \frac{i}{2}\gamma^3\gamma^1$, $s_3 := \frac{i}{2}\gamma^1\gamma^2$. Let $u_s(\mathbf{p}) = (u_s^l(\mathbf{p}))_{l=1}^4 \in \mathbb{C}^4$ describe the positive energy part with spin $s = \pm 1/2$ and $v_s(\mathbf{p}) = (v_s^l(\mathbf{p}))_{l=1}^4 \in \mathbb{C}^4$ the negative energy part with spin s, that is,

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta M) u_s(\mathbf{p}) = E_M(\mathbf{p}) u_s(\mathbf{p}), \quad (\mathbf{s} \cdot \mathbf{p}) u_s(\mathbf{p}) = s|\mathbf{p}| u_s(\mathbf{p}), \tag{6.31}$$

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta M) v_s(\mathbf{p}) = -E_M(\mathbf{p}) v_s(\mathbf{p}), \quad (\mathbf{s} \cdot \mathbf{p}) v_s(\mathbf{p}) = s|\mathbf{p}| v_s(\mathbf{p}), \quad \mathbf{p} \in \mathbb{R}^3. \tag{6.32}$$

These form an orthogonal base of $\mathbb{C}^4$,

$$u_s(\mathbf{p})^* u_{s'}(\mathbf{p}) = v_s(\mathbf{p})^* v_{s'}(\mathbf{p}) = \delta_{ss'} E_M(\mathbf{p}), \quad u_s(\mathbf{p})^* v_{s'}(\mathbf{p}) = 0, \tag{6.33}$$

and satisfy the completeness,

$$\sum_s (u_s^l(\mathbf{p}) u_s^{l'}(\mathbf{p})^* + v_s^l(\mathbf{p}) v_s^{l'}(\mathbf{p})^*) = 2\delta_{ll'} E_M(\mathbf{p}).$$

The annihilation operator $B(G)$ with $G \in \mathcal{H}_{\rm el}$ is defined to be a *bounded* operator on $\mathcal{F}_{\rm el}$ whose adjoint is given by

$$(B(G)^*\Psi)^{(0)} = 0, \quad (B(G)^*\Psi)^{(n)} = \sqrt{n} A_n(G \otimes \Psi^{(n-1)}), \quad n \geq 1, \quad \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\rm el}, \tag{6.34}$$

where A_n denotes the anti-symmetrization operator on $\otimes^n \mathcal{H}_{\rm el}$, i.e. $A_n(\otimes^n \mathcal{H}_{\rm el}) = \wedge^n \mathcal{H}_{\rm el}$. $B(G)$ is anti-linear in G and $B(G')^*$ linear in G'. It is well known that the operator norm of $B(G)$ and $B(G')^*$ is given by [1]

$$\|B(G)\| = \|G\|_{\mathcal{H}_{\rm el}}, \quad \|B(G')^*\| = \|G'\|_{\mathcal{H}_{\rm el}}. \tag{6.35}$$

For each $g \in L^2(\mathbb{R}^3_{\mathbf{p}})$, we use the notation

$$b_{1/2}(g) := B(g,0,0,0), \qquad b_{-1/2}(g) := B(0,g,0,0),$$
$$d_{1/2}(g) := B(0,0,g,0), \qquad d_{-1/2}(g) := B(0,0,0,g),$$

and $b^*_s(g) := b_s(g)^*$, $d^*_s(g) := d_s(g)^*$, $(s = \pm 1/2)$. Then, we have the canonical anti-commutation relations:

$$\{b_s(g), b^*_{s'}(g')\} = \{d_s(g), d^*_{s'}(g')\} = \delta_{ss'} \langle g, g' \rangle_{L^2(\mathbb{R}^3_{\mathbf{p}})}, \tag{6.36}$$

$$\{b_s(g), b_{s'}(g')\} = \{d_s(g), d_{s'}(g')\} = \{b_s(g), d_{s'}(g')\} = \{b_s(g), d^*_{s'}(g')\} = 0. \tag{6.37}$$

For all $g \in L^2(\mathbb{R}^3_{\mathbf{x}})$, we set

$$\psi_l(0,g) := \sum_{s=\pm 1/2} \left(b_s\left(\frac{\widehat{g^*} \cdot u^{l\,*}_s}{\sqrt{2E_M}} \right) + d^*_s\left(\frac{\widehat{g} \cdot \overline{v}^l_s}{\sqrt{2E_M}} \right) \right), \tag{6.38}$$

where $\overline{v}^l_s(\mathbf{p}) := v^l_s(-\mathbf{p})$. The functional $\mathcal{S}(\mathbb{R}^3_{\mathbf{x}}) \ni g \mapsto \psi_l(0,g)$ is called the quantized Dirac field at time $t = 0$. Now, fix $\chi_{\rm el} \in L^2(\mathbb{R}^3_{\mathbf{x}})$. We set

$$\psi_l(\mathbf{x}) := \psi_l(0, \chi^{\mathbf{x}}_{\rm el}), \tag{6.39}$$

$$\chi^{\mathbf{x}}_{\rm el}(\mathbf{y}) := \chi_{\rm el}(\mathbf{y} - \mathbf{x}), \quad \mathbf{y} \in \mathbb{R}^3. \tag{6.40}$$

$\psi_l(\mathbf{x})$ is called the point-like quantized Dirac field with momentum cutoff $\widehat{\chi_{\rm el}}$. For each $\mathbf{x} \in \mathbb{R}^3$ and $\mu = 0,1,2,3$, we define the current operator $j^\mu(\mathbf{x})$ by

$$j^\mu(\mathbf{x}) := \sum_{l,l'=1}^{4} \psi_l(\mathbf{x})^* \alpha^\mu_{ll'} \psi_{l'}(\mathbf{x}). \tag{6.41}$$

Then $j^\mu(\mathbf{x})$ is bounded and self-adjoint.

6.3 Total Hamiltonian

The Hilbert space of state vectors for QED in Lorenz gauge is taken to be

$$\mathcal{F}_{\rm tot} := \mathcal{F}_{\rm el} \otimes \mathcal{F}_{\rm ph}. \tag{6.42}$$

This Hilbert space can be identified as

$$\mathcal{F}_{\rm tot} = \bigoplus_{n=0}^{\infty} \left(\mathcal{F}_{\rm el} \otimes \left(\bigotimes_s^n \mathcal{H}_{\rm ph} \right) \right). \tag{6.43}$$

We freely use this identification. The free Hamiltonian is

$$H_{\rm fr} := H_{\rm el} \otimes I + I \otimes H_{\rm ph}, \tag{6.44}$$

where the subscript fr in $H_{\rm fr}$ means *free*.

We introduce an indefinite metric on $\mathcal{F}_{\rm tot}$ by

$$\langle \Psi | \Phi \rangle := \langle \Psi, I \otimes \eta \Phi \rangle_{\mathcal{F}_{\rm tot}}, \quad \Psi, \Phi \in \mathcal{F}_{\rm tot}. \tag{6.45}$$

Then, η-adjointness, η-symmetricity, η-self-adjointness and η-unitarity are defined on $\mathcal{F}_{\mathrm{tot}}$, in the same way as in subsection 6.1, by replacing η with $I \otimes \eta$.

We introduce the minimal interaction between the quantized Dirac field and the quantized radiation field. We denote the charge of the Dirac particle by $e \in \mathbb{R}$. Let $N_{\mathrm{b}} := \mathrm{d}\Gamma_{\mathrm{b}}(I_{H_{\mathrm{ph}}})$ be the photon number operator and $\chi_{\mathrm{sp}} \in L^1(\mathbb{R}^3)$ be a real-valued function on $\mathbb{R}^3$ playing the role of a spacial cut-off. Our interaction Hamiltonian H_{int} is defined as

$$D(H_{\mathrm{int}}) = D(I \otimes N_{\mathrm{b}}^{\frac{1}{2}}), \tag{6.46}$$

$$H_{\mathrm{int}}\Psi = e \int_{\mathbb{R}^3} d\mathbf{x} \chi_{\mathrm{sp}}(\mathbf{x}) j^{\mu}(\mathbf{x}) \otimes A_{\mu}(\mathbf{x})\Psi, \quad \Psi \in D(H_{\mathrm{int}}), \tag{6.47}$$

where the integral on the right hand side is taken in the sense of strong Bochner integral, and we used the standard Einstein notation in which the summation over repeated indices with one upper and the other lower is understood. As will be seen in later, the operator H_{int} is well-defined since

$$\int_{\mathbb{R}^3} d\mathbf{x} \, |\chi_{\mathrm{sp}}(\mathbf{x})| \, \| j^{\mu}(\mathbf{x}) \otimes A_{\mu}(\mathbf{x})\Psi\| < \infty, \quad \Psi \in D(I \otimes N_{\mathrm{b}}^{\frac{1}{2}}). \tag{6.48}$$

Moreover, H_{int} is essentially η-self-adjoint.

The quantum system under consideration is described by the Hamiltonian

$$H_{\mathrm{QED}} := H_{\mathrm{fr}} + \overline{H_{\mathrm{int}}} \quad \text{in} \quad \mathcal{F}_{\mathrm{tot}}. \tag{6.49}$$

The time evolution of the quantum fields A_{μ}, ψ_l is generated by the Heisenberg equations:

$$\frac{d}{dt}A^{\mu}(t,f) = [iH_{\mathrm{QED}}, A^{\mu}(t,f)], \tag{6.50}$$

$$\frac{d}{dt}\psi_l(t,g) = [iH_{\mathrm{QED}}, \psi_l(t,g)]. \tag{6.51}$$

It is easy to find *formal* solutions of these equations:

$$A^{\mu}(t,f) = e^{itH_{\mathrm{QED}}}A^{\mu}(0,f)e^{-itH_{\mathrm{QED}}}, \tag{6.52}$$

$$\psi_l(t,g) = e^{itH_{\mathrm{QED}}}\psi_l(0,g)e^{-itH_{\mathrm{QED}}}. \tag{6.53}$$

However, this does not immediately make sense because our QED Hamiltonian in Lorenz gauge is neither self-adjoint nor normal, and therefore we cannot define the time evolution operational $e^{-itH_{\mathrm{QED}}}$ through the operational calculus.

6.4 η-self-adjointness

In this subsection, we will prove the η-self-adjointness of $A_{\mu}(0,f)$, H_{int}, and H_{QED} under some suitable conditions.

Lemma 6.2. *(i) For all $f \in L^2(\mathbb{R}^3_{\mathbf{k}})$ and $\Psi \in D(N_{\mathrm{b}}^{1/2})$,*

$$\left\| a_{\mu}(f)\Psi \right\|_{\mathcal{F}_{\mathrm{ph}}} \leq \|f\|_{L^2(\mathbb{R}^3_{\mathbf{k}})} \left\| N_{\mathrm{b}}^{1/2}\Psi \right\|_{\mathcal{F}_{\mathrm{ph}}}, \tag{6.54}$$

$$\left\| a_{\mu}^{\dagger}(f)\Psi \right\|_{\mathcal{F}_{\mathrm{ph}}} \leq \|f\|_{L^2(\mathbb{R}^3_{\mathbf{k}})} \left\| (N_{\mathrm{b}}+1)^{1/2}\Psi \right\|_{\mathcal{F}_{\mathrm{ph}}}. \tag{6.55}$$

(ii) For all $f \in D(\omega^{-1/2})$ and $\Psi \in D(H_{\rm ph}^{1/2})$,

$$\left\|a_\mu(f)\Psi\right\|_{\mathcal{F}_{\rm ph}} \leq \left\|f/\sqrt{\omega}\right\|_{L^2(\mathbb{R}^3_{\mathbf{k}})} \left\|H_{\rm ph}^{1/2}\Psi\right\|_{\mathcal{F}_{\rm ph}}, \tag{6.56}$$

$$\left\|a_\mu^\dagger(f)\Psi\right\|_{\mathcal{F}_{\rm ph}} \leq \left\|f/\sqrt{\omega}\right\|_{L^2(\mathbb{R}^3_{\mathbf{k}})} \left\|H_{\rm ph}^{1/2}\Psi\right\|_{\mathcal{F}_{\rm ph}} + \|f\|_{L^2(\mathbb{R}^3_{\mathbf{k}})} \|\Psi\|_{\mathcal{F}_{\rm ph}}. \tag{6.57}$$

Proof. The estimates (i) and (ii) are easily proved by applying well known estimations (see [1] Proposition 4-24, [12] Section 13.3), and we omit the proof. □

Note that the spectrum of the photon number operator $N_{\rm b}$ is a purely discrete set $\{0, 1, 2, \dots\}$, and that for all integer $N \geq 0$,

$$R(E_{N_{\rm b}}([0, N])) = \bigoplus_{n=0}^{N} \overset{n}{\underset{\rm s}{\otimes}} \mathcal{H}_{\rm ph} \subset \mathcal{F}_{\rm ph}. \tag{6.58}$$

For each $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}_{{\rm b},0}(\mathcal{H}_{\rm ph})$, we denote by N_Ψ the maximum photon number of Ψ, that is,

$$N_\Psi := \max\{n \geq 0 \,\big|\, \Psi^{(n)} \neq 0\} < \infty. \tag{6.59}$$

Lemma 6.3. *Let f be a real-valued function satisfying $f \in L^2(\mathbb{R}^3_{\mathbf{x}})$ and $\widehat{f}/\sqrt{\omega} \in L^2(\mathbb{R}^3_{\mathbf{k}})$. Then, the quantized radiation field $A_\mu(0, f)$ is essentially η-self-adjoint, i.e., $\overline{A_\mu(0, f)} = \left(\overline{A_\mu(0, f)}\right)^\dagger$.*

Proof. By Lemma 6.1 (iii), it is sufficient to prove that $\eta A_\mu(0, f)$ is essentially self-adjoint.

Put $f_- := \widehat{f}/\sqrt{2\omega}$. Since $\eta^* = \eta$, we have

$$\begin{aligned}
(\eta A_\mu(0, f))^* &= A_\mu(0, f)^* \eta \\
&\supset (a_\mu(f_-)^* + a_\mu^\dagger(f_-)^*)\eta \\
&= \eta\eta(a_\mu(f_-)^* + (\eta a_\mu^*(f_-)\eta)^*)\eta \\
&= \eta(a_\mu^\dagger(f_-) + a_\mu(f_-)) \\
&= \eta A_\mu(0, f),
\end{aligned} \tag{6.60}$$

which means that $A_\mu(0, f)$ is η-symmetric.

We prove the η-self-adjointness by Nelson's analytic vector theorem ([11], Theorem X.39 and its corollaries). Clearly, $A_\mu(0, f)$ and η leaves $\mathcal{F}_{{\rm b},0}(\mathcal{H}_{\rm ph})$ invariant. By Lemma 6.2 (i), we have for all $\Psi \in D(N_{\rm b}^{1/2})$,

$$\|A_\mu(0, f)\Psi\| \leq 2\|f_-\| \, \|(N_{\rm b} + 1)^{1/2}\Psi\|. \tag{6.61}$$

By the fact that η and $N_{\rm b}$ are strongly commuting, one finds for each $\Psi \in \mathcal{F}_{{\rm b},0}(\mathcal{H}_{\rm ph})$ and $n = 1, 2, \dots,$

$$\|(\eta A_\mu(0, f))^n\Psi\| \leq (N_\Psi + n)^{1/2} \dots (N_\Psi + 1)^{1/2}(2\|f_-\|)^n\|\Psi\|. \tag{6.62}$$

Thus, one obtains for all $t > 0$,

$$\begin{aligned}
\sum_{n=0}^\infty \frac{t^n}{n!}\|(\eta A_\mu(0, f))^n\Psi\| &\leq \sum_{n=0}^\infty \frac{t^n}{n!}(N_\Psi + n)^{1/2} \dots (N_\Psi + 1)^{1/2}(2\|f_-\|)^n\|\Psi\| \\
&< \infty,
\end{aligned} \tag{6.63}$$

by using d'Alembert's ratio test. Therefore, $\mathcal{F}_{{\rm b},0}(\mathcal{H}_{\rm ph})$ is a $\eta A_\mu(0, f)$-invariant analytic vector space, and we have the essentially self-adjointness of $\eta A_\mu(0, f)$ by Nelson's analytic vector theorem. By Lemma 6.1 (v), the assertion follows. □

We denote the closure of $A_\mu(0,f)$ by the same symbol.

Lemma 6.4. *Let $\Psi \in D((I\otimes N_{\rm b})^{1/2})$. Then, the followings hold.*

(i)

$$\|j^\mu(\mathbf{x})\otimes A_\mu(\mathbf{x})\Psi\| \le M_{\rm el}M_{\rm ph}\|(I\otimes N_{\rm b}+1)^{1/2}\Psi\|, \tag{6.64}$$

where,

$$M_{\rm el} := \sum_{\mu=0}^{3}\sup_{\mathbf{x}\in\mathbb{R}^3}\|j^\mu(\mathbf{x})\|, \quad M_{\rm ph} := 2\left\|\frac{\widehat{\chi_{\rm ph}}}{\sqrt{2\omega}}\right\|. \tag{6.65}$$

(ii) The vector-valued function $\mathbf{x}\mapsto j^\mu(\mathbf{x})\otimes A_\mu(\mathbf{x})\Psi$ is strongly continuous.

(iii) Let $\chi_{\rm sp}$ be a real-valued function satisfying $\chi_{\rm sp}\in L^1(\mathbb{R}^3_{\mathbf{x}})$. Then,

$$\int_{\mathbb{R}^3} d\mathbf{x}\,|\chi_{\rm sp}(\mathbf{x})|\,\|j^\mu(\mathbf{x})\otimes A_\mu(\mathbf{x})\Psi\| \le \|\chi_{\rm sp}\|_{L^1(\mathbb{R}^3)}M_{\rm el}M_{\rm ph}\|(I\otimes N_{\rm b}+1)^{1/2}\Psi\| < \infty. \tag{6.66}$$

Proof. First of all, notice that the operator identity $(I\otimes N_{\rm b})^{1/2} = I\otimes N_{\rm b}^{1/2}$ follows from operational calculus.

(i) Note that the current operator $j^\mu(\mathbf{x})$ is uniformly bounded in $\mathbf{x}\in\mathbb{R}^3$, and thus $M_{\rm el}<\infty$. By using Lemma 6.2 (i), one finds for all $\Psi\in\mathcal{F}_{\rm el}\widehat{\otimes}D(N_{\rm b}^{1/2})$,

$$\begin{aligned}\|j^\mu(\mathbf{x})\otimes A_\mu(\mathbf{x})\Psi\| &= \|(j^\mu(\mathbf{x})\otimes I)(I\otimes A_\mu(\mathbf{x}))\Psi\|\\ &\le M_{\rm el}M_{\rm ph}\|(I\otimes N_{\rm b}+1)^{1/2}\Psi\|.\end{aligned} \tag{6.67}$$

Since $\mathcal{F}_{\rm el}\widehat{\otimes}D(N_{\rm b}^{1/2})$ is a core of $(I\otimes N_{\rm b})^{1/2}$, we obtain (6.64).

(ii) Let $\mathbf{P} := (P^1,P^2,P^3)$, $P^j := \overline{{\rm d}\Gamma_{\rm f}(p^j)\otimes I + I\otimes{\rm d}\Gamma_{\rm b}(k^j)}$, where p^j and k^j are the multiplication operators in $\mathcal{H}_{\rm el}$ and $\mathcal{H}_{\rm ph}$ respectively. Then, P^j, $j=1,2,3$ are self-adjoint. Note that the operator $j^\mu(\mathbf{x})\otimes A_\mu(\mathbf{x})$ can be rewritten as $e^{-i\mathbf{P}\cdot\mathbf{x}}j^\mu(\mathbf{0})\otimes A_\mu(\mathbf{0})e^{i\mathbf{P}\cdot\mathbf{x}}$. Since P^j and $I\otimes N_{\rm b}$ are strongly commuting, we have for all $\Psi\in D((I\otimes N_{\rm b})^{1/2})$,

$$j^\mu(\mathbf{x})\otimes A_\mu(\mathbf{x})\Psi = e^{-i\mathbf{P}\cdot\mathbf{x}}j^\mu(\mathbf{0})\otimes A_\mu(\mathbf{0})(I\otimes N_{\rm b})^{-1/2}e^{i\mathbf{P}\cdot\mathbf{x}}(I\otimes N_{\rm b})^{1/2}\Psi, \tag{6.68}$$

and the right hand side is strongly continuous since $j^\mu(\mathbf{0})\otimes A_\mu(\mathbf{0})(I\otimes N_{\rm b})^{-1/2}$ is a bounded operator by (6.64).

(iii) The inequality (6.66) immediately follows from (6.64).

□

Set

$$\mathcal{F}_{\rm b,0} := \widehat{\bigoplus_{n=0}^{\infty}}\left(\mathcal{F}_{\rm el}\otimes\left(\bigotimes_s^n\mathcal{H}_{\rm ph}\right)\right) \tag{6.69}$$

where $\widehat{\oplus}_{n=0}^{\infty}$ denotes the algebraic direct sum. Let N_Ψ denote the maximum number of photons of $\Psi\in\mathcal{F}_{\rm b,0}$, namely, for $\Psi=\{\Psi^{(N)}\}_{N=0}^{\infty}\in\oplus_{n=0}^{\infty}(\mathcal{F}_{\rm el}\otimes(\otimes_{\rm s}^n H_{\rm ph}))$, N_Ψ is the largest natural number N satisfying $\Psi^{(N)}\neq 0$. Define

$$\mathcal{F}_N := \bigoplus_{n=0}^{N}\left(\mathcal{F}_{\rm el}\otimes\left(\bigotimes_s^n\mathcal{H}_{\rm ph}\right)\right), \quad N=0,1,2,\dots. \tag{6.70}$$

Then, for all integer $N \geq 0$, one finds

$$R(E_{I\otimes N_{\rm b}}([0,N]) = \mathcal{F}_N. \tag{6.71}$$

Since the photon field $A_\mu(\mathbf{x})$ creates at most one photon, it follows that if $\Psi \in \mathcal{F}_{\rm b,0}$ belongs to $\mathcal{F}_N$, then $A_\mu(\mathbf{x})\Psi \in \mathcal{F}_{N+1}$ for all $\mathbf{x} \in \mathbb{R}^3$. Thus, since $\mathcal{F}_{N+1}$ is a closed subspace, we find that $H_{\rm int}\mathcal{F}_N$ is contained in $\mathcal{F}_{N+1}$.

Lemma 6.5. *The interaction Hamiltonian $H_{\rm int}$ is essentially η-self-adjoint.*

Proof. Firstly, we show the η-symmetricity of $H_{\rm int}$. By direct calculation, one finds for each $\Psi, \Phi \in \mathcal{F}_{\rm el}\widehat{\otimes}D(N_{\rm b}^{1/2})$,

$$\langle \Psi, \eta H_{\rm int}\Phi\rangle = \langle \eta H_{\rm int}\Psi, \Phi\rangle\,. \tag{6.72}$$

Since $\mathcal{F}_{\rm el}\widehat{\otimes}D(N_{\rm b}^{1/2})$ is a core of $(I\otimes N_{\rm b})^{1/2}$, and since $H_{\rm int}$ is $(I\otimes N_{\rm b})^{1/2}$- bounded, (6.72) holds for all $\Psi, \Phi \in D((I\otimes N_{\rm b})^{1/2})$, thus $H_{\rm int}$ is η-symmetric.

Next, we show the η-self-adjointness by Nelson's analytic vector theorem, similarly to the proof of Lemma 6.3. Note that $H_{\rm int}$ and $I\otimes\eta$ leaves $\mathcal{F}_{\rm b,0}$ invariant. Put $M_{\rm int} := |e|\,\|\chi_{\rm sp}\|_{L^1(\mathbb{R}^3_{\mathbf{x}})}M_{\rm el}M_{\rm ph}$. By Lemma 6.4 (iii), and by the fact that $I\otimes\eta$ and $I\otimes N_{\rm b}$ are strongly commuting, one finds for each $\Psi \in \mathcal{F}_{\rm b,0}$ and $n = 1, 2, \ldots,$

$$\|(\eta H_{\rm int})^n\Psi\| \leq (N_\Psi + n)^{1/2}\ldots(N_\Psi+1)^{1/2}M_{\rm int}^n\|\Psi\|. \tag{6.73}$$

Thus, one obtains for all $t > 0$,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}\|(\eta H_{\rm int})^n\Psi\| &\leq \sum_{n=0}^{\infty}\frac{t^n}{n!}(N_\Psi+n)^{1/2}\ldots(N_\Psi+1)^{1/2}M_{\rm int}^n\|\Psi\| \\ &< \infty,\end{aligned} \tag{6.74}$$

by using d'Alembert's ratio test. Therefore, $\mathcal{F}_{\rm b,0}$ is an $\eta H_{\rm int}$-invariant analytic vector space, and the essentially self-adjointness of $\eta H_{\rm int}$ follows. □

We denote the closure of $H_{\rm int}$ by the same symbol.

Proposition 6.1. *Let $\widehat{\chi_{\rm ph}} \in D(\omega^{-1})$. Then, $H_{\rm QED}$ is η-self-adjoint.*

Proof. By the Kato-Rellich theorem and Lemma 6.1, it is sufficient to show that the symmetric operator $\eta H_{\rm int}$ is $\eta H_{\rm fr}$- bounded with a relative bound less than 1.

By Lemma 6.2, we have for $\Psi \in D(I\otimes H_{\rm ph}^{1/2})$,

$$\|I\otimes A_\mu(\mathbf{x})\Psi\| \leq \sqrt{2}\|\widehat{\chi_{\rm ph}}/\omega\|_{L^2(\mathbb{R}^3_{\mathbf{k}})}\,\|I\otimes H_{\rm ph}^{1/2}\Psi\| + \frac{1}{\sqrt{2}}\|\widehat{\chi_{\rm ph}}/\sqrt{\omega}\|_{L^2(\mathbb{R}^3_{\mathbf{k}})}\,\|\Psi\|. \tag{6.75}$$

Thus, for all $\Psi \in \mathcal{F}_{\rm el}\widehat{\otimes}D(H_{\rm ph}^{1/2})$, one finds

$$\begin{aligned}\|j^\mu(\mathbf{x})\otimes A_\mu(\mathbf{x})\Psi\| &= \|(j^\mu(\mathbf{x})\otimes I)(I\otimes A_\mu(\mathbf{x}))\Psi\| \\ &\leq M_{\rm el}\left(\sqrt{2}\|\widehat{\chi_{\rm ph}}/\omega\|\,\|I\otimes H_{\rm ph}^{1/2}\Psi\| + \frac{1}{\sqrt{2}}\|\widehat{\chi_{\rm ph}}/\sqrt{\omega}\|\,\|\Psi\|\right).\end{aligned} \tag{6.76}$$

By (6.76), we find that for all $\Psi \in \mathcal{F}_{\rm el}\widehat{\otimes}(D(H_{\rm ph}^{1/2})\cap D(N_{\rm b}^{1/2}))$ ($\subset D(H_{\rm int})$),

$$\|H_{\rm int}\Psi\| \leq |e|\,\|\chi_{\rm sp}\|_{L^1(\mathbb{R}^3_{\mathbf{x}})}\,M_{\rm el}\left(\sqrt{2}\|\widehat{\chi_{\rm ph}}/\omega\|\,\|I\otimes H_{\rm ph}^{1/2}\Psi\| + \frac{1}{\sqrt{2}}\|\widehat{\chi_{\rm ph}}/\sqrt{\omega}\|\,\|\Psi\|\right). \tag{6.77}$$

Since $\mathcal{F}_{\rm el}\widehat{\otimes}(D(H_{\rm ph}^{1/2})\cap D(N_{\rm b}^{1/2}))$ is a core of $I\otimes H_{\rm ph}^{1/2}$, we conclude that $H_{\rm int}$ is $I\otimes H_{\rm ph}^{1/2}$- bounded. As is easily proved, one finds $I\otimes H_{\rm ph}^{1/2}$ is infinitesimally small with respect to $H_{\rm fr}$. Therefore, it follows that $\eta H_{\rm int}$ is infinitesimally small with respect to $\eta H_{\rm fr}$, hence we get the desired result. □

6.5 Existence of dynamics

In order to prove that there exists dynamics of these quantum fields, we have only to see that Theorem 2.7 can be applied to our case by checking that Assumptions 2.1, 2.2 and 2.3 are valid. We see in what follows that this is indeed the case with $A = I \otimes N_{\mathrm{b}}$, $H_0 = H_{\mathrm{fr}}$, $H_1 = H_{\mathrm{int}}$, $D = \mathcal{F}_{\mathrm{b},0}$.

Hereafter, we omit trivial tensor product like $\otimes I$ or $I \otimes$ when no confusion may occur and just write, for instance, H_{el} instead of $H_{\mathrm{el}} \otimes I$ and so forth.

Lemma 6.6. *(I) N_{b} is self-adjoint and non-negative.*

(II) N_{b} and H_{fr} are strongly commuting.

(III) The interaction Hamiltonian H_{int} and its adjoint H_{int}^* are relatively bounded with respect to $N_{\mathrm{b}}^{1/2}$.

(IV) For all $L \geq 0$ and $\Psi \in R(E_{N_{\mathrm{b}}}([0, L]))$, $H_{\mathrm{int}}\Psi$ and $H_{\mathrm{int}}^*\Psi$ belong to $R(E_{N_{\mathrm{b}}}([0, L+1]))$.

Proof. The statements (I) and (II) are well known facts.

We prove (III). As in the proof of Lemma 6.4, it can be seen that H_{int} is $N_{\mathrm{b}}^{1/2}$- bounded. We prove that H_{int}^* is also $H_{\mathrm{b}}^{1/2}$- bounded. Take arbitrary $\Psi \in \mathcal{F}_{\mathrm{b},0}$. By Lemma 6.5, we obtain

$$\begin{aligned} \|H_{\mathrm{int}}^*\Psi\| &= \|H_{\mathrm{int}}\eta\Psi\| \\ &\leq M_{\mathrm{int}}\|(N_{\mathrm{b}}+1)^{\frac{1}{2}}\eta\Psi\| \\ &\leq M_{\mathrm{int}}\|(N_{\mathrm{b}}+1)^{\frac{1}{2}}\Psi\|. \end{aligned} \tag{6.78}$$

Since $\mathcal{F}_{\mathrm{b},0}$ is a core of $N_{\mathrm{b}}^{1/2}$, we have the assertion (III).

Finally, we prove (IV). Notice that the spectrum of the photon number operator N_{b} is a purely discrete set $\{0, 1, 2, \dots\}$, and that for all $L \geq 0$ and $[L]$, the largest integer satisfying $[L] \leq L$,

$$R(E_{N_{\mathrm{b}}}([0, L])) = R(E_{N_{\mathrm{b}}}([0, [L]])) = \mathcal{F}_{[L]}. \tag{6.79}$$

Suppose Ψ belongs to $R(E_{N_{\mathrm{b}}}([0, L]))$. Then, it is clear that Ψ belongs to $\mathcal{F}_{[L]+1} = R(E_{N_{\mathrm{b}}}([0, L+1]))$, since the interaction Hamiltonian creates at most one photon. By Lemma 6.5, and the fact that η and N_{b} are strongly commuting and thus η preserves photon number, it immediately follows that $H_{\mathrm{int}}^*\Psi \in \mathcal{F}_{[L]+1} = R(E_{N_{\mathrm{b}}}([0, L+1]))$. □

From Lemma 6.6, we can apply the general theory constructed in the earlier sections to obtain:

Theorem 6.1. *For each $\Psi \in \mathcal{F}_{\mathrm{b},0}$, the series*

$$U(t, t')\Psi := \Psi + (-i)\int_{t'}^{t} d\tau_1\, H_{\mathrm{int}}(\tau_1)\Psi + (-i)^2 \int_{t'}^{t} d\tau_1 \int_{t'}^{\tau_1} d\tau_2\, H_{\mathrm{int}}(\tau_1)H_{\mathrm{int}}(\tau_2)\Psi + \cdots \tag{6.80}$$

converges absolutely, where each of integrals is strong integral. Furthermore, $U(t, t')$ has properties stated in Theorems 2.1, 2.2, 2.3 and 2.5, with H_1 replaced by H_{int}, H_0 by H_{fr} and D by $\mathcal{F}_{\mathrm{b},0}$.

As we have already seen in the general theory, once $U(t, t')$ is constructed, we immediately obtain a time evolution which is generated by $W(t) := e^{-itH_{\mathrm{fr}}}U(t, 0)$. Here, recall that we omit the overline to mean the operator closure. In the present application to physics, it should be made sure that this time evolution is physically acceptable, that is, the probability amplitude is conserved

$$\langle \Psi | \Phi \rangle = \langle W(t)\Psi | W(t)\Phi \rangle\,,$$

for suitable vectors Ψ, Φ. The following theorem is concerned with this aspect:

Theorem 6.2. *The time evolution operator*

$$W(t) := e^{-itH_{\rm fr}} U(t,0) \tag{6.81}$$

satisfies

$$W(t)^\dagger \supset W(t)^{-1}. \tag{6.82}$$

In particular,

$$\langle W(t)\Psi | W(t)\Phi \rangle = \langle \Psi | \Phi \rangle \tag{6.83}$$

for all $\Psi, \Phi \in D(W(t))$.

To prove Theorem 6.2, we prepare some facts.

Lemma 6.7. *For each* $\Psi \in \mathcal{F}_{\rm b,0}$,

$$\eta U(t,t')\eta\Psi = U(t',t)^*\Psi \tag{6.84}$$

holds.

Proof. From Lemma 6.5, the operator identities $\eta H_{\rm int}^* \eta = H_{\rm int}$ and $\eta H_{\rm fr}\eta = H_{\rm fr}$ hold, which imply

$$\eta H_{\rm int}(t)^* \eta = H_{\rm int}(t). \tag{6.85}$$

We claim that (6.85) implies the identity

$$\eta U_n(t,t')\eta\Psi = U_n(t',t)^*\Psi \tag{6.86}$$

for all $n = 0, 1, 2, \dots$ and $\Psi \in \mathcal{F}_{\rm b,0}$. In fact, by Lemma 3.8, one finds

$$\begin{aligned}
\eta U_n(t,t')\eta\Psi &= (-i)^n \int_{t'}^{t} d\tau_1 \dots \int_{t'}^{\tau_{n-1}} d\tau_n \, H_{\rm int}(\tau_1)^* \dots H_{\rm int}(\tau_n)^*\Psi \\
&= (-i)^n \int_{t'}^{t} d\tau_n \dots \int_{t'}^{\tau_2} d\tau_1 \, H_{\rm int}(\tau_n)^* \dots H_{\rm int}(\tau_1)^*\Psi \\
&= U_n(t',t)^*\Psi.
\end{aligned} \tag{6.87}$$

Hence (6.86) is true for all n, and by summing up over $n = 0, 1, 2, \dots$, we obtain (6.84). □

Proposition 6.2. *For all* $t, t' \in \mathbb{R}$, $U(t,t')$ *is invertible and the operator equality*

$$U(t,t') = U(t',t)^{-1} \tag{6.88}$$

holds.

Proof. Fix t, t'. Firstly, we prove that $U(t,t')$ is injective. Note that if for all $\Psi \in \mathcal{F}_{\rm b,0}$, $\langle \Psi | \Phi \rangle = 0$, then it follows that $\Phi = 0$. From Lemma 6.7, we have the operator relation

$$\eta U(t,t') \subset U(t',t)^* \eta, \tag{6.89}$$

since $\mathcal{F}_{\rm b,0}$ is a core of $U(t,t')$, and since $U(t,t')^*$ is closed. Now, suppose that Φ satisfies $U(t,t')\Phi = 0$. Let $\Psi \in \mathcal{F}_{\rm b,0}$ be arbitrary. It follows that

$$\begin{aligned}
0 &= \langle U(t,t')\Psi | U(t,t')\Phi \rangle \\
&= \langle U(t,t')\Psi, \eta U(t,t')\Phi \rangle \\
&= \langle U(t,t')\Psi, U(t',t)^*\eta\Phi \rangle \\
&= \langle U(t',t)U(t,t')\Psi | \Phi \rangle \\
&= \langle \Phi | \Psi \rangle ,
\end{aligned} \tag{6.90}$$

where we have used (6.89) in the third equality, and Theorem 2.3 (2.9) in the last equality. Since $\Psi \in \mathcal{F}_{\mathrm{b},0}$ is arbitrarily taken, we find $\Phi = 0$. This proves that $U(t,t')$ is injective.

Secondly, we prove

$$U(t,t') \subset U(t',t)^{-1}. \tag{6.91}$$

Take arbitrary $\Psi \in \mathcal{F}_{\mathrm{b},0}$. Then, we have from Theorem 2.3, (2.9)

$$U(t',t)U(t,t')\Psi = \Psi,$$

which implies $\Psi \in D(U(t',t)^{-1})$ and

$$U(t,t')\Psi = U(t',t)^{-1}\Psi, \quad \Psi \in \mathcal{F}_{\mathrm{b},0}.$$

But since $\mathcal{F}_{\mathrm{b},0}$ is a core of $U(t,t')$, the relation (6.91) follows.

Finally, we prove

$$U(t,t') \supset U(t',t)^{-1}. \tag{6.92}$$

Let $\Psi \in D(U(t',t)^{-1}) = R(U(t',t))$. Then, there is some $\Phi \in D(U(t',t))$ with $\Psi = U(t,t')\Phi$. Since $\mathcal{F}_{\mathrm{b},0}$ is a core of $U(t',t)$, we can choose a sequence $\{\Phi_n\}_n \subset \mathcal{F}_{\mathrm{b},0}$ satisfying

$$\Phi_n \to \Phi, \quad U(t',t)\Phi_n \to U(t',t)\Phi = \Psi, \tag{6.93}$$

as n tends to infinity. Therefore, we have

$$U(t,t')U(t',t)\Phi_n = \Phi_n \to \Phi, \quad n \to \infty.$$

Since $U(t,t')$ is closed, we conclude that $\Psi = U(t',t)\Phi \in D(U(t,t'))$ and

$$U(t,t')\Psi = U(t,t')U(t',t)\Phi = \Phi = U(t',t)^{-1}\Psi.$$

This proves (6.92). □

Proposition 6.3. *For all $t \in \mathbb{R}$, the operator $W(t)$ is injective and*

$$W(t)^{-1} = W(-t) \tag{6.94}$$

holds as an operator equality.

Proof. Since $e^{-itH_{\mathrm{fr}}}$ is unitary and $U(t,0)$ is injective by Lemma 6.2, one finds $W(t)$ is injective.

We prove (6.94). Fix $t \in \mathbb{R}$. For all $\Psi \in D(W(t)) = D(U(t,0))$, we have by Theorem 2.3 (2.9)

$$\begin{aligned} W(t)W(-t)\Psi &= e^{-itH_{\mathrm{fr}}}U(t,0)e^{itH_{\mathrm{fr}}}U(-t,0)\Psi \\ &= U(0,-t)U(-t,0)\Psi \\ &= \Psi. \end{aligned} \tag{6.95}$$

This means $\Psi \in D(W(t)^{-1})$ and

$$W(-t)\Psi = W(t)^{-1}\Psi.$$

Since $\mathcal{F}_{\mathrm{b},0}$ is a core of $W(t)$, we conclude $W(-t) \subset W(t)^{-1}$. The proof of the inverse inclusion is very similar to the proof of (6.92), and we omit it. □

Proof of Theorem 6.2. Let $\Psi \in \mathcal{F}_{\mathrm{b},0}$. Then, we have

$$\eta W(t)^* \eta \Psi = W(-t)\Psi. \tag{6.96}$$

Thus, by taking closure, it follows that

$$\eta W(t)^* \eta \supset W(-t). \tag{6.97}$$

But, we know $W(-t) = W(t)^{-1}$ from Proposition 6.3. This proves (6.82).

Moreover, since $W(t)\Phi \in D(W(t)^{-1}) \subset D(\eta W(t)^* \eta)$, we obtain

$$\begin{aligned} \langle W(t)\Psi | W(t)\Phi \rangle &= \langle W(t)\Psi, \eta W(t)\Phi \rangle \\ &= \langle \Psi, W(t)^* \eta W(t)\Phi \rangle \\ &= \left\langle \Psi | W(t)^{-1} W(t)\Phi \right\rangle \\ &= \langle \Psi | \Phi \rangle . \end{aligned} \tag{6.98}$$

This completes the proof. □

We next consider the Heisenberg equations of motion for quantum fields A_μ and ψ_l.

Lemma 6.8. *(I) $A_\mu(0, f)$ and $A_\mu(0, f)^*$ are $N_{\mathrm{b}}^{1/2}$- bounded and closed.*

(II) $\psi_l(0, g)$ and $\psi_l(0, g)^*$ are $N_{\mathrm{b}}^{1/2}$- bounded and closed.

(III) For all $L \geq 0$, $\Psi \in R(E_{N_{\mathrm{b}}}([0, L]))$ implies that $A_\mu(0, f)\Psi$, $A_\mu(0, f)^*\Psi$, $\psi_l(0, g)\Psi$, and $\psi_l(0, g)^*\Psi$ belong to $R(E_{N_{\mathrm{b}}}([0, L + 1]))$.

Proof. The assertion (I) follows from Lemma 6.2. The closedness is obvious.

The assertion (II) is an immediate consequence of the fact that Dirac fields are bounded operators.

The claim (III) immediately follows from the fact that photon field operators $A_\mu(0, f)$ and $A_\mu(0, f)^*$ create at most one photon and Dirac fields $\psi_l(0, g)$ and $\psi_l(0, g)$ create no photon. □

Put for $t \in \mathbb{R}$ and $\mu = 0, 1, 2, 3$, $l = 1, 2, 3, 4$,

$$A_{0,\mu}(t, f)\Psi := e^{itH_{\mathrm{fr}}} A_\mu(0, f) e^{-itH_{\mathrm{fr}}}\Psi, \quad f \in \mathcal{S}(\mathbb{R}^3), \tag{6.99}$$

$$\psi_{0,l}(t, g)\Psi := e^{itH_{\mathrm{fr}}} \psi_l(0, g) e^{-itH_{\mathrm{fr}}}\Psi, \quad g \in \mathcal{S}(\mathbb{R}^3). \tag{6.100}$$

Lemma 6.9. *(I) For each $\Psi \in D(N_{\mathrm{b}}^{1/2})$ and each $f, g \in \mathcal{S}(\mathbb{R}^3)$, $A_{0,\mu}(t, f)\Psi$ and $\psi_{0,l}(t, g)\Psi$ are strongly continuously differentiable in $t \in \mathbb{R}$.*

(II) The strong derivatives of $A_{0,\mu}(t, f)\Psi$ and $\psi_{0,l}(t, g)\Psi$, which are denoted by $A'_{0,\mu}(t, f)\Psi$ and $\psi'_{0,l}(t, g)\Psi$, are closable for all $t \in \mathbb{R}$.

(III) The operators $A'_{0,\mu}(t, f)\Psi$ and $\psi'_{0,l}(t, g)\Psi$ are $A^{1/2}$-bounded uniformly in $t \in \mathbb{R}$. That is, there exist constants $c_0, c_1, d_0, d_1 \geq 0$ such that for all $t \in \mathbb{R}$ and $\xi \in D(A^{1/2})$,

$$\|A'_{0,\mu}(t, f)\Psi\| \leq c_0 \|N_{\mathrm{b}}^{1/2}\Psi\| + c_1 \|\Psi\|, \tag{6.101}$$

$$\|\psi'_{0,l}(t, g)\Psi\| \leq d_0 \|N_{\mathrm{b}}^{1/2}\Psi\| + d_1 \|\Psi\|. \tag{6.102}$$

Proof. (I) It follows from the general theorem [1, Lemma 4-49] that $t \mapsto a_\mu(e^{it\omega} f)\Psi$ and $t \mapsto a_\mu(e^{it\omega} f)^*\Psi$ are strongly differentiable and the strong derivatives become

$$\frac{d}{dt} a_\mu(e^{it\omega} f)\Psi = a_\mu(i\omega e^{it\omega} f)\Psi, \tag{6.103}$$

$$\frac{d}{dt} a_\mu(e^{it\omega} f)^*\Psi = a_\mu(i\omega e^{it\omega} f)^*\Psi, \tag{6.104}$$

for $\Psi \in D(N_{\rm b}^{1/2})$ and $f \in \mathcal{S}(\mathbb{R}^3)$. From Lemma 6.2 (i) and (6.103), (6.104), the continuity of the mappings

$$t \mapsto \frac{d}{dt} a_\mu(e^{it\omega} f)\Psi, \quad t \mapsto \frac{d}{dt} a_\mu(e^{it\omega} f)^* \Psi$$

are obvious. For the fermion creation and annihilation operators, the operator valued functions

$$t \mapsto B(e^{itE_M} G), \quad t \mapsto B(e^{itE_M} G)^*, \quad G \in D(E_M) \tag{6.105}$$

are continuously differentiable in the operator norm and the derivatives are

$$\frac{d}{dt} B(e^{itE_M} G) = B(iE_M e^{itE_M} G), \tag{6.106}$$

$$\frac{d}{dt} B(e^{itE_M} G)^* = B(iE_M e^{itE_M} G)^*. \tag{6.107}$$

In fact, one finds from the well known estimates for the fermion creation and annihilation operators (6.35) that

$$\left\| \frac{B(e^{i(t+h)E_M} G) - B(e^{itE_M} G)}{h} - B(iE_M e^{itE_M} G) \right\| \leq \left\| \frac{e^{i(t+h)E_M} G - e^{itE_M} G}{h} - iE_M e^{itE_M} G \right\|_{\mathcal{H}_{\rm el}} \to 0, \tag{6.108}$$

as $h \to 0$, and the mappings

$$t \mapsto B(iE_M e^{itE_M} G), \quad t \mapsto B(iE_M e^{itE_M} G)^*$$

are continuous in the operator norm. On the other hand, from these facts, the assertion (I) immediately follows.

(II) From (6.103) and (6.104), we have

$$A'_{0,\mu}(t, f)\Psi = a_\mu\left(\frac{i\omega e^{it\omega} \widehat{f^*}}{\sqrt{2\omega}} \right)\Psi + a^\dagger\left(\frac{i\omega e^{it\omega} \widehat{f}}{\sqrt{2\omega}} \right)\Psi, \quad \Psi \in D(N_{\rm b}^{1/2}). \tag{6.109}$$

It is clear from this expression that the adjoint operator of $A'_{0,\mu}(t, f)$ is defined on dense subspace $\mathcal{F}_{\rm b,0}$ and therefore it is closable. From (6.106) and (6.107), the closability of $\psi'_{0,l}(t, g)$ is obvious.

(III) This statement clearly follows from Lemma 6.2 (i) and (6.35).

□

Finally, we have arrived at the existence of strong solutions for the Heisenberg equations of motion for quantum fields, by combining all the results obtained so far. We can define from Lemma 6.6, Lemma 6.8 and Theorem 2.7 the following filed operators:

$$D(A_\mu(t, f)) := \mathcal{F}_{\rm b,0}, \quad A_\mu(t, f)\Psi := W(-t)A_\mu(0, f)W(t)\Psi, \quad \Psi \in \mathcal{F}_{\rm b,0}, \quad f \in \mathcal{S}(\mathbb{R}^3), \tag{6.110}$$

$$D(\psi_l(t, g)) := \mathcal{F}_{\rm b,0}, \quad \psi_l(t, g)\Psi := W(-t)\psi_l(0, g)W(t)\Psi, \quad \Psi \in \mathcal{F}_{\rm b,0}, \quad g \in \mathcal{S}(\mathbb{R}^3), \tag{6.111}$$

and one concludes from Lemma 6.9:

Theorem 6.3. *For all $f, g \in \mathcal{S}(\mathbb{R}^3)$, $\mu = 0, 1, 2, 3$ and $l = 1, 2, 3, 4$, the operator valued functions $\mathbb{R} \ni t \mapsto A_\mu(t, f)$ and $\mathbb{R} \ni t \mapsto \psi_l(t, g)$ are strong solutions of the differential equations*

$$\frac{d}{dt} A_\mu(t, f)\Psi = [iH_{\rm QED}, A_\mu(t, f)]\Psi, \quad \Psi \in D(H_{\rm fr}) \cap \mathcal{F}_{\rm b,0}, \tag{6.112}$$

$$\frac{d}{dt} \psi_l(t, g)\Psi = [iH_{\rm QED}, \psi_l(t, g)]\Psi, \quad \Psi \in D(H_{\rm fr}) \cap \mathcal{F}_{\rm b,0}. \tag{6.113}$$

We remark that from Theorem 6.2 the above quantum fields are also written as

$$A_\mu(t, f)\Psi = W(t)^\dagger A_\mu(0, f)W(t)\Psi, \quad \Psi \in \mathcal{F}_{\rm b,0}, \tag{6.114}$$

$$\psi_l(t, g)\Psi = W(t)^\dagger \psi_l(0, g)W(t)\Psi, \quad \Psi \in \mathcal{F}_{\rm b,0}. \tag{6.115}$$

A A little about the function $t \mapsto U(t,t')\xi$

In Section 3, we have constructed the solution of the differential equation

$$\frac{d}{dt}\phi(t) = -iH_1(t)\phi(t), \quad \phi(0) = \xi, \tag{A.1}$$

via the evolution operator $U(t,t')$,

$$\phi(t) = U(t,t')\xi.$$

In this appendix, we investigate the properties of $\phi(t) = U(t,t')\xi$ ($\xi \in D$) as a function of $t \in \mathbb{R}$. Let us denote by I a closed interval in $\mathbb{R}$, fix $t' \in I$ and set

$$K := (A+1)^{1/2},$$

for short. Throughout the appendix, we assume only Assumption 2.1, which is sufficient to ensure that there is a solution of (A.1).

Lemma A.1. *Suppose that Assumption 2.1 holds. Let $u : I \to D$ be a strongly continuous mapping satisfying*

$$u(t) \in V_L, \quad t \in I \tag{A.2}$$

for some constant $L \geq 0$. Then, the mapping

$$I \ni t \mapsto H_1(t)u(t) \in \mathcal{H} \tag{A.3}$$

is strongly continuous. Moreover, for all $t \in I$, the strong Riemann integral

$$\int_{t'}^{t} ds\, H_1(s)u(s). \tag{A.4}$$

belongs to V_{L+b}.

Proof. Let $s, t \in I$.

$$\begin{aligned} \|H_1(t)u(t) - H_1(s)u(s)\| &\leq \|(H_1(t) - H_1(s))u(t)\| + \|H_1(s)(u(t) - u(s))\| \\ &\leq \|(H_1(t) - H_1(s))K^{-1} \cdot Ku(t)\| + \|H_1(s)K^{-1}\|\,\|K(u(t) - u(s))\|. \end{aligned} \tag{A.5}$$

Firstly, note that the mapping

$$t \mapsto e^{itH_0}H_1K^{-1}e^{-itH_0} \tag{A.6}$$

is strongly continuous, since H_1K^{-1} is a bounded operator. Thus, the first term of (A.5) tends to zero as $s \to t$.

Secondly, note also that there is some $L \geq 0$ such that

$$u(t) - u(s) \in V_L, \quad s, t \in I$$

Therefore, the second term of (A.5) also vanishes as s tends to t.

From Assumption 2.1, $H_1(t)u(t) \in V_{L+b}$ ($t \in I$). Since V_{L+b} is closed, one finds

$$\int_{t'}^{t} ds\, H_1(s)u(s) \in V_{L+b}.$$

□

From Lemma A.1, it follows that under Assumption 2.1 we may define a linear transformation J in the linear space

$$C_\infty := \{u : I \to D \,|\, u \text{ is strongly continuous and there is an } L \text{ such that } u(t) \in V_L \text{ for all } t \in I\} \tag{A.7}$$

by

$$(Ju)(t) := \int_{t'}^{t} ds\, H_1(s)u(s), \quad t \in I, \tag{A.8}$$

where $t' \in I$ is fixed ($[t', t] \subset I$ if $t' < t$; $[t, t'] \subset I$ if $t < t'$).

Lemma A.2. *Under assumption 2.1, for all $n = 1, 2, \dots$ and $u \in C_\infty$, the estimate*

$$\|(J^n u)(t)\| \leq \frac{|t - t'|^n}{n!} C^n (L + (n-1)b + 1)^{1/2} (L + (n-2)b + 1)^{1/2} \dots (L+1)^{1/2} \sup_{s \in I} \|u(s)\| \tag{A.9}$$

holds, where $L \geq 0$ is a constant depending on u satisfying

$$u(t) \in V_L, \quad t \in I.$$

Moreover,

$$(J^n u)(t) \in V_{L+nb}, \quad t \in I. \tag{A.10}$$

Proof. We prove by induction. When $n = 1$, the assertion is clear.

Suppose the lemma is true for some n. Then, by noting that

$$(J^n u)(t) \in V_{L+nb}, \quad t \in I,$$

we have

$$\begin{aligned}
\|(J \circ J^n u)(t)\| &\leq \int_{t'}^{t} ds\, \|H_1(s)(J^n u)(s)\| \\
&\leq \int_{t'}^{t} ds\, \|H_1 K^{-1}\| \, \|K(J^n u)(s)\| \\
&\leq C(L + nb + 1)^{1/2} \int_{t'}^{t} ds\, \|(J^n u)(s)\| \\
&\leq C(L + nb + 1)^{1/2} \int_{t'}^{t} ds\, \frac{|s - t'|^n}{n!} C^n (L + (n-1)b + 1)^{1/2} \dots (L+1)^{1/2} \sup_{s \in I} \|u(s)\| \\
&= \frac{|t - t'|^{n+1}}{(n+1)!} C^{n+1} (L + nb + 1)^{1/2} (L + (n-1)b + 1)^{1/2} \dots (L+1)^{1/2} \sup_{s \in I} \|u(s)\|,
\end{aligned} \tag{A.11}$$

and, by Lemma A.1,

$$(J^{n+1} u)(t) \in V_{L+(n+1)b}, \quad t \in I. \tag{A.12}$$

Hence, by induction, the assertion follows. □

Denote the approximated solution by ϕ_n:

$$\phi_n(t) = \sum_{k=0}^{n} S_n(t, t')\xi = \sum_{k=0}^{n} \left((-iJ)^n \xi\right)(t), \quad \xi \in D,$$

where in the last expression, ξ is identified with a constant function which belongs to C_∞. As mentioned above, ϕ_n belongs to C_∞ for all $n \in \mathbb{N}$. But, the limit function ϕ does not have to belong to C_∞. We identify the set of functions to which ϕ belongs.

Definition A.1. *Let T be a self-adjoint operator. We say that a function $u : I \to D(T)$ is T-uniformly integrable if the following two conditions are satisfied:*

(1) $t \mapsto Tu(t)$ *is strongly continuous.*

(2) $\lim_{\Lambda\to\infty} \sup_{t\in I} \|TE_T([\Lambda,\infty))u(t)\| = 0$.

For $\alpha \geq 0$, we denote

$$C_\alpha := \{u : I \to D(K^\alpha) \,|\, u \text{ is } K^\alpha\text{- uniformly integrable}\}. \tag{A.13}$$

The set C_α becomes a vector space by naturally defined addition and scalar multiplication. For each $u \in C_\alpha$, we introduce

$$\|u\|_{\alpha,\infty} := \sup_{t\in I} \|K^\alpha u(t)\|. \tag{A.14}$$

This is finite, since $K^\alpha u(\cdot)$ is strongly continuous. It is clear that $\|\cdot\|_{\alpha,\infty}$ is a norm of C_α. Further, we have

Proposition A.1. *The normed space $(C_\alpha, \|\cdot\|_{\alpha,\infty})$ is a Banach space.*

Proof. We show only the completeness. Take a Cauchy sequence in $(C_\alpha, \|\cdot\|_{\alpha,\infty})$, $\{u_n\}_n$. Then, for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n \geq m \geq N$ implies that

$$\|K^\alpha(u_n(t) - u_m(t))\| < \epsilon, \quad t \in I. \tag{A.15}$$

Since $\{K^\alpha u_n(t)\}_n$ is Cauchy in $\mathcal{H}$, there exists some $v(t) \in \mathcal{H}$ such that

$$\lim_{n\to\infty} K^\alpha u_n(t) = v(t).$$

On the other hand, since $K^\alpha \geq 1$, one finds for $n, m \geq N$

$$\begin{aligned}\|u_n(t) - u_m(t)\| &\leq \|K^{-\alpha}\|\,\|K^\alpha(u_n(t) - u_m(t))\| \\ &\leq \|K^\alpha(u_n(t) - u_m(t))\| < \epsilon.\end{aligned} \tag{A.16}$$

Therefore, $\{u_n(t)\}_n$ is also Cauchy and there is some $u(t) \in \mathcal{H}$ such that

$$\lim_{n\to\infty} u_n(t) = u(t).$$

Hence, we have $u(t) \in D(K^\alpha)$ and $K^\alpha u(t) = v(t)$ $(t \in I)$. Take $m \to \infty$ in (A.15). Then, one has for all $n \geq N$,

$$\|K^\alpha(u_n(t) - u(t))\| \leq \epsilon, \quad t \in I. \tag{A.17}$$

This means

$$\|u_n - u\|_{\alpha,\infty} \leq \epsilon$$

for all $n \geq N$.

We show $u \in C_\alpha$, that is, u is K^α- uniformly integrable. Note that $K^\alpha u(\cdot)$ — a uniform limit of $K^\alpha u_n(\cdot)$ — is strongly continuous. Take an arbitrary $\epsilon > 0$. We can choose $N \in \mathbb{N}$ in such a way that $n \geq N$ implies that

$$\sup_{t\in I} \|K^\alpha(u_n(t) - u(t))\| < \epsilon.$$

Then for an arbitrary $\Lambda \geq 0$, one has

$$\begin{aligned}\|K^\alpha E_{K^\alpha}([\Lambda,\infty))u(t)\| &\leq \|K^\alpha E_{K^\alpha}([\Lambda,\infty))(u_n(t) - u(t))\| + \|K^\alpha E_{K^\alpha}([\Lambda,\infty))u_n(t)\| \\ &< \epsilon + \|K^\alpha E_{K^\alpha}([\Lambda,\infty))u_n(t)\|.\end{aligned} \tag{A.18}$$

Taking the limit in which $\Lambda \to \infty$, we have

$$\sup_{t\in I_T} \|K^\alpha E_{K^\alpha}([\Lambda,\infty))u(t)\| < \epsilon + \sup_{t\in I_T} \|K^\alpha E_{K^\alpha}([\Lambda,\infty))u_n(t)\| \to \epsilon, \quad \Lambda \to \infty. \tag{A.19}$$

Since $\epsilon > 0$ is arbitrary, one has

$$\lim_{\Lambda\to\infty} \sup_{t\in I_T} \|K^\alpha E_{K^\alpha}([\Lambda,\infty))u(t)\| = 0,$$

namely, $u \in C_\alpha$. □

The next lemma shows the relation between C_α and C_∞.

Lemma A.3. *For all $\alpha \geq 0$, $C_\infty \subset C_\alpha$ as normed spaces. Moreover, C_∞ is dense in C_α with respect to the $\|\cdot\|_{\alpha,\infty}$ norm topology.*

Proof. First, we prove that $C_\infty \subset C_\alpha$. Let $u \in C_\infty$. To show that u is K^α-uniformly integrable, note that there is an $L \geq 0$ such that

$$u(t) \in V_L, \quad t \in I,$$

by definition of C_∞. One has

$$\|K^\alpha(u(t) - u(s))\| \leq (L+1)^{\alpha/2}\|u(t) - u(s)\|,$$

which shows that $K^\alpha u(\cdot)$ is strongly continuous. Let $\Lambda > (L+1)^{\alpha/2}$. Then,

$$\|K^\alpha E_{K^\alpha}([\Lambda,\infty))u(t)\| = \|K^\alpha E_A([\Lambda^{2/\alpha} - 1,\infty))u(t)\| = 0. \tag{A.20}$$

Hence, u is K^α-uniformly integrable, that is, $u \in C_\alpha$.

Next, we prove that C_∞ is dense in C_α. Take any $u \in C_\alpha$. Define $\{u_n\}_n \subset C_\infty$ by

$$u_n(t) := E_A([0,n))u(t), \quad n = 1, 2, \ldots.$$

Then it follows that

$$\begin{aligned}
\|u_n - u\|_{\alpha,\infty} &= \sup_{t\in I_T} \|K^\alpha(E_A([0,n)) - 1)u(t)\| \\
&= \sup_{t\in I_T} \|K^\alpha E_A([n,\infty))u(t)\| \\
&= \sup_{t\in I_T} \|K^\alpha E_{K^\alpha}([(1+n)^{\alpha/2},\infty))u(t)\| \to 0, \quad n \to \infty.
\end{aligned} \tag{A.21}$$

This proves that C_∞ is dense. □

This lemma tells us that C_α is the completion of C_∞ with respect to the norm $\|\cdot\|_{\alpha,\infty}$.

Lemma A.4. *Let $0 \leq \alpha \leq \beta$. Then $C_\beta \subset C_\alpha$ and the inclusion mapping*

$$\iota : C_\beta \to C_\alpha$$

is continuous.

Proof. Let $u \in C_\beta$. Then, $K^{\alpha-\beta}$ is bounded and thus the mapping

$$t \mapsto K^\alpha u(t) = K^{\alpha-\beta} K^\beta u(t)$$

is strongly continuous. Further, we have

$$\begin{aligned}\sup_{t\in I_T} \|K^\alpha E_{K^\alpha}([\Lambda,\infty))u(t)\| &\le \|K^{\alpha-\beta}\| \sup_{t\in I_T} \|K^\beta E_{K^\alpha}([\Lambda,\infty))u(t)\| \\ &= \|K^{\alpha-\beta}\| \sup_{t\in I_T} \|K^\beta E_{K^\beta}([\Lambda^{\beta/\alpha},\infty))u(t)\| \to 0 \quad \Lambda\to\infty. \end{aligned} \tag{A.22}$$

These imply that $u \in C_\alpha$. Hence $C_\beta \subset C_\alpha$.

The second assertion immediately follows from the fact that

$$\|K^\alpha u(t)\| \le \|K^{\alpha-\beta}\| \, \|K^\beta u(t)\|, \quad u \in C_\beta.$$

□

Proposition A.2. *The approximated solution $\{\phi_n\}_n \subset C_\infty$ is a Cauchy sequence in $\|\cdot\|_{\alpha,\infty}$ for all $\alpha \ge 0$.*

Proof. Fix $\alpha \ge 0$. Choose L for $\xi \in D$ in such a way that

$$\xi \in V_L$$

holds. Since

$$\phi_n = \sum_{k=0}^{n} (-iJ)^k \xi,$$

and from Lemma A.2, we obtain $\phi_n(t) \in V_{L+nb}$. Therefore, we have

$$\begin{aligned}\|\phi_{n+1} - \phi_n\|_{\alpha,\infty} &= \sup_{t\in I_T} \|K^\alpha(\phi_{n+1}(t) - \phi_n(t))\| \\ &\le \sup_{t\in I} (L + nb + 1)^{\alpha/2} \|\phi_{n+1}(t) - \phi_n(t)\| \\ &\le \frac{|I|^{n+1}}{(n+1)!} C^n (L+nb+1)^{\alpha/2} (L+nb+1)^{1/2} \dots (L+1)^{1/2} \|\xi\|, \end{aligned} \tag{A.23}$$

where $|I|$ denotes the Lebesgue measure of the interval I. By d'Alembert's ratio test, one obtains

$$\sum_{n=0}^{\infty} \|\phi_{n+1} - \phi_n\|_{\alpha,\infty} < \infty.$$

Hence, for any $\epsilon > 0$, there is an $N \in \mathbb{N}$, such that $n \ge m \ge N$ implies

$$\|\phi_n - \phi_m\|_{\alpha,\infty} \le \sum_{k=m}^{n-1} \|\phi_{k+1} - \phi_k\|_{\alpha,\infty} < \epsilon.$$

□

Theorem A.1. *For any $\xi \in D$, the solution ϕ, which is the limit of $\phi_n = S_n(\cdot, t')\xi$, belongs to $\bigcap_{\alpha\ge 0} C_\alpha$. That is, $U(\cdot, t')\xi$ is K^α-uniformly integrable for all $\alpha \ge 0$.*

Proof. Fix $\alpha \ge 0$. By Proposition A.1 and Proposition A.2, there is an element $u_\alpha \in C_\alpha$ satisfying

$$\|\phi_n - \phi_\alpha\|_{\alpha,\infty} \to 0.$$

Take $\beta > 0$. Without loss of generality, we may assume $\alpha \le \beta$. Then, we obtain

$$\begin{aligned}\|\phi_\alpha - \phi_\beta\|_{\alpha,\infty} &\le \|\phi_\alpha - \phi_n\|_{\alpha,\infty} + \|\phi_n - \phi_\beta\|_{\alpha,\infty} \\ &\le \|\phi_\alpha - \phi_n\|_{\alpha,\infty} + \|K^{\alpha-\beta}\| \, \|\phi_n - \phi_\beta\|_{\beta,\infty} \to 0, \quad n \to \infty, \end{aligned} \tag{A.24}$$

which shows that $\phi_\alpha = \phi_\beta$. If $\alpha = 0$, then $\|\phi_n - \phi\|_{0,\infty} \to 0$. Therefore, we conclude that for all $\alpha \ge 0$, $\phi = \phi_\alpha \in C_\alpha$. □

Acknowledgement

The authors would like to thank Professor Asao Arai for fruitful comments and discussions and for a critical reading of this manuscript. They also thank Dr. Kazuyuki Wada of Hokkaido University for valuable comments.

References

[1] A. Arai. *Fock spaces and Quantum Fields I, II (in Japanese)*. Nippon-Hyoronsha, Tokyo, 2000.

[2] A. Arai. Heisenberg operators, invariant domains and Heisenberg equations of motion. *Rev. Math. Phys.*, 19(10):1045–1069, 2007.

[3] G. Da Prato, P. C. Kunstmann, I. Lasiecka, A. Lunardi, R. Schnaubelt, and L. Weis. *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004. Edited by M. Iannelli, R. Nagel and S. Piazzera.

[4] J. D. Dollard and C. N. Friedman. On strong product integration. *J. Funct. Anal.*, 28(3):309–354, 1978.

[5] J. D. Dollard and C. N. Friedman. *Product integration with applications to differential equations*, volume 10 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1979. With a foreword by Felix E. Browder, With an appendix by P. R. Masani.

[6] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[7] F. Hiroshima and A. Suzuki. Physical state for nonrelativistic quantum electrodynamics. *Ann. Henri Poincaré*, 10(5):913–953, 2009.

[8] R. J. Hughes. Singular perturbations in the interaction representation. II. *J. Funct. Anal.*, 49(3):293–314, 1982.

[9] R. J. Hughes and I. E. Segal. Singular perturbations in the interaction representation. *J. Funct. Anal.*, 38(1):71–98, 1980.

[10] M. E. Peskin and D. V. Schroeder. *An introduction to quantum field theory*. Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1995. Edited and with a foreword by David Pines.

[11] M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[12] H. Spohn. *Dynamics of charged particles and their radiation field*. Cambridge University Press, Cambridge, 2004.

[13] A. Suzuki. Physical subspace in a model of the quantized electromagnetic field coupled to an external field with an indefinite metric. *J. Math. Phys.*, 49(4):042301, 24, 2008.

[14] T. Takaesu. On the spectral analysis of quantum electrodynamics with spatial cutoffs. I. *J. Math. Phys.*, 50(6):062302, 28, 2009.

[15] S. Weinberg. *The quantum theory of fields. Vol. I*. Cambridge University Press, Cambridge, 2005. Foundations.

[16] S. Weinberg. *The quantum theory of fields. Vol. II*. Cambridge University Press, Cambridge, 2005. Modern applications.