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On the special values of certain L-series related to half-integral weight modular forms

Hidenori Katsurada*

Abstract

Let h be a cuspidal Hecke eigenform of half-integral weight, and $E_{n/2+1/2}$ be Cohen's Eisenstein series of weight $n/2+1/2$. For a Dirichlet character χ we define a certain linear combination $R^{(\chi)}(s, h, E_{n/2+1/2})$ of the Rankin-Selberg convolution products of h and $E_{n/2+1/2}$ twisted by Dirichlet characters related with χ . We then prove a certain algebraicity result for $R^{(\chi)}(l, h, E_{n/2+1/2})$ with l integers.

0 Introduction

For two modular forms $h_1(z)$ and $h_2(z)$ of half-integral weights $k_1 + 1/2$ and $k_2 + 1/2$, respectively, for $\Gamma_0(4)$, and a primitive character χ we define the Rankin-Selberg convolution product $\tilde{R}(s, h_1, h_2, \chi)$ twisted by χ as

$$\tilde{R}(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi_{-1}^{k_1-k_2} \chi^2) \sum_{m=1}^{\infty} \frac{c_1(m)c_2(m)\chi(m)}{m^s},$$

where $c_1(m)$ and $c_2(m)$ denote the m -th Fourier coefficients of h_1 and h_2 , respectively, and $L(s, \chi_{-1}^{k_1-k_2} \chi^2)$ is the Dirichlet L -function for $\chi_{-1}^{k_1-k_2} \chi^2$ (for the precise definition of χ_{-1} see Section 1.)

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The analytic properties of this Dirichlet series were investigated by Shimura [Sh2]. Furthermore the algebraicity of the values of this Dirichlet series at half-integers was deeply investigated by Shimura [Sh2]. However, as far as we know, there is no literature on the algebraicity of its special values at integers except for [K-M]. Therefore we naturally ask the following question:

Question. What can one say about the algebraicity of $\tilde{R}(m, h_1, h_2, \chi)$ with m an integer?

In [K-M], we gave a partial answer to the above question in the case h_1 is a cuspidal Hecke eigenform in Kohnen's plus subspace for $\Gamma_0(4)$ and h_2 is Zagier's Eisenstein series of weight $3/2$. In this paper, we consider the above question in the case h_1 is a cuspidal Hecke eigenform in Kohnen's plus subspace for $\Gamma_0(4)$ and h_2 is Cohen's Eisenstein series. This paper is a summary of our paper [Ka], which will be published elsewhere. To state our main result more explicitly, we define another Dirichlet series $R(s, h_1, h_2, \chi)$ by

$$R(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi^2) \sum_{m=1}^{\infty} c_{h_1}(m) c_{h_2}(m) \chi(m) m^{-s}.$$

Assume that $k_1 + k_2$ is even, and that the conductor of χ is odd. Then, as will be explained in Section 1, it suffices to consider the above question for $R(m, h_1, h_2, \chi)$ with integer m . Now let k and n be even integers such that $n \geq 4$ and $2k - n \geq 12$. Let h be a Hecke eigenform of weight $k - n/2 + 1/2$ for $\Gamma_0(4)$ belonging to Kohnen's plus subspace, and $S(h)$ the normalized Hecke eigenform of weight $2k - n$ for $SL_2(\mathbf{Z})$ corresponding to h under the Shimura correspondence. Moreover let $E_{n/2+1/2}$ be Cohen's Eisenstein series of weight $n/2 + 1/2$ (for the precise definition of $E_{n/2+1/2}$, see Section 2). Let χ be a primitive character of conductor N . We assume that N is square free and let $N = p_1 \cdots p_r$ be the prime decomposition of N . Put $l_j = l_{n, p_j} = \text{G.C.D}(n, p_j - 1)$. For an r -tuple $(i_1, i_2, \dots, i_r)$ of integers put

$$\chi_{(i_1, \dots, i_r)} = \chi \prod_{j=1}^r \left(\frac{*}{p_j} \right)_{l_j}^{i_j},$$

where $\left(\frac{*}{p_j} \right)_{l_j}$ denotes the l_j -th power residue symbol mod p_j . For two Dirich-

let characters η_1 and $\eta_2 \bmod N$, we define $J_m(\eta_1, \eta_2)$ by

$$J_m(\eta_1, \eta_2) = \sum_Z \eta_1(\det Z) \eta_2(1 - \text{tr}(Z)),$$

where Z runs over all symmetric matrices of degree m with entries in $\mathbf{Z}/N\mathbf{Z}$ and $\text{tr}(Z)$ denotes the trace of a matrix Z . We note that $J_1(\eta_1, \eta_2)$ is the Jacobi sum $J(\eta_1, \eta_2)$ associated with η_1 and η_2 . We also put $J_m(\eta_1) = J_m(\eta_1 \left(\frac{*}{N}\right)^{m-1}, \eta_1)$, where $\left(\frac{*}{N}\right)$ is the Jacobi symbol. We then define

$$R^{(\chi)}(s, h, E_{n/2+1/2}) = \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\chi_{(i_1, \dots, i_r)}(2^n)} R(s, h, E_{n/2+1/2}, \chi_{(i_1, \dots, i_r)}) \\ \times \overline{J(\chi_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right))} \overline{J_{n-1}(\chi_{(i_1, \dots, i_r)})} \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \chi_{(i_1, \dots, i_r)}^2),$$

where $L(s, S(h), \chi_{(i_1, \dots, i_r)}^2)$ is Hecke's L -function of $S(h)$ twisted by $\chi_{(i_1, \dots, i_r)}^2$. Then our main result (Theorem 2.1) can be stated as follows:

There exists a finite dimensional $\overline{\mathbf{Q}}$ -vector space $W_{h, E_{n/2+1/2}}$ in $\mathbf{C}$ such that

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in W_{h, E_{n/2+1/2}}$$

for any integer m such that $n/2 + 1 \leq m \leq k - n/2 - 1$ and a character χ of odd square free conductor such that χ^n is primitive.

From the above result we easily obtain the following (cf. Theorem 2.2):

Let $r > \dim_{\overline{\mathbf{Q}}} W_{h, E_{n/2+1/2}}$. Let $m_1, m_2, \dots, m_r$ be integers such that $n/2 + 1 \leq m_1, m_2, \dots, m_r \leq k - n/2 - 1$ and $\chi_1, \chi_2, \dots, \chi_r$ be Dirichlet characters of odd square free conductors $N_1, N_2, \dots, N_r$, respectively such that χ_i^n is primitive for any $i = 1, 2, \dots, r$. Then the values

$$\frac{R^{(\chi_1)}(m_1, h, E_{n/2+1/2})}{\pi^{m_1 n}}, \dots, \frac{R^{(\chi_r)}(m_r, h, E_{n/2+1/2})}{\pi^{m_r n}}$$

are linearly dependent over $\overline{\mathbf{Q}}$.

This is a certain generalization of a main result in [K-M] as will be explained later.

A main tool for proving Theorem 2.1 is the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift of h . To explain this, we define the twisted Koecher-Maaß series of a Siegel modular form in a more general setting. Let $F(Z)$ be a modular form of weight k with respect to the symplectic group $Sp_n(\mathbf{Z})$. For a positive integer N let $SL_{n,N}(\mathbf{Z}) = \{U \in SL_n(\mathbf{Z}) \mid U \equiv 1_n \pmod{N}\}$, and $e_N(T) = \#\{U \in SL_{n,N}(\mathbf{Z}) \mid T[U] = T\}$. For a primitive Dirichlet character $\chi \pmod{N}$ we define the Koecher-Maaß series $L(s, F, \chi)$ of F twisted by χ as

$$L(s, F, \chi) = \sum_T \frac{\chi(\text{tr}(T))c_F(T)}{e_N(T)(\det T)^s},$$

where T runs over a complete set of representatives of $SL_{n,N}(\mathbf{Z})$ -equivalence classes of positive definite half-integral matrices of degree n , and $c_F(T)$ denotes the T -th Fourier coefficient of F . We note that this Dirichlet series coincides with the Hecke L -function associated to F twisted by χ in case $n = 1$. Though we are mainly concerned with $L(s, F, \chi)$ in this paper, we also define another type of twisted Koecher-Maaß series $L^*(s, F, \chi)$ as

$$L^*(s, F, \chi) = \sum_T \frac{\chi(\det(2T))c_F(T)}{e(T)(\det T)^s},$$

where T runs over a complete set of representatives of $SL_n(\mathbf{Z})$ -equivalence classes of positive definite half-integral matrices of degree n , and $e(T) = e_1(T)$. These two Dirichlet series $L(s, F, \chi)$ and $L^*(s, F, \chi)$ essentially coincide with each other in case $n = 1$, but they don't in general. To distinguish these two Dirichlet series, we sometimes call $L(s, F, \chi)$ and $L^*(s, F, \chi)$ the twisted Koecher-Maaß series of the first and second kind, respectively. In Section 3, we will discuss a relation between these two Dirichlet series (cf. Theorem 3.5.) Now for the integers k and n stated above, let h a cuspidal Hecke eigenform h in Kohnen's plus subspace of weight $k - n/2 + 1/2$ for $\Gamma_0(4)$. Let $I_n(h)$ be the Duke-Imamoglu-Ikeda lift of h to the space of Siegel cusp forms of degree n . Then, in Section 4, first we give an explicit formula of $L^*(s, I_n(h), \eta)$ in terms of the Rankin-Selberg series $R(s, h, E_{n/2+1/2}, \eta)$ and shifted products of Hecke's L -functions of $S(h)$ twisted by η^2 in the case η is a primitive character (cf. Theorem 4.1.) Next, by this result combined with Theorem 3.5, we give an explicit formula of $L(s, I_n(h), \chi^n)$ in terms of $R^{(\chi)}(s, h, E_{n/2+1/2})$ and a sum of the shifted products $\prod_{j=1}^{n/2-1} L(2s - 2j + 1, S(h), \chi_{(i_1, \dots, i_r)}^2)$ (cf.

Theorem 4.2 and its corollary.) This implies that $R^{(\chi)}(s, h, E_{n/2+1/2})$ can be expressed in terms of $L(s, I_n(h), \chi^n)$ and the sum of the shifted products. Thus we can prove our main result using the algebraicity of Hecke's L -function of $S(h)$ (cf. Theorem 1.1) combined with the arithmetic properties of $L(s, I_n(h), \chi^n)$, which were investigated by Choie and Kohnen [C-K] in a more general setting (cf. Theorem 3.2). We can also prove a functional equation for $R^{(\chi)}(s, h, E_{n/2+1/2})$ in case $n \equiv 2 \pmod{4}$ using the functional equation for $L(s, F, \chi^n)$ (cf. Theorem 2.3.)

Notation. We denote by $e(x) = \exp(2\pi\sqrt{-1}x)$ for a complex number x . For a commutative ring R , we denote by $M_{mn}(R)$ the set of (m, n) -matrices with entries in R . For an (m, n) -matrix X and an (m, m) -matrix A , we write $A[X] = {}^tXAX$, where tX denotes the transpose of X . Let a be an element of R . Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the coset $X \bmod aM_{mn}(R)$. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, and $SL_m(R) = \{A \in M_m(R) \mid \det A = 1\}$, where $\det A$ denotes the determinant of a square matrix A and R^* is the unit group of R . We denote by $S_n(R)$ the set of symmetric matrices of degree n with entries in R . In particular, if S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. The group $SL_n(\mathbf{Z})$ acts on the set $S_n(\mathbf{R})$ in the following way:

$$SL_n(\mathbf{Z}) \times S_n(\mathbf{R}) \ni (g, A) \longrightarrow {}^tgAg \in S_n(\mathbf{R}).$$

Let G be a subgroup of $GL_n(\mathbf{Z})$. For a subset $\mathcal{B}$ of $S_n(\mathbf{R})$ stable under the action of G we denote by $\mathcal{B}/G$ the set of equivalence classes of $\mathcal{B}$ with respect to G . We sometimes identify $\mathcal{B}/G$ with a complete set of representatives of $\mathcal{B}/G$. Two symmetric matrices A and A' with entries in R are said to be equivalent with each other with respect to G and write $A \sim_G A'$ if there is an element X of G such that $A' = A[X]$. Let $\mathcal{L}_n$ denote the set of half-integral matrices of degree n over $\mathbf{Z}$, that is, $\mathcal{L}_n$ is the set of symmetric matrices of degree n whose (i, j) -component belongs to $\mathbf{Z}$ or $\frac{1}{2}\mathbf{Z}$ according as $i = j$ or not.

1 Review on the algebraicity of L-values of elliptic modular forms of integral and half-integral weights

Before stating our main results, we review on the special values of L functions of elliptic modular forms of integral and half-integral weights. Put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where 1_n and O_n denotes the unit matrix and the zero matrix of degree n , respectively. Furthermore, put

$$Sp_n(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \mid J_n[M] = J_n\}.$$

Let l be an integer or a half-integer, and let Γ be a congruence subgroup of $Sp_n(\mathbf{Z})$. We then denote by $M_l(\Gamma)$ the space of modular forms of weight l with respect to Γ , and by $S_l(\Gamma)$ the subspace of $M_l(\Gamma)$ consisting of cusp forms. We also denote by $\Gamma_0(4)$ the subgroup of $SL_2(\mathbf{Z})$ consisting of matrices whose left lower entries are congruent to 0 mod N . Let

$$f(z) = \sum_{m=1}^{\infty} c_f(m) \mathbf{e}(mz)$$

be a normalized Hecke eigenform in $S_k(SL_2(\mathbf{Z}))$, and χ be a primitive Dirichlet character. Then let us define Hecke's L -function $L(s, f, \chi)$ of f twisted by χ as

$$L(s, f, \chi) = \sum_{m=1}^{\infty} c_f(m) \chi(m) m^{-s}.$$

Then we have the following result (cf. [Sh1]):

Theorem 1.1 *There exist complex numbers $u_{\pm}(f)$ uniquely determined up to $\overline{\mathbf{Q}}^{\times}$ multiple such that*

$$L(m, f, \chi) (\pi^m u_j(f))^{-1} \in \overline{\mathbf{Q}}$$

for any integer $0 < m \leq k-1$ and a primitive character χ , where $j = +$ or $-$ according as $(-1)^m \chi(-1) = 1$ or -1 .

We remark that we have $L(m, f, \chi) \neq 0$ if $m \neq k/2$, and $L(k/2, f, \chi) \neq 0$ for infinitely many χ .

Next let us consider the half-integral weight case. Let

$$h_1(z) = \sum_{m=1}^{\infty} c_{h_1}(m) \mathbf{e}(mz)$$

be a Hecke eigenform in $S_{k_1+1/2}(\Gamma_0(4))$, and

$$h_2(z) = \sum_{m=0}^{\infty} c_{h_2}(m) \mathbf{e}(mz)$$

be an element of $M_{k_2+1/2}(\Gamma_0(4))$. For positive integers e and l , let $\chi_{(-1)^l e}$ be the Dirichlet character corresponding to the extension $\mathbf{Q}(\sqrt{(-1)^l e}/\mathbf{Q})$. Let χ be a primitive character mod N . Then we define

$$\tilde{R}(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \omega) \sum_{m=1}^{\infty} c_{h_1}(m) c_{h_2}(m) \chi(m) m^{-s},$$

where $\omega(d) = \chi_{-1}^{k_1-k_2} \chi^2(d)$. Now let $S(h_1)$ be the normalized Hecke eigenform in $S_{2k_1}(SL_2(\mathbf{Z}))$ corresponding to h_1 under the Shimura correspondence. Then the following result is due to Shimura [Sh2].

Theorem 1.2 *Assume that $k_1 > k_2$. Under the above notation we have*

$$\tilde{R}(m + 1/2, h_1, h_2, \chi) (u_-(S(h_1)) \pi^{-k_2+1+2m})^{-1} \in \overline{\mathbf{Q}}(h_1) \overline{\mathbf{Q}}(h_2)$$

for any integer $k_2 \leq m \leq k_1 - 1$ and a primitive character χ , where $\overline{\mathbf{Q}}(h_i)$ is the field, generated over $\overline{\mathbf{Q}}$, by all the Fourier coefficients of h_i .

Corollary *Let the notation be as above. Assume that $k_1 > k_2$ and that $c_{h_1}(n), c_{h_2}(n) \in \overline{\mathbf{Q}}$ for any $n \in \mathbf{Z}_{\geq 0}$. Then there exists a one-dimensional $\overline{\mathbf{Q}}$ -vector space U_{h_1, h_2} in $\mathbf{C}$ such that*

$$\tilde{R}(m + 1/2, h_1, h_2, \chi) \pi^{-2m} \in U_{h_1, h_2}$$

for any integer $k_2 \leq m \leq k_1 - 1$ and a primitive character χ .

Now we consider the values of $\tilde{R}(s, h_1, h_2, \chi)$ at integers. Let

$$R(s, h_1, h_2, \chi) = L(2s - k_1 - k_2 + 1, \chi^2) \sum_{m=1}^{\infty} c_{h_1}(m) c_{h_2}(m) \chi(m) m^{-s}.$$

be the Dirichlet series defined in Section 0. Assume that $k_1 + k_2$ is even, and that the conductor of χ is odd. Then we have

$$R(s, h_1, h_2, \chi) = (1 - 2^{-2s+k_1+k_2-1} \chi^2(2))^{-1} \tilde{R}(s, h_1, h_2, \chi).$$

Hence it suffices to consider the question in Section 0 for $R(m, h_1, h_2, \chi)$ with integer m .

2 Main results

For a non-negative integer m and a positive integer l , Cohen's function $H(l, m)$ is given by $H(l, m) = L_{-m}(1 - l)$. Here

$$L_D(s) = \begin{cases} \zeta(2s - 1), & D = 0 \\ L(s, \chi_{D_K}) \sum_{a|f} \mu(a) \chi_{D_K}(a) a^{-s} \sigma_{1-2s}(f/a), & D \neq 0, D \equiv 0, 1 \pmod{4} \\ 0, & D \equiv 2, 3 \pmod{4}, \end{cases}$$

where the positive integer f is defined by $D = D_K f^2$ with the discriminant D_K of $K = \mathbf{Q}(\sqrt{D})$, χ_{D_K} is the Kronecker symbol, μ is the Möbius function and $\sigma_s(n) = \sum_{d|n} d^s$. Furthermore we define Cohen's Eisenstein series $E_{l+1/2}(z)$ by

$$E_{l+1/2}(z) = \sum_{m=0}^{\infty} H(l, m) \mathbf{e}(mz).$$

It is known that $E_{l+1/2}(z)$ is a modular form of weight $l + 1/2$ belonging to Kohnen's plus space. Let k and n be positive even integers such that $n \geq 4$, $2k - n \geq 12$. Let $h(z)$ be a Hecke eigenform in Kohnen's plus subspace $S_{k-n/2+1/2}^+(\Gamma_0(4))$ (cf. [Ko]), and $S(h)$ be the normalized Hecke eigenform in $S_{2k-n}(SL_2(\mathbf{Z}))$ corresponding to h under the Shimura correspondence. Let p be a prime number and l be a positive integer dividing $p - 1$. Take an l -th root of unity ζ_l and a prime ideal $\mathfrak{p}$ of $\mathbf{Q}(\zeta_l)$ lying above p . Let a be an integer prime to p . Then we have $a^{(p-1)/l} \equiv \zeta_l^i \pmod{\mathfrak{p}}$ with some $i \in \mathbf{Z}$. We then put $\left(\frac{a}{p}\right)_l = \zeta^i$. We call $\left(\frac{*}{p}\right)_l$ the l -th power residue symbol mod p . In the case

$l = 2$, this is the Legendre symbol, and we write it as $\left(\frac{*}{p}\right)$ as usual. We

note that this definition of the power residue symbol is different from the usual one, and depends on the choice of $\mathfrak{p}$ and ζ_l except the case $l = 2$. We denote by $\left(\frac{*}{N}\right)$ the Jacobi symbol for a positive odd integer M . Let χ be a primitive Dirichlet character of conductor N . We assume that N is a square free odd integer, and write $N = p_1 \cdots p_r$ with $p_1, \dots, p_r$ prime numbers. Put $l_j = l_{n,p_j} = \text{G.C.D}(n, p_j - 1)$. For an r -tuple $(i_1, i_2, \dots, i_r)$ of integers put

$$\chi_{(i_1, \dots, i_r)} = \chi \prod_{j=1}^r \left(\frac{*}{p_j} \right)_{l_j}^{i_j}.$$

For two Dirichlet characters η_1 and $\eta_2 \bmod N$, let $J_m(\eta_1, \eta_2)$ and $J_m(\eta_1)$ be as those defined in Section 0. By definition, $J_m(\eta_1, \eta_2)$ is an algebraic number. As in Section 0, we define

$$\begin{aligned} & R^{(\chi)}(s, h, E_{n/2+1/2}) \\ &= \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\chi_{(i_1, \dots, i_r)}(2^n) J(\chi_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right) J_{n-1}(\chi_{(i_1, \dots, i_r)})} \\ & \quad \times R(s, h, E_{n/2+1/2}, \chi_{(i_1, \dots, i_r)}) \mathbf{L}_n(s, S(h), \chi_{(i_1, \dots, i_r)}), \end{aligned}$$

where

$$\mathbf{L}_n(s, S(h), \eta) = \prod_{j=1}^{n/2-1} L(2s - 2j, S(h), \eta^2)$$

for a primitive character η . We note that $R^{(\chi)}(s, h, E_{n/2+1/2})$ does not depend on the choice of an l_i -th root of unity ζ_{l_i} and an prime ideal $\mathfrak{p}_i$ of $\mathbf{Q}(\zeta_{l_i})$ lying above p_i .

Remark. (1) Let m be an integer s.t. $n/2 + 1 \leq m \leq k - n/2 - 1$. Then the value $\frac{\mathbf{L}_n(m, S(h), \chi_{(i_1, \dots, i_r)}^2)}{\pi^{m(n-2)}}$ belongs to $\overline{\mathbf{Q}}u_+(S(h))^{n/2-1} \pi^{-n^2/4+n/2}$ for any χ .

In particular if $n \equiv 2 \pmod{4}$, then it is nonzero for any χ , and if $n \equiv 0 \pmod{4}$, then it is nonzero for infinitely many χ .

(2) As will be stated in Section 3, $J_{n-1}(\chi_{(i_1, \dots, i_r)})$ is expressed as a product of Jacobi sums, and it is non-zero algebraic number if χ^n is rewrote.

Theorem 2.1 *There exists a finite dimensional $\overline{\mathbf{Q}}$ -vector space $W_{h, E_{n/2+1/2}}$ in $\mathbf{C}$ such that*

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in W_{h, E_{n/2+1/2}}$$

for any integer $n/2 + 1 \leq m \leq k - n/2 - 1$ and a character χ of odd square free conductor such that χ^n is rewrote.

Theorem 2.2 Let $r > \dim_{\overline{\mathbf{Q}}} W_{h, E_{n/2+1/2}}$. Let $m_1, m_2, \dots, m_r$ be integers such that $n/2 + 1 \leq m_1, m_2, \dots, m_r \leq k - n/2 - 1$ and $\chi_1, \chi_2, \dots, \chi_r$ be Dirichlet characters of odd square free conductors $N_1, N_2, \dots, N_r$, respectively such that χ_i^n is primitive for any $i = 1, 2, \dots, r$. Then the values $\frac{R^{(\chi_1)}(m_1, h, E_{n/2+1/2})}{\pi^{m_1 n}}, \dots, \frac{R^{(\chi_r)}(m_r, h, E_{n/2+1/2})}{\pi^{m_r n}}$ are linearly dependent over $\overline{\mathbf{Q}}$.

Corollary Assume that $n \equiv 2 \pmod{4}$. Let r and $m_1, m_2, \dots, m_r$ be as above. Let $\chi_1, \chi_2, \dots, \chi_r$ be Dirichlet characters of odd prime conductors $p_1, p_2, \dots, p_r$, respectively such that χ_i^n is non-trivial for any $i = 1, 2, \dots, r$. Put $l_i = \text{GCD}(n, p_i - 1)$. Then the values $\left\{ \frac{R(m_i, h, E_{n/2+1/2}, \chi_{i(j)})}{\pi^{2m_i}} \right\}_{1 \leq i \leq r, 0 \leq j \leq l_i - 1}$ are linearly dependent over $\overline{\mathbf{Q}}$.

We also have a functional equation for $R^{(\chi)}(s, h, E_{n/2+1/2})$:

Theorem 2.3 Let h be as above. Let χ be a primitive character of odd square free conductor N . Assume that $n \equiv 2 \pmod{4}$, and that χ^n is primitive. Put

$$\mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2}) = N^{2s} \tau(\chi^n)^{-1} \gamma_n(s) R^{(\chi)}(s, h, E_{n/2+1/2}),$$

where $\tau(\chi^n)$ is the Gauss sum of χ^n , and

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^n \pi^{(i-1)/2} \Gamma(s - (i-1)/2).$$

Then $\mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2})$ has an analytic continuation to the whole s -plane, and has the following functional equation:

$$\mathcal{R}^{(\chi)}(k - s, h, E_{n/2+1/2}) = \mathcal{R}^{(\chi)}(s, h, E_{n/2+1/2}).$$

Remark. (1) The series $\{R(s, h, E_{n/2+1/2}, \chi_{i(j)})\}_{1 \leq i \leq r, 0 \leq j \leq l_i - 1}$ are linearly independent over $\mathbf{C}$ as functions of s .

(2) In the case of $n = 2$, this type of result was given for $R(m, h, E_{3/2})$ with $E_{3/2}$ Zagier's Eisenstein series of weight $3/2$ by [K-M]. Cohen's Eisenstein

series is a holomorphic modular form, where as Zagier's Eisenstein series is not. Nevertheless, the former can be regarded as a generalization of the latter. Therefore, our present result can be regarded as a generalization of [K-M]. (3) The meromorphy of this type of series was derived in [Sh2] by using so called the Rankin-Selberg integral expression in a more general setting, but we don't know whether the functional equation of the above type can be directly proved without using the above method.

3 Twisted Koecher-Maaß series

To prove the main results, in this section and the next, we consider the twisted Koecher-Maaß series of a Siegel modular form. Let $F(Z) \in M_k(Sp_n(\mathbf{Z}))$. Then $F(Z)$ has the Fourier expansion:

$$F(Z) = \sum_{T \in \mathcal{L}_n \geq 0} c_F(T) e(\text{tr}(TZ)),$$

where $\text{tr}(X)$ denotes the trace of a matrix X . For $N \in \mathbf{Z}_{>0}$, put $SL_{n,N}(\mathbf{Z}) = \{U \in SL_n(\mathbf{Z}) \mid U \equiv 1_n \pmod{N}\}$, and for $T \in \mathcal{L}_{n>0}$ put $e_N(T) = \#\{U \in SL_{n,N}(\mathbf{Z}) \mid T[U] = T\}$. For a primitive Dirichlet character $\chi \pmod{N}$ Let

$$L(s, F, \chi) = \sum_{T \in \mathcal{L}_{n>0}/SL_{n,N}(\mathbf{Z})} \frac{\chi(\text{tr}(T)) c_F(T)}{e_N(T) (\det T)^s}$$

be the twisted Koecher-Maaß series of F of the first kind as in Section 0. The following two theorems are due to Choie and Kohnen [C-K].

Theorem 3.1 *Let $F \in S_k(Sp_n(\mathbf{Z}))$, and χ a primitive character of conductor N . Put*

$$\gamma_n(s) = (2\pi)^{-ns} \prod_{i=1}^n \pi^{(i-1)/2} \Gamma(s - (i-1)/2),$$

and

$$\Lambda(s, F, \chi) = N^{2s} \tau(\chi)^{-1} \gamma_n(s) L(s, F, \chi) \quad (\text{Re}(s) \gg 0),$$

where $\tau(\chi)$ is the Gauss sum of χ . Then $\Lambda(s, F, \chi)$ has an analytic continuation to the whole s -plane and has the following functional equation:

$$\Lambda(k - s, F, \chi) = (-1)^{nk/2} \chi(-1) \Lambda(s, F, \bar{\chi}).$$

Theorem 3.2 *Let F and χ be as above. Then there exists a finite dimensional $\overline{\mathbf{Q}}$ -vector space V_F in $\mathbf{C}$ such that*

$$L(m, F, \chi)\pi^{-nm} \in V_F$$

for any primitive character χ and any integer m such that $(n+1)/2 \leq m \leq k - (n+1)/2$.

Example. Let $n = 1$. Take a basis $\{f_1, \dots, f_d\}$ of $S_k(SL_2(\mathbf{Z}))$ consisting of normalized Hecke eigenforms. Write $f \in S_k(SL_2(\mathbf{Z}))$ as

$$f = a_1 f_1 + \dots + a_d f_d$$

with $a_1, \dots, a_d \in \mathbf{C}$. Then put $w_i = a_i u_+(f_i)$, $w_{d+i} = a_i u_-(f_i)$ ($i = 1, \dots, d$)

and $V_f = \sum_{i=1}^{2d} \overline{\mathbf{Q}} w_i$. Then V_f satisfies the required property for f .

Now let

$$L^*(s, F, \chi) = \sum_{T \in \mathcal{L}_{n>0}/SL_n(\mathbf{Z})} \frac{\chi(\det(2T))c_F(T)}{e(T)(\det T)^s}$$

be the twisted Koecher-Maaß series of F of the second kind as in Section 0. We will discuss a relation between these two Dirichlet series. Let N be a positive integer. Let g be a periodic function on $\mathbf{Z}$ with a period N and ϕ a polynomial in $t_1, \dots, t_r$. Then for an element $u = (a_1 \bmod N, \dots, a_r \bmod N) \in (\mathbf{Z}/N\mathbf{Z})^r$, the value $g(\phi(a_1, \dots, a_r))$ does not depend on the choice of the representative u . Therefore we denote this value by $g(\phi(u))$. Now let χ be a primitive character mod N . For $A \in \mathcal{L}_{n>0}$, put

$$h(A, \chi) = \sum_{U \in SL_n(\mathbf{Z}/N\mathbf{Z})} \chi(\text{tr}(A[U])).$$

The following proposition is due to [[K-M], Proposition 3.1].

Proposition 3.3 *Let*

$$F(Z) = \sum_{A \in \mathcal{L}_{n \geq 0}} c_F(A) \mathbf{e}(\text{tr}(AZ))$$

be an element of $M_k(Sp_n(\mathbf{Z}))$. Let χ be a Dirichlet character mod N . Assume $N \neq 2$. Then we have

$$L(s, F, \chi) = \sum_{A \in \mathcal{L}_{n>0}/SL_n(\mathbf{Z})} \frac{c_F(A)h(A, \chi)}{e(A)(\det A)^s}.$$

For a Dirichlet character $\chi \bmod N$, let $\chi^{(p)}$ be the p -factor of χ so that $\chi = \prod_{p|N} \chi^{(p)}$. For a prime number p put

$$\gamma_{n,p} = p^{n^2-n(n+1)/2} (1 - p^{-n/2}) \prod_{e=1}^{(n-2)/2} (1 - p^{-2e})$$

or

$$\gamma_{n,p} = p^{n^2-n(n+1)/2} \prod_{e=1}^{(n-1)/2} (1 - p^{-2e})$$

according as n is even or odd. The following result is a technical tool for proving our main result.

Theorem 3.4 *Let $A \in \mathcal{L}_{n>0}$. Let N be a square free odd integer, and let $N = \prod_{i=1}^r p_i$ be the prime decomposition of N . Let χ be a primitive Dirichlet character mod N . For each positive integer $i \leq r$, put $l_i = \text{G.C.D}(n, p_i - 1)$ and let $u_{0,i}$ be a primitive l_i -th root of unity mod p_i .*

- (1). *If $\chi^{(p_i)}(u_{0,i}) \neq 1$ for some i . Then we have $h(A, \chi) = 0$.*
- (2). *Assume that $\chi^{(p_i)}(u_{0,i}) = 1$ for any i . Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^n = \chi$.*

(2.1) *Let n be even. Then we have*

$$h(A, \chi) = \prod_{i=1}^r (-1)^{n(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\tilde{\chi}_{(i_1, \dots, i_r)}(2^n) \tilde{\chi}_{(i_1, \dots, i_r)}(\det(2A)) J(\tilde{\chi}_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right)) J_{n-1}(\tilde{\chi}_{(i_1, \dots, i_r)})}.$$

(2.2) *Let n be odd, and assume that χ^2 is primitive. Then we have*

$$h(A, \chi) = \prod_{i=1}^r (-1)^{(n-1)(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\tilde{\chi}_{(i_1, \dots, i_r)}(2^n) \tilde{\chi}_{(i_1, \dots, i_r)}(\det(2A)) J_{n-1}(\tilde{\chi}_{(i_1, \dots, i_r)})}.$$

The proof of the above theorem is elementary but is rather lengthy. The details will be given in [Ka].

Remark. Let η be a primitive Dirichlet character of odd prime conductor p . Assume that $\eta^2 \neq 1$. Then we can prove that we have

$$J(\eta, \left(\frac{*}{p}\right)) J(\eta \left(\frac{*}{p}\right), \eta \left(\frac{*}{p}\right)) = \left(\frac{-1}{p}\right) \bar{\eta}(4)p.$$

(This is not so trivial. For the details, see [Ka].) Hence for $A \in \mathcal{L}_{2>0}$ and a primitive character χ of odd square free conductor N such that $\chi^{(p)}(-1) = 1$ for any prime divisor p of N , we have

$$h(A, \chi) = \prod_{p|N} \left\{ \left(1 + \left(\frac{4 \det A}{p} \right) \right) \left(1 - \left(\frac{-1}{p} \right) p^{-1} \right) \right\} N^2 \left(\frac{-1}{N} \right) \tilde{\chi}(4 \det A),$$

where $\tilde{\chi}$ is a character such that $\tilde{\chi}^2 = \chi$. This coincides with (2) of Theorem 3.8 in [K-M].

By Theorem 3.4 and Proposition 3.3 we easily obtain:

Theorem 3.5 *Let $N, p_i, l_i, u_{0,i}$ ($i = 1, \dots, r$) and χ be as in Theorem 3.4, and let F be an element of $M_k(Sp_n(\mathbf{Z}))$.*

- (1). *If $\chi^{(p_i)}(u_{0,i}) \neq 1$ for some i . Then we have $L(s, F, \chi) = 0$.*
- (2). *Assume that $\chi^{(p_i)}(u_{0,i}) = 1$ for any i . Fix a character $\tilde{\chi}$ such that $\tilde{\chi}^n = \chi$.*
- (2.1) *Let n be even. Then we have*

$$L(s, F, \chi) = \prod_{i=1}^r (-1)^{n(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\tilde{\chi}_{(i_1, \dots, i_r)}(2^n) J(\tilde{\chi}_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right)) J_{n-1}(\tilde{\chi}_{(i_1, \dots, i_r)})} L^*(s, F, \tilde{\chi}_{(i_1, i_2, \dots, i_r)}).$$

- (2.2) *Let n be odd, and assume that χ^2 is primitive. Then we have*

$$L(s, F, \chi) = \prod_{i=1}^r (-1)^{(n-1)(p_i-1)/4} \gamma_{n,p_i}$$

$$\times \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\tilde{\chi}_{(i_1, \dots, i_r)}(2^n) J_{n-1}(\tilde{\chi}_{(i_1, i_2, \dots, i_r)})} L^*(s, F, \tilde{\chi}_{(i_1, i_2, \dots, i_r)}).$$

To give an explicit formula of $J_m(\chi, \eta)$ for primitive characters $\chi, \eta \bmod N$, we define $I_m(\chi, \eta)$ as

$$I_m(\chi, \eta) = \sum_{Z \in S_m(\mathbf{Z}/N\mathbf{Z})} \chi(\det Z) \eta(\operatorname{tr}(Z)).$$

Then we have the following two propositions, whose proof will be given precisely in [Ka].

Proposition 3.6 *Let χ and η be primitive character mod an odd prime number p . Assume that $\chi^2 \neq 1$ and that η is non-trivial. Put $c_m(\chi, \eta) = 1$ or 0 according as $\chi^{m-1}\eta = 1$ or not.*

(1) *Assume that m is odd. Then*

$$I_m(\chi, \eta) = c_m(\chi, \eta) \left(\frac{-1}{p} \right)^{(m-1)/2} p^{(m-1)/2} (p-1) J_{m-1} \left(\chi \left(\frac{*}{p} \right), \eta \right).$$

(2) *Assume that m is even. Then*

$$I_m(\chi, \eta) = c_m(\chi, \eta) \left(\frac{-1}{p} \right)^{m/2} p^{(m-2)/2} (p-1) \chi(-1) J \left(\chi, \left(\frac{*}{p} \right) \right) J_{m-1} \left(\chi \left(\frac{*}{p} \right), \eta \right).$$

Proposition 3.7 *Let χ, η and p be as in Proposition 3.6.*

(1) *Assume that m is odd. Then*

$$J_m(\chi, \eta) = \left(\frac{-1}{p} \right)^{(m-1)/2} p^{(m-1)/2} \times \{ J(\chi, \chi^{m-1}\eta) J_{m-1} \left(\chi \left(\frac{*}{p} \right), \eta \right) + \eta(-1) I_{m-1} \left(\chi \left(\frac{*}{p} \right), \eta \right) \}.$$

(2) *Assume that m is even. Then*

$$J_m(\chi, \eta) = \left(\frac{-1}{p} \right)^{m/2} p^{(m-2)/2} J \left(\chi, \left(\frac{*}{p} \right) \right) \times \{ J(\chi, \chi^{m-1} \left(\frac{*}{p} \right) \eta) J_{m-1} \left(\chi \left(\frac{*}{p} \right), \eta \right) + \eta(-1) I_{m-1} \left(\chi \left(\frac{*}{p} \right), \eta \right) \}.$$

From the above two propositions we have the following:

Theorem 3.8 *Let χ be a primitive character with a prime conductor p such that $\chi^2 \neq 1$.*

(1) *Let m be odd.*

(1.1) *Assume that $\chi^m \neq 1$. Then*

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi) = \left(\frac{-1}{p}\right)^{(m-1)/2} p^{(m-1)/2} J(\chi \left(\frac{*}{p}\right)^i, \chi^m) J_{m-1}(\chi \left(\frac{*}{p}\right)^{i+1}, \chi).$$

(1.2) *Assume that $\chi^m = 1$. Then*

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi) = p^{m-1} \left(\frac{-1}{p}\right)^{i+1} J(\chi \left(\frac{*}{p}\right)^{i+1}, \left(\frac{*}{p}\right)) J_{m-2}(\chi \left(\frac{*}{p}\right)^i, \chi).$$

(2) *Let m be even.*

(2.1) *Assume that $\chi^m \left(\frac{*}{p}\right)^{i+1} \neq 1$. Then*

$$\begin{aligned} & J_m(\chi \left(\frac{*}{p}\right)^i, \chi) \\ &= \left(\frac{-1}{p}\right)^{m/2-1} J(\chi \left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right)) J(\chi \left(\frac{*}{p}\right)^{i+1}, \chi^m \left(\frac{*}{p}\right)^{i+1}) J_{m-1}(\chi \left(\frac{*}{p}\right)^{i+1}, \chi). \end{aligned}$$

(2.2) *Assume that $\chi^m \left(\frac{*}{p}\right)^{i+1} = 1$. Then*

$$J_m(\chi \left(\frac{*}{p}\right)^i, \chi) = \chi(-1) p^{m-1} J(\chi \left(\frac{*}{p}\right)^i, \left(\frac{*}{p}\right)) J_{m-2}(\chi \left(\frac{*}{p}\right)^i, \chi).$$

Corollary *Let χ be a primitive character with an odd square free conductor N . Assume that χ^2 is primitive. Then the value $J_m(\chi)$ is nonzero.*

4 An explicit formula for the twisted Koecher-Maaß series of the D-I-I lift

Throughout this section and the next, we assume that n and k are even positive integers. Let h be a Hecke eigenform of weight $k-n/2+1/2$ belonging to Kohnen's plus space. Then h has the following Fourier expansion:

$$h(z) = \sum_e c_h(e) e(ez),$$

where e runs over all positive integers such that $(-1)^{k-n/2}e \equiv 0, 1 \pmod{4}$. Let

$$S(h)(z) = \sum_{m=1}^{\infty} c_{S(h)}(m) \mathbf{e}(mz)$$

be the normalized Hecke eigenform of weight $2k - n$ with respect to $SL_2(\mathbf{Z})$ corresponding to h under the Shimura correspondence. For a prime number p let β_p be a non-zero complex number such that $\beta_p + \beta_p^{-1} = p^{-k+n/2+1/2} c_{S(h)}(p)$. For a prime number p , let $\mathbf{Q}_p$, and $\mathbf{Z}_p$ be the field of p -adic numbers, and the ring of p -adic integers, respectively. We denote by ν_p the additive valuation on $\mathbf{Q}_p$ normalized so that $\nu_p(p) = 1$, and by $\mathbf{e}_p$ the continuous homomorphism from the additive group $\mathbf{Q}_p$ to $\mathbf{C}^\times$ such that $\mathbf{e}_p(x) = \mathbf{e}(x)$ for $x \in \mathbf{Z}[p^{-1}]$. For a positive definite half integral matrix T of degree n write $(-1)^{n/2} \det(2T)$ as $(-1)^{n/2} \det(2T) = \mathfrak{d}_T \mathfrak{f}_T^2$ with $\mathfrak{d}_T$ a fundamental discriminant and $\mathfrak{f}_T$ a positive integer. We then define the local Siegel series $b_p(T, s)$ by

$$b_p(T, s) = \sum_{R \in S_n(\mathbf{Q}_p)/S_n(\mathbf{Z}_p)} \mathbf{e}_p(\mathrm{tr}(TR)) p^{-\nu_p(\mu_p(R))s} \quad (s \in \mathbf{C})$$

for each prime number p , where $\mu_p(R) = [R\mathbf{Z}_p^n + \mathbf{Z}_p^n : \mathbf{Z}_p^n]$. Then there exists a polynomial $F_p(T, X)$ in X such that

$$b_p(T, s) = F_p(T, p^{-s})(1 - p^{-s})(1 - \left(\frac{\mathfrak{d}_T}{p}\right) p^{n/2-s})^{-1} \prod_{i=1}^{n/2} (1 - p^{2i-2s})$$

(cf. [Ki].) We then put

$$c_{I_n(h)}(T) = c_h(|\mathfrak{d}_T|) \prod_p (p^{k-n/2-1/2} \beta_p)^{\nu_p(\mathfrak{f}_T)} F_p(T, p^{-(n+1)/2} \beta_p^{-1}).$$

We note that $c_{I_n(h)}(T)$ does not depend on the choice of β_p . Define a Fourier series $I_n(h)(Z)$ by

$$I_n(h)(Z) = \sum_{T \in \mathcal{L}_{n>0}} c_{I_n(h)}(T) \mathbf{e}(\mathrm{tr}(TZ)).$$

In [I] Ikeda showed that $I_n(h)(Z)$ is a cuspidal Hecke eigenform in $S_k(Sp_n(\mathbf{Z}))$ and its standard L -function $L(s, I_n(h), \mathrm{St})$ is given by

$$L(s, I_n(h), \mathrm{St}) = \zeta(s) \prod_{i=1}^n L(s + k - i, S(h)).$$

We call $I_n(h)$ the Duke-Imamoglu-Ikeda lift (D-I-I lift) of h . Now using the same argument as in the proof of Theorem 1 of [I-K] we obtain the following. For the details see [Ka].

Theorem 4.1 *Let χ be a primitive Dirichlet character mod N . Then we have*

$$L^*(s, F, \chi) = 2^{ns} \{ c_n R(s, h, E_{n/2+1/2}, \chi) \prod_{j=1}^{n/2-1} L(2s-2j, S(h), \chi^2) \\ + d_n c_h(1) \prod_{j=1}^{n/2} L(2s-2j+1, S(h), \chi^2) \},$$

where c_n and d_n are non-zero rational numbers depending only on n .

Now by the above theorem combined with Theorem 3.5 we obtain:

Theorem 4.2 *Let N be a square free odd integer, and $N = p_1 \cdots p_r$ be the prime decomposition of N . For each $i = 1, \dots, r$ let $l_i = \text{G.C.D}(n, p_i - 1)$ and $u_0 \in \mathbf{Z}$ be a primitive l_i -th root of unity mod p_i .*

(1) *Assume $\chi^{(p_i)}(u_i) \neq 1$ for some i . Then $L(s, I_n(h), \chi) = 0$.*

(2) *Assume $\chi^{(p_i)}(u_i) = 1$ for any i . Then*

$$L(s, I_n(h), \chi) = 2^{ns} \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\widetilde{\chi}_{(i_1, \dots, i_r)}(2^n) J(\widetilde{\chi}_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right)) \overline{J_{n-1}(\widetilde{\chi}_{(i_1, \dots, i_r)})}} \\ \times \{ c_{n,N} R(s, h, E_{n/2+1/2}, \widetilde{\chi}_{(i_1, \dots, i_r)}) \prod_{j=1}^{n/2-1} L(2s-2j, S(h), \widetilde{\chi}_{(i_1, \dots, i_r)}^2) \\ + d_{n,N} c_h(1) \prod_{j=1}^{n/2} L(2s-2j+1, S(h), \widetilde{\chi}_{(i_1, \dots, i_r)}^2) \},$$

where $c_{n,N}$ and $d_{n,N}$ are non-zero rational numbers depending only on n and N , and $\widetilde{\chi}$ is a character s.t. $\widetilde{\chi}^n = \chi$.

Remark. In the case $n = 2$, an explicit formula for $L(s, I_2(h), \chi)$ was given by Katsurada-Mizuno [K-M].

Corollary *Let χ be a Dirichlet character of odd square free conductor N such that χ^n is primitive. Then for any integer $n/2 + 1 \leq m \leq k - n/2 - 1$*

$$\frac{L(m, I_n(h), \chi^n)}{\pi^{mn}} = \left\{ \gamma_{n,N} \frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} + \delta_{n,N} c_h(1) \frac{\mathbf{M}^{(\chi)}(m, S(h))}{\pi^{mn}} \right\},$$

where $\gamma_{n,N}$ and $\delta_{n,N}$ are non-zero numbers, and

$$\begin{aligned} \mathbf{M}^{(\chi)}(m, S(h)) &= \sum_{i_1=0}^{l_1-1} \cdots \sum_{i_r=0}^{l_r-1} \overline{\chi_{(i_1, \dots, i_r)}(2^n) J(\chi_{(i_1, \dots, i_r)}, \left(\frac{*}{N}\right)) J_{n-1}(\chi_{(i_1, \dots, i_r)})} \\ &\quad \times \prod_{j=1}^{n/2} L(2m - 2j + 1, S(h), (\chi_{(i_1, \dots, i_r)})^2). \end{aligned}$$

5 Proof of main results and some comments

We prove the results in Section 2.

Proof of Theorem 2.1. Assume that $n \equiv 2 \pmod{4}$. Then we have $c_h(1) = 0$, and by Theorem 3.1 and Corollary to Theorem 4.2, we have

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in \overline{\mathbf{Q}} u_1 \otimes_{\overline{\mathbf{Q}}} V_{I_n(h)}$$

with some complex number u_1 , where $V_{I_n(h)}$ is the $\overline{\mathbf{Q}}$ -vector space associated with $I_n(h)$ in Theorem 3.1. Assume that $n \equiv 0 \pmod{4}$. By Theorem 1.1 we have

$$\frac{\mathbf{M}^{(\chi)}(m, S(h))}{\pi^{mn}} \in \overline{\mathbf{Q}} u_{-}(S(h))^{n/2} \pi^{-n^2/4}.$$

Hence, again by Theorem 3.1 and Corollary to Theorem 4.2,

$$\frac{R^{(\chi)}(m, h, E_{n/2+1/2})}{\pi^{mn}} \in \overline{\mathbf{Q}} u_1 \otimes_{\overline{\mathbf{Q}}} V_{I_n(h)} + \overline{\mathbf{Q}} u_2$$

with complex numbers u_1 and u_2 . This proves the assertion.

Proof of Theorem 2.2 and its corollary. Theorem 2.2 follows directly from Theorem 2.1. We note that $J_{n-1}(\chi_{(i_1, \dots, i_r)})$ is a non-zero algebraic number by virtue of Corollary to Proposition 3.8. We also note that $\frac{\mathbf{L}_n(m, S(h), \eta)}{\pi^{m(n-2)}}$ belongs to $\overline{\mathbf{Q}}_{u_+}(S(h))^{n/2-1} \pi^{-n^2/4+n/2}$, and nonzero for any integer $n/2+1 \leq m \leq k-n/2-1$ and primitive character η . This proves the corollary.

Proof of Theorem 2.3. The assertion follows from Theorem 3.2.

Now we give some comments. First we are interested in the dimension of $W_{h, E_{n/2+1/2}}$ over $\overline{\mathbf{Q}}$. Therefore we propose the following problem.

Problem 1. Give $\dim_{\overline{\mathbf{Q}}} W_{h, E_{n/2+1/2}}$ explicitly or estimate it.

This problem is reduced to the following problem:

Problem 2. Give $\dim_{\overline{\mathbf{Q}}} V_{I_n(h)}$ explicitly or estimate it.

Next we consider a generalization or a refinement of Theorem 2.1. Namely we propose the following conjecture.

Conjecture. Let $h_1(z)$ be a Hecke eigenform in $S_{k_1+1/2}^+(\Gamma_0(4))$ and $h_2(z) \in M_{k_2+1/2}(\Gamma_0(4))$ with $k_1 \geq k_2 + 2$. Assume that $c_{h_2}(m) \in \overline{\mathbf{Q}}$ for any $m \in \mathbf{Z}_{\geq 0}$. Then there exists a finite dimensional $\overline{\mathbf{Q}}$ -vector space $W_{h_1, h_2} \subset \mathbf{C}$ such that

$$R(m, h_1, h_2, \chi) \pi^{-2m} \in W_{h_1, h_2}$$

for any $k_2 + 1 \leq m \leq k_1 - 1$ and any primitive character χ .

Problem 3. Prove Theorem 2.1 without using the relation between the twisted Koecher-Maaß series of the Duke-Imamoglu-Ikeda lift and the twisted Rankin-Selberg series of modular forms of half-integral weight.

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