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# Computability and complexity of Julia sets: a review

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**Abstract:** Since A. M. Turing introduced the notion of computability in 1936, various theories of real number computation have been studied [1][10][13]. Some are of interest in nonlinear and statistical physics while others are extensions of the mathematical theory of computation. In this review paper, we introduce a recently developed computability theory for Julia sets in complex dynamical systems by Braverman and Yampolsky [3].

## 1 Computability and complexity

Chaos and fractals have been studied from the viewpoint of computability in physics [6][12][1][2]. Investigation has focused on the nature of complexity arising from simple nonlinear equations. Unpredictability in chaotic attractors and final state sensitivity in fractal basins are discussed in terms of computability and complexity in the theory of computation.

We summarize the leading results of a recently developed computability theory for Julia sets by Braverman and Yampolsky [3].

First, we introduce the classical notions of computability introduced by Turing [14], and computable real functions introduced by Pour-El [13]. Turing computability is defined by rather a physical model of human computation, called a Turing machine, which is an automaton which consisting of finite internal states and a head to read/write symbols on an external tape. The length of the tape is not restricted, but the number of alphabets is finite. It can manipulate individual symbols on a tape, according to a transition diagram, which tells the machine what action to undertake depending on the current internal state

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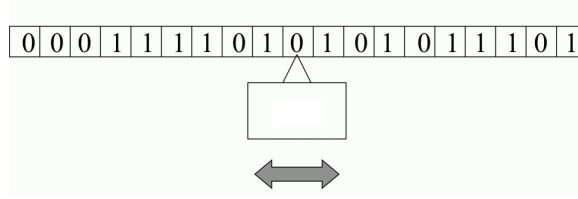


Figure 1: Turing machine

and the current symbol read by the head. After symbol manipulation, it moves left or right on the tape and changes its internal state, again according to the transition diagram. It has a special internal state, which is called the halting state and if the machine meets halting state, it stops and outputs the symbols on the tape (see Fig. 1). Given this automaton, we define computability as follows.

**Def.1.1** (Computability)

*A function  $f(x)$  is computable if there is a Turing machine  $M$  such that  $M$  takes  $x$  as an input and outputs  $f(x)$  represented by  $M(x)$ .*

Despite its simplicity, a Turing machine can calculate a broad class of functions, which form computable functions. Although it is a very intuitive and constructive physical model, Turing computability is consistent with the more mathematically oriented notion of computation such as lambda-definability introduced by Alonzo Church [4]. This fact is referred to as the “Church-Turing thesis.” Note that, however, most functions are uncomputable, because there are uncountably many functions from  $\mathbb{N}$  to  $\mathbb{N}$ , while there are only countably many Turing machines by definition. A typical example of an uncomputable function is the so-called the halting problem: “Given a Turing machine  $M$  and input  $x$ , determine whether  $M(x)$  halts.”

There is a special Turing machine that can emulate all other Turing machines, called universal Turing machine. Let  $U$  be a universal Turing machine taking a Turing machine  $M$  and input  $x$ , both represented on an input tape, then  $U$  outputs  $U(M, x) = M(x)$ . By using  $U$ , the halting problem can be stated in the following form: “Given a universal Turing machine  $U$  and an input  $w$ , determine whether  $U(w)$  halts,” where  $w$  is a representation of both  $M$  and  $x$ .

We can also consider polynomial time computability based on Turing machines. We say that the number of steps  $T_M(w)$  that  $M$  makes before terminating with an input  $w$  is the running time. This is a basis of time complexity in computational complexity theory.

**Def.1.2** (Time complexity)

*For a Turing machine  $M$  on input  $w$ , the time complexity  $T_M$  is the function*

$T_M : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$T_M(n) = \max_{|w|=n} \{\text{the running time of } M(w)\},$$

where  $|w|$  denotes the length of  $w$ . In other words,  $T_M(n)$  is the worst case running time for inputs of length  $n$ .

A decision problem is said to be “tractable,” if its time complexity is polynomial order in  $n$ .

**Def.1.3** (Polynomial time computability)

Let  $L$  be a language class,  $w \in L$  be of length  $n$  words, and  $p$  be a polynomial. A function  $f : L \rightarrow \{0, 1\}$  is said to be polynomial time computable, if there is a Turing machine  $M$  computing  $f$  such that  $T_M(n) \leq p(n)$  for all  $n$ .

Proving the lower bound of time complexity is difficult including the well known “P=?NP” problem [5]. A typical example of such computable but “hard” problem is SAT: “Given a Boolean formula  $\phi(x)$  in  $n$  variables, determine whether there is an assignment  $x'$  of variables that satisfies  $\phi$ , i.e., such that  $\phi(x') = 1$ .” SAT is a NP-complete problem, which means that it belongs to a language class such that we can check a solution in polynomial time (but finding a solution might be very hard to have exponential time complexity). The class NP is also defined as a set of problems, which is polynomial time computable using a non-deterministic Turing machine. The class of (deterministic) polynomial time computable problems is called class P, and most researchers believe that  $P \neq NP$  holds.

A definition of a computable real number was also given by Turing [14].

**Def.1.4** (Computable real number)

A real number  $\alpha$  is said to be computable if there is a computable function  $f : \mathbb{N} \rightarrow \mathbb{Q}$  such that for every natural number  $n$

$$|\alpha - f(n)| < 2^{-n}.$$

In brief, a real number is computable if it is the limit of an effectively converging computable sequence of rational numbers. It is known that most real numbers are uncomputable. Examples of computable real numbers are  $e$ ,  $\pi$ , and  $\sqrt{2}$ , as there exist constructive series expansions, e.g.,  $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots$ . We denote the set of computable numbers  $\mathbb{R}_C$ .

**Def.1.6** (Right-computable real number)

A real number  $\alpha$  is said to be right-computable if there is a computable function  $f : \mathbb{N} \rightarrow \mathbb{Q}$  such that the sequence  $\{f(n)\}$  satisfies (i)  $f(1) \geq f(2) \geq \dots f(n) \geq \dots$  and (ii)  $\lim_{n \rightarrow \infty} f(n) = \alpha$ .

Left-computable real numbers are also defined in the same way with allowance of slow convergence. Note that a computable number is also right-computable, but the inverse does not necessarily hold. Right-computable numbers form a dense subset in  $\mathbb{R} \setminus \mathbb{R}_C$ .

The Chaitin number  $\Omega$  is an example of an uncomputable real number defined as follows.

**Def.1.5** (Chaitin number)

Let  $M$  be a Turing machine and  $w$  be an input, the Chaitin number is given as

$$\Omega = \sum_{M(w) \text{ halts.}} 2^{-|M|+|w|}.$$

The Chaitin number is the ratio of halting Turing machine over all possible machine  $M$  with all possible input symbol sequence  $w$ . It is sometimes called halting probability. Generalized shifts given by Moore is an application of computability theory in nonlinear physics [12]. There is a two-dimensional piecewise linear map  $U$  that includes a universal Turing machine. A classical billiard system in three-dimensional space with a finitely complex boundary and an escape hole can include  $U$  as a Poincaré map of the dynamical system. Then, the escape rate of the billiard system is uncomputable because it gives the Chaitin number.

A definition of a computable real function is given by M. Pour-El in the context of computable analysis [13].

**Def.1.7** (Computable real function)

Let  $I^q = \{a_i \leq x_i \leq b_i, 1 \leq i \leq q\} \subseteq \mathbb{R}^q$ , where  $a_i$  and  $b_i$  are computable reals, be a computable rectangle. A real function  $f : I^q \rightarrow \mathbb{R}^q$  is said to be computable if

- (i)  $f$  is sequentially computable, i.e.  $f$  maps every computable sequence of points  $x_k \in I^q$  into a computable sequence  $\{f(x_k)\}$  of real numbers.
- (ii)  $f$  is effectively uniformly continuous, i.e. there is a computable function  $d : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $x, y \in I^q$  and all  $N$ :

$$|x - y| \leq \frac{1}{d(N)} \text{ implies } |f(x) - f(y)| \leq 2^{-N},$$

where  $|\cdot|$  denotes Euclidean norm.

$L^p$ -computability is a natural generalization of the computable real function given by Def. 1.7, and is defined as follows.

**Def.1.8** ( $L^p$ -computability)

A function  $f \in L^p[a, b]$  is  $L^p$ -computable if there exists a sequence  $\{g_k\}$  of continuous functions which is computable in the sense of Def. 1.7 and such that the  $L^p$ -norms  $\|g_k - f\|_p$  converge to zero effectively as  $k \rightarrow \infty$ .

Roughly speaking, a real function is computable if there is a finite interpolation approximated by an effectively converging computable series expansion of functions. It is known that most real functions are uncomputable. Most elementary continuous real functions, such as  $e^x$ ,  $\sin(x)$ , and  $\log(x)$ , are computable, as there exist constructive series expansions, e.g.  $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$ , and so on. It is known that solutions of PDE described by computable real function with computable boundary conditions can be uncomputable.

An physical example is given with a linear wave equation.

$$\begin{aligned} \frac{\partial^2 u(x, t)}{\partial t^2} &= \nabla^2 u(x, t) \\ u(x, 0) &= f(x) \text{ (computable), } \quad \frac{\partial u(x, 0)}{\partial t} = 0 \end{aligned} \tag{1}$$

An unbounded linear operator in a function space can take an input computable real function into uncomputable outputs [13].

## 2 Complex dynamical systems and Julia sets

In this section, we define the Julia set of complex dynamical systems [11]. By a complex dynamical system we mean a (polynomial or rational) dynamical systems on the complex plane  $\mathbb{C}$ , or on the Riemann sphere  $\hat{\mathbb{C}}$ . It is well known that they produce abundant fractal structures. To say simply, the Riemann sphere is the union of a complex plane and a point at infinity, i.e.,  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ .

We will denote the  $n$ -th iterate of mapping  $R$  by  $R^n$ . Let  $z_0$  be a periodic point of period  $n \in \mathbb{N}$ , that is, we have  $R^n(z_0) = z_0$ . Its orbit is called a periodic orbit of period  $n$ , and is also called a cycle. A periodic orbit is called attracting (or repelling) if we have  $|DR^n(z_0)| < 1$  (or  $|DR^n(z_0)| > 1$ ) where  $D$  denotes the derivative. When  $|DR^n(z_0)| = 0$ , we say it is super-attracting. In the case when  $|DR^n(z_0)| = 1$ , so that we have  $|DR^n(z_0)| = e^{2\pi i\theta}$  where  $\theta \in \mathbb{R}$ , we say the cycle is parabolic if  $\theta \in \mathbb{Q}$ ; otherwise it is called irrationally indifferent.

### **Def.2.1** (Fatou set and Julia set)

*Let  $R$  be a rational function of degree  $d \geq 2$  on the Riemann sphere  $\hat{\mathbb{C}}$ . The Fatou set is the set of points which have an open neighborhood  $U(z)$  on which the family of iterates  $R^n|_{U(z)}$  is equicontinuous. The Fatou set is denoted by  $F(R)$ . The set  $J(R) = \hat{\mathbb{C}} \setminus F(R)$  is called the Julia set.*

In the case of polynomial functions of degree  $d \geq 2$ , the filled Julia set is defined as follows.

### **Def.2.2** (Filled Julia set)

*Let  $P$  be a polynomial function of degree  $d \geq 2$ . Then its filled Julia set  $K(P)$*

is defined to be the set of points  $z \in \mathbb{C}$  whose orbit is bounded. That is,  $K(P) = \{z \in \mathbb{C} \mid \{P^n(z)\} \text{ is bounded}\}$ .

It is known that the Julia set is the boundary of the filled Julia set, namely,  $J(P) = \partial K(P)$ .

We now consider a family  $P_c(z) = z^2 + c$  of polynomial maps of degree 2 depending on a parameter  $c \in \mathbb{C}$  and its Julia set  $J_c = J(P_c)$ .

The simplest parameter in this family is  $c = 0$ , for which the origin and the point at infinity are attracting points. Let  $z$  be a point in the interior of the closed unit disk  $\mathbb{U}$ . Then we have  $P_0^n(z) \rightarrow 0$  as  $n \rightarrow \infty$ . It is easy to see that the family of iterations is equicontinuous. By the same argument, in the case that  $z \in \hat{\mathbb{C}} \setminus \mathbb{U}$ , the family of iterations is also equicontinuous. On the other hand, if  $z$  is on the unit circle, on any neighborhood of  $z$  the family of iterations can not be equicontinuous. Thus, the unit circle  $\{z \in \hat{\mathbb{C}} \mid |z| = 1\}$  is the Julia set of  $P_0$ . Depending of the parameter,  $J_c$  may take various different geometric shapes. Some of them are illustrated in Figure 2.

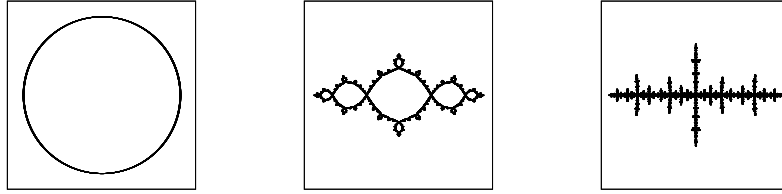


Figure 2: Julia sets  $J_c$  of  $P_c(z) = z^2 + c$  with  $c = 0$  (left),  $c = -1$  (center), and  $c = -1.543689$  (right). When  $c = 0$ , the origin and the point at infinity are attracting points, and the unit circle is Julia set, that is, a basin boundary of the origin and the point at infinity.

As we have seen above, the Julia and filled Julia sets exist in the  $z$ -plane, that is, the phase space of the system. Next, we focus on the  $c$ -plane, the parameter space of the system. The most important object in this space, from the dynamical point of view, is the Mandelbrot defined as follows.

**Def.2.3** (Mandelbrot set)

Let  $P_c(z) = z^2 + c$ . The Mandelbrot set  $\mathcal{M}$  is the set of parameters  $c$  which the orbit of the origin remains bounded. That is,  $\mathcal{M} := \{c \in \mathbb{C} \mid \{P_c^n(0)\}_{n \in \mathbb{N}} \text{ is bounded}\}$ .

Although the Mandelbrot set  $\mathcal{M}$  has a very complicated geometric structure, it is known to be connected (however, it is not known if  $\mathcal{M}$  is locally connected). Furthermore, we can prove that the complement of the Mandelbrot set is homeomorphic to the complement of the unit disc. It is surprising to see that while the Mandelbrot set is defined using only the information of the orbit

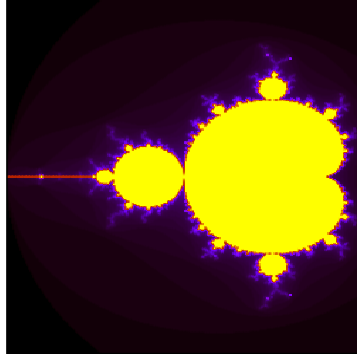


Figure 3: Mandelbrot set  $\mathcal{M}$

of the origin, it describes the dynamical behavior of the map  $P_c$  on the Julia set  $J_c$ . More precisely, we have the following dichotomy: if  $c \in \mathcal{M}$ , then  $J_c$  is connected; if  $c \notin \mathcal{M}$ , then  $J_c$  is not connected and in fact, it is a Cantor set (a totally disconnected perfect compact metric space). In the latter case, we also have the complete description of the dynamics on  $J_c$ , namely,  $P_c : J_c \rightarrow J_c$  is topologically conjugate to the shift map on the one-sided shift space of two symbols.

Consider a rational mapping  $R(z)$  and suppose that  $z_0$  is a periodic point of period  $n$ , i.e.,  $z_0 = R^n(z_0)$ . Then its multiplier  $\lambda$  is the derivative  $\lambda = DR^n(z_0)$ . Note that the value of  $\lambda$  is the same for all points on the orbit of  $z_0$  by the chain rule. If  $|\lambda| < 1$ , then the cycle is attracting. If  $|\lambda| > 1$ , then it is repelling. In both cases, the mapping is locally linearizable near the periodic point as follows:

$$\phi(R^n(z)) = \lambda \cdot \phi(z),$$

where  $\phi$  is a conformal mapping (a holomorphic mapping whose derivative is non-zero everywhere on its domain) of a small neighborhood of  $z_0$  to a disk around the origin.

We say a rational mapping  $R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  is **hyperbolic** if the orbit of every critical point of  $R$  is either periodic, or converges to an attracting (or a super-attracting) cycle.

In the case of our quadratic polynomial  $z^2 + c$ , the well known Fatou-Shishikura bound theorem implies that it has at most one non-repelling cycle in the complex plane. Another important fact here is that the basin of an attract-

ing or parabolic cycle of a rational map must contain a critical point of the map. Therefore, we can classify the dynamics with respect to the eigenvalue  $\lambda$  of the unique non-repelling cycle as follows. First, if there is no such a non-repelling cycle, then the dynamics is hyperbolic, by definition. When  $|\lambda| < 1$ , the corresponding Julia set is hyperbolic. When  $|\lambda| = 1$ , we have two cases: the unique non-repelling cycle is parabolic, or irrationally indifferent. If it is parabolic,  $R^n$  is not locally linearizable. If it is irrationally indifferent, there are two other possibilities, that is, Siegel Julia sets and Cremer Julia sets. In the former case,  $R^n$  is locally linearizable while in the latter case,  $R^n$  is not linearizable. In the Siegel case, the local linearizing equation above holds and in a neighborhood of the cycle, the dynamics is a rotation around the cycle by an irrational angle; this neighborhood of the cycle is called **Siegel disk**. We will see later that each class of Julia sets belongs to different complexity classes in terms of real number computation.

To understand the geometry of a Siegel disk, we consider some number-theoretic properties of irrational numbers using continued fractional expansion. For a real number  $\theta \in [0, 1)$ , we denote  $[r_1, r_2, \dots, r_n, \dots]$  by its possibly finite continued fractional expansion:

$$\theta = [r_1, r_2, \dots, r_n, \dots] := \frac{1}{r_1 + \frac{1}{r_2 + \frac{1}{\dots + \frac{1}{r_n + \dots}}}},$$

where  $r_i \in \mathbb{N} \cup \{\infty\}$ . Note that if  $\theta \notin \mathbb{Q}$  the representation is uniquely defined. For irrational  $\theta$ , the  $n$ -th convergents  $p_n/q_n = [r_1, r_2, \dots, r_n]$  are the closest rational approximation of  $\theta$  among the numbers whose denominators do not exceed  $q_n$ .

**Def.2.4** (Diophantine number)

*The Diophantine number of order  $k$ , denoted by  $\mathcal{D}(k)$ , is the class of irrational numbers satisfying following condition:  $\theta \in \mathcal{D}(k)$  if there exists  $c > 0$  such that*

$$q_{n+1} < cq_n^{k-1}$$

*where  $q_n$  is the denominator of  $n$ -th convergent of  $\theta$ .*

In particular,  $\theta \in \mathcal{D}(2)$  if and only if the sequence  $\{r_n\}$  is bounded. The class  $\mathcal{D}(2)$  is called bounded type. An irrational numbers which is not Diophantine is called a Liouville number.

An important class  $\mathcal{D}(2+)$  is defined as follows:

$$\mathcal{D}(2+) := \bigcap_{k>2} \mathcal{D}(k).$$

The class  $\mathcal{D}(2+)$  has full measure in  $[0, 1]$ .

**Thm.2.5**

*Let  $R$  be an analytic map with a periodic point  $z_0$  of period  $p$ . Suppose that the multiplier  $\lambda$  of the cycle is*

$$\lambda = e^{2\pi i\theta}, \theta \in \mathcal{D}(2+),$$

*then the local linearization equation holds.*

Since  $\mathcal{D}(2+)$  has full measure in  $[0, 1]$ , an immediate consequence is that the most irrational  $\theta$  generate Siegel disks. Another class of irrational numbers is the Brjuno number defined as follows.

**Def.2.6** (Brjuno number)

*For an irrational number  $\theta$  and its rational convergents  $p_n/q_n$ , let  $B(\theta) = \sum_n \frac{\log(q_{n+1})}{q_n}$ . An irrational number  $\theta$  is called a Brjuno number if  $B(\theta) < \infty$ . The set of Brjuno numbers are denoted by  $\mathcal{B}$ .*

It is easy to verify that  $\cup \mathcal{D}(k) \subset \mathcal{B}$ . Because  $\cup \mathcal{D}(k)$  has full measure in the interval  $[0, 1]$ , there are very few real numbers in  $(\cup \mathcal{D}(k))^c \cap \mathcal{B}$ . The parameter  $\theta$  to generate uncomputable Julia sets belongs to this class of real numbers.

**Thm.2.7**

*Let  $R$  be an analytic mapping with a periodic point  $z_0$ . If the multiplier of the cycle is  $\lambda = e^{2\pi i\theta}$  with Brjuno number  $\theta$ , then the local linearization equation holds. Furthermore,  $z_0$  is a center of Siegel disk.*

Consider a quadratic polynomial  $P_\theta(z) = z^2 + e^{2\pi i\theta}z$ , which can be identified with  $P_c(z) = z^2 + c$  by changing the coordinates. Note that the origin is a fixed point of  $P_\theta$  with multiplier  $e^{2\pi i\theta}$ .

**Thm.2.8**

*Suppose that  $P_\theta$  has a Siegel disk around the origin. Then  $\theta \in \mathcal{B}$ .*

A schematic view of the class of irrational numbers is given in Fig. 4.

We can characterize the size of a Siegel disk with the notion of conformal radius.

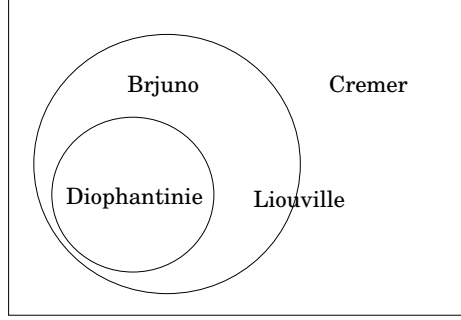


Figure 4: Classes of irrational numbers. Irrational numbers that do not belong to Brjuno number is said to be Cremer number. The set of irrational numbers, except Diophantine is called Liouville number.

**Def.2.9** (Conformal radius)

Let  $W \in \mathbb{C}$  be a simply connected domain and  $w \in W$  be an arbitrary point. Consider the unique conformal isomorphism  $\phi_{(W,w)} : \mathbb{U} \rightarrow W$  which satisfies  $\phi_{(W,w)}(0) = w$  and has a positive real derivative at 0, that is,  $\phi'_{(W,w)}(0) > 0$ . Then the conformal radius of a marked domain  $(W, w)$  is

$$r(W, w) = \phi'_{(W,w)}(0).$$

When  $P_\theta(z) = z^2 + e^{2\pi i\theta}z$  have a Siegel disk  $\Delta_\theta \ni 0$ , the conformal radius of  $\Delta_\theta$  is defined by  $r(\theta) = r(\Delta_\theta, 0)$ . If  $\theta$  does not generate a Siegel disk, we say  $r(\theta) = 0$ .

### 3 Definition of computability over the real

In this section, we discuss the computability and complexity of Julia sets in the framework developed by Braverman and Yampolsky [3]. First, we introduce a framework of computable real numbers and functions proposed by Ko [10]. Let  $\mathbb{D}$  be the set of dyadic numbers given by

$$\mathbb{D} = \left\{ \frac{k}{2^l}; k \in \mathbb{Z}, l \in \mathbb{N} \right\}.$$

We now define an oracle approximating a real number with precision  $n$  in a single step.

**Def.3.1** (Oracle)

A function  $\phi : \mathbb{N} \rightarrow \mathbb{D}$  is called an oracle for  $x \in \mathbb{R}$  if it satisfies  $|\phi(m) - x| < 2^{-m}$  for every  $m \in \mathbb{N}$ .

Intuitively, an oracle is a “equipped device” for computers and cannot be described as an algorithm. We denote a Turing Machine with an oracle for a real number  $x$  by  $M^\phi$ . When we write  $M^\phi(n)$ ,  $n$  represents the precision of the approximation of  $x$ .

**Def.3.2** (Computability of function)

Let  $S \subset \mathbb{R}$  and let  $f : S \rightarrow \mathbb{R}$ . Then  $f$  is said to be computable if there is an oracle Turing machine  $M^\phi(n)$  such that the following holds. If  $\phi(m)$  is an oracle for  $x \in S$ , then for all  $n \in \mathbb{N}$ ,  $M^\phi(n)$  returns  $q \in \mathbb{D}$  such that  $|q - f(x)| < 2^{-n}$ .

In this framework, a computable function must be continuous, so that the characteristic function of  $S \subset \mathbb{R}^k$  given by

$$\chi_S(x) = \begin{cases} 1 & (x \in S) \\ 0 & (x \notin S) \end{cases}, \quad (2)$$

is not computable unless  $S = \emptyset$  or  $\mathbb{R}^k$  itself.

We now give the following definitions of computability of real functions.

**Def.3.3** (Regular computability)

A set  $K \subset \mathbb{R}^k$  is computable if a Turing machine  $M$  computing a function  $f_K(d, r)$  from the family

$$f_K(d, r) = \begin{cases} 1 & \text{if } B(d, r) \cap K \neq \emptyset \\ 0 & \text{if } B(d, 2r) \cap K = \emptyset \\ 0 \text{ or } 1 & \text{otherwise} \end{cases}$$

exists.

A schematic view of regular computability is given in Fig. 5.

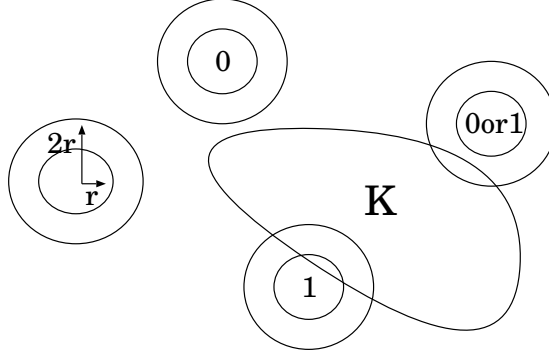


Figure 5: Regular computability. Radius is  $r = 2^{-n}$ .

**Def.3.4** (Weak computability)

The set  $K \subset \mathbb{R}^k$  is weakly computable if there is an oracle Turing machine  $M^\phi(n)$  such that, if  $\phi = (\phi_1, \phi_2, \dots, \phi_k)$  represents  $x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k$ , then the outputs of  $M^\phi(n)$  is

$$M^\phi(n) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } B(x, 2^{-(n-1)}) \cap K = \emptyset \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases}$$

A schematic view of weak computability is given in Fig. 6. The value of  $M^\phi(n)$  is not 1 unless the center of the pixel is contained in  $K$ .

To investigate the computability of Julia sets, we extend the definition of regular computability to those for set-valued functions and give a geometric interpretation for the characteristic function based on the Hausdorff metric [8]. Intuitively, this is a notion of computability based on “drawability” of a picture of  $K$  with round pixels on the computer screen.

**Def.3.5** (Regular computability of set-valued function)

Let  $S \subset \mathbb{R}^k$ , and  $F : S \rightarrow K_2^*$  be a function which maps a points in  $S$  to  $K_2^*$ , which is a set of compact subsets of  $\mathbb{R}^2$ . Then  $F$  is said to be computable on  $S$  if there is an oracle Turing machine  $M^{\phi_1, \dots, \phi_k}(d, r)$  which, for the oracles representing a point  $x = (x_1, \dots, x_k) \in S$ , computes a function  $f^{\phi_1, \dots, \phi_k} : \mathbb{D}^2 \times \mathbb{D} \rightarrow \{0, 1\}$  from the family

$$f^{\phi_1, \dots, \phi_k}(d, r) = \begin{cases} 1 & \text{if } B(d, r) \cap F(x) \neq \emptyset \\ 0 & \text{if } B(d, 2r) \cap F(x) = \emptyset \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases}$$

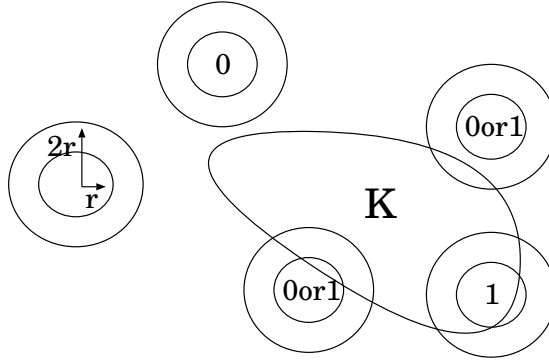


Figure 6: Weak computability. Radius is  $r = 2^{-n}$ .

Although the definitions of regular and weak computability differ from each other, they produce the same results in computability. As for computational complexity, we can define time complexity as

**Def.3.6** (Running time)

The running time  $T_{M^\phi}(n)$  is the longest time it takes to compute  $f^{\phi_1, \dots, \phi_k}(d, r)$  where  $r = 2^{-n}$  and  $d \in (\mathbb{Z}/2^{2n})$ .

In terms of computational complexity of running time, the definitions of regular and weak computability may produce different results.

## 4 Computability of Julia sets

The following theorems hold with regular computability for the membership problem of Julia sets [3].

**Thm.4.1** (Computability of hyperbolic Julia sets)

Fix  $d \geq 2$ . There exists a Turing machine  $M^\phi$  with oracle access to the coefficients of a rational mapping of degree  $d$  which computes the Julia sets of every hyperbolic rational map of degree  $d$ . Moreover, the Julia sets can be computed in polynomial time.

The property of hyperbolicity enables us to approximate the Julia sets quickly by using an inverse map. By using the technique of distance estimation [11], we can compute hyperbolic Julia sets in polynomial time.

**Thm.4.2** (Computability of parabolic Julia sets)

Given a rational function  $R(z)$  such that every critical orbit of  $R$  converges to an

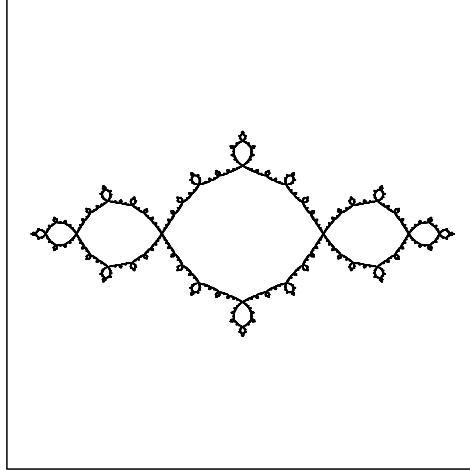


Figure 7: Example of hyperbolic Julia set. The polynomial is  $P(z) = z^2 - 1$ . The origin is a periodic point of period 2.

*attracting or a parabolic orbit and some finite combinatorial information about the parabolic orbit of  $R$ , there is an algorithm  $M$  that produces an image of the Julia set  $J(R)$  in polynomial time.*

It takes a longer time for the distance estimation algorithm to verify whether a given point belongs to the Julia set. However, we can compute it in polynomial time by using an algorithm accelerated by the renormalization group technique.

**Thm.4.3** (Computability of filled Julia sets)

*For any polynomial  $p(z)$  there is an oracle Turing machine  $M^\phi(n)$  that, given an oracle access to the coefficients of  $p(z)$ , outputs a  $2^{-n}$ -approximation of the filled Julia set.*

In the case of filled Julia sets, it is always easy to solve the membership problem. Even in the Siegel case, which we mention later, the computation will be done in polynomial time. In fact, the algorithm to compute filled Julia sets is much simpler than those for Julia sets.

**Thm.4.4** (Uncomputable Julia sets)

*There exists computable parameter  $c$ , such that the Julia set  $J_c$  is not computable by a Turing machine  $M^\phi$  with oracle access to  $c$ .*

Uncomputable Julia sets must have Siegel disks, which is a class of Julia sets, such that

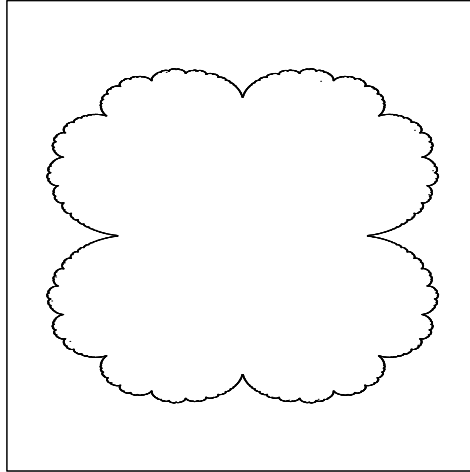


Figure 8: Example of parabolic Julia set. The polynomial is  $P(z) = z^2 + z$ . The origin is contained in Julia set.

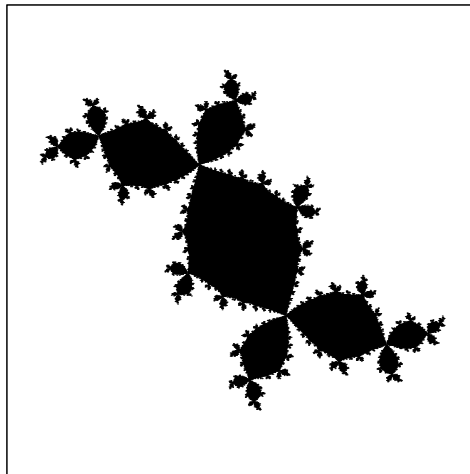


Figure 9: Filled Julia set of  $P(z) = z^2 - 0.122 + 0.745i$ . These Julia sets are called “Douady’s rabbit.”

$\phi(R^n(z)) = \lambda\phi(z)$  (linearizable) and  $\lambda = e^{2\pi i\theta}$  ( $\theta$  : irrational).

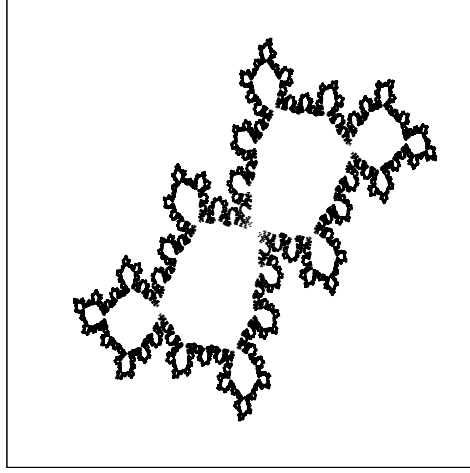


Figure 10: Uncomputable Julia sets must have Siegel disks. Example of Julia set with Siegel disks, which is computable, is depicted above. The polynomial is  $P(z) = z^2 + e^{2\pi i\theta}z$ , where  $\theta = \frac{\sqrt{5}-1}{2}$  (inverse golden mean).

This negative result is based on the following lemmas. Recall that  $r(\theta)$  is the conformal radius of the Siegel disk  $\Delta_\theta \ni 0$  of  $P_\theta(z) = z^2 + e^{2\pi i\theta}z$ .

**Lemma.4.5**

*The conformal radius  $r(\theta)$  is computable by a Turing machine with an oracle for  $\theta$  if and only if the Julia set  $J_\theta$  is computable.*

**Lemma.4.6**

*Let  $r \in (0, r_{sup})$ . Then  $r = r(\theta)$  is the conformal radius of a Siegel disk of the Julia set  $J_\theta$  for some computable number  $\theta$  if and only if  $r$  is right-computable.*

Outline of proof of “if” direction

Given a right-computable number  $r = r(\theta)$  and a sequence  $\{r_n\}$  converging to  $r$ , we must construct a sequence  $\{\theta_n\}$ , which satisfies

- (i)  $\{\theta_n\}$  converges to  $\theta$  effectively, i.e.,  $|\theta_n - \theta| < 2^{-n}$ ;
- (ii) behavior of the sequence  $\{r(\theta_n)\}$  is similar to the sequence  $\{r_n\}$ , i.e.,  $r(\theta_n) \approx r_n$ ;
- (iii)  $r(\theta) = r(\lim \theta_n) = \lim r(\theta_n) = \lim r_n = r$ .

Assume that for all  $n$ ,  $\theta_n$  have a form of continued fractional expansion like

$$\theta_n = [I_n, 1, 1, \dots],$$

where  $I_n$  represents the initial segment. When we obtain  $\theta_{n-1} = [I_{n-1}, 1, 1, \dots]$  which satisfies  $r(\theta_{n-1}) \approx r_{n-1}$ , the next step of construction will be done in the following manner. Suppose that the initial segment  $I_{n-1}$  has  $k$  elements. We choose a position  $m > k$  in the continued fractional expansion of  $\theta_{n-1}$  and denoted by

$$\theta_{n-1}^N = [I_{n-1}, 1, 1, \dots, 1, \underbrace{N}_{m\text{-th}}, 1, \dots].$$

The number  $m$  should be chosen so that for any  $N$ ,

$$|\theta_{n-1}^N - \theta_{n-1}| < 2^{-n}.$$

According to the value of  $N$ , the property of  $\theta_{n-1}^N$  changes dramatically. For instance, when  $N = 1$ ,  $\theta_{n-1}^N = \theta_{n-1}$  and  $r(\theta_{n-1}^N) = r(\theta_{n-1})$ . On the other hand, when  $N$  tends to  $\infty$ ,  $\theta_{n-1}^N$  will be a rational number and conformal radius will vanish. By gradually increasing the value of  $N$ , the value of  $r(\theta_{n-1}^N)$  decreases gradually. By using this strategy, we can find the value of  $N^*$  that satisfies  $r(\theta_{n-1}^{N^*}) \approx r_n$ . Then we set  $\theta_{n-1}^{N^*} = \theta_n$ .  $\square$

We now give a proof of theorem 4.4 with lemmas 4.5 and 4.6.

#### Proof of Thm.4.4

Recall that there exists a right-computable number  $r^* \in (0, r_{sup})$ , which is not a computable number. By Lemma 4.6,  $r^* = r(\theta_*)$  for some computable number  $\theta_*$ . By Lemma 4.5, the Julia set  $J_{\theta_*}$  is uncomputable by a Turing machine with an oracle access to  $\theta_*$ . Because  $J_c$  and  $J_\theta$  are equivalent by changing the coordinates, the proof is done.  $\square$

A practical consequence of uncomputable Julia sets  $J_c$  is that we will never see their pictures on the computer screen. When  $d = 2$ , Theorem 4.1 and the hyperbolicity conjecture, which states that hyperbolic parameters are dense in the Mandelbrot set  $\mathcal{M}$ , asserts  $\mathcal{M}$  is computable [7][9].

## 5 Summary

A class of Julia sets with Siegel disks has the most complex geometry compared with the other Julia sets, and they are sometime uncomputable. On the other hand, filled Julia sets are simpler than Julia sets, and they are always computable even in the Siegel case. We give an intuitive explanation of these facts.

In general, there are many fjords towards a periodic point, which is the center of the Julia sets. For hyperbolic Julia sets, the fjords do not reach the center, and for parabolic Julia sets, they always reach the center, while for Siegel disks, the fjord's depth varies and sometime reaches very close to the center.

This makes the membership problem difficult in the case of Siegel disks. Also, if we consider a filled Julia set, the fjords can be ignored to draw it. However, for a Julia set, they cannot be ignored because the true picture is very different from the approximated picture on the computer screen in terms of the Hausdorff metric. This is the reason uncomputable Julia sets belong to the class of Julia sets with Siegel disks.

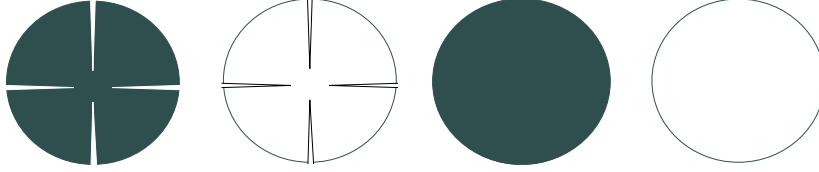


Figure 11: Simple model for filled Julia sets and Julia sets. True picture (left) and approximated picture on computer screen (right).

We give a toy model of Julia sets and prove uncomputability of the membership problem of it.

Let us consider a unit disk with many fjords. We denote the closed wedge around direction  $\theta$  with width  $w$  by  $W(\theta, w)$ . This wedge penetrates the unit disk to depth  $1/2$ . Recall that a function  $A : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ , which takes Boolean values is called a predicate. This predicate takes an input  $(x, y)$  and views  $x$  as an encoding  $(M, w)$  of a Turing machine and its input. If  $x$  is a valid encoding of  $(M, w)$  and  $M$  halts on an input  $w$  in just  $y$  steps,  $A(x, y) = 1$ , where  $A$  is a computable predicate. However, the following problem

$$B(x) = \exists y A(x, y)$$

is equivalent to the halting problem. Therefore,  $B$  is not computable.

We now consider a computable predicate  $A : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$  such that the predicate  $B(x) = \exists y A(x, y)$  is not computable.

Let

$$S_A = \mathbb{U} \setminus \bigcup_{A(x, y)=1} W\left(\frac{2\pi}{x}, \frac{1}{10x^2y}\right).$$

Under this condition, we state two propositions.

**Prop.5.1**

$\partial S_A$  is not computable.

*Proof.*

Take a point  $p_x = (\frac{1}{2} \cos \frac{2\pi}{x}, \frac{1}{2} \sin \frac{2\pi}{x})$ , where  $p_x$  is located on the tip of the wedge  $W(x, \circ)$ , which is a wedge around the direction  $x$ . Let us consider the part of  $\partial S_A$  around  $p_x$ . If  $B(x) = 1$ , i.e.,  $\exists y$  such that  $A(x, y) = 1$ , then  $p_x \in$

$\partial S_A$ . Otherwise, the ball  $B(p_x, \frac{1}{10x^2})$  and  $\partial S_A$  have no intersection. Therefore, if we assume that  $\partial S_A$  is computable,  $B(x)$  becomes computable. This is a contradiction.  $\square$

**Prop.5.2**

$S_A$  is computable.

*Proof.*

Let us compute  $S_A$  with precision  $2^{-n}$ . Under this condition, we can ignore wedges whose width is smaller than  $\frac{1}{m} = 2^{-(n+1)}$ . Hence, when we draw a picture of  $S_A$  on the computer screen, we will only need to evaluate  $A(x, y)$  for values of  $x$  and  $y$  such that  $x^2y \leq m$ . There is a finite number of such pairs. Therefore,  $S_A$  is computable.  $\square$

We reviewed the results of the computability and complexity of Julia sets developed by Braverman and Yampolsky. Extending the results to real dynamical systems in nonlinear physics may shed light on the complexity of real world nonlinear phenomena. Applications to cryptographic systems or formal language theory with this framework will be promising future works.

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