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GLOBAL WELL POSEDNESS FOR A TWO-FLUID MODEL

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ABSTRACT. We study a two fluid system which models the motion of a charged fluid. Local in time solutions of this system were proven by Giga-Yoshida [15]. In this paper, we improve this result in terms of requiring less regularity on the electromagnetic field. We also prove that small solutions are global in time.

1. INTRODUCTION

We consider the following two-fluid incompressible Navier-Stokes-Maxwell system (NSM):

$$(1) \quad \begin{cases} nm_- \partial_t v_- = \nu_- \Delta v_- - nm_- v_- \cdot \nabla v_- - en(E + v_- \times B) - R - \nabla p_- \\ nm_+ \partial_t v_+ = \nu_+ \Delta v_+ - nm_+ v_+ \cdot \nabla v_+ + eZn(E + v_+ \times B) + R - \nabla p_+ \\ \partial_t E = \frac{1}{\varepsilon_0 \mu_0} \nabla \times B - \frac{ne}{\varepsilon_0} (Zv_+ - v_-) \\ \partial_t B = -\nabla \times E \\ R := -\alpha(v_+ - v_-) \\ \operatorname{div} v_- = \operatorname{div} v_+ = \operatorname{div} B = \operatorname{div} E = 0 \end{cases}$$

with the initial data

$$v_-|_{t=0} = v_{1,0}, \quad v_+|_{t=0} = v_{2,0}, \quad B|_{t=0} = B_0, \quad E|_{t=0} = E_0.$$

The system models the motion of a plasma of cations (positively charged) and anions (negatively charged) particles with approximately equal masses $m_{\pm}$. The constants n stands for the number density, and e is the elementary charge. The charge number is given by Z and ε_0 represents the vacuum dielectric constant and μ_0 is the vacuum permeability. The constant α is a positive parameter.

The vectors v_- and $v_+ : \mathbb{R}_t^+ \times \mathbb{R}_x^d \rightarrow \mathbb{R}^3$ represent then the velocities of the anions and cations, respectively. The electromagnetic field is represented by $E, B : \mathbb{R}_t^+ \times \mathbb{R}_x^d \rightarrow \mathbb{R}^3$. Here the space dimension is $d = 3$ (or 2), and $\nu_{\pm}$ is the kinematic viscosity of the two-fluid and the scalar function $p^{\pm}$ stands for its pressure. We refer to [8] for more details about the model. The third equation is the Ampère-Maxwell equation for an electric field E .

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The equations on the velocities are the momentum equation, and the fourth equation is nothing but Farady's law. For a detailed introduction to the NSM, we refer to Davidson [3] and Biskamp [6].

First of all, we will assume that all the above physical constants are normalized to one. This yields nice cancelations and thus avoids some extra technical difficulties. Then, we refer to Section 6 for more details on how our results extend to the original physical system (1). A such simplification yields the following system of equations

$$(2) \quad \begin{cases} \partial_t v_- + v_- \cdot \nabla v_- - \Delta v_- + \nabla p_- = -(E + v_- \times B) - R \\ \partial_t v_+ + v_+ \cdot \nabla v_+ - \Delta v_+ + \nabla p_+ = (E + v_+ \times B) + R \\ \partial_t E - \nabla \times B = -(v_+ - v_-) \\ \partial_t B + \nabla \times E = 0 \\ R := -\alpha(v_+ - v_-) \\ \operatorname{div} v_- = \operatorname{div} v_+ = \operatorname{div} B = \operatorname{div} E = 0 \end{cases}$$

with the initial data

$$v_-|_{t=0} = v_{1,0}, \quad v_+|_{t=0} = v_{2,0}, \quad B|_{t=0} = B_0, \quad E|_{t=0} = E_0.$$

Here, α is a positive constant and R represents the momentum transfer between the two components of the fluid.

Defining the bulk velocity $u = \frac{v_- + v_+}{2}$ and the electrical current $j = \frac{v_+ - v_-}{2}$ we can rewrite system (2), as in [1], in the following equivalent form

$$(3) \quad \begin{cases} \partial_t u + u \cdot \nabla u + j \cdot \nabla j - \Delta u &= -\nabla p + j \times B \\ \partial_t j + u \cdot \nabla j + j \cdot \nabla u - \Delta j + 2\alpha j &= -\nabla \bar{p} + E + u \times B \\ \partial_t E - \nabla \times B &= -2j \\ \partial_t B + \nabla \times E &= 0 \\ \operatorname{div} u = \operatorname{div} j = \operatorname{div} E = \operatorname{div} B &= 0. \end{cases}$$

System (3) has appeared in [1] including its dependence upon the speed of light that shows up in Maxwell's equations. The non-relativistic asymptotic of that system (i.e. as the speed of light tends to infinity), was then analyzed and a convergence towards the standard Magneto Hydrodynamic equations was shown. We refer to [1] for full details.

In this paper, we mainly study the existence of global in time solutions. First, we construct global weak solutions *à la Leray* for system (2). Although the proof goes along the same lines as for the incompressible Navier-Stokes equations, we outline it in this paper for the sake of completeness. We also emphasize that this is in a striking difference with the following slightly modified Navier-Stokes-Maxwell one fluid model studied in [9] and [7].

$$(4) \quad \begin{cases} \frac{\partial v}{\partial t} + v \cdot \nabla v - \nu \Delta v + \nabla p &= j \times B \\ \partial_t E - \operatorname{curl} B &= -j \\ \partial_t B + \operatorname{curl} E &= 0 \\ \operatorname{div} v = \operatorname{div} B &= 0 \\ \sigma(E + v \times B) &= j. \end{cases}$$

It is important to mention that the existence of global weak solutions of (4) is still an outstanding important open problem in both space dimension two and three. The local well-posedness and the existence of global small solutions were studied in [9] and [7] for initial data in $u_0, E_0, B_0 \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{\frac{1}{2}} \times \dot{H}^{\frac{1}{2}}$.

A particularity of our two fluid model is that the term R in (2) brings exponential decay of the energies of both v_- and v_+ .

Second, we construct local in time mild-type solution. The proof combines *à priori* estimate techniques to the Banach fixed point theorem. Then, we prove the global existence of these solutions when the initial data is small. To do so, we introduce a truncated NSM system (in the spirit of [8]). Then we show that the truncated solution with small initial data does solve our original NSM globally in time with its value in L^2 . Then we show that for a fixed time $T > 0$, the truncated velocity field solution on $[0, T]$ is also in $u \in L_T^\infty H^{\frac{1}{2}} \cap L_T^2 H^{\frac{3}{2}}$, where we have used a short-hand notation $L_T^\infty H^{\frac{1}{2}} = L^\infty(0, T, H^{\frac{1}{2}})$, $L_T^2 H^{\frac{3}{2}} = L^2(0, T, H^{\frac{3}{2}})$. Finally, we show how our method can be applied to the original physical model (see appendix in [8]) where the key point is to get the energy estimates.

The following is our first result.

Theorem 1.1. For $v_{1,0}, v_{2,0}, B_0, E_0 \in L^2$ with $\operatorname{div} v_{1,0} = \operatorname{div} v_{2,0} = 0$, there exists a weak solution

$$\begin{cases} v_- \in L^\infty(0, \infty, L^2) \cap L^2(0, \infty, \dot{H}^1) \subset L^{10/3}((0, \infty) \times \mathbb{R}^3) \\ v_+ \in L^\infty(0, \infty, L^2) \cap L^2(0, \infty, \dot{H}^1) \subset L^{10/3}((0, \infty) \times \mathbb{R}^3) \\ B \in L^\infty(0, \infty, L^2) \\ E \in L^\infty(0, \infty, L^2), \end{cases}$$

for system (2) on $(0, \infty) \times \mathbb{R}^3$. Moreover the energy of the velocity vector fields v_- and v_+ enjoys an exponential decay with a rate depending on α :

$$\|v_-(t)\|_{L^2}^2 + \|v_+(t)\|_{L^2}^2 \leq 2(\|v_{1,0}\|_{L^2}^2 + \|v_{2,0}\|_{L^2}^2)e^{-2\alpha t}.$$

Our second result concerns the local existence of mild-type solutions for arbitrary large initial data. More precisely we have the following:

Theorem 1.2. For any $\alpha \geq 0$ and $\delta_0 > 0$, there exists small $T > 0$ (in this case T is independent of α) such that if the initial conditions (u_0, j_0, E_0, B_0) of NSM (3) are such that

$$\|(u_0, j_0, E_0, B_0)\|_{\dot{H}^{\frac{1}{2}} \times H^{\frac{1}{2}} \times L^2 \times L^2} \leq \delta_0,$$

then system (3) has a unique solution (u, j, E, B) satisfying

$$\begin{aligned} u &\in L_T^\infty \dot{H}^{\frac{1}{2}} \cap L_T^2 \dot{H}^{\frac{3}{2}}, \\ j &\in L_T^\infty \dot{H}^{\frac{1}{2}} \cap L_T^2 \dot{H}^{\frac{3}{2}}, \\ E &\in L_T^\infty L^2, \\ B &\in L_T^\infty L^2. \end{aligned}$$

The following theorem shows that the solutions given by the previous theorem are global if the initial data is small enough. More precisely,

Theorem 1.3. If the initial conditions $(u_{-,0}, u_{+,0}, E_0, B_0)$ of NSM (2) (or the physical model (1)) are small enough in $H^{\frac{1}{2}} \times H^{\frac{1}{2}} \times L^2 \times L^2$, then there exists a unique global solution (u_-, u_+, E, B) of (2) with its value in L^2 . Furthermore,

$$\begin{aligned} u_\pm &\in L^\infty(H^{\frac{1}{2}}) \cap L_{\text{loc}}^2(H^{\frac{3}{2}}) \\ E, B &\in L^\infty(L^2). \end{aligned}$$

Remark 1.1. • This is the first global existence result for small initial data for this system. Our result improves Giga-Yoshida [8] by requiring less regularity on the data. Moreover, in [8] a global existence result for (1) is proved for small initial data under the additional assumption that E_0, B_0 are small compared with $v_{\pm,0}$. Also half a derivative in less is required on the electro-magnetic field (E, B) compared with the similar results of [9], [7] and [10] for the one fluid model (4). This is naturally given the “regularizing” effect caused by the term R .

- Note also that we have the same result for NSM(3) since the two systems NSM(2) and NSM(3) are equivalent.

The next theorem is the local existence extended to the physical model.

Theorem 1.4. Let all physical constants be fixed (and thus, a priori, the lifespan depends on them). If the initial conditions of the physical model (1) are such that $(v_{-,0}, v_{+,0}, E_0, B_0) \in H^{\frac{1}{2}} \times H^{\frac{1}{2}} \times L^2 \times L^2$, then there exists $T > 0$ and a unique solution (v_-, v_+, E, B) of system (1) such that

$$\begin{aligned} v_\pm &\in L_T^\infty \dot{H}^{\frac{1}{2}} \cap L_T^2 \dot{H}^{\frac{3}{2}} \\ E, B &\in L_T^\infty L^2. \end{aligned}$$

Remark 1.2. Compared to Theorem 1.2, the result of Theorem 1.4 requires a better control of the low frequencies of the initial data $v_{-,0}$ and $v_{+,0}$.

The proof of Theorem 1.2 uses a fixed point argument in an appropriate space. First, we list several *à priori* estimates given by Lemmas 2.1-Lemma 2.4. To prove Theorem 1.3, we proceed in several steps. First, a truncated system (cf. [8]) is introduced to approximate the original system; Second, we

set up a contradiction argument to prove that the truncated system coincides with the original system when the initial condition is small enough; Then the global solution to the truncated system is the solution to the original one. In this proof, we use NSM(2) because the truncated system of NSM(2) is much more simple than that for NSM(3). However, the result remains true for both.

The following is the notation used throughout this paper.

$A(t) \lesssim B(t)$ means $A(t) \leq CB(t)$, where C is a universal constant. Let $\mathcal{P}$ denote Leray (or Helmholtz) projection onto divergence free vector fields. More precisely, if u is a smooth vector field on $\mathbb{R}^d$, then u can be uniquely written as the sum of divergence free vector v and a gradient(also called Hodge decomposition):

$$u = v + \nabla \omega, \quad \operatorname{div} v = 0.$$

Then, Leray projection of u is $\mathcal{P}u = v$. Denote by Δ_q the frequency localization operator, defined as follows:

Assume $\mathcal{C}$ is the ring of center 0 with small radius $1/2$ and great radius 2, B is the ball centered at 0 with radius 1. There exist two nonnegative radial function $\chi \in \mathcal{D}(B)$ and $\phi \in \mathcal{D}(\mathcal{C})$, s.t

$$\chi(\xi) + \sum_{p \geq 0} \phi(2^{-p}\xi) = 1.$$

Then define

$$\Delta_q u = \mathcal{F}^{-1}(\phi(2^{-q}\xi)\mathcal{F}u)$$

where $\mathcal{F}$ is the Fourier transform.

We define the space: $L^p(0, T; H^1)$, usually denoted as $L_T^p H^1$ by its norm

$$\|f(t, x)\|_{L_T^p H^1}^p := \int_0^T \|f\|_{H^1}^p(t) dt < \infty.$$

Finally, the spaces $\tilde{L}_T^r H^s$ and $\tilde{L}_T^r \dot{H}^s$ are the set of tempered distributions u such that

$$\|u\|_{\tilde{L}_T^r H^s} := \|(1 + 2^q)^s \|\Delta_q u\|_{L_T^r}\|_{l^2(\mathbb{Z})} < \infty$$

$$\|u\|_{\tilde{L}_T^r \dot{H}^s} := \|2^{qs} \|\Delta_q u\|_{L_T^r}\|_{l^2(\mathbb{Z})} < \infty.$$

This kind of spaces were first introduced by Chemin and Lerner in [5]. This paper is organized as follows. Section 2 is devoted to list some preliminary lemmas. In sections 3 to 5, we provide the detail proof of Theorem 1.1 and Theorem 1.3. In section 6, we extend well-posedness result to original physical model.

2. PARABOLIC REGULARIZATION, PRODUCT ESTIMATES AND ENERGY ESTIMATES

Lemma 2.1 (parabolic regularization). let u be a smooth divergence-free vector field which solves

$$(5) \quad \begin{aligned} \partial_t u - \Delta u + \nabla p &= f_1 + f_2 \\ u|_{t=0} &= u_0 \end{aligned}$$

or

$$(6) \quad \begin{aligned} \partial_t u + \alpha u - \Delta u + \nabla p &= f_1 + f_2 \\ u|_{t=0} &= u_0 \end{aligned}$$

on some interval $[0, T]$, where α is a nonnegative constant. Then, for every $p \geq r_1 \geq 1$, $p \geq r_2 \geq 1$ and $s \in \mathbb{R}$,

$$\|u\|_{C([0,T];\dot{H}^s) \cap \tilde{L}_T^p \dot{H}^{s+2/p}} \lesssim \|u_0\|_{\dot{H}^s} + \|f_1\|_{\tilde{L}_T^{r_1} \dot{H}^{s-2+2/r_1}} + \|f_2\|_{\tilde{L}_T^{r_2} \dot{H}^{s-2+2/r_2}}.$$

We also have a similar result in nonhomogeneous spaces but with T -dependent constants. More specifically,

$$\|u\|_{\tilde{L}_T^p H^{s+2/p}} \leq C_T \left(\|u_0\|_{H^s} + \|f_1\|_{\tilde{L}_T^{r_1} H^{s-2+2/r_1}} + \|f_2\|_{\tilde{L}_T^{r_2} H^{s-2+2/r_2}} \right)$$

where $C_T = C \max\{1, T\}$, C is an absolute constant.

Proof. We only show the reason why (5) and (6) share the same estimate. For more details of the proof, we refer to [14]. (6) can be written as the following:

$$\partial_t u - (\Delta - \alpha I)u + \nabla p = f.$$

By Duhamel's formula, we have

$$(7) \quad u(t) = e^{t(\Delta - \alpha I)} u_0 + \int_0^t e^{(t-s)(\Delta - I)} \mathbb{P} f(s) ds.$$

Applying Δ_q , the frequency localization operator, to (7), taking L^2 norm in space, and using the standard estimate for Δ_q (see for instance [14] and [8]), we get

$$\begin{aligned} \|\Delta_q u(t)\|_{L^2} &\lesssim \|e^{-\alpha t} \Delta_q u_0\|_{L^2} e^{-c2^{2q}t} + \int_0^t e^{c2^{2q}(t-s)} \|e^{-\alpha t} \Delta_q \mathbb{P} f(s)\|_{L^2} ds \\ &\lesssim \|\Delta_q u_0\|_{L^2} e^{-c2^{2q}t} + \int_0^t e^{c2^{2q}(t-s)} \|\Delta_q \mathbb{P} f(s)\|_{L^2} ds, \end{aligned}$$

with some universal constant in the estimate. Then we can follow the same method which is used to get the estimate for (5) (See [14]). Taking L^p norm in time and using Young's inequality (in time) we obtain

$$\begin{aligned} \|\Delta_q u\|_{L_T^p L^2} &\lesssim 2^{\frac{2q}{p}} \|\Delta_q u_0\|_{L^2} + \|e^{-2^{2q}t} 1_{t>0}\|_{L^\alpha} \|\Delta_q f\|_{L^r L^2} \\ &\lesssim 2^{\frac{-2q}{p}} \|\Delta_q u_0\|_{L^2} + C 2^{\frac{-2q}{\alpha}} \|\Delta_q f\|_{L^r L^2} \end{aligned}$$

where $\frac{1}{p} + 1 = \frac{1}{\alpha} + \frac{1}{r}$. At last, multiplying by $2^{q(s+2+\frac{2}{p}-\frac{2}{r})}$ and taking l^2 norm over $q \in \mathbb{Z}$, we get the desired result. $\square$

Remark 2.1. We should note that the estimates in the the above Lemma are uniform with respect to the parameter α . When we deal with the original physical model (1), we loose the cancellation property of the term R and thus we cannot write the system (1) in the form of (3). However, in this case we consider this R as a source term, and therefore estimates become dependent upon α .

The next lemma is a standard H^s -energy estimate for Maxwell's system.

Lemma 2.2. Let (E, B) solve

$$\begin{cases} \partial_t E - \nabla \times B &= -2j \\ \partial_t B + \nabla \times E &= 0 \\ E(t=0) &= E_0 \\ B(t=0) &= B_0 \end{cases}$$

on some interval $[0, T]$. Then, for every $s \in \mathbb{R}$,

$$\|E\|_{C([0,T];H^s)} + \|B\|_{C([0,T];H^s)} \leq \|E_0\|_{H^s} + \|B_0\|_{H^s} + 2\|j\|_{L_T^1 H^s}.$$

We refer to [14] for the proof. Next, we set the nonlinear estimates necessary to derive the *a priori* bounds.

Lemma 2.3 (products estimate).

$$(8) \quad \|u \cdot \nabla v\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \lesssim \|u\|_{L_T^\infty \dot{H}^{\frac{1}{2}}} \|v\|_{L_T^2 \dot{H}^{\frac{3}{2}}}$$

$$(9) \quad \|u \cdot \nabla v\|_{\tilde{L}_T^{\frac{4}{3}} L^2} \lesssim \|u\|_{L_T^\infty \dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|u\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \|v\|_{L_T^2 \dot{H}^{\frac{3}{2}}}$$

$$(10) \quad \|u \times B\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \lesssim \|u\|_{L_T^2 \dot{H}^1} \|B\|_{L_T^\infty L^2}$$

$$(11) \quad \|j \times B\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \lesssim T^{\frac{1}{4}} \|j\|_{L_T^\infty \dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|j\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \|B\|_{L_T^\infty L^2}$$

Proof. We only show the proof of (9) as the others are similar.

$$\begin{aligned} \|u \cdot \nabla v\|_{\tilde{L}_T^{\frac{4}{3}} L^2} &\lesssim \|u \cdot \nabla v\|_{L_T^{\frac{4}{3}} L^2} \\ &\lesssim \|u\|_{L_T^4 L^6} \|\nabla v\|_{L_T^2 L^3} \\ &\lesssim \|u\|_{L_T^4 \dot{H}^1} \|\nabla v\|_{L_T^2 \dot{H}^{\frac{1}{2}}}. \end{aligned}$$

By interpolation, we have

$$\begin{aligned} \|u \cdot \nabla v\|_{\tilde{L}_T^{\frac{4}{3}} L^2} &\lesssim \left(\int_0^T \|u\|_{\dot{H}^1}^4 \right)^{\frac{1}{4}} \|\nabla v\|_{L_T^2 \dot{H}^{\frac{1}{2}}} \\ &\lesssim \left(\int_0^T \|u\|_{\dot{H}^{\frac{1}{2}}}^2 \|u\|_{\dot{H}^{\frac{3}{2}}}^2 \right)^{\frac{1}{4}} \|\nabla v\|_{L_T^2 \dot{H}^{\frac{1}{2}}}, \end{aligned}$$

and using Hölder inequality, we have

$$\begin{aligned} \|u \cdot \nabla v\|_{\tilde{L}_T^{\frac{4}{3}} L^2} &\lesssim \left(\|u\|_{L_T^\infty \dot{H}^{\frac{1}{2}}}^2 \|u\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^2 \right)^{\frac{1}{4}} \|v\|_{L_T^2 \dot{H}^{\frac{3}{2}}} \\ &\lesssim \|u\|_{L_T^\infty \dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|u\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \|v\|_{L_T^2 \dot{H}^{\frac{3}{2}}}. \end{aligned}$$

□

The next lemma is a standard energy estimate for the whole NSM system.

Lemma 2.4. For NSM system (2), we have the following energy estimate.

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \left(|v_+|^2 + |v_-|^2 + |B|^2 + |E|^2 \right) dx &\leq -\nu \int_{\mathbb{R}^3} \left(|\nabla v_+|^2 + |\nabla v_-|^2 \right) dx \\ &\quad - \alpha \int_{\mathbb{R}^3} \left((v_+ - v_-)^2 \right) dx \end{aligned}$$

or, equivalently for NSM system (3), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \left(2|u|^2 + 2|j|^2 + |B|^2 + |E|^2 \right) dx &\leq -\nu \int_{\mathbb{R}^3} \left(2|\nabla u|^2 + 2|\nabla j|^2 \right) dx \\ &\quad - 4\alpha \int_{\mathbb{R}^3} |j|^2 dx. \end{aligned}$$

Proof. The proof is standard. In (2), multiply the first equation by v_- , the second one by v_+ , the third one by E and the fourth one by B , then integrating by parts will give us the desired estimate. The second energy estimate is derived from (3) by the same method. □

3. EXISTENCE OF GLOBAL WEAK SOLUTIONS

We shall prove Theorem 1.1. According to Lemma 2.4, we can see that sequences $v_\pm^j$, E^j and B^j are $L_T^\infty L^2$ bounded, $v_\pm^j$ is moreover $L_T^2 \dot{H}^1$ bounded. This means that there are subsequences of $\{B^j\}_j$ and $\{v^j\}_j$ such that

$$B^j \rightarrow B \text{ in weak* of } L_T^\infty L^2,$$

$$v_\pm^j \rightarrow v_\pm \text{ in weak* of } L_T^\infty L^2$$

and

$$v_\pm^j \rightarrow v_\pm \text{ in weak of } L_T^2 \dot{H}^1.$$

By the following inclusion:

$$L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1 \subset L_t^\infty L_x^2 \cap L_t^2 L_x^6 \subset L_t^{\frac{10}{3}} L_x^{\frac{10}{3}},$$

we see that v_-^j and v_+^j are uniformly bounded in $L^{10/3}$ in both space and time. This means that (by compact inclusion) $v_-^j \otimes v_-^j$ locally converges to $v_- \otimes v_-$ strongly in L^1 in both space and time (also v_+). Therefore

$$\nabla(v_-^j \otimes v_-^j) \rightarrow \nabla(v_- \otimes v_-)$$

in distribution sense (also v_+). Next, note that the uniform bounds on v_-^j and B^j imply that

$$(12) \quad v_-^j \times B^j \in L_t^\infty L_x^1 \cap L_t^2 L_x^{\frac{3}{2}},$$

whence, for any $2 < p < \infty$

$$v_-^j \times B^j \rightarrow v_- \times B \quad \text{in } L_t^p L_x^{\frac{3p}{3p-2}}.$$

The above inclusion also gives us that

$$v_-^j \times B^j \rightarrow v_- \times B$$

in distribution sense (also v_+). Therefore we can construct a weak solution to (2). Recall the energy inequality.

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} (|v_-|^2 + |v_+|^2 + |B|^2 + |E|^2) dx &\leq -\nu \int_{\mathbb{R}^3} (|\nabla v_-|^2 + |\nabla v_+|^2) dx \\ &\quad - \alpha \int_{\mathbb{R}^3} (v_- - v_+)^2. \end{aligned}$$

We show that $\int |v_-|^2 + |v_+|^2$ has exponential decay. Recall that

$$\frac{v_- - v_+}{2} =: j \quad \text{and} \quad \frac{v_- + v_+}{2} =: u.$$

A direct calculation yields that

$$v_-^2 + v_+^2 = 2(j^2 + u^2).$$

From the energy identity,

$$\frac{d}{dt} (\|j\|_2^2 + \|u\|_2^2) \leq -2\alpha \|j\|_2^2.$$

Thus

$$\|j(t)\|_2^2 + \|u(t)\|_2^2 \leq -2\alpha \int_0^t \|j(s)\|_2^2 ds + \|j(0)\|_2^2 + \|u(0)\|_2^2.$$

This means that

$$\|j(t)\|_2^2 \leq (\|j(0)\|_2^2 + \|u(0)\|_2^2) \exp(-2\alpha t).$$

Once we get the exponential decay of $\|j\|_2^2$,

$$\frac{d}{dt} (\|j\|_2^2 + \|u\|_2^2) \leq -2\alpha (\|j(0)\|_2^2 + \|u(0)\|_2^2) \exp(-2\alpha t).$$

Then we have that

$$(\|j(t)\|_2^2 + \|u(t)\|_2^2) \leq -2\alpha (\|j(0)\|_2^2 + \|u(0)\|_2^2) \int_0^t \exp(-2\alpha s) ds + (\|j(0)\|_2^2 + \|u(0)\|_2^2).$$

This means that

$$\|u(t)\|_2^2 \leq (\|j(0)\|_2^2 + \|u(0)\|_2^2) e^{-2\alpha t}.$$

Thus we have the exponential decay of $\int |v_-|^2 + |v_+|^2$.

4. EXISTENCE OF A UNIQUE LOCAL-IN-TIME SOLUTION

Now we prove Theorem 1.2. The proof goes into a few steps.

STEP 1: À PRIORI ESTIMATE.

By Lemma 2.1, for any (u, j, E, B) solution of (3), we have

$$(13) \quad \|u\|_{L_T^\infty \dot{H}^{\frac{1}{2}} \cap L_T^2 \dot{H}^{\frac{3}{2}}} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|j \times B\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} + \|u \cdot \nabla u + j \cdot \nabla j\|_{\tilde{L}_T^{\frac{4}{3}} L^2}$$

and similarly,

$$(14) \quad \|j\|_{L_T^\infty \dot{H}^{\frac{1}{2}} \cap L_T^2 \dot{H}^{\frac{3}{2}}} \lesssim \|j_0\|_{\dot{H}^{\frac{1}{2}}} + \|u \times B\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} + \|u \cdot \nabla j + j \cdot \nabla u\|_{\tilde{L}_T^{\frac{4}{3}} L^2} + \|E\|_{\tilde{L}_T^{\frac{4}{3}} L^2}.$$

Now again, Lemma 2.1 gives

$$(15) \quad \begin{aligned} \|j\|_{L_T^1 L^2} &\leq T \|j\|_{L_T^\infty L^2} \\ &\leq T \|j_0\|_{L^2} + T^{\frac{5}{4}} \|u \cdot \nabla j + j \cdot \nabla u + u \times B\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} + T \|E\|_{\tilde{L}_T^1 L^2} \\ &\leq T \|j_0\|_{L^2} + T^{\frac{5}{4}} \|u \cdot \nabla j + j \cdot \nabla u + u \times B\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} + T \|E\|_{L_T^1 L^2} \end{aligned}$$

and by Lemma 2.2, we have:

$$(16) \quad \|E\|_{L_T^\infty L^2} + \|B\|_{L_T^\infty L^2} \leq \|E_0\|_{L^2} + \|B_0\|_{L^2} + \|j\|_{L_T^1 L^2}.$$

Putting together (15) and (16), we obtain the *à priori* estimate for (E, B) :

$$(17) \quad \begin{aligned} \|E\|_{L_T^\infty L^2} + \|B\|_{L_T^\infty L^2} &\leq \|E_0\|_{L^2} + \|B_0\|_{L^2} + T \|j_0\|_{L^2} \\ &\quad + T^{\frac{5}{4}} \|u \cdot \nabla j + j \cdot \nabla u + u \times B\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} + T \|E\|_{L_T^1 L^2}. \end{aligned}$$

STEP 2: CONTRACTION ARGUMENT.

Let $\Gamma := (u, j, E, B)^T$ be such that

$$\begin{aligned} u &\in X^u := L_T^\infty \dot{H}^{\frac{1}{2}} \cap L_T^2 \dot{H}^{\frac{3}{2}} \\ j &\in X^j := L_T^\infty \dot{H}^{\frac{1}{2}} \cap L_T^2 \dot{H}^{\frac{3}{2}} \\ E &\in X^E := L_T^\infty L^2 \\ B &\in X^B := L_T^\infty L^2, \end{aligned}$$

and set $X = X^u \times X^j \times X^E \times X^B$. Then the norm of Γ can be defined as $\|\Gamma\|_X := \|u\|_{X^u} + \|j\|_{X^j} + \|E\|_{X^E} + \|B\|_{X^B}$. We look for a solution in the following integral form

$$\Gamma(t) = e^{tA} \Gamma(0) + \int_0^t e^{(t-s)A} f(\Gamma(s)) ds$$

with

$$A = \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta - \alpha I & 0 & 0 \\ 0 & 0 & 0 & \nabla \times \\ 0 & 0 & -\nabla \times & 0 \end{pmatrix}$$

and

$$f(\Gamma) := (\mathcal{P}(-u \cdot \nabla u - j \cdot \nabla j + j \times B), \mathcal{P}(-u \cdot \nabla j - j \cdot \nabla u + u \times B) + E, -j, 0)^T.$$

Define a map $\Phi : X \rightarrow X$ as

$$\Phi(\Gamma) := \int_0^t e^{(t-s)A} f(e^{sA}\Gamma_0 + \Gamma(s)) ds$$

where $\Gamma_0 = (u_0, j_0, E_0, B_0)^T$. We denote the components

$$\Phi(\Gamma) = (\Phi(\Gamma)^u, \Phi(\Gamma)^j, \Phi(\Gamma)^E, \Phi(\Gamma)^B).$$

Note that $\Phi(-e^{tA}\Gamma_0) = 0$, and by Lemma 2.1

$$\|e^{tA}\Gamma_0\|_X \leq C\|\Gamma_0\|_{\dot{H}^{\frac{1}{2}} \times H^{\frac{1}{2}} \times L^2 \times L^2}$$

where C is a universal constant. Moreover, setting $r = C\|\Gamma_0\|_{\dot{H}^{\frac{1}{2}} \times H^{\frac{1}{2}} \times L^2 \times L^2}$ and denoting by B_r the ball of space X centered at 0 and with radius r , we claim that $\Phi(B_r) \subset B_r$ if T is sufficiently small.

Assume $\Gamma \in B_r$, and set $e^{tA}\Gamma_0 - \Gamma =: \bar{\Gamma}$ (we also define $\bar{u}$, $\bar{j}$, $\bar{B}$ and $\bar{E}$ in the same manner). Then $\|\bar{\Gamma}\|_X \leq 2r$ and $Y = \Phi(\Gamma)$ solves the following equation with zero initial data:

$$\partial_t Y - AY = f(e^{tA}\Gamma_0 - \Gamma(t)).$$

By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned} (18) \quad \|\Phi(\Gamma)^u\|_{X^u} &\lesssim \|\bar{u} \cdot \nabla \bar{u} + \bar{j} \cdot \nabla \bar{j} + \bar{j} \times \bar{B}\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \\ &\lesssim \left(\|\bar{u}\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + \|\bar{j}\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + T^{\frac{1}{4}} \|\bar{j}\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \|\bar{j}\|_{L_T^\infty \dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \right) \|\bar{\Gamma}\|_X \\ &\lesssim 2 \left(\|\bar{u}\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + \|\bar{j}\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + T^{\frac{1}{4}} r^{\frac{1}{2}} \|\bar{j}\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) r. \end{aligned}$$

Note that r is a constant independent of T and that $\|\bar{u}\|_{L_T^2 \dot{H}^{\frac{3}{2}}}, \|\bar{j}\|_{L_T^2 \dot{H}^{\frac{3}{2}}}$ can be taken arbitrarily small by choosing T small enough, if we denote by C the universal constant in the last inequality above, then we could choose T small such that

$$2C \left(\|\bar{u}\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + \|\bar{j}\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + T^{\frac{1}{4}} r^{\frac{1}{2}} \|\bar{j}\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) < \frac{1}{4}.$$

So that the following inequality holds

$$\|\Phi(\Gamma)^u\|_{X^u} \leq \frac{1}{4}r.$$

Similarly, we also have

$$\|\Phi(\Gamma)^j\|_{X^j} \leq \frac{1}{4}r,$$

$$\|\Phi(\Gamma)^E\|_{X^E} + \|\Phi(\Gamma)^B\|_{X^B} \leq \frac{1}{2}r.$$

This shows that, $\Phi(\Gamma) \in B_r$ as desired. Furthermore, if Φ is a contraction on B_r , then there is a unique fixed point: Γ^* and hence, $e^{tA}\Gamma_0 - \Gamma^*$ must be our desired solution. We will discuss the contraction in the next section.

STEP 3: CONTRACTION.

To prove that Φ is a contraction on B_r if T is small enough, assume given Γ_1 and Γ_2 belonging to B_r and set $e^{tA}\Gamma_0 - \Gamma_i = \bar{\Gamma}_i$, $i = 1, 2$. By the *à priori* estimate (13), we have

$$(19) \quad \begin{aligned} \|\Phi(\Gamma_1)^u - \Phi(\Gamma_2)^u\|_{X^u} &\lesssim \|(\bar{j}_1 - \bar{j}_2) \times \bar{B}_2 + \bar{j}_1 \times (\bar{B}_1 - \bar{B}_2)\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \\ &\quad + \|(\bar{u}_1 - \bar{u}_2) \cdot \nabla \bar{u}_2 + \bar{u}_1 \cdot \nabla (\bar{u}_1 - \bar{u}_2)\|_{\tilde{L}_T^{\frac{4}{3}} L^2} \\ &\quad + \|(\bar{j}_1 - \bar{j}_2) \cdot \nabla \bar{j}_2 + \bar{j}_1 \cdot \nabla (\bar{j}_1 - \bar{j}_2)\|_{\tilde{L}_T^{\frac{4}{3}} L^2}. \end{aligned}$$

Using (8) and (9) in Lemma 2.3, (19) becomes

$$\begin{aligned} \|\Phi(\Gamma_1)^u - \Phi(\Gamma_2)^u\|_{X^u} &\lesssim T^{\frac{1}{4}} \left(\|\bar{B}_2\|_{L_T^\infty L^2} + \|\bar{j}_1\|_{L_T^\infty \dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|\bar{j}_1\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) \|\Gamma_1 - \Gamma_2\|_X \\ &\quad + \left(\|\bar{u}_2\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + \|\bar{u}_1\|_{L_T^\infty \dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|\bar{u}_1\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) \|\Gamma_1 - \Gamma_2\|_X \\ &\quad + \left(\|\bar{j}_2\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + \|\bar{j}_1\|_{L_T^\infty \dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|\bar{j}_1\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) \|\Gamma_1 - \Gamma_2\|_X \\ &\lesssim T^{\frac{1}{4}} (4r) \|\Gamma_1 - \Gamma_2\|_X \\ &\quad + \left(\|\bar{u}_2\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + (2r)^{\frac{1}{2}} \|\bar{u}_1\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) \|\Gamma_1 - \Gamma_2\|_X \\ &\quad + \left(\|\bar{j}_2\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + (2r)^{\frac{1}{2}} \|\bar{j}_1\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) \|\Gamma_1 - \Gamma_2\|_X. \end{aligned}$$

We choose T small enough so that

$$(20) \quad \|\Phi(\Gamma_1)^u - \Phi(\Gamma_2)^u\|_{X^u} \leq \frac{1}{8} \|\Gamma_1 - \Gamma_2\|_X.$$

Now, estimating $\|\Phi(\Gamma_1) - \Phi(\Gamma_2)\|_{X^j}$, $\|\Phi(\Gamma_1) - \Phi(\Gamma_2)\|_{X^E}$ and $\|\Phi(\Gamma_1) - \Phi(\Gamma_2)\|_{X^B}$ in the same way gives

$$\begin{aligned} \|\Phi(\Gamma_1)^j - \Phi(\Gamma_2)^j\|_{X^j} &\lesssim T^{\frac{1}{4}} (4r) \|\Gamma_1 - \Gamma_2\|_X \\ &\quad + \left(\|\bar{j}_2\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + (2r)^{\frac{1}{2}} \|\bar{u}_1\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) \|\Gamma_1 - \Gamma_2\|_X \\ &\quad + \left(\|\bar{u}_2\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + (2r)^{\frac{1}{2}} \|\bar{j}_1\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) \|\Gamma_1 - \Gamma_2\|_X \\ &\quad + T^{\frac{3}{4}} \|\Gamma_1 - \Gamma_2\|_X \\ &\leq \frac{1}{8} \|\Gamma_1 - \Gamma_2\|_X, \end{aligned}$$

$$\begin{aligned}
& \|\Phi(\Gamma_1)^E - \Phi(\Gamma_2)^E\|_{X^E} + \|\Phi(\Gamma_1)^B - \Phi(\Gamma_2)^B\|_{X^B} \\
& \lesssim T^{\frac{5}{4}} \|(u_- - u_+) \cdot \nabla \bar{j}_1 + \bar{u}_2 \cdot \nabla(j_1 - j_2)\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \\
& \quad + T^{\frac{5}{4}} \|(j_1 - j_2) \cdot \nabla \bar{u}_1 + \bar{j}_2 \cdot \nabla(u_- - u_+)\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \\
& \quad + T^{\frac{5}{4}} \|(u_- - u_+) \cdot \nabla \bar{B}_1 + \bar{u}_2 \cdot \nabla(B_1 - B_2)\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \\
& \quad + T^2 \|E_1 - E_2\|_{L_T^\infty L^2} \\
& \lesssim (12rT^{\frac{5}{4}} + T^2) \|\Gamma_1 - \Gamma_2\|_X \\
& \leq \frac{1}{4} \|\Gamma_1 - \Gamma_2\|_X.
\end{aligned}$$

Finally, together with the above estimates, we have

$$\|\Phi(\Gamma_1) - \Phi(\Gamma_2)\|_X \leq \frac{1}{2} \|\Gamma_1 - \Gamma_2\|_X.$$

The proof is complete.

Remark 4.1. • We emphasize that in order to estimate $j \in L^1(L^2)$, we needed $j_0 \in L^2$. This explains why the initial condition j_0 should be in the nonhomogeneous space $H^{\frac{1}{2}}$ while for u_0 , we only need $u_0 \in \dot{H}^{\frac{1}{2}}$.
• We can also show existence of global-in-time unique solution of (3) for large initial velocities (but small initial electromagnetic field) provided that α is large enough. This is due to parabolic regularization in Lemma 2.1 (see also Remark 2.1), energy inequality in Lemma 2.4, global existence for small initial data (Theorem 1.3) and the interpolation inequality between $\dot{H}^{1/2}$, L^2 and $\dot{H}^1$.

5. GLOBAL EXISTENCE FOR SMALL INITIAL DATA

We shall give a proof of Theorem 1.3. First, we would like to introduce a truncated NSM system as in [8]. Let

$$\begin{aligned}
\gamma &:= (v_-, v_+, E, B)^T \\
L\gamma &:= (\Delta v_- - E, \Delta v_+ + E, \nabla \times B - v_+ + v_-, -\nabla \times E)^T \\
R\gamma &:= (-v_+ + v_-, v_+ - v_-, 0, 0)^T \\
N\gamma &:= (\mathcal{P}(-v_- \cdot \nabla v_-) - \mathcal{P}(v_- \times B), \mathcal{P}(-v_+ \cdot \nabla v_+) + \mathcal{P}(v_+ \times B), 0, 0)^T
\end{aligned}$$

and $\Psi_M(s)$ be a cut-off function defined by

$$\Psi_M(s) := \begin{cases} 1 & 0 \leq s \leq M/2 \\ 2 - \frac{2}{M}s & M/2 < s \leq M \\ 0 & M < s. \end{cases}$$

Set

$$\Psi_M(\gamma) := \Psi_M(\|u_-\|_{\dot{H}^1}) \Psi_M(\|u_+\|_{\dot{H}^1}) \Psi_M(\|B\|_{L^2}).$$

Now define a truncated operator S_M by:

$$S_M \gamma := L\gamma + \Psi_M(\gamma) N\gamma + R\gamma$$

and the truncated NSM system is

$$\partial_t \gamma = S_M \gamma.$$

We give a few properties of this truncated NSM system. Let L_σ^2 denote the solenoidal L^2 space, i.e.,

$$L_\sigma^2 = \{v \in L^2(\mathbb{R}^3) \mid \operatorname{div} v = 0 \text{ in } \mathbb{R}^3\}.$$

We consider γ in the Hilbert space $H := L_\sigma^2 \times L_\sigma^2 \times L^2 \times L^2$. The operator S_M is assumed to have its domain

$$D = (H^2 \cap L_\sigma^2) \times (H^2 \cap L_\sigma^2) \times \{E \in L^2 \mid \nabla \times E \in L^2\} \times \{B \in L^2 \mid \nabla \times B \in L^2\},$$

which is dense in H .

Lemma 5.1. The operator S_M generates a strongly continuous nonlinear semigroup $T_M(t)$ in H , in particular, $\gamma(t) = T_M(t)\gamma_0$, $\gamma_0 \in H$ is in $C([0, T], H)$. Moreover, if $\gamma_0 \in D$, then $\gamma(t)$ is absolutely continuous in $[\delta, T]$ for all $\delta, T > 0$ ($\delta < T$) with values in H and that it solves

$$\partial_t \gamma = S_M \gamma \quad (\text{a.e. } t \geq 0), \quad \gamma(0) = \gamma_0.$$

Moreover, $\nabla v_-(t)$ and $\nabla v_+(t)$ are continuous from $[0, \infty)$ to L^2 .

The proof is essentially given in [8], where the problem is considered in a bounded domain. It elaborates the method introduced by [15] for the Navier-Stokes equations. We give a sketch of the proof for the reader's convenience and completeness.

The first conclusion of Lemma 5.1 is a generation result. It follows from a general generation theorem due to Kōmura [11] (see also [4]) once we prove that $-(S_M - \omega I)$ is a m -accretive operator in H with some $\omega \geq 0$, where I denotes the identity operator.

To show that $-(S_M - \omega I)$ is m -accretive, we have to prove that

(i) (monotonicity)

$$\langle (S_M - \omega I)\gamma - (S_M - \omega I)\gamma', \gamma - \gamma' \rangle \leq 0$$

for all $\gamma, \gamma' \in D$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in H .

(ii) (solvability) The range of $I - \lambda(S_M - \omega I)$ equals H for some $\lambda > 0$.

The proof of the monotonicity is the same as in the proof of [8, Lemma 2.1].

We first observe that

$$\begin{aligned} & \langle \psi_M(\gamma)N\gamma - \psi_M(\gamma')N\gamma', \gamma - \gamma' \rangle \\ & \leq \|v_+ - v'_+\|_{\dot{H}^1}^2 + \|v_- - v'_-\|_{\dot{H}^1}^2 + \omega (\|B - B'\|_{L^2}^2 + \|v_+ - v'_+\|_{L^2}^2 + \|v_- - v'_-\|_{L^2}^2) \end{aligned}$$

with $\omega = cM^4$, where c is a numerical constant. Here $\gamma = (v_+, v_-, B, E)$ and $\gamma' = (v'_+, v'_-, B', E')$. This can be proved as in [15], where the Navier-Stokes system is first treated by the theory of nonlinear semigroup [15, Lemma 2.1] at least to handle $\mathcal{P}(v_\pm \cdot \nabla v_\pm)$. To handle B terms we use

$$\left\| (-\Delta)^{-1/4} \mathcal{P}(v_\pm \times B) \right\|_{L^2} \leq C \|v_\pm\|_{\dot{H}^1} \|B\|_{L^2}.$$

Since $\langle R\gamma, \gamma \rangle \leq 0$, $\langle L\gamma, \gamma \rangle = \|v_+\|_{\dot{H}^1}^2 + \|v_-\|_{\dot{H}^1}^2$, the above estimate for $\langle \psi_M(\gamma)N\gamma - \psi_M(\gamma')N\gamma', \gamma - \gamma' \rangle$ yields the desired estimate (i).

The proof of (ii) is more involved not a direct application of [8, Lemma 2.2, Step 2] since our problem is in $\mathbb{R}^3$ not in a bounded domain. It suffices to solve

$$-(L + \mu)\gamma + \psi_M(\gamma)N\gamma + R\gamma = f,$$

for general $f \in H$ and $\mu > 0$. We shall first solve this equation in a ball B_R of radius R centered at the origin. Arguing as in [15, Lemma 2.2] by a fixed point argument based on the Leray-Schauder theory (where we need compactness), we find a solution γ_R

$$V_R = \{\gamma = (v_+, v_-, B, E) \mid v_{\pm} \in H^1(B_R), v_{\pm} = 0 \text{ on } \partial B_R, \operatorname{div} v_{\pm} = 0 \text{ in } B_R\}$$

which is a subspace of $H_R = L^2_{\sigma}(B_R) \times L^2_{\sigma}(B_R) \times L^2(B_R) \times L^2(B_R)$.

The desired solution γ is obtained as a limit of γ_R as $R \rightarrow \infty$. In fact, as in the proof of [15, Lemma 2.2, Step 2] we have a uniform $W^{2,3/2}$ estimates for γ . Thus the limit γ solves the integral equation

$$-\gamma + (L + \mu)^{-1}(\psi_M(\gamma)N\gamma + R\gamma) = f.$$

We thus conclude that $-(S_M - \omega I)$ satisfies the solvability (ii).

It remains to prove the continuity $\nabla v_{\pm}$ from $[0, \infty)$ to L^2 . The proof is essentially the same as that of [15, Lemma 3.2]. We first prove that $A^{1/2}v_{\pm}$ in $L^{\infty}(0, T, L^2)$ for all $T > 0$ where $A = -\mathcal{P}\Delta$ is the Stokes operator. This is easy since

$$\|v_+\|_{\dot{H}^1}^2 + \|v_-\|_{\dot{H}^1}^2 = \|A^{1/2}v_+\|_{L^2}^2 + \|A^{1/2}v_-\|_{L^2}^2 \leq e^{\omega t} \|S_m \gamma_0\|_H \|\gamma_0\|_H.$$

We next prove that $Av_{\pm} \in L^{\infty}(0, T, L^2)$ for all $T > 0$ which is more involved. We have to control nonlinear terms $F_{\pm}$. We observe that $F_{\pm} \in L^{\infty}(0, T, L^2)$ to conclude this result as in the proof of [15, Lemma 3.2] since $A^{1/2}v_{\pm}$ is in $L^{\infty}(0, T, L^2)$. The continuity of $\nabla v_{\pm}$ follows from a simple interpolation

$$\begin{aligned} \frac{1}{2} \|v_{\pm}(t) - v_{\pm}(s)\|_{\dot{H}^1}^2 &= \int_s^t \left\langle v_{\pm}, \frac{dv_{\pm}}{dt} \right\rangle_{\dot{H}^1} = \int_s^t \left\langle Av_{\pm}, \frac{dv_{\pm}}{dt} \right\rangle_{L^2} ds \\ &\leq \|Av_{\pm}\|_{L^2(s,t,L^2)} \cdot \left\| \frac{dv_{\pm}}{dt} \right\|_{L^2(s,t,L^2)} \end{aligned}$$

since $dv_{\pm}/dt \in L^{\infty}(0, T, L^2)$.

Lemma 5.2 (energy inequality). Let $\gamma(t) = T_M(t)\gamma_0$, then

$$\|\gamma(t)\|_{L^2}^2 + 2 \left(\|v_-\|_{L_t^2 \dot{H}^1}^2 + \|v_+\|_{L_t^2 \dot{H}^1}^2 \right) \leq \|\gamma(0)\|_{L^2}^2.$$

Proof of Theorem 1.3. Let M be an arbitrary positive constant, denote by

$$\gamma_0 = (v_-(0), v_+(0), E(0), B(0))^T$$

and set

$$C_0 := \|\gamma_0\|_{H^{\frac{1}{2}} \times H^{\frac{1}{2}} \times L^2 \times L^2}.$$

Define the truncated solution $\gamma(t) = T_M(t)\gamma_0 = (v_-(t), v_+(t), E(t), B(t))$, where v stands for v_- or v_+ . Let $M' = M/2$ and define

$$T := \sup\{t \geq 0; \|v(s)\|_{\dot{H}^1} \leq M' \quad 0 \leq \forall s \leq t\}.$$

Our goal is to proof if initial data is small, this T goes to ∞ , so that $\|v\|_{\dot{H}^1(t)} \leq M'$ always holds. By energy estimate (Lemma 5.2), $\|B\|_{L^2} \leq M'$. Then the truncated system becomes the original system NSM(2) according to it's definition. Thus the solution of truncated system is also a solution of NSM(2). Applying Lemma 5.1, we will finish the proof.

Without loss of generality we assume that T is finite and define

$$S := \sup\{t > T; \|v(s)\|_{\dot{H}^1} > M' \quad T < \forall s \leq t\}.$$

Since $\nabla v(s)$ is continuous (Lemma 5.1), $\|v(s)\|_{\dot{H}^1}$ is continuous too. Then the above set is non-empty and therefore, S exists¹ and $S > T$. So that $S - T$ can not be arbitrarily small.

By the definition of S , we have

$$M' < \|u(S)\|_{\dot{H}^1}.$$

Integrating from T to S

$$\begin{aligned} (S - T)M' &< \int_T^S \|u(t)\|_{\dot{H}^1} dt \\ &\leq \int_0^S \|u(t)\|_{\dot{H}^1} dt \\ (21) \quad &\leq \int_0^S \frac{1}{2} \|u\|_{\dot{H}^{\frac{1}{2}}} + \frac{1}{2} \|u\|_{\dot{H}^{\frac{3}{2}}} dt \\ &\leq \frac{1}{2} S \|u\|_{L_S^\infty \dot{H}^{\frac{1}{2}}} + \frac{1}{2} \sqrt{S} \|u\|_{L_S^2 \dot{H}^{\frac{3}{2}}} \\ &\leq \frac{1}{2} S \|u\|_{L_S^\infty H^{\frac{1}{2}}} + \frac{1}{2} \sqrt{S} \|u\|_{L_S^2 H^{\frac{3}{2}}}. \end{aligned}$$

Next step, we estimate $\|u\|_{L_S^\infty H^{\frac{1}{2}} \cap L_S^2 H^{\frac{3}{2}}}$. Because the truncated operator only affect the nonlinear term of the original NSM and $|\Psi_M(\gamma)| \leq 1$, $\gamma(t) := T_M(t)\gamma_0$ still satisfies all the estimates we did before. By Lemma 2.1, Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} \|u\|_{L_S^\infty H^{\frac{1}{2}} \cap L_S^2 H^{\frac{3}{2}}} &\leq C_S (\|u(0)\|_{H^{\frac{1}{2}}} + \|E\|_{L_S^{\frac{4}{3}} L^2} + \|R\|_{L_S^{\frac{4}{3}} L^2} \\ &\quad + \|u \cdot \nabla u\|_{L_S^2 H^{-\frac{1}{2}}} + \|u \times B\|_{L_S^2 H^{-\frac{1}{2}}}) \\ &\leq C_S C_0 (1 + C_0 + S^{\frac{3}{4}} + S^{\frac{1}{4}}) + C_S \|u\|_{L_S^\infty H^{\frac{1}{2}}} \|u\|_{L_S^2 H^{\frac{3}{2}}} \\ &\leq C_S C_0 (1 + C_0 + S^{\frac{3}{4}} + S^{\frac{1}{4}}) + C_S \|u\|_{L_S^\infty H^{\frac{1}{2}} \cap L_S^2 H^{\frac{3}{2}}}^2, \end{aligned}$$

where $C_S = C \max\{1, S\}$. We rewrite the inequality above in the following form

$$\|u\|_{L_S^\infty H^{\frac{1}{2}} \cap L_S^2 H^{\frac{3}{2}}} \left(1 - C_S \|u\|_{L_S^\infty H^{\frac{1}{2}} \cap L_S^2 H^{\frac{3}{2}}} \right) \leq C_S C_0 (1 + C_0 + S^{\frac{3}{4}} + S^{\frac{1}{4}})$$

¹Of course, S can be ∞ , for this case, we redefine S as an arbitrary number between T and ∞ .

which means if we choose C_0 small, then $\|u\|_{L_S^\infty H^{\frac{1}{2}} \cap L_S^2 H^{\frac{3}{2}}}$ should satisfies

$$(22) \quad 0 \leq \|u\|_{L_S^\infty H^{\frac{1}{2}} \cap L_S^2 H^{\frac{3}{2}}} \leq \frac{1 - \sqrt{1 - 4C_S^2 C_0 \left(1 + C_0 + S^{\frac{3}{4}} + S^{\frac{1}{4}}\right)}}{2C_S}$$

or

$$(23) \quad \|u\|_{L_S^\infty H^{\frac{1}{2}} \cap L_S^2 H^{\frac{3}{2}}} \geq \frac{1 + \sqrt{1 - 4C_S^2 C_0 \left(1 + C_0 + S^{\frac{3}{4}} + S^{\frac{1}{4}}\right)}}{2C_S}.$$

Because of the continuity of $\gamma(t)$ (Lemma 5.1) and small initial data, we know the case of (23) is impossible. So, (22) is the estimate of u . Substituting (22) into (21), we get

$$(S - T)M' < \frac{S + \sqrt{S}}{4C} \left(1 - \sqrt{1 - 4C_S^2 C_0 \left(1 + C_0 + S^{\frac{3}{4}} + S^{\frac{1}{4}}\right)}\right),$$

which is equivalent to (if C_0 is small enough):

$$\begin{aligned} S - T &< \frac{S + \sqrt{S}}{M'} \cdot \frac{C_S C_0 \left(1 + C_0 + S^{\frac{3}{4}} + S^{\frac{1}{4}}\right)}{1 + \sqrt{1 - 4C_S^2 C_0 \left(1 + C_0 + S^{\frac{3}{4}} + S^{\frac{1}{4}}\right)}} \\ &\leq \frac{S + \sqrt{S}}{M'} \cdot C_S C_0 \left(1 + C_0 + S^{\frac{3}{4}} + S^{\frac{1}{4}}\right). \end{aligned}$$

Finally, we choose C_0 small enough, and $S - T$ can be arbitrarily small which is a contradiction. So that T goes to ∞ . Thus, $\gamma(t)$ globally and uniquely solves the NSM (2) provided the initial data small enough. Furthermore, if time T is fixed, according to (22), we can set C_0 small so that $u(t)$ is bounded in $L_T^\infty H^{\frac{1}{2}} \cap L_T^2 H^{\frac{3}{2}}$.

6. THE PHYSICAL MODEL

In this section we expect to extend the wellposedness result to the physical model.

6.1. Global existence. Rewriting back the original system with all physical parameters, we have

$$(24) \quad \begin{cases} m_- n \partial_t v_- = \nu_- \Delta v_- - m_- n v_- \cdot \nabla v_- - en(E + v_- \times B) - R - \nabla p_- \\ m_+ n \partial_t v_+ = \nu_+ \Delta v_+ - m_+ n v_+ \cdot \nabla v_+ + eZn(E + v_+ \times B) + R - \nabla p_+ \\ \partial_t E = \frac{1}{\varepsilon_0 \mu_0} \nabla \times B - \frac{ne}{\varepsilon_0} (Zv_+ - v_-) \\ \partial_t B = -\nabla \times E \\ R := -\alpha(v_+ - v_-) \\ \operatorname{div} v_- = \operatorname{div} v_+ = \operatorname{div} B = \operatorname{div} E = 0. \end{cases}$$

We can also define the corresponding truncated system in the same way as before we get.

$$(25) \quad \begin{cases} nm_- \partial_t v_- = \nu_- \Delta v_- - nm_- \Psi(\gamma) v_- \cdot \nabla v_- - en(E + \Psi(\gamma) v_- \times B) - R - \nabla p_- \\ nm_+ \partial_t v_+ = \nu_+ \Delta v_+ - nm_+ \Psi(\gamma) v_+ \cdot \nabla v_+ + eZn(E + \Psi(\gamma) v_+ \times B) + R - \nabla p_+ \\ \partial_t E = \frac{1}{\varepsilon_0 \mu_0} \nabla \times B - \frac{ne}{\varepsilon_0} (Zv_+ - v_-) \\ \partial_t B = -\nabla \times E \\ R := -\alpha(v_+ - v_-) \\ \operatorname{div} v_- = \operatorname{div} v_+ = \operatorname{div} B = \operatorname{div} E = 0, \end{cases}$$

where γ and Ψ are defined in the previous section. In order to get the global existence, the key is to ensure that we still have the energy estimates for (24) and (25) which are similar to Lemma 2.4 and Lemma 5.2. The differences of other estimates like in Lemma 2.1 are just a matter of constant. To get the energy estimate for (1), multiply $\frac{v_-}{\varepsilon_0}$, $\frac{v_+}{\varepsilon_0}$, E , $\frac{B}{\varepsilon_0 \mu_0}$ to the first four equations of (1) respectively, integrate over space and add them together. With the divergence free condition, we get

$$\begin{aligned} \frac{m_- n}{2\varepsilon_0} \frac{d}{dt} \|v_-\|_{L^2}^2 + \frac{m_+ n}{2\varepsilon_0} \frac{d}{dt} \|v_+\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|E\|_{L^2}^2 + \frac{1}{2\varepsilon_0 \mu_0} \frac{d}{dt} \|B\|_{L^2}^2 \\ + \frac{\nu_-}{\varepsilon_0} \|\nabla v_-\|_{L^2}^2 + \frac{\nu_+}{\varepsilon_0} \|\nabla v_+\|_{L^2}^2 + \frac{\alpha}{\varepsilon_0} \|v_- - v_+\|_{L^2}^2 = 0. \end{aligned}$$

Integrating in time from 0 to t , the above identity becomes

$$(26) \quad \begin{aligned} \frac{nm_-}{2\varepsilon_0} \|v_-\|_{L^2}^2 + \frac{nm_+}{2\varepsilon_0} \|v_+\|_{L^2}^2 + \frac{1}{2} \|E\|_{L^2}^2 + \frac{1}{2\varepsilon_0 \mu_0} \|B\|_{L^2}^2 \\ + \frac{\nu_-}{\varepsilon_0} \|v_-\|_{L_t^2 \dot{H}^1}^2 + \frac{\nu_+}{\varepsilon_0} \|v_+\|_{L_t^2 \dot{H}^1}^2 + \frac{\alpha}{\varepsilon_0} \|v_- - v_+\|_{L_t^2 L^2}^2 \\ = \\ \frac{nm_-}{2\varepsilon_0} \|v_-(0)\|_{L^2}^2 + \frac{nm_+}{2\varepsilon_0} \|v_+(0)\|_{L^2}^2 + \frac{1}{2} \|E(0)\|_{L^2}^2 + \frac{1}{2\varepsilon_0 \mu_0} \|B(0)\|_{L^2}^2, \end{aligned}$$

which, after setting

$$\lambda_1 = \min\left\{\frac{nm_-}{2\varepsilon_0}, \frac{nm_+}{2\varepsilon_0}, \frac{1}{2}, \frac{1}{2\varepsilon_0 \mu_0}, \frac{\nu_-}{\varepsilon_0}, \frac{\nu_+}{\varepsilon_0}, \frac{\alpha}{\varepsilon_0}\right\}$$

and

$$\lambda_2 = \max\left\{\frac{nm_-}{2\varepsilon_0}, \frac{nm_+}{2\varepsilon_0}, \frac{1}{2}, \frac{1}{2\varepsilon_0 \mu_0}\right\}$$

gives the following energy type estimate can then be written as

$$(27) \quad \begin{aligned} \|v_-\|_{L^2}^2 + \|v_+\|_{L^2}^2 + \|E\|_{L^2}^2 + \|B\|_{L^2}^2 \\ + \|v_-\|_{L_t^2 \dot{H}^1}^2 + \|v_+\|_{L_t^2 \dot{H}^1}^2 + \|v_- - v_+\|_{L_t^2 L^2}^2 \leq \\ \frac{\lambda_2}{\lambda_1} (\|v_-(0)\|_{L^2}^2 + \|v_+(0)\|_{L^2}^2 + \|E(0)\|_{L^2}^2 + \|B(0)\|_{L^2}^2). \end{aligned}$$

This is our desired result. For the truncated system(25), we can do the same thing and get exactly the same result as (27) since the truncated function Ψ is only a function of time and satisfy $0 \leq \Psi \leq 1$.

6.2. Local existence. In the physical model, since the coefficients are no longer the same, we won't have the nice cancellation. However, we can still apply the fixed point argument and get theorem 1.4.

Proof of Theorem 1.4. In the sequel and since there is no cancellation of terms anymore, all constants may depend upon all physical parameters but for simplicity we assume all constants are 1. The proof is similar to the proof of Theorem 1.2.

STEP 1: À PRIORI ESTIMATE.

By Lemma 2.1, for any (v_-, v_+, E, B) solution of (1), we have

$$(28) \quad \|v\|_{L_T^\infty \dot{H}^{\frac{1}{2}} \cap L_T^2 \dot{H}^{\frac{3}{2}}} \lesssim \|v(0)\|_{\dot{H}^{\frac{1}{2}}} + \|v \times B + R\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} + \|E + v \cdot \nabla v\|_{\tilde{L}_T^{\frac{4}{3}} L^2}$$

where v stands for v_- or v_+ . And again, Lemma 2.1 gives

$$(29) \quad \begin{aligned} \|v\|_{L_T^1 L^2} &\leq T \|v\|_{L_T^\infty L^2} \\ &\leq T \|v(0)\|_{L^2} + T^{\frac{5}{4}} \|v \cdot \nabla v + v \times B + R\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} + T \|E\|_{\tilde{L}_T^1 L^2}. \end{aligned}$$

By lemma 2.2, we have:

$$(30) \quad \|E\|_{L_T^\infty L^2} + \|B\|_{L_T^\infty L^2} \leq \|E(0)\|_{L^2} + \|B(0)\|_{L^2} + \|v_-\|_{L_T^1 L^2} + \|v_+\|_{L_T^1 L^2}.$$

Together with (29) and (30), we obtain the *à priori estimate* for (E, B) :

$$(31) \quad \begin{aligned} \|E\|_{L_T^\infty L^2} + \|B\|_{L_T^\infty L^2} &\leq \|E(0)\|_{L^2} + \|B(0)\|_{L^2} + T \|v_-(0)\|_{L^2} + T \|v_+(0)\|_{L^2} \\ &\quad + T^{\frac{5}{4}} \|v_- \cdot \nabla v_- + v_- \times B + R\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \\ &\quad + T^{\frac{5}{4}} \|v_+ \cdot \nabla v_+ + v_+ \times B + R\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \\ &\quad + 2T \|E\|_{\tilde{L}_T^1 L^2}. \end{aligned}$$

STEP 2: CONTRACTION ARGUMENT.

The space setting and notations are similar to Step 2 of the proof of theorem 1.2, just replacing u, j by v_-, v_+ respectively. Besides, the operator A and function $f(\Gamma)$ are changed to

$$\begin{aligned} A &= \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon_0 \mu_0} \nabla \times \\ 0 & 0 & -\nabla \times & 0 \end{pmatrix} \\ f(\Gamma) &= \begin{pmatrix} \mathcal{P}(-v_- \cdot \nabla v_- - \frac{e}{m_-}(E + v_- \times B) - \frac{R}{nm_-}) \\ \mathcal{P}(-v_+ \cdot \nabla v_+ + \frac{eZ}{m_+}(E + v_- \times B) + \frac{R}{nm_+}) \\ -\frac{ne}{\varepsilon_0}(Zv_+ - v_-) \\ 0 \end{pmatrix}. \end{aligned}$$

The map Φ stays the same. And $e^{tA}\Gamma_0$ is now controlled by

$$\|e^{tA}\Gamma_0\| \leq C\|\Gamma_0\|_{H^{\frac{1}{2}} \times H^{\frac{1}{2}} \times L^2 \times L^2}$$

Where C is the universal constant. Moreover, setting $r = C\|\Gamma_0\|_{H^{\frac{1}{2}} \times H^{\frac{1}{2}} \times L^2 \times L^2}$ and denoting by B_r the ball of space X centered at 0 with radius r . Our goal is to proof if T is small enough then $\Phi(B_r) \subset B_r$.

Assume $\Gamma \in B_r$ and set $\bar{\Gamma} := e^{tA}\Gamma_0 - \Gamma$ (also define $\bar{v}_-, \bar{v}_+, \bar{E}, \bar{B}, \bar{R}$ in the same manner). Then by lemma 2.1 and lemma 2.3

$$\begin{aligned} \|\Phi(\Gamma)\|_{X^v} &\lesssim \|\bar{v} \times \bar{B} + \bar{v} \cdot \nabla \bar{v}\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} + \|\bar{E}\|_{\tilde{L}_T^{\frac{4}{3}} L^2} + \|\bar{R}\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \\ &\leq C \left(\|\bar{v}\|_{L_T^2 \dot{H}^1} + \|\bar{v}\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + T^{\frac{3}{4}} + 2T^{\frac{1}{2}} \right) \|\bar{\Gamma}\|_X \\ &\leq 2C \left(\|\bar{v}\|_{L_T^2 \dot{H}^1} + \|\bar{v}\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + T^{\frac{3}{4}} + 2T^{\frac{1}{2}} \right) r. \end{aligned}$$

Thus we could choose T small such that

$$2C \left(\|\bar{v}\|_{L_T^2 \dot{H}^1} + \|\bar{v}\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + T^{\frac{3}{4}} + 2T^{\frac{1}{2}} \right) < \frac{1}{4}.$$

So

$$\|\Phi(\Gamma)\|_{X^{v-}} < \frac{1}{4}r, \quad \|\Phi(\Gamma)\|_{X^{v+}} < \frac{1}{4}r.$$

Similarly, we could also get

$$\|\Phi(\Gamma)\|_X^E + \|\Phi(\Gamma)\|_X^E < \frac{1}{2}r.$$

Thus

$$\|\Phi(\Gamma)\|_X < r.$$

And $\Phi(\Gamma) \in B_r$.

STEP 3: CONTRACTION.

The notations and settings are the same as that in the proof of theorem 1.2 (of course, replacing u, j by v_-, v_+). Let v be v_- or v_+ , $\Gamma_1, \Gamma_2 \in B_r$ and $\bar{\Gamma}_i = e^{tA}\Gamma_0 - \Gamma_i$, $i = 1, 2$. By the *à priori estimate* (28):

$$\begin{aligned} \|\Phi(\Gamma_1) - \Phi(\Gamma_2)\|_{X^v} &\lesssim \|(\bar{v}_1 - \bar{v}_2) \times \bar{B}_2 + \bar{v}_1 \times (\bar{B}_1 - \bar{B}_2)\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \\ (32) \quad &+ \|(\bar{v}_1 - \bar{v}_2) \cdot \nabla \bar{v}_2 + \bar{v}_1 \cdot \nabla (\bar{v}_1 - \bar{v}_2)\|_{\tilde{L}_T^{\frac{4}{3}} L^2} \\ &\quad + \|\bar{E}_1 - \bar{E}_2\|_{\tilde{L}_T^{\frac{4}{3}} L^2} + \|\bar{R}_1 - \bar{R}_2\|_{L_T^2 \dot{H}^{-\frac{1}{2}}}. \end{aligned}$$

The fact that

$$\begin{aligned} \|\bar{R}_1 - \bar{R}_2\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} &\leq T^{\frac{1}{2}} \|v_{+,1} - v_{+,2} + v_{-,2} - v_{-,1}\|_{L_T^\infty \dot{H}^{\frac{1}{2}}} \\ (33) \quad &\leq 2T^{\frac{1}{2}} \|\Gamma_1 - \Gamma_2\|_X \end{aligned}$$

together with (9) and (11) in lemma 2.3, (32) becomes

$$\begin{aligned}
\|\Phi(\Gamma_1) - \Phi(\Gamma_2)\|_{X^v} &\lesssim T^{\frac{1}{4}} \left(\|\bar{B}_2\|_{L_T^\infty L^2} + \|\bar{v}_1\|_{L_T^\infty \dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|\bar{v}_1\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) \|\Gamma_1 - \Gamma_2\|_X \\
&\quad + \left(\|\bar{v}_2\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + \|\bar{v}_1\|_{L_T^\infty \dot{H}^{\frac{1}{2}}}^{\frac{1}{2}} \|\bar{v}_1\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) \|\Gamma_1 - \Gamma_2\|_X \\
&\quad + (T^{\frac{3}{4}} + 2T^{\frac{1}{2}}) \|\Gamma_1 - \Gamma_2\|_X \\
&\lesssim T^{\frac{1}{4}} (4r) \|\Gamma_1 - \Gamma_2\|_X \\
&\quad + \left(\|\bar{v}_2\|_{L_T^2 \dot{H}^{\frac{3}{2}}} + (2r)^{\frac{1}{2}} \|\bar{v}_1\|_{L_T^2 \dot{H}^{\frac{3}{2}}}^{\frac{1}{2}} \right) \|\Gamma_1 - \Gamma_2\|_X \\
&\quad + (T^{\frac{3}{4}} + 2T^{\frac{1}{2}}) \|\Gamma_1 - \Gamma_2\|_X.
\end{aligned}$$

We choose T small enough so that

$$\|\Phi(\Gamma_1) - \Phi(\Gamma_2)\|_{X^v} \leq \frac{1}{8} \|\Gamma_1 - \Gamma_2\|_X.$$

Similarly, we can estimate $\|\Gamma_1 - \Gamma_2\|_{X^E} + \|\Gamma_1 - \Gamma_2\|_{X^B}$ when T is small:

$$\begin{aligned}
\|\Gamma_1 - \Gamma_2\|_{X^E} + \|\Gamma_1 - \Gamma_2\|_{X^B} &\leq (16rT^{\frac{5}{4}} + 2T^{\frac{7}{4}} + 2T^2) \|\Gamma_1 - \Gamma_2\|_X \\
&\leq \frac{1}{4} \|\Gamma_1 - \Gamma_2\|_X.
\end{aligned}$$

Finally, together with the above estimates

$$\|\Phi(\Gamma_1) - \Phi(\Gamma_2)\|_X \leq \frac{1}{2} \|\Gamma_1 - \Gamma_2\|_X.$$

This complete the proof.

Remark 6.1. In this proof, for simplicity we set all parameters to 1. One should note that the small time T is dependent of α which appears in R . Because if we recover α , (33) becomes $\|\bar{R}_1 - \bar{R}_2\|_{L_T^2 \dot{H}^{-\frac{1}{2}}} \leq 2\alpha T^{\frac{1}{2}} \|\Gamma_1 - \Gamma_2\|_X$. However, if $\alpha \leq C_0$, where C_0 is a positive constant, then T will be independent of α .

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