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ON L^∞-BMO ESTIMATES FOR DERIVATIVES OF THE STOKES SEMIGROUP

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Abstract. We consider the Stokes equations in a class of domains that we will call admissible domains including bounded domains, the half space and exterior domains. We will prove new L^∞ estimates for derivatives of velocity and pressure. The estimates will be given in terms of a BMO-type norm of the initial data.

1. Introduction

We consider the Stokes equations

$$\begin{aligned} v_t - \Delta v + \nabla q &= 0 && \text{in } (0,T)\times\Omega \\ \operatorname{div} v &= 0 && \text{in } (0,T)\times\Omega \\ v &= 0 && \text{on } (0,T)\times\partial\Omega \\ v(0) &= v_0 \end{aligned} \tag{1.1}$$

in a uniformly C^3-domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). Our goal is to study the regularizing effect of the solution semigroup defined by $S(t)v_0 = v(\cdot,t)$ and establish $L^\infty - BMO$ estimates for its derivatives.

The regularizing effect in the scale of the L^r-norm is well understood. Let $L^r_\sigma(\Omega)$ ($1 < r < \infty$) be the L^r-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all solenoidal vector fields with compact support in Ω. We know for example that

$$t^{1/2}\|\nabla S(t)v_0\|_r \leq c\|v_0\|_r \quad \text{and} \quad t\|\frac{\mathrm{d}}{\mathrm{d}t}S(t)v_0\|_r \leq c\|v_0\|_r$$

for all $v_0 \in L^r_\sigma(\Omega)$ and $t \in (0,T_0)$ with c and $T_0 > 0$ independent of v_0 for various type of domains; see e.g. [Gig81] for the case of a bounded domain and [GHHS12] for general domains admitting the L^r-Helmholtz decomposition.

We will establish estimates of the derivatives of the L^∞-norm of the solution $S(t)v_0$ by a BMO-type norm of v_0, $\|v_0\|_{BMO^{\mu,\nu}_b}$ consisting of the seminorm $[v_0]^\mu_{BMO}$ measuring the mean oscillation in a ball of radius less than μ and the seminorm $[v_0]^\nu_b$ measuring the mean absolute value in a ball of radius less than ν near the boundary. These estimates are for example of the form

$$t^{1/2}\|\nabla S(t)v_0\|_\infty \leq c\|v_0\|_{BMO^{\mu,\nu}_b} \quad \text{and} \quad t\|\frac{\mathrm{d}}{\mathrm{d}t}S(t)v_0\|_\infty \leq c\|v_0\|_{BMO^{\mu,\nu}_b}$$

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for all $t \in (0, T_0)$ and all $v_0 \in VMO^{\mu,\nu}_{0,b,\sigma}(\Omega)$, which is the closure of $C^\infty_{c,\sigma}(\Omega)$ under the $BMO^{\mu,\nu}_b$-norm. See Section 2 for a precise definition of the $BMO^{\mu,\nu}_b$-norm.

In order to motivate our results we first have a look at the heat equation in $\mathbb{R}^n$. A. Carpio ([Car96]) as well as the second author, S. Matsui and Y. Shimizu ([GMS99]) established $\mathcal{H}^1$-L^1 estimates from which one can deduce that the heat semigroup $G_t * \cdot$ has the regularizing effect

$$t^{1/2}\|\nabla G_t * v_0\|_\infty \le C[v_0]_{BMO} \quad (t > 0). \tag{1.2}$$

Here G_t denotes the Gauss kernel and $*$ denotes the convolution in $\mathbb{R}^n$. We will give a proof of this fact in the appendix. This regularizing effect cannot be generalized to a domain with nonempty boundary under the Dirichlet boundary condition since the solution will not be spatially constant even if v_0 is constant. Furthermore $\|G_t * v_0\|_\infty$ is not bounded by the seminorm $[v_0]_{BMO}$.

Similar results on regularizing effects for (1.1) have been obtained for L^∞ instead of BMO by K. Abe and the second author for a large class of domains called admissible domains. Let

$$\tilde{N}(v,q)(x,t) := t^{1/2}|\nabla v(x,t)| + t|\nabla^2 v(x,t)| + t|v_t(x,t)| + t|\nabla p(x,t)|,$$

$$N(v,q)(x,t) := |v(x,t)| + \tilde{N}(v,q)(x,t).$$

They proved that there exist positive constants C and T_0 depending only on Ω such that

$$\sup_{0<t<T_0} \|N(S(t)v_0, q)(\cdot,t)\|_\infty \le C\|v_0\|_\infty$$

holds for the solution operator S of (1.1), when $v_0 \in C_{0,\sigma}(\Omega)$, the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$ ([AG13]). In particular, S is an analytic semigroup on $C_{0,\sigma}(\Omega)$. Note that there is an alternative proof of analyticity in $C_{0,\sigma}(\Omega)$ by resolvent estimates ([AGH15]) showing that the angle of analyticity is $\pi/2$. For a bounded domain ([AG13]) and an exterior domain ([AG14],[AGH15]) analyticity can be extended to the bigger function space $L^\infty_\sigma(\Omega)$.

If we replace $\|v_0\|_\infty$ by $\|v_0\|_{BMO^{\mu,\nu}_b}$, we cannot expect $\sup_{0<t<T_0}\|v(t)\|_\infty$ to be bounded by the BMO-type norm. Therefore we have to replace $N(v,q)$ by $\tilde{N}(v,q)$. We are then able to obtain a bound for $\tilde{N}(v,q)$ in terms of $\|v_0\|_{BMO^{\mu,\nu}_b}$. Our main theorem reads as follows.

Theorem 1.1. *Let Ω be an admissible, uniformly C^3-domain in $\mathbb{R}^n$, $\mu, \nu \in (0, \infty]$. Then there exist a solution operator S to (1.1) and constants $C, T_0 > 0$ depending only on μ, ν and Ω such that*

$$\sup_{0<t<T_0} \|\tilde{N}(v,q)(\cdot,t)\|_\infty \le C\|v_0\|_{BMO^{\mu,\nu}_b}$$

for each $v_0 \in VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$ with $S(t)v_0 = v$ and a suitable choice of q. The solution operator S is taken so that it agrees with the L^2-Stokes semigroup on $C^\infty_{c,\sigma}(\Omega)$.

Since the Stokes semigroup is the same as the heat semigroup with $q = 0$ when $\Omega = \mathbb{R}^n$, the estimate (1.2) implies

$$\sup_{t>0} \|\tilde{N}(v,0)(\cdot,t)\|_\infty \le c[v_0]_{BMO}$$

for all $v_0 \in BMO(\mathbb{R}^n)$, where $[v_0]_{BMO} = [v_0]^\infty_{BMO}$. Theorem 1.1 is therefore still valid for $\mathbb{R}^n$ and $\mu = \infty$ without any restriction on the time interval. For finite μ

one cannot take $T_0 = \infty$. In fact, if $v_0 = x_1$ in (1.2) such that $G_t * v_0 = x_1$, then $\nabla G_t * v_0$ does not decay as $t \to \infty$ despite the fact that $[v_0]^{\mu}_{BMO} < \infty$ for finite μ.

The notions of an admissible domain and a Helmholtz domain, i.e. a domain admitting L^r-Helmholtz decomposition, are different. A bounded domain ([AG13]), an exterior domain ([AG14]) as well as the half space ([AG13]) are admissible and they are also Helmholtz domains. However, a layer domain in $\mathbb{R}^n$ with $n \geq 3$ is not admissible ([Bel14]) but a Helmholtz domain ([Miy94]). In ([AGSS15]) it is proved that there is a planar non-Helmholtz domain which is admissible.

The main idea of the proof is using a blow-up argument already used in [AG13] for the proof of the estimates in L^∞. We will assume that the Theorem does not hold. Then we will get a sequence of solutions to (1.1) with decreasing initial data. After normalizing and rescaling a compactness argument yields that a subsequence of the sequence of solutions needs to converge to a weak solution of (1.1) in the whole space $\mathbb{R}^n$ or the half space $\mathbb{R}^n_+$ with weak initial data $v_0 = 0$. A uniqueness result for the weak formulation of (1.1) in those spaces will finally show that the limit needs to be constant which will lead to a contradiction.

The difficulties that appear in the proof compared to the proof in [AG13] are that one has in our case no sufficient control about the size of the function itself but just on its derivatives. Therefore we need estimates for even higher than second derivatives in order to obtain compactness. Furthermore we need more general uniqueness theorems allowing unbounded functions in order to get our result.

Y. Shimizu ([Shi03]) claimed a similar result for a half space by a duality argument using corresponding estimates in Hardy spaces obtained in [GMS99]. He claims that in a half space

$$t^{1/2}\|\nabla S(t)v_0\|_\infty \leq C[v_0]_{BMO} \quad (t > 0)$$

holds. The definition of the BMO-seminorm in $\mathbb{R}^n_+$ used there is based on extensions to $\mathbb{R}^n$. However, as we point out at the end of the article, the statement is not given accurately since it is not made clear what is meant by initial data in BMO if one considers the quotient space $BMO/\mathbb{R}$ as done there.

It should be noted that the result obtained in this paper as well as the result of [AG13] mentioned above is local in time. The boundedness of the Stokes semigroup and its derivatives in BMO-type spaces remains an open question. For global L^∞ estimates there are some global estimates available. For bounded domains the semigroup is even exponentially decaying, see [AG13]. For exterior domains the global boundedness was proved in [Mar14], which was extended to a global time derivative estimate in [HM15] and both results were even further extended to global boundedness in sectors of angle less than $\pi/2$ in [BH15]. For the case of a half space these results were proved in [Sol03] and [DHP01].

This paper is organized as follows. In section 2 we will define the notion of admissibility and the BMO-type norm and space. In section 3 we will give a uniqueness result on the heat equation in $\mathbb{R}^n$ and $\mathbb{R}^n_+$. In section 4 we will give a uniqueness result on the Stokes equations in a half space $\mathbb{R}^n_+$. In section 5 we will establish local Hölder estimates which we will need for the compactness argument. Finally, in section 6 we will give a proof of Theorem 1.1 by a blow-up argument.

2. Admissible Domains and BMO-type norm

In this section we will introduce notations and define the BMO-type norms and spaces. Furthermore we will give the definition of an admissible domain. Let

$$U_{\alpha,\beta,h} := \{(y', y_n) \in \mathbb{R}^n : h(y') - \beta < y_n < h(y') + \beta \text{ and } |y'| < \alpha\}$$

be a neighbourhood of 0. A domain $\Omega \subset \mathbb{R}^n$ will be called uniformly C^k-domain of type (α, β, K) if for each $x_0 \in \partial\Omega$ there are a rotation R_{x_0} and a C^k-function $h = h_{x_0}$ of $n-1$ variables with

$$\sup_{|l|\leq k, |y'|<\alpha} |\partial_{y'}^l h(y')| \leq K, \quad \nabla' h(0) = 0, \quad h(0) = 0,$$

such that

$$\begin{aligned}&(R_{x_0} U_{\alpha,\beta,h} + x_0) \cap \Omega \\ =&R_{x_0}\{(y', y_n) \in \mathbb{R}^n : h(y') < y_n < h(y') + \beta \text{ and } |y'| < \alpha\} + x_0,\end{aligned}$$

$$(R_{x_0} U_{\alpha,\beta,h} + x_0) \cap \partial\Omega = R_{x_0}\{(y', y_n) \in \mathbb{R}^n : y_n = h(y') \text{ and } |y'| < \alpha\} + x_0.$$

Let $L^r_\sigma(\Omega) := \overline{C^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_r}$ and $G^r(\Omega) := \{\nabla f \in L^r(\Omega) : f \in L^r_{\mathrm{loc}}(\Omega)\}$, where $C^\infty_{c,\sigma}(\Omega) = \{f \in C^\infty_c(\Omega) : \operatorname{div} f = 0\}$. For $r \in [2, \infty)$ let $\tilde{L}^r(\Omega)$ be defined as $L^2(\Omega) \cap L^r(\Omega)$ equipped with the norm $\|\cdot\|_{\tilde{L}^r} = \max\{\|\cdot\|_2, \|\cdot\|_r\}$ and let $\tilde{L}^r_\sigma(\Omega) := L^2_\sigma(\Omega) \cap L^r_\sigma(\Omega)$ and $\tilde{G}^r(\Omega) := G^2(\Omega) \cap G^r(\Omega)$. For $r \in (1, 2)$ the space $\tilde{L}^r(\Omega)$ is defined as $L^2(\Omega) + L^r(\Omega)$ and $\tilde{G}^r(\Omega) := G^2(\Omega) + G^r(\Omega)$. R. Farwig, H. Kozono and H. Sohr ([FKS07]) established the Helmholtz decomposition in $\tilde{L}^r(\Omega)$.

Proposition 2.1. *Let $n \geq 2$, $r \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be a uniformly C^1-domain of type (α, β, K). Then the Helmholtz projection $P_r : \tilde{L}^r(\Omega) \to \tilde{L}^r_\sigma(\Omega)$ is a continuous linear operator with kernel $\tilde{G}^r(\Omega)$.*

For $r \in (1, \infty)$ we define further $Q_r = I - P_r$, where I is the identity. In [FKS05], [FKS09] it was proved that for each $v_0 \in \tilde{L}^r_\sigma(\Omega)$ there is a unique solution (v, q) of (1.1) satisfying $v(t), v_t(t), \nabla v(t), \nabla^2 v(t) \in \tilde{L}^r_\sigma(\Omega)$ for all $t > 0$ such that the operator $S(t) : v_0 \to v(t)$ is an analytic semigroup in $\tilde{L}^r_\sigma(\Omega)$. We will call such a solution $\tilde{L}^r$-solution.

We will further define the notion of an admissible domain in the sense of [AG13]. Let $d_\Omega(x)$ denote the distance between $\partial\Omega$ and x,

$$d_\Omega(x) = \inf\{|x - y| : y \in \partial\Omega\}.$$

Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^1-domain with nonempty boundary. Then Ω is called admissible if there exist $r > n$ and C depending only on Ω such that

$$\sup_{x\in\Omega} d_\Omega(x) |Q_r(\operatorname{div} f)(x)| \leq C \|f\|_{L^\infty(\partial\Omega)}$$

holds for all $f = (f_{ij})_{1\leq i,j\leq n} \in C^1(\overline{\Omega})$ which satisfy $\operatorname{div} f \in \tilde{L}^r(\Omega)$, $\operatorname{tr} f = 0$ and $\partial_l f_{ij} = \partial_j f_{il}$ for $1 \leq i, j, l \leq n$. For $\tilde{L}^r$-solutions (v, q) we will use this estimate for $\nabla q = Q_r(\Delta v) = Q_r(\operatorname{div} \nabla v)$ with $f = \nabla v$.

Lemma 2.2. *Let Ω be a uniformly C^2-domain of type (α, β, K). Then there exists a constant $R_p(\alpha, \beta, K)$ such that for $x \in \Omega$ with $d_\Omega(x) < R_p$ there is a unique projection to $x_p \in \partial\Omega$ with*

$$x = x_p - d_\Omega(x)\mathbf{n}_\Omega(x_p),$$

where $\mathbf{n}_\Omega(x_p)$ is the exterior normal of Ω at x_p. The projection is continuously differentiable.

Proof. For a proof see [GT77, appendix] and [KP02, §4.4]. □

Let for $f \in L^1_{\mathrm{loc}}(\Omega)$ and $B \subset \Omega$

$$f_B := \frac{1}{|B|}\int_B f(y)\,dy$$

and let for $\mu \in (0, \infty]$

$$[f]^\mu_{BMO} := \sup\{\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - f_{B_r(x)}|\,dy : B_r(x) \subset \Omega, r < \mu\}.$$

The space $BMO^\mu(\mathbb{R}^n)$ is then defined as

$$BMO^\mu(\mathbb{R}^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : [f]^\mu_{BMO} < \infty\}$$

and we set $BMO(\mathbb{R}^n) := BMO^\infty(\mathbb{R}^n)$.

Let for $\nu \in (0, \infty]$

$$U_\nu(\partial\Omega) := \bigcup_{x_0 \in \partial\Omega} (R_{x_0} U_{\alpha,\beta,h_{x_0}} + x_0) \bigcap \{x \in \mathbb{R}^n : d(x, \partial\Omega) < \nu\},$$

$$[f]^\nu_b := \sup\{r^{-n}\int_{\Omega \cap B_r(x)} |f(y)|\,dy : x \in \partial\Omega, r > 0, B_r(x) \subset U_\nu(\partial\Omega)\}.$$

Then the norm of BMO-type is defined by

$$\|f\|_{BMO^{\mu,\nu}_b} := [f]^\mu_{BMO} + [f]^\nu_b$$

and the BMO-type space by

$$BMO^{\mu,\nu}_b(\Omega) := \{f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|_{BMO^{\mu,\nu}_b} < \infty\}$$

for $\mu, \nu \in (0, \infty]$. The arbitrary choice of μ and ν will be useful for later applications. A standard choice is for example $\mu = \infty$ and $\nu = R^*_p$, where R^*_p is the supremum of all R_p with properties as in Lemma 2.2.

We define further the closure of $C^\infty_c(\Omega)$ and $C^\infty_{c,\sigma}(\Omega)$ in $BMO^{\mu,\nu}_b(\Omega)$ with respect to the $\|\cdot\|_{BMO^{\mu,\nu}_b}$-norm by $VMO^{\mu,\nu}_{b,0}(\Omega)$ and $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$, respectively.

3. Uniqueness for the heat equation

This section deals with the uniqueness problem for the heat equation

$$\begin{array}{rcll} \partial_t u - \Delta u &=& f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(0) &=& u_0 & \text{in } \mathbb{R}^n. \end{array} \tag{3.1}$$

We will reduce this to the problem if the homogeneous equation

$$\begin{array}{rcll} \partial_t u - \Delta u &=& 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(0) &=& 0 & \text{in } \mathbb{R}^n \end{array} \tag{3.2}$$

has 0 as its only solution under certain assumptions. It is well known that

$$\begin{aligned} u(t) &= \int_0^t \mathrm{e}^{(t-s)\Delta} f(s)\,dt + \mathrm{e}^{t\Delta} u_0 \\ &= \int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y) f(y,s)\,dy\,dt + \int_{\mathbb{R}^n} G_t(x-y) u_0(y)\,dy \end{aligned}$$

with $G_t(x) = (4\pi t)^{-n/2} \mathrm{e}^{-\frac{|x|^2}{4t}}$ solves the heat equation (3.1) and that this solution is not unique if we allow sufficiently fast growth of the solution. As a first step we will prove the following Proposition.

Proposition 3.1. *Let $f \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$. Then there is a solution $\psi \in C^\infty(\mathbb{R}^n \times [0,\infty))$ of (3.1) with $u_0 = 0$ such that for any $T > 0$ and any multiindex l there are $C > 0$, $b > 0$ such that*

$$\sup_{0<t\le T} |\partial_x^l \psi(x,t)| \le C\mathrm{e}^{-b|x|^2}, \ \sup_{0<t\le T} |\partial_t \psi(x,t)| \le C\mathrm{e}^{-b|x|^2}. \tag{3.3}$$

The solution additionally satisfies $\lim_{t\to 0} \|\psi(t)\|_\infty = 0$.

Proof. We follow the proof of [GGS10, Proposition 4.3.2]. We define

$$\psi(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} f(s)\,ds, \quad t > 0$$

and for $0 < \rho < t$ we cut off the singularity of the integrand

$$\psi^\rho(t) = \int_0^{t-\rho} \mathrm{e}^{(t-s)\Delta} f(s)\,ds.$$

Then we may change differentiation and integration and get $\psi^\rho \in C^\infty(\mathbb{R}^n \times (\rho,\infty))$. By the chain rule we obtain

$$\partial_t \psi^\rho(t) = \mathrm{e}^{\rho\Delta} f(t-\rho) + \int_0^{t-\rho} \Delta \mathrm{e}^{(t-s)\Delta} f(s)\,ds := I_1^\rho + I_2^\rho.$$

Let

$$I_1 = f(t), \quad I_2 = \int_0^t \mathrm{e}^{(t-s)\Delta} \Delta f(s)\,ds.$$

We now want to show that I_i^ρ converges uniformly to I_i $(i = 1, 2)$. We get

$$\begin{aligned} I_1^\rho - I_1 &= (\mathrm{e}^{\rho\Delta} - \mathrm{I}) f(t-\rho) + (f(t-\rho) - f(t)) \\ &= \int_0^\rho \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{s\Delta} f(t-\rho)\,ds + (f(t-\rho) - f(t)) \\ &= \int_0^\rho \mathrm{e}^{s\Delta} \Delta f(t-\rho)\,ds + (f(t-\rho) - f(t)) \end{aligned}$$

and

$$I_2 - I_2^\rho = \int_{t-\rho}^t \mathrm{e}^{-(t-s)\Delta} \Delta f(s)\,ds.$$

Thus we get from the mean value theorem and the boundedness of the heat semigroup in L^∞

$$\begin{aligned}
&\|I_1 + I_2 - (I_1^\rho + I_2^\rho)\|_\infty \\
\leq &\int_{t-\rho}^{t} \|\Delta f(t-\rho)\|_\infty \, ds + \int_0^\rho \|\Delta f(t-\rho)\|_\infty \, ds + \rho \sup_{0<s\leq T} \|\partial_t f(s)\|_\infty \\
\leq &\rho \left(2 \sup_{0<s\leq T} \|\Delta f(s)\|_\infty + \sup_{0<s\leq T} \|\partial_t f(s)\|_\infty \right).
\end{aligned}$$

Thus $I_1^\rho + I_2^\rho$ converges uniformly to $I_1 + I_2$ on $\mathbb{R}^n \times [\rho_0, T]$ with $\rho_0 > 0$ if $\rho \to 0$ and therefore $I_1 + I_2$ is continuous on $\mathbb{R}^n \times [\rho_0, T]$. Finally we get $\partial_t \psi = I_1 + I_2 \in C(\mathbb{R}^n \times (0,\infty))$. In a similar way we can obtain this result for spatial derivatives and higher order derivatives. Thus $\psi \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and we further get by the boundedness of the heat semigroup in L^∞

$$\lim_{t\to 0} \|\psi(t)\|_\infty = \lim_{t\to 0} \int_0^t \|f(s)\|_\infty \, ds \leq \lim_{t\to 0} t \sup_{0<s\leq T} \|f(s)\|_\infty = 0.$$

For $f \in C_c^\infty(\mathbb{R}^n \times [0,\infty))$ a direct calculation shows that for any l, T there are $C, b > 0$ such that $\sup_{0<t\leq T} |\mathrm{e}^{t\Delta} \partial_x^l f(x)| \leq C\mathrm{e}^{-b|x|^2}$. Therefore we have

$$\begin{aligned}
\sup_{0<t\leq T} |\partial_x^l \psi(x,t)| &\leq \sup_{0<t\leq T} |\partial_x^l \int_0^t \mathrm{e}^{(t-s)\Delta} f(s) \, ds| \\
&\leq \sup_{0<t\leq T} \int_0^t |\mathrm{e}^{(t-s)\Delta} \partial_x^l f(s)| \, ds \\
&\leq CT\mathrm{e}^{-b|x|^2}
\end{aligned}$$

and the same estimate holds for $\partial_t \psi = \Delta\psi + f$. By similar arguments one can show that for any $k \in \mathbb{N}_0$, $T > 0$ and multiindex l $\sup_{0\leq t\leq T} \|\partial_t^k \partial_x^l \psi(t)\|_\infty < \infty$ and thus $\psi \in C^\infty(\mathbb{R}^n \times [0,\infty))$. □

Having these estimates we can now prove a uniqueness result for weak solutions of the heat equation by a duality argument.

Theorem 3.2. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n \times (0,T))$ satisfy*

$$\int_0^T \int_{\mathbb{R}^n} u(\partial_t \varphi + \Delta\varphi) \, dx \, dt = 0 \tag{3.4}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \times [0,T))$ and let u satisfy the estimate

$$|u(t,x)| \leq Ct^{-\gamma} \mathrm{e}^{a|x|} \tag{3.5}$$

for some $\gamma \in [0,1)$, $C > 0$, $a > 0$. Then $u = 0$.

This theorem also holds if we assume u in (3.5) to be bounded by $C\mathrm{e}^{(\frac{a}{t})^{-\alpha} + a|x|^2}$ with $a, C > 0$ and $0 < \alpha < 1$; compare [Chu99, Theorem 3.1 and Theorem 3.2]. Since it is sufficient for our needs, we will give a simpler proof under the more restrictive growth condition (3.5).

Proof. We can extend the weak formulation (3.4) by the estimate (3.5) to $\psi \in C^\infty(\mathbb{R}^n \times [0,T))$ with $\psi(T) = 0$ satisfying

$$\sup_{0\leq t<T, |l|\leq 2} |\partial_x^l \psi(x,t)| + \sup_{0\leq t<T} |\partial_t \psi(x,t)| \leq C_0 \mathrm{e}^{-b|x|^2}.$$

Let $f \in C_c^\infty(\mathbb{R}^n \times (0,T))$. Then by substituting $\tau = T - t$ in Proposition 3.1 there is a function ψ satisfying the above mentioned conditions with $\partial_t \psi + \Delta \psi = f$. Inserting this in (3.4) we obtain

$$\int_0^T \int_{\mathbb{R}^n} uf \, dx \, dt = 0$$

for all $f \in C_c^\infty(\mathbb{R}^n \times (0,T))$. By the fundamental lemma of calculus of variations we get $u = 0$. □

Corollary 3.3. *Let $u \in C(\overline{\mathbb{R}^n_+} \times (0,T))$ be a function satisfying*

$$\int_0^T \int_{\mathbb{R}^n_+} u(\partial_t \varphi + \Delta \varphi) \, dx \, dt = 0 \tag{3.6}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n_+ \times [0,T))$. Let furthermore u satisfy the estimate

$$|u(x,t)| \le Ct^{-\gamma} \mathrm{e}^{a|x|} \tag{3.7}$$

for some $\gamma \in [0,1)$, $C > 0$, $a > 0$. Then $u = 0$.

Proof. We define $\bar{u}$ as the odd extension of u, i.e. $\bar{u}(t, x', -x_n) = -u(t, x', x_n)$ for $x_n > 0$. Since $\bar{u}$ is continuous in $x_n = 0$ and by (3.6) we obtain

$$\int_0^T \int_{\mathbb{R}^n} \bar{u}(\partial_t \varphi + \Delta \varphi) \, dx \, dt = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \times [0,T))$ and thus $\bar{u} = 0$ by Theorem 3.2. □

4. Uniqueness for the Stokes equations in a half space

In this section we consider the uniqueness problem for the Stokes equations in a half space

$$\begin{aligned} v_t - \Delta v + \nabla q &= 0 && \text{in } \mathbb{R}^n_+ \times (0,T) \\ \operatorname{div} v &= 0 && \text{in } \mathbb{R}^n_+ \times (0,T) \\ v &= 0 && \text{on } \partial\mathbb{R}^n_+ \times (0,T) \\ v(\cdot,0) &= v_0 && \text{on } \mathbb{R}^n_+. \end{aligned} \tag{4.1}$$

V. A. Solonnikov ([Sol03]) proved a uniqueness theorem for this equation under the assumption that v is bounded in space and time. We will apply a more general uniqueness theorem which allows growth in time near 0.

Theorem 4.1. *Let $v \in C^{2,1}(\mathbb{R}^n_+ \times (0,T))$ and $\nabla q \in C(\mathbb{R}^n_+ \times (0,T))$ satisfy $(4.1)_{1-3}$ and let $(v, \nabla q)$ satisfy*

$$\int_0^T \int_{\mathbb{R}^n_+} v \cdot (\varphi_t + \Delta \varphi) - \varphi \cdot \nabla q \, dx \, dt = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n_+ \times [0,T))$. Furthermore let v satisfy the estimate

$$\sup_{0<t\le T} t^{1/2} \|v\|_\infty \le \infty, \tag{4.2}$$

let $\nabla v(t) \in L^\infty(\mathbb{R}^n_+)$ for every $t > 0$ and let ∇q satisfy the estimate

$$\sup_{0<t\le T, x \in \mathbb{R}^n_+} t^{1/2}(x_n^2 + t)|\nabla q(x,t)| < \infty. \tag{4.3}$$

Then $u = 0$ and $\nabla q = 0$.

Proof. The theorem was proved by K. Abe in [Abe14], where the weak formulation is assumed to hold for all

$$\varphi \in \mathcal{S} = \{\varphi \in C^\infty(\overline{\mathbb{R}^n_+} \times [0,1]) : \varphi, \nabla\varphi, \nabla^2\varphi, \varphi_t \in L^\infty((0,1), L^1(\mathbb{R}^n_+)),$$
$$\varphi|_{t=1} = 0, \varphi|_{x_n=0} = 0, \partial_n\varphi \in L^\infty((0,1), L^\infty((0,\infty), L^1(\mathbb{R}^{n-1})))\}.$$

By the pressure estimate (4.3) we can extend the weak formulation stated in the theorem to the weak formulation with $\varphi \in \mathcal{S}$.

We will give a sketch of the proof of K. Abe's uniqueness result. As a first step consider the dual problem

$$\begin{aligned} -\varphi_t - \Delta\varphi + \nabla\pi &= \partial_{\tan} f && \text{in } \mathbb{R}^n_+ \times (0,T) \\ \operatorname{div}\varphi &= 0 && \text{in } \mathbb{R}^n_+ \times (0,T) \\ \varphi &= 0 && \text{on } \{x_n = 0\} \times (0,T) \\ \varphi(\cdot,T) &= 0 && \text{on } \mathbb{R}^n_+. \end{aligned} \tag{4.4}$$

for $f \in C^\infty_{c,\sigma}(\Omega)$ and show that there exists a solution $(\varphi, \nabla\pi)$ with $\varphi \in \mathcal{S}$ and $\nabla\pi \in L^\infty((0,T), L^1(\mathbb{R}^n_+))$. This solution can be obtained by an explicit formula. Then test v with $\partial_{\tan} f$ to get

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n_+} v \cdot \partial_{\tan} f \, dx \, dt &= \int_0^T \int_{\mathbb{R}^n_+} v \cdot (-\varphi_t - \Delta\varphi + \nabla\pi) \, dx \, dt \\ &= -\int_0^T \int_{\mathbb{R}^n_+} \nabla q \cdot \varphi \, dx \, dt \\ &= 0 \end{aligned}$$

by integration by parts and $\nabla\pi(t) \in L^1(\mathbb{R}^n_+)$, $\nabla q(t) \in L^\infty(\mathbb{R}^n_+)$ for almost all $t \in (0,T)$. Together with de Rham's theorem one gets a potential $\partial_j v = \nabla\Phi^j$ $(1 \le j \le n-1)$, where Φ^j is harmonic by the divergence condition and $\nabla\Phi^j$ is in addition bounded by assumption. Thus $\nabla\Phi^j$ is constant and by the boundary conditions $\partial_j v = \nabla\Phi^j = 0$ for $1 \le j \le n-1$. Thus (v,q) needs to be a solution of Poiseuille type flow of the form $v = (v_{\tan}(x_n,t), 0)$ and $\nabla q = a(t)$, where $a(t) = 0$ by the pressure estimate (4.3). Then each v_i $(1 \le i \le n-1)$ solves the heat equation in $\mathbb{R}_+$, i.e. v_i solves (3.6) and thus by Corollary 3.3 one gets $v_i = 0$. □

5. Hölder estimates

In this section we want to derive local Hölder estimates for the solution of the Stokes equations. Following [LSU68] we denote for $\mu \in (0,1)$, $Q = \Omega \times (0,T)$

$$[f]^{(\mu)}_{(0,T]}(x) = \sup\left\{\frac{|f(x,t) - f(x,s)|}{|t-s|^\mu} : t,s \in (0,T], s \ne t\right\},$$

$$[f]^{(\mu)}_{\Omega}(t) = \sup\left\{\frac{|f(x,t) - f(y,t)|}{|x-y|^\mu} : x,y \in \Omega, x \ne y\right\},$$

$$[f]^{(\mu)}_{t,Q} = \sup_{x\in\Omega}[f]^{(\mu)}_{(0,T]}(x), \quad [f]^{(\mu)}_{x,Q} = \sup_{t\in(0,T]}[f]^{(\mu)}_{\Omega}(t).$$

For $\gamma \in (0,1)$ we denote $[f]^{(\gamma,\gamma/2)}_Q = [f]^{(\gamma)}_{x,Q} + [f]^{(\gamma/2)}_{t,Q}$. Finally we define the parabolic Hölder norm by

$$\|f\|^{(\gamma,\gamma/2)}_Q = \|f\|_{L^\infty(Q)} + [f]^{(\gamma,\gamma/2)}_Q.$$

In [AG13, Theorem 3.2 and Theorem 3.4] K. Abe and the second author proved the following theorems.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^2-domain, $\gamma \in (0,1)$, $T > \delta > 0$, $R > 0$. Then there exists a constant $C(\Omega, \delta, R, d, \gamma, T) > 0$ independent of translation, rotation and dilation to a larger scale of Ω such that*

$$[\nabla^2 v]_Q^{(\gamma,\frac{\gamma}{2})} + [v_t]_Q^{(\gamma,\frac{\gamma}{2})} + [\nabla q]_Q^{(\gamma,\frac{\gamma}{2})} \le C \sup_{0<t\le T} \|N(v,q)(t)\|_\infty \tag{5.1}$$

holds for all $\tilde{L}^r$-solutions (v,q) of (1.1) provided $x_0 \in \Omega$ and $B_R(x_0) \subset \Omega$, where $Q = B_R(x_0) \times (\delta, T]$, $d := \operatorname{dist}(B_R(x_0), \partial\Omega)$.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be an admissible, uniformly C^3-domain of type (α, β, K). Then there exists $R_0(\alpha, \beta, K) > 0$ such that for any $\gamma \in (0,1)$, $T > \delta > 0$ and $R \le \frac{R_0}{2}$ there is a constant $C(\Omega, \alpha, \beta, K, \delta, \gamma, T, R) > 0$ independent of translation, rotation and dilation to a larger scale of Ω such that*

$$[\nabla^2 v]_{Q'}^{(\gamma,\frac{\gamma}{2})} + [v_t]_{Q'}^{(\gamma,\frac{\gamma}{2})} + [\nabla q]_{Q'}^{(\gamma,\frac{\gamma}{2})} \le C \sup_{0<t\le T} \|N(v,q)(t)\|_\infty \tag{5.2}$$

holds for all $\tilde{L}^r$-solutions (v,q) of (1.1) with $Q' = (B_R(x_0) \cap \Omega) \times (\delta, T]$ and $x_0 \in \partial\Omega$.

We will improve the term on the right hand side of those inequalities.

Theorem 5.3. *Under the assumptions of Theorem 5.1 the estimates*

$$\begin{aligned} &[\nabla^2 v]_Q^{(\gamma,\frac{\gamma}{2})} + [v_t]_Q^{(\gamma,\frac{\gamma}{2})} + [\nabla q]_Q^{(\gamma,\frac{\gamma}{2})} \\ \le& C\Big(\sup_{0<t\le T} \|\tilde{N}(v,q)(\cdot,t)\|_\infty + \sup_{\delta/2<t\le T} \|v(,t)\|_{L^\infty(B_{R+d/2}(x_0))}\Big), \end{aligned} \tag{5.3}$$

$$\|\nabla^3 v\|_Q^{(\gamma,\frac{\gamma}{2})} + \|\nabla v_t\|_Q^{(\gamma,\frac{\gamma}{2})} + \|\nabla^2 q\|_Q^{(\gamma,\frac{\gamma}{2})} \le C \sup_{0<t\le T} \|\tilde{N}(v,q)(\cdot,t)\|_\infty \tag{5.4}$$

hold for all $\tilde{L}^r$-solutions (v,q) of (1.1). The constants C depend on $d, \Omega, \gamma, R, \delta$ and T but are independent of translation, rotation and dilation to a larger scale of Ω. Additionally, the constants are decreasing in d.

Proof. The proof follows the lines of the proof of [AG13, Theorem 3.1 and Theorem 3.2]. Since ∇q is harmonic, we get by Cauchy estimates for harmonic functions

$$\|\nabla^2 q(t)\|_{L^\infty(B_{R+d/2}(x_0))} \le \frac{C}{d} \|\nabla q(t)\|_{L^\infty(B_{R+d}(x_0))}, \tag{5.5}$$

$$\|\nabla^3 q(t)\|_{L^\infty(B_{R+d/2}(x_0))} \le \frac{C}{d^2} \|\nabla q(t)\|_{L^\infty(B_{R+d}(x_0))}. \tag{5.6}$$

We claim that there exists a constant $M > 0$ depending on Ω but independent of dilation and translation of Ω such that

$$[d_\Omega(x) \nabla q]_{t,\Omega\times(\delta,T)}^{(1/2)} \le \frac{M}{\delta} \sup_{\delta<t\le T} \{(\|v_t(\cdot,t)\|_\infty + \|\nabla^2 v(\cdot,t)\|_\infty) t\} \tag{5.7}$$

holds for all $\tilde{L}^r$-solutions (v,q) of (1.1) and $\delta \in (0,T)$. This fact was proved in [AG13, Lemma 3.1]. For the sake of completeness we will give the proof here.

By interpolation (see [Tan97, Theorem 3.1]) there are constants C_1, C_2 such that for any $\varepsilon > 0$

$$\begin{aligned}\|\nabla v(t)\|_\infty &\le C_1 \|v(t)\|_\infty^{1/2} \|\nabla^2 v(t)\|_\infty^{1/2} \\ &\le \varepsilon \|\nabla^2 v(t)\|_\infty + \frac{C_2}{\varepsilon} \|v(t)\|_\infty.\end{aligned}$$

Since v is an $\tilde{L}^r$-solution, $\nabla q = Q_r(\Delta v)$ and by the admissibility of the domain, we obtain

$$\begin{aligned}&d_\Omega(x)|\nabla q(x,t) - \nabla q(x,s)| \\ \le& C(\Omega)\|\nabla v(t) - \nabla v(s)\|_\infty \\ \le& C(\Omega)(\varepsilon \max\{\|\nabla^2 v(t)\|_\infty, \|\nabla^2 v(s)\|_\infty\} + \frac{C}{\varepsilon}\|v(t) - v(s)\|_\infty).\end{aligned}$$

By

$$\|v(t) - v(s)\|_\infty \le |t-s| \sup_{\tau \in [t,s]} \|v_t(\tau)\|_\infty \le \frac{|t-s|}{\delta} \sup_{\delta \le \tau \le T} \tau \|v_t\|_\infty \tag{5.8}$$

and choosing $\varepsilon = |t-s|^{1/2}$ we obtain (5.7).

For the estimate (5.3) we combine the estimates (5.5) and (5.7) to obtain

$$\begin{aligned}[\nabla q]_{Q''}^{(\gamma,\gamma/2)} &\le \sup_{\delta < t \le T} (\frac{C}{d\delta} t \|\nabla q(t)\|_\infty + \frac{4M}{d\delta} t(\|v_t\|_\infty + \|\nabla^2 v\|_\infty)) \\ &\le C \sup_{0 < t \le T} \|\tilde{N}(v,q)(\cdot,t)\|_\infty\end{aligned}$$

for $Q'' = B_{R+d/2}(x_0) \times (\delta/2, T]$. By standard local Hölder estimates for the heat equation

$$v_t - \Delta v = -\nabla q$$

as in [LSU68, Chapter IV, Theorem 10.1] we get that

$$\begin{aligned}&[\nabla^2 v]_Q^{(\gamma,\frac{\gamma}{2})} + [v_t]_Q^{(\gamma,\frac{\gamma}{2})} \\ \le& C(\sup_{0<t\le T} \|\tilde{N}(v,q)(\cdot,t)\|_\infty + \sup_{\delta/2 < t \le T} \|v(t)\|_{L^\infty(B_{R+d/2(x_0)})}).\end{aligned}$$

We will now prove the second estimate (5.4). We use the fact that u and p are smooth in the interior of Ω. Since the function $\nabla q(\cdot,t) - \nabla q(\cdot,s)$ is harmonic for $t, s \in (\delta/2, T]$ we get by Cauchy's estimate

$$\|\nabla^2(q(\cdot,t) - q(\cdot,s))\|_{L^\infty(B_{R+d/2}(x_0))} \le \frac{C}{d}\|\nabla q(\cdot,t) - \nabla q(\cdot,s)\|_\infty.$$

By admissibility of the domain we obtain

$$\begin{aligned}&\|d_\Omega(\cdot)(\nabla q(\cdot,t) - \nabla q(\cdot,s))\|_\infty \\ \le& C(\Omega)\|\nabla v(t) - \nabla v(s)\|_\infty \\ \le& C(\Omega)(\varepsilon \max\{\|\nabla^2 v(t)\|_\infty, \|\nabla^2 v(s)\|_\infty\} + \frac{C}{\varepsilon}\|v(t) - v(s)\|_\infty).\end{aligned}$$

By (5.8) and choosing $\varepsilon = |t-s|^{1/2}$ again we obtain

$$\begin{aligned}&\|d_\Omega(\cdot)(\nabla q(\cdot,t) - \nabla q(\cdot,s))\|_\infty\\ \leq&\frac{C_0}{\delta}|t-s|^{1/2} \sup_{\delta/2<\tau\leq T} \tau(\|v_t(\tau)\|_\infty + \|\nabla^2 v(\tau)\|_\infty)\end{aligned}$$

for $\delta/2 < t, s \leq T$ with a constant C_0 depending only on Ω. Thus we get

$$\begin{aligned}&\|\nabla^2(q(\cdot,t) - q(\cdot,s))\|_{L^\infty(B_{R+d/2}(x_0))}\\ \leq&\frac{C}{d}\|\nabla q(\cdot,t) - \nabla q(\cdot,s)\|_\infty\\ \leq&\frac{C_1}{\delta d^2}|t-s|^{1/2} \sup_{\delta/2<\tau\leq T} \tau(\|v_t(\tau)\|_\infty + \|\nabla^2 v(\tau)\|_\infty).\end{aligned}$$

Thus for every $x \in B_{R+d/2}(x_0)$

$$[\nabla^2 q]^{(1/2)}_{(\delta/2,T]}(x) \leq \frac{C_1}{\delta d^2} \sup_{\delta/2<\tau\leq T} \tau(\|v_t(\tau)\|_\infty + \|\nabla^2 v(\tau)\|_\infty),$$

where C_1 just depends on Ω. This combined with the estimate for $\nabla^3 q$ yields

$$[\nabla^2 q]^{(\gamma,\gamma/2)}_{Q''} \leq \frac{C}{\delta d^2} \sup_{\delta/2<\tau\leq T} \tau(\|v_t(\tau)\|_\infty + \|\nabla^2 v(\tau)\|_\infty + \|\nabla q(\tau)\|_\infty).$$

Then we get by local Hölder estimates for

$$(\partial_i v)_t - \Delta(\partial_i v) = -\nabla(\partial_i q) \quad (1 \leq i \leq n)$$

the estimate

$$\begin{aligned}&\|\nabla^2 \partial_i v\|^{(\gamma,\frac{\gamma}{2})}_Q + \|(\partial_i v)_t\|^{(\gamma,\frac{\gamma}{2})}_Q\\ \leq&\|\nabla \partial_i q\|^{(\gamma,\frac{\gamma}{2})}_{Q''} + \sup_{\delta/2<t\leq T} \|\partial_i v(t)\|_{L^\infty(B_{R+d/2}(x_0))}\\ \leq&C \sup_{\delta/2<t\leq T} (\|\nabla v(t)\|_\infty + t\|\nabla q(t)\|_\infty + t\|v_t(t)\|_\infty + t\|\nabla^2 v(t)\|_\infty)\\ \leq&C \sup_{0<t\leq T} \|\tilde{N}(v,q)(\cdot,t)\|_\infty,\end{aligned}$$

which proves the second estimate. □

Theorem 5.2 is proved in [AG13, Theorem 3.4] and the proof is quite technical. The main steps are localizing the equation, using the Bogovskiĭ operator to reobtain a solenoidal vector field and then using Schauder estimates for the Stokes equations together with the Helmholtz decomposition in Hölder spaces. An inspection of the proof in [AG13] shows that we can replace $\sup_{0<t\leq T} \|v(t)\|_\infty$ on the right hand side of the estimate (5.2) by $\sup_{\delta/2<t\leq T} \|v(t)\|_{L^\infty(\Omega\cap B_{2R}(x_0))}$. Although not explicitly stated, the Hölder estimate for the pressure q in [AG13, Lemma 3.5] is there actually proved for $\sup_{\delta<t\leq T}(t(\|v_t(t)\|_\infty + \|\nabla^2 v(t)\|_\infty + \|\nabla q(t)\|_\infty))$ on the right hand side. After localization one needs to estimate $v\xi_t$, $v\Delta\xi$, q with derivatives of q and derivatives of v to get the result, where ξ is a cutoff function in space and time with support in $(\delta, T] \times B_{2R}(x_0)$, see [AG13, Section 3.4]. Then $v\xi_t$ and $v\Delta\xi$ can be estimated by $\sup_{\delta/2<t\leq T} \|v(t)\|_{L^\infty(\Omega\cap B_{2R}(x_0))}$ and the derivatives of v and the terms depending on q by $\sup_{\delta/2<t\leq T} \|\tilde{N}(v,q)(\cdot,t)\|_\infty$.

By the homogeneous boundary conditions we get for $t > \delta/2$

$$\|v(\cdot,t)\|_{L^\infty(\Omega\cap B_{2R}(x_0))} \le 2R\|\nabla v(\cdot,t)\|_\infty \le C(R,\delta)\|\tilde{N}(v,q)(\cdot,t)\|_\infty$$

and thus we can improve Theorem 5.2 to the following theorem.

Theorem 5.4. *Under the assumptions of Theorem 5.2 there is a constant $C > 0$ depending on Ω, α, β, K, δ, γ, T and R but independent of translation, rotation and dilation to a larger scale of Ω such that*

$$[\nabla^2 v]_{Q'}^{(\gamma,\frac{\gamma}{2})} + [v_t]_{Q'}^{(\gamma,\frac{\gamma}{2})} + [\nabla q]_{Q'}^{(\gamma,\frac{\gamma}{2})} \le C \sup_{0<t\le T} \|\tilde{N}(v,q)(t)\|_\infty \tag{5.9}$$

holds for all $\tilde{L}^r$-solutions (v,q) of (1.1).

6. Blow-up argument

In this section we will prove our main result by a blow-up argument. Due to [AG13, Proposition 5.2] we can assume the following regularity of $\tilde{L}^r$-solutions. The proof relies on estimates derived from the analyticity of the solution operator S in $\tilde{L}^r(\Omega)$ and local higher regularity theory for elliptic systems.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a uniformly C^3-domain, $T > 0$, $r > n$ and (v,q) an $\tilde{L}^r$-solution with $v_0 \in C^\infty_{c,\sigma}(\Omega)$. Then $v(\cdot,t) \in C^2(\overline{\Omega})$ for all $t > 0$ and*

$$\sup_{0<t<T} \|N(v,q)(\cdot,t)\|_\infty < \infty.$$

This finiteness result enables us to prove the next key theorem which yields Theorem 1.1.

Theorem 6.2. *Let Ω be an admissible C^3-domain in $\mathbb{R}^n$, $\mu,\nu \in (0,\infty]$, $r > n$. Then there exist $C, T_0 > 0$ depending only on μ,ν and Ω such that*

$$\sup_{0<t<T_0} \|\tilde{N}(v,q)(\cdot,t)\|_\infty \le C\|v_0\|_{BMO_b^{\mu,\nu}} \tag{6.1}$$

holds for all $\tilde{L}^r$-solutions (v,q) of (1.1) with $v_0 \in C^\infty_{c,\sigma}(\Omega)$.

Proof of Theorem 6.2. Assume that there is no such T_0 and C. Then there is a sequence $v_{0m} \in C^\infty_{c,\sigma}(\Omega)$ with $\tilde{L}^r$-solutions (v_m,q_m) solving (1.1) with v_{0m} as initial data and $\tau_m \to 0$ such that

$$\|\tilde{N}(v_m,q_m)(\cdot,\tau_m)\|_\infty > m\|v_{0m}\|_{BMO_b^{\mu,\nu}}. \tag{6.2}$$

Let

$$M_m := \sup_{0<t<\tau_m} \|\tilde{N}(v_m,q_m)(\cdot,t)\|_\infty. \tag{6.3}$$

We normalize (v_m,q_m) by setting

$$\tilde{v}_m := v_m/M_m \text{ and } \tilde{q}_m := q_m/M_m.$$

Then

$$\sup_{0<t<\tau_m} \|\tilde{N}(\tilde{v}_m,\tilde{q}_m)(\cdot,t)\|_\infty = 1 \tag{6.4}$$

and therefore there exist t_m with $0 < t_m < \tau_m$ and $x_m \in \Omega$ such that

$$\tilde{N}(\tilde{v}_m,\tilde{q}_m)(x_m,t_m) > \frac{1}{2}. \tag{6.5}$$

We rescale $(\tilde{v}_m, \tilde{q}_m)$ with respect to x_m and t_m by

$$u_m(x,t) := \tilde{v}_m(x_m + t_m^{1/2}x, t_m t)$$

$$p_m(x,t) := t_m^{1/2}\tilde{q}_m(x_m + t_m^{1/2}x, t_m t)$$

$$\Omega_m := \{x \in \mathbb{R}^n : x = (y - x_m)/t_m^{1/2}, y \in \Omega\}$$

such that (u_m, p_m) solves (1.1) in the rescaled domain $\Omega_m \times (0,1)$. For (u_m, p_m) the following estimates hold by (6.2), (6.4) and (6.5)

$$\sup_{0<t\le 1} \|\tilde{N}(u_m, p_m)(\cdot, t)\|_\infty \le 1 \tag{6.6}$$

$$\tilde{N}(u_m, p_m)(0,1) \ge 1/2 \tag{6.7}$$

$$\|u_{0m}\|_{BMO_b^{\hat{\mu}_m, \hat{\nu}_m}} \le 1/m \to 0 \quad (m \to \infty), \tag{6.8}$$

where $\hat{\mu}_m = \mu/(t_m^{1/2})$ and $\hat{\nu}_m = \nu/(t_m^{1/2})$. By the definition of an admissible domain and (6.6) we further get

$$\sup_{x\in\Omega_m, 0<t\le 1} t^{1/2} d_{\Omega_m}(x)|\nabla p_m(x,t)| < C. \tag{6.9}$$

We will distinguish between two cases. Either Ω_m converges to $\mathbb{R}^n$ or to $\mathbb{R}^n_{+,-c_0} := \{x \in \mathbb{R}^n : x_n > -c_0\}$ with some $c_0 \ge 0$. In order to do this we define

$$c_m := d_\Omega(x_m)/t_m^{1/2} = d_{\Omega_m}(0).$$

Case 1: $\limsup_{m\to\infty} c_m = \infty$
Without loss of generality we can assume $\lim_{m\to\infty} c_m = \infty$. Then Ω_m is expanding to $\mathbb{R}^n$. Thus we have for any $\varphi \in C_c^\infty(\mathbb{R}^n \times [0,1))$ and sufficiently large m

$$\int_0^1 \int_{\mathbb{R}^n} (u_m \cdot (\varphi_t + \Delta\varphi) - \nabla p_m \cdot \varphi)\, dx\, dt = -\int_{\mathbb{R}^n} u_m(x,0)\cdot\varphi(x,0)\, dx \tag{6.10}$$

and get a similar formulation for $(\partial_i u_m, \nabla\partial_i p_m)$ $(i = 1, \dots, n)$

$$\int_0^1 \int_{\mathbb{R}^n} (\partial_i u_m \cdot (\varphi_t + \Delta\varphi) + \nabla p_m \cdot \partial_i\varphi)\, dx\, dt = \int_{\mathbb{R}^n} u_m(x,0)\cdot\partial_i\varphi(x,0)\, dx. \tag{6.11}$$

The right hand side converges to 0 since $[u_{0m}]_{BMO}^{\hat{\mu}_m} \to 0$, $\partial_i\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \partial_i\varphi\, dx = 0$ and the support of $\partial_i\varphi$ will be contained in balls of radii $r < \hat{\mu}_m$ for $m \ge m_0$ with some m_0.

By the local Hölder estimates obtained in Theorem 5.3 and estimate (6.6) we get that there is a subsequence of (u_m, p_m) such that $\partial_i u_m$, $\nabla\partial_i u_m$, $\nabla^2\partial_i u_m$, $(\partial_i u_m)_t$, ∇p_m converge to some $(w, \nabla p)$ and its derivatives locally uniformly in $\mathbb{R}^n \times (0,1]$. By (6.9) we obtain $\nabla p = 0$. Thus the weak formulation becomes in the limit

$$\int_0^1 \int_{\mathbb{R}^n} w(\varphi_t + \Delta\varphi)\, dx\, dt = 0 \tag{6.12}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n \times [0,1))$. Then $w = 0$ by uniqueness result of Theorem 3.2.

Therefore $\nabla u_m \to 0$ and thus $\nabla^2 u_m \to 0$ $(m \to \infty)$ locally uniformly. Furthermore $\nabla p_m \to 0$ $(m \to \infty)$ locally uniformly. Thus

$$(u_m)_t = \Delta u_m - \nabla p_m$$

converges locally uniformly to 0 as well. Therefore $\tilde{N}(u_m, q_m)(x,t)$ needs to converge to 0 for any $x \in \mathbb{R}^n, 0 < t \leq 1$ which is a contradiction to (6.7).

Case 2: $\limsup_{m\to\infty} c_m < \infty$

Without loss of generality we can assume $\lim_{m\to\infty} c_m = c_0$ and $t_m < 1$. For sufficiently large m there is by Lemma 2.2 for each x_m a projection to $(x_m)_p \in \partial\Omega$. By rotation and translation we get a sequence of domains $\tilde{\Omega}_m$ such that $(x_m)_p = 0$ and that the exterior normal is $\mathbf{n}_{\tilde{\Omega}_m}(0) = (0, \dots, 0, -1)$. In particular $x_m = (0, \dots, 0, d_\Omega(x_m))$. Since Ω and therefore $\tilde{\Omega}_m$ are uniformly C^3-domains of type (α, β, K) we can represent $\tilde{\Omega}_m$ in a neighbourhood $(\tilde{\Omega}_m)_{\text{loc}}$ of $(x_m)_p = 0$ by

$$(\tilde{\Omega}_m)_{\text{loc}} = \{(x', x_n) \in \mathbb{R}^n : h_m(x') < x_n < h_m(x') + \beta \text{ and } |x'| < \alpha\}$$

with h_m a C^3-function satisfying $\sup_{|k|\leq 3, |x'|<\alpha} |\partial^k_{x'} h_m(x')| \leq K$, $h_m(0) = 0$ and $\nabla' h_m(0) = 0$. If one now rescales as explained before with respect to $t_m^{1/2}$ and x_m we can represent the rescaled domain $(\Omega_m)_{\text{loc}}$ as

$$\begin{aligned}&\{y \in \mathbb{R}^n : h_m(t_m^{1/2} y' + x_m') < t_m^{1/2} y_n + (x_m)_n < h_m(t_m^{1/2} y' + x_m') + \beta \text{ and } |t_m^{1/2} y'| < \alpha\}\\ =&\{y \in \mathbb{R}^n : \frac{h_m(t_m^{1/2} y') - (x_m)_n}{t_m^{1/2}} < y_n < \frac{h_m(t_m^{1/2} y') + \beta - (x_m)_n}{t_m^{1/2}} \text{ and } |y'| < \frac{\alpha}{t_m^{1/2}}\}.\end{aligned}$$

By Taylor expansion

$$\left| \frac{h_m(t_m^{1/2} y')}{t_m^{1/2}} \right| \leq \frac{h_m(0) + \nabla' h_m(0) \cdot |t_m^{1/2} y'| + \frac{K}{2} |t_m^{1/2} y'|^2}{t_m^{1/2}} = \frac{K}{2} t_m^{1/2} |y'|^2 \to 0$$

for $m \to \infty$ and fixed $y' \in \mathbb{R}^{n-1}$. This together with $\frac{(x_m)_n}{t_m^{1/2}} \to c_0$, $\frac{\alpha}{t_m^{1/2}} \to \infty$, $\frac{\beta}{t_m^{1/2}} \to \infty$ for $m \to \infty$ yields that $(\Omega_m)_{\text{loc}}$ expands to a half space $\mathbb{R}^n_{+,-c_0}$.

Then (u_m, p_m) solves $(1.1)_1$ in $(\Omega_m)_{\text{loc}} \times (0,1)$. Due to the boundary conditions and (6.6) we can estimate

$$\begin{aligned}\sup_{\delta<t\leq 1} \|u_m(\cdot,t)\|_{L^\infty(B_{R+d/2}(x_0))} &\leq \sup_{\delta<t\leq 1} (2R + 3d/2) \|\nabla u_m(\cdot,t)\|_\infty\\ &\leq (2R + 3d/2)\delta^{-1/2}\end{aligned}$$

for $\delta > 0$ and d the distance between $B_R(x_0)$ and $\partial\Omega_m$, which means that u_m is locally uniformly bounded. Together with Theorem 5.4, Theorem 5.3 and (6.6) we then obtain that there is a subsequence of (u_m, p_m) such that u_m, ∇u_m, $\nabla^2 u_m$, $(u_m)_t$ and ∇p_m converge to some $(u, \nabla p)$ and its derivatives locally uniformly in $\overline{\mathbb{R}^n_{+,-c_0}} \times (0,1]$. Then (u_m, p_m) solves

$$\int_0^1 \int_{\mathbb{R}^n_{+,-c_0}} u_m \cdot (\varphi_t + \Delta\varphi) - \varphi \cdot \nabla p_m \, dx \, dt = -\int_{\mathbb{R}^n_{+,-c_0}} u_m(x,0) \cdot \varphi(x,0) \, dx$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n_{+,-c_0} \times [0,1))$ and m sufficiently large. Let $a_0 := (0, \dots, 0, -c_0)$ and $a_m := (0, \dots, 0, -d_\Omega(x_m))$. For fixed $\varphi \in C_c^\infty(\mathbb{R}^n_{+,-c_0} \times [0,1))$ there exist R and m_0 depending only on $(\hat{\nu}_m)_{m\in\mathbb{N}}$, $((\Omega_m)_{\text{loc}})_{m\in\mathbb{N}}$ and φ such that for all $m \geq m_0$

$$\operatorname{supp} \varphi(\cdot, 0) \subset\subset B_{R/2}(a_0) \cap \Omega_m \text{ and } B_R(a_m) \subset U_{\hat{\nu}_m}(\partial\Omega_m).$$

Since $[u_{0m}]_b^{\hat{\nu}_m}$ converges to 0 we get for $a_m \in \partial\Omega_m$ sufficiently near to a_0

$$\begin{aligned}\left|\int_{\mathbb{R}^n_+} u_m(x,0)\cdot\varphi(x,0)\,dx\right| &\le \int_{B_{R/2}(a_0)\cap\Omega_m} |u_m(x,0)\cdot\varphi(x,0)|\,dx\\ &\le R^n R^{-n}\int_{B_R(a_m)\cap\Omega_m} |u_m(x,0)|\,dx\|\varphi(\cdot,0)\|_\infty\\ &\le R^n [u_{0m}]_b^{\hat{\nu}_m}\|\varphi(\cdot,0)\|_\infty \to 0 \quad (m\to\infty).\end{aligned}$$

The weak formulation then becomes in the limit

$$\int_0^1\int_{\mathbb{R}^n_{+,-c_0}} u\cdot(\varphi_t+\Delta\varphi)-\varphi\cdot\nabla p\,dx\,dt = 0 \tag{6.13}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n_{+,-c_0}\times[0,1))$ with $(u,\nabla p)\in C^{2,1}(\mathbb{R}^n_{+,-c_0}\times(0,1))\times C(\mathbb{R}^n_{+,-c_0}\times(0,1))$ satisfying the following estimates

$$\sup_{0<t<1}\|\tilde{N}(u,p)(\cdot,t)\|_\infty \le 1, \tag{6.14}$$

$$\sup_{0<t<1, x\in\mathbb{R}^n_{+,-c_0}} t^{1/2}(x_n+c_0)|\nabla p(x,t)| \le C. \tag{6.15}$$

Then (u,p) solves $(1.1)_{1-3}$ even in the classical sense. We now want to use the uniqueness result of Theorem 4.1 in order to prove that $u=0$. Since our estimates are not exactly those required there we define for ρ_ε ($\varepsilon>0$) the standard mollifier on $\mathbb{R}^{n-1}$ with support in $B_\varepsilon(0)$ and $*$ the convolution in $\mathbb{R}^{n-1}$

$$\begin{aligned}u_\varepsilon(\cdot,x_n,t) &= u(\cdot,x_n,t)*\rho_\varepsilon \quad (x_n\ge -c_0, t>0),\\ p_\varepsilon(\cdot,x_n,t) &= p(\cdot,x_n,t)*\rho_\varepsilon \quad (x_n\ge -c_0, t>0).\end{aligned}$$

Then $(u_\varepsilon,p_\varepsilon)\in C^{2,1}(\mathbb{R}^n_{+,-c_0}\times(0,1))\times C^1(\mathbb{R}^n_{+,-c_0}\times(0,1))$ and $(\partial_{\tan}u_\varepsilon,\partial_{\tan}p_\varepsilon)\in C^{2,1}(\mathbb{R}^n_{+,-c_0}\times(0,1))\times C^1(\mathbb{R}^n_{+,-c_0}\times(0,1))$ solve the Stokes equations in a half space and satisfy the weak formulation (6.13). Furthermore we have the following estimates.

$$\begin{aligned}\|\partial_{\tan}u_\varepsilon\|_\infty &\le \|\partial_{\tan}u*\rho_\varepsilon\|_\infty\\ &\le C\|\nabla u\|_\infty\\ &\le Ct^{-1/2}\end{aligned}$$

by (6.14). By (6.15) and the Cauchy estimate

$$|\nabla^2 p(x)| \le C(d_{\mathbb{R}^n_{+,-c_0}}(x))^{-1}|\nabla p(x)| = C(x_n+c_0)^{-1}|\nabla p(x)|$$

we get

$$\begin{aligned}&\sup_{0<t<1, x_n>-c_0} t^{1/2}(x_n+c_0)^2\|\nabla\partial_{\tan}p_\varepsilon(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\\ \le&\sup_{x_n>-c_0} t^{1/2}\|\rho_\varepsilon\|_{L^1(\mathbb{R}^{n-1})}(x_n+c_0)^2\|\nabla^2 p(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\\ \le&\sup_{x_n>-c_0} C\|\rho_\varepsilon\|_{L^1(\mathbb{R}^{n-1})}t^{1/2}(x_n+c_0)\|\nabla p(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\\ \le& C\end{aligned}$$

Furthermore

$$\begin{aligned} t^{3/2}\|\nabla\partial_{\tan}p_\varepsilon\|_\infty &\le t\|\nabla\partial_{\tan}p_\varepsilon\|_\infty \\ &\le t\sup_{x_n>-c_0}\|\nabla p(\cdot,x_n)*\partial_{\tan}\rho_\varepsilon\|_{L^\infty(\mathbb{R}^{n-1})} \\ &\le C\|\partial_{\tan}\rho_\varepsilon\|_{L^1(\mathbb{R}^{n-1})} \end{aligned}$$

for $t<1$ by (6.14). Thus (4.3) and (4.2) hold for $(\partial_{\tan}u_\varepsilon,\partial_{\tan}p_\varepsilon)$ in the shifted half space $\mathbb{R}^n_{+,-c_0}$. Additionally $\nabla\partial_{\tan}u_\varepsilon$ is bounded for each $0<t<1$ by (6.14). Thus by Theorem 4.1 we can conclude that $\partial_{\tan}u_\varepsilon=0$ and $\nabla\partial_{\tan}p_\varepsilon=0$. Since ε was arbitrary, we get $\partial_{\tan}u=0$ and $\nabla\partial_{\tan}p=0$. Thus $u=u(x_n,t)$ and $\nabla p=a(x_n,t)$.

By the divergence condition we then get $\partial_n u_n=0$ from which we can deduce that u_n may only depend on time. By the boundary conditions we obtain $u_n=0$. Thus $u=(u_{\tan}(t,x_n),0)$ and therefore $\nabla p=a(t)$. The pressure term ∇p then needs to vanish by (6.15).

Then each u_i $(1\le i\le n-1)$ fulfills

$$\int_0^1\int_{-c_0}^\infty u_i(x_n,t)(\partial_t\varphi(x_n,t)+\partial_n^2\varphi(x_n,t))\,dx_n\,dt=0 \tag{6.16}$$

for all $\varphi\in C_c^\infty((-c_0,\infty)\times[0,1))$ and satisfies the estimate

$$|u_i(t,x_n)|\le(x_n+c_0)\|\nabla u\|_\infty\le t^{-1/2}(x_n+c_0)$$

for $t\in(0,1]$ and $x_n>-c_0$. Thus $u_i=0$ by Corollary 3.3. We obtained $u=0$ and $\nabla p=0$. Therefore $\tilde N(u_m,p_m)$ needs to converge to 0 for every $t\in(0,1]$, $x\in\mathbb{R}^n_{+,-c_0}$, which is a contradiction to (6.7). □

This theorem can be extended to the closure of $C^\infty_{c,\sigma}(\Omega)$ in $BMO_b^{\mu,\nu}$, which is $VMO^{\mu,\nu}_{b,0,\sigma}(\Omega)$, and thus Theorem 1.1 is proved.

Remark 6.3. In the case of $\Omega=\mathbb{R}^n$, $\mu\in(0,\infty]$ there are T_0 and C only depending on μ such that

$$\sup_{0<t<T_0}\|\tilde N(v,0)(\cdot,t)\|_\infty\le C[v_0]^\mu_{BMO}$$

for each $v_0\in VMO^\mu$ with $S(t)v_0=v$. The proof is similar to case 1 in the proof of the Theorem 1.1. We will show in the appendix that for $\mu=\infty$ this estimate holds without any restriction on time.

Remark 6.4. The statement of Theorem 1.1 is not only valid for the class of BMO_b-norms and the corresponding spaces we defined but also for a definition of BMO_b used by A. Miyachi ([Miy90]) that resembles ours. The definition used there is

$$\begin{aligned} [f]_{BMO}&=\sup\{\frac{1}{|B_r(x)|}\int_{B_r(x)}|f-f_{B_r(x)}|\,dy : B_{2r}(x)\subset\Omega\} \\ [f]_b&=\sup\{\frac{1}{|B_r(x)|}\int_{B_r(x)}|f|\,dy : B_{2r}(x)\subset\Omega \text{ and } B_{5r}(x)\cap\Omega^c\neq\emptyset\}. \end{aligned}$$

Case 1 of the proof of the main theorem can then be treated in the same way. For case 2 one has to reprove the convergence of

$$\int_{\mathbb{R}^n_{+,-c_0}}u_m(x,0)\varphi(x,0)\,dx \tag{6.17}$$

for $\varphi \in C_c^\infty(\mathbb{R}^n_{+,-c_0} \times [0,1))$ since the balls do not contain the boundary any more. The strategy is to cover $\operatorname{supp}\varphi$ by finitely many balls satisfying $B_{2r} \subset \mathbb{R}^n_{+,-c_0}$ and $B_{5r} \cap (\mathbb{R}^n_{+,-c_0})^c \neq \emptyset$. Then one can show by a similar argumentation as in case 2 that the integral over each of these balls converges to 0. Since there are only finitely many balls the integral (6.17) converges to 0.

Example 6.5. Finally we will give an expample that the additional term $[\cdot]_b$ in the above estimate is in fact necessary. Consider (1.1) in a half space $\mathbb{R}^n_+$ and assume constant initial data $v_0 = (c_1, \dots, c_{n-1}, 0) \neq 0$. Consider the solution $\bar{v}$ of the heat equation in $\mathbb{R}^n$ obtained by the heat kernel with initial data $\bar{v_0}$, the odd extension of v_0. Then $v(t) = \bar{v}(t)|_{\mathbb{R}^n_+}$ with $q = 0$ is a solution to (1.1) with $\partial_{\tan} v = 0$ and $[v_0]^\mu_{BMO} = 0$ ($\mu \in (0, \infty]$) but non-vanishing normal derivative $\partial_n v \neq 0$ and non-vanishing time derivative $v_t \neq 0$.

Appendix: $L^\infty - BMO$ estimate for the heat semigroup

In this section we will give a simple proof of (1.2) for the reader's convenience.

Theorem A.6. *For a given $v_0 \in BMO(\mathbb{R}^n)$ ($n \geq 1$) the L^∞-norm of $\nabla G_t * v_0$ is estimated as*

$$t^{1/2}\|\nabla G_t * v_0\|_\infty \leq C_*[v_0]_{BMO} \quad (t > 0)$$

with a constant C_* depending only on n.

Proof. It suffices to prove that

$$t^{1/2}\|\nabla G_t * u_0\|_{\mathcal{H}^1} \leq C\|u_0\|_1 \tag{A.1}$$

for $u_0 \in L^1(\mathbb{R}^n)$ since $(\mathcal{H}^1)^* = BMO$ by [FS72], where $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^n)$ denotes the Hardy space consisting of all $f \in L^1(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{H}^1} = \|\sup_{s>0}|G_s * f|\|_1 = \int_{\mathbb{R}^n} \sup_{s>0}|G_s * f|(x)\,dx < \infty.$$

Indeed by duality

$$\|\partial_j G_t * v_0\|_\infty = \sup\left\{\left|\int_{\mathbb{R}^n} u_0(\partial_j G_t) * v_0\,dx\right| : \|u_0\|_1 \leq 1\right\}$$

for $j = 1, \dots, n$. Since $(\mathcal{H}^1)^* = BMO$ and by the antisymmetry of $\partial_j G_t$ we observe that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} u_0((\partial_j G_t) * v_0)\,dx\right| &= \left|-\int_{\mathbb{R}^n}((\partial_j G_t) * u_0)v_0\,dx\right| \\ &\leq \|\partial_j G_t * u_0\|_{\mathcal{H}^1}[v_0]_{BMO}.\end{aligned}$$

The estimate (A.1) now yields

$$\|\partial_j G_t * v_0\|_\infty \leq Ct^{-1/2}[v_0]_{BMO}.$$

The proof of (A.1) is given in [Car96] and [GMS99]. We will give it here for the sake of completeness. We first show (A.1) for $t = 1$. By definition we observe that

$$\begin{aligned}\|\partial_j G_1 * u_0\|_{\mathcal{H}^1} &= \|\sup_{s>0} |G_s * \partial_j G_1 * u_0|\|_1 \\ &\le \|\sup_{s>0} (|\partial_j G_{s+1}| * |u_0|)\|_1 \\ &\le \|(\sup_{s>0} |\partial_j G_{s+1}|) * |u_0|\|_1 \\ &\le \|\sup_{s>0} |\partial_j G_{s+1}|\|_1 \|u_0\|_1.\end{aligned}$$

Since $\partial_j G_t = -x_j G_t/(2t)$, we observe that

$$|\partial_j G_t(x)| \le \frac{2|x_j|}{|x|^{n+2}} \frac{1}{\pi^{n/2}} \varrho^{\frac{n+2}{n}} \mathrm{e}^{-\varrho} \quad \text{with} \quad \varrho = \frac{|x|^2}{4t}.$$

This implies

$$|\partial_j G_t(x)| \le \frac{C_0}{|x|^{n+1}}$$

with $C_0 = 2\pi^{-n/2} \sup_{\varrho>0} \varrho^{\frac{n+2}{n}} \mathrm{e}^{-\varrho}$. Additionally we can estimate $|\partial_j G_t(x)| \le (4\pi)^{-n/2}$ for $t \ge 1$ and thus observe that

$$|\partial_j G_{s+1}(x)| \le \min\{\frac{C_0}{|x|^{n+1}}, \frac{1}{(4\pi)^{n/2}}\} =: a(x) \text{ for } s > 0, x \in \mathbb{R}^n.$$

The right hand side is integrable in x and therefore

$$\|\partial_j G_1 * u_0\|_{\mathcal{H}^1} \le C_* \|u_0\|_1,$$

where $C_* = \int_{\mathbb{R}^n} a(x)\,dx$. We have now proved (A.1) with $t = 1$. To obtain (A.1) for general t we apply a scaling transformation. For $\lambda > 0$ and a function f in $\mathbb{R}^n$ let f_λ be defined by $f_\lambda(x) = \lambda^n f(\lambda x)$. We notice that

$$(\partial_j G_1) * (u_0)_\lambda = \lambda((\partial_j G_{\lambda^2}) * u_0)_\lambda.$$

Since both the $\mathcal{H}^1$-norm and the L^1-norm are invariant under this scaling transformation, the estimate (A.1) with $t = 1$ yields

$$\lambda \|\partial_j G_{\lambda^2} * u_0\|_{\mathcal{H}^1} \le C_* \|u_0\|_1$$

which yields (A.1) by taking $\lambda = t^{1/2}$.

Note that our proof implies

$$\|\partial_j G_1\|_{\mathcal{H}^1} \le C_* \text{ and } \|\partial_j G_t\|_{\mathcal{H}^1} \le C_* t^{-1/2},$$

which was established by [Car96]. □

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