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Title	On isomorphism for the space of solenoidal vector fields and its application to the Stokes problem
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Citation	Hokkaido University Preprint Series in Mathematics, 1076, 1-15
Issue Date	2015-08-27
DOI	https://doi.org/10.14943/84220
Doc URL	https://hdl.handle.net/2115/69880
Type	departmental bulletin paper
File Information	re1076.pdf



On isomorphism for the space of solenoidal vector fields and its application to the Stokes problem

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Abstract

We consider the space of solenoidal vector fields in an unbounded domain $\Omega \subset \mathbb{R}^n$ whose boundary is given as a Lipschitz graph. It is shown that, under suitable functional setting, the space of solenoidal vector fields is isomorphic to the $n - 1$ product space of the space of scalar functions. As an application, we introduce a natural and systematic reduction of the equations describing the motion of incompressible flows. This gives a new perspective of the derivation of Ukai's solution formula for the Stokes equations in the half space, and provides a key step for the generalization of Ukai's approach to the Stokes semigroup in the case of the curved boundary.

Keywords: Solenoidal vector fields, incompressible flows, Helmholtz projection, reduction argument, factorization of elliptic operators

2010 Mathematics Subject Classification: 35J05, 35J15, 35J25, 35Q35, 76B99, 76D07

1 Introduction

In the analysis of incompressible flows the space of solenoidal vector fields in a given domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is clearly a fundamental class to be studied. Under suitable conditions on Ω and its boundary $\partial\Omega$ the space is characterized as the set of all vector fields $u = (u_1, \dots, u_n)$ such that

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \quad u \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where $\mathbf{n}$ is the unit exterior normal vector to $\partial\Omega$. Since (1.1) is considered as a boundary value problem of one partial differential equation, it is heuristically expected that the degree of freedom is $n - 1$ for the space of solenoidal vector fields. For example, when $\Omega = \mathbb{R}_+^n = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n > 0\}$ the equation (1.1) is written as $\partial_n u_n = -\nabla' \cdot u'$ in $\mathbb{R}_+^n$ and $\gamma u_n = 0$ on $\partial\mathbb{R}_+^n$, where $\nabla' = (\partial_1, \dots, \partial_{n-1})$, $u' = (u_1, \dots, u_{n-1})^\top$, and γ is the trace

operator to the boundary $\partial\mathbb{R}_+^n$. Thus the normal component u_n is formally recovered as $u_n = -\int_0^{x_n} \nabla' \cdot u' dy_n$, and then the solenoidal vector field is given as an image of the map

$$u' \mapsto (u', -\int_0^{x_n} \nabla' \cdot u' dy_n). \quad (1.2)$$

However, this characterization is not useful in practice, for the map (1.2) is in general not bounded from X^{n-1} to X^n , where X is a Banach space of functions in Ω such as the standard Lebesgue spaces $L^p(\Omega)$.

The aim of this paper is to construct an isomorphism between X^{n-1} and the space of solenoidal vector fields in X^n . To be precise, it will be convenient to set up our problem in an abstract manner. Let Ω be a domain in $\mathbb{R}^n$ and let $X(\Omega)$ be a Banach space of functions in Ω satisfying $C_0^\infty(\Omega) \subset X(\Omega) \subset L_{loc}^1(\Omega)$. The space of solenoidal vector fields in $(X(\Omega))^n$, denoted by $X_\sigma(\Omega)$, is defined as

$$X_\sigma(\Omega) = \overline{\{u \in (C_0^\infty(\Omega))^n \mid \operatorname{div} u = 0 \text{ in } \Omega\}}^{\|u\|_{X(\Omega)}}. \quad (1.3)$$

Here we have written $\|u\|_{X(\Omega)}$ for $\|u\|_{(X(\Omega))^n}$ to simplify the notation. We call two Banach spaces X and Y *isomorphic* if there is a bounded and bijective linear operator $L : X \rightarrow Y$. We write $X \simeq Y$ when X and Y are isomorphic. Then our problem is to show that $X_\sigma(\Omega) \simeq (X(\Omega))^{n-1}$.

In the following paragraph we assume that $\Omega = \mathbb{R}^n$ or

$$\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n > \eta(x')\}, \quad (1.4)$$

where $\eta \in L_{loc}^1(\mathbb{R}^{n-1})$ is a given Lipschitz function, i.e., $\|\nabla' \eta\|_{L^\infty} < \infty$. When Ω is of the form (1.4) we introduce the anisotropic Lebesgue spaces $Y^{q,r}(\Omega)$ as in [15] by using the homeomorphism $\Phi : \Omega \ni x \mapsto y = \Phi(x) \in \mathbb{R}_+^n$:

$$\Phi_j(x) = \begin{cases} x_j, & 1 \leq j \leq n-1, \\ x_n - \eta(x'), & j = n. \end{cases} \quad (1.5)$$

That is, for $1 < q, r < \infty$ the Banach space $Y^{q,r}(\Omega)$ is defined as

$$Y^{q,r}(\Omega) = \{f \in L_{loc}^1(\Omega) \mid \|f\|_{Y^{q,r}(\Omega)} = \|f \circ \Phi^{-1}\|_{L_{y_n}^q(\mathbb{R}_+; L_{y'}^r(\mathbb{R}^{n-1}))} < \infty\} \quad (1.6)$$

with the norm $\|\cdot\|_{Y^{q,r}(\Omega)}$. The space $Y^{q,r}(\Omega)$ can be naturally defined also for $\Omega = \mathbb{R}^n$ as

$$Y^{q,r}(\mathbb{R}^n) = \{f \in L_{loc}^1(\mathbb{R}^n) \mid \|f\|_{Y^{q,r}(\mathbb{R}^n)} = \|f\|_{L_{x_n}^q(\mathbb{R}; L_{x'}^r(\mathbb{R}^{n-1}))} < \infty\}. \quad (1.7)$$

We note that $Y^{q,q}(\Omega)$ coincides with $L^q(\Omega)$. Our result is stated as follows.

Theorem 1.1. *Let $1 < q, r < \infty$ and assume that Ω is of the form (1.4) with some $\eta \in L_{loc}^1(\mathbb{R}^{n-1})$ satisfying $\|\nabla' \eta\|_{L^\infty} < \infty$. Then $Y_\sigma^{q,r}(\Omega) \simeq (Y^{q,r}(\Omega))^{n-1}$.*

The construction of the isomorphism in Theorem 1.1 is motivated by Ukai [22], and we have its explicit representation in terms of the Riesz transform $\nabla'(-\Delta')^{-1/2}$ and the Poisson semigroup $\{e^{-x_n(-\Delta')^{1/2}}\}_{x_n \geq 0}$ in $L^r(\mathbb{R}^{n-1})$. Furthermore, in the case $\Omega = \mathbb{R}^n, \mathbb{R}_+^n$ this isomorphism from $Y_\sigma^{q,r}(\Omega)$ to $(Y^{q,r}(\Omega))^{n-1}$ has an additional structure; it is a restriction of a bounded linear operator $\mathbf{W} : (Y^{q,r}(\Omega))^n \rightarrow (Y^{q,r}(\Omega))^{n-1}$ which enjoys the property

$$\{\nabla p \in (Y^{q,r}(\Omega))^n \mid p \in L_{loc}^1(\Omega), \Delta p = 0 \text{ in } \Omega\} \subset \operatorname{Ker} \mathbf{W} = \{f \in (Y^{q,r}(\Omega))^n \mid \mathbf{W}f = 0\}. \quad (1.8)$$

In fact, the kernel property such as (1.8) plays an important role in the analysis of the Stokes operator in [22]. Unfortunately, for general case $\Omega \neq \mathbb{R}^n, \mathbb{R}_+^n$, the isomorphism in Theorem 1.1 is not a restriction of the operator satisfying (1.8). Therefore, a natural question is whether we can construct an operator $\mathbf{W} : (Y^{q,r}(\Omega))^n \rightarrow (Y^{q,r}(\Omega))^{n-1}$ so that (1.8) holds and its restriction to $Y_\sigma^{q,r}(\Omega)$ defines an isomorphism from $Y_\sigma^{q,r}(\Omega)$ to $(Y^{q,r}(\Omega))^{n-1}$. Our next result is as follows.

Theorem 1.2. *Let $1 < q < \infty$ and assume that Ω is of the form (1.4) with some $\eta \in L_{loc}^1(\mathbb{R}^{n-1})$ satisfying $\|\nabla' \eta\|_{L^\infty} < \infty$. Then there is a bounded linear operator $\mathbf{W} : (Y^{q,2}(\Omega))^n \rightarrow (Y^{q,2}(\Omega))^{n-1}$ enjoying the following properties.*

- (i) $\mathbf{W}$ satisfies (1.8) for $r = 2$.
- (ii) The restriction $\mathbf{W}|_{Y_\sigma^{q,2}(\Omega)} : Y_\sigma^{q,2}(\Omega) \rightarrow (Y^{q,2}(\Omega))^{n-1}$ is an isomorphism.

When $\Omega = \mathbb{R}^n, \mathbb{R}_+^n$ the above assertion holds for $Y^{q,r}(\Omega)$ with $1 < q, r < \infty$.

In Theorem 1.2 so far we need a strong condition $r = 2$ for the space $Y^{q,r}(\Omega)$ except for the case $\Omega = \mathbb{R}^n, \mathbb{R}_+^n$. This is because the regularity of η assumed here is rather mild, and moreover, η in Theorem 1.2 is allowed to behave wildly at infinity. For example, the boundary need not to be asymptotically flat (this means $|\nabla' \eta(x)| \rightarrow 0$ as $|x'| \rightarrow \infty$) and η may even grow linearly as $|x'| \rightarrow \infty$. It seems that the existence of $\mathbf{W}$ in Theorem 1.2 is closely related with the validity of the Helmholtz decomposition in $(Y^{q,r}(\Omega))^n$ for $r = 2$ (cf. [15]), and therefore, the assertion as in Theorem 1.2 might fail for some $r \neq 2$ if one does not impose any other condition than $\|\nabla' \eta\|_{L^\infty} < \infty$; see, e.g., [2] for a counterexample of the Helmholtz decomposition in $L^p(\Omega)$ when Ω is of the form (1.4).

In these theorems we concretely construct an isomorphism in terms of the Poisson semi-group and the Dirichlet-Neumann map associated with the Laplace equations: $\Delta u = 0$ in Ω , $u = g$ on $\partial\Omega$. This construction is particularly nontrivial when Ω is of the form (1.4) with $\eta \neq 0$. The key tool here is a factorization of divergence form elliptic operators in [13, 14], which is considered as an operator theoretical description of the classical Rellich identity [20].

Thanks to (1.8), the isomorphism obtained in Theorem 1.2 is useful in the analysis of fluid equations. Indeed, it reduces the equations describing the motion of incompressible flows, which usually consists of $n + 1$ equations due to the unknowns of the solenoidal velocity $u = (u_1, \dots, u_n)$ and the scalar pressure p , into the equations of $n - 1$ dependent variables. As a typical example, let us consider the Stokes equations

$$\left\{ \begin{array}{ll} \partial_t u - \nu \Delta u + \nabla p = 0, & t > 0, \quad x \in \Omega, \\ \operatorname{div} u = 0, & t \geq 0, \quad x \in \Omega, \\ u = 0, & t > 0, \quad x \in \partial\Omega, \\ u|_{t=0} = a, & x \in \Omega. \end{array} \right. \quad (\text{S})$$

Here $\nu > 0$ is a viscosity coefficient and a is a given solenoidal vector field. By formally introducing the Helmholtz projection $\mathbf{P}$ and the Stokes operator $\mathbf{A} = -\mathbf{P}\Delta_D$ with the homogeneous Dirichlet boundary condition (these are well-defined at least in the L^2 functional framework), (S) is written in the abstract form

$$\frac{du}{dt} + \nu \mathbf{A}u = 0, \quad t > 0, \quad u|_{t=0} = a. \quad (1.9)$$

Let $\mathbf{V} : (Y^{q,r}(\Omega))^{n-1} \rightarrow Y_\sigma^{q,r}(\Omega)$ be an isomorphism. Then by setting

$$w(t) = (w_1(t), \dots, w_{n-1}(t))^\top = \mathbf{V}^{-1}u(t),$$

we obtain the reduced equations

$$\frac{dw}{dt} + \nu \mathbf{B}w = 0, \quad t > 0, \quad w|_{t=0} = \mathbf{V}^{-1}a, \quad (\text{RS})$$

where $\mathbf{B} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$. If $\mathbf{V}^{-1}$ is a restriction of $\mathbf{W}$ on $Y_{\sigma}^{q,r}(\Omega)$ satisfying (1.8) then we formally have

$$\mathbf{B} = -\mathbf{W}\mathbf{P}\Delta_D\mathbf{V} = -\mathbf{W}(\Delta_D - \mathbf{Q}\Delta_D)\mathbf{V} = -\mathbf{W}\Delta_D\mathbf{V},$$

where $\mathbf{Q} = I - \mathbf{P}$ and Δ_D is the Laplace operator subject to the homogeneous Dirichlet boundary condition. As will be shown in [16], when Ω is of the form (1.4) with a smooth η our isomorphism provides

$$\mathbf{B} = -\Delta_D + \text{lower order operators} \quad \text{in } (Y^{q,2}(\Omega))^{n-1}.$$

Furthermore, we have $\mathbf{B} = -\Delta_D$ in $(Y^{q,r}(\Omega))^{n-1}$ when $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$; see Section 3.4. In any cases, this reduction significantly simplifies the original equations. The idea to achieve such a reduction is inspired by the derivation of the solution formula for the Stokes problem in $\mathbb{R}_+^n$ in [22], although the characterization of the space of solenoidal vector fields as in Theorems 1.1 and 1.2 is not observed in [22]; see Remark 1.4 (ii) below. When $\Omega = \mathbb{R}_+^n$ (and $\mathbb{R}^n$), due to the relation $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = -\Delta_D$, the Stokes semigroup $\{e^{-t\mathbf{A}}\}_{t \geq 0}$ associated with $-\mathbf{A}$ is expressed as $e^{-t\mathbf{A}} = \mathbf{V}e^{t\Delta_D}\mathbf{V}^{-1}$. In terms of the general semigroup theory the relation $e^{-t\mathbf{A}} = \mathbf{V}e^{t\Delta_D}\mathbf{V}^{-1}$ represents the similarity or isomorphy between the Stokes semigroup and the heat semigroup (we say that two semigroups $\{S(t)\}_{t \geq 0}$ in a Banach space X and $\{T(t)\}_{t \geq 0}$ in another Banach space Y are *isomorphic* if there is an isomorphism $L : X \rightarrow Y$ such that $S(t) = L^{-1}T(t)L$; see [4, I. 5.10], and we will write $(\{S(t)\}_{t \geq 0}, X) \simeq (\{T(t)\}_{t \geq 0}, Y)$ in this case). That is, we have

Theorem 1.3. *Let $1 < q, r < \infty$ and let $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$. Then the Stokes semigroup in $Y_{\sigma}^{q,r}(\Omega)$ and the heat semigroup in $(Y^{q,r}(\Omega))^{n-1}$ are isomorphic. That is,*

$$(\{e^{-t\mathbf{A}}\}_{t \geq 0}, Y_{\sigma}^{q,r}(\Omega)) \simeq (\{e^{t\Delta_D}\}_{t \geq 0}, (Y^{q,r}(\Omega))^{n-1}).$$

Remark 1.4. (i) When $\Omega = \mathbb{R}^n$ the Helmholtz projection commutes with the Laplace operator. Thus the Stokes semigroup in $Y_{\sigma}^{q,r}(\mathbb{R}^n)$ coincides with the heat semigroup in the invariant subspace $Y_{\sigma}^{q,r}(\mathbb{R}^n) \subset (Y^{q,r}(\mathbb{R}^n))^n$. It should be emphasized that Theorem 1.3 is *not* the same assertion as this fundamental fact. As far as the authors know, the isomorphic relation stated in Theorem 1.3 is not found in the literature even in the case $\Omega = \mathbb{R}^n$.

(ii) In [22, Theorem 1.1] the solution formula for $u(t) = e^{-t\mathbf{A}}a$ in $\mathbb{R}_+^n$ is given as

$$u'(t) = e^{t\Delta_D}(a' + Sa_n) - SUe^{t\Delta_D}(-S \cdot a' + a_n), \quad (1.10)$$

$$u_n(t) = Ue^{t\Delta_D}(-S \cdot a' + a_n), \quad (1.11)$$

where $S = \nabla'(-\Delta')^{-1/2}$ and $(U\varphi)(x_n) = (-\Delta')^{1/2} \int_0^{x_n} e^{-(x_n-y_n)(-\Delta')^{1/2}} \varphi(\cdot, y_n) dy_n$. In fact, the map $\mathbf{W} : (Y^{q,r}(\mathbb{R}_+^n))^n \rightarrow (Y^{q,r}(\mathbb{R}_+^n))^{n-1}$ is given as $\mathbf{W} = E' + SE_n$, where $E'u := u'$ and $E_nu := u_n$ when $\Omega = \mathbb{R}_+^n$. In the argument of [22] the relation $(E' + SE_n)e^{-t\mathbf{A}} = e^{t\Delta_D}(E' + SE_n)$ is already found and it is a key to derive (1.10) - (1.11) in [22]. On the other hand, our argument in Theorem 1.2 reveals that $E' + SE_n$ actually defines an isomorphism from $Y_{\sigma}^{q,r}(\mathbb{R}_+^n)$ onto $(Y^{q,r}(\mathbb{R}_+^n))^{n-1}$, and it is also shown that such a description is generic for a wide class of domains like (1.4) including the whole space. Moreover, our approach leads to the isomorphic formulation between the Stokes semigroup in $Y_{\sigma}^{q,r}(\Omega)$ and the heat (or perturbed heat) semigroup in $(Y^{q,r}(\Omega))^{n-1}$ as in Theorem 1.3 (or in [16]).

(iii) When $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$, it is not difficult to see that the Laplace operator generates an analytic semigroup in $(Y^{q,r}(\Omega))^{n-1}$. Therefore from Theorem 1.3 we see that the Stokes operator also generates an analytic semigroup in $Y_\sigma^{q,r}(\Omega)$. For the case $r = q$, this fact has already proved in many literature, e.g., [21, 17, 7, 22, 5, 3]. In fact, in the same reason, it is proved that the Stokes operator admits a bounded H^∞ -calculus in $Y_\sigma^{q,r}(\Omega)$. See [19, 3] for the fact that the Laplace (or Stokes) operator in $L^p(\Omega)$ (or $L_\sigma^p(\Omega)$ respectively) admits a bounded H^∞ -calculus.

(iv) As a by product of our construction of the isomorphism, we obtain a projection onto the space of solenoidal vector fields in the domain of the form (1.4), which was found in [22, Remark 1.5] for $\Omega = \mathbb{R}_+^n$ and is different from the standard Helmholtz projection; see Remark 3.6.

This paper is organized as follows. In Section 2.1 we give a characterization of $Y_\sigma^{q,r}(\Omega)$ and in Section 2.2 we recall the result of [13] on the factorization of some class of elliptic operators. In Sections 3.1 and 3.2 we prove Theorem 1.2. Theorem 1.1 is proved in Section 3.3, while Theorem 1.3 is proved in Section 3.4.

2 Preliminaries

2.1 Characterization of $Y_\sigma^{q,r}(\Omega)$

In this section we show

Lemma 2.1. *Let $1 < q, r < \infty$ and assume that Ω is of the form (1.4) with some $\eta \in L_{loc}^1(\mathbb{R}^{n-1})$ satisfying $\|\nabla' \eta\|_{L^\infty} < \infty$. Then*

$$Y_\sigma^{q,r}(\Omega) = \{u \in (Y^{q,r}(\Omega))^n \mid \operatorname{div} u = 0 \text{ in } \Omega, \quad u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}. \quad (2.1)$$

Proof. It suffices to show the inclusion $\supset$, for the opposite one is trivial. For convenience we denote by $L^{q,r}(\Omega)$ the set in the right-hand side of (2.1). The case $\Omega = \mathbb{R}^n$ is easy and we omit the details. Firstly let us consider the case $\Omega = \mathbb{R}_+^n$. For $u \in L^{q,r}(\mathbb{R}_+^n)$ we introduce its extension U to $\mathbb{R}^n$ as

$$\begin{aligned} U(x) &= u(x) \quad \text{if } x_n > 0, \\ U'(x', x_n) &= u'(x', -x_n) \quad \text{and} \quad U_n(x', x_n) = -u_n(x', -x_n) \quad \text{if } x_n < 0. \end{aligned}$$

Then the mollification $U_\epsilon = J_\epsilon * U$ of U by the radial symmetric mollifier J_ϵ is smooth in $\mathbb{R}^n$ and converges to U in $(Y^{q,r}(\mathbb{R}^n))^n$ as $\epsilon \rightarrow 0$. Moreover, U_ϵ satisfies $\operatorname{div} U_\epsilon = 0$ in $\mathbb{R}^n$ and $U_{\epsilon,n} = 0$ for $x_n = 0$ by the symmetry. Therefore, by considering the restriction of U_ϵ on $\mathbb{R}_+^n$, the problem is now reduced to construct the approximation $\{v_j\}_{j=1}^\infty \in C_{0,\sigma}^\infty(\mathbb{R}^n)$ of u in $L^{q,r}(\mathbb{R}_+^n)$ when $\nabla^\alpha u$ belongs to $(Y^{q,r}(\mathbb{R}_+^n))^n$ for any multi-index α . For such u we set

$$w = u' + Su_n, \quad S = \nabla'(-\Delta')^{-\frac{1}{2}}.$$

Then $\nabla^\alpha w \in (Y^{q,r}(\mathbb{R}_+^n))^{n-1}$ for any α since the Riesz transform S is bounded in $L^r(\mathbb{R}^{n-1})$ for $1 < r < \infty$. Let us take a sequence $\{\varphi_j\}_{j=1}^\infty \subset (C_0^\infty(\mathbb{R}_+^n))^{n-1}$ such that $\varphi_j \rightarrow w$ in $(Y^{q,r}(\mathbb{R}_+^n))^{n-1}$ holds. Then we define $\psi_j = (\psi'_j, \psi_{j,n})^\top = \mathbf{V}[\varphi_j]$ as

$$\begin{aligned} \psi'_j &= \varphi'_j + S(-\Delta')^{\frac{1}{2}} \int_0^{x_n} e^{-(x_n-y_n)(-\Delta')^{\frac{1}{2}}} (-\Delta')^{-\frac{1}{2}} \nabla' \cdot \varphi_j(y_n) dy_n, \\ \psi_{j,n} &= -(-\Delta')^{\frac{1}{2}} \int_0^{x_n} e^{-(x_n-y_n)(-\Delta')^{\frac{1}{2}}} (-\Delta')^{-\frac{1}{2}} \nabla' \cdot \varphi_j(y_n) dy_n. \end{aligned}$$

As is seen in Section 3.2, $\mathbf{V}$ is bounded from $(Y^{p,s}(\mathbb{R}_+^n))^{n-1}$ to $(Y^{p,s}(\mathbb{R}_+^n))^n$ for $1 < p, s < \infty$ and we can directly check that $\nabla^\alpha \psi_j \in (Y^{p,s}(\mathbb{R}_+^n))^n$ for any α , and ψ_j also satisfies $\operatorname{div} \psi_j = 0$ in $\mathbb{R}_+^n$. Moreover, since $\varphi_j = 0$ near $\partial\mathbb{R}_+^n$, we have $\operatorname{dist}(\operatorname{supp} \psi_j, \partial\mathbb{R}_+^n) > 0$ by the definition of ψ_j . Then we can construct $\{v_{j,l}\}_{l=1}^\infty \subset C_{0,\sigma}^\infty(\mathbb{R}_+^n)$ so that $v_{j,l} \rightarrow \psi_j$ in $(W^{m,p}(\mathbb{R}_+^n))^n$ as $l \rightarrow \infty$ for any $1 < p < \infty$ and $m \in \mathbb{N}$ (see, e.g., [6, Section III]). By the embedding $W^{m,p}(\mathbb{R}_+^n) \hookrightarrow Y^{q,r}(\mathbb{R}_+^n)$ for $p = \min\{q, r\}$ and sufficiently large m , we observe that $v_{j,l}$ is the desired approximation of u in $(Y^{q,r}(\mathbb{R}_+^n))^n$, since ψ_j converges to $\mathbf{V}[w]$ in $(Y^{q,r}(\mathbb{R}_+^n))^n$, while $\mathbf{V}[w] = u$ by the result of Section 3.2.

Next we consider the case when Ω is of the form (1.4) with nontrivial η . Let $u \in L^{q,r}(\Omega)$. Then $\tilde{u} = u \circ \Phi^{-1} \in (Y^{q,r}(\mathbb{R}_+^n))^n$, where $\Phi : \Omega \rightarrow \mathbb{R}_+^n$ be as in (1.5), satisfies $\operatorname{div} B^\top \tilde{u} = 0$ in $\mathbb{R}_+^n$ and $(B^\top \tilde{u})_n = 0$ on $\partial\mathbb{R}_+^n$ with the invertible matrix $B = (b_{i,j})_{1 \leq i,j \leq n}$ defined as $b_{i,j} = \delta_{ij}$ for $1 \leq i, j \leq n-1$, $b_{i,n} = -\partial_i \eta$ for $1 \leq i \leq n-1$, $b_{n,j} = 0$ for $1 \leq j \leq n-1$, and $b_{n,n} = 1$. Hence, $B^\top \tilde{u} \in Y_{\sigma}^{q,r}(\mathbb{R}_+^n)$ and there is $\{\tilde{\varphi}_j\}_{j=1}^\infty \subset C_{0,\sigma}^\infty(\mathbb{R}_+^n)$ such that $\tilde{\varphi}_j \rightarrow B^\top \tilde{u}$ in $(Y^{q,r}(\mathbb{R}_+^n))^n$. Set $\tilde{\psi}_j = (B^\top)^{-1} \tilde{\varphi}_j$, which converges to $\tilde{u}$ in $(Y^{q,r}(\mathbb{R}_+^n))^n$ as $j \rightarrow \infty$. Then $\operatorname{div} B^\top \tilde{\psi}_j = 0$ in $\mathbb{R}_+^n$, $\operatorname{supp} \tilde{\psi}_j$ is compact, and $\tilde{\psi}_j = 0$ near $\partial\mathbb{R}_+^n$. Thus, $v_j = \tilde{\psi}_j \circ \Phi \in (Y^{q,r}(\Omega))^n$ is compactly supported, and also satisfies $\operatorname{div} v_j = 0$ in Ω and $v_j = 0$ near $\partial\Omega$. Then, by acting the standard mollifier $v_{j,\epsilon} = J_\epsilon * v_j$, $\epsilon > 0$, we see $v_{j,\epsilon} \in C_{0,\sigma}^\infty(\Omega)$ for sufficiently small $\epsilon > 0$. By the construction $v_{j,\epsilon}$ converges to v_j in $(Y^{q,r}(\Omega))^n$ as $\epsilon \rightarrow 0$, while v_j converges to u in $(Y^{q,r}(\Omega))^n$. The proof is complete.

2.2 Elliptic operator of divergence form

In this section we recall the result of [13] for some class of second order elliptic operators of divergence form. By the coordinate transform (1.5) the Laplace operator Δ is transformed to an elliptic operator of divergence form whose coefficients are independent of one variable. Taking this into mind, we consider the second order elliptic operator in $\mathbb{R}^n = \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}\}$,

$$\mathcal{A} = -\nabla \cdot A \nabla, \quad A = A(x') = (a_{i,j}(x'))_{1 \leq i,j \leq n}. \quad (2.2)$$

Here $n \in \mathbb{N}$, $\nabla = (\nabla', \partial_n)^\top$ with $\nabla' = (\partial_1, \dots, \partial_{n-1})^\top$, and each $a_{i,j}$ is always assumed to be t -independent. We further assume that A is a real symmetric matrix and each component $a_{i,j}$ is a measurable function satisfying the uniformly elliptic condition

$$\langle A(x')\eta, \eta \rangle \geq \nu_1 |\eta|^2, \quad |\langle A(x')\eta, \zeta \rangle| \leq \nu_2 |\eta| |\zeta| \quad (2.3)$$

for all $\eta, \zeta \in \mathbb{R}^n$ and for some constants ν_1, ν_2 with $0 < \nu_1 \leq \nu_2 < \infty$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^n$, i.e., $\langle \eta, \zeta \rangle = \sum_{j=1}^n \eta_j \zeta_j$ for $\eta, \zeta \in \mathbb{R}^n$. For later use we set $b = a_{n,n}$, which satisfies $\nu_1 \leq b \leq \nu_2$ due to (2.3). We also denote by $\mathbf{a}$ the vector $\mathbf{a}(x') = (a_{1,n}(x'), \dots, a_{n-1,n}(x'))^\top$.

We write $D_H(T)$ for the domain of a linear operator T in a Banach space H . Under the condition (2.3) the standard theory of sesquilinear forms gives a realization of $\mathcal{A}$ in $L^2(\mathbb{R}^n)$, denoted again by $\mathcal{A}$, such as

$$D_{L^2}(\mathcal{A}) = \{w \in H^1(\mathbb{R}^n) \mid \text{there is } F \in L^2(\mathbb{R}^n) \text{ such that} \\ \langle A \nabla w, \nabla v \rangle_{L^2(\mathbb{R}^n)} = \langle F, v \rangle_{L^2(\mathbb{R}^n)} \text{ for all } v \in H^1(\mathbb{R}^n)\}, \quad (2.4)$$

and $\mathcal{A}w = F$ for $w \in D_{L^2}(\mathcal{A})$. Here $H^1(\mathbb{R}^n)$ is the usual Sobolev space and $\langle w, v \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} w(x', t) v(x', t) dx' dt$.

Definition 2.2. (i) For a given $h \in \mathcal{S}'(\mathbb{R}^{n-1})$ we denote by $M_h : \mathcal{S}(\mathbb{R}^{n-1}) \rightarrow \mathcal{S}'(\mathbb{R}^{n-1})$ the multiplication $M_h u = hu$.

(ii) We denote by $E_{\mathcal{A}} : H^{1/2}(\mathbb{R}^{n-1}) \rightarrow \dot{H}^1(\mathbb{R}_+^n)$ the $\mathcal{A}$ -extension operator, i.e., $w = E_{\mathcal{A}}\varphi$ is the solution to the Dirichlet problem

$$\begin{cases} \mathcal{A}w = 0 & \text{in } \mathbb{R}_+^n, \\ w = \varphi & \text{on } \partial\mathbb{R}_+^n = \mathbb{R}^{n-1}. \end{cases} \quad (2.5)$$

The one parameter family of linear operators $\{E_{\mathcal{A}}(t)\}_{t \geq 0}$, defined by $E_{\mathcal{A}}(t)\varphi = (E_{\mathcal{A}}\varphi)(\cdot, t)$ for $\varphi \in H^{1/2}(\mathbb{R}^{n-1})$, is called the Poisson semigroup associated with $\mathcal{A}$.

(iii) We denote by $\Lambda_{\mathcal{A}} : H^{1/2}(\mathbb{R}^{n-1}) \rightarrow \dot{H}^{-1/2}(\mathbb{R}^{n-1}) = (\dot{H}^{1/2}(\mathbb{R}^{n-1}))^*$ the Dirichlet-Neumann map associated with $\mathcal{A}$, which is defined through the sesquilinear form

$$\langle \Lambda_{\mathcal{A}}\varphi, g \rangle_{\dot{H}^{-1/2}, \dot{H}^{1/2}} = \langle A \nabla E_{\mathcal{A}}\varphi, \nabla E_{\mathcal{A}}g \rangle_{L^2(\mathbb{R}_+^n)}, \quad \varphi, g \in H^{1/2}(\mathbb{R}^{n-1}). \quad (2.6)$$

Here $\langle \cdot, \cdot \rangle_{\dot{H}^{-1/2}, \dot{H}^{1/2}}$ denotes the duality coupling of $\dot{H}^{-1/2}(\mathbb{R}^{n-1})$ and $\dot{H}^{1/2}(\mathbb{R}^{n-1})$.

Remark 2.3. (i) Eq. (2.5) is considered in a weak sense; cf. [13, Section 2.1]. The proof of the existence of the extension operator $E_{\mathcal{A}}$ is well known. As is shown in [13, Proposition 2.4], $\{E_{\mathcal{A}}(t)\}_{t \geq 0}$ is a strongly continuous and analytic semigroup in $H^{1/2}(\mathbb{R}^{n-1})$. We denote its generator by $-\mathcal{P}_{\mathcal{A}}$, and $\mathcal{P}_{\mathcal{A}}$ is called a Poisson operator associated with $\mathcal{A}$. (ii) Since A is Hermite and satisfies the uniformly elliptic condition (2.3), the theory of the sesquilinear forms [11, Chapter VI. §2] shows that $\Lambda_{\mathcal{A}}$ is extended as a self-adjoint operator in $L^2(\mathbb{R}^{n-1})$.

The next result plays a fundamental role in our argument.

Theorem 2.4. Let $\mathcal{A}$ be the elliptic operator defined in (2.2) with a real symmetric matrix A satisfying (2.3). Then $D_{L^2}(\Lambda_{\mathcal{A}}) = H^1(\mathbb{R}^{n-1})$ holds with equivalent norms, and the Poisson semigroup $\{E_{\mathcal{A}}(t)\}_{t \geq 0}$ in $H^{1/2}(\mathbb{R}^{n-1})$ is extended as a strongly continuous and analytic semigroup in $L^2(\mathbb{R}^{n-1})$, where its generator $-\mathcal{P}_{\mathcal{A}}$ satisfies

$$D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^{n-1}), \quad -\mathcal{P}_{\mathcal{A}}\varphi = -M_{1/b}\Lambda_{\mathcal{A}}\varphi - M_{\mathbf{a}/b} \cdot \nabla' \varphi, \quad \varphi \in H^1(\mathbb{R}^{n-1}). \quad (2.7)$$

Furthermore, the realization $\mathcal{A}'$ in $L^2(\mathbb{R}^{n-1})$ and the realization $\mathcal{A}$ in $L^2(\mathbb{R}^n)$ are respectively factorized as

$$\mathcal{A}' = M_b \mathcal{Q}_{\mathcal{A}} \mathcal{P}_{\mathcal{A}}, \quad \mathcal{Q}_{\mathcal{A}} = M_{1/b} (M_b \mathcal{P}_{\mathcal{A}})^*, \quad (2.8)$$

$$\mathcal{A} = -M_b (\partial_t - \mathcal{Q}_{\mathcal{A}}) (\partial_t + \mathcal{P}_{\mathcal{A}}). \quad (2.9)$$

Here $(M_b \mathcal{P}_{\mathcal{A}})^*$ is the adjoint of $M_b \mathcal{P}_{\mathcal{A}}$ in $L^2(\mathbb{R}^{n-1})$.

For the proof of Theorem 2.4, see, e.g. [13, Theorem 1.3, Theorem 4.2]. The identities (2.8) and (2.9) are considered as an operator-theoretical description of the classical Rellich identity [20], but when the matrix A is not real symmetric and possesses a limited smoothness the verification of this identity becomes a delicate problem. The Rellich type identity is verified and used by [9] when A is real symmetric and by [1] when $\mathbf{r}_2 = 0$ without any extra regularity condition on A . See also [18, 10, 8] for the study of the elliptic boundary value problem in relation to the Rellich identity.

3 Proof of main theorems

In this section we prove Theorems 1.1, 1.2, and 1.3. We will use the notation given in Section 2.

3.1 Proof of Theorem 1.2 for general Ω

When Ω is of the form (1.4), through the standard transformation

$$u = \tilde{u} \circ \Phi^{-1}, \quad g = \tilde{g} \circ \Phi^{-1}, \quad \Phi \text{ is as in (1.5)},$$

the Laplace equations, $-\Delta \tilde{u} = 0$ in Ω and $\tilde{u} = \tilde{g}$ on $\partial\Omega$, are transformed to the elliptic equations in the half space

$$\mathcal{A}u = 0 \quad \text{in } \mathbb{R}_+^n, \quad u = g \quad \text{on } \partial\mathbb{R}_+^n. \quad (3.1)$$

Here $\mathcal{A} = -\nabla \cdot A \nabla$ and A is a real symmetric and positive definite matrix with $\mathbf{a} = -\nabla' \eta$, $b = 1 + |\nabla' \eta|^2$, and $A' = (a_{i,j})_{1 \leq i,j \leq n-1} = I'$ (the identity matrix). Note that each coefficient of A is independent of the y_n variable, and hence we can apply the result of Section 2. It is straightforward to see that the matrix A is written as $A = B^\top B$, where $B = (b_{i,j})_{1 \leq i,j \leq n}$ with $b_{i,j} = \delta_{ij}$ for $1 \leq i, j \leq n-1$, $b_{i,n} = -\partial_i \eta$ for $1 \leq i \leq n-1$, $b_{n,j} = 0$ for $1 \leq j \leq n-1$, and $b_{n,n} = 1$. The matrix $B^\top$ is the transpose of B . The key point here is that the solenoidal property in the original variables

$$\operatorname{div} \tilde{u} = 0 \quad \text{in } \Omega, \quad \tilde{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega$$

is equivalent with

$$\operatorname{div} B^\top u = 0 \quad \text{in } \mathbb{R}_+^n, \quad \gamma(B^\top u)_n = 0 \quad \text{on } \partial\mathbb{R}_+^n \quad (3.2)$$

in the new variables, where γ is the trace to the boundary $\partial\mathbb{R}_+^n$. Thus it is natural to introduce the space $Y_\sigma^{q,r}(\mathbb{R}_+^n)$ as

$$\begin{aligned} Y_\sigma^{q,r}(\mathbb{R}_+^n) &= \{ \tilde{u} \circ \Phi^{-1} \in (Y^{q,r}(\mathbb{R}_+^n))^n \mid \tilde{u} \in Y_\sigma^{q,r}(\Omega) \} \\ &= \{ u \in (Y^{q,r}(\mathbb{R}_+^n))^n \mid \operatorname{div} B^\top u = 0 \text{ in } \mathbb{R}_+^n, \quad \gamma(B^\top u)_n = 0 \text{ on } \partial\mathbb{R}_+^n \}, \end{aligned} \quad (3.3)$$

due to Lemma 2.1.

For a vector $v = (v', v_n)^\top \in \mathbb{R}^{n-1} \times \mathbb{R}$ we define the $(n-1) \times n$ matrix E' and the $1 \times n$ matrix E_n by the relation

$$E'v = (B^\top v)' = v', \quad E_nv = (B^\top v)_n = v_n - M_{\nabla' \eta} \cdot v'. \quad (3.4)$$

Set

$$S = (\nabla' + M_{\nabla' \eta} \mathcal{P}_A) \Lambda_A^{-1}, \quad K = -\Lambda_A^{-1} \nabla' \cdot E' + \mathcal{P}_A \Lambda_A^{-1} E_n. \quad (3.5)$$

As is proved in the next key lemma, S and K are extended as bounded operators in $L^2(\mathbb{R}^{n-1})$.

Lemma 3.1. *Let $j = 1, \dots, n-1$. Then the operators $\partial_j \Lambda_A^{-1}$, $\Lambda_A^{-1} \partial_j$, and $\mathcal{P}_A \Lambda_A^{-1}$ are extended as bounded operators from $L^2(\mathbb{R}^{n-1})$ to $L^2(\mathbb{R}^{n-1})$. Moreover, for any $f \in L^2(\mathbb{R}^{n-1})$ and $v \in (L^2(\mathbb{R}^{n-1}))^n$ we have*

$$\nabla' \cdot S f = -\Lambda_A \mathcal{P}_A \Lambda_A^{-1} f \quad \text{in } \dot{H}^{-1}(\mathbb{R}^{n-1}). \quad (3.6)$$

$$K v = -\Lambda_A^{-1} \nabla' \cdot (E' + S E_n) v. \quad (3.7)$$

Proof of Lemma 3.1. The fact that $\partial_j \Lambda_{\mathcal{A}}^{-1}$ and $\mathcal{P}_{\mathcal{A}} \Lambda_{\mathcal{A}}^{-1}$ are extended as bounded operators from $L^2(\mathbb{R}^{n-1})$ to $L^2(\mathbb{R}^{n-1})$ follows from Theorem 2.4 and the relations

$$\|M_{\sqrt{b}} \mathcal{P}_{\mathcal{A}} f\|_{L^2(\mathbb{R}^{n-1})} = \|\nabla' f\|_{L^2(\mathbb{R}^{n-1})}, \quad (3.8)$$

$$C_1 \|\nabla' f\|_{L^2(\mathbb{R}^{n-1})} \leq \|\Lambda_{\mathcal{A}} f\|_{L^2(\mathbb{R}^{n-1})} \leq C_2 \|\nabla' f\|_{L^2(\mathbb{R}^{n-1})}, \quad (3.9)$$

which are known as variants of the classical Rellich identity [20, 18, 9, 12]; see also [15, Proposition 2] for a short proof in relation with the Helmholtz decomposition. The boundedness of $\Lambda_{\mathcal{A}}^{-1} \partial_j$ in $L^2(\mathbb{R}^{n-1})$ is then follows from the adjoint relation

$$\langle \Lambda_{\mathcal{A}}^{-1} \partial_j f, \varphi \rangle_{L^2(\mathbb{R}^{n-1})} = -\langle f, \partial_j \Lambda_{\mathcal{A}}^{-1} \varphi \rangle_{L^2(\mathbb{R}^{n-1})}.$$

In particular, we have shown that $\mathcal{P}_{\mathcal{A}} \Lambda_{\mathcal{A}}^{-1}$ and S are respectively bounded from $L^2(\mathbb{R}^{n-1})$ to $L^2(\mathbb{R}^{n-1})$ and from $L^2(\mathbb{R}^{n-1})$ to $(L^2(\mathbb{R}^{n-1}))^{n-1}$. The identity (3.6) follows from (2.8). Indeed, in this case $\mathcal{A}' = -\Delta' = -\sum_{j=1}^{n-1} \partial_j^2$, and (2.8) is written as

$$\begin{aligned} \langle \nabla' g, \nabla' \varphi \rangle_{L^2(\mathbb{R}^{n-1})} &= \langle \mathcal{P}_{\mathcal{A}} g, M_b \mathcal{P}_{\mathcal{A}} \varphi \rangle_{L^2(\mathbb{R}^{n-1})} \\ &= \langle \mathcal{P}_{\mathcal{A}} g, \Lambda_{\mathcal{A}} \varphi - M_{\nabla' \eta} \cdot \nabla' \varphi \rangle_{L^2(\mathbb{R}^{n-1})} \end{aligned}$$

for all $g, \varphi \in H^1(\mathbb{R}^{n-1})$. Here we have used (2.7). Thus we have

$$\langle \mathcal{P}_{\mathcal{A}} \Lambda_{\mathcal{A}}^{-1} \Lambda_{\mathcal{A}} g, \Lambda_{\mathcal{A}} \varphi \rangle_{L^2(\mathbb{R}^{n-1})} = \langle (\nabla' + M_{\nabla' \eta}) \Lambda_{\mathcal{A}}^{-1} \Lambda_{\mathcal{A}} g, \nabla' \varphi \rangle_{L^2(\mathbb{R}^{n-1})} = \langle S \Lambda_{\mathcal{A}} g, \nabla' \varphi \rangle_{L^2(\mathbb{R}^{n-1})}.$$

Since the range of $\Lambda_{\mathcal{A}}$ is dense in $L^2(\mathbb{R}^{n-1})$, we finally obtain

$$\langle \mathcal{P}_{\mathcal{A}} \Lambda_{\mathcal{A}}^{-1} f, \Lambda_{\mathcal{A}} \varphi \rangle_{L^2(\mathbb{R}^{n-1})} = \langle S f, \nabla' \varphi \rangle_{L^2(\mathbb{R}^{n-1})},$$

for $f \in L^2(\mathbb{R}^{n-1})$ and $\varphi \in H^1(\mathbb{R}^{n-1})$. Then (3.9) shows (3.6). The identity (3.7) directly follows from (3.6). The proof is complete.

For $\varphi(\cdot, z_n) = \varphi(z', z_n)$ we introduce the operator U defined by

$$(U\varphi)(\cdot, y_n) = \Lambda_{\mathcal{A}} \int_0^{y_n} e^{-(y_n - z_n) \mathcal{P}_{\mathcal{A}}} \varphi(z_n) dz_n. \quad (3.10)$$

Hence, for $\varphi \in Y^{q,2}(\mathbb{R}_+^n)$ the function $U\varphi$ satisfies the equation

$$\partial_n U\varphi + \Lambda_{\mathcal{A}} \mathcal{P}_{\mathcal{A}} \Lambda_{\mathcal{A}}^{-1} U\varphi = \Lambda_{\mathcal{A}} \varphi \quad \text{in } \mathcal{D}'(\mathbb{R}_+^n). \quad (3.11)$$

Now we define the operator $Z = (Z', Z_n)^{\top} : (C_0^\infty(\mathbb{R}_+^n))^{n-1} \rightarrow (\mathcal{D}'(\mathbb{R}_+^n))^n$ as

$$Z'[h] = h + S U \Lambda_{\mathcal{A}}^{-1} \nabla' \cdot h, \quad (3.12)$$

$$Z_n[h] = -U \Lambda_{\mathcal{A}}^{-1} \nabla' \cdot h + M_{\nabla' \eta} \cdot Z'[h]. \quad (3.13)$$

We will sometimes write Zh for $Z[h]$ to simplify the notation. By the definition of E_n we have

$$E_n Z[h] = -U \Lambda_{\mathcal{A}}^{-1} \nabla' \cdot h, \quad (3.14)$$

which will be often used in the argument below. Lemma 3.1 leads to some important properties of the operator Z , which are described in Lemmas 3.2, 3.3, and Corollary 3.4 below.

Lemma 3.2. *Let $1 < q < \infty$. Then the operator Z defined by (3.12) - (3.13) is extended as a bounded operator from $(Y^{q,2}(\mathbb{R}_+^n))^{n-1}$ to $(Y^{q,2}(\mathbb{R}_+^n))^n$. Moreover, we have*

$$Z[h] \in Y_{\sigma}^{q,2}(\mathbb{R}_+^n) \quad \text{and} \quad (E' + SE_n)Z[h] = h \quad \text{for } h \in (Y^{q,2}(\mathbb{R}_+^n))^{n-1}. \quad (3.15)$$

Proof. We firstly note that the maximal regularity estimate

$$\|\mathcal{P}_{\mathcal{A}} \int_0^{y_n} e^{-(y_n-z_n)\mathcal{P}_{\mathcal{A}}} \varphi(z_n) dz_n\|_{Y^{q,2}(\mathbb{R}_+^n)} \leq C \|\varphi\|_{Y^{q,2}(\mathbb{R}_+^n)}$$

holds, which is observed by [15, Remark 5, Proposition 2]. Thus, from (3.8) and (3.9) we have

$$\|U\varphi\|_{Y^{q,2}(\mathbb{R}_+^n)} \leq C \|\varphi\|_{Y^{q,2}(\mathbb{R}_+^n)}. \quad (3.16)$$

Hence, combining (3.16) with Lemma 3.1, we see that Z is extended as a bounded operator from $(Y^{q,2}(\mathbb{R}_+^n))^{n-1}$ to $(Y^{q,2}(\mathbb{R}_+^n))^n$. Next we observe that

$$\nabla' \cdot Z'[h] = \nabla' \cdot h - \Lambda_{\mathcal{A}} \mathcal{P}_{\mathcal{A}} \Lambda_{\mathcal{A}}^{-1} U \Lambda_{\mathcal{A}}^{-1} \nabla' \cdot h \quad \text{in } \mathcal{D}'(\mathbb{R}_+^n), \quad (3.17)$$

where Lemma 3.1 was used, and that

$$\partial_n E_n Z[h] = -\partial_n U \Lambda_{\mathcal{A}}^{-1} \nabla' \cdot h = -\nabla' \cdot h + \Lambda_{\mathcal{A}} \mathcal{P}_{\mathcal{A}} \Lambda_{\mathcal{A}}^{-1} U \Lambda_{\mathcal{A}}^{-1} \nabla' \cdot h \quad \text{in } \mathcal{D}'(\mathbb{R}_+^n), \quad (3.18)$$

by the equation (3.11). The equations (3.17) and (3.18) imply $\operatorname{div} B^{\top} Z[h] = 0$ in the sense of distributions. Moreover, from (3.14) the trace of $E_n Z[h]$ to the boundary $\partial\mathbb{R}_+^n$ must vanish. The second identity in (3.15) follows from the definition of Z and (3.14). This completes the proof.

We have a converse of Lemma 3.2, which leads to a characterization of solenoidal vector fields in terms of the operator Z .

Lemma 3.3. *Let $1 < q < \infty$. Suppose that $v \in Y_{\sigma}^{q,2}(\mathbb{R}_+^n)$. Then $E_n v = UKv$ and $v = Z(E' + SE_n)v$ hold.*

Proof. By a density argument we may assume that $v \in (L^q(\mathbb{R}_+; H^1(\mathbb{R}^{n-1})))^n$. We first show $E_n v = UKv$. To this end we introduce the approximation K_{ϵ} of the operator K , which is defined by

$$K_{\epsilon} = -(\epsilon + \Lambda_{\mathcal{A}})^{-1} \nabla' \cdot E' + \mathcal{P}_{\mathcal{A}} (\epsilon + \Lambda_{\mathcal{A}})^{-1} E_n. \quad (3.19)$$

Note that $\operatorname{div} B^{\top} v = 0$ implies $\partial_n E_n v = -\nabla' \cdot E' v \in Y^{q,2}(\mathbb{R}_+^n)$. Then we have

$$K_{\epsilon} v = \partial_n (\epsilon + \Lambda_{\mathcal{A}})^{-1} E_n v + \mathcal{P}_{\mathcal{A}} (\epsilon + \Lambda_{\mathcal{A}})^{-1} E_n v = (\partial_n + \mathcal{P}_{\mathcal{A}}) (\epsilon + \Lambda_{\mathcal{A}})^{-1} E_n v.$$

Hence, taking into account $\gamma E_n v = 0$ on $\partial\mathbb{R}_+^n$, we have

$$(E_n v)(\cdot, x_n) = (\Lambda_{\mathcal{A}} + \epsilon) \int_0^{x_n} e^{-(x_n-y_n)\mathcal{P}_{\mathcal{A}}} K_{\epsilon} v(\cdot, y_n) dy_n, \quad \text{for all } \epsilon > 0. \quad (3.20)$$

We note that, for any $f \in L^2(\mathbb{R}^{n-1})$, $\Lambda_{\mathcal{A}}(\epsilon + \Lambda_{\mathcal{A}})^{-1} f$ converges to f in $L^2(\mathbb{R}^{n-1})$ as $\epsilon \rightarrow 0$. Then, since $\mathcal{P}_{\mathcal{A}} \Lambda_{\mathcal{A}}^{-1}$ is bounded in $L^2(\mathbb{R}^{n-1})$ and $\Lambda_{\mathcal{A}}^{-1} \nabla' \cdot$ is bounded from $(L^2(\mathbb{R}^{n-1}))^{n-1}$ to $L^2(\mathbb{R}^{n-1})$ we see from (3.19) that $(K_{\epsilon} v)(\cdot, y_n)$ converges to $(Kv)(\cdot, y_n)$ in $L^2(\mathbb{R}^{n-1})$ for

a.e. $y_n \in \mathbb{R}_+$. Moreover, it is easy to see $\sup_{\epsilon > 0} \|K_\epsilon v\|_{L^q(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))} \leq C\|v\|_{L^q(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))}$. Hence for any $x_n \in \mathbb{R}_+$ the term $(\Lambda_{\mathcal{A}} + \epsilon) \int_0^{x_n} e^{-(x_n - y_n)\mathcal{P}_{\mathcal{A}}} K_\epsilon v(\cdot, y_n) dy_n$ converges to the limit $\Lambda_{\mathcal{A}} \int_0^{x_n} e^{-(x_n - y_n)\mathcal{P}_{\mathcal{A}}} K v(\cdot, y_n) dy_n = (UKv)(\cdot, x_n)$ in $H^{-1}(\mathbb{R}^{n-1})$ by the Lebesgue convergence theorem. Thus we have $E_n v = UKv$ due to the equality (3.20), which then implies from (3.7) that

$$Z'(E' + SE_n)v = (E' + SE_n)v - SUKv = (E' + SE_n)v - SE_n v = v'.$$

To show $Z_n(E' + SE_n)v = v_n$ it suffices to prove $E_n Z(E' + SE_n)v = E_n v$. Again from the definitions of E_n , Z , and the identity (3.7), we have

$$E_n Z(E' + SE_n)v = UKv = E_n v.$$

This completes the proof.

As an immediate corollary of Lemmas 3.2 - 3.3 we have

Corollary 3.4. *Let $1 < q < \infty$. Then the operator Z defined by (3.12) - (3.13) is bounded and bijective from $(Y^{q,2}(\mathbb{R}_+^n))^{n-1}$ onto $Y_{\sigma}^{q,2}(\mathbb{R}_+^n)$. Moreover, we have $Z^{-1} = E' + SE_n$.*

The operator $E' + SE_n$ has a special property about its kernel, which plays an important role in the relation between the Stokes operator and the Laplace operator.

Lemma 3.5. *Let $1 < q < \infty$. Assume that $p \in L_{loc}^1(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))$ satisfies $\nabla p \in (Y^{q,2}(\mathbb{R}_+^n))^n$ and $\mathcal{A}p = 0$ in $\mathbb{R}_+^n$ in the sense of distributions. Then $(E' + SE_n)B\nabla p = 0$.*

Proof. Firstly we see $p(\cdot, x_n) \in H^1(\mathbb{R}^{n-1})$ for a.e. x_n and $\|p(x_n)\|_{L^2(\mathbb{R}^{n-1})} \leq Cx_n^{1-1/q}$ for $x_n \geq 1$. Fix $z_n \in (0, 1)$ such that $p(\cdot, z_n) \in H^1(\mathbb{R}^{n-1})$ and set $P(x_n) = e^{-x_n\mathcal{P}_{\mathcal{A}}}p(z_n)$. From the property of the Poisson semigroup we see $\sup_{x_n > 0} \|P(x_n)\|_{L^2(\mathbb{R}^{n-1})} < \infty$ and $\nabla P \in L^q(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))$. Moreover, $\theta(\cdot, x_n) = p(\cdot, x_n + z_n) - P(x_n)$ solves $\mathcal{A}\theta = 0$ in $\mathbb{R}_+^n$ in the sense of distribution and $\theta = 0$ on $\partial\mathbb{R}_+^n$. It is not difficult to show the Liouville theorem within the class of θ above, which leads to $\theta = 0$. Thus we have a representation $p(x_n + z_n) = e^{-x_n\mathcal{P}_{\mathcal{A}}}p(z_n)$. Then we see from the definition of S and (2.7),

$$E'B\nabla p(x_n + z_n) = S\Lambda_{\mathcal{A}}p(x_n + z_n), \quad E_n B\nabla p(x_n + z_n) = -\Lambda_{\mathcal{A}}p(x_n + z_n).$$

Hence we have $(E' + SE_n)B\nabla p = 0$ in $\{(x, x_n) \in \mathbb{R}_+^n \mid x_n \geq z_n\}$. Since $z_n \in (0, 1)$ is arbitrary small, we have $(E' + SE_n)B\nabla p = 0$ in $\mathbb{R}_+^n$, as desired. The proof is complete.

Proof of Theorem 1.2. Let us define the bounded linear operators $\mathbf{V} : (Y^{q,2}(\Omega))^{n-1} \rightarrow (Y^{q,2}(\Omega))^n$ and $\mathbf{W} : (Y^{q,2}(\Omega))^n \rightarrow (Y^{q,2}(\Omega))^{n-1}$ as

$$(\mathbf{V}w)(x) = (Z[w \circ \Phi^{-1}])(\Phi(x)). \quad (3.21)$$

$$(\mathbf{W}u)(x) = ((E' + SE_n)[u \circ \Phi^{-1}])(\Phi(x)). \quad (3.22)$$

Here Z is defined as (3.12) - (3.13), while E' , E_n , and S are defined as (3.4) - (3.5). Then by Corollary 3.4 it follows that $\text{Ran}(\mathbf{V}) = Y_{\sigma}^{q,2}(\Omega)$ and $\mathbf{V}$ is also invertible. In particular, $\mathbf{V}^{-1} = \mathbf{W}$ on $Y_{\sigma}^{q,2}(\Omega)$. Finally, $\mathbf{W}$ satisfies (i) of Theorem 1.2 by Lemma 3.5. The proof of Theorem 1.2 for general Ω is complete.

Remark 3.6 (Ukai's projection). Let $\mathbf{V}$ and $\mathbf{W}$ be the operators given as (3.21) and (3.22). Theorem 1.2 implies that the operator

$$\mathbf{P}_0 = \mathbf{V}\mathbf{W} : (Y^{q,2}(\Omega))^n \rightarrow Y_{\sigma}^{q,2}(\Omega) \quad (3.23)$$

is a continuous projection from $(Y^{q,2}(\Omega))^n$ onto $Y_{\sigma}^{q,2}(\Omega)$. In the case $\eta = 0$ (i.e., $\Omega = \mathbb{R}_+^n$), through a short calculation, this projection coincides with the one found by [22, Remark 1.5]:

$$(\mathbf{P}_0 u)' = u' + S u_n - S U(-S \cdot u' + u_n), \quad (\mathbf{P}_0 u)_n = U(-S \cdot u' + u_n),$$

where S and U are defined as in Remark 1.4 (ii). In the case of general η we have used a factorization of the elliptic operators. The projection $\mathbf{P}_0$ is different from the well-known Helmholtz projection, which is orthogonal in $(L^2(\Omega))^n$ while $\mathbf{P}_0$ is not, as is observed in [22] for the case $\eta = 0$.

3.2 Proof of Theorem 1.2 for $\Omega = \mathbb{R}^n, \mathbb{R}_+^n$

When $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$ we can simply take $\Phi(x) = x$, and hence, the isomorphism $\mathbf{V}$ coincides with Z (which is defined as (3.28) - (3.29) below) in both cases. Moreover, the matrices E' , E_n , and the operators S , K are respectively defined as

$$E'v = v', \quad E_nv = v_n, \quad v = (v', v_n)^\top \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad (3.24)$$

$$S = \nabla'(-\Delta')^{-\frac{1}{2}}, \quad K = -(-\Delta')^{-\frac{1}{2}} \nabla' \cdot E' + E_n. \quad (3.25)$$

When $\Omega = \mathbb{R}^n$ the operator U is defined as

$$(U\varphi)(\cdot, y_n) = (-\Delta')^{\frac{1}{2}} \int_{-\infty}^{y_n} e^{-(y_n - z_n)(-\Delta')^{\frac{1}{2}}} \varphi(z_n) dz_n, \quad (3.26)$$

while when $\Omega = \mathbb{R}_+^n$ we set

$$(U\varphi)(\cdot, y_n) = (-\Delta')^{\frac{1}{2}} \int_0^{y_n} e^{-(y_n - z_n)(-\Delta')^{\frac{1}{2}}} \varphi(z_n) dz_n. \quad (3.27)$$

With these operators $\mathbf{V} = (V', V_n)$ is given as in (3.12) - (3.13), that is,

$$V'[w] = w + S U(-\Delta')^{-\frac{1}{2}} \nabla' \cdot w, \quad (3.28)$$

$$V_n[w] = -U(-\Delta')^{-\frac{1}{2}} \nabla' \cdot w. \quad (3.29)$$

On the other hand, the operator $\mathbf{W}$ is given as

$$\mathbf{W} = E' + S E_n. \quad (3.30)$$

Note that, when $\Omega = \mathbb{R}_+^n$, the Dirichlet-Neumann map and the Poisson operator coincide with the fractional Laplacian $(-\Delta')^{1/2}$. It is classical that $\nabla'(-\Delta')^{-1/2}$ and $(-\Delta')^{-1/2}\nabla'$ define the singular integral operators. Hence, for any $1 < r < \infty$, the operators S and K are respectively bounded from $L^r(\mathbb{R}^{n-1})$ to $(L^r(\mathbb{R}^{n-1}))^{n-1}$ and from $(L^r(\mathbb{R}^{n-1}))^{n-1}$ to $L^r(\mathbb{R}^{n-1})$. Moreover, it is also well known that the Poisson semigroup $\{e^{-t(-\Delta')^{1/2}}\}_{t \geq 0}$ admits the maximal regularity estimates in $L^q(\mathbb{R}_+; L^r(\mathbb{R}^{n-1}))$, $1 < q, r < \infty$. Hence we have

$$\|U\varphi\|_{Y^{q,r}(\Omega)} \leq C\|\varphi\|_{Y^{q,r}(\Omega)}, \quad 1 < q, r < \infty, \quad \Omega = \mathbb{R}^n \text{ or } \mathbb{R}_+^n. \quad (3.31)$$

Then it is easy to see that the counterparts of Lemmas 3.2 and 3.3 hold with $Y^{q,r}(\Omega)$ when $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$ as follows.

Lemma 3.7. *Let $1 < q, r < \infty$ and let $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$. Then the operator $\mathbf{V}$ defined by (3.28) - (3.29) is extended as a bounded operator from $(Y^{q,r}(\Omega))^{n-1}$ to $(Y^{q,r}(\Omega))^n$. Moreover, we have*

$$\mathbf{V}w \in Y_{\sigma}^{q,r}(\Omega) \quad \text{and} \quad \mathbf{W}\mathbf{V}w = w \quad \text{for } w \in (Y^{q,r}(\Omega))^{n-1}. \quad (3.32)$$

Lemma 3.8. *Let $1 < q, r < \infty$ and let $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$. Suppose that $u \in Y_{\sigma}^{q,r}(\Omega)$. Then $E_n u = UKu$ and $u = \mathbf{VW}u$ hold.*

The proof of these lemmas is the same as in Lemmas 3.2 and 3.3, so it is omitted. Theorem 1.2 for $\Omega = \mathbb{R}^n, \mathbb{R}_+^n$ follows from the next corollary.

Corollary 3.9. *Let $1 < q, r < \infty$ and let $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$. Then the operator $\mathbf{V}$ defined as (3.28) - (3.29) is bounded and bijective from $(Y_{\sigma}^{q,r}(\Omega))^{n-1}$ onto $Y_{\sigma}^{q,r}(\Omega)$. Moreover, we have $\mathbf{V}^{-1} = \mathbf{W}$, where $\mathbf{W}$ is defined as (3.30).*

3.3 Proof of Theorem 1.1

It suffices to show $Y_{\sigma}^{q,r}(\mathbb{R}_+^n) \simeq Y^{q,r}(\mathbb{R}_+^n)^{n-1}$, where $Y_{\sigma}^{q,r}(\mathbb{R}_+^n)$ is characterized as (3.3) and is isomorphic to $Y_{\sigma}^{q,r}(\Omega)$. By (3.3) the correspondence $Y_{\sigma}^{q,r}(\mathbb{R}_+^n) \ni u \mapsto B^{\top}u \in Y_{\sigma}^{q,r}(\mathbb{R}_+^n)$ defines an isomorphism from $Y_{\sigma}^{q,r}(\mathbb{R}_+^n)$ to $Y_{\sigma}^{q,r}(\mathbb{R}_+^n)$. Since $Y_{\sigma}^{q,r}(\mathbb{R}_+^n) \simeq (Y^{q,r}(\mathbb{R}_+^n))^{n-1}$ by Theorem 1.2, where the isomorphism is explicit as is stated in Section 3.2, we have $Y_{\sigma}^{q,r}(\mathbb{R}_+^n) \simeq Y^{q,r}(\mathbb{R}_+^n)^{n-1}$. The proof of Theorem 1.1 is complete.

3.4 Proof of Theorem 1.3

In this section we prove Theorem 1.3. Firstly let us consider the case $\Omega = \mathbb{R}_+^n$. Since the Poisson operator $(-\Delta')^{1/2}$ and the semigroup $\{e^{-t(-\Delta')^{1/2}}\}_{t \geq 0}$ satisfy all of the conditions (i), (ii), (iii) in [15, Section 3.1] it follows that the space $(Y^{q,r}(\mathbb{R}_+^n))^n$ admits the Helmholtz decomposition, and the Helmholtz projection $\mathbf{P} : (Y^{q,r}(\mathbb{R}_+^n))^n \rightarrow Y_{\sigma}^{q,r}(\mathbb{R}_+^n)$ is well-defined. Let us define the Laplace operator Δ_D in $Y^{q,r}(\mathbb{R}_+^n)$ and Stokes operator $\mathbf{A}$ in $Y_{\sigma}^{q,r}(\mathbb{R}_+^n)$ as

$$\begin{aligned} D_{Y^{q,r}}(\Delta_D) &= \{f \in Y^{q,r}(\mathbb{R}_+^n) \mid \nabla^{\alpha} f \in Y^{q,r}(\mathbb{R}_+^n) \text{ for } |\alpha| \leq 2, \quad \gamma f = 0 \text{ on } \partial\mathbb{R}_+^n\}, \\ D_{Y_{\sigma}^{q,r}}(\mathbf{A}) &= Y_{\sigma}^{q,r}(\mathbb{R}_+^n) \cap (D_{Y^{q,r}}(\Delta_D))^n, \\ \Delta_D f &= \Delta f, \quad f \in D_{Y^{q,r}}(\Delta_D), \quad \mathbf{A}u = -\mathbf{P}\Delta_D u, \quad u \in D_{Y_{\sigma}^{q,r}}(\mathbf{A}). \end{aligned}$$

From (3.31) and the $L^r(\mathbb{R}^{n-1})$ boundedness of the Riesz transform $\partial_j(-\Delta')^{-1/2}$, $j = 1, \dots, n-1$, it is easy to see from Corollary 3.9 that $w \in (D_{Y^{q,r}}(\Delta_D))^{n-1}$ if and only if $\mathbf{V}w \in D_{Y_{\sigma}^{q,r}}(\mathbf{A})$. Thus we have $D_{Y_{\sigma}^{q,r}}(\mathbf{A}) = \{\mathbf{V}w \mid w \in (D_{Y^{q,r}}(\Delta_D))^{n-1}\}$. Moreover, since $\operatorname{div} \Delta_D \mathbf{V}w = 0$ in $\mathbb{R}_+^n$ in the sense of distributions, we see $(I - \mathbf{P})\Delta_D \mathbf{V}w$ is a harmonic pressure, which implies $\mathbf{W}(I - \mathbf{P})\Delta_D \mathbf{V}w = 0$ by Lemma 3.5. Note that Δ_D commutes with $\mathbf{W} = E' + SE_n$ by the definitions of E' , S , E_n in (3.24) - (3.25). Thus we have

$$\begin{aligned} \mathbf{V}^{-1}\mathbf{A}\mathbf{V}w &= -\mathbf{W}\mathbf{P}\Delta_D \mathbf{V}w = -\mathbf{W}\Delta_D \mathbf{V}w \\ &= -\Delta_D \mathbf{W}\mathbf{V}w \\ &= -\Delta_D w. \end{aligned}$$

Hence $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = -\Delta_D$ holds as operators in $(Y^{q,r}(\mathbb{R}_+^n))^{n-1}$, and equivalently, we have $-\mathbf{A} = \mathbf{V}\Delta_D \mathbf{V}^{-1}$ as operators in $Y_{\sigma}^{q,r}(\mathbb{R}_+^n)$. It is not difficult to see that Δ_D generates a strongly continuous and analytic semigroup in $Y^{q,r}(\mathbb{R}_+^n)$, which has an explicit representation in terms of the n dimensional Gaussian and its reflection with respect to the vertical variable. Therefore, $-\mathbf{A}$ also generates a strongly continuous and analytic semigroup in $Y_{\sigma}^{q,r}(\mathbb{R}_+^n)$. In particular, the formula $e^{-t\mathbf{A}} = \mathbf{V}e^{t\Delta_D}\mathbf{V}^{-1}$ holds. The case $\Omega = \mathbb{R}^n$ is proved in the similar way, though this case is simpler due to the fact

$$(I - \mathbf{P})\Delta_D \mathbf{V}w = \Delta_D \mathbf{V}w - \Delta_D \mathbf{P}\mathbf{V}w = \Delta_D \mathbf{V}w - \Delta_D \mathbf{V}w = 0,$$

where $\mathbf{P}$ and Δ_D are now realized in $(Y^{q,r}(\mathbb{R}^n))^n$. We omit the details here. The proof is complete.

Acknowledgements. The first author is partially supported by Grant-in-Aid for Young Scientists (B) 25800079. The second author is partially supported by Grant-in-Aid for Young Scientists (A) 25707005.

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