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ON ANALYTICITY OF THE L^p -STOKES SEMIGROUP FOR SOME NON-HELMHOLTZ DOMAINS

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ABSTRACT. Consider the Stokes equations in a sector-like C^3 domain $\Omega \subset \mathbf{R}^2$. It is shown that the Stokes operator generates an analytic semigroup in $L_\sigma^p(\Omega)$ for $p \in [2, \infty)$. This includes domains where the L^p -Helmholtz decomposition fails to hold. To show our result we interpolate results of the Stokes semigroup in VMO and L^2 by constructing a suitable non-Helmholtz projection to solenoidal spaces.

1. INTRODUCTION

In this paper, as a continuation of [5], [6] and [10], we study the Stokes semigroup, i.e., the solution operator $S(t) : v_0 \mapsto v(\cdot, t)$ of the initial-boundary problem for the Stokes system

$$v_t - \Delta v + \nabla q = 0, \quad \operatorname{div} v = 0 \quad \text{in} \quad \Omega \times (0, \infty)$$

with the zero boundary condition

$$v = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty)$$

and the initial condition $v|_{t=0} = v_0$, where Ω is a domain in $\mathbf{R}^n$ with $n \geq 2$. It is by now well-known that $S(t)$ forms a C_0 -analytic semigroup in L_σ^p ($1 < p < \infty$) for various domains like smooth bounded domains ([21], [35]). Here $L_\sigma^p = L_\sigma^p(\Omega)$ denotes the L^p -closure of $C_{c,\sigma}^\infty(\Omega)$, the space of all solenoidal vector fields with compact support in Ω . More recently, it has been proved in [20] that $S(t)$ always forms a C_0 -analytic semigroup in $L_\sigma^p(\Omega)$ for any uniformly C^2 -domain Ω provided that $L^p(\Omega)$ admits a topological direct sum decomposition called the Helmholtz decomposition of the form

$$L^p(\Omega) = L_\sigma^p(\Omega) \oplus G^p(\Omega)$$

where $G^p(\Omega) = \{\nabla q \in L^p(\Omega) \mid q \in L_{loc}^1(\Omega)\}$. In [20] the L^q maximal regularity in time with values in $L_\sigma^p(\Omega)$ was also established.

The Helmholtz decomposition holds for any domain if $p = 2$. The L^p -Helmholtz decomposition holds for various domains like bounded or exterior domains with

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smooth boundary for $1 < p < \infty$ ([19]). However, it is also known ([9], [28]) that there is an improper smooth sector-like planar domain such that the L^p -Helmholtz decomposition fails to hold. Let us state one of the results in [28] more precisely. Let $C(\vartheta)$ denote the cone of the form

$$C(\vartheta) = \{x = (x', x_n) \in \mathbf{R}^n \mid -x_n \geq |x| \cos(\vartheta/2)\},$$

where $\vartheta \in (0, 2\pi)$ is the opening angle. When $n = 2$, we simply say that $C(\vartheta)$ is a sector. We say that a planar domain Ω is a *sector-like domain* with opening angle ϑ if $\Omega \setminus B_R(0) = C(\vartheta) \setminus B_R(0)$ for some $R > 0$ (up to rotation and translation), where $B_R(0)$ is an open disk of radius R centered at the origin.

It is known that the L^p -Helmholtz decomposition fails for a sector-like domain Ω when $p > q'_\vartheta$ or $p < q_\vartheta$ with $q_\vartheta = 2/(1 + \pi/\vartheta)$, $1/q_\vartheta + 1/q'_\vartheta = 1$ even if the boundary $\partial\Omega$ is smooth [28, Example 2, Fig. 5] while for $p \in (q_\vartheta, q'_\vartheta)$ the L^p -Helmholtz decomposition holds. This means that if the opening angle ϑ is larger than π , there always exists $p > 2$ such that the L^p -Helmholtz decomposition fails.

It has been a longstanding open question whether or not the existence of the L^p -Helmholtz decomposition is necessary for L^p analyticity of $S(t)$. In this paper, we give a negative answer for this question by proving that there is a domain Ω for which $S(t)$ is analytic in L^p_σ while the L^p -Helmholtz decomposition fails. This is a subtle problem since the existence of the L^p -Helmholtz projection is known to be necessary for L^p solvability of the resolvent equation ([33]). However, in this statement the external force term is allowed to be in the more general space L^p instead of L^p_σ . Our problem is different from that in [33].

We say that Ω has a C^k graph boundary if Ω is of the form

$$\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x_n > h(x')\}$$

(up to translation and rotation) with some real-valued C^k function h with variable $x' \in \mathbf{R}^{n-1}$.

Theorem 1.1. *Let Ω be a sector-like domain in $\mathbf{R}^2$ having a C^3 graph boundary. Then $S(t)$ forms a C_0 -analytic semigroup in $L^p_\sigma(\Omega)$ for all $p \in [2, \infty)$.*

Here is our strategy to prove Theorem 1.1. It is by now well-known that $S(t)$ forms an analytic semigroup in $\tilde{L}^p_\sigma$, i.e., $\tilde{L}^p_\sigma = L^p_\sigma \cap L^2_\sigma$ ($p \geq 2$), $\tilde{L}^p = L^p_\sigma + L^2_\sigma$ ($1 < p < 2$) ([14], [15], [16]). Thus $S(t)v_0$ is well-defined for $v_0 \in C^\infty_{c,\sigma}(\Omega)$. To show Theorem 1.1, a key step is to prove the two estimates

$$(1.1) \quad \|S(t)v_0\|_p \leq C\|v_0\|_p$$

$$(1.2) \quad t \left\| \frac{d}{dt} S(t)v_0 \right\|_p \leq C\|v_0\|_p$$

for all $v_0 \in C^\infty_{c,\sigma}(\Omega)$, $t \in (0, 1)$, where $\|v_0\|_p$ denotes the L^p -norm of v_0 . The constant C should be taken independent of t and v_0 . We shall establish (1.1) and (1.2) by interpolation since both estimates are known for $p = 2$.

We are tempted to interpolate the L^∞ type result obtained in [5] with the L^2 -result. In fact, in [5] the estimates (1.1) and (1.2) with $p = \infty$ are established for all $v_0 \in C_{0,\sigma}(\Omega)$, the L^∞ -closure of $C^\infty_{c,\sigma}(\Omega)$ for a C^2 sector-like domain Ω in $\mathbf{R}^2$. However, it is not clear that the complex interpolation space $[L^2_\sigma, C_{0,\sigma}]_\rho$ agrees with L^p_σ with $2/p = 1 - \rho$ although it is well-known as the Riesz-Thorin theorem that $[L^2, L^\infty]_\rho = L^p$. To interpolate, we would need a projection to solenoidal spaces

which is almost impossible since such a projection involves the singular integral operator which is not bounded in L^∞ .

To circumvent this difficulty, we consider the Stokes semigroup $S(t)$ in BMO -type spaces as studied in [10], [11], [12]. For $p \in [1, \infty)$, $\mu \in (0, \infty]$ we define the BMO seminorm

$$[f : BMO_p^\mu(\Omega)] := \sup \left\{ \left(\int_{B_r(x)} |f(y) - f_{B_r(x)}|^p dy \right)^{1/p} \mid B_r(x) \subset \Omega, r < \mu \right\},$$

where $f_B = \int_B f dx$, the average of f over B and $B_r(x)$ denotes the closed ball of radius r centered at x . It is well-known that one gets an equivalent seminorm when the ball B_r is replaced by a cube. We also need to control the boundary behavior. For $\nu \in (0, \infty]$ we define

$$[f : b_p^\nu(\Omega)] := \sup \left\{ \left(\frac{1}{r^n} \int_{B_r(x_0) \cap \Omega} |f(y)|^p dy \right)^{1/p} \mid x_0 \in \partial\Omega, r > 0, B_r(x_0) \subset U_\nu(\partial\Omega) \right\},$$

where $U_\nu(E)$ is a ν -open neighborhood of E , i.e.,

$$U_\nu(E) = \{x \in \mathbf{R}^n \mid \text{dist}(x, E) < \nu\}.$$

We shall often assume that $\nu < R^*$, where R^* is the reach from the boundary. The BMO norm we use is

$$\|f : BMO_{b,p}^{\mu,\nu}(\Omega)\| = [f : BMO_p^\mu(\Omega)] + [f : b_p^\nu(\Omega)].$$

If $p = 1$, we often drop p . The BMO space we consider is

$$BMO_{b,p}^{\mu,\nu}(\Omega) = \left\{ f \in L_{loc}^1(\Omega) \mid \|f : BMO_{b,p}^{\mu,\nu}(\Omega)\| < \infty \right\}.$$

This space is independent of p for sufficiently small ν , i.e., $\nu < R^*$ ([11], [12]) and $BMO_b^{\infty,\infty}$ agrees with Miyachi BMO space ([29]) for various domains including a half space and bounded C^2 domains ([12]). Although the $BMO_b^{\infty,\nu}(\Omega)$ norm is equivalent to the $BMO_b^{\infty,\infty}(\Omega)$ norm when Ω is bounded, there are many unbounded domains for which the $BMO_b^{\infty,\nu}(\Omega)$ norm is actually weaker than the $BMO_b^{\infty,\infty}(\Omega)$ norm when ν is finite. We define the solenoidal space $VMO_{b,0,\sigma}^{\mu,\nu}$ as the $BMO_{b,0,\sigma}^{\mu,\nu}$ -closure of $C_{c,\sigma}^\infty(\Omega)$. In [10], [11] among other results the analyticity of $S(t)$ in $VMO_{b,0,\sigma}^{\infty,\nu}$ has been established for a uniformly C^3 domain which is admissible in the sense of [2] provided that ν is sufficiently small.

Theorem 1.2 ([10], [11]). *Let Ω be an admissible uniformly C^3 domain in $\mathbf{R}^n$. Then $S(t)$ forms a C_0 -analytic semigroup in $VMO_{b,0,\sigma}^{\mu,\nu}$ for any $\mu \in (0, \infty]$ and $\nu \in (0, \nu_0)$ with some ν_0 depending only on μ and regularity of $\partial\Omega$.*

Moreover, we obtain not only estimates of the form (1.1) and (1.2), where we replace L^p by L^∞ or $BMO_b^{\infty,\nu}$, but even an estimate stronger than (1.2) with $p = \infty$, i.e.,

$$(1.3) \quad t \left\| \frac{dS(t)}{dt} v_0 \right\|_\infty \leq C \|v_0 : BMO_b^{\mu,\nu}(\Omega)\|, \quad \mu, \nu \in (0, \infty]$$

which shows a regularizing effect.

It has been proved in [5] that a C^2 sector-like domain in $\mathbf{R}^2$ is admissible and thus Theorem 1.2 applies to the setting of Theorem 1.1. Note that a C^2 sector-like

domain in $\mathbf{R}^2$ is expected to be not strictly admissible in the sense of [3]. In fact, a bounded domain ([2]), a half space ([2]), an exterior domain ([3], [4]) and a bent half space ([1]) are strictly admissible if the boundary is uniformly C^3 . On the other hand, an infinite cylinder is admissible but not strictly admissible ([6]) and a layer domain with $n \geq 3$ is not admissible ([8]).

In order to get the L^p estimates we need an interpolation result. Let $C_c(\Omega)$ denote the space of all continuous functions with compact support in Ω .

Theorem 1.3. *Let Ω be a Lipschitz half-space in $\mathbf{R}^n$, i.e., a domain having Lipschitz graph boundary. Let T be a linear operator from $C_c(\Omega)$ to $L^2(\Omega)$. Assume that there is a constant C such that*

$$\|Tu\|_2 \leq C\|u\|_2$$

$$[Tu : BMO^\infty(\Omega)] \leq C\|u\|_\infty$$

for $u \in C_c(\Omega)$. Then $\|Tu\|_p \leq C_*\|u\|_p$ for $u \in C_c(\Omega)$ with C_* depending only on C , h and $p \in (2, \infty)$.

There are a couple of such interpolation results between BMO and L^2 , which go back to Campanato and Stampacchia; in [22, Theorem 2.14] the interpolation between L^p and BMO is discussed when Ω is a cube. However, in these results the original inequalities are assumed to hold for $L^2(\Omega) \cap BMO(\Omega)$ and not for $C_c(\Omega)$. Thus ours are not included in the literature. In [13] Duong and Yan showed a similar result (Theorem 5.2) with $BMO_A(\mathcal{X})$, where A is some operator. They worked on metric measure spaces of homogeneous type $(\mathcal{X}, d, \mu)$. In particular, in the case $\mathcal{X} = \Omega$, $d(x, y) = |x - y|$ and $\mu(E) = |E|$, we can see that $BMO_A(\Omega) \subset BMO^\infty(\Omega)$.

Unfortunately, Theorem 1.2 and Theorem 1.3 are not enough to derive (1.1) and (1.2) by interpolation. Similarly to the L^∞ case we do not know whether or not the complex interpolation space $[L_\sigma^2, VMO_{b,0,\sigma}^{\infty,\nu}]_\rho$ with $2/p = 1 - \rho$ agrees with L_σ^p , although we know that $[L^2, BMO]_\rho = L^p$ for $\Omega = \mathbf{R}^n$ as discussed in [25].

To circumvent this difficulty, we construct the following projection operator.

Theorem 1.4. *Let Ω be a Lipschitz half-space in $\mathbf{R}^n$. Assume that $\nu \in (0, \infty]$. There is a linear operator Q from $C_c(\Omega)$ to $VMO_{b,0,\sigma}^{\infty,\nu}(\Omega) \cap L_\sigma^2(\Omega)$ such that*

$$\|Qu : BMO_b^{\infty,\nu}(\Omega)\| \leq C\|u\|_\infty$$

$$\|Qu\|_2 \leq C\|u\|_2$$

for all $u \in C_c(\Omega)$. Moreover, $Qu = u$ for $u \in C_c(\Omega) \cap L_\sigma^2(\Omega)$.

Since there may be no L^p -Helmholtz decomposition our Q should be different from the Helmholtz projection. We shall construct such an operator Q using the solution operator of the equation $\operatorname{div} u = f$ given by Solonnikov [36]. Although deriving the L^2 estimate is easy, to derive the BMO estimate is more involved since we have to estimate the b^ν type seminorm.

To derive (1.1), we actually interpolate

$$\|S(t)Qu\|_2 \leq C\|u\|_2$$

and

$$\|S(t)Qu : BMO_b^{\infty,\nu}\| \leq C\|u\|_\infty$$

for $u \in C_c(\Omega)$. Similarly, we derive (1.2) by interpolating the estimate for $t \frac{dS}{dt} Q$.

This paper is organized as follows. In Section 2, we establish an interpolation inequality of Campanato-Stampacchia type. In Section 3, we construct the projection operator Q . In Section 4, we give a complete proof of Theorem 1.1.

2. $L^2 - BMO$ INTERPOLATION ON A LIPSCHITZ HALF-SPACE

In this section, we give a proof of Theorem 1.3 for a Lipschitz half-space, i.e.,

$$\Omega := \{(x', x_n) \in \mathbf{R}^n | x_n > h(x')\}$$

with a Lipschitz function h on $\mathbf{R}^{n-1}$.

By Q we mean a closed cube with sides parallel to the coordinate axes. Let $\ell(Q)$ be the side length of Q , and for $\tau > 0$, τQ a cube with the same center as Q and side length $\tau\ell(Q)$.

2.1. Reduction to the half-space and extension. Here, we prepare lemmas that are basic estimates for the proof. Since h is Lipschitz continuous, $F(x) := (x', x_n - h(x'))$ is a bi-Lipschitz map from Ω to $\mathbf{R}_+^n$. For a function u defined on $\mathbf{R}_+^n$ the pull-back function $F^*(u)$ of u on Ω is defined by $u \circ F$. We start with estimates for $(F^{-1})^*$ which is the pull-back function $(F^{-1})^*(v)$ of v on $\mathbf{R}_+^n$ defined by $v \circ F^{-1}$.

Lemma 2.1. *Let Ω be a Lipschitz half-space.*

(i):

$$[(F^{-1})^*v : BMO^\infty(\mathbf{R}_+^n)] \leq c[v : BMO^\infty(\Omega)].$$

(ii):

$$\|(F^{-1})^*v\|_{L^2(\mathbf{R}_+^n)} \leq c\|v\|_{L^2(\Omega)}.$$

Here c is a constant depending only on Lipschitz bound of h and n .

Proof. (i): Because $\mathbf{R}_+^n$ is an open subset of $\mathbf{R}^n$, we know that for any $\tau > 2$,

$$[(F^{-1})^*v : BMO^\infty(\mathbf{R}_+^n)] \leq c_\tau \sup_{\tau Q \subset \mathbf{R}_+^n} \inf_{d \in \mathbf{R}} \int_Q |(F^{-1})^*v - d| \, dy,$$

where the supremum is taken over cubes Q , for which τQ is contained in $\mathbf{R}_+^n$, see [37]. Since F is a bi-Lipschitz map, it holds

$$c_1 \operatorname{dist}(y, \partial\mathbf{R}_+^n) \leq \operatorname{dist}(F^{-1}(y), \partial\Omega) \leq c_2 \operatorname{dist}(y, \partial\mathbf{R}_+^n)$$

with some constants $c_1, c_2 > 0$ for all $y \in \mathbf{R}_+^n$. Since $(\tau - 1)\ell(Q)/2 \leq \operatorname{dist}(Q, \partial\mathbf{R}_+^n)$ for such cubes Q , we have the lower bound

$$c\tau\ell(Q) \leq \operatorname{dist}(F^{-1}(Q), \partial\Omega)$$

with some $c > 0$, which depends on n and h . Therefore, taking large τ , we can find cubes $\{R_k\}_{k=1}^{c_*} \subset \Omega$, which have no intersection of interiors, so that $\cup_{k=1}^{c_*} R_k$ is connected and

$$\begin{cases} \circ \ell(R_k) = \ell(Q), \\ \circ F^{-1}(Q) \subset \cup_{k=1}^{c_*} R_k, \text{ where } c_* \in \mathbf{N} \text{ depends only on } h, \text{ and} \\ \circ \text{if } R_j \cap R_k \neq \emptyset, \text{ the smallest cube } R_{j,k} \text{ including } R_j \text{ and } R_k \text{ is in } \Omega. \end{cases}$$

From these, one obtains that for cubes Q with $\tau Q \subset \mathbf{R}_+^n$,

$$\inf_{d \in \mathbf{R}} \frac{1}{|Q|} \int_Q |(F^{-1})^*v - d| \, dy \leq c \sum_{k=1}^{c_*} \frac{1}{|R_k|} \int_{R_k} |v - v_{R_1}| \, dy.$$

It is enough to show that

$$(2.1) \quad \frac{1}{|R_k|} \int_{R_k} |v - v_{R_j}| dy \leq c[v : BMO^\infty(\Omega)]$$

for the case $R_j \cap R_k \neq \emptyset$. To do this, we follow the argument of [26, Lemma 2.2 and 2.3]. Let $\tilde{R}_k$ and $\tilde{R}_j$ be subcubes of R_k and R_j respectively so that $\ell(\tilde{R}_k) = \ell(R_k)/2$, $\ell(\tilde{R}_j) = \ell(R_j)/2$ and they touch each other. Moreover, denote by $\tilde{R}_{j,k}$ a cube satisfying $\ell(\tilde{R}_{j,k}) = \ell(\tilde{R}_j) + \ell(\tilde{R}_k)$ and $\tilde{R}_j \cup \tilde{R}_k \subset \tilde{R}_{j,k} \subset R_{j,k}$. Hence, we have

$$\begin{aligned} \frac{1}{|R_k|} \int_{R_k} |v - v_{R_j}| dy &\leq \frac{1}{|R_k|} \int_{R_k} |v - v_{R_k}| dy + |v_{R_k} - v_{R_j}| \\ &\leq c[v : BMO^\infty(\Omega)] + c|v_{\tilde{R}_j} - v_{\tilde{R}_k}| \\ &\leq c[v : BMO^\infty(\Omega)] + c \frac{1}{|\tilde{R}_{j,k}|} \int_{\tilde{R}_{j,k}} |v - v_{\tilde{R}_{j,k}}| dy \\ &\leq c[v : BMO^\infty(\Omega)]. \end{aligned}$$

(ii): This is verified as follows

$$\|(F^{-1})^* v\|_{L^2(\mathbf{R}_+^n)}^2 = \int_{\Omega} |v|^2 J_F dx \leq c \int_{\Omega} |v|^2 dx,$$

where J_F is the modulus of the Jacobian of F which is bounded, because h is Lipschitz continuous. $\square$

Next, we consider the even extension of functions on the half space. For a function f on $\mathbf{R}_+^n$, we extend f outside $\mathbf{R}_+^n$ by

$$E[f](x', -x_n) := f(x', x_n) \text{ for } x_n > 0.$$

From elementary geometrical observation, we can see that the extension operator E is a BMO -extension operator for $\mathbf{R}_+^n$.

Lemma 2.2.

$$[E[f] : BMO^\infty(\mathbf{R}^n)] \leq c[f : BMO^\infty(\mathbf{R}_+^n)].$$

Proof. It is sufficient to consider cubes $Q \subset \mathbf{R}^n$ with $Q \cap \mathbf{R}_+^n \neq \emptyset$ and $Q \cap \mathbf{R}_-^n \neq \emptyset$. For such Q , let Q' be a cube so that its center lies on $\partial \mathbf{R}_+^n$, $\ell(Q') = 2\ell(Q)$ and $Q \subset Q'$. Further, let Q^* be the smallest cube in $\mathbf{R}_+^n$ containing the upper half of Q' . With these notations, the desired inequality is proved from

$$\inf_{d \in \mathbf{R}} \frac{1}{|Q|} \int_Q |E[f] - d| dy \leq c \inf_{d \in \mathbf{R}} \frac{1}{|Q^*|} \int_{Q^*} |f - d| dy.$$

$\square$

2.2. Sharp maximal operator. For the proof of Theorem 1.3, we make use of the sharp maximal operator $M^\sharp$ due to Fefferman and Stein ([18]). We define for $x \in \mathbf{R}^n$ and $f \in L_{loc}^1(\mathbf{R}^n)$ the function $M^\sharp f$ by

$$M^\sharp f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

It is immediate from the definition that $[f : BMO^\infty(\mathbf{R}^n)] = \|M^\sharp f\|_{L^\infty(\mathbf{R}^n)}$. It is well-known that if $f \in L^{p_0}(\mathbf{R}^n)$ for some $p_0 \in (1, \infty)$, then for $p \in [p_0, \infty)$

$$(2.2) \quad \|f\|_{L^p(\mathbf{R}^n)} \leq c \|M^\sharp f\|_{L^p(\mathbf{R}^n)},$$

which is applied below. (Both sides of (2.2) may be infinite.) This follows from $\|f\|_{L^p(\mathbf{R}^n)} \leq \|Mf\|_{L^p(\mathbf{R}^n)}$ and $\|Mf\|_{L^p(\mathbf{R}^n)} \leq c \|M^\sharp f\|_{L^p(\mathbf{R}^n)}$, where M is the Hardy-Littlewood maximal operator [18].

2.3. Marcinkiewicz interpolation. Here, we give a variant of the Marcinkiewicz interpolation theorem.

Proposition 2.3. *Let D be an open subset of $\mathbf{R}^n$ and S a sublinear operator from $C_c(D)$ to $L^2(\mathbf{R}^n)$. If*

$$\|S[f]\|_{L^2(\mathbf{R}^n)} \leq c \|f\|_{L^2(D)}$$

$$\|S[f]\|_{L^\infty(\mathbf{R}^n)} \leq c \|f\|_{L^\infty(D)}$$

for $f \in C_c(D)$, then $\|S[f]\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^p(D)}$ for $f \in C_c(D)$ with C depending only on c and $p \in (2, \infty)$.

Proof. For $\lambda > 0$ and $\alpha > 0$, we decompose f into two parts; $f = f_2 + f_\infty$ where

$$f_2(x) = \begin{cases} 0 & \text{if } |f(x)| \leq \alpha\lambda \\ f(x) - \alpha\lambda \text{sign}(f(x)) & \text{if } |f(x)| > \alpha\lambda, \end{cases}$$

where $\text{sign } \xi = \xi/|\xi|$ for $\xi \neq 0$ and $\text{sign } \xi = 0$ for $\xi = 0$. Observe that $f_2, f_\infty \in BC(D)$, and then $f_2, f_\infty \in C_c(D)$. Therefore, the two inequalities of our assumption hold for f_2 and f_∞ , respectively. We set $\alpha = (2\|S\|_{L^\infty(D) \rightarrow L^\infty(\mathbf{R}^n)})^{-1}$ and observe that $|\{x \in \mathbf{R}^n \mid |S[f_\infty](x)| > \lambda/2\}| = 0$. We now conclude that

$$\begin{aligned} \int_{\mathbf{R}^n} |S[f]|^p dx &\leq p \int_0^\infty \lambda^{p-1} |\{x \in \mathbf{R}^n \mid |S[f](x)| > \lambda\}| d\lambda \\ &\leq p \int_0^\infty \lambda^{p-1} |\{x \in \mathbf{R}^n \mid |S[f_2](x)| > \lambda/2\}| d\lambda \\ &\leq p \int_0^\infty \lambda^{p-1} \left(\frac{2}{\lambda} \|S\|_{L^2(D) \rightarrow L^2(\mathbf{R}^n)} \|f_2\|_{L^2(D)} \right)^2 d\lambda \\ &\leq c \int_0^\infty \lambda^{p-3} \int_{\{|f| > \alpha\lambda\}} |f(x)|^2 dx d\lambda \\ &= 2c \int_0^\infty \lambda^{p-3} \left(\int_{\alpha\lambda}^\infty t |\{x \in \mathbf{R}^n \mid |f(x)| > t\}| dt \right) d\lambda \\ &= 2c \int_0^\infty t |\{x \in \mathbf{R}^n \mid |f(x)| > t\}| \left(\int_0^{t/\alpha} \lambda^{p-3} d\lambda \right) dt \\ &\leq c \|f\|_{L^p(D)}^p. \end{aligned}$$

□

2.4. Proof of Theorem 1.3. For simplicity, we write $g := Tf$. By changing variables, one obtains

$$\int_{\Omega} |g|^p dx \leq c \int_{\mathbf{R}_+^n} |(F^{-1})^* g|^p dy \leq c \int_{\mathbf{R}^n} |E[(F^{-1})^* g]|^p dy \leq c \int_{\mathbf{R}^n} |\Phi[f]|^p dy,$$

where $\Phi[f] := M^\sharp(E[(F^{-1})^* g])$. Here, because $E[(F^{-1})^* g] \in L^2(\mathbf{R}^n)$, we have applied (2.2) in the third inequality. With the help of Proposition 2.3, it is enough to see $L^2(\Omega) - L^2(\mathbf{R}^n)$ and $L^\infty(\Omega) - L^\infty(\mathbf{R}^n)$ estimates for Φ . The former estimate can be seen by L^2 -boundedness of Hardy-Littlewood maximal operator and (ii) of Lemma 2.1. The later one follows from (i) of Lemma 2.1 and Lemma 2.2. Then the proof of Theorem 1.3 is completed.

3. NON-HELMHOLTZ PROJECTION

Our goal in this section is to prove Theorem 1.4.

3.1. A solution operator to the divergence problem. As in Section 2, let $\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, x_n > h(x')\}$ be a Lipschitz half-space in $\mathbf{R}^n$ with a Lipschitz continuous function h on $\mathbf{R}^{n-1}$. Then, there is a closed cone of the form

$$C_1 = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, -x_n \geq |x| \cos(2\theta)\}$$

with an angle $\theta \in (0, \pi/4)$ (depending on the Lipschitz constant of h) such that

$$x + C_1 = \{y \in \mathbf{R}^n \mid y - x \in C_1\} \subset \Omega^c \quad (:= \mathbf{R}^n \setminus \Omega) \quad \text{for all } x \in \Omega^c.$$

In the notion of the introduction $C_1 = C(4\theta)$ so that the opening angle equals 4θ . With this angle we define a closed cone $C_0 = C(2\theta)$, i.e.,

$$C_0 = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, -x_n \geq |x| \cos \theta\}.$$

The closed cone C_0 also satisfies

$$(3.1) \quad x + C_0 \subset \Omega^c \quad \text{for all } x \in \Omega^c.$$

Let $L \in C_c^\infty(\mathbf{R}^n)$ be a function such that

$$(3.2) \quad \text{supp } L \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} L(\sigma) d\mathcal{H}^{n-1}(\sigma) = 1.$$

Here $-C_0 = \{-y \mid y \in C_0\}$ and S^{n-1} is the unit sphere in $\mathbf{R}^n$. Then we define a vector field $K = (K_1, \dots, K_n)$ as

$$(3.3) \quad K(x) := \frac{x}{|x|^n} L\left(\frac{x}{|x|}\right), \quad x \in \mathbf{R}^n \setminus \{0\}.$$

Definition 3.1. For $f \in C_c^\infty(\Omega)$, we define a vector field $u = Sf$ as

$$u(x) = Sf(x) := (K * \bar{f})(x) = \int_{\mathbf{R}^n} K(x-y) \bar{f}(y) dy, \quad x \in \mathbf{R}^n.$$

Here $\bar{f}$ denotes the zero extension of f to $\mathbf{R}^n$ given by

$$\bar{f}(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \Omega^c. \end{cases}$$

This operator was introduced by Solonnikov [36]. For a fixed $x \in \mathbf{R}^n$, since

$$\frac{x-y}{|x-y|} \in \text{supp } L|_{S^{n-1}} \subset S^{n-1} \cap (-C_0)$$

implies $y \in x + C_0$, we can write

$$u(x) = \int_{x+C_0} K(x-y) \bar{f}(y) dy.$$

This formula and the property (3.1) of Ω imply that $u(x) = 0$ for all $x \in \Omega^c$. In particular, u vanishes on $\partial\Omega$. However, the support of u may become unbounded although f is compactly supported in Ω .

By the change of variables $x-y = r\sigma$ with $r > 0$ and $\sigma \in S^{n-1}$ we have

$$u(x) = \int_0^\infty \int_{S^{n-1}} L(\sigma) \bar{f}(x-r\sigma) r^{n-1} d\mathcal{H}^{n-1}(\sigma) dr.$$

Hence if $f \in C_c^\infty(\Omega)$ is supported in $B_R(0)$ and $x \in B_a(0)$ ($R, a > 0$), then

$$u(x) = \int_0^{R+a} \int_{S^{n-1}} L(\sigma) \bar{f}(x-r\sigma) r^{n-1} d\mathcal{H}^{n-1}(\sigma) dr,$$

which implies that $u = Sf$ is smooth in Ω . Moreover, $u = Sf$ vanishes near $\partial\Omega$ and thus it is smooth in the whole space $\mathbf{R}^n$, since f is compactly supported in Ω .

Lemma 3.2. *Let $p \in (1, \infty)$. There exists a constant $c > 0$ such that*

$$\|\nabla u\|_{L^p(\Omega)} \leq c \|f\|_{L^p(\Omega)}$$

for all $f \in C_c^\infty(\Omega)$ and $u = Sf$.

Proof. Let u_i be the i -th component of u :

$$u_i(x) = (K_i * \bar{f})(x) = \int_{\mathbf{R}^n} K_i(z) \bar{f}(x-z) dz.$$

Differentiating both sides with respect to the j -th variable, we have

$$\partial_j u_i(x) = \int_{\mathbf{R}^n} K_i(z) (\partial_j \bar{f})(x-z) dz = \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n \setminus B_\varepsilon(0)} K_i(z) (\partial_j \bar{f})(x-z) dz$$

and, by changing variables $y = x-z$ and integrating by parts,

$$\begin{aligned} \partial_j u_i(x) &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\partial B_\varepsilon(x)} K_i(x-y) \frac{x_j - y_j}{|x-y|} \bar{f}(y) d\mathcal{H}^{n-1}(y) + \int_{\mathbf{R}^n \setminus B_\varepsilon(x)} (\partial_j K_i)(x-y) \bar{f}(y) dy \right). \end{aligned}$$

On the one hand, we change variables $x-y = \varepsilon\sigma$ with $\sigma \in S^{n-1}$ to get

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{|x-y|=\varepsilon} K_i(x-y) \frac{x_j - y_j}{|x-y|} \bar{f}(y) d\mathcal{H}^{n-1}(y) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{|x-y|=\varepsilon} \frac{x_i - y_i}{|x-y|} \frac{x_j - y_j}{|x-y|} L\left(\frac{x-y}{|x-y|}\right) \bar{f}(y) \frac{1}{|x-y|^{n-1}} d\mathcal{H}^{n-1}(y) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \bar{f}(x-\varepsilon\sigma) d\mathcal{H}^{n-1}(\sigma) \\ &= \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) d\mathcal{H}^{n-1}(\sigma), \end{aligned}$$

where the last equality follows from the fact that L is integrable on S^{n-1} and $\bar{f}$ is continuous at x . On the other hand, we differentiate K_i to obtain

$$(3.4) \quad \begin{aligned} K_{ij}(z) &:= \partial_j K_i(z) = \frac{k_{ij}(z/|z|)}{|z|^n}, \\ k_{ij}(z) &:= (\delta_{ij} - nz_i z_j) L(z) + z_i (\partial_j L)(z) - z_i z_j \sum_{\ell=1}^n z_\ell (\partial_\ell L)(z) \end{aligned}$$

for $z \in \mathbf{R}^n \setminus \{0\}$. Then K_{ij} is homogeneous of degree $-n$ and there is a constant $c > 0$ such that

$$|K_{ij}(z)| \leq \frac{c}{|z|^n} \quad \text{for all } z \in \mathbf{R}^n \setminus \{0\}$$

by the smoothness of L on S^{n-1} . Moreover, for every R_1 and R_2 with $0 < R_1 < R_2$,

$$\begin{aligned} \int_{R_1 < |z| < R_2} K_{ij}(z) \, dz &= \int_{R_1 < |z| < R_2} \partial_j K_i(z) \, dz \\ &= \int_{|z|=R_2} K_i(z) \frac{z_j}{|z|} \, d\mathcal{H}^{n-1}(z) - \int_{|z|=R_1} K_i(z) \frac{z_j}{|z|} \, d\mathcal{H}^{n-1}(z) \\ &= \int_{|z|=R_2} \frac{z_i}{|z|} \frac{z_j}{|z|} L\left(\frac{z}{|z|}\right) \frac{1}{|z|^{n-1}} \, d\mathcal{H}^{n-1}(z) - \int_{|z|=R_1} \frac{z_i}{|z|} \frac{z_j}{|z|} L\left(\frac{z}{|z|}\right) \frac{1}{|z|^{n-1}} \, d\mathcal{H}^{n-1}(z) \\ &= \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) - \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 0. \end{aligned}$$

In the fourth equality we changed variables $z = R_2 \sigma$ and $z = R_1 \sigma$ with $\sigma \in S^{n-1}$, respectively. This equality is equivalent to

$$(3.5) \quad \int_{S^{n-1}} k_{ij}(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = 0.$$

Thus we can apply the Calderón-Zygmund theory (see eg. [23, Theorem 5.2.7 and Theorem 5.2.10]) of singular integral operators to the kernel K_{ij} and obtain the formula

$$(3.6) \quad \partial_j u_i(x) = \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} K_{ij}(x-y) \bar{f}(y) \, dy,$$

where the second integral is considered in the sense of the Cauchy principal value.

Finally, the inequality

$$\left| \bar{f}(x) \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) \right| \leq |\bar{f}(x)| \int_{S^{n-1}} L(\sigma) \, d\mathcal{H}^{n-1}(\sigma) = |\bar{f}(x)|$$

and the Calderón-Zygmund theory imply that

$$\|\partial_j u_i\|_{L^p(\Omega)} \leq c \|\bar{f}\|_{L^p(\mathbf{R}^n)} = c \|f\|_{L^p(\Omega)}$$

with a positive constant c independent of f . Hence the lemma follows. $\square$

Lemma 3.3. *For every $f \in C_c^\infty(\Omega)$ the vector field $u = Sf$ satisfies*

$$\operatorname{div} u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Proof. We have already observed that u vanishes on the boundary. Let us compute $\operatorname{div} u = \sum_{i=1}^n \partial_i u_i$ in Ω . By the formula (3.6) in the proof of Lemma 3.2,

$$\operatorname{div} u(x) = \bar{f}(x) \int_{S^{n-1}} \sum_{i=1}^n \sigma_i^2 L(\sigma) d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} \sum_{i=1}^n K_{ii}(x-y) \bar{f}(y) dy.$$

In this formula, we have

$$\int_{S^{n-1}} \sum_{i=1}^n \sigma_i^2 L(\sigma) d\mathcal{H}^{n-1}(\sigma) = \int_{S^{n-1}} L(\sigma) d\mathcal{H}^{n-1}(\sigma) = 1$$

by (3.2) and, for all $z \in \mathbf{R}^n \setminus \{0\}$,

$$\begin{aligned} \sum_{i=1}^n K_{ii}(z) &= \frac{1}{|z|^n} L\left(\frac{z}{|z|}\right) \sum_{i=1}^n \left(1 - n \frac{z_i^2}{|z|^2}\right) \\ &\quad + \frac{1}{|z|^n} \sum_{i=1}^n \frac{z_i}{|z|} (\partial_i L)\left(\frac{z}{|z|}\right) - \sum_{i=1}^n \frac{z_i^2}{|z|^{n+2}} \sum_{k=1}^n \frac{z_k}{|z|} (\partial_k L)\left(\frac{z}{|z|}\right) = 0. \end{aligned}$$

Hence $\operatorname{div} u(x) = \bar{f}(x) = f(x)$ for all $x \in \Omega$. $\square$

Lemma 3.3 means that the operator S is a solution operator to the divergence problem with Dirichlet boundary condition. Note that S is not a unique solution operator because a solution to the divergence problem is not unique.

Next we define a linear operator that plays a main role in this section.

Definition 3.4. For a vector field $u \in C_c^\infty(\Omega)$, we define a vector field Tu as

$$Tu(x) := \int_{\mathbf{R}^n} K(x-y) \overline{\operatorname{div} u}(y) dy, \quad x \in \mathbf{R}^n.$$

Here K is given by (3.3) and $\overline{\operatorname{div} u}$ denotes the zero extension of $\operatorname{div} u$ to $\mathbf{R}^n$.

The above definition means that T is given by $T = S \circ \operatorname{div}$. Since $u \in C_c^\infty(\Omega)$, its divergence is in $C_c^\infty(\Omega)$ and thus Tu is smooth in the whole space $\mathbf{R}^n$ and vanishes outside of Ω , as discussed right after Definition 3.1. Also, by Lemma 3.3 we have

$$\operatorname{div} Tu = \operatorname{div} u \quad \text{in } \Omega, \quad Tu = 0 \quad \text{on } \partial\Omega.$$

Clearly $Tu = 0$ in $\mathbf{R}^n$ for $u \in C_{c,\sigma}^\infty(\Omega)$. Note that, as in the case of the operator S , the support of Tu may be unbounded.

Theorem 3.5. *Let Ω be a Lipschitz half-space. Let $p \in (1, \infty)$. There exists a constant $c > 0$ such that*

$$\|Tu\|_{L^p(\Omega)} \leq c \|u\|_{L^p(\Omega)}$$

for all $u \in C_c^\infty(\Omega)$.

Proof. Let us compute the i -th component $(Tu)_i$ of Tu with $i = 1, \dots, n$ for compactly supported vector field u in Ω . As in the proof of Lemma 3.2, we integrate

by parts to get

$$\begin{aligned} (Tu)_i(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(x)} K_i(x-y) \frac{x-y}{|x-y|} \cdot \bar{u}(y) d\mathcal{H}^{n-1}(y) \\ &\quad + \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n \setminus B_\varepsilon(x)} (\nabla K_i)(x-y) \cdot \bar{u}(y) dy \\ &= \int_{S^{n-1}} \sigma_i L(\sigma) \{\sigma \cdot \bar{u}(x)\} d\mathcal{H}^{n-1}(\sigma) + \int_{\mathbf{R}^n} (\nabla K_i)(x-y) \cdot \bar{u}(y) dy, \end{aligned}$$

or equivalently,

$$(3.7) \quad (Tu)_i(x) = \sum_{j=1}^n \{a_{ij} \bar{u}_j(x) + S_{ij} \bar{u}_j(x)\}, \quad x \in \mathbf{R}^n.$$

Here u_j is the j -th component of u and

$$a_{ij} = \int_{S^{n-1}} \sigma_i \sigma_j L(\sigma) d\mathcal{H}^{n-1}(\sigma), \quad S_{ij} \bar{u}_j(x) = \int_{\mathbf{R}^n} K_{ij}(x-y) \bar{u}_j(y) dy,$$

where $K_{ij} = \partial_j K_i$ is given by (3.4). Since a_{ij} is a constant satisfying

$$(3.8) \quad |a_{ij}| \leq \int_{S^{n-1}} L(\sigma) d\mathcal{H}^{n-1}(\sigma) = 1$$

and $S_{ij} \bar{u} = K_{ij} * \bar{u}$ is a singular integral (see the proof of Lemma 3.2), the Calderón-Zygmund theory yields the boundedness of the operator T on $L^p(\Omega)$. $\square$

By Theorem 3.5, the operator T extends uniquely to a bounded linear operator on $L^p(\Omega)$ with each $p \in (1, \infty)$, which we again refer to as T .

Our next goal is to estimate the $BMO_b^{\infty, \nu}(\Omega)$ -norm of Tu for $u \in C_c^\infty(\Omega)$ and $\nu \in (0, \infty]$. To this end, we estimate each term of the right-hand side in (3.7) for $u = (u_1, \dots, u_n) \in C_c^\infty(\Omega)$. By (3.8) we have

$$[a_{ij} \bar{u}_j : BMO^\infty(\Omega)] \leq [u_j : BMO^\infty(\Omega)], \quad [a_{ij} \bar{u}_j : b^\nu(\Omega)] \leq [u_j : b^\nu(\Omega)]$$

and thus

$$\|a_{ij} \bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| \leq \|u_j : BMO_b^{\infty, \nu}(\Omega)\|.$$

Moreover, since

$$[u_j : BMO^\infty(\Omega)] \leq 2\|u_j\|_{L^\infty(\Omega)}, \quad [u_j : b^\nu(\Omega)] \leq \omega_n \|u_j\|_{L^\infty(\Omega)},$$

where $\omega_n = 2\pi^{n/2}/n\Gamma(n/2)$ is the volume of the unit ball $B_1(0)$ in $\mathbf{R}^n$ with the Gamma function $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$, we have

$$(3.9) \quad \|a_{ij} \bar{u}_j : BMO_b^{\infty, \nu}(\Omega)\| \leq (2 + \omega_n) \|u_j\|_{L^\infty(\Omega)}.$$

Let us estimate $S_{ij} \bar{u}_j = K_{ij} * \bar{u}_j$, $i, j = 1, \dots, n$ in $BMO_b^{\infty, \nu}(\Omega)$. Recall that the integral kernel K_{ij} is of the form

$$K_{ij}(x) = \frac{k_{ij}(x/|x|)}{|x|^n}, \quad x \in \mathbf{R}^n \setminus \{0\},$$

where $k_{ij} \in C_c^\infty(\mathbf{R}^n)$ is given by (3.4) and satisfies

$$\text{supp } k_{ij} \subset (B_2(0) \setminus B_{1/2}(0)) \cap (-C_0), \quad \int_{S^{n-1}} k_{ij}(\sigma) d\mathcal{H}^{n-1}(\sigma) = 0,$$

see (3.2) and (3.5). We first estimate the BMO^∞ -seminorm of $S_{ij} \bar{u}_j$.

Lemma 3.6. *Let K be a function defined on $\mathbf{R}^n \setminus \{0\}$ such that*

$$(3.10) \quad |K(x-y) - K(x)| \leq A|y|^\delta |x|^{-n-\delta} \quad \text{whenever} \quad |x| \geq 2|y| > 0$$

for some $A, \delta > 0$. Suppose that a convolution operator S with K is bounded on $L^2(\mathbf{R}^n)$ with a norm B . Then, there exists a dimensional constant c_n such that

$$[Sf : BMO^\infty(\mathbf{R}^n)] \leq c_n(A+B)\|f\|_{L^\infty(\mathbf{R}^n)}$$

for all $f \in L^2(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$.

Proof. See [24, Theorem 3.4.9 and Corollary 3.4.10]. □

Lemma 3.7. *There exists a constant $c > 0$ such that*

$$(3.11) \quad [S_{ij}\bar{u}_j : BMO^\infty(\Omega)] \leq c\|u_j\|_{L^\infty(\Omega)}$$

for all $u = (u_1, \dots, u_n) \in C_c^\infty(\Omega)$ and $i, j = 1, \dots, n$.

Proof. We shall apply Lemma 3.6 to $S = S_{ij}$. For this purpose it is sufficient to show that the function $K = K_{ij}$ satisfies (3.10), since we already know that the convolution operator S_{ij} is bounded on $L^2(\mathbf{R}^n)$, see the proof of Lemma 3.2. To this end, we differentiate K_{ij} to get

$$\nabla K_{ij}(x) = -\frac{nk_{ij}(x/|x|)}{|x|^{n+1}} \frac{x}{|x|} + \frac{1}{|x|^{n+1}} \left(I_n - \frac{1}{|x|^2} x \otimes x \right) \nabla k_{ij} \left(\frac{x}{|x|} \right)$$

for $x \in \mathbf{R}^n \setminus \{0\}$, where I_n is the identity matrix of size n and $x \otimes x := (x_i x_j)_{i,j}$ is the tensor product of x . Since k_{ij} is smooth on S^{n-1} , we have

$$|\nabla K_{ij}(x)| \leq \frac{c}{|x|^{n+1}}, \quad x \in \mathbf{R}^n \setminus \{0\}.$$

Hence, for all $x, y \in \mathbf{R}^n \setminus \{0\}$ with $|x| \geq 2|y| > 0$,

$$\begin{aligned} |K(x-y) - K(x)| &= \left| \int_0^1 \frac{d}{dt} (K(x-ty)) dt \right| = \left| \int_0^1 (-y) \cdot \nabla K(x-ty) dt \right| \\ &\leq |y| \int_0^1 \frac{c}{|x-ty|^{n+1}} dt \leq |y| \int_0^1 \frac{c}{(|x|-|y|)^{n+1}} dt \\ &\leq \frac{c|y|}{(|x|-|x|/2)^{n+1}} = \frac{2^{n+1}c|y|}{|x|^{n+1}}. \end{aligned}$$

Thus K_{ij} satisfies (3.10) with $\delta = 1$ and we can apply Lemma 3.6 to obtain

$$(3.12) \quad [S_{ij}\bar{u}_j : BMO^\infty(\mathbf{R}^n)] \leq c\|\bar{u}_j\|_{L^\infty(\mathbf{R}^n)} = c\|u_j\|_{L^\infty(\Omega)}$$

with some constant $c > 0$.

By definition of the BMO^∞ -seminorm, we have

$$[S_{ij}\bar{u}_j : BMO^\infty(\Omega)] \leq [S_{ij}\bar{u}_j : BMO^\infty(\mathbf{R}^n)].$$

Hence the inequality (3.11) follows from (3.12). □

Next, let us estimate the b^ν -part of $S_{ij}\bar{u}_j$. Recall the two closed cones

$$C_j = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, -x_n \geq |x| \cos(2^j \theta)\}, \quad j = 0, 1$$

with opening angle $\theta \in (0, \pi/4)$. For $r > 0$ and $x_0 \in \mathbf{R}^n$, we define

$$(3.13) \quad A_r(x_0) := \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)^c} (x + C_0) \cap (x_0 + C_1)^c \subset \mathbf{R}^n.$$

Here $x_0 + C_1 = \{y \in \mathbf{R}^n \mid y - x_0 \in C_1\}$ and $x + C_0$ is defined similarly.

Lemma 3.8. *For all $r > 0$ and $x_0 \in \mathbf{R}^n$ we have $A_r(x_0) \subset B_{r/\sin \theta}(x_0)$.*

Proof. By translation, we may assume that $x_0 = 0$. Let $a := (0, \dots, 0, r/\sin \theta) \in \mathbf{R}^n$. Suppose that

- (1) $B_r(0) \subset a + C_0$,
- (2) $x + C_0 \subset a + C_0$ for all $x \in a + C_0$,
- (3) $(a + C_0) \cap C_1^c \subset B_{r/\sin \theta}(0)$.

Then, the statements (1) and (2) imply

$$A_r(0) = \bigcup_{x \in B_r(0) \cap C_1^c} (x + C_0) \cap C_1^c \subset (a + C_0) \cap C_1^c.$$

Hence the statement (3) yields $A_r(0) \subset B_{r/\sin \theta}(0)$. Now let us prove the statements (1)-(3). Note that, since $\theta \in (0, \pi/4)$, the cones C_0 and C_1 are represented as

$$C_j = \{x = (x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1}, x_n \leq 0, |x'| \leq (-x_n) \tan(2^j \theta)\}, \quad j = 0, 1.$$

(1) Let $x = (x', x_n) \in B_r(0)$. Then, $x - a = (x', x_n - r/\sin \theta)$ satisfies

$$(x - a)_n = x_n - \frac{r}{\sin \theta} \leq r - \frac{r}{\sin \theta} < 0$$

and

$$\left(\frac{r}{\sin \theta} - x_n\right)^2 \tan^2 \theta - |x'|^2 \geq \frac{(r - x_n \sin \theta)^2}{\cos^2 \theta} - (r^2 - x_n^2) = \frac{(r \sin \theta - x_n)^2}{\cos^2 \theta} \geq 0,$$

or equivalently,

$$|x'| \leq \left(\frac{r}{\sin \theta} - x_n\right) \tan \theta = -(x - a)_n \tan \theta.$$

Hence $x - a \in C_0$, that is, $x \in a + C_0$ and the statement (1) holds.

(2) Let $x \in a + C_0$. If $y \in x + C_0$, then $(y - a)_n = (y - x)_n + (x - a)_n \leq 0$ and

$$|y'| \leq |x'| + |y' - x'| \leq -(x - a)_n \tan \theta - (y - x)_n \tan \theta = -(y - a)_n \tan \theta,$$

which means that $y \in a + C_0$. Hence the statement (2) holds.

(3) Let $x \in (a + C_0) \cap C_1^c$. Then we have

$$(3.14) \quad (x - a)_n = x_n - r/\sin \theta \leq 0, \quad |x'| \leq \left(\frac{r}{\sin \theta} - x_n\right) \tan \theta.$$

Hence

$$|x|^2 \leq \left(\frac{r}{\sin \theta} - x_n\right)^2 \tan^2 \theta + x_n^2 =: f(x_n).$$

To estimate the right-hand side in the above inequality for $x \in (a + C_0) \cap C_1^c$, we derive the range of x_n for $x \in (a + C_0) \cap C_1^c$. If $x_n \geq 0$, then $x \in (a + C_0) \cap C_1^c$ holds if and only if the condition (3.14) is satisfied. Thus x_n must satisfy

$$0 \leq x_n \leq \frac{r}{\sin \theta}.$$

On the other hand, if $x_n < 0$, then $x \in (a + C_0) \cap C_1^c$ holds if and only if

$$(-x_n) \tan(2\theta) < |x'| \leq \left(\frac{r}{\sin \theta} - x_n\right) \tan \theta.$$

Hence, in particular, if $x \in (a + C_0) \cap C_1^c$ and $x_n < 0$, then x_n must satisfy

$$(-x_n) \tan(2\theta) < \left(\frac{r}{\sin \theta} - x_n\right) \tan \theta,$$

which yields the inequality

$$-\frac{r}{\cos \theta} < (\tan(2\theta) - \tan \theta) x_n.$$

Since

$$\begin{aligned} \tan(2\theta) - \tan \theta &= \tan(2\theta) - \frac{1}{2} \tan(2\theta)(1 - \tan^2 \theta) \\ &= \frac{1}{2} \tan(2\theta)(1 + \tan^2 \theta) = \frac{\tan(2\theta)}{2 \cos^2 \theta} > 0 \quad \left(0 < \theta < \frac{\pi}{4}\right), \end{aligned}$$

the above inequality is equivalent to

$$-\frac{2r \cos \theta}{\tan(2\theta)} < x_n (< 0).$$

In summary, the range of x_n for $x \in (a + C_0) \cap C_1^c$ is

$$\alpha := -\frac{2r \cos \theta}{\tan(2\theta)} < x_n \leq \frac{r}{\sin \theta} =: \beta$$

and thus we obtain

$$|x|^2 \leq f(x_n) \leq \sup_{s \in (\alpha, \beta]} f(s) = \max\{f(\alpha), f(\beta)\},$$

where the last equality follows from the fact that $f(x_n)$ is a concave parabola. On the one hand, we have $f(\beta) = \beta^2 = r^2 / \sin^2 \theta$. On the other hand, since

$$\alpha = -\frac{2r \cos \theta \cos(2\theta)}{\sin(2\theta)} = -\frac{r \cos(2\theta)}{\sin \theta} = \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta},$$

we have

$$\begin{aligned} f(\alpha) &= \left(\frac{r}{\sin \theta} - \frac{r(1 - 2 \cos^2 \theta)}{\sin \theta} \right)^2 \tan^2 \theta + \frac{r^2 \cos^2(2\theta)}{\sin^2 \theta} \\ &= \frac{r^2}{\sin^2 \theta} \{4 \tan^2 \theta \cos^4 \theta + \cos^2(2\theta)\} = \frac{r^2}{\sin^2 \theta}. \end{aligned}$$

Hence $|x|^2 \leq r^2 / \sin^2 \theta$ and thus $x \in B_{r/\sin \theta}(0)$ for every $x \in (a + C_0) \cap C_1^c$. Therefore, the statement (3) holds and the lemma follows. $\square$

Now we can estimate the b^ν -part of $S_{ij} \bar{u}_j$.

Lemma 3.9. *Let $\nu \in (0, \infty]$. There exists a constant $c > 0$ such that*

$$(3.15) \quad [S_{ij} \bar{u}_j : b^\nu(\Omega)] \leq \frac{c}{\sin^{n/2} \theta} \|u_j\|_{L^\infty(\Omega)}$$

for all $u = (u_1, \dots, u_n) \in C_c^\infty(\Omega)$ and $i, j = 1, \dots, n$.

Proof. First we note that for all $f \in L_{loc}^1(\Omega)$ the inequality

$$[f : b^\nu(\Omega)] \leq \omega_n^{1/2} [f : b_2^\nu(\Omega)]$$

holds by Hölder's inequality. Hence, to prove (3.15), it is sufficient to show the inequality

$$(3.16) \quad [S_{ij} \bar{u}_j : b_2^\nu(\Omega)] \leq \frac{c}{\sin^{n/2} \theta} [u_j : b_2^{\nu/\sin \theta}(\Omega)] \leq \frac{c \omega_n^{1/2}}{\sin^{n/2} \theta} \|u_j\|_{L^\infty}.$$

The second inequality of (3.16) follows from the definition of $[\cdot : b_2^{\nu/\sin\theta}(\Omega)]$. Let us show the first inequality. The singular integral $S_{ij}\bar{u}_j$ is of the form

$$S_{ij}\bar{u}_j(x) = (K_{ij} * \bar{u}_j)(x) = \int_{\mathbf{R}^n} K_{ij}(x-y)\bar{u}_j(y) dy, \quad x \in \mathbf{R}^n.$$

Since $\text{supp } K_{ij} \subset -C_0$ (see (3.4) and (3.2)) and $\text{supp } u \subset \Omega$, we can write

$$S_{ij}\bar{u}_j(x) = \int_{(x+C_0)\cap\Omega} K_{ij}(x-y)\bar{u}_j(y) dy, \quad x \in \mathbf{R}^n.$$

Hence, if we set

$$W_r(x_0) := \bigcup_{x \in B_r(x_0) \cap \Omega} (x + C_0) \cap \Omega$$

for each $x_0 \in \partial\Omega$ and $r > 0$ with $B_r(x_0) \subset U_\nu(\partial\Omega)$, then we have

$$S_{ij}\bar{u}_j(x) = \int_{(x+C_0)\cap\Omega} K_{ij}(x-y)(\bar{u}_j|_{W_r(x_0)})(y) dy = [K_{ij} * (\bar{u}_j|_{W_r(x_0)})](x)$$

for all $x \in B_r(x_0) \cap \Omega$, where

$$(\bar{u}_j|_{W_r(x_0)})(x) := \begin{cases} \bar{u}_j(x), & x \in W_r(x_0), \\ 0, & x \notin W_r(x_0). \end{cases}$$

Since K_{ij} is a singular kernel (see the proof of Lemma 3.2), the Calderón-Zygmund theory implies that

$$\begin{aligned} \int_{B_r(x_0)\cap\Omega} |S_{ij}\bar{u}_j(x)|^2 dx &= \int_{B_r(x_0)\cap\Omega} |[K_{ij} * (\bar{u}_j|_{W_r(x_0)})](x)|^2 dx \\ &\leq c \int_{\mathbf{R}^n} |(\bar{u}_j|_{W_r(x_0)})(x)|^2 dx = c \int_{W_r(x_0)} |\bar{u}_j(x)|^2 dx \end{aligned}$$

with some constant $c > 0$. Now we recall the property of the infinite cone C_1 :

$$x + C_1 \subset \Omega^c \Leftrightarrow \Omega \subset (x + C_1)^c \quad \text{for all } x \in \Omega^c.$$

By this property we have

$$W_r(x_0) \subset \bigcup_{x \in B_r(x_0) \cap (x_0 + C_1)^c} (x + C_0) \cap ((x_0 + C_1)^c \cap \Omega) = A_r(x_0) \cap \Omega,$$

where $A_r(x_0)$ is given by (3.13), and thus Lemma 3.8 yields

$$W_r(x_0) \subset A_r(x_0) \cap \Omega \subset B_{r/\sin\theta}(x_0) \cap \Omega.$$

Hence we have

$$\begin{aligned} \frac{1}{r^n} \int_{B_r(x_0)\cap\Omega} |S_{ij}\bar{u}_j(x)|^2 dx &\leq \frac{c}{r^n} \int_{W_r(x_0)} |\bar{u}_j(x)|^2 dx \\ &\leq \frac{c}{r^n} \int_{B_{r/\sin\theta}(x_0)\cap\Omega} |\bar{u}_j(x)|^2 dx = \frac{c}{\sin^n\theta} \left(\frac{\sin\theta}{r}\right)^n \int_{B_{r/\sin\theta}(x_0)\cap\Omega} |u_j(x)|^2 dx \\ &\leq \frac{c}{\sin^n\theta} \left[u_j : b_2^{\nu/\sin\theta}(\Omega) \right]^2 \end{aligned}$$

for every $x_0 \in \partial\Omega$ and $r > 0$ with $B_r(x_0) \subset U_\nu(\partial\Omega)$, which yields

$$[S_{ij}\bar{u}_j : b_2^\nu(\Omega)]^2 \leq \frac{c}{\sin^n\theta} \left[u_j : b_2^{\nu/\sin\theta}(\Omega) \right]^2.$$

The proof is complete. $\square$

Now we obtain an estimate for the $BMO_b^{\infty,\nu}(\Omega)$ -norm of Tu .

Theorem 3.10. *Let $\nu \in (0, \infty]$. There exists a constant $c > 0$ such that*

$$\|Tu : BMO_b^{\infty,\nu}(\Omega)\| \leq c\|u\|_{L^\infty(\Omega)}$$

for all $u \in C_c^\infty(\Omega)$.

Proof. Since the i -th component of Tu , $i = 1, \dots, n$, is of the form (3.7), we have by (3.9), (3.11) and (3.15) that

$$\begin{aligned} & \|Tu : BMO_b^{\infty,\nu}(\Omega)\| \\ & \leq c \sum_{i,j=1}^n (\|a_{ij}\bar{u}_j : BMO_b^{\infty,\nu}(\Omega)\| + [S_{ij}\bar{u}_j : BMO^\infty(\Omega)] + [S_{ij}\bar{u}_j : b^\nu(\Omega)]) \\ & \leq c \sum_{j=1}^n \|u_j\|_{L^\infty(\Omega)} \leq c\|u\|_{L^\infty(\Omega)} \end{aligned}$$

with a positive constant c . □

3.2. Non-Helmholtz projection. As in the previous subsection, let Ω denote a Lipschitz half-space in $\mathbf{R}^n$.

Definition 3.11. For a vector field $u \in C_c^\infty(\Omega)$, we define a vector field $Q'u$ on $\mathbf{R}^n$ as $Q'u := u - Tu$. Here the operator T is given in Definition 3.4.

For a vector field $u \in C_c^\infty(\Omega)$, the vector field Tu is smooth in $\mathbf{R}^n$ and

$$\operatorname{div} Tu = \operatorname{div} u \quad \text{in } \Omega, \quad Tu = 0 \quad \text{on } \partial\Omega.$$

Moreover, $Tu = 0$ for all $u \in C_{c,\sigma}^\infty(\Omega)$, see the argument after Definition 3.4. Thus $Q'u = u - Tu$ is also smooth in $\mathbf{R}^n$ and

$$(3.17) \quad \operatorname{div} Q'u = 0 \quad \text{in } \Omega, \quad Q'u = 0 \quad \text{on } \partial\Omega$$

for all $u \in C_c^\infty(\Omega)$, and $Q'u = u$ for all $u \in C_{c,\sigma}^\infty(\Omega)$. Note that Q' is not a projection from $C_c^\infty(\Omega)$ onto $C_{c,\sigma}^\infty(\Omega)$, since the support of Tu may be unbounded and thus $Q'u$ is not in $C_{c,\sigma}^\infty(\Omega)$ in general. However, Q' maps $C_c^\infty(\Omega)$ into $L_\sigma^p(\Omega)$.

Lemma 3.12. *For all $u \in C_c^\infty(\Omega)$ and $p \in (1, \infty)$, we have $Q'u \in L_\sigma^p(\Omega)$.*

We shall first prove an auxiliary proposition for the above lemma. For $p \in (1, \infty)$, let $G_p(\Omega) = \{\nabla q \in L^p(\Omega) \mid q \in L_{loc}^1(\Omega)\}$.

Proposition 3.13. *Let $p \in (1, \infty)$. For every $\nabla q \in G_p(\Omega)$, there exists a sequence $\{q_k\}_{k=1}^\infty$ of functions in $C_c^\infty(\mathbf{R}^n)$ such that*

$$(3.18) \quad \lim_{k \rightarrow \infty} \|\nabla q - \nabla q_k\|_{L^p(\Omega)} = 0.$$

Proof. Since the restriction of $C_c^\infty(\mathbf{R}^n)$ on Ω is dense in $W^{1,p}(\Omega)$, it is sufficient to show that for every $\nabla q \in G_p(\Omega)$ there is a sequence $\{q_k\}_{k=1}^\infty$ of functions in $W^{1,p}(\Omega)$ such that (3.18) holds. Let us prove this claim.

(1) First we assume that the claim is valid for the half space $\mathbf{R}_+^n$ and show the claim for general Lipschitz half-spaces $\Omega = \{(x', x_n) \in \mathbf{R}^n \mid x_n > h(x')\}$. As in Section 2, let $F(x) := (x', x_n - h(x'))$ be a bi-Lipschitz map from Ω to $\mathbf{R}_+^n$. Let $\nabla q \in G_p(\Omega)$ and $\tilde{q} := q \circ F^{-1}$, where $F^{-1}(y) := (y', y_n + h(y'))$ is the inverse mapping of F . Then, since $\nabla \tilde{q}(y) = \nabla F^{-1}(y) \nabla q(F^{-1}(y))$ for $y \in \mathbf{R}_+^n$ and each component

of ∇F^{-1} is bounded (because h is Lipschitz continuous), we have $\nabla \tilde{q} \in G_p(\mathbf{R}_+^n)$. Hence, by our assumption that the claim is valid for $\mathbf{R}_+^n$, there is a sequence $\{\tilde{q}_k\}_{k=1}^\infty$ of functions in $W^{1,p}(\mathbf{R}_+^n)$ such that $\lim_{k \rightarrow \infty} \|\nabla \tilde{q} - \nabla \tilde{q}_k\|_{L^p(\mathbf{R}_+^n)} = 0$.

Let $q_k := \tilde{q}_k \circ F$ for each $k \in \mathbf{N}$. Then, since

$$\nabla q(x) = \nabla F(x) \nabla \tilde{q}(F(x)), \quad \nabla q_k(x) = \nabla F(x) \nabla \tilde{q}_k(F(x)), \quad x \in \Omega$$

and each component of ∇F is bounded, we have $q_k \in W^{1,p}(\Omega)$ and

$$\|\nabla q - \nabla q_k\|_{L^p(\Omega)} \leq c \|\nabla \tilde{q} - \nabla \tilde{q}_k\|_{L^p(\mathbf{R}_+^n)} \rightarrow 0$$

as $k \rightarrow \infty$. Thus the claim is valid for general Lipschitz half-spaces Ω .

(2) Now we prove the claim for $\Omega = \mathbf{R}_+^n$. We follow the idea of the proof of the claim in the case $\Omega = \mathbf{R}^n$, see [34, Lemma 2.5.4]. Let $\varphi \in C_c^\infty(\mathbf{R}^n)$ be a function such that

$$0 \leq \varphi \leq 1 \quad \text{in } \mathbf{R}^n, \quad \varphi = 1 \quad \text{in } B_1(0), \quad \varphi = 0 \quad \text{in } \mathbf{R}^n \setminus B_2(0)$$

and $\varphi_k(x) := \varphi(k^{-1}x)$ for $k \in \mathbf{N}$ and $x \in \mathbf{R}^n$. Then, $\lim_{k \rightarrow \infty} \varphi_k(x) = 1$ for all $x \in \mathbf{R}^n$ and $\text{supp } \varphi_k \subset B_{2k}(0)$, $\text{supp } \nabla \varphi_k \subset B_{2k}(0) \setminus B_k(0)$ for $k \in \mathbf{N}$.

Let $\nabla q \in G_p(\mathbf{R}_+^n)$. Then $q \in W_{loc}^{1,p}(\overline{\mathbf{R}_+^n})$, that is, $q \in W^{1,p}(U)$ for every bounded subset U of $\mathbf{R}_+^n$; see the proof of [31, Theorem 7.6 in Chapter 2]. Hence by setting $G_k := \mathbf{R}_+^n \cap (B_{2k}(0) \setminus B_k(0))$ for $k \in \mathbf{N}$, we have $q \in W^{1,p}(G_k)$ and thus there is a constant a_k such that $\int_{G_k} (q - a_k) dx = 0$ for each $k \in \mathbf{N}$. From this equality and the change of variables $x = ky$ for $x \in G_k$ and $y \in G_1$ we have

$$\int_{G_1} (q(ky) - a_k) dy = k^{-n} \int_{G_k} (q(x) - a_k) dx = 0.$$

Hence we can apply Poincaré's inequality to $q(ky) - a_k$ on G_1 and get

$$\left(\int_{G_1} |q(ky) - a_k|^p dy \right)^{1/p} \leq c \left(\int_{G_1} |\nabla(q(ky))|^p dy \right)^{1/p}$$

with a constant $c > 0$ independent of k . In this inequality, we observe that

$$\begin{aligned} \int_{G_1} |q(ky) - a_k|^p dy &= k^{-n} \int_{G_k} |q(x) - a_k|^p dx, \\ \int_{G_1} |\nabla(q(ky))|^p dy &= k^p \int_{G_1} |(\nabla q)(ky)|^p dy = k^{p-n} \int_{G_k} |\nabla q(x)|^p dx \end{aligned}$$

by the change of variables $x = ky$ and thus

$$(3.19) \quad \|q - a_k\|_{L^p(G_k)} \leq ck \|\nabla q\|_{L^p(G_k)}, \quad k \in \mathbf{N}.$$

For each $k \in \mathbf{N}$, let $q_k := \varphi_k(q - a_k)$ on $\mathbf{R}_+^n$. Then since $\text{supp } q_k \subset \mathbf{R}_+^n \cap B_{2k}(0)$ holds by the relation $\text{supp } \varphi_k \subset B_{2k}(0)$, it follows that $q_k \in W^{1,p}(\mathbf{R}_+^n)$ and

$$(3.20) \quad \|\nabla q - \nabla q_k\|_{L^p(\mathbf{R}_+^n)} \leq \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbf{R}_+^n)} + \|(\nabla \varphi_k)(q - a_k)\|_{L^p(\mathbf{R}_+^n)}.$$

Since $0 \leq \varphi_k(x) \leq 1$ and $\lim_{k \rightarrow \infty} \varphi_k(x) = 1$ for all $x \in \mathbf{R}_+^n$ and $\nabla q \in L^p(\mathbf{R}_+^n)$, the dominated convergence theorem yields

$$(3.21) \quad \lim_{k \rightarrow \infty} \|\nabla q - \varphi_k \nabla q\|_{L^p(\mathbf{R}_+^n)} = 0.$$

On the other hand, since $\nabla \varphi_k = k^{-1}(\nabla \varphi)_k$ and $\text{supp } \nabla \varphi_k|_{\mathbf{R}_+^n} \subset \overline{G_k}$ for each $k \in \mathbf{N}$, it follows from (3.19) and the dominated convergence theorem that

$$(3.22) \quad \|(\nabla \varphi_k)(q - a_k)\|_{L^p(\mathbf{R}_+^n)} \leq ck^{-1} \|q - a_k\|_{L^p(G_k)} \leq c \|\nabla q\|_{L^p(G_k)} \rightarrow 0$$

as $k \rightarrow \infty$. Applying (3.21) and (3.22) to (3.20) we have

$$\lim_{k \rightarrow \infty} \|\nabla q - \nabla q_k\|_{L^p(\mathbf{R}_+^n)} = 0,$$

where $q_k \in W^{1,p}(\mathbf{R}_+^n)$ for all $k \in \mathbf{N}$. Hence the claim is valid when $\Omega = \mathbf{R}_+^n$ and the proposition follows. $\square$

Proof of Lemma 3.12. Let $u \in C_c^\infty(\Omega)$ and $p \in (1, \infty)$. Then, since $Tu \in L_p(\Omega)$ by Theorem 3.5, we have $Q'u = u - Tu \in L^p(\Omega)$. To show $Q'u \in L_\sigma^p(\Omega)$, we employ a characterization of elements of $L_\sigma^p(\Omega)$ ([19, Lemma III.2.1]): a vector field $v \in L^p(\Omega)$ is in $L_\sigma^p(\Omega)$ if and only if

$$\int_\Omega v \cdot \nabla q \, dx = 0 \quad \text{for all } \nabla q \in G_{p'}(\Omega) \quad \left(p' := \frac{p}{p-1}\right).$$

Let ∇q be any element of $G_{p'}(\Omega)$. From Proposition 3.13, there is a sequence $\{q_k\}_{k=1}^\infty$ of functions in $C_c^\infty(\mathbf{R}^n)$ such that the equality (3.18) with p replaced by p' holds. Since $Q'u$ is defined and smooth in $\mathbf{R}^n$ for $u \in C_c^\infty(\Omega)$ and $q_k \in C_c^\infty(\mathbf{R}^n)$, integration by parts yields

$$\int_\Omega Q'u \cdot \nabla q_k \, dx = - \int_\Omega q_k \operatorname{div} Q'u \, dx + \int_{\partial\Omega} q_k Q'u \cdot \nu \, d\mathcal{H}^{n-1}$$

for all $k \in \mathbf{N}$, where ν denotes the unit outer normal vector field of $\partial\Omega$. We apply (3.17) to the right-hand side of this equality to get $\int_\Omega Q'u \cdot \nabla q_k \, dx = 0$ for all $k \in \mathbf{N}$. Since $Q'u \in L^p(\Omega)$ and (3.18) with p replaced by p' holds, the above equality implies that

$$\int_\Omega Q'u \cdot \nabla q \, dx = \lim_{k \rightarrow \infty} \int_\Omega Q'u \cdot \nabla q_k \, dx = 0.$$

Hence by the characterization of elements of $L_\sigma^p(\Omega)$ we conclude that $Q'u \in L_\sigma^p(\Omega)$ for all $u \in C_c^\infty(\Omega)$. The proof is complete. $\square$

Remark 3.14.

- (1) Let $p \in (1, \infty)$. By Theorem 3.5 and Lemma 3.12, we have $Q'u \in L_\sigma^p(\Omega)$ and $\|Q'u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}$ for all $u \in C_c^\infty(\Omega)$. Moreover, $Q'u = u$ holds for all $u \in C_{c,\sigma}^\infty(\Omega)$. Hence, by the density argument, Q' extends uniquely to a bounded linear operator on $L^p(\Omega)$ that is a projection onto $L_\sigma^p(\Omega)$.
- (2) The projection onto $L_\sigma^p(\Omega)$ given as above is NOT the Helmholtz projection. Indeed, if it were the Helmholtz projection, then for each $u \in C_c^\infty(\Omega)$ there would exist $\pi \in L_{loc}^1(\Omega)$ such that $(I - Q')u = \nabla\pi$ holds. Since $(I - Q')u = Tu = K * \operatorname{div} u$ for $u \in C_c^\infty(\Omega)$, the existence of such π would imply that $\partial_j(K_i * \operatorname{div} u) = \partial_i(K_j * \operatorname{div} u)$ for all $i, j = 1, \dots, n$. For each $f \in C_c^\infty(\Omega)$ with $\int_\Omega f \, dx = 0$ there is $u \in C_c^\infty(\Omega)$ satisfying $f = \operatorname{div} u$. This is possible since we are able to apply Bogovskii's lemma to a bounded Lipschitz domain $D \subset \Omega$ containing the support of f (see [19, Theorem III.3.3]). Thus the above equality would imply that $\partial_j K_i = \partial_i K_j + c$ with some constant c for all $i, j = 1, \dots, n$ as a distribution. This contradicts the fact that $\partial_j K_i \neq \partial_i K_j + c$ for $i \neq j$ as observed in (3.4).
- (3) It is possible to prove the characterization

$$L_\sigma^p(\Omega) = \{u \in L^p(\Omega) \mid \operatorname{div} u = 0 \text{ in } \Omega, \ u \cdot \nu = 0 \text{ on } \partial\Omega\}$$

if we use Proposition 3.13 and an integration by parts formula. This characterization is well-known for bounded ([17]), exterior and other domains

(see [19, Section III.2]). However, for a Lipschitz half-space, it is less popular. A proof can be found in [30, Lemma 2.1].

The linear operator Q' also maps $C_c^\infty(\Omega)$ into $VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$.

Lemma 3.15. *Let Ω be a Lipschitz half-space. For all $u \in C_c^\infty(\Omega)$ and $\nu \in (0, \infty]$, we have $Q'u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$.*

We shall prove two auxiliary propositions for the above lemma. For $p \in (1, \infty)$, let $W_{0,\sigma}^{1,p}(\Omega)$ be the $W^{1,p}$ -closure of $C_{c,\sigma}^\infty(\Omega)$.

Proposition 3.16. *Let Ω be a Lipschitz half-space. For all $p \in (1, \infty)$ we have $L_\sigma^p(\Omega) \cap W_0^{1,p}(\Omega) \subset W_{0,\sigma}^{1,p}(\Omega)$. Thus $L_\sigma^p(\Omega) \cap W_0^{1,p}(\Omega) = W_{0,\sigma}^{1,p}(\Omega)$.*

Proof. Let $\rho \in C_c^\infty(\mathbf{R}^n)$ be a function such that

$$0 \leq \rho \leq 1 \quad \text{in } \mathbf{R}^n, \quad \text{supp } \rho \subset B_1(0), \quad \int_{B_1(0)} \rho \, dx = 1$$

and $\rho_\delta(x) := \delta^{-n} \rho(\delta^{-1}x)$ for $\delta > 0$, $x \in \mathbf{R}^n$. Let $u \in L_\sigma^p(\Omega) \cap W_0^{1,p}(\Omega)$. Then there is a sequence $\{u_k\}_{k=1}^\infty$ of functions in $C_{c,\sigma}^\infty(\Omega)$ such that $\lim_{k \rightarrow \infty} \|u - u_k\|_{L^p(\Omega)} = 0$. For $a > 0$, we define a vector field u^a on Ω as

$$u^a(x) := \begin{cases} u(x', x_n - a), & x_n > h(x') + a, \\ 0, & h(x') < x_n \leq h(x') + a \end{cases}$$

and $u_k^a = (u_k)^a$ similarly. Then it is clear that $u^a \in W_0^{1,p}(\Omega)$ and $u_k^a \in C_{c,\sigma}^\infty(\Omega)$ for all $a > 0$. Moreover, we have

$$\|u^a - u_k^a\|_{L^p(\Omega)} = \|u - u_k\|_{L^p(\Omega)} \quad \text{for all } a > 0, \quad \lim_{a \rightarrow 0} \|u - u^a\|_{W^{1,p}(\Omega)} = 0.$$

By the second equality and the fact that $W_{0,\sigma}^{1,p}(\Omega)$ is closed in $W^{1,p}(\Omega)$, it is sufficient for showing $u \in W_{0,\sigma}^{1,p}(\Omega)$ to prove $u^a \in W_{0,\sigma}^{1,p}(\Omega)$ for all $a > 0$.

For each $a > 0$, there is a constant $d = d(a) > 0$ such that $\text{dist}(\text{supp } u_k^a, \partial\Omega) \geq d$ for all $k \in \mathbf{N}$. Then, for a given $\varepsilon > 0$, we can take $\delta \in (0, d/2)$ so small that

$$\|u^a - u^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \frac{\varepsilon}{2},$$

since $u^a \in W_0^{1,p}(\Omega)$. Also, since $\nabla \rho_\delta = \delta^{-1}(\nabla \rho)_\delta$, we have

$$\begin{aligned} & \|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} \\ & \leq c(\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{L^p(\Omega)} + \|u^a * \nabla \rho_\delta - u_k^a * \nabla \rho_\delta\|_{L^p(\Omega)}) \\ & = c(\|(u^a - u_k^a) * \rho_\delta\|_{L^p(\Omega)} + \delta^{-1} \|(u^a - u_k^a) * (\nabla \rho)_\delta\|_{L^p(\Omega)}) \\ & \leq c(1 + \delta^{-1}) \|u^a - u_k^a\|_{L^p(\Omega)} = c(1 + \delta^{-1}) \|u - u_k\|_{L^p(\Omega)} \end{aligned}$$

with a constant $c > 0$ independent of ε and δ . Hence by taking $k \in \mathbf{N}$ so large that

$$\|u - u_k\|_{L^p(\Omega)} < \frac{\varepsilon}{2c(1 + \delta^{-1})},$$

we have $\|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon/2$ and thus

$$\|u^a - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} \leq \|u^a - u^a * \rho_\delta\|_{W^{1,p}(\Omega)} + \|u^a * \rho_\delta - u_k^a * \rho_\delta\|_{W^{1,p}(\Omega)} < \varepsilon.$$

On the other hand, since $\text{dist}(\text{supp } u_k^a, \partial\Omega) > d$ and $\delta \in (0, d/2)$, the function $u_k^a * \rho_\delta$ is smooth and compactly supported in Ω . Moreover, we have

$$\text{div}(u_k^a * \rho_\delta) = (\text{div } u_k^a) * \rho_\delta = 0 \quad \text{in } \Omega.$$

Thus $u_k^a * \rho_\delta \in C_{c,\sigma}^\infty(\Omega)$ and u^a is approximated by elements of $C_{c,\sigma}^\infty(\Omega)$ in $W^{1,p}(\Omega)$, which means that $u^a \in W_{0,\sigma}^{1,p}(\Omega)$. Hence $u \in W_{0,\sigma}^{1,p}(\Omega)$ and the proof is now complete. $\square$

Proposition 3.17. *Let $\nu \in (0, \infty]$. If $p > n$, then $W_{0,\sigma}^{1,p}(\Omega) \subset VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$.*

Proof. Let $u \in W_{0,\sigma}^{1,p}(\Omega)$ and $u_k \in C_{c,\sigma}^\infty(\Omega)$ such that $\lim_{k \rightarrow \infty} \|u - u_k\|_{W^{1,p}(\Omega)} = 0$. Since $p > n$ and $u, u_k \in W_0^{1,p}(\Omega)$, Morrey's inequality (see e.g. [7, Theorem 4.12]) implies

$$\|u - u_k\|_{L^\infty(\Omega)} \leq c \|u - u_k\|_{W^{1,p}(\Omega)}$$

with a positive constant c independent of u and u_k . Thus we have

$$\|u - u_k : BMO_b^{\infty,\nu}(\Omega)\| \leq (2 + \omega_n) \|u - u_k\|_{L^\infty(\Omega)} \leq c \|u - u_k\|_{W^{1,p}(\Omega)} \rightarrow 0$$

as $k \rightarrow \infty$. Hence $u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ and the proof is now complete. $\square$

Proof of Lemma 3.15. Since $u \in C_c^\infty(\Omega)$ and thus $\partial_i u \in C_c^\infty(\Omega)$ for all $i = 1, \dots, n$, it follows from Lemma 3.12 that $Q'u \in L_\sigma^r(\Omega)$ and $\partial_i Q'u = Q'(\partial_i u) \in L^r(\Omega)$ for all $r \in (1, \infty)$ and $i = 1, \dots, n$. From this fact and the equality (3.17), we have $Q'u \in L_\sigma^r(\Omega) \cap W_0^{1,r}(\Omega)$ for all $r \in (1, \infty)$. Hence, by taking $r > n$, we can apply Proposition 3.16 and Proposition 3.17 to obtain $Q'u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$. $\square$

Remark 3.18. Let $\nu \in (0, \infty]$. Theorem 3.10 and Lemma 3.15 imply that $Q'u \in VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ and $\|Q'u : BMO_b^{\infty,\nu}(\Omega)\| \leq c \|u\|_{L^\infty(\Omega)}$ for all $u \in C_c^\infty(\Omega)$. Also, we have $Q'u = u$ for all $u \in C_{c,\sigma}^\infty(\Omega)$. Hence Q' extends uniquely to a bounded linear operator (again referred to as Q') from $C_0(\Omega)$, which is the L^∞ -closure of $C_c^\infty(\Omega)$, into $VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ that satisfies $Q'u = u$ for all $u \in C_{0,\sigma}(\Omega)$.

Now let us extend Q' to a linear operator that gives the projection mentioned in Theorem 1.4. For $p \in (1, \infty)$, we define a Banach space X_p and its norm as

$$X_p := L^p(\Omega) \cap C_0(\Omega), \quad \|u\|_{X_p} := \max\{\|u\|_{L^p(\Omega)}, \|u\|_{L^\infty(\Omega)}\}.$$

Note that the Banach space $C_0(\Omega)$ consists of all continuous functions f on Ω such that the set $\{x \in \Omega \mid |f(x)| \geq \varepsilon\}$ is compact in Ω for every $\varepsilon > 0$ (see e.g. [32, Theorem 3.17]).

Lemma 3.19. *For each $p \in (1, \infty)$, the linear subspace $C_c^\infty(\Omega)$ is dense in X_p .*

Proof. The proof is more or less standard (see e.g. [27, Corollary 19.24]). We give it for completeness. Let $u \in X_p$ and $\Omega_k := \{x \in \Omega \mid |x| \leq k, \text{dist}(x, \partial\Omega) \geq 1/k\}$ for $k \in \mathbf{N}$. For any given $\varepsilon > 0$, the set $\{x \in \Omega \mid |u(x)| \geq \varepsilon/2\}$ is compact in Ω since $u \in C_0(\Omega)$. Moreover, since $u \in L^p(\Omega)$, we can take $k \in \mathbf{N}$ so large that

$$(3.23) \quad \|u\|_{L^p(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}, \quad \|u\|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}.$$

Let $\varphi \in C_c^\infty(\Omega)$ be a continuous cut-off function such that

$$0 \leq \varphi \leq 1 \quad \text{in } \Omega, \quad \varphi = 1 \quad \text{in } \Omega_k, \quad \varphi = 0 \quad \text{in } \Omega \setminus \Omega_{2k}.$$

Since $u - \varphi u = 0$ in Ω_k and $|u - \varphi u| \leq |u|$ in $\Omega \setminus \Omega_k$, it follows from (3.23) that

$$(3.24) \quad \|u - \varphi u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}, \quad \|u - \varphi u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega \setminus \Omega_k)} < \frac{\varepsilon}{2}.$$

Let ρ_δ be a mollifier as in the beginning of the proof of Proposition 3.16. Since

$$\varphi u \in L^p(\Omega), \quad \text{dist}(\text{supp}(\varphi u), \partial\Omega) \geq \frac{1}{2k},$$

we can take $\delta \in (0, 1/4k)$ so small that

$$(3.25) \quad u_\delta := \rho_\delta * (\varphi u) \in C_c^\infty(\Omega), \quad \|\varphi u - u_\delta\|_{L^p(\Omega)} < \frac{\varepsilon}{2}.$$

On the other hand, since φu is uniformly continuous on Ω_{4k} , we can again choose $\delta \in (0, 1/4k)$ so small that $\|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \varepsilon/2$. Moreover, since $\text{supp}(\varphi u) \subset \Omega_{2k}$ and $\delta \in (0, 1/4k)$, we have $\varphi u = u_\delta = 0$ outside of Ω_{4k} and thus

$$(3.26) \quad \|\varphi u - u_\delta\|_{L^\infty(\Omega)} = \|\varphi u - u_\delta\|_{L^\infty(\Omega_{4k})} < \frac{\varepsilon}{2}.$$

Combining (3.24), (3.25) and (3.26), we obtain $u_\delta \in C_c^\infty(\Omega)$ and

$$\|u - u_\delta\|_{X_p} = \max\{\|u - u_\delta\|_{L^p(\Omega)}, \|u - u_\delta\|_{L^\infty(\Omega)}\} < \varepsilon.$$

Hence the lemma follows. $\square$

Let $Y_p := L_\sigma^p(\Omega) \cap VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ for $p \in (1, \infty)$, $\nu \in (0, \infty]$. Since $L_\sigma^p(\Omega)$ and $VMO_{b,0,\sigma}^{\infty,\nu}(\Omega)$ are closed in $L^p(\Omega)$ and $BMO_b^{\infty,\nu}(\Omega)$, respectively, Y_p becomes a Banach space under the norm $\|v\|_{Y_p} := \max\{\|v\|_{L^p(\Omega)}, \|v\|_{BMO_b^{\infty,\nu}(\Omega)}\}$.

Theorem 3.20. *Let $p \in (1, \infty)$ and $\nu \in (0, \infty]$. The linear operator Q' given in Definition 3.11 extends uniquely to a bounded linear operator Q_p from X_p into Y_p . Moreover, there exists a constant $c > 0$ such that*

$$(3.27) \quad \|Q_p u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q_p u\|_{BMO_b^{\infty,\nu}(\Omega)} \leq c\|u\|_{L^\infty(\Omega)}$$

for all $u \in X_p$ and $Q_p u = u$ holds for all u in the X_p -closure of $C_{c,\sigma}^\infty(\Omega)$.

Proof. Let $u \in C_c^\infty(\Omega)$. Then we have $Q'u \in Y_p$ by Lemma 3.12 and Lemma 3.15. Moreover, by Theorem 3.5 and Theorem 3.10, there is a constant $c > 0$ independent of u such that

$$(3.28) \quad \|Q'u\|_{L^p(\Omega)} \leq c\|u\|_{L^p(\Omega)}, \quad \|Q'u\|_{BMO_b^{\infty,\nu}(\Omega)} \leq c\|u\|_{L^\infty(\Omega)}.$$

Hence we have $Q'u \in Y_p$ and $\|Q'u\|_{Y_p} \leq c\|u\|_{X_p}$ for all $u \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in X_p by Lemma 3.19, the operator Q' extends uniquely to a bounded linear operator Q_p from X_p into Y_p . Also, it follows from (3.28) that the inequality (3.27) holds for all $u \in X_p$. Since $Q'u = u$ holds for all $u \in C_{c,\sigma}^\infty(\Omega)$ as observed after Definition 3.11, by the density argument we have $Q_p u = u$ for all u in the X_p -closure of $C_{c,\sigma}^\infty(\Omega)$. The proof is complete. $\square$

Finally, Theorem 1.4 follows from Theorem 3.20 with $p = 2$, that is, the linear operator Q in Theorem 1.4 is given by $Q = Q_2$.

4. ANALYTICITY IN L^p

In this section we shall give a complete proof of Theorem 1.1.

Proof of Theorem 1.1. Let $S(t)$ be the Stokes semigroup in $\tilde{L}_\sigma^p$ constructed by [14], [16]. To show that $S(t)$ forms an analytic semigroup in L_σ^p ($2 \leq p < \infty$) it suffices to prove that there exists a constant C that

$$(4.1) \quad \|S(t)v_0\|_p \leq C\|v_0\|_p$$

$$(4.2) \quad \left\| t \frac{d}{dt} S(t) v_0 \right\|_p \leq C \|v_0\|_p$$

for all $v_0 \in C_{c,\sigma}^\infty(\Omega)$ and for all $t \in (0, 1)$. Let Q be the operator in Theorem 1.4. Since Q is bounded in L^2 and maps L^2 to L_σ^2 and $S(t)$ fulfills (4.1) and (4.2) for $p = 2$, we have

$$(4.3) \quad \|S(t)Qu\|_2 \leq C\|u\|_2$$

$$(4.4) \quad \left\| t \frac{d}{dt} S(t)Qu \right\|_2 \leq C\|u\|_2$$

for all $u \in C_c(\Omega)$ and $t \in (0, 1)$. Since Ω is admissible as proved in [5], $S(t)$ forms an analytic semigroup in $VMO_{b,0,\sigma}^{\infty,\nu}$ by Theorem 1.2. We conclude that

$$(4.5) \quad \|S(t)Qu : BMO_b^{\infty,\nu}(\Omega)\| \leq C\|u\|_\infty$$

$$(4.6) \quad \left\| t \frac{d}{dt} S(t)Qu : BMO_b^{\infty,\nu}(\Omega) \right\| \leq C\|u\|_\infty$$

for all $u \in C_c(\Omega)$ and $t \in (0, 1)$ since Q fulfills

$$\|Qu : BMO_b^{\infty,\nu}(\Omega)\| \leq C\|u\|_\infty, \quad Qu \in VMO_{b,0,\sigma}^{\infty,\nu}$$

for all $u \in C_c(\Omega)$ by Theorem 1.4. (Note that we have a stronger statement than (4.6) by replacing the BMO_b type norm by the L^∞ norm since we have the regularizing estimate (1.3).) We apply an interpolation result (Theorem 1.3) to (4.3) and (4.5) and to (4.4) and (4.6) to get, respectively

$$(4.7) \quad \|S(t)Qu\|_p \leq C\|u\|_p$$

$$(4.8) \quad \left\| t \frac{d}{dt} S(t)Qu \right\|_p \leq C\|u\|_p$$

for all $u \in C_c(\Omega)$ and for all $t \in (0, 1)$. Since $Qu = u$ for $u \in C_{c,\sigma}^\infty(\Omega)$ this yields (4.1) and (4.2).

It remains to prove that $S(t)$ is a C_0 -semigroup in L_σ^p . Since $C_{c,\sigma}^\infty(\Omega)$ is dense in L_σ^p , for $v_0 \in L_\sigma^p$ there is $v_{0m} \in C_{c,\sigma}^\infty$ such that $\|v_0 - v_{0m}\|_p \rightarrow 0$ as $m \rightarrow \infty$. By (4.1) we observe that

$$\begin{aligned} \|S(t)v_0 - v_0\|_p &\leq \|S(t)(v_0 - v_{0m})\|_p + \|S(t)v_{0m} - v_{0m}\|_p + \|v_{0m} - v_0\|_p \\ &\leq C\|v_0 - v_{0m}\|_p + \|S(t)v_{0m} - v_{0m}\|_p. \end{aligned}$$

Sending $t \downarrow 0$, we get

$$\overline{\lim}_{t \downarrow 0} \|S(t)v_0 - v_0\|_p \leq C\|v_0 - v_{0m}\|_p,$$

since $S(t)v_{0m} \rightarrow v_{0m}$ in $\tilde{L}_\sigma^p$ as $t \downarrow 0$ by [14], [16]. Sending $m \rightarrow \infty$, we conclude that $S(t)v_0 \rightarrow v_0$ in L_σ^p as $t \downarrow 0$. $\square$

Remark 4.1. In a similar way as we derived (4.5) and (4.6) we are able to derive from the L^∞ - BMO estimates in [10] that

$$t \|\nabla^2 S(t)Qu : BMO_b^{\infty,\nu}(\Omega)\| \leq C\|u\|_\infty$$

$$t^{1/2} \|\nabla S(t)Qu : BMO_b^{\infty,\nu}(\Omega)\| \leq C\|u\|_\infty$$

for all $u \in C_c(\Omega)$ and $t \in (0, 1)$.

Note that L^2 results

$$t \|\nabla^2 S(t)Qu\|_2 \leq C\|u\|_2$$

$$t^{1/2} \|\nabla S(t)Qu\|_2 \leq C\|u\|_2$$

easily follow from the analyticity of $S(t)$ in L^2_σ and L^2 -boundedness of Q if one observes that $\|\nabla u\|_2^2 = (Au, u)_{L^2}$ and

$$\|\nabla^2 u\|_2 \leq C (\|Au\|_2 + \|\nabla u\|_2 + \|u\|_2)$$

(see e.g. [34, Chapter III, Theorem 2.1.1 (d)]), where A is the Stokes operator in L^2_σ .

Interpolating the L^2 results and the above L^∞ -BMO results, we are able to prove that there is $C_p > 0$ satisfying

$$t \|\nabla^2 S(t)v_0\|_p \leq C_p \|v_0\|_p$$

$$t^{1/2} \|\nabla S(t)v_0\|_p \leq C_p \|v_0\|_p$$

for all $v_0 \in L^p_\sigma(\Omega)$ and $t \in (0, 1)$ with $p \in (2, \infty)$.

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