HOKKAIDO UNIVERSITY

Title	Existence conditions of framed curves for smooth curves
Author(s)	Fukunaga, Tomonori; Takahashi, Masatomo
Citation	Hokkaido University Preprint Series in Mathematics, 1088, 1-10
Issue Date	2016-05-12
DOI	https://doi.org/10.14943/84232
Doc URL	https://hdl.handle.net/2115/69892
Type	departmental bulletin paper
File Information	re1088.pdf



Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

Existence conditions of framed curves for smooth curves

Tomonori Fukunaga and Masatomo Takahashi

May 9, 2016

Abstract

A framed curve is a smooth curve in the Euclidean space with a moving frame. We call the smooth curve in the Euclidean space the framed base curve. In this paper, we give an existence condition of framed curves. Actually, we construct a framed curve such that the image of the framed base curve coincides with the image of a given smooth curve under a condition. As a consequence, polygons in the Euclidean plane can be realised as not only a smooth curve but also a framed base curve.

1 Introduction

A framed curve in the Euclidean space is a curve with a moving frame. It is a generalisation of not only regular curves with the linear independent condition (cf. [7]), but also regular curves with Bishop frame (cf. [2]). Moreover, framed curves may have singular points. It is also a generalisation of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$ (cf. [1, 4]).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i$, where $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$. For $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1} \in \mathbb{R}^n$, we define the vector product,

$$\boldsymbol{a}_1 \times \cdots \times \boldsymbol{a}_{n-1} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n-11} & \cdots & a_{n-1n} \\ e_1 & \cdots & e_n \end{vmatrix} = \sum_{i=1}^{n} \det(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}, e_i) e_i,$$

where $\boldsymbol{a}_i = (a_{i1}, \ldots, a_{in})$ for $i = 1, \ldots, n-1$ and $e_1, \ldots, e_n$ are the canonical basis on $\mathbb{R}^n$. Then we have $(\boldsymbol{a}_1 \times \cdots \times \boldsymbol{a}_{n-1}) \cdot \boldsymbol{a}_i = 0$ for $i = 1, \ldots, n-1$. We denote the set Δ_{n-1},

$$\begin{aligned} \Delta_{n-1} &= \{\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \mid \nu_i \cdot \nu_j = \delta_{ij}, i, j = 1, \ldots, n-1\} \\ &= \{\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{n-1}) \in S^{n-1} \times \cdots \times S^{n-1} \mid \nu_i \cdot \nu_j = 0, i \neq j, i, j = 1, \ldots, n-1\}. \end{aligned}$$

Then Δ_{n-1} is an $n(n-1)/2$-dimensional smooth manifold. If $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{n-1}) \in \Delta_{n-1}$, we define the unit vector $\boldsymbol{\mu} = \nu_1 \times \cdots \times \nu_{n-1}$ of $\mathbb{R}^n$. It follows that the pair $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in \Delta_n$. By definition, we have $\det(\nu_1, \ldots, \nu_{n-1}, \boldsymbol{\mu}) = 1$. Note that $\Delta_2 = S^1$.

Let I be an interval or $\mathbb{R}$.

2010 Mathematics Subject classification: 58K05, 53A04, 57R45

Key Words and Phrases. framed curve, framed base curve,smooth curve, polygon

Definition 1.1 We say that a smooth map $(\gamma, \boldsymbol{\nu}) : I \to \mathbb{R}^n \times \Delta_{n-1}$ is a *framed curve* if $\dot{\gamma}(t) \cdot \nu_i(t) = 0$ for all $t \in I$ and $i = 1, \ldots, n-1$. We also say that a smooth map $\gamma : I \to \mathbb{R}^n$ is a *framed base curve* if there exists a smooth map $\boldsymbol{\nu} : I \to \Delta_{n-1}$ such that $(\gamma, \boldsymbol{\nu})$ is a framed curve.

For a framed curve $(\gamma, \boldsymbol{\nu}) : I \to \mathbb{R}^n \times \Delta_{n-1}$, the framed base curve γ may have singular points. We denote the set of singular points of γ by $\Sigma(\gamma)$, that is, we set $\Sigma(\gamma) = \{t \in I \mid \dot{\gamma}(t) = \boldsymbol{0}\}$. The framed curves can be characterised by the moving frame $\{\boldsymbol{\nu}(t), \boldsymbol{\mu}(t)\}$ of the framed base curve $\gamma(t)$ and the curvature of the framed curve, in detail see [6].

In the case of $n = 2$, the framed curve is nothing but a Legendre curve with respect to the canonical contact structure on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$. We have shown that analytic curves are at least locally framed base curves in the cases of plane curves ($n = 2$) and space curves ($n = 3$), see [4] and [6], respectively.

For a function f, we denote $f(a-0)$ (respectively, $f(a+0)$) as one sided limit $\lim_{t \to a-0} f(t)$ (respectively, $\lim_{t \to a+0} f(t)$). We denote $\boldsymbol{t}(t)$ as the unit tangent vector of $\gamma(t)$ at regular points, that is, $\boldsymbol{t}(t) = \dot{\gamma}(t)/\|\dot{\gamma}(t)\|$ if $\dot{\gamma}(t) \neq \boldsymbol{0}$.

The main result in this paper is as follows. We give an existence condition of a framed curve such that the image of the framed base curve coincides with the image of a given smooth curve.

Theorem 1.2 *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a C^∞-curve. Suppose that the singular set $\Sigma(\gamma)$ is finite, and the limit of the derivatives of the tangent vectors $\boldsymbol{t}^{(k)}(s-0)$ and $\boldsymbol{t}^{(k)}(s+0)$ exist for all $s \in \Sigma(\gamma)$ and $k \in \mathbb{N} \cup \{0\}$. Then there exists a framed curve $(\widetilde{\gamma}, \widetilde{\boldsymbol{\nu}}) : [0, 1] \to \mathbb{R}^n \times \Delta_{n-1}$ such that $\widetilde{\gamma}([0, 1]) = \gamma([a, b])$.*

In section 2, we give a proof of the main result by using flat functions. In section 3, we give examples of a polygon and a 3/2-cusp singularity. We also give an example that the smooth curve does not admit as a framed curve.

All maps and manifolds considered here are differential of class C^∞ unless the contrary is explicitly stated.

Acknowledgement. The first author was partially supported by JSPS KAKENHI Grant Number 15K17457 and the second author was partially supported by JSPS KAKENHI Grant Number 26400078.

2 Proof of the main result

We introduce notations as preparations. Let $\varphi : [0, 1] \to \mathbb{R}$ be a non-analytic smooth function defined by

$$\varphi(t) = \begin{cases} e^{-1/t} & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0. \end{cases}$$

We also define a smooth function $\psi : [0, 1] \to \mathbb{R}$ by

$$\psi(t) = \frac{\varphi(t)}{\varphi(t) + \varphi(1-t)}.$$

The function ψ provides a smooth transition from 0 to 1 on the interval $[0, 1]$ and $\psi^{(n)}(0+0) = \psi^{(n)}(1-0) = 0$ for all $n \in \mathbb{N}$. Moreover, we define a smooth function $\psi_{a,b} : [0, 1] \to \mathbb{R}$ by $\psi_{a,b}(t) = \psi(t)b + (1 - \psi(t))a$, where $a, b \in \mathbb{R}$ with $a < b$. Note that $\psi_{0,1} = \psi$.

Lemma 2.1 *The function $\psi_{a,b} : [0,1] \to \mathbb{R}$ provides a smooth transition from a to b in the interval $[0,1]$.*

Proof. By definition, $\psi_{a,b}(0) = \psi(0)b + (1-\psi(0))a = a$ and $\psi_{a,b}(1) = \psi(1)b + (1-\psi(1))a = b$. Moreover, we have $\dot{\psi}_{a,b}(t) > 0$ for $0 < t < 1$. Since $\psi_{a,b}^{(n)}(t) = \psi^{(n)}(t)(b-a)$ for all $n \in \mathbb{N}$, we have $\psi_{a,b}^{(n)}(0+0) = \psi^{(n)}(0+0)(b-a) = 0$ and $\psi_{a,b}^{(n)}(1-0) = \psi^{(n)}(1-0)(b-a) = 0$ for all $n \in \mathbb{N}$. □

Let X be a topological space. For two maps on the unit interval $f_1 : [0,1] \to X$ and $f_2 : [0,1] \to X$ with $f_1(1) = f_2(0)$, we define a concatenation map $f_2 * f_1 : [0,1] \to X$ by

$$(f_2 * f_1)(t) = \begin{cases} f_1(2t) & \text{if } 0 \leq t \leq 1/2, \\ f_2(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Note that the operator $*$ is not associative. The concatenation map of two continuous maps turns out a continuous map again (see [8], for example). On the other hand, in general, the concatenation map of two C^∞-maps does not turn out a C^∞-map. However, we can concatenate two C^∞-maps smoothly by using the smooth transition function.

Lemma 2.2 *Let M be a smooth manifold. Assume $f_1 : [0,1] \to M$ and $f_2 : [0,1] \to M$ are C^∞-maps with $f_1(1) = f_2(0)$. Then the concatenation map $(f_2 \circ \psi) * (f_1 \circ \psi) : [0,1] \to M$ is a C^∞-map.*

Proof. Since the map $(f_2 \circ \psi) * (f_1 \circ \psi)$ is C^∞ on $t \neq 1/2$, it is sufficient to show that $\{(f_2 \circ \psi) * (f_1 \circ \psi)\}^{(n)}(1/2 - 0) = \{(f_2 \circ \psi) * (f_1 \circ \psi)\}^{(n)}(1/2 + 0)$ for all $n \in \mathbb{N}$. By definition of the concatenation map, we have

$$\{(f_2 \circ \psi) * (f_1 \circ \psi)\}^{(n)} \left(\frac{1}{2} - 0\right) = (f_1 \circ \psi)^{(n)}(1-0)$$

and

$$\{(f_2 \circ \psi) * (f_1 \circ \psi)\}^{(n)} \left(\frac{1}{2} + 0\right) = (f_2 \circ \psi)^{(n)}(0+0).$$

By the chain rule, we can write each component of $(f_1 \circ \psi)^{(n)}$ (respectively, $(f_2 \circ \psi)^{(n)}$) as a sum of products of each component of $f_1^{(k)}$ (respectively, $f_2^{(k)}$) and $\psi^{(k)}$ for $k \in \{1, \cdots, n\}$. By Lemma 2.1, $\psi^{(k)}(1-0) = 0$ and $\psi^{(k)}(0+0) = 0$ for $k = 1, \cdots, n$. Hence we have $(f_1 \circ \psi)^{(n)}(1-0) = \mathbf{0}$ and $(f_2 \circ \psi)^{(n)}(0+0) = \mathbf{0}$. Therefore, the map $(f_2 \circ \psi) * (f_1 \circ \psi) : [0,1] \to M$ is a C^∞-map. □

Remark 2.3 By Lemma 2.2, piece-wise C^∞-curves can be realised as a C^∞-curve such that the same image. Especially, polygons in the Euclidean plane may be considered as the image of a C^∞-curve.

Proof of the Theorem 1.2. Let $\{s_0, \cdots, s_n\}$ be the set of singular points except a and b.

First step: We define a smooth map $\widetilde{\gamma}_{a,s_0} : [0,1] \to \mathbb{R}^n$ by $\widetilde{\gamma}_{a,s_0}(t) = \gamma(\psi_{a,s_0}(t))$. We show this map has the following properties:

(i) $\widetilde{\gamma}_{a,s_0}(0) = \gamma(a)$ and $\widetilde{\gamma}_{a,s_0}(1) = \gamma(s_0)$,

(ii) $\widetilde{\gamma}_{a,s_0}^{(n)}(0+0) = \widetilde{\gamma}_{a,s_0}^{(n)}(1-0) = \mathbf{0}$ for all $n \in \mathbb{N}$,

(iii) $\widetilde{\gamma}_{a,s_0}([0,1]) = \gamma([a,s_0])$.

By Lemma 2.1, we obtain $\widetilde{\gamma}_{a,s_0}(0) = \gamma(\psi_{a,s_0}(0)) = \gamma(a)$ and $\widetilde{\gamma}_{a,s_0}(1) = \gamma(\psi_{a,s_0}(1)) = \gamma(s_0)$. By the chain rule, we can calculate $\widetilde{\gamma}^{(n)}_{a,s_0}$ as a sum of products of $\gamma^{(k)}$ and $\psi^{(k)}_{a,s_0}$ for $k \in \{1, \cdots, n\}$. By Lemma 2.1, we have $\widetilde{\gamma}^{(n)}_{a,s_0}(0+0) = \widetilde{\gamma}^{(n)}_{a,s_0}(1-0) = \mathbf{0}$ for all $n \in \mathbb{N}$. Since ψ_{a,s_0} is a bijection from $[0,1]$ to $[a,s_0]$, we have $\widetilde{\gamma}_{a,s_0}([0,1]) = \gamma([a,s_0])$. Therefore, (i), (ii) and (iii) hold.

Second step: We construct a map $\widetilde{\boldsymbol{\nu}}_{a,s_0} : [0,1] \to \Delta_{n-1}$ such that $(\widetilde{\gamma}_{a,s_0}, \widetilde{\boldsymbol{\nu}}_{a,s_0}) : [0,1] \to \mathbb{R}^n \times \Delta_{n-1}$ is a framed curve. By the assumption, we have $\boldsymbol{t}(a+0)$. Consider an orthonormal $n-1$ frame $\boldsymbol{\nu}_- = (\nu_{-,1}, \cdots, \nu_{-,n-1})$ with $({}^T\boldsymbol{t}(a+0), {}^T\boldsymbol{\nu}_-) \in SO(n)$, where ${}^T\mathbf{a}$ is the transpose of a vector $\mathbf{a}$ and $SO(n)$ is the $n \times n$ special orthogonal group. Since $\boldsymbol{t}$ is the smooth unit tangent vector field along γ on $[a,s_0)$, there exists a smooth map $A \in C^\infty([a,s_0), SO(n))$ such that $\boldsymbol{t}(t) = \boldsymbol{t}(a+0)A(t)$. By the assumption, the one side derivatives $\boldsymbol{t}^{(k)}(s_0 - 0)$ exists for all $k \in \mathbb{N} \cup \{0\}$. We can extend A to $t = s_0$, that is, $A \in C^\infty([a,s_0], SO(n))$. Now we define $\boldsymbol{\nu}_{a,s_0} : [a,s_0] \to \Delta_{n-1}$ by $\nu_i(t) = \nu_{-,i}A(t)$ for each component $i = 1, \cdots, n-1$. Then $\widetilde{\boldsymbol{\nu}}_{a,s_0} : [0,1] \to \Delta_{n-1}$ defined by $\widetilde{\boldsymbol{\nu}}_{a,s_0}(t) = \boldsymbol{\nu}_{a,s_0}(\psi_{a,s_0}(t))$ is the required map. In fact, we have $(d/dt)\widetilde{\gamma}_{a,s_0}(t) \in \langle \widetilde{\boldsymbol{\nu}}_{a,s_0}(t) \rangle^\perp$, since

$$\begin{aligned}
\frac{d}{dt}\widetilde{\gamma}_{a,s_0}(t) \cdot \widetilde{\nu}_{a,s_0;i}(t) &= \dot{\gamma}(\psi_{a,s_0}(t))\dot{\psi}_{a,s_0}(t) \cdot \nu_{a,s_0;i}(\psi_{a,s_0}(t)) \\
&= \|\dot{\gamma}(\psi_{a,s_0}(t))\|\boldsymbol{t}(\psi_{a,s_0}(t))\dot{\psi}_{a,s_0}(t) \cdot \nu_{a,s_0;i}(\psi_{a,s_0}(t)) \\
&= \|\dot{\gamma}(\psi_{a,s_0}(t))\|\dot{\psi}_{a,s_0}(t)\boldsymbol{t}(\psi_{a,s_0}(0+0))A(\psi_{a,s_0}(t)) \cdot \nu_{-,i}A(\psi_{a,s_0}(t)) \\
&= \|\dot{\gamma}(\psi_{a,s_0}(t))\|\dot{\psi}_{a,s_0}(t)\boldsymbol{t}(\psi_{a,s_0}(0+0)) \cdot \nu_{-,i} \\
&= 0
\end{aligned}$$

for all $i = 1, \cdots, n-1$, where $\widetilde{\boldsymbol{\nu}}_{a,s_0} = (\widetilde{\nu}_{a,s_0;1}, \cdots, \widetilde{\nu}_{a,s_0;n-1})$, $\boldsymbol{\nu}_{a,s_0} = (\nu_{a,s_0;1}, \cdots, \nu_{a,s_0;n-1})$ and $\langle \widetilde{\boldsymbol{\nu}}_{a,s_0}(t) \rangle^\perp$ is the orthogonal complement of the linear space spanned by $\widetilde{\boldsymbol{\nu}}_{a,s_0}(t)$.

Third step: We define $\widetilde{\gamma}_{s_0} : [0,1] \to \mathbb{R}^n$ by a constant map $\widetilde{\gamma}_{s_0}(t) = \gamma(s_0)$ for all $t \in [0,1]$.

Fourth step: Let $\boldsymbol{\nu}_+$ be an element of Δ_{n-1} with $({}^T\boldsymbol{t}(s_0+0), {}^T\boldsymbol{\nu}_+) \in SO(n)$. We denote $({}^T\boldsymbol{t}(s_0+0), {}^T\boldsymbol{\nu}_+)$ by S_+, and $({}^T\boldsymbol{t}(s_0-0), {}^T\widetilde{\boldsymbol{\nu}}_{a,s_0}(1))$ by S_-. Note that $S_- \in SO(n)$ by the definition of $\widetilde{\boldsymbol{\nu}}_{a,s_0}$ in the second step.

We construct a map $\widetilde{\boldsymbol{\nu}}_{s_0} : [0,1] \to \Delta_{n-1}$ which connects ${}^T\widetilde{\boldsymbol{\nu}}_{a,s_0}(1)$ and ${}^T\boldsymbol{\nu}_+$. By the linear algebra, there is a C^∞-map $P_1 : [0,1] \to SO(n)$, which connects S_- and I_n, where I_n is the unit element of $SO(n)$ (see [5] for example). Further, there is a C^∞-map $P_2 : [0,1] \to SO(n)$, which connects I_n and S_+. We define $\widetilde{P}_i : [0,1] \to SO(n)$ by $\widetilde{P}_i(t) = P_i(\psi(t))$ for $i = 1, 2$. Then we obtain the required map $\widetilde{\boldsymbol{\nu}}_{s_0} : [0,1] \to \Delta_{n-1}$ by $\widetilde{\boldsymbol{\nu}}_{s_0}(t) = ({}^T(\widetilde{P}_2 * \widetilde{P}_1)_2(t), \cdots, {}^T(\widetilde{P}_2 * \widetilde{P}_1)_n(t))$, where $(\widetilde{P}_2 * \widetilde{P}_1)_k$ is the k-th column of the matrix $(\widetilde{P}_2 * \widetilde{P}_1)$. By Lemma 2.2, the map $\widetilde{\boldsymbol{\nu}}_{s_0}$is a C^∞-map. Since $\widetilde{\gamma}_{s_0}$ is a constant map, $(\widetilde{\gamma}_{s_0}, \widetilde{\boldsymbol{\nu}}_{s_0}) : [0,1] \to \mathbb{R}^n \times \Delta_{n-1}$ is also a framed curve.

Fifth step: Similar to the first step to the fourth step, we construct $\widetilde{\gamma}_{s_i,s_{i+1}}$, $\widetilde{\boldsymbol{\nu}}_{s_i,s_{i+1}}$, $\widetilde{\gamma}_{s_{i+1}}$, $\widetilde{\boldsymbol{\nu}}_{s_{i+1}}$, $\widetilde{\gamma}_{s_n,b}$ and $\widetilde{\boldsymbol{\nu}}_{s_n,b}$ for all $i = 1, \cdots, n-1$. Note that we can take $\widetilde{\boldsymbol{\nu}}_{s_i,s_{i+1}}$ (respectively, $\widetilde{\boldsymbol{\nu}}_{s_{i+1},b}$) such that $\widetilde{\boldsymbol{\nu}}_{s_i,s_{i+1}}(0) = \widetilde{\boldsymbol{\nu}}_{s_i}(1)$ for all $i = 1, \ldots, n-1$ (respectively, $\widetilde{\boldsymbol{\nu}}_{s_n,b}(0) = \widetilde{\boldsymbol{\nu}}_{s_n}(1)$).

Sixth step: We concatenate on the all maps, that is, we define a C^∞-map $(\widetilde{\gamma}, \widetilde{\boldsymbol{\nu}}) : [0,1] \to \mathbb{R}^n \times \Delta_{n-1}$ by

$$\widetilde{\gamma}(t) = (\widetilde{\gamma}_{s_n,b} * (\widetilde{\gamma}_{s_n} * (\cdots * (\widetilde{\gamma}_{s_0} * \widetilde{\gamma}_{a,s_0}))))(t),\ \widetilde{\boldsymbol{\nu}}(t) = (\widetilde{\boldsymbol{\nu}}_{s_n,b} * (\widetilde{\boldsymbol{\nu}}_{s_n} * (\cdots * (\widetilde{\boldsymbol{\nu}}_{s_0} * \widetilde{\boldsymbol{\nu}}_{a,s_0}))))(t).$$

By the construction, we have $\langle \dot{\widetilde{\gamma}}(t) \rangle \subset \langle \widetilde{\boldsymbol{\nu}}(t) \rangle^\perp$ for all $t \in [0,1]$. It follows that the map $(\widetilde{\gamma}, \widetilde{\boldsymbol{\nu}}) : [0,1] \to \mathbb{R}^n \times \Delta_{n-1}$ is a framed curve such that $\widetilde{\gamma}([0,1]) = \gamma([a,b])$. □

Remark 2.4 By the above construction, the boundaries 0 and 1 in the unit interval $[0,1]$ are singular points of $\widetilde{\gamma}$ in spite of a and b may be regular points of γ. On the other hand, if we use $\varphi_{s_0,a}(1-t)$ (respectively, $\varphi_{s_n,b}(t)$) instead of $\psi_{a,s_0}(t)$ (respectively $\psi_{s_n,b}(t)$), where $\varphi_{a,b}:[0,1]\to[a,b]$ is defined by $\varphi_{a,b}(t)=(e\varphi(t))b+\{1-(e\varphi(t))\}a$, then 0 (respectively, 1) is a regular point of $\widetilde{\gamma}$ if and only if a (respectively, b) is a regular point of γ.

The assumption that the limit of the derivatives of the tangent vectors $\boldsymbol{t}^{(k)}(s-0)$ and $\boldsymbol{t}^{(k)}(s+0)$ exist for all $s\in\Sigma(\gamma)$ and $k\in\mathbb{N}\cup\{0\}$ is essential. We can construct a C^∞-curve which is not the image of the framed base curves, see Example 3.4.

In the case of the domain of γ is an open interval or $\mathbb{R}$, we also have the following result.

Corollary 2.5 (1) *Let $\gamma:(a,b)\to\mathbb{R}^n$ be a C^∞-curve. Suppose that the singular set $\Sigma(\gamma)$ is finite, and the limit of the derivatives of the tangent vectors $\boldsymbol{t}^{(k)}(s-0)$ and $\boldsymbol{t}^{(k)}(s+0)$ exist for all $s\in\Sigma(\gamma)$ and $k\in\mathbb{N}\cup\{0\}$. Then there exists a framed curve $(\widetilde{\gamma},\widetilde{\boldsymbol{\nu}}):(0,1)\to\mathbb{R}^n\times\Delta_{n-1}$ such that $\widetilde{\gamma}((0,1))=\gamma((a,b))$.*

(2) *Let $\gamma:\mathbb{R}\to\mathbb{R}^n$ be a C^∞-curve. Suppose that the singular set $\Sigma(\gamma)$ is finite, and the limit of the derivatives of the tangent vectors $\boldsymbol{t}^{(k)}(s-0)$ and $\boldsymbol{t}^{(k)}(s+0)$ exist for all $s\in\Sigma(\gamma)$ and $k\in\mathbb{N}\cup\{0\}$. Then there exists a framed curve $(\widetilde{\gamma},\widetilde{\boldsymbol{\nu}}):\mathbb{R}\to\mathbb{R}^n\times\Delta_{n-1}$ such that $\widetilde{\gamma}(\mathbb{R})=\gamma(\mathbb{R})$.*

Proof. (1) By a similar construction in the proof of Theorem 1.2, we have the result.

(2) Parameter changes preserve the conditions of the framed curves. By using (1) and a diffeomorphism between $\mathbb{R}$ and an open interval, we have the result. □

3 Examples

We give concrete examples of the construction of framed curves in the proof of Theorem 1.2. Furthermore, we give an example of a C^∞-curve which is not the image of the framed base curves.

Example 3.1 Let $\gamma:(-1,1)\to\mathbb{R}^2$ be a C^∞-curve given by

$$\gamma(t)=\begin{cases}(e^{-\frac{1}{t^2}},0) & \text{if } -1<t<0,\\ (0,0) & \text{if } t=0,\\ (0,e^{-\frac{1}{t^2}}) & \text{if } 0<t<1.\end{cases}$$

Note that this curve is not a frontal (see [4, 6]). However, we can construct a framed curve $(\widetilde{\gamma},\widetilde{\nu}):(0,1)\to\mathbb{R}^2\times S^1$ such that $\widetilde{\gamma}((0,1))=\gamma((-1,1))$ by using the method in the proof of Theorem 1.2, since the singular set $\Sigma(\gamma)=\{0\}$ and the limit of the derivatives of the tangent vectors exists at the origin.

First, we define $\widetilde{\gamma}_{-1,0}:(0,1]\to\mathbb{R}^2$ by

$$\widetilde{\gamma}_{-1,0}(t)=\gamma(\psi_{-1,0}(t))=\begin{cases}\left(\exp\left(-\frac{1}{\psi_{-1,0}(t)^2}\right),0\right) & \text{if } 0<t<1,\\ (0,0) & \text{if } t=1.\end{cases}$$

Second, we define $\widetilde{\nu}_{-1,0}:(0,1]\to S^1$ as follows. By a direct calculation, we have $\boldsymbol{t}(-1+0)=(-1,0)$ and $\nu_-=(0,-1)$. The unit tangent vector is given by $\boldsymbol{t}(t)=(-1,0)$ for all $t\in(-1,0]$.

Hence, we have $\boldsymbol{t}(t) = \boldsymbol{t}(-1+0)I_2$, for all $t \in (-1, 0]$, where I_2 is the 2×2 unit matrix. Then we have the constant map $\nu_{-1,0} : (-1, 0] \to S^1, \nu_{-1,0}(t) = \nu_- I_2 = \nu_-$. Now we define $\widetilde{\nu}_{-1,0} : (0, 1] \to S^1$ by $\widetilde{\nu}_{-1,0}(t) = \nu_{-1,0}(\psi_{-1,0}(t)) = (0, -1)$.

Third, we define a map $\widetilde{\gamma}_0 : [0, 1] \to \mathbb{R}^2$ by $\widetilde{\gamma}_0(t) = \gamma(0) = (0, 0)$ for all $t \in [0, 1]$.

Fourth, we define a map $\widetilde{\nu}_0 : [0, 1] \to S^1$ as follows. By a direct calculation, we have $\boldsymbol{t}(0+0) = (0, 1)$, $\nu_+ = (-1, 0)$, $\boldsymbol{t}(0-0) = (-1, 0)$ and $\widetilde{\nu}_{-1,0}(1) = (0, -1)$. Hence,

$$S_+ = ({}^T\boldsymbol{t}(0+0), {}^T\nu_+) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}$$

and

$$S_- = ({}^T\boldsymbol{t}(0-0), {}^T\widetilde{\nu}_{-1,0}(1)) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}.$$

We define maps P_1 (respectively, P_2) from S_- to I_2 (respectively, from I_2 to S_+) by

$$P_1(t) = \begin{pmatrix} \cos(1-t)\pi & -\sin(1-t)\pi \\ \sin(1-t)\pi & \cos(1-t)\pi \end{pmatrix}, P_2(t) = \begin{pmatrix} \cos\frac{t\pi}{2} & -\sin\frac{t\pi}{2} \\ \sin\frac{t\pi}{2} & \cos\frac{t\pi}{2} \end{pmatrix}.$$

Then we have $\widetilde{P_i}(t) = P_i(\psi(t))$, that is,

$$\widetilde{P_1}(t) = \begin{pmatrix} \cos(1-\psi(t))\pi & -\sin(1-\psi(t))\pi \\ \sin(1-\psi(t))\pi & \cos(1-\psi(t))\pi \end{pmatrix}, \widetilde{P_2}(t) = \begin{pmatrix} \cos\frac{\psi(t)\pi}{2} & -\sin\frac{\psi(t)\pi}{2} \\ \sin\frac{\psi(t)\pi}{2} & \cos\frac{\psi(t)\pi}{2} \end{pmatrix}.$$

Now we define

$$\widetilde{\nu}_0(t) = {}^T(\widetilde{P_2} * \widetilde{P_1})_2(t) = \begin{cases} (-\sin(1-\psi(2t))\pi, \cos(1-\psi(2t))\pi) & \text{if } 0 \le t \le 1/2, \\ \left(-\sin\frac{\psi(2t-1)\pi}{2}, \cos\frac{\psi(2t-1)\pi}{2}\right) & \text{if } 1/2 \le t \le 1. \end{cases}$$

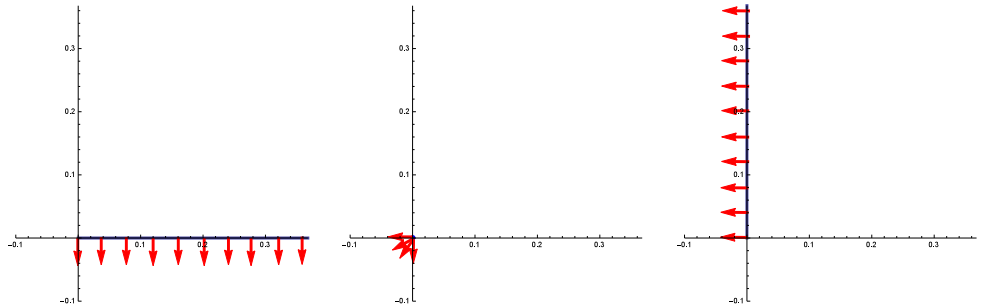
Fifth, we define $\widetilde{\gamma}_{0,1} : [0, 1) \to \mathbb{R}^2$ by

$$\widetilde{\gamma}_{0,1}(t) = \gamma(\psi(t)) = \begin{cases} \left(0, \exp\left(-\frac{1}{\psi(t)^2}\right)\right) & \text{if } 0 < t < 1, \\ (0, 0) & \text{if } t = 0. \end{cases}$$

Sixth, we define $\widetilde{\nu}_{0,1} : [0, 1) \to S^1$ as follows. By a direct calculation, we have $\boldsymbol{t}(0+0) = (0, 1)$ and $\nu_- = (-1, 0)$. The unit tangent vector is given by $\boldsymbol{t}(t) = (0, 1)$ for all $t \in [0, 1)$. Hence, we have $\boldsymbol{t}(t) = \boldsymbol{t}(0+0)I_2$, for all $t \in [0, 1)$. Then we have the constant map $\nu_{0,1} : [0, 1) \to S^1$, $\nu_{0,1}(t) = \nu_- I_2 = \nu_-$. Now we define $\widetilde{\nu}_{0,1} : [0, 1) \to S^1$ by $\widetilde{\nu}_{0,1}(t) = \nu_{0,1}(\psi(t)) = (-1, 0)$.

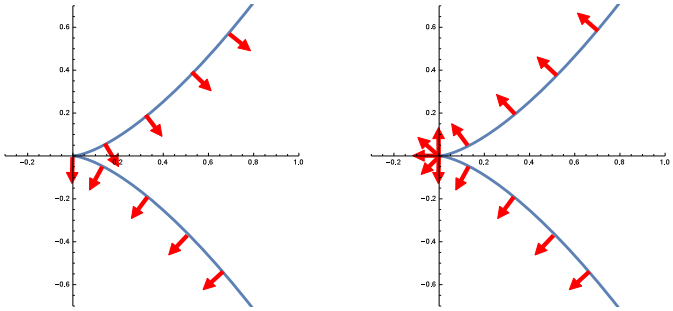
Finally, we concatenate on the all maps, that is, we define $\widetilde{\gamma} : (0, 1) \to \mathbb{R}^2$ and $\widetilde{\nu} : (0, 1) \to S^1$ by $\widetilde{\gamma}(t) = (\widetilde{\gamma}_{0,1} * (\widetilde{\gamma}_0 * \widetilde{\gamma}_{-1,0}))(t)$ and $\widetilde{\nu}(t) = (\widetilde{\nu}_{0,1} * (\widetilde{\nu}_0 * \widetilde{\nu}_{-1,0}))(t)$. Then we obtain a framed curve $(\widetilde{\gamma}, \widetilde{\nu}) : (0, 1) \to \mathbb{R}^2 \times S^1$ such that $\widetilde{\gamma}((0, 1)) = \gamma((-1, 1))$, see Figure 1.

Remark 3.2 Since piece-wise smooth curves can be realised as a C^∞-curve, see Remark 2.3, it is also realised as a framed base curve by Theorem 1.2 if the conditions satisfy. It follows that polygons in the Euclidean plane can be realised as the image of a framed base curve.



$(\widetilde{\gamma}_{-1,0}(t), \widetilde{\nu}_{-1,0}(t))$ $(\widetilde{\gamma}_{0}(t), \widetilde{\nu}_{0}(t))$ $(\widetilde{\gamma}_{0,1}(t), \widetilde{\nu}_{0,1}(t))$

Figure 1: Legendre curve $(\widetilde{\gamma}, \widetilde{\nu})$. Note that the length of the unit normal vectors is modified.



$(\gamma(t), \nu(t))$ $(\widetilde{\gamma}(t), \widetilde{\nu}(t))$

Figure 2: Images of the 3/2-cusp and unit normal vector fields. Note that the length of the unit normal vectors is modified.

Example 3.3 Let $\gamma : (-1, 1) \to \mathbb{R}^2$ be a 3/2-cusp $\gamma(t) = (t^2/2, t^3/3)$ (cf. [4]). As well known, the 3/2-cusp is a front. In fact, if we take $\nu(t) = (1/\sqrt{t^2+1})(-t, 1)$ (respectively, $-\nu$), then (γ, ν) (respectively, $(\gamma, -\nu)$) is a framed curve and (γ, ν) (respectively, $(\gamma, -\nu)$) is an immersion. Both cases, the unit normal vectors change inner (outer) to outer (inner) of the curve γ around the origin, see Figure 2 left. However, we can construct a framed curve $(\widetilde{\gamma}, \widetilde{\nu}) : (0, 1) \to \mathbb{R}^2 \times S^1$ such that $\widetilde{\gamma}((0, 1)) = \gamma((-1, 1))$ and the unit normal $\widetilde{\nu}$ does not change inner and outer of the curve γ, by using the method of the proof in Theorem 1.2, see Figure 2 right.

By definition of γ, the singular set $\Sigma(\gamma) = \{0\}$ and the limit of the derivatives of the tangent vectors exists at the origin.

First, we define $\widetilde{\gamma}_{-1,0} : (0, 1] \to \mathbb{R}^2$ by

$$\widetilde{\gamma}_{-1,0}(t) = \gamma(\psi_{-1,0}(t)) = \left(\frac{1}{2}\psi_{-1,0}(t)^2, \frac{1}{3}\psi_{-1,0}(t)^3\right).$$

Second, we define $\widetilde{\nu}_{-1,0} : (0, 1] \to S^1$ as follows. By a direct calculation, we have

$$\boldsymbol{t}(-1+0) = \lim_{t \to -1+0} \frac{1}{|t|\sqrt{t^2+1}}(t, t^2) = \frac{1}{\sqrt{2}}(-1, 1)$$

and $\nu_- = (1/\sqrt{2})(-1, -1)$. The unit tangent vector is given by $\boldsymbol{t}(t) = (-1/\sqrt{t^2+1})(1, t)$ for all

$t \in (-1, 0]$. Hence, we have $\boldsymbol{t}(t) = \boldsymbol{t}(-1+0)A(t)$, where

$$A(t) = \frac{-\sqrt{2}}{2\sqrt{t^2+1}} \begin{pmatrix} t-1 & -t-1 \\ t+1 & t-1 \end{pmatrix}$$

for all $t \in (-1, 0]$. Then we have a map $\nu_{-1,0} : (-1, 0] \to S^1$,

$$\nu_{-1,0}(t) = \nu_- A(t) = \frac{1}{\sqrt{2}}(-1,-1)\frac{-\sqrt{2}}{2\sqrt{t^2+1}} \begin{pmatrix} t-1 & -t-1 \\ t+1 & t-1 \end{pmatrix} = \frac{-1}{\sqrt{t^2+1}}(-t, 1).$$

Now we define $\widetilde{\nu}_{-1,0} : (0, 1] \to S^1$ by

$$\widetilde{\nu}_{-1,0}(t) = \nu_{-1,0}(\psi_{-1,0}(t)) = \frac{-1}{\sqrt{\psi_{-1,0}(t)^2+1}}(-\psi_{-1,0}(t), 1).$$

Third, we define a map $\widetilde{\gamma}_0 : [0, 1] \to \mathbb{R}^2$ by $\widetilde{\gamma}_0(t) = \gamma(0) = (0, 0)$ for all $t \in [0, 1]$.

Fourth, we define a map $\widetilde{\nu}_0 : [0, 1] \to S^1$ as follows. By a direct calculation, we have $\boldsymbol{t}(0+0) = (1, 0)$, $\nu_+ = (0, 1)$, $\boldsymbol{t}(0-0) = (-1, 0)$ and $\widetilde{\nu}_{-1,0}(1) = (0, -1)$. Hence,

$$S_+ = ({}^T\boldsymbol{t}(0+0), {}^T\nu_+) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}$$

and

$$S_- = ({}^T\boldsymbol{t}(0-0), {}^T\widetilde{\nu}_{-1,0}(1)) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}.$$

We define a map P_1 from S_- to I_2 by

$$P_1(t) = \begin{pmatrix} \cos(1-t)\pi & -\sin(1-t)\pi \\ \sin(1-t)\pi & \cos(1-t)\pi \end{pmatrix},$$

and we define a map P_2 from I_2 to S_+ by $P_2(t) = I_2$ for all $t \in [0, 1]$. Then we have $\widetilde{P}_i(t) = P_i(\psi(t))$, that is,

$$\widetilde{P_1}(t) = \begin{pmatrix} \cos(1-\psi(t))\pi & -\sin(1-\psi(t))\pi \\ \sin(1-\psi(t))\pi & \cos(1-\psi(t))\pi \end{pmatrix}, \ \widetilde{P_2}(t) = I_2$$

for all $t \in [0, 1]$. Now we define

$$\widetilde{\nu_0}(t) = {}^T(\widetilde{P_2} * \widetilde{P_1})_2(t) = \begin{cases} (-\sin(1-\psi(2t))\pi, \cos(1-\psi(2t))\pi) & \text{if } 0 \le t \le 1/2, \\ (0, 1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Fifth, we define $\widetilde{\gamma}_{0,1} : [0, 1) \to \mathbb{R}^2$ by

$$\widetilde{\gamma}_{0,1} = \gamma(\psi(t)) = \left(\frac{1}{2}\psi_{-1,0}(t)^2, \frac{1}{3}\psi_{-1,0}(t)^3\right).$$

Sixth, we define $\widetilde{\nu}_{0,1} : [0, 1) \to S^1$ as follows. By a direct calculation, we have $\boldsymbol{t}(0+0) = (1, 0)$ and $\nu_- = (0, 1)$. The unit tangent vector is given by $\boldsymbol{t}(t) = (1/\sqrt{t^2+1})(1, t)$ for all $t \in [0, 1)$. Hence, we have $\boldsymbol{t}(t) = \boldsymbol{t}(0+0)A(t)$, where

$$A(t) = \frac{1}{\sqrt{t^2+1}} \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix}$$

for all $t \in [0,1)$. Then we have a map $\nu_{0,1} : (-1,0] \to S^1$,

$$\nu_{0,1}(t) = \nu_- A(t) = (0,1)\frac{1}{\sqrt{t^2+1}}\begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix} = \frac{1}{\sqrt{t^2+1}}(-t,1).$$

Now we define $\widetilde{\nu}_{0,1} : [0,1) \to S^1$ by

$$\widetilde{\nu}_{0,1}(t) = \nu_{0,1}(\psi(t)) = \frac{1}{\sqrt{\psi(t)^2+1}}(-\psi(t),1).$$

Finally, we concatenate all maps, that is, we define $\widetilde{\gamma} : (0,1) \to \mathbb{R}^2$ and $\widetilde{\nu} : (0,1) \to S^1$ by $\widetilde{\gamma}(t) = (\widetilde{\gamma}_{0,1} * (\widetilde{\gamma}_0 * \widetilde{\gamma}_{-1,0}))(t)$ and $\widetilde{\nu}(t) = (\widetilde{\nu}_{0,1} * (\widetilde{\nu}_0 * \widetilde{\nu}_{-1,0}))(t)$, respectively. Then we obtain a framed curve $(\widetilde{\gamma}, \widetilde{\nu}) : (0,1) \to \mathbb{R}^2 \times S^1$ such that $\gamma((-1,1)) = \widetilde{\gamma}((0,1))$.

Example 3.4 Let $\gamma : [0,1] \to \mathbb{R}^2$ be given by

$$\gamma(t) = \begin{cases} \left(e^{-1/t}\cos\frac{1}{t}, e^{-1/t}\sin\frac{1}{t}\right) & \text{if } 0 < t \leq 1, \\ (0,0) & \text{if } t = 0, \end{cases}$$

see Figure 3. Since $\gamma^{(n)}$ is given by a sum of products of $\varphi^{(k)}$, $\sin^{(k)}$, $\cos^{(k)}$, $(1/t)^{(k)}$ for $k \in \{0,1,\cdots,n\}$ and $\gamma^{(n)}(0+0) = \mathbf{0}$ for all $n \in \mathbb{N}$, γ is a C^∞-curve. The singular set $\Sigma(\gamma) = \{0\}$. However, the unit tangent vector is given by

$$\boldsymbol{t}(t) = \frac{1}{\sqrt{2}}\left(\cos\frac{1}{t} + \sin\frac{1}{t}, \sin\frac{1}{t} - \cos\frac{1}{t}\right)$$

on $(0,1]$. The limit of the tangent vector $\boldsymbol{t}(0+0)$ and hence the limit of a unit normal vector $\boldsymbol{\nu}(0+0)$ oscillate. Therefore, we can not extend the unit normal vector $\boldsymbol{\nu}$ to $[0,1]$. This means that there are no framed curves $(\widetilde{\gamma}, \widetilde{\boldsymbol{\nu}}) : I \to \mathbb{R}^2 \times S^1$ such that $\widetilde{\gamma}(I) = \gamma([0,1])$.

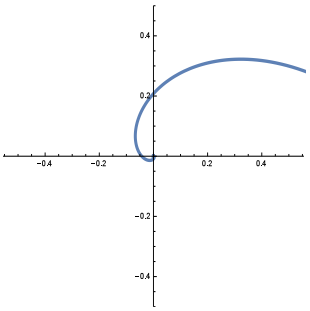


Figure 3: An example of the image of a curve which can not be the image of a framed base curve.

References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol.* I, Birkhäuser, 1986.

[2] R. L. Bishop, There is more than one way to frame a curve, *Amer. Math. Monthly.* **82** (1975), 246–251.

[3] J. W. Bruce and P. J. Giblin, *Curves and singularities. A geometrical introduction to singularity theory. Second edition*, Cambridge University Press, Cambridge, 1992.

[4] T. Fukunaga and M. Takahashi, Existence and uniqueness for Legendre curves, *J. Geometry.* **104** (2013), 297–307.

[5] B. C. Hall, *Lie groups, Lie algebras, and representations*, Graduate Texts in Mathematics, Springer, 2003.

[6] S. Honda and M. Takahashi, Framed curves in the Euclidean space, *to appear in Advances in Geometry.* (2016).

[7] A. Gray, E. Abbena, and S. Salamon, *Modern differential geometry of curves and surfaces with Mathematica. Third edition.* Studies in Advanced Mathematics. Chapman and Hall/CRC, Boca Raton, FL, 2006.

[8] I. M. Singer and J. A. Thorpe, *Lecture notes on elementary topology and geometry*, Undergraduate Texts in Mathematics, Springer, 1977.

Tomonori Fukunaga,
Kyushu Sangyo University, Fukuoka 813-8503, Japan,
E-mail address: tfuku@ip.kyusan-u.ac.jp

Masatomo Takahashi,
Muroran Institute of Technology, Muroran 050-8585, Japan,
E-mail address: masatomo@mmm.muroran-it.ac.jp