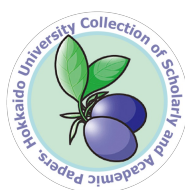




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Envelopes of Legendre curves in the unit tangent bundle over the Euclidean plane

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Abstract

For singular plane curves, the classical definitions of envelopes are vague. In order to define envelopes for singular plane curves, we introduce a one-parameter family of Legendre curves in the unit tangent bundle over the Euclidean plane and the curvature. Then we give a definition of an envelope for the one-parameter family of Legendre curves. We investigate properties of the envelopes. For instance, the envelope is also a Legendre curve. Moreover, we consider bi-Legendre curves and give a relationship between envelopes.

1 Introduction

Envelopes are classical object in the differential geometry. There are many applications of envelopes to differential geometry, differential equations and physics, for instance [4, 5, 7, 9, 10, 15, 16, 18, 20]. An envelope of a family of curves in the plane is a curve that is "tangent" to each member of the family at some point. If the curves are regular, then the tangent is well-defined. However, the definitions of envelopes are vague for singular plane curves (smooth curves with singular points). In this paper, we would like to clarify the definition of the envelope for a family of singular curves. As singular curves, we consider Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, see Appendix A (cf. [8]). The basic results on the singularity theory see [2, 4, 14, 17]. In §2, we quickly review on the definitions of envelopes which are given by implicit functions [3, 4, 12] and parametric curves [11, 19]. In §3, we consider one-parameter families of Legendre curves. We give a moving frame and the curvature of the one-parameter family of Legendre curves. Then we show that the existence and uniqueness theorem for one-parameter families of Legendre curves. In §4, we define an envelope of a one-parameter family of Legendre curves. Then the envelope is also a Legendre curve and hence we give a curvature of the envelope as a Legendre curve. Moreover, we give relationships between the envelopes given by implicit functions and one-parameter family of Legendre curves. In §5, we define a bi-Legendre curve as a special class of one-parameter family of Legendre curves and give a relationship between envelopes.

All maps and manifolds considered here are differential of class C^∞ .

2010 Mathematics Subject classification: 58K05, 53A04

Key Words and Phrases. envelope, one-parameter family of Legendre curves, Legendre curve.

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2 Previous results

Let $\mathbb{R}^2$ be the Euclidean plane equipped with the inner product $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$, where $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$.

We review two definitions of envelopes for one-parameter family of plane curves. These are given by implicit functions and parametrized curves. Here we denote these envelopes by E_I and E_P respectively. Other related definitions of envelopes see [3, 19, 21].

Let $F : V \times \Lambda \rightarrow \mathbb{R}, (x, y, \lambda) \mapsto F(x, y, \lambda)$ be a smooth function, where V is a domain in $\mathbb{R}^2$, and Λ is an interval or $\mathbb{R}$. A family of curves in the plane is given by $\Gamma_\lambda = \{(x, y) \in V \mid F(x, y, \lambda) = 0\}$ for each $\lambda \in \Lambda$. Then one of the classical definition of the envelope is as follows, see for instance [3, 4]:

Definition 2.1 The *envelope* of the family F is the set E_I of points given by

$$E_I = \left\{ (x, y) \in V \mid \text{for some } \lambda \in \Lambda, F(x, y, \lambda) = \frac{\partial F}{\partial \lambda}(x, y, \lambda) = 0 \right\}.$$

If $F(x, y, \lambda) = (\partial F / \partial \lambda)(x, y, \lambda) = 0$, we say that $(x, y) \in E_I$ with respect to λ .

On the other hand, let $\gamma : I \times \Lambda \rightarrow \mathbb{R}^2$ be a one-parameter family of smooth parametrized curves, and let $e_p : U \rightarrow I \times \Lambda, e_p(u) = (t(u), \lambda(u))$ be a regular curve, where I, Λ and U are intervals or $\mathbb{R}$. We denote $\Gamma_\lambda(t) = \gamma(t, \lambda)$ and $E_P(u) = \gamma \circ e_p(u)$.

Definition 2.2 ([11, Page 138]) We call E_P an *envelope* (and e_p a *pre-envelope*) for the family γ , when the following conditions are satisfied.

- (i) The function λ is non-constant on any non-trivial subinterval of U . (The Variability Condition.)
- (ii) For all u , the curve E_P is tangent at u to the curve $\Gamma_{\lambda(u)}$ at the parameter $t(u)$, meaning that the tangent vectors $E'_P(u) = (dE_P/du)(u)$ and $\dot{\Gamma}_{\lambda(u)}(t(u)) = (d\Gamma_{\lambda(u)}/dt)(t(u))$ are linearly dependent. (The Tangency Condition.)

We say that the *singular set* of $\gamma : I \times \Lambda \rightarrow \mathbb{R}^2, \gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$ is the subset of the domain $I \times \Lambda$ defined by

$$\det(\gamma_t(t, \lambda), \gamma_\lambda(t, \lambda)) = \det \begin{pmatrix} x_t(t, \lambda) & y_t(t, \lambda) \\ x_\lambda(t, \lambda) & y_\lambda(t, \lambda) \end{pmatrix} = 0. \quad (1)$$

Here we denote $\gamma_t(t, \lambda) = (\partial \gamma / \partial t)(t, \lambda) = (x_t(t, \lambda), y_t(t, \lambda))$ and $\gamma_\lambda(t, \lambda) = (\partial \gamma / \partial \lambda)(t, \lambda) = (x_\lambda(t, \lambda), y_\lambda(t, \lambda))$. Then the envelope theorem is as follows:

Theorem 2.3 ([11, Page 140]) Let $\gamma : I \times \Lambda \rightarrow \mathbb{R}^2$ be a family of parametrized curves, and let $e_p : U \rightarrow I \times \Lambda$ be a regular curve satisfying the variability condition. Then e_p is a pre-envelope of γ (and E_P is an envelope) if and only if the trace of e_p lies in the singular set of γ .

We consider one-parameter families of 3/2-cusps as examples. Other examples see [3, 11].

Example 2.4 Let $F : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}, F(x, y, \lambda) = (x - \lambda)^3 - y^2$. Since $(\partial F / \partial \lambda)(x, y, \lambda) = -3(x - \lambda)^2$, the envelope is given by $E_I = \{(\lambda, 0) | \lambda \in \mathbb{R}\}$.

Let $\gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2, \gamma(t, \lambda) = (t^2 + \lambda, t^3)$. Since (1), we have $-3t^2 = 0$. By Theorem 2.3, the pre-envelope and the envelope are given by $e_p : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, e_p(u) = (0, u)$ and $E_P : \mathbb{R} \rightarrow \mathbb{R}^2, E_P(u) = (u, 0)$.

Both cases, the envelopes are given by the x -axis, see Figure 1.

Example 2.5 Let $F : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}, F(x, y, \lambda) = x^3 - (y - \lambda)^2$. Since $(\partial F / \partial \lambda)(x, y, \lambda) = -2(y - \lambda)$, the envelope is given by $E_I = \{(0, \lambda) | \lambda \in \mathbb{R}\}$.

Let $\gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2, \gamma(t, \lambda) = (t^2, t^3 + \lambda)$. Since (1), we have $2t = 0$. By Theorem 2.3, the pre-envelope and the envelope are given by $e_p : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, e_p(u) = (0, u)$ and $E_P : \mathbb{R} \rightarrow \mathbb{R}^2, E_P(u) = (0, u)$.

Both cases, the envelopes are given by the y -axis, see Figure 2. However, in the sense of limit tangent of the 3/2-cusp, y -axis is not tangent to the 3/2-cusps. Moreover, as a solution of differential equations, the x -axis in Figure 1 is a singular solution of the ODE $-y + ((2/3)y')^3 = 0$ and y -axis in Figure 2 is not a singular solution of the ODE $-x + ((2/3)y')^2 = 0$ (cf. [15, 16, 20]). We would like to distinguish as envelopes, see Examples 4.2 and 4.3 below.

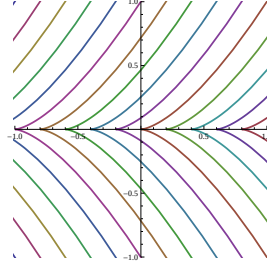


Figure 1.

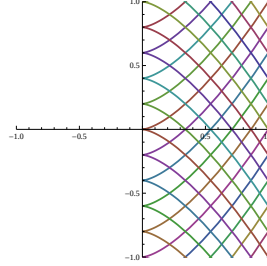


Figure 2.

3 One parameter families of Legendre curves

In this section, we consider one-parameter families of Legendre curves in the unit tangent bundle $T_1 S^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$. The fundamental results for Legendre curves in the unit tangent bundle over $\mathbb{R}^2$ see the Appendix or [8].

Definition 3.1 Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a smooth mapping. We say that (γ, ν) is a *one-parameter family of Legendre curves* if $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in I \times \Lambda$.

Then $(\gamma(\cdot, \lambda), \nu(\cdot, \lambda)) : I \rightarrow \mathbb{R}^2 \times S^1$ is a Legendre curve for each fixed parameter $\lambda \in \Lambda$, that is, $(\gamma(\cdot, \lambda), \nu(\cdot, \lambda))$ is an integrable curve with respect to the canonical contact 1-form on $\mathbb{R}^2 \times S^1$. Therefore, $\gamma : I \times \Lambda \rightarrow \mathbb{R}^2$ is a one-parameter family of frontals.

We denote $J(\mathbf{a}) = (-a_2, a_1)$ the anticlockwise rotation by $\pi/2$ of a vector $\mathbf{a} = (a_1, a_2)$. We define $\boldsymbol{\mu}(t, \lambda) = J(\nu(t, \lambda))$. Since $\{\nu(t, \lambda), \boldsymbol{\mu}(t, \lambda)\}$ is a moving frame along $\gamma(t, \lambda)$ on $\mathbb{R}^2$, we have the Frenet type formula.

$$\begin{pmatrix} \nu_t(t, \lambda) \\ \boldsymbol{\mu}_t(t, \lambda) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t, \lambda) \\ -\ell(t, \lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t, \lambda) \\ \boldsymbol{\mu}(t, \lambda) \end{pmatrix},$$

$$\begin{pmatrix} \nu_\lambda(t, \lambda) \\ \boldsymbol{\mu}_\lambda(t, \lambda) \end{pmatrix} = \begin{pmatrix} 0 & m(t, \lambda) \\ -m(t, \lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t, \lambda) \\ \boldsymbol{\mu}(t, \lambda) \end{pmatrix}$$

and

$$\gamma_t(t, \lambda) = \beta(t, \lambda)\boldsymbol{\mu}(t, \lambda),$$

where $\ell(t, \lambda) = \nu_t(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda)$, $m(t, \lambda) = \nu_\lambda(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda)$ and $\beta(t, \lambda) = \gamma_t(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda)$. By the integrability condition $\nu_{t\lambda}(t, \lambda) = \nu_{\lambda t}(t, \lambda)$, ℓ and m satisfies the condition

$$\ell_\lambda(t, \lambda) = m_t(t, \lambda) \quad (2)$$

for all $(t, \lambda) \in I \times \Lambda$. We call the pair (ℓ, m, β) with the integrability condition (2) a *curvature of the one-parameter family of Legendre curves* (γ, ν) .

Remark 3.2 Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with the curvature (ℓ, m, β) . Then $(\gamma, -\nu)$ is also a one-parameter family of Legendre curves with the curvature $(\ell, m, -\beta)$.

Example 3.3 (Example 2.4) Let $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1$, $\gamma(t, \lambda) = (t^2 + \lambda, t^3)$, $\nu(t, \lambda) = (4 + 9t^2)^{-1/2}(-3t, 2)$. Since $\gamma_t(t, \lambda) = (2t, 3t^2)$, $\nu_t(t, \lambda) = 6(4 + 9t^2)^{-3/2}(-2, -3t)$, $\nu_\lambda(t, \lambda) = 0$ and $\boldsymbol{\mu}(t, \lambda) = (4 + 9t^2)^{-1/2}(-2, -3t)$, (γ, ν) is a one-parameter family of Legendre curves with the curvature $(\ell(t, \lambda), m(t, \lambda), \beta(t, \lambda)) = (6(4 + 9t^2)^{-1}, 0, -t(4 + 9t^2)^{1/2})$.

Example 3.4 (Example 2.5) Let $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1$, $\gamma(t, \lambda) = (t^2, t^3 + \lambda)$, $\nu(t, \lambda) = (4 + 9t^2)^{-1/2}(-3t, 2)$. Since $\gamma_t(t, \lambda) = (2t, 3t^2)$, $\nu_t(t, \lambda) = 6(4 + 9t^2)^{-3/2}(-2, -3t)$, $\nu_\lambda(t, \lambda) = 0$ and $\boldsymbol{\mu}(t, \lambda) = (4 + 9t^2)^{-1/2}(-2, -3t)$, (γ, ν) is a one-parameter family of Legendre curves with the curvature $(\ell(t, \lambda), m(t, \lambda), \beta(t, \lambda)) = (6(4 + 9t^2)^{-1}, 0, -t(4 + 9t^2)^{1/2})$.

Definition 3.5 Let (γ, ν) and $(\tilde{\gamma}, \tilde{\nu}) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be one-parameter families of Legendre curves. We say that (γ, ν) and $(\tilde{\gamma}, \tilde{\nu})$ are *congruent as one-parameter family of Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a smooth translation mapping $\mathbf{a} : \Lambda \rightarrow \mathbb{R}^2$ such that $\tilde{\gamma}(t, \lambda) = A(\gamma(t, \lambda)) + \mathbf{a}(\lambda)$ and $\tilde{\nu}(t, \lambda) = A(\nu(t, \lambda))$ for all $(t, \lambda) \in I \times \Lambda$.

We give the existence and uniqueness theorems for one-parameter families of Legendre curves.

Theorem 3.6 (The Existence Theorem for one-parameter families of Legendre curves.) *Let $(\ell, m, \beta) : I \times \Lambda \rightarrow \mathbb{R}^3$ be a smooth mapping with the integrability condition. There exists a one-parameter family of Legendre curves $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ whose associated curvature is (ℓ, m, β) .*

Proof. Let $(t_0, \lambda_0) \in I \times \Lambda$ be fixed. We define a smooth mapping $\theta : I \times \Lambda \rightarrow \mathbb{R}$ by

$$\theta(t, \lambda) = \int_{t_0}^t \ell(t, \lambda) dt + \int_{\lambda_0}^\lambda m(t_0, \lambda) d\lambda.$$

Then θ satisfy the conditions $\theta_t(t, \lambda) = \ell(t, \lambda)$ and $\theta_\lambda(t, \lambda) = m(t, \lambda)$. We define a smooth mapping $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ by

$$\begin{aligned} \gamma(t, \lambda) &= \left(- \int \beta(t, \lambda) \sin \theta(t, \lambda) dt, \int \beta(t, \lambda) \cos \theta(t, \lambda) dt \right), \\ \nu(t, \lambda) &= (\cos \theta(t, \lambda), \sin \theta(t, \lambda)). \end{aligned}$$

By a direct calculation, (γ, ν) is a one-parameter family of Legendre curves with the curvature (ℓ, m, β) . $\square$

Theorem 3.7 (The Uniqueness Theorem for one-parameter families of Legendre curves.) *Let (γ, ν) and $(\tilde{\gamma}, \tilde{\nu}) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be one-parameter families of Legendre curves with the curvatures (ℓ, m, β) and $(\tilde{\ell}, \tilde{m}, \tilde{\beta})$ respectively. Then (γ, ν) and $(\tilde{\gamma}, \tilde{\nu})$ are congruent as one-parameter family of Legendre curves if and only if (ℓ, m, β) and $(\tilde{\ell}, \tilde{m}, \tilde{\beta})$ coincides.*

Proof. Suppose that (γ, ν) and $(\tilde{\gamma}, \tilde{\nu})$ are congruent as one-parameter families of Legendre curves. By a direct calculation, we have

$$\begin{aligned}\tilde{\gamma}_t(t, \lambda) &= \frac{\partial}{\partial t}(A(\gamma(t, \lambda)) + \mathbf{a}(\lambda)) = A(\gamma_t(t, \lambda)) = \beta(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = \beta(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda), \\ \tilde{\nu}_t(t, \lambda) &= \frac{\partial}{\partial t}(A(\nu(t, \lambda))) = A(\nu_t(t, \lambda)) = \ell(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = \ell(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda), \\ \tilde{\nu}_\lambda(t, \lambda) &= \frac{\partial}{\partial \lambda}(A(\nu(t, \lambda))) = A(\nu_\lambda(t, \lambda)) = m(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = m(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda).\end{aligned}$$

Therefore the curvatures (ℓ, m, β) and $(\tilde{\ell}, \tilde{m}, \tilde{\beta})$ coincides.

Conversely, suppose that (ℓ, m, β) and $(\tilde{\ell}, \tilde{m}, \tilde{\beta})$ coincides. Let $(t_0, \lambda_0) \in I \times \Lambda$ be fixed. By using a congruence as one-parameter family of Legendre curves, we may assume $\gamma(t_0, \lambda_0) = \tilde{\gamma}(t_0, \lambda_0)$ and $\nu(t_0, \lambda_0) = \tilde{\nu}(t_0, \lambda_0)$. Moreover, we have $\theta(t, \lambda) = \tilde{\theta}(t, \lambda)$ for all $(t, \lambda) \in I \times \Lambda$ in the proof of Theorem 3.6. It follows from the construction that we have $\nu(t, \lambda) = \tilde{\nu}(t, \lambda)$, and $\gamma(t, \lambda) = \tilde{\gamma}(t, \lambda)$ up to a smooth translation mapping $\mathbf{a}(\lambda)$ for all $(t, \lambda) \in I \times \Lambda$. $\square$

4 Envelopes of Legendre curves

Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with the curvature (ℓ, m, β) , and let $e_L : U \rightarrow I \times \Lambda, e_L(u) = (t(u), \lambda(u))$ be a smooth curve. We denote $\Gamma_\lambda(t) = \gamma(t, \lambda)$ and $E_L = \gamma \circ e_L(u)$. Note that we don't assume e_L is a regular curve, see section 2.

Definition 4.1 We call E_L an *envelope* (and e_L a *pre-envelope*) for the family of Legendre curves (γ, ν) , when the following conditions are satisfied.

- (i) The function λ is non-constant on any non-trivial subinterval of U . (The Variability Condition.)
- (ii) For all u the curve E_L is tangent at u to the curve $\Gamma_{\lambda(u)}$ at the parameter $t(u)$, meaning that $E'_L(u)$ and $\boldsymbol{\mu}(t(u), \lambda(u))$ are linearly dependent. (The Tangency Condition.)

Note that the tangency condition is equivalent to the condition $E'_L(u) \cdot \nu(e_L(u)) = 0$ for all $u \in U$.

Example 4.2 (Example 3.3) Let $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1, \gamma(t, \lambda) = (t^2 + \lambda, t^3), \nu(t, \lambda) = (4 + 9t^2)^{-1/2}(-3t, 2)$. Let $e_L : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, e_L(u) = (t(u), \lambda(u)) = (0, u)$. Then $E_L(u) = \gamma \circ e_L(u) = (u, 0)$. Since $\lambda'(u) = 1$ and $E'_L(u) \cdot \nu(0, u) = 0$, E_L is an envelope of (γ, ν) .

Example 4.3 (Example 3.4) Let $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1, \gamma(t, \lambda) = (t^2, t^3 + \lambda), \nu(t, \lambda) = (4 + 9t^2)^{-1/2}(-3t, 2)$. Let $e_L : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, e_L(u) = (t(u), \lambda(u)) = (0, u)$. Then $E_L(u) = \gamma \circ e_L(u) = (0, u)$ and $\lambda'(u) = 1$. Since $E'_L(u) \cdot \nu(0, u) = 1 \neq 0$, E_L is not an envelope of (γ, ν) .

Proposition 4.4 Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves with the curvature (ℓ, m, β) . Suppose that $e_L : U \rightarrow I \times \Lambda$ is a pre-envelope and $E_L = \gamma \circ e_L : U \rightarrow \mathbb{R}^2$ is an envelope of (γ, ν) . Then E_L is a frontal. More preciously, $(E_L, \nu \circ e_L) : U \rightarrow \mathbb{R}^2 \times S^1$ is a Legendre curve with the curvature

$$\begin{aligned}\ell_E(u) &= t'(u)\ell(e_L(u)) + \lambda'(u)m(e_L(u)), \\ \beta_E(u) &= t'(u)\beta(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u)) \cdot \boldsymbol{\mu}(e_L(u)).\end{aligned}$$

Proof. We denote $e_L(u) = (t(u), \lambda(u))$. Since E_L is an envelope, $E'_L(u) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. It follows that $(E_L, \nu \circ e_L) : U \rightarrow \mathbb{R}^2 \times S^1$ is a Legendre curve. Then $\ell_E(u) = (d/du)(\nu(e_L(u))) \cdot \boldsymbol{\mu}(e_L(u)) = (t'(u)\nu_t(e_L(u)) + \lambda'(u)\nu_\lambda(e_L(u))) \cdot \boldsymbol{\mu}(e_L(u)) = t'(u)\ell(e_L(u)) + \lambda'(u)m(e_L(u))$ and $\beta_E(u) = (d/du)(\gamma(e_L(u))) \cdot \boldsymbol{\mu}(e_L(u)) = (t'(u)\gamma_t(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u))) \cdot \boldsymbol{\mu}(e_L(u)) = t'(u)\beta(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u)) \cdot \boldsymbol{\mu}(e_L(u))$. $\square$

We give the envelope theorem for one-parameter family of Legendre curves.

Theorem 4.5 Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves, and let $e_L : U \rightarrow I \times \Lambda$ be a smooth curve satisfying the variability condition. Then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope) if and only if $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$.

Proof. Suppose that e_L is a pre-envelope of (γ, ν) . By the tangency condition, there exists a function $c(u) \in \mathbb{R}$ such that $E'_L(u) = c(u)\boldsymbol{\mu}(e_L(u))$. By differentiate $E_L(u) = \gamma \circ e_L(u)$, we have $E'_L(u) = t'(u)\gamma_t(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u))$. It follows from $\gamma_t(t, \lambda) = \beta(t, \lambda)\boldsymbol{\mu}(t, \lambda)$ that $(t'(u)\beta(e_L(u)) - c(u))\boldsymbol{\mu}(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u)) = 0$. Then we have $\lambda'(u)\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$. By the variability condition, we have $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$.

Conversely, suppose that $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. Since $E'_L(u) \cdot \nu(e_L(u)) = (t'(u)\gamma_t(e_L(u)) + \lambda'(u)\gamma_\lambda(e_L(u))) \cdot \nu(e_L(u)) = 0$, e_L is a pre-envelope of (γ, ν) . $\square$

Example 4.6 Let i, j, m, n, j, k be natural numbers with $j = i + h, n = m + k$. Moreover, we take $h = 1$ or $k = 1$, or h, k are relatively prime numbers. Let $(\gamma, \nu) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1$,

$$\gamma(t, \lambda) = \left(\frac{t^m}{m} + \frac{\lambda^i}{i}, \frac{t^n}{n} + \frac{\lambda^j}{j} \right), \quad \nu(t, \lambda) = \frac{1}{\sqrt{t^{2k} + 1}}(-t^k, 1).$$

Since $\gamma_t(t, \lambda) = (t^{m-1}, t^{n-1})$, we have $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in \mathbb{R} \times \mathbb{R}$. Moreover, since $\gamma_\lambda(t, \lambda) = (\lambda^{i-1}, \lambda^{j-1})$, we have $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = (\lambda^{i-1}/\sqrt{t^{2k} + 1})(-t^k + \lambda^h)$. If we take $e_L : \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$, $e_L(u) = (u^h, u^k)$, then the variability condition holds. Furthermore, since

$$\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = \frac{u^{k(i-1)}}{\sqrt{u^{2kh} + 1}}(-u^{hk} + u^{hk}) = 0,$$

e_L is a pre-envelope of (γ, ν) by Theorem 4.5. Hence, the envelope $(E_L, \nu_L) : \mathbb{R} \rightarrow \mathbb{R}^2 \times S^1$ is given by

$$E_L(u) = \left(\frac{u^{mh}}{m} + \frac{u^{ik}}{i}, \frac{u^{nh}}{n} + \frac{u^{jk}}{j} \right), \quad \nu_L(u) = \frac{1}{\sqrt{u^{2kh} + 1}}(-u^{kh}, 1).$$

Proposition 4.7 Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves. Suppose that $e_L : U \rightarrow I \times \Lambda$ is a pre-envelope and $E_L = \gamma \circ e_L$ is an envelope of (γ, ν) . Then $e_L : U \rightarrow I \times \Lambda$ is also a pre-envelope and $E_L = \gamma \circ e_L$ is also an envelope of $(\gamma, -\nu)$.

Proof. By Remark 3.2, $(\gamma, -\nu)$ is also a one-parameter family of Legendre curves. It follows from Theorem 4.5 that we have the same pre-envelopes and the envelopes of (γ, ν) and $(\gamma, -\nu)$. $\square$

Definition 4.8 We say that a map $\Phi : \tilde{I} \times \tilde{\Lambda} \rightarrow I \times \Lambda$ is a one-parameter family of parameter change if Φ is a diffeomorphism and given by the form $\Phi(s, k) = (\phi(s, k), \varphi(k))$.

Proposition 4.9 Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves. Suppose that $e_L : U \rightarrow I \times \Lambda$ is a pre-envelope, $E_L = \gamma \circ e_L$ is an envelope and $\Phi : \tilde{I} \times \tilde{\Lambda} \rightarrow I \times \Lambda$ is a one-parameter family of parameter change. Then $(\tilde{\gamma}, \tilde{\nu}) = (\gamma \circ \Phi, \nu \circ \Phi) : \tilde{I} \times \tilde{\Lambda} \rightarrow \mathbb{R}^2 \times S^1$ is also a one-parameter family of Legendre curves. Moreover, $\Phi^{-1} \circ e_L : U \rightarrow \tilde{I} \times \tilde{\Lambda}$ is a pre-envelope and E_L is also an envelope of $(\tilde{\gamma}, \tilde{\nu})$.

Proof. Since $\tilde{\gamma}_s(s, k) = \phi_s(s, k)\gamma_t(\Phi(s, k))$ and $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in I \times \Lambda$, we have $\tilde{\gamma}_s(s, k) \cdot \tilde{\nu}(s, k) = 0$ for all $(s, k) \in \tilde{I} \times \tilde{\Lambda}$. Therefore, $(\tilde{\gamma}, \tilde{\nu})$ is a one-parameter family of Legendre curves. By the form of the diffeomorphism $\Phi(s, k) = (\phi(s, k), \varphi(k))$, $\Phi^{-1} : I \times \Lambda \rightarrow \tilde{I} \times \tilde{\Lambda}$ is given by the form $\Phi^{-1}(t, \lambda) = (\psi(t, \lambda), \varphi^{-1}(\lambda))$. It follows that $\Phi^{-1} \circ e_L(u) = (\phi(t(u), \lambda(u)), \varphi^{-1}(\lambda(u)))$. Since $(d/du)\varphi^{-1}(\lambda(u)) = \varphi_\lambda^{-1}(\lambda(u))\lambda'(u)$, the variability condition holds. Moreover, we have $\tilde{\gamma}_k(s, k) \cdot \tilde{\nu}(s, k) = (\gamma_t(\Phi(s, k))\phi_k(s, k) + \gamma_\lambda(\Phi(s, k))\varphi'(k)) \cdot \nu(\Phi(s, k)) = \varphi'(k)\gamma_\lambda(\Phi(s, k)) \cdot \nu(\Phi(s, k))$. It follows that $\tilde{\gamma}_k(\Phi^{-1} \circ e_L(u)) \cdot \tilde{\nu}(\Phi^{-1} \circ e_L(u)) = \varphi'(\varphi^{-1}(\lambda(u)))\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$. By Theorem 4.5, $\Phi^{-1} \circ e_L$ is a pre-envelope of $(\tilde{\gamma}, \tilde{\nu})$. Therefore, $\tilde{\gamma} \circ \Phi^{-1} \circ e_L = \gamma \circ \Phi \circ \Phi^{-1} \circ e_L = \gamma \circ e_L = E_L$ is also an envelope of $(\tilde{\gamma}, \tilde{\nu})$. $\square$

We give a relationship between envelopes which are given by implicit functions (Definition 2.1) and one-parameter families of Legendre curves.

Proposition 4.10 Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves, and let $F(x, y, \lambda) = 0$ be an implicit function of the one-parameter family of frontals, that is, $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, where $\gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$. If $e_L : U \rightarrow I \times \Lambda$ is a pre-envelope and $E_L : U \rightarrow \mathbb{R}^2$ is an envelope of (γ, ν) , then $E_L(U) \subset E_I$.

Proof. By differentiate $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, we have

$$x_t(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_t(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) = 0$$

and

$$x_\lambda(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_\lambda(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0.$$

Equivalently, $\gamma_t(t, \lambda) \cdot (F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) = 0$ and $\gamma_\lambda(t, \lambda) \cdot (F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0$. Since (γ, ν) is a one-parameter family of Legendre curves, there exists a function $c(t, \lambda)$ such that $(F_x, F_y)(x(t, \lambda), y(t, \lambda), \lambda) = c(t, \lambda)\nu(t, \lambda)$. Moreover, $e_L(u) = (t(u), \lambda(u))$ is a pre-envelope of (γ, ν) , we have $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. It follows that $F_\lambda(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) = 0$. Therefore, we have $E(u) = \gamma \circ e_L(u) \in E_I$ with respect to $\lambda(u)$ for all $u \in U$. $\square$

In order to consider the converse result, we need the following lemma and proposition.

Lemma 4.11 Let $\mathbf{a}, \mathbf{b} : U \rightarrow \mathbb{R}^2$ be smooth maps. Suppose that the set of non-zero points of smooth function $k : U \rightarrow \mathbb{R}$ is dense in U . If $k(u)\mathbf{a}(u)$ and $\mathbf{b}(u)$ are linearly dependent, then $\mathbf{a}(u)$ and $\mathbf{b}(u)$ are linearly dependent for all $u \in U$.

Proof. Since $\det(k(u)\mathbf{a}(u), \mathbf{b}(u)) = 0$, we have $k(u)\det(\mathbf{a}(u), \mathbf{b}(u)) = 0$. By the condition and continuous property, we have $\det(\mathbf{a}(u), \mathbf{b}(u)) = 0$ for all $u \in U$. $\square$

Proposition 4.12 *Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2$ be a one-parameter family of Legendre curves, and let $e_L : U \rightarrow I \times \Lambda$ be a smooth curve satisfying the variability condition. If the set of regular points of γ on $e_L(U)$ is dense in U and the trace of e_L lies in the singular set of γ , then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope).*

Proof. Since $e_L(u)$ belong to the singular set of γ , we have $\det(\gamma_t(e_L(u)), \gamma_\lambda(e_L(u))) = 0$ for all $u \in U$. Therefore $\gamma_t(e_L(u)) = \beta(e_L(u))\boldsymbol{\mu}(e_L(u))$ and $\gamma_\lambda(e_L(u))$ are linearly dependent. By the assumption, the set of non-zero points of $\beta \circ e_L$ is dense in U . It follows from Lemma 4.11 that $\boldsymbol{\mu}(e_L(u))$ and $\gamma_\lambda(e_L(u))$ are linearly dependent. Therefore $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. By Theorem 4.5, e_L is a pre-envelope of (γ, ν) . $\square$

Proposition 4.13 *Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a one-parameter family of Legendre curves, and let $F(x, y, \lambda) = 0$ be an implicit function of the one-parameter family of frontals, that is, $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, where $\gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda))$. Let $e_L : U \rightarrow I \times \Lambda, e(u) = (t(u), \lambda(u))$ be a smooth curve satisfying the variability condition. If the set of regular points of γ on $e_L(U)$ is dense in U , $E_L(u) = \gamma \circ e_L(u) \in E_I$ with respect to $\lambda(u)$ and*

$$(F_x, F_y)(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \neq (0, 0)$$

for all $u \in U$, then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope).

Proof. By differentiate $F(x(t, \lambda), y(t, \lambda), \lambda) = 0$, we have

$$x_t(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_t(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) = 0$$

and

$$x_\lambda(t, \lambda)F_x(x(t, \lambda), y(t, \lambda), \lambda) + y_\lambda(t, \lambda)F_y(x(t, \lambda), y(t, \lambda), \lambda) + F_\lambda(x(t, \lambda), y(t, \lambda), \lambda) = 0.$$

Since $E_L(u) = \gamma \circ e_L(u) \in E_I$ with respect to $\lambda(u)$, we have $F_\lambda(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) = 0$. It follows that

$$\begin{pmatrix} x_t(t(u), \lambda(u)) & y_t(t(u), \lambda(u)) \\ x_\lambda(t(u), \lambda(u)) & y_\lambda(t(u), \lambda(u)) \end{pmatrix} \begin{pmatrix} F_x(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \\ F_y(x(t(u), \lambda(u)), y(t(u), \lambda(u)), \lambda(u)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then the trace of e_L lies in the singular set of γ . By Proposition 4.12, e_L is a pre-envelope of (γ, ν) . $\square$

We give interesting examples of envelopes of one-parameter families of Legendre curves by using two Legendre curves. Also see [6, 9, 10].

Let $(\mathbf{p}, \nu_p) : I \rightarrow \mathbb{R}^2 \times S^1$ and $(\mathbf{q}, \nu_q) : \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be Legendre curves with the curvature (ℓ_p, β_p) and (ℓ_q, β_q) respectively, see Appendix A. We denote

$$\mathbf{p}(t) = (p_1(t), p_2(t)), \nu_p(t) = (\nu_{p1}(t), \nu_{p2}(t)), \boldsymbol{\mu}_p(t) = (-\nu_{p2}(t), \nu_{p1}(t)),$$

$$\mathbf{q}(\lambda) = (q_1(\lambda), q_2(\lambda)), \nu_q(\lambda) = (\nu_{q1}(\lambda), \nu_{q2}(\lambda)), \boldsymbol{\mu}_q(\lambda) = (-\nu_{q2}(\lambda), \nu_{q1}(\lambda)),$$

respectively. Suppose that $\mathbf{p}(0) = (0, 0)$ and $\nu_p(0) = (0, 1)$.

We define $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ by

$$\gamma(t, \lambda) = \mathbf{q}(\lambda) + A(\theta(\lambda))\mathbf{p}(t), \nu(t, \lambda) = A(\theta(\lambda))\nu_p(t), \quad (3)$$

where $\theta : \Lambda \rightarrow \mathbb{R}$ and

$$A(\theta(\lambda)) = \begin{pmatrix} \cos \theta(\lambda) & -\sin \theta(\lambda) \\ \sin \theta(\lambda) & \cos \theta(\lambda) \end{pmatrix}.$$

By a direct calculation, (γ, ν) is a one-parameter family of Legendre curves.

First, we consider a Legendre curve $\mathbf{p}$ along a Legendre curve $\mathbf{q}$ which satisfying the both unit normal vectors coincide. Suppose that $\nu(0, \lambda) = \nu_q(\lambda)$. This means that the unit normal vector of γ at $(0, \lambda)$ coincide with the unit normal vector of $\mathbf{q}$ at λ . It follows that $\cos \theta(\lambda) = \nu_{q2}(\lambda)$ and $\sin \theta(\lambda) = -\nu_{q1}(\lambda)$. By a direct calculation, we have $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = -\beta_q(\lambda)\nu_{p1}(t) - \ell_q(\lambda)\mathbf{p}(t) \cdot \boldsymbol{\mu}_p(t)$. By a corollary of Theorem 4.5, we have the following.

Corollary 4.14 *Under the above notations, let (γ, ν) be given by (3) with the conditions $\mathbf{p}(0) = (0, 0)$, $\nu_p(0) = (0, 1)$ and $\nu(0, \lambda) = \nu_q(\lambda)$. Let $e_L : U \rightarrow I \times \Lambda$ be a smooth curve satisfying the variability condition. If $\beta_q(\lambda(u))\nu_{p1}(t(u)) + \ell_q(\lambda(u))\mathbf{p}(t(u)) \cdot \boldsymbol{\mu}_p(t(u)) = 0$, then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope).*

Note that $e_L(u) = (0, u)$ is a pre-envelope of (γ, ν) . Thus, $E_L(u) = \mathbf{q}(\lambda)$ is always an envelope of (γ, ν) .

Second, we consider a Legendre curve $\mathbf{p}$ along a Legendre curve $\mathbf{q}$ which satisfying the unit normal vector of $\mathbf{p}$ coincide with the tangent vector of $\mathbf{q}$. Suppose that $\nu(0, \lambda) = \boldsymbol{\mu}_q(\lambda)$. This means that the unit normal vector of γ at $(0, \lambda)$ coincide with the unit tangent vector of $\mathbf{q}$ at λ . It follows that $\cos \theta(\lambda) = \nu_{q1}(\lambda)$ and $\sin \theta(\lambda) = \nu_{q2}(\lambda)$. By a direct calculation, we have $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = \beta_q(\lambda)\nu_{p2}(t) - \ell_q(\lambda)\mathbf{p}(t) \cdot \boldsymbol{\mu}_p(t)$. By a corollary of Theorem 4.5, we have the following.

Corollary 4.15 *Under the above notations, let (γ, ν) be given by (3) with the conditions $\mathbf{p}(0) = (0, 0)$, $\nu_p(0) = (0, 1)$ and $\nu(0, \lambda) = \boldsymbol{\mu}_q(\lambda)$. Let $e_L : U \rightarrow I \times \Lambda$ be a smooth curve satisfying the variability condition. If $\beta_q(\lambda(u))\nu_{p2}(t(u)) - \ell_q(\lambda(u))\mathbf{p}(t(u)) \cdot \boldsymbol{\mu}_p(t(u)) = 0$, then e_L is a pre-envelope of (γ, ν) (and E_L is an envelope).*

Example 4.16 Let $(\mathbf{p}, \nu_p) : [0, 2\pi) \rightarrow \mathbb{R}^2 \times S^1$ be an astroid $\mathbf{p}(t) = (\cos^3 t - 1, \sin^3 t)$, $\nu_p(t) = (\sin t, \cos t)$ and $(\mathbf{q}, \nu_q) : [0, 2\pi) \rightarrow \mathbb{R}^2 \times S^1$ be the unit circle $\mathbf{q}(\lambda) = (\cos \lambda, \sin \lambda)$, $\nu_q(\lambda) = (\cos \lambda, \sin \lambda)$, see Figure 3. Then we have $\beta_p(t) = 3 \cos t \sin t$, $\ell_p(t) = -1$, $\beta_q(\lambda) = 1$ and $\ell_q(\lambda) = 1$. Moreover, the conditions $\mathbf{p}(0) = (0, 0)$ and $\nu_p(0) = (0, 1)$ are satisfied.

First, we consider a Legendre curve $\mathbf{p}$ along a Legendre curve $\mathbf{q}$ which satisfying the both unit normal vectors coincide. By (3) and the condition $\nu(0, \lambda) = \nu_q(\lambda)$, the one-parameter family of Legendre curves $(\gamma, \nu) : [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{R}^2 \times S^1$ is given by

$$\begin{aligned} \gamma(t, \lambda) &= \begin{pmatrix} \cos \lambda \\ \sin \lambda \end{pmatrix} + \begin{pmatrix} \sin \lambda & \cos \lambda \\ -\cos \lambda & \sin \lambda \end{pmatrix} \begin{pmatrix} \cos^3 t - 1 \\ \sin^3 t \end{pmatrix}, \\ \nu(t, \lambda) &= \begin{pmatrix} \sin \lambda & \cos \lambda \\ -\cos \lambda & \sin \lambda \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \end{aligned}$$

By a direct calculation, we have $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = -4 \cos(t - (\pi/4)) \cos((t/2) - (\pi/4)) \sin t/2$. It follows that $e_L : [0, 2\pi) \rightarrow [0, 2\pi) \times [0, 2\pi)$, $e_L(u) = (0, u), (3\pi/4, u), (3\pi/2, u), (7\pi/4, u)$ are pre-envelopes of (γ, ν) respectively, by Corollary 4.14. Therefore, the envelopes $E_L : [0, 2\pi) \rightarrow \mathbb{R}^2$ of (γ, ν) are given by $E_L(u) = (\cos u, \sin u), (\sqrt{2} + (1/2))(\cos(u + \pi/4), \sin(u + \pi/4)), (\cos(u + \pi/2), \sin(u + \pi/2)), (\sqrt{2} - (1/2))(\cos(u + \pi/4), \sin(u + \pi/4))$, respectively see Figure 4 left.

Second, we consider a Legendre curve $\mathbf{p}$ along a Legendre curve $\mathbf{q}$ which satisfying the unit normal vector of $\mathbf{p}$ coincide with the tangent vector of $\mathbf{q}$. By (3) and the condition $\nu(0, \lambda) = \mu_q(\lambda) = (-\sin \lambda, \cos \lambda)$, the one-parameter family of Legendre curves $(\gamma, \nu) : [0, 2\pi) \times [0, 2\pi) \rightarrow \mathbb{R}^2 \times S^1$ is given by

$$\begin{aligned}\gamma(t, \lambda) &= \begin{pmatrix} \cos \lambda \\ \sin \lambda \end{pmatrix} + \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} \cos^3 t - 1 \\ \sin^3 t \end{pmatrix}, \\ \nu(t, \lambda) &= \begin{pmatrix} \cos \lambda & -\sin \lambda \\ \sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.\end{aligned}$$

By a direct calculation, we have $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = \cos 2t$. It follows that $e_L : [0, 2\pi) \rightarrow [0, 2\pi) \times [0, 2\pi)$, $e_L(u) = (\pi/4, u), (3\pi/4, u), (5\pi/4, u), (7\pi/4, u)$ are pre-envelopes of (γ, ν) respectively, by Corollary 4.15. Therefore the envelopes $E_L : [0, 2\pi) \rightarrow \mathbb{R}^2$ of (γ, ν) are given by $E_L(u) = (1/2)(\cos(u + \pi/4), \sin(u + \pi/4)), (1/2)(\cos(u + 3\pi/4), \sin(u + 3\pi/4)), (1/2)(\cos(u + 5\pi/4), \sin(u + 5\pi/4)), (1/2)(\cos(u + 7\pi/4), \sin(u + 7\pi/4))$, respectively see Figure 4 right.

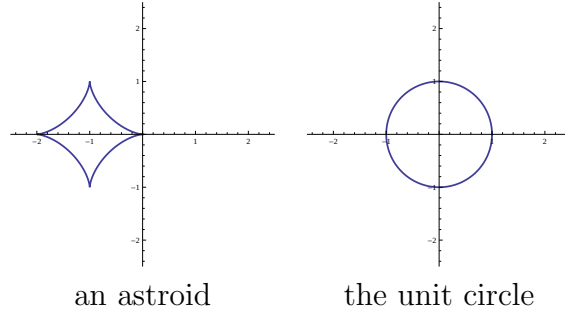


Figure 3.

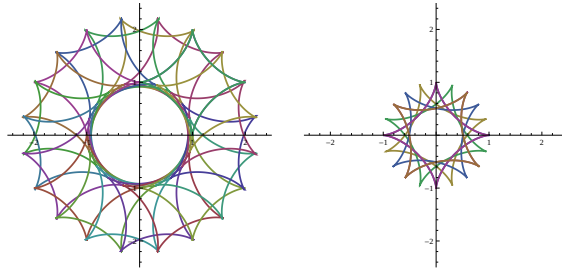


Figure 4.

5 Bi-Legendre curves and envelopes

We consider a special class of one-parameter families of Legendre curves. Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a smooth mapping.

Definition 5.1 We say that $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ is a *bi-Legendre curve* if $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$ and $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in I \times \Lambda$.

Then (γ, ν) is a one-parameter family of Legendre curves with respect to both parameters t and λ . We define $\boldsymbol{\mu}(t, \lambda) = J(\nu(t, \lambda))$. Since $\{\nu(t, \lambda), \boldsymbol{\mu}(t, \lambda)\}$ is a moving frame along $\gamma(t, \lambda)$, we have the Frenet type formula.

$$\begin{aligned} \begin{pmatrix} \nu_t(t, \lambda) \\ \boldsymbol{\mu}_t(t, \lambda) \end{pmatrix} &= \begin{pmatrix} 0 & \ell(t, \lambda) \\ -\ell(t, \lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t, \lambda) \\ \boldsymbol{\mu}(t, \lambda) \end{pmatrix}, \\ \begin{pmatrix} \nu_\lambda(t, \lambda) \\ \boldsymbol{\mu}_\lambda(t, \lambda) \end{pmatrix} &= \begin{pmatrix} 0 & m(t, \lambda) \\ -m(t, \lambda) & 0 \end{pmatrix} \begin{pmatrix} \nu(t, \lambda) \\ \boldsymbol{\mu}(t, \lambda) \end{pmatrix}, \\ \gamma_t(t, \lambda) &= \beta(t, \lambda)\boldsymbol{\mu}(t, \lambda), \\ \gamma_\lambda(t, \lambda) &= \alpha(t, \lambda)\boldsymbol{\mu}(t, \lambda), \end{aligned}$$

where

$$\begin{aligned} \ell(t, \lambda) &= \nu_t(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda), \quad m(t, \lambda) = \nu_\lambda(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda), \\ \beta(t, \lambda) &= \gamma_t(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda), \quad \alpha(t, \lambda) = \gamma_\lambda(t, \lambda) \cdot \boldsymbol{\mu}(t, \lambda). \end{aligned}$$

By the integrability conditions $\nu_{t\lambda}(t, \lambda) = \nu_{\lambda t}(t, \lambda)$, $\gamma_{t\lambda}(t, \lambda) = \gamma_{\lambda t}(t, \lambda)$, ℓ, m, β, α satisfies the conditions

$$\ell_\lambda(t, \lambda) = m_t(t, \lambda), \beta_\lambda(t, \lambda) = \alpha_t(t, \lambda), \ell(t, \lambda)\alpha(t, \lambda) = m(t, \lambda)\beta(t, \lambda) \quad (4)$$

for all $(t, \lambda) \in I \times \Lambda$. We call the pair (ℓ, m, β, α) with the integrability conditions (4) a *curvature of the bi-Legendre curve* (γ, ν) .

Definition 5.2 Let (γ, ν) and $(\tilde{\gamma}, \tilde{\nu}) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be bi-Legendre curves. We say that (γ, ν) and $(\tilde{\gamma}, \tilde{\nu})$ are *congruent as bi-Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a translation $\mathbf{a}$ on $\mathbb{R}^2$ such that $\tilde{\gamma}(t, \lambda) = A(\gamma(t, \lambda)) + \mathbf{a}$ and $\tilde{\nu}(t, \lambda) = A(\nu(t, \lambda))$ for all $(t, \lambda) \in I \times \Lambda$.

Theorem 5.3 (The Existence Theorem for bi-Legendre curves.) *Let $(\ell, m, \beta, \alpha) : I \times \Lambda \rightarrow \mathbb{R}^4$ be a smooth mapping with the integrability conditions. There exists a bi-Legendre curve $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ whose associated curvature is (ℓ, m, β, α) .*

Proof. Let $(t_0, \lambda_0) \in I \times \Lambda$ be fixed. We define a smooth mapping $\theta : I \times \Lambda \rightarrow \mathbb{R}$ by

$$\theta(t, \lambda) = \int_{t_0}^t \ell(t, \lambda) dt + \int_{\lambda_0}^\lambda m(t_0, \lambda) d\lambda.$$

Then θ satisfy the conditions $\theta_t(t, \lambda) = \ell(t, \lambda)$ and $\theta_\lambda(t, \lambda) = m(t, \lambda)$. We also define a smooth mapping $(x, y) : I \times \Lambda \rightarrow \mathbb{R}^2$ by

$$\begin{aligned} x(t, \lambda) &= - \int_{t_0}^t \beta(t, \lambda) \sin \theta(t, \lambda) dt - \int_{\lambda_0}^\lambda \alpha(t_0, \lambda) \sin \theta(t_0, \lambda) d\lambda \\ y(t, \lambda) &= \int_{t_0}^t \beta(t, \lambda) \cos \theta(t, \lambda) dt + \int_{\lambda_0}^\lambda \alpha(t_0, \lambda) \cos \theta(t_0, \lambda) d\lambda. \end{aligned}$$

By the integrability condition (4), we have

$$x_t(t, \lambda) = -\beta(t, \lambda) \sin \theta(t, \lambda), \quad x_\lambda(t, \lambda) = -\alpha(t, \lambda) \sin \theta(t, \lambda),$$

$$y_t(t, \lambda) = \beta(t, \lambda) \cos \theta(t, \lambda), y_\lambda(t, \lambda) = \alpha(t, \lambda) \cos \theta(t, \lambda).$$

We define a smooth mapping $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ by

$$\gamma(t, \lambda) = (x(t, \lambda), y(t, \lambda)), \nu(t, \lambda) = (\cos \theta(t, \lambda), \sin \theta(t, \lambda)).$$

By a direct calculation, (γ, ν) is a bi-Legendre curve with the curvature (ℓ, m, β, α) . $\square$

Theorem 5.4 (The Uniqueness Theorem for bi-Legendre curves.) *Let (γ, ν) and $(\tilde{\gamma}, \tilde{\nu}) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be bi-Legendre curves with the curvatures (ℓ, m, β, α) and $(\tilde{\ell}, \tilde{m}, \tilde{\beta}, \tilde{\alpha})$ respectively. Then (γ, ν) and $(\tilde{\gamma}, \tilde{\nu})$ are congruent as bi-Legendre curves if and only if (ℓ, m, β, α) and $(\tilde{\ell}, \tilde{m}, \tilde{\beta}, \tilde{\alpha})$ coincides.*

Proof. Suppose that (γ, ν) and $(\tilde{\gamma}, \tilde{\nu})$ are congruent as bi-Legendre curves. By a direct calculation, we have

$$\begin{aligned} \tilde{\gamma}_t(t, \lambda) &= \frac{\partial}{\partial t}(A(\gamma(t, \lambda)) + \mathbf{a}) = A(\gamma_t(t, \lambda)) = \beta(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = \beta(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda), \\ \tilde{\gamma}_\lambda(t, \lambda) &= \frac{\partial}{\partial \lambda}(A(\gamma(t, \lambda)) + \mathbf{a}) = A(\gamma_\lambda(t, \lambda)) = \alpha(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = \alpha(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda), \\ \tilde{\nu}_t(t, \lambda) &= \frac{\partial}{\partial t}A(\nu(t, \lambda)) = A(\nu_t(t, \lambda)) = \ell(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = \ell(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda), \\ \tilde{\nu}_\lambda(t, \lambda) &= \frac{\partial}{\partial \lambda}A(\nu(t, \lambda)) = A(\nu_\lambda(t, \lambda)) = m(t, \lambda)A(\boldsymbol{\mu}(t, \lambda)) = m(t, \lambda)\tilde{\boldsymbol{\mu}}(t, \lambda). \end{aligned}$$

Therefore the curvatures (ℓ, m, β, α) and $(\tilde{\ell}, \tilde{m}, \tilde{\beta}, \tilde{\alpha})$ coincides.

Conversely, suppose that (ℓ, m, β, α) and $(\tilde{\ell}, \tilde{m}, \tilde{\beta}, \tilde{\alpha})$ coincides. Let $(t_0, \lambda_0) \in I \times \Lambda$ be fixed. By using a congruence as bi-Legendre curves, $\gamma(t_0, \lambda_0) = \tilde{\gamma}(t_0, \lambda_0)$ and $\nu(t_0, \lambda_0) = \tilde{\nu}(t_0, \lambda_0)$. Moreover, we have $\theta(t, \lambda) = \tilde{\theta}(t, \lambda)$ in the proof of Theorem 5.3. It follows from the construction that $\nu(t, \lambda) = \tilde{\nu}(t, \lambda)$ and $\gamma(t, \lambda) = \tilde{\gamma}(t, \lambda)$ for all $(t, \lambda) \in I \times \Lambda$. $\square$

Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a bi-Legendre curve. Then (γ, ν) is a one-parameter family of Legendre curves with respect to the parameter λ . We denote a smooth map $e_L : U \rightarrow I \times \Lambda$, $e_L(u) = (t(u), \lambda(u))$. Since $\gamma_\lambda(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in I \times \Lambda$, we have $\gamma_\lambda(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. If the function λ is non-constant on any non-trivial subinterval of U , then $E_L = \gamma \circ e_L$ is an envelope of (γ, ν) with respect to the parameter λ by Theorem 4.5. Moreover, (γ, ν) is also a one-parameter family of Legendre curves with respect to the parameter t . Since $\gamma_t(t, \lambda) \cdot \nu(t, \lambda) = 0$ for all $(t, \lambda) \in I \times \Lambda$, we have $\gamma_t(e_L(u)) \cdot \nu(e_L(u)) = 0$ for all $u \in U$. If the function t is non-constant on any non-trivial subinterval of U , then $E_L = \gamma \circ e_L$ is an envelope of (γ, ν) with respect to the parameter t by Theorem 4.5. Summary we have the following result.

Proposition 5.5 *Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a bi-Legendre curve. If $e_L : U \rightarrow I \times \Lambda$, $e_L(u) = (t(u), \lambda(u))$ satisfy the conditions that the functions t and λ are non-constant on any non-trivial subinterval of U , then $E_L = \gamma \circ e_L$ is an envelope of (γ, ν) with respect to the both parameter t and λ respectively.*

Let $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$ be a bi-Legendre curve. Since $\gamma_t(t, \lambda) = \beta(t, \lambda)\boldsymbol{\mu}(t, \lambda)$ and $\gamma_\lambda(t, \lambda) = \alpha(t, \lambda)\boldsymbol{\mu}(t, \lambda)$, we have $\det(\gamma_t(t, \lambda), \gamma_\lambda(t, \lambda)) = 0$ for all $(t, \lambda) \in I \times \Lambda$. It follows that for any $(t, \lambda) \in I \times \Lambda$ are singular points of $\gamma : I \times \Lambda \rightarrow \mathbb{R}^2$. Hence, at a rank 1 point, the image of γ is a curve at least locally. We give a concrete example of bi-Legendre curves.

Example 5.6 Let k, n be natural numbers. We define $(\ell, m, \beta, \alpha) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^4$ by $\ell(t, \lambda) = \lambda^{k+1}t^k$, $m(t, \lambda) = \lambda^k t^{k+1}$, $\beta(t, \lambda) = \lambda^{n+1}t^n$, $\alpha(t, \lambda) = \lambda^n t^{n+1}$. Then the integrability conditions $\ell_\lambda(t, \lambda) = m_t(t, \lambda)$, $\beta_\lambda(t, \lambda) = \alpha_t(t, \lambda)$, $\alpha(t, \lambda)\ell(t, \lambda) = \beta(t, \lambda)m(t, \lambda)$ hold for all $(t, \lambda) \in \mathbb{R} \times \mathbb{R}$. It follows that $\theta(t, \lambda) = \lambda^{k+1}t^{k+1}/(k+1)$. By the construction in the proof of Theorem 5.3, we give a bi-Legendre curve $(\gamma, \nu) : I \times \Lambda \rightarrow \mathbb{R}^2 \times S^1$,

$$\begin{aligned}\gamma(t, \lambda) &= (x(t, \lambda), y(t, \lambda)) = \left(-\int_0^t \lambda^{n+1}t^n \sin\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) dt, \int_0^\lambda \lambda^{n+1}t^n \cos\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) dt \right), \\ \nu(t, \lambda) &= (\cos \theta(t, \lambda), \sin \theta(t, \lambda)) = \left(\cos\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right), \sin\left(\frac{\lambda^{k+1}t^{k+1}}{k+1}\right) \right).\end{aligned}$$

A Legendre curves in the unit tangent bundle

We quickly review on the theory of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, see detail [8]. We say that $(\gamma, \nu) : I \rightarrow \mathbb{R}^2 \times S^1$ is a *Legendre curve* if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$ (cf. [1, 2]). This condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. We say that $\gamma : I \rightarrow \mathbb{R}^2$ is a *frontal* if there exists $\nu : I \rightarrow S^1$ such that (γ, ν) is a Legendre curve. Examples of Legendre curves see [13, 14]. We have the Frenet formula of a frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. Then we call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame of a frontal* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of a frontal (or, Legendre curve),

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \quad \dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t),$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$ and $\beta(t) = \dot{\gamma}(t) \cdot \boldsymbol{\mu}(t)$. We call the pair (ℓ, β) the *curvature of the Legendre curve*.

Definition A.1 Let (γ, ν) and $(\tilde{\gamma}, \tilde{\nu}) : I \rightarrow \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\tilde{\gamma}, \tilde{\nu})$ are *congruent as Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a translation $\mathbf{a}$ on $\mathbb{R}^2$ such that $\tilde{\gamma}(t) = A(\gamma(t)) + \mathbf{a}$ and $\tilde{\nu}(t) = A(\nu(t))$ for all $t \in I$.

Theorem A.2 (The Existence Theorem for Legendre curves.) *Let $(\ell, \beta) : I \rightarrow \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \rightarrow \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ, β) .*

Theorem A.3 (The Uniqueness Theorem for Legendre curves.) *Let (γ, ν) and $(\tilde{\gamma}, \tilde{\nu}) : I \rightarrow \mathbb{R}^2 \times S^1$ be Legendre curves with the curvatures of Legendre curves (ℓ, β) and $(\tilde{\ell}, \tilde{\beta})$. Then (γ, ν) and $(\tilde{\gamma}, \tilde{\nu})$ are congruent as Legendre curves if and only if (ℓ, β) and $(\tilde{\ell}, \tilde{\beta})$ coincides.*

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