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EVOLUTION OF SPIRAL-SHAPED POLYGONAL CURVE BY CRYSTALLINE CURVATURE FLOW WITH A PINNED TIP*

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Abstract. Evolution of convex polygonal spiral with fixed center by crystalline eikonal-curvature flow is considered. In this evolution we consider a new facet of the polygonal curve generates from center when a facet associated with the center evolves with enough length, which is equal to the length of the facet in Wulff shape of energy density function. We prove the existence, uniqueness and intersection free of solution to our formulation globally-in-time. In the proof of the existence we also prove that new facets are generated repeatedly in time. The important property for intersection-free result is monotonicity property such that the normal velocity of every facets are positive after the next new facet is generated, so that the center is always behind of the moving facets.

Key words. Crystalline eikonal-curvature flow, Evolution of convex polygonal spiral

AMS subject classifications. 34A34, 53A04, 53C44

1. Introduction. Burton, Cabrera and Frank [1] proposed a theory of crystal growth with aid of screw dislocations in 1951. According to the theory, a monomolecular step is provided by a screw dislocation across with the crystal surface. Atoms on the surface are caught by kinks, which is a corner of atoms in the step, with a heigher probability when they are close to the steps, and then results in an evolution of steps. The dynamics of steps in this setting is given as

$$V = U - H \tag{1}$$

in [1], where V and H, respectively, are the normal velocity and the curvature of the curve drawn by the steps, and U is a constant denoting the driving force of the evolution. Note that the directions of V and H are inverse, so that the above equation should be a parabolic type equation. There is a nice review paper [2] on its mathematical modelling as well as computational methods.

One often can find a polygonal spiral steps on the growing crystal surface: see e.g. [21]. It is caused by the anisotropy of the surface energy by the geometry of the structure of atoms. Anisotropic surface energy of a curve S should be given by

$$E_\gamma(S) = \int_S \gamma(\mathbf{n}) \mathrm{d}\sigma \tag{2}$$

by a density function $\gamma\colon S^1 \to (0,\infty)$, where $\mathrm{d}\sigma$ is the line element. Then, we obtain the weighted curvature H_γ of curve S with surface energy density γ as the first variation of E_γ, i.e.,

$$H_\gamma(\mathbf{n}, \nabla\mathbf{n}) = \frac{\delta E_\gamma}{\delta S}(S).$$

Moreover, we consider the mobility of the evolution reflects the anisotropy of the lattice of atoms as the coefficient in front of the normal velocity. Then, we obtain the

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generalized evolution equation

$$\beta(\mathbf{n})V = U - H_\gamma(\mathbf{n}, \nabla\mathbf{n}) \tag{3}$$

by anisotropic curvature and velocity. The equation (1) is regarded as the above equation with $\gamma \equiv 1$ and $\beta \equiv 1$. When the polygonal curve appears provided that the curve S evolves by (3), then its stationary solution, called "Wulff shape",

$$\mathcal{W}_\gamma = \{\mathbf{x} \in \mathbb{R}^2;\ \mathbf{x}\cdot\mathbf{n} \le \gamma(\mathbf{n}) \quad \text{for } \mathbf{n} \in S^1\} \tag{4}$$

should be a polygon, where $\mathbf{x}\cdot\mathbf{y}$ denotes the usual inner product for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. However, it is well-known that γ is not only not smooth, but also possibly not convex even if $\mathcal{W}_\gamma$ is a convex polygon. At least it is also well-known that piecewise linear γ implies convex polygonal $\mathcal{W}_\gamma$. We call the energy (2) having convex polygonal $\mathcal{W}_\gamma$ as crystalline surface energy, and the motion of polygonal curves by crystalline surface energy as crystalline motion. In this paper, we are interested in evolution of spirals with crystalline surface energy.

If γ is smooth and convex, then several PDE approachs formulating interface evolution equation with (3) are proposed. There is a nice review book [5] for the theory of interface evolution equation with PDE approach. However, because of the non-smoothness or non-convexity of γ, a lot of PDE approach tracking the evolution of curves does not work well for the crystalline motion. Taylor [22] proposed an ODE approach tracking the evolution of polygonal curve with (3) by a system of ordinary differential equation on the length of each facet. In the theory by [22] we introduce the crystalline curvature as the ratio of the evolving facets and that in $\mathcal{W}_\gamma$ with the same orientation. Then, we calculate the extending speed of the facets from the displacement of the evolving facets by (3). Then, we describe the evolution of curves by the length and tangential direction of facets of the evolving curves. There is a good book [8] for details of the ODE approach to the crystalline motion by interfacial curves. Mathematical analysis for the crystalline motion of interfacial curves has been done well. Ushijima, Yagishita, Yazaki and the first author [14] proved the existence of non-convex self-similar solution to the crystalline curvature flow, which violates the convexity phenomena as in the isotropic curvature flow by [7]. Then, the first author [10] classified the motion of closed polygonal curves by the crystalline curvature flow. For the crystalline eikonal-curvature flow, the first author [12, 11] investigated the behavior of V-shaped solution. On one hand there is a few works on level set approach [4, 6] for evolution of polygonal curves or polyhedral surfaces by crystalline curvature flow.

When we consider evolution of polygonal spirals with (3) with $U \neq 0$, then we are faced to the following characteristic problem:

- Does the center of the spiral move or not?
- Generation of new facets from the center of the spiral.

As the simple example, we now consider the situation such that a straight line with infinite length evolves by (3) with $U > 0$. If the center moves associating with the facet, then the "spiral" does not evolve since the line just move to the normal direction. Then, for the "evolution of spiral", one can find that the center should stay around the initial location with generating new facets so that the polygonal curve forms a spiral (see Figure 1 for the illustration of the generation of new facets). Imai, Ishimura and Ushijima [9] proposed a formulation for evolving polygonal spirals with the idea by [22] without generation of new facets. However, they investigate the evolution of curves only with $U = 0$, then they are concerned on the extinction of facets. On the

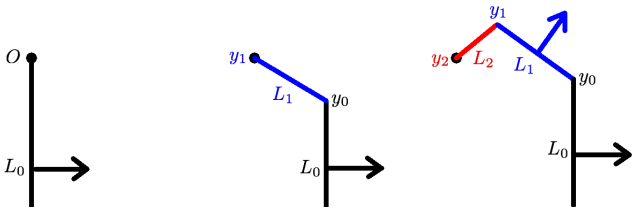

FIG. 1. *Evolution of polygonal spiral. In the above figures, L_j and y_j denotes the facets and vertices of the polygonal spiral.*

other hand, the first author [13] proposed the evolution of spirals with generation of new facets. In this theory the center of spiral is moved on a pre-determined polygonal trajectory of the center corresponding with $\mathcal{W}_\gamma$, and then a new facet is generated when the center go through the corner on the trajectory.

In this paper, we propose a new formulation of evolving polygonal spiral by (3) with pinned center at the origin. In our algorithm a new facet should be generated when the facet associated with the center has enough long length for the facet of $\mathcal{W}_\gamma$ which has the same direction as the evolving facets. Then, as mathematical analysis of our scheme, we consider not only the existence and uniqueness of solutions to the system of facet length, but also the existence of an infinite time sequence $\{T_n\}$ at when a new facet generates. Moreover, we show an important property for evolution of spiral: $\lim_{n\to\infty} T_n = \infty$ under a suitable setting of admissible initial curve. We also prove that the evolving curve has no self-intersections. As results in this paper, a polygonal spiral generates new facets repeatedly and then spiral steadily grows.

If the evolution is isotropic or anisotropic with smooth and convex density, then we have several PDE approach for evolution of spirals. It is well-known that phase-field method or level set method is powerful option to describe the motion of interfacial curve by (3). Then, there are several developments to apply their method to the evolution of spirals. Karma–Plapp [15], Kobayashi [16], and Miura–Kobayashi [17] proposed a formulation of evolving spirals by phase-field method with multiple-well potential and a pre-determined function (called sheet structure function) reflecting the sheet structure of the crystal lattice. On the other hand, Smereka [20] proposed a level set method with two auxiliary functions. The second author also proposed a level set formulation with a single auxiliary function and the sheet structure function due to [16] for single or several evolving spirals in [18] or [19]. However, the formulations in [15, 16, 18, 19, 17] requires to remove an open neighborhood of each center from the domain, since their equation have strong singularity at each center. In other words, their methods regard the centers of spirals as the open neighborhoods. Forcadel, Imbert and Monneau [3] give a formulation for a single evolving spiral with a pinned center as the origin.

This paper is organized as follows. We first prepare some notations and definitions in §2. As the new feature, we divide the definition of admissibility as in [13] into semi-admissibility (only the continuity of the direction of facets), and admissibility (semi-admissibility with intersection free). We also define the generation time and the rule of the generation of new facet. Then, we introduce a scheme of the evolution of spirals, and define its semi-solution (solution of the length system with semi-admissibility) and solution (the solution with admissibility).

According to the definition defined in §2, we prove the existence and uniqueness of solutions of our scheme in §3. We divide the proof into two parts. The first part

is the existence and uniqueness of solution to the length system, which is presented in §3.1. In the proof of existence, the continuity of the direction of facets are derived automatically. The second part is the intersection free property. However, the rigorous proof of the intersection-free property is established with just a precise calculation of inner products of some vector investigating the direction of facets, or detection of interior or exterior of a domain whose boundary is hyperplane. Thus, we just present an idea of the proof in §3.2. The rigorous proofs are presented in Appendix (§5.2).

We also present some numerical results in §4. In §5, we mention on some remarks of our scheme. In §5.1 we mention on how to determine the direction of new facets. In §5.2, we give some rigorous proof of some properties on intersection.

2. Mathematical formulation.

2.1. Preliminaries.

We prepare some notations and assumptions.

We now recall a Wulff shape of an anisotropic surface energy. Let $\mathcal{W}_\gamma$ be a set defined by (4), which is called Wulff shape of the surface energy defined by (2) with a continuous density function $\gamma\colon S^1 \to (0,\infty)$.

We here impose that $\mathcal{W}_\gamma$ is a N_γ sided convex polygon. The j-th facet of $\mathcal{W}_\gamma$ has an outer unit normal vector $\mathbf{N}_j$ with angle φ_j for $j \in \mathbb{Z}/(N_\gamma\mathbb{Z})$, and set the unit tangential vector $\mathbf{T}_j$ of the j-th facet as well as the definition of the Frenet frame, i.e.,

$$\mathbf{N}_j = (\cos\varphi_j, \sin\varphi_j), \quad \mathbf{T}_j = (\sin\varphi_j, -\cos\varphi_j).$$

We here consider a generalized number of facets $j \in \mathbb{Z}/(N_\gamma\mathbb{Z})$ i.e., we regard $j + nN_\gamma$ as j for every $n \in \mathbb{Z}$. From the convexity of $\mathcal{W}_\gamma$ we assume that

(W1) $0 = \varphi_0 < \varphi_1 < \varphi_2 < \cdots < \varphi_{N_\gamma - 1} < 2\pi$,

(W2) $(\varphi_j <)\varphi_{j+1} < \varphi_j + \pi$ for every $j \in \mathbb{Z}/(N_\gamma\mathbb{Z})$.

We denote the length of the j-th facet of $\mathcal{W}_\gamma$ by $\ell_j > 0$.

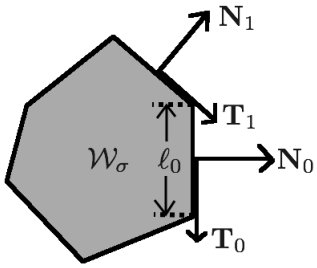

FIG. 2. *Notations to the Wulff shape $\mathcal{W}_\gamma$.*

We next prepare some notations for an evolving spiral-shaped polygonal curve by a crystalline curvature flow with respect to $\mathcal{W}_\gamma$. Let $\Gamma(t)$ be a piecewise linear curve which has $k+1$ facets denoted by $L_j(t)$ for $j = 0, 1, 2, \ldots, k$; set $\Gamma(t) = \bigcup_{j=0}^{k} L_j(t)$. The j-th facet $L_j(t)$ is given as

$$L_j(t) = \begin{cases} \{\lambda y_j(t) + (1-\lambda) y_{j-1}(t);\ \lambda \in [0,1]\} & \text{for } j = k, k-1, \ldots, 1, \\ \{y_0(t) + \lambda \boldsymbol{\tau}_0;\ \lambda \geq 0\} & \text{if } j = 0 \end{cases} \tag{5}$$

with the center of $\Gamma(t)$ denoted by $y_k(t)$, vertices denoted by $y_j(t)$ for $j = 0, 1, \ldots, k-1$

of $\Gamma(t)$, and a given unit tangential vector $\boldsymbol{\tau}_0 \in S^1$ of $L_0(t)$. Let

$$\boldsymbol{\tau}_j = \frac{y_{j-1}(t) - y_j(t)}{|y_{j-1}(t) - y_j(t)|}$$

be the unit tangential vector of $L_j(t)$ $j = 1, 2, \ldots, k$. We now impose that, for $j = 0, 1, 2, \ldots, k$, there exists $\nu(j) \in \mathbb{Z}/(N_\gamma \mathbb{Z})$ such that

$$\boldsymbol{\tau}_j = \mathbf{T}_{\nu(j)}. \tag{6}$$

We call $\nu(j)$ on the above a corresponding number of j-th facet to $\mathcal{W}_\gamma$. Thus, we now remark that $L_j(t)$ has other description as

$$L_j(t) = \{y_j(t) + r\mathbf{T}_{\nu(j)};\ r \in [0, d_j(t)]\} \tag{7}$$

with $d_j(t) = |y_j(t) - y_{j-1}(t)|$.

We denote the direction of the evolution of $\Gamma(t)$ by the unit normal vector $\mathbf{n}_j \in S^1$ of $L_j(t)$ for $j = 0, 1, 2, \ldots, k$. For each curve $\Gamma(t)$ we set an orientation coefficient $\alpha \in \{\pm 1\}$ such that

$$\mathbf{n}_j = \alpha \mathbf{N}_{\nu(j)} \tag{8}$$

for $j = 0, 1, 2, \ldots, k$ provided that (6) holds. From the above context we have

$$V_j(t) = \dot{s}_j(t)$$

for $s_j(t) = y_j(t) \cdot \mathbf{n}_j$, where $\dot{s}_j$ denotes the time derivative of $s_j(t)$. The function s_j is called *support function* of $L_j(t)$. Note that $L_j(t) \subset \{x \in \mathbb{R}^2;\ x \cdot \mathbf{n}_j = s_j(t)\}$.

Evolving spiral has a rotational orientation with respect to the center provided that its normal velocity is positive. See [19] for the definition of rotational orientations on the evolving smooth curve. We now introduce them to the evolving polygonal curve. We also introduce some geometric properties of $\Gamma(t)$. See also [13].

DEFINITION 1. *(i) We say $\Gamma(t) = \bigcup_{j=0}^{k} L_j(t)$ is an oriented spiral if there exists $\alpha \in \{\pm 1\}$ satisfying (8) for $j = 0, 1, 2, \ldots, k$. We also say $\Gamma(t)$ is a positive (resp. negative) spiral if $\alpha = 1$ (resp. $\alpha = -1$).*

(ii) Let $\Gamma(t) = \bigcup_{j=0}^{k} L_j(t)$ be an oriented spiral. We say $\Gamma(t)$ is admissible with respect to $\mathcal{W}_\gamma$ if the followings hold:

(A1) $d_j(t) = |y_j(t) - y_{j-1}(t)| > 0$ for $j = 1, 2, \ldots, k-1$,

(A2) $\mathbf{n}_{j\pm 1} \in \{\alpha \mathbf{N}_{\nu(j)-1}, \alpha \mathbf{N}_{\nu(j)+1}\}$,

(A3) $\Gamma(t)$ has no self intersections.

We also say $\Gamma(t)$ is semi-admissible if (A1) and (A2) hold.

(iii) Let $\Gamma(t)$ be an admissible with respect to $\mathcal{W}_\gamma$. We now call $L_j(t)$ for $j = 1, 2, \ldots, k-1$ an intermediate facet. We say an intermediate facet $L_j(t)$ is convex (resp. concave) if

$$\mathbf{n}_{j-1} = \alpha \mathbf{N}_{\nu(j)-\alpha},\ \text{and}\ \mathbf{n}_{j+1} = \alpha \mathbf{N}_{\nu(j)+\alpha}$$
$$(\text{resp. } \mathbf{n}_{j-1} = \alpha \mathbf{N}_{\nu(j)+\alpha}, \text{and } \mathbf{n}_{j+1} = \alpha \mathbf{N}_{\nu(j)-\alpha}).$$

We say an admissible spiral $\Gamma(t)$ is a convex (resp. concave) if the all intermediate facets of Γ_t are convex (resp. concave), i.e., $\mathbf{n}_j = \alpha \mathbf{N}_{\nu(0)+\alpha j}$ (resp. $\mathbf{n}_j = \alpha \mathbf{N}_{\nu(0)-\alpha j}$) holds for for $j = 0, 1, 2, \ldots, k$.

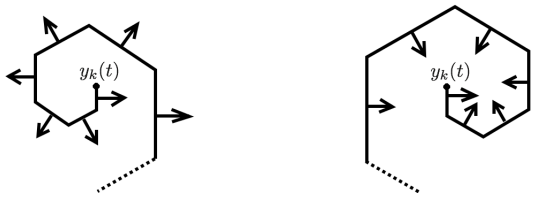

FIG. 3. *Example of positive convex (left) and positive concave(right) spiral for hexagonal* $\mathcal{W}_\gamma$.

See Figure 3 for examples of the positive convex and concave spirals.

We now introduce an crystalline curvature to an intermediate facet of an admissible spiral as well as [13]. See also [8] or [22].

DEFINITION 2. *Let* $\Gamma(t) = \bigcup_{j=0}^{k} L_j(t)$ *be an admissible spiral with respect to* $\mathcal{W}_\gamma$, *and oriented with the coefficient* $\alpha \in \{\pm 1\}$, *and assume that* $\mathbf{n}_j = \alpha \mathbf{N}_{\nu(j)}$ *with* $\nu(j) \in \mathbb{Z}/(N_\gamma \mathbb{Z})$. *We define the crystalline curvature* H_j *of an intermediate facet* $L_j(t)$ *of* $\Gamma_j(t)$ *with respect to* $\mathcal{W}_\gamma$ *as*

$$H_j = \sigma_j \frac{\ell_{\nu(j)}}{d_j},$$

$$\sigma_j = \begin{cases} 1 & \text{if } L_j \text{ is convex}, \\ -1 & \text{if } L_j \text{ is concave}, \\ 0 & \text{otherwise}, \end{cases}$$

where $d_j = |y_j - y_{j-1}|$ *is the length of* L_j, *and* $\ell_{\nu(j)}$ *is the length of the* $\nu(j)$*-th facet of* $\mathcal{W}_\gamma$ *for* $j \in \mathbb{Z}/(N_\gamma \mathbb{Z})$.

2.2. Evolution system and scheme. Let $\Gamma(t) = \bigcup_{j=0}^{k} L_j(t)$ be an admissible spiral evolve with the normal velocity V_j for $j = 0, 1, 2, \ldots, k$. We now introduce a scheme of the evolution of $\Gamma(t)$ by (3) with a crystalline surface energy and the pinned center at the origin, and a rule of the generation of a new facet at the center of $\Gamma(t)$. Note that each $L_j(t)$ is given as (5), and then the center of $\Gamma(t)$ is $y_k(t)$, which is fixed at the origin. Then, we impose that

$$V_k = 0. \tag{9}$$

This condition should be taken over to a new facet L_{k+1} when it is generated. We now consider only the evolution of a positive spiral to avoid complication of formulation, then we set $\alpha = 1$. We also may assume that $U > 0$. (The case when $\alpha = -1$ or $U < 0$ will be mentioned later.) We shall use the same hypothesis on §2.1, then we observe that

$$\mathbf{n}_j = \mathbf{N}_{\nu(j)}, \quad \tau_j = \mathbf{T}_{\nu(j)} \quad \text{for } j = 0, 1, 2, \ldots, k$$

with the corresponding facet number $\nu(j)$ defined in (6) for $j = 0, 1, 2, \ldots, k$.

We first derive the evolution system of $\Gamma(t)$ by (3) with a crystalline surface energy. According to [8], the evolution of $\Gamma(t)$ with the normal velocity V_j of $L_j(t)$ is

described by

$$\begin{aligned}\dot{d}_j(t) &= \frac{1}{\sin(\theta_{j+1}-\theta_j)}V_{j+1} - \frac{1}{\tan(\theta_{j+1}-\theta_j)}V_j \\ &\quad + \frac{1}{\sin(\theta_j-\theta_{j-1})}V_{j-1} - \frac{1}{\tan(\theta_j-\theta_{j-1})}V_j \\ &= -\left(\frac{1}{\tan(\theta_{j+1}-\theta_j)} + \frac{1}{\tan(\theta_j-\theta_{j-1})}\right)V_j \\ &\quad + \frac{1}{\sin(\theta_{j+1}-\theta_j)}V_{j+1} + \frac{1}{\sin(\theta_j-\theta_{j-1})}V_{j-1},\end{aligned}$$

where $d_j(t) = |y_j(t) - y_{j-1}(t)|$ (See also the Figure 4). If $\Gamma(t)$ evolves by (3) with the

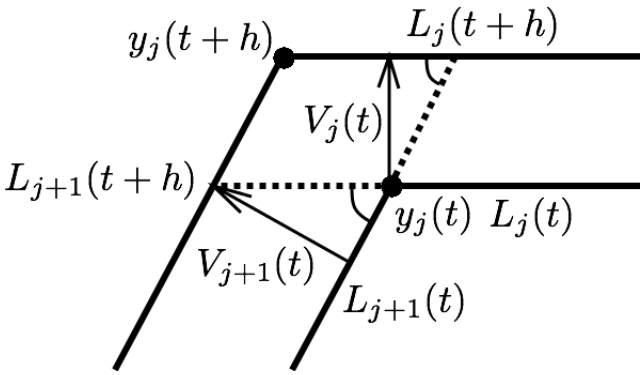

FIG. 4. *Evolution of polygonal curves. The angle of the corners with a round curve is* $\theta_{j+1} - \theta_j$.

crystalline energy, then V_j is given as

$$\beta_j V_j = U - H_j \quad \text{for } j = 1, 2, \dots, k-1, \tag{10}$$

where $\beta_j = \beta(\mathbf{n}_j)$ and H_j is the crystalline curvature with respect to $\mathcal{W}_\gamma$ defined in Definition 2. Note that $L_0(t)$ has infinite length as in (5), and then we regard $H_0 = 0$. By combining above and (9) we now obtain the system of the ordinary differential equations on the length $d_j(t)$ of $L_j(t)$;

$$\begin{cases} \dot{d}_k = c_k^- \left(U - \sigma_{k-1}\dfrac{\ell_{\nu(k-1)}}{d_{k-1}}\right), \\ \dot{d}_{k-1} = -b_{k-1}\left(U - \sigma_{k-1}\dfrac{\ell_{\nu(k-1)}}{d_{k-1}}\right) + c_{k-1}^-\left(U - \sigma_{k-2}\dfrac{\ell_{\nu(k-2)}}{d_{k-2}}\right), \\ \dot{d}_j = -b_j\left(U - \sigma_j\dfrac{\ell_{\nu(j)}}{d_j}\right) \\ \qquad\qquad + c_j^+\left(U - \sigma_{j+1}\dfrac{\ell_{\nu(j+1)}}{d_{j+1}}\right) + c_j^-\left(U - \sigma_{j-1}\dfrac{\ell_{\nu(j-1)}}{d_{j-1}}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } j = 2, \dots, k-2, \\ \dot{d}_1 = -b_1\left(U - \sigma_1\dfrac{\ell_{\nu(1)}}{d_1}\right) + c_1^+\left(U - \sigma_2\dfrac{\ell_{\nu(2)}}{d_2}\right) + c_1^- U, \end{cases} \tag{11}$$

where $b_j, c_j^{\pm} \in \mathbb{R}$ are constants given by

$$b_j = \frac{1}{\beta_j}\left(\frac{1}{\tan(\theta_{j+1}-\theta_j)} + \frac{1}{\tan(\theta_j-\theta_{j-1})}\right), \quad c_j^{\pm} = \pm\frac{1}{\beta_{j\pm1}\sin(\theta_{j\pm1}-\theta_j)}$$

for $j = 0, 1, 2, \dots, k$. We solve the system (11) at least where $d_j > 0$ for $j = 1, 2, \dots, k-1$, and describe $\Gamma(t)$ with setting

$$y_{j-1}(t) = y_j(t) + d_j(t)\boldsymbol{\tau}_j \quad \text{for } j = k, k-1, \dots, 1 \tag{12}$$

to obtain the time-local evolution of $\Gamma(t)$. Note that first formula in (11) is formally out of the system. However, d_k has a role deciding a time of the generation of a new facet explaned below.

For the evolution of a "spiral" generation of a new facet at the center of $\Gamma(t)$ is necessary. In this paper we give a rule of the generation as resultant of the evolution of the present facet associated with the center. Let $T_k \in \mathbb{R}$ be a time when $L_k(t)$ is generated, i.e., $d_k(T_k) = 0$ and $\tau_k = \mathbf{T}_{\nu(k)}$ is given. We define a *generation time* of $L_{k+1}(t)$ as

$$T_{k+1} = \sup\{T > T_k;\ d_k(t) < \ell_{\nu(k)}/U \quad \text{for } t \in [T_k, T]\}. \tag{13}$$

(Set $T_{k+1} = \sup\{T > T_k;\ d_k(t) < -\ell_{\nu(k)}/U \quad \text{for } t \in [T_k, T]\}$ if $U < 0$.) We summarize the generation rule of a new facet as follows.

Generation rule(G) When $t = T_{k+1} < \infty$, we add a new vertex $y_{k+1}(t)$ fixed at the origin and a new facet $L_{k+1}(t)$ to $\Gamma(t)$, and set $y_k(t)$ by (12) with $j = k+1$ for the movement of $y_k(t)$ for $t \geq T_{k+1}$. We determine the orientation of $L_{k+1}(t)$ as follows;

- If $\Gamma(t)$ is positive and $U > 0$ then we set $\nu(k+1) = \nu(k) + 1$, i.e., $\mathbf{n}_{k+1} = \mathbf{N}_{\nu(k)+1}$ and $\boldsymbol{\tau}_{k+1} = \mathbf{T}_{\nu(k)+1}$. (Similarly, we set $\mathbf{n}_{k+1} = -\mathbf{N}_{\nu(k)+1}$ and $\boldsymbol{\tau}_{k+1} = \mathbf{T}_{\nu(k)+1}$ if $\Gamma(t)$ is negative and $U \leq 0$.)
- If $\Gamma(t)$ is positive and $U \leq 0$ (and similarly negative and $U > 0$), then we set $\nu(k+1) = \nu(k) - 1$.

We mention the reason why we discard the other choices on the orientation of $L_{k+1}(t)$ on the above in §5. We now summarize our scheme of the evolution of spirals as the following.

Summary of the scheme(SP)

Step 1. Give an suitable initial curve $\Gamma(T_{k_0}) = \bigcup_{j=0}^{k_0} L_j(T_{k_0})$ at $t = T_{k_0} \in \mathbb{R}$. (We impose that $d_{k_0}(T_{k_0}) = 0$.) Let $k = k_0$.

Step 2. Solve (11) with the initial data $d_j(T_k) = |y_j(T_k) - y_{j-1}(T_k)|$ for $j = 1, 2, \dots, k$, and obtain the solution $(d_1, \dots, d_k)$ for $t > T_k$.

Step 3. Describe $\Gamma(t) = \bigcup_{j=0}^{k} L_j(t)$ for $t \geq T_k$ by drawing $L_j(t)$ given by (5) with $y_k(t) = O$ and $y_j(t)$ given by (12) for $j = k, k-1, \dots, 0$.

Step 4. If $T_{k+1} < \infty$, where T_{k+1} is defined as (13), then generate L_{k+1} with the rule (G). and $y_{k+1} = O$ at $t = T_k$ and return to Step 2 with updating the number of facets from k to $k+1$, and initial time T_{k+1}.

We define the some classes of solution $\Gamma(t)$ evolving by (3) with a crystalline surface energy.

DEFINITION 3. *(i) Let $I \subset \mathbb{R}$ be an interval. We say $\Gamma(t) = \bigcup_{j=1}^{k} L_j(t)$, where $L_j(t)$ is given as (5), is a local semi-solution to (3) in I if the followings hold;*

(a) *$\Gamma(t)$ is an oriented semi-admissible spiral with respect to $\mathcal{W}_\gamma$ for $t \in I$,*
(b) *$(d_1(t), \dots, d_k(t)) \in (C^1(I))^k$ is a solution to (11) in I, where $d_j(t) = |y_j(t) - y_{j-1}(t)|$ for $j = 1, 2, \dots, k$.*

We say $\Gamma(t)$ is a local solution to (3) in I if $\Gamma(t)$ is a local semi-solution in I and admissible for every $t \in I$.

(ii) Let $I_k = [T_k, T_{k+1})$ and $J = \bigcup_{j=k_0}^{k_M - 1} I_k \subset \mathbb{R}$ be an interval. for $k \in [k_0, k_M)$ with $k_M \in (k_0, \infty]$. We say $\Gamma(t)$ is a maximal semi-solution (resp. solution) to (3) on J with the scheme (SP) if

(a) *$\Gamma(t) = \bigcup_{j=0}^{k} L_j(t)$ for $t \in I_k$,*
(b) *$\Gamma(t)$ is a local semi-solution (resp. local solution) on each I_k,*
(c) *$d_j \in C^1[T_j, T_{k_M})$, where $T_j = T_{k_0}$ provided that $j \le k_0$,*
(d) *T_k is the generation time of L_k for $k \in [k_0, k_M)$, and $T_{k_M} = \infty$ or T_{k_M} is the life span of the system (11).*

(iii) We say $\Gamma(t)$ is a global semi-solution (resp. solution) to (3) with the scheme (SP) if $\Gamma(t)$ is a maximal semi-solution (resp. solution) to (3) with the scheme (SP) on $J = [T_{k_0}, \infty) = \bigcup_{k \ge k_0} [T_k, T_{k+1})$.

Remark 4. There are possibilities of maximal solution that (i)$T_{k_M} < \infty$ or (ii)$T_{k_M} = \infty$ provided that $k_M < \infty$, and (iii)$T_\infty := \lim_{k\to\infty} T_k < \infty$ or (iv)$T_\infty = \infty$ provided that $k_M = \infty$. For each case the spiral possibly behave the followings.

(i) A facet extincts.
(ii) A solution exists globally-in-time, but just the curve stays there.
(iii) Spiral scrolls infinitely at the center.
(iv) Spiral grows up.

3. Global existence and admissibility. In this section we prove the existence and uniqueness of a global solution $\Gamma(t)$ evolving by (3) with a pinned center. The main result of this section is as the following. Throughout this section we assume that $U > 0$ and consider an evolving positive spiral.

THEOREM 5. *Let $\Gamma(T_{k_0}) = \bigcup_{j=0}^{k_0} L_j(T_{k_0})$ be an positive spiral curve satisfying either the following (I1) or (I2) holds;*

(I1) *(for $k_0 = 1$) $\Gamma(T_1)$ is a single line, i.e., $\Gamma(T_1) = \bigcup_{j=0}^{1} L_j(T_1) = \{\lambda \mathbf{T}_0;\ \lambda \ge 0\}$, $y_1(T_1) = y_0(T_1) = O$.*

(I2) *(for $k_0 \ge 2$) $\Gamma(T_{k_0}) = \bigcup_{j=0}^{k_0} L_j(T_{k_0})$ is a convex spiral satisfying the following three conditions.*

(i) $d_{k_0}(T_{k_0}) = 0$, and $\delta_j := d_j(T_{k_0}) \ge \ell_{\nu(j)}/U$ for $j = 1, 2, \dots, k_0 - 1$, where $\nu(j)$ is the corresponding facet number for $j = 1, 2, \dots, k_0 - 1$ defined as (6).

(ii) The quantity $\kappa_j := \ell_{\nu(j)}/\delta_j$ satisfies

$$-b_j\,(U - \kappa_j) + c_j^+\,(U - \kappa_{j+1}) + c_j^-\,(U - \kappa_{j-1}) > 0 \qquad \text{for } j = 1, 2, \dots, k_0 - 1;$$

here we have set $\kappa_{k_0} = U$, $\kappa_0 = 0$.

(iii) $s_j(T_{k_0}) \geq 0$ *for* $j = 0, 1, 2, \ldots, k_0$.

Then, there exists a sequence of the generation time $\{T_k\}_{k=k_0}^{\infty}$ *and a global solution* $\Gamma(t)$ *to* (3) *by the scheme (SP) such that* $\Gamma(t)$ *is a positive convex for* $t \geq T_{k_0}$ *and* $\lim_{k\to\infty} T_k = \infty$.

3.1. Existence and uniqueness. To prove Theorem 5 we divide the proof into two steps; the existence of semi-solution and intersection-free result. In this subsection we verify the following existence result on semi-solution as the first step.

THEOREM 6. *Let* $\Gamma(T_{k_0}) = \bigcup_{j=0}^{k_0} L_j(T_{k_0})$ *be a positive, convex and admissible spiral satisfying either* (I1) *or* (I2i)–(I2ii). *Then there exists a positive convex global semi-solution* $\Gamma(t)$ *to* (3) *by the scheme* (SP) *for* $t \geq T_{k_0}$.

We first consider the local semi-solution to the system (11) for $t \geq T_k$ with fixed $k \geq k_0$ and a positive convex semi-admissible $\Gamma(T_k)$. Note that $\Gamma(T_k)$ is positive, then we may assume that

$$\nu(j) = j, \quad \text{i.e.,} \quad \mathbf{n}_j = \mathbf{N}_j \tag{14}$$

without loss of generality. Moreover, by the convexity of $\Gamma(T_k)$, the system (11) is simplified as

$$\dot{d}_k = c_k^- \left(U - \frac{\ell_{k-1}}{d_{k-1}} \right), \tag{15}$$

$$\begin{cases} \dot{d}_{k-1} = -b_{k-1} \left(U - \dfrac{\ell_{k-1}}{d_{k-1}} \right) + c_{k-1}^- \left(U - \dfrac{\ell_{k-2}}{d_{k-2}} \right), \\ \dot{d}_j = -b_j \left(U - \dfrac{\ell_j}{d_j} \right) + c_j^+ \left(U - \dfrac{\ell_{j+1}}{d_{j+1}} \right) + c_j^- \left(U - \dfrac{\ell_{j-1}}{d_{j-1}} \right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } j = 2, \ldots, k-2, \\ \dot{d}_1 = -b_1 \left(U - \dfrac{\ell_1}{d_1} \right) + c_1^+ \left(U - \dfrac{\ell_2}{d_2} \right) + c_1^- U \end{cases} \tag{16}$$

with constants $b_j, c_j^{\pm} \in \mathbb{R}$ at least in a short time for $t \geq T_k$. Note that $c_j^{\pm} > 0$ by (14). We now demonstrate that the solution $(d_1, \ldots, d_{k-1}, d_k)$ to (15)–(16) exists uniquely and globally-in-time provided that $d_1(T_1) = 0$ if $k = 1$, or (I2i) and (I2ii) with $\nu(j) = j$ hold if $k \geq 2$, i.e.,

$$\delta_j := d_j(T_k) \geq \ell_j / U \quad \text{for } j = 1, 2, \ldots, k-1, \tag{17}$$

$$-b_j (U - \kappa_j) + c_j^+ (U - \kappa_{j+1}) + c_j^- (U - \kappa_{j-1}) > 0 \quad \text{for } j = 1, 2, \ldots, k-1, \tag{18}$$

where $\kappa_j = \ell_j / \delta_j$ provided that $j = 1, 2, \ldots, k-1$, and $\kappa_k = U$, $\kappa_0 = 0$. Moreover, we also demonstrate that the facet L_{k+1} generates at a finite time $T_{k+1} < \infty$, and $d_j(T_{k+1})$ or $\dot{d}_j(T_{k+1})$ for $j = 1, 2, \ldots, k$ keep (I2i) and (I2ii). The result on the above should be summarized as the following.

LEMMA 7. *Assume that $d_1(T_1) = 0$ provided that $k = 1$, or $\delta_1, \dots, \delta_{k-1} > 0$ satisfy (17), (18) provided that $k > 1$. Then, there exists a unique solution $(d_1, \dots, d_{k-1}, d_k) \in (C^1[T_k, \infty))^k$ to (15)–(16) with $(d_1(T_k), \dots, d_{k-1}(T_k), d_k(T_k)) = (\delta_1, \dots, \delta_{k-1}, 0)$. Moreover, the followings hold.*

(i) $d_j > \ell_j/U$, in (T_k, ∞) for $j = 1, 2, \dots, k-1$,
(ii) $\sup_{t\in[T_k,\infty)} \dot{d}_j < \infty$ for $j = 1, 2, \dots, k$,
(iii) $\dot{d}_j > 0$ in (T_k, ∞) for $j = 1, 2, \dots, k$,
(iv) $T_{k+1} := \sup\{\hat{t};\ d_k < \ell_k/U \text{ in } (T_k, \hat{t})\} < \infty$, and there exists $R_k > 0$ for $k \in \mathbb{Z}/(N_\gamma\mathbb{Z})$ such that $T_{k+1} - T_k \geq R_k$.

Proof. We first note that the $k = 1$ is clear since this case is just

$$\dot{d}_1 = c_1^- U > 0 \quad \text{for } t > T_0,\ d_1(T_1) = 0.$$

Then, we obtain $d_1(t) = c_1^- U(t - T_1)$, and thus $T_2 = \ell_1/(c_1^- U^2) + T_1 < \infty$.

For the case $k \geq 2$ we divide the proof into the three steps.

Step 1. We first demonstrate that there exists a solution $(d_1, \dots, d_{k-1})$ to (16) on $[T_k, \infty)$ with an initial data $(d_1(T_k), \dots, d_{k-1}(T_k)) = (\delta_1, \dots, \delta_{k-1})$ satisfying (17) and (18), which implies (i) and (ii). For this purpose we represent (16) as

$$\dot{\mathbf{d}} = \mathbf{F}(\mathbf{d}) \quad \text{for } t \geq T_k, \tag{19}$$

$$\mathbf{d}(T_k) = \boldsymbol{\delta} := (\delta_1, \dots, \delta_{k-1}) \in Q_0 := \prod_{j=1}^{k-1} [\ell_j/U, \infty) \tag{20}$$

with $\mathbf{d} = (d_1, \dots, d_{k-1})$ and $\mathbf{F}\colon Q := \prod_{j=1}^{k-1}[\ell_j/(2U), \infty) \to \mathbb{R}^{k-1}$ denoting the right-hand side formula of (16). Then, since $\mathbf{F}$ is bounded and Lipschitz continuous on Q, there exists a solution $\mathbf{d} \in (C^1[T_k, T_k + \tilde{\rho}])^{k-1}$ to (19)–(20) for a constant $\tilde{\rho} > 0$ by the usual iteration in the theory of ordinary differential equations. Moreover, the constant $\tilde{\rho} > 0$ is independent of $\boldsymbol{\delta} \in Q_0$. Then, we now demonstrate that

$$d_j \geq \frac{\ell_j}{U} \quad \text{on } [T_k, T_k + \tilde{\rho}] \tag{21}$$

for $j = 1, 2, \dots, k-1$ to derive the existence of the global solution. In fact, we can extend $\mathbf{d}$ to the solution on $[T_k, T_k + 3/2\tilde{\rho}]$ by solving (19) with an initial time $t = T_k + \tilde{\rho}/2$ and an initial data $\boldsymbol{\delta} = \mathbf{d}(T_k + \tilde{\rho}/2)$. The above implies that we can extend the solution to (19)–(20) on $[T_k, \infty)$.

To prove (21) set

$$\hat{T} = \sup\{\hat{t} > T_k;\ d_j > \ell_j/U \text{ on } (T_k, \hat{t}) \text{ for } j = 1, 2, \dots, k-1\}.$$

Note that $\hat{T}$ is well-defined by (18), which implies $\dot{d}_j(T_k) > 0$. If $\hat{T} \leq T_k + \tilde{\rho}$, then

- there exists $j_0 \in \{1, 2, \dots, k-1\}$ such that $d_{j_0}(\hat{T}) = \ell_{j_0}/U$, and
- $d_j \geq \ell_j/U$ on $[T_k, \hat{T}]$ for every $j = 1, 2, \dots, k-1$.

The above properties imply $\dot{d}_{j_0}(\hat{T}) \leq 0$, and thus $j_0 \neq 1$. In fact, if $j_0 = 1$ then we have

$$0 \geq \dot{d}_1(\hat{T}) \geq c_1^- U > 0$$

by the third formula of (16), which is the contradiction. Thus we obtain $d_1(\hat{T}) > \ell_1/U$, which yields $j_0 \neq 2$. In fact, if $j_0 = 2$, then

$$0 \geq \dot{d}_2(\hat{T}) \geq c_2^- \left(U - \frac{\ell_1}{d_1(\hat{T})} \right) > 0$$

by the second formula of (16), which is the contradiction. Then, we obtain $j_0 \notin \{1, 2, \ldots, k-1\}$ by the inductive argument of the above, which contradicts to the definition of $\hat{T}$. Hence, we obtain (21), which implies the existence of the global solution to (19)–(20). Moreover, we obtain

$$\left.\begin{aligned} &d_j > \ell_j/U \\ &\dot{d}_j \leq \max_{j \in \mathbb{Z}/(N_\gamma \mathbb{Z})} (|b_j| + c_j^+ + c_j^-)U \end{aligned}\right\} \quad \text{in } (T_k, \infty) \text{ for } j = 1, 2, \ldots, k-1,$$

which are (i) and (ii) by the same argument of the above (for (i)) and the straight forward calculation (for (ii)).

Step 2. We next demonstrate (iii), which is

$$\dot{d}_j > 0 \quad \text{in } (T_k, \infty) \text{ for } j = 1, 2, \ldots, k.$$

Set

$$\bar{T} = \sup\{\bar{t} > T_k;\ \dot{d}_j > 0 \text{ in } (T_k, \bar{t}] \text{ for } j = 1, 2, \ldots, k-1\},$$

and assume $\bar{T} < \infty$. If $k = 2$, we observe that $d_1 \equiv d_1(\bar{T})$ by the uniqueness of the solution to (16), which contradicts to (18). Thus, we obtain $\bar{T} = \infty$ when $k = 2$.

If $k \geq 3$, then we observe that

(a) $\dot{d}_j \geq 0$ on $[T_k, \bar{T}]$ for $j = 1, 2, \ldots, k-1$,

(b) there exists $j_1, j_2 \in \{1, 2, \ldots, k-1\}$ such that $\dot{d}_{j_1}(\bar{T}) = 0$ and $\dot{d}_{j_2}(\bar{T}) > 0$.

We now mention on (b) of the above; there would be a possibility of $\dot{d}_j(\bar{T}) = 0$ for all $j = 1, 2, \ldots, k-1$. However, if so, then we observe that $d_j \equiv d_j(\bar{T})$ by the uniqueness of solution to (16), which contradicts to (18). Hence, we obtain (b) which implies that there exists $j_0 \in \{1, 2, \ldots, k-1\}$ such that

- $\dot{d}_{j_0}(\bar{T}) = 0$,
- $\dot{d}_{j_0+1}(\bar{T}) > 0$ or $\dot{d}_{j_0-1}(\bar{T}) > 0$.

Since the proof is parallel, we now assume that $\dot{d}_{j_0-1}(\bar{T}) > 0$. Then, for a fixed $m \in (0, \dot{d}_{j_0-1}(\bar{T})/(\dot{d}_{j_0-1}(\bar{T}))^2)$ there exists $\mu_0 > 0$ such that

$$\frac{1}{d_{j_0-1}(t)} \geq \frac{1}{d_{j_0-1}(\bar{T})} - m(t - \bar{T}) \quad \text{for } t \in [\bar{T} - \mu_0, \bar{T}].$$

We also find from $\dot{d}_{j_0}(\bar{T}) = 0$ and $\dot{d}_{j_0+1}(\bar{T}) \geq 0$ provided that $j_0 + 1 < k$, for every $\varepsilon > 0$ there exists $\mu \in (0, \mu_0]$ such that

$$\left.\begin{aligned} \left| \frac{1}{d_{j_0}(t)} - \frac{1}{d_{j_0}(\bar{T})} \right| &\leq -\varepsilon(t - \bar{T}), \\ \frac{1}{d_{j_0+1}(t)} &\geq \frac{1}{d_{j_0+1}(\bar{T})} + \varepsilon(t - \bar{T}) \end{aligned}\right\} \quad \text{on } [\bar{T} - \mu, \bar{T}].$$

Then, by combining the above we obtain

$$\dot{d}_{j_0}(t) \leq \dot{d}_{j_0}(\bar{T}) + (m c_{j_0}^- \ell_{j_0-1} + O(\varepsilon))(t - \bar{T}) \quad \text{for } t \in [\bar{T} - \mu, \bar{T}]$$

as $\varepsilon \to 0$. In fact, if $j_0 \in (1, k-1)$, then we observe that

$$\begin{aligned}
\dot{d}_{j_0}(t) &= -b_{j_0}\left(U - \frac{\ell_{j_0}}{d_{j_0}(t)}\right) + c_{j_0}^+\left(U - \frac{\ell_{j_0+1}}{d_{j_0+1}(t)}\right) + c_{j_0}^-\left(U - \frac{\ell_{j_0-1}}{d_{j_0-1}(t)}\right) \\
&= -b_{j_0}\left(U - \frac{\ell_{j_0}}{d_{j_0}(\bar{T})}\right) + b_{j_0}\ell_{j_0}\left(\frac{1}{d_{j_0}(t)} - \frac{1}{d_{j_0}(\bar{T})}\right) \\
&\quad + c_{j_0}^+\left(U - \frac{\ell_{j_0+1}}{d_{j_0+1}(\bar{T})}\right) + c_{j_0}^+\ell_{j_0+1}\left(\frac{1}{d_{j_0+1}(\bar{T})} - \frac{1}{d_{j_0+1}(t)}\right) \\
&\quad + c_{j_0}^-\left(U - \frac{\ell_{j_0-1}}{d_{j_0-1}(\bar{T})}\right) + c_{j_0}^-\ell_{j_0-1}\left(\frac{1}{d_{j_0-1}(\bar{T})} - \frac{1}{d_{j_0-1}(t)}\right) \\
&\le \dot{d}_{j_0}(\bar{T}) - |b_{j_0}|\ell_{j_0}\varepsilon(t-\bar{T}) + c_{j_0}^+\ell_{j_0+1}\varepsilon(t-\bar{T}) + c_{j_0}^-\ell_{j_0-1}m(t-\bar{T})
\end{aligned}$$

for $t \in [\bar{T}-\mu, \bar{T}]$. We also observe that

$$\dot{d}_{k-1}(t) \le \dot{d}_{k-1}(\bar{T}) - |b_{k-1}|\ell_{k-1}\varepsilon(t-\bar{T}) + c_{k-1}^-\ell_{k-2}m(t-\bar{T})$$

for $t \in [\bar{T}-\mu, \bar{T}]$ if $j_0 = k-1$, or

$$\dot{d}_1(t) \le \dot{d}_1(\bar{T}) - |b_1|\ell_1\varepsilon(t-\bar{T}) + c_1^+\ell_2 m(t-\bar{T})$$

for $t \in [\bar{T}-\mu, \bar{T}]$ if $j_0 = 1$ provided that $\dot{d}_1(\bar{T}) = 0$ and $\dot{d}_2(\bar{T}) > 0$. Therefore, if we choose ε enough small such that $mc_{j_0}^-\ell_{j_0-1} + O(\varepsilon) > 0$, then we have $\dot{d}_{j_0}(t) < 0$ in $[\bar{T}-\mu, \bar{T})$, which contradicts to the definition of $\bar{T}$. Hence, we obtain $\bar{T} = \infty$, which implies

$$\dot{d}_j > 0 \quad \text{on } [T_k, \infty) \text{ for } j = 1, 2, \dots, k-1.$$

It remains that $\dot{d}_k > 0$, which is derived from $d_{k-1} > \ell_{k-1}/U$ for $t \in (T_k, \infty)$ since $\dot{d}_{k-1} > 0$ in (T_k, ∞). Hence we obtain (iii).

Step 3. Finally, we demonstrate (iv). From (iii) we have $d_{k-1} \ge d_{k-1}(T_k + \mu)$ on $[T_k + \mu, \infty)$ for some $\mu > 0$, which implies

$$\dot{d}_k \ge \dot{d}_k(T_k + \mu) = c_k^-\left(U - \frac{\ell_{k-1}}{d_{k-1}(T_k+\mu)}\right) > 0 \quad \text{on } [T_k+\mu, \infty).$$

Then, we obtain $T_{k+1} < \infty$ for $k \ge k_0$. On the other hand,

$$\dot{d}_k \le c_k^- U \quad \text{on } [T_k, \infty),$$

which implies $d_k(t) \le c_k^- U(t - T_k)$. Then, we obtain $T_{k+1} - T_k \ge \ell_k/(c_k^- U^2) =: R_k$. Recall that $c_k^- = c_{k+nN_\gamma}^-$, $\ell_k = \ell_{k+nN_\gamma}$, and thus $R_k = R_{k+nN_\gamma}$ for $n \in \mathbb{Z}$. Hence, we obtain (iv). ∎

By the result of Lemma 7 we obtain the following monotonicity principle.

COROLLARY 8. *Let $\Gamma(T_{k_0})$ be a positive spiral curve satisfying* (I1) *or* (I2). *Let $s_j(t) = y_j(t) \cdot N_j$ be a support function of $L_j(t)$ for $t \ge T_j$, where T_j is the generation time of $L_j(t)$ and accordingly $T_j = T_{k_0}$ if $j \le k_0$. Then, the followings hold.*

(i) $s_j(t), \dot{s}_j(t) > 0$ for $t > T_j$.

(ii) $s_i(t) = s_j(t) = 0$ *provided* $i < j$ *if and only if* $i + 1 = j \geq k_0$ *and* $t = T_j$.

We omit the proof of Corollary 8 since it is obtained from the straightforward calculation of $\dot{s}_j(t) = \dot{y}_j(t) \cdot \mathbf{N}_j$. We are in the position to prove Theorem 6.

Proof of Theorem 6. As we already mentioned in this subsection, we may assume that $\nu(j) = j$. We divide the proof into 2 steps. We shall demonstrate that there exist the generation time T_k and $\mathbf{d}_k = (d_{k,1}, \dots, d_{k,k}) \in C^1[T_k, \infty)^k$ for $k \geq k_0$ satisfying

(i) $\mathbf{d}_k$ is a solution to (15)–(16) on $[T_k, \infty)$,
(ii) $\mathbf{d}_k(T_k) = (\mathbf{d}_{k-1}(T_k), 0)$ if $k > k_0$, $\mathbf{d}_{k_0}(T_{k_0}) = (\delta_1, \dots, \delta_{k_0-1}, 0)$,
(iii) $T_{k+1} := \sup\{\hat{t} > T_k;\ d_{k,k} < \ell_k/U \text{ in } (T_k, \hat{t})\} < \infty$,
(iv) $\mathbf{d}_k(T_{k+1})$ satisfies (17)–(18) with $\delta_j = \delta_{k,j} := d_{k,j}(T_{k+1})$ and $\kappa_j = \kappa_{k,j} := \ell_j/\delta_{k,j}$ for $j = 1, 2, \dots, k-1$

in Step 1 by inductive argument. Then, we obtain the result of Theorem 6 in Step 2 by combining $\mathbf{d}_k$.

Step 1. We first demonstrate that there exist $\mathbf{d}_{k_0}$ and T_{k_0+1} satisfying (i)–(iv) provided that either (I1) or (I2) holds.

We first consider the case of (I1); let $k_0 = 1$ and $\Gamma(T_1) = \{\lambda \mathbf{T}_0;\ \lambda \geq 0\}$. Then, we now find a solution $\mathbf{d}_1 = d_{1,1}$ to (15)–(16) with $k = 1$ and $d_1(T_1) = 0$, i.e.,

$$\dot{d}_{1,1} = c^- U \quad \text{for } t \geq T_1, \quad d_{1,1}(T_1) = 0.$$

It is given by

$$d_{1,1}(t) = c_1^- U(t - T_1).$$

Then, we observe that

$$T_2 = T_1 + \frac{\ell_1}{c_1^- U^2}.$$

We also find $d_{1,1}$ holds (iv). In fact, $d_{1,1}(T_2) = \ell_1/U$ and

$$-b_1(U - \kappa_{2,1}) + c_1^- U = c_1^- U > 0$$

for $\kappa_{2,1} = \ell_1/d_{1,1}(T_2) = U$. Here we have used $\kappa_2 = U$ in (18) for $k = 2$.

Similarly, we consider the case (I2); let $k_0 \geq 2$ and assume that $\Gamma(T_{k_0}) = \bigcup_{j=0}^{k_0} L_j(T_{k_0})$ satisfy (I2). Then, there exists a solution $\mathbf{d}_{k_0} \in C^1[T_{k_0}, \infty)^{k_0}$ to (15)–(16) provided $k = k_0$ with the initial data $\mathbf{d}_{k_0}(T_{k_0}) = (\delta_1, \dots, \delta_{k_0-1}, 0)$ by Lemma 7. Moreover, we obtain

$$T_{k_0+1} := \sup\{\hat{t} > T_{k_0};\ d_{k_0,k_0} < \ell_j/U \text{ in } (T_{k_0}, \hat{t})\} < \infty,$$

and $\mathbf{d}_{k_0}(T_{k_0+1}) = (\delta_{k_0+1,1}, \dots \delta_{k_0+1,k_0})$ satisfies (17)–(18).

Once we obtain $\mathbf{d}_k \in C^1[T_k, \infty)^k$ satisfying (i)–(iv) for $k \geq k_0$, then we obtain $\mathbf{d}_{k+1} \in C^1[T_{k+1}, \infty)^{k+1}$ satisfying (i)–(iii) by Lemma 7. Moreover, $\mathbf{d}_{k+1}(T_{k+2})$ satisfies (iv). In fact, (17) follows from Lemma 7 (i) and $d_{k+1,k+1}(T_{k+2}) = \ell_{k+1}/U$. Then, (18) follows from (16) and Lemma 7 (iii), i.e.,

$$-b_j(U - \kappa_{k+1,j}) + c_j^+(U - \kappa_{k+1,j+1}) + c_j^-(U - \kappa_{k+1,j-1}) = \dot{d}_{k+1,j}(T_{k+2}) > 0.$$

Hence, we obtain $\mathbf{d}_k \in C^1[T_k, \infty)^k$ for $k \geq k_0$ satisfying (i)–(iv).

Step 2. We now construct a semi-solution $\Gamma(t)$ for $t \geq T_{k_0}$. Set $I_k = [T_k, T_{k+1})$ for $k \geq k_0$. Define

$$d_j(t) := d_{k,j}(t) \quad \text{if } t \in I_k \cap [T_j, \infty),$$

for $k \geq k_0$ and $j \in \mathbb{N}$ where T_j is what we obtained in Step 1 for $j > k_0$, and accordingly $T_j = T_{k_0}$ if $j \leq k_0$. Note that $d_j \in C^1[T_j, \infty)$. In fact, $d_j \in C[T_j, \infty)$ and $d_j \in C^1(T_k, T_{k+1})$ for $k \geq k_0$ by its definition and the property (ii) of $\mathbf{d}_k$. Then, it suffices to prove

$$\lim_{t \uparrow T_k} \dot{d}_j(t) = \lim_{t \downarrow T_k} \dot{d}_j(t) \tag{22}$$

in particular for $k = j+1, j+2$ to prove $d \in C^1[T_j, \infty)$, since the equation of d_j for $j \geq k_0 - 1$ is exactly changed at $t = T_{j+1}$ and $t = T_{j+2}$; see (15) and (16). We now verify the above for T_{j+1}. On one hand,

$$\lim_{t \uparrow T_{j+1}} \dot{d}_j(t) = \lim_{t \uparrow T_{j+1}} \dot{d}_{j,j}(t) = c_j^- \left(U - \frac{\ell_{j-1}}{d_{j,j-1}(T_{j+1})} \right)$$

by (15). On the other hand,

$$\begin{aligned} \lim_{t \downarrow T_{j+1}} \dot{d}_j(t) &= \lim_{t \downarrow T_{j+1}} \dot{d}_{j+1,j}(t) \\ &= -b_j \left(U - \frac{\ell_j}{d_{j+1,j}(T_{j+1})} \right) + c_j^- \left(U - \frac{\ell_{j-1}}{d_{j+1,j-1}(T_{j+1})} \right) \\ &= c_j^- \left(U - \frac{\ell_{j-1}}{d_{j,j-1}(T_{j+1})} \right) \end{aligned}$$

by the first equation of (16) and the property (ii) and (iii) of $\mathbf{d}_k$, i.e., $d_{j+1,j}(T_{j+1}) = \ell_j / U$ and $d_{j+1,j-1}(T_{j+1}) = d_{j,j-1}(T_{j+1})$. Hence, we obtain (22) at $t = T_{j+1}$, which implies the differentiability and continuity of $\dot{d}_j$ at $t = T_{j+1}$. We also obtain (22) at $t = T_{j+2}$ by the similar argument with $d_{j+2,j+1}(T_{j+2}) = \ell_{j+1}/U$.

We now construct $\Gamma(t)$ for $t \geq T_{k_0}$ as a family of polygonal curves in each interval I_k; set $\Gamma(t) := \bigcup_{j=0}^k L_j(t)$ if $t \in I_k$ for each interval I_k for $k \geq k_0$ with $L_j(t)$ given by (5) and

$$\begin{aligned} &y_k(t) := O, \\ &y_j(t) := y_{j+1} + d_j(t)\mathbf{T}_j \quad \text{for } j = k-1, k-2, \dots, 0. \end{aligned}$$

Then, $\Gamma(t)$ is a global semi-solution to (3) with the scheme (SP). ∎

3.2. Self-intersection free. In the previous section we prove the existence of a semi-solution $\Gamma(t)$ provided that initial curve $\Gamma(T_{k_0})$ satisfies (I1) or (I2i)–(I2ii). We shall prove the admissibility of $\Gamma(t)$ to obtain the solution to (3) with scheme (SP). Note that the properties (A1) and (A2) of the admissibility are guaranteed by Lemma 7 and the generation rule (G). Then, we now prove that $\Gamma(t)$ is self-intersection free for $t \geq T_{k_0}$.

To verify the self-intersection free result, it is convenient to introduce semi-open and open line segment description of $L_j(t)$ as (5) to $\Gamma(t) = \bigcup_{j=0}^k L_j(t)$. Define

$$\begin{aligned} &\Lambda_j(t) = \begin{cases} \emptyset & \text{if } y_{j-1}(t) = y_j(t), \\ \{\lambda y_{j-1}(t) + (1-\lambda) y_j(t);\ \lambda \in [0,1)\} & \text{otherwise,} \end{cases} \\ &\Lambda_j^\circ(t) = \Lambda_j(t) \setminus \{y_j(t)\}. \end{aligned}$$

Then, we observe that $\bigcup_{j=0}^{k} L_j(t) = \bigcup_{j=0}^{k} \Lambda_j(t)$. We now classify the kinds of self-intersection of a polygonal curve from the usual definition of self-intersection.

DEFINITION 9. *We say $\Gamma(t) = \bigcup_{j=0}^{k} L_j(t) = \bigcup_{j=0}^{k} \Lambda_j(t)$ has a self-intersection if there exist $i, j \in \{0, 1, \ldots, k\}$ such that $i \neq j$ and $\Lambda_i(t) \cap \Lambda_j(t) \neq \emptyset$. Moreover, we classify the kinds of self-intersections between $\Lambda_i(t)$ and $\Lambda_j(t)$ as the following;*

(i) (vertex-vertex) $\Lambda_i(t) \cap \Lambda_j(t) \neq \emptyset$ with $y_i(t) = y_j(t)$,
(ii) (facet-vertex) $\Lambda_i^\circ(t) \cap \Lambda_j(t) = \{y_j(t)\}$,
(iii) (facet-facet) $\Lambda_i^\circ(t) \cap \Lambda_j^\circ(t) \neq \emptyset$, and either $\mathbf{n}_i = \mathbf{n}_j$ or $\mathbf{n}_i = -\mathbf{n}_j$ holds,
(iv) (cross) $\Lambda_i^\circ(t) \cap \Lambda_j^\circ(t) \neq \emptyset$ provided that $\mathbf{n}_i \neq \alpha \mathbf{n}_j$ with $\alpha \in \{\pm 1\}$.

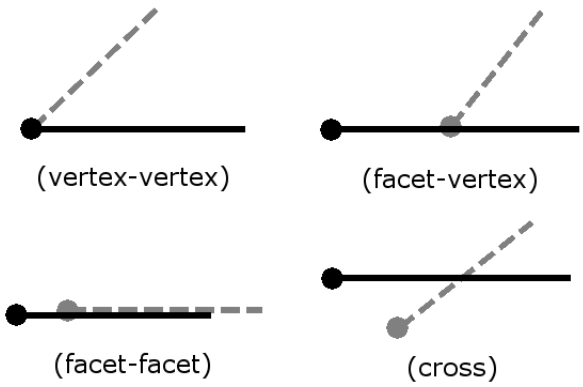

FIG. 5. *Examples of self-intersection between $\Lambda_i(t)$ (solid line) and $\Lambda_j(t)$ (dashed line). Note that dots means $y_i(t)$ or $y_j(t)$, which is a vertex belongs to $\Lambda_i(t)$ or $\Lambda_j(t)$, respectively.*

Then, we obtain the following self-intersection free result.

THEOREM 10. *Let $\Gamma(t) = \bigcup_{j=0}^{k} L_j(t) = \bigcup_{j=0}^{k} \Lambda_j(t)$ be a positive semi-solution (3) by the scheme (SP) with an initial admissible curve $\Gamma(T_{k_0})$ satisfying either (I1) or (I2). Then, $\Gamma(t)$ has no self-intersections for $t \geq T_{k_0}$.*

To prove Theorem 10 we introduce a first touch time $\bar{t} > T_{k_0}$;

$$\bar{t} = \sup\{\bar{T};\ \Gamma(t) \text{ has no self-intersections for } t \in [T_{k_0}, \bar{T})\}. \tag{23}$$

Note that $\Gamma(\bar{t})$ has a self-intersection if $\bar{t} < \infty$. In fact, there exist $i \in \{0, 1, \ldots, k\}$ and $j \in \{0, 1, \ldots, k\} \setminus \{i-1, i, i+1\}$ such that $\mathrm{dist}(\Lambda_i(\bar{t}), \Lambda_j(\bar{t})) = 0$, which implies $(\Lambda_i \cup \Lambda_{i-1}) \cap (\Lambda_j \cup \Lambda_{j-1}) \neq \emptyset$ at $t = \bar{t}$; here we have set $\Lambda_{-1} = \emptyset$, and $\mathrm{dist}(A, B) = \inf\{|x - y|;\ x \in A, y \in B\}$ for $A, B \subset \mathbb{R}^2$. Then, we first demonstrate the following Lemma on some basic properties of intersections. Throughout this subsection we also assume (14) since $\Gamma(t)$ is positive convex as in Theorem 6.

LEMMA 11. *Assume that $\bar{t} < \infty$ and a self-intersection appears between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ with $i < j$. Then the following properties hold.*

(i) The intersection between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ is not the cross intersection.

(ii) If the intersection between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ is the facet-facet intersection, then $\mathbf{N}_i = \mathbf{N}_j$.

(iii) If the intersection between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ is not the facet-vertex intersection with $\Lambda_i(\bar{t}) \cap \Lambda_j^\circ(\bar{t}) = \{y_i(\bar{t})\}$, then both $\Lambda_{i+1}(t)$ and $\Lambda_{j+1}(t)$ are not empty in a neighborhood of $\bar{t}$.

Proof. We first demonstrate (i). Let us choose $y \in \Lambda_i^\circ(\bar{t}) \cap \Lambda_j^\circ(\bar{t})$. We fix the small constant $\bar{\mu} > 0$ and the open neighborhood U of y satisfying

$$\{x \in \mathbb{R}^2;\ x \cdot \mathbf{N}_i = s_i(t)\} \cap U = \Lambda_i^\circ(t) \cap U \quad \text{for } t \in [\bar{t} - \bar{\mu}, \bar{t}],$$

which implies $\partial\{x \in \mathbb{R}^2;\ x \cdot \mathbf{N}_i > s_i(t)\} \cap U = \partial\{x \in \mathbb{R}^2;\ x \cdot \mathbf{N}_i < s_i(t)\} \cap U = \Lambda_i^\circ(\bar{t}) \cap U$ for $t \in [\bar{t} - \bar{\mu}, \bar{t}]$. Then, there exist $r^+ > r^- > 0$ such that

$$\begin{aligned} &z_j(t, r) := y_j(t) + r\mathbf{T}_j \in U \cap \Lambda_j^\circ(t) \quad \text{for } r \in [r^-, r^+],\ t \in [\bar{t} - \bar{\mu}, \bar{t}], \\ &z_j(\bar{t}, r^-) \cdot \mathbf{N}_i < s_i(\bar{t}) \text{ and } z_j(\bar{t}, r^+) \cdot \mathbf{N}_i > s_i(\bar{t}) \\ &(\text{or } z_j(\bar{t}, r^-) \cdot \mathbf{N}_i > s_i(\bar{t}) \text{ and } z_j(\bar{t}, r^+) \cdot \mathbf{N}_i < s_i(\bar{t})). \end{aligned} \tag{24}$$

We may assume that $z_j(\bar{t}, r^-) \cdot \mathbf{N}_i < s_i(\bar{t})$ and $z_j(\bar{t}, r^+) \cdot \mathbf{N}_i > s_i(\bar{t})$ without loss of generality. Then, there exists $\mu \in (0, \bar{\mu})$ such that

$$z_j(\bar{t} - \mu, r^-) \cdot \mathbf{N}_i < s_i(\bar{t} - \mu) \text{ and } z_j(\bar{t} - \mu, r^+) \cdot \mathbf{N}_i > s_i(\bar{t} - \mu).$$

This yields $z_j(\bar{t} - \mu, r_0) \in \Lambda_i^\circ(\bar{t} - \mu) \cap \Lambda_j^\circ(\bar{t} - \mu)$ for some $r_0 \in (r_-, r_+)$ by (24), which contradicts to the definition of $\bar{t}$. Hence, we obtain the conclusion (i).

We next verify (ii). Assume that the facet-facet intersection appears between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ with $\mathbf{N}_i = -\mathbf{N}_j$. Then, we observe that $s_i(\bar{t}) = -s_j(\bar{t}) = y \cdot \mathbf{N}_i$ with some $y \in \Lambda_i(\bar{t}) \cap \Lambda_j(\bar{t})$. Thus, if $s_i(\bar{t}) \neq 0$, then $s_i(\bar{t}) s_j(\bar{t}) < 0$ which contradicts to $s_i \geq 0$ and $s_j \geq 0$ on $[T_{k_0}, \infty)$ by Corollary 8. This yields that $s_j(\bar{t}) = s_j(\bar{t}) = 0$. However, this situation appears only when $\bar{t} = T_j$ by (ii) in Corollary 8. This implies $\Lambda_j(\bar{t}) = \emptyset$ which contradicts to the definitions of the self-intersections. Hence, we obtain (ii).

We demonstrate (iii). Note that $j \geq i + 1 > i$ and thus each $\Lambda_j(\bar{t}), \Lambda_{i+1}(\bar{t})$ or $\Lambda_i(\bar{t})$ is not empty. Then, we now lead a contradiction with assuming $\Lambda_{j+1}(\bar{t}) = \emptyset$. In fact, this implies that $\bar{t} \in (T_j, T_{j+1}]$ by (i) in Corollary 8, and then $y_j(\bar{t}) = O$. We now remark that the situation of the intersection between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ should be divided into the following three cases;

- facet-facet intersection with $\mathbf{N}_i = \mathbf{N}_j$ (see (ii)),
- vertex-vertex intersection, i.e., $y_i(\bar{t}) = y_j(\bar{t}) = O$,
- facet-vertex intersection with $\Lambda_j(\bar{t}) \cap \Lambda_i^\circ(\bar{t}) = \{y_j(\bar{t})\}$.

However, we observe that $s_i(\bar{t}) = s_j(\bar{t}) = 0$ for each above case. In fact, for the first case of the above we find $s_j(\bar{t}) = y_j \cdot \mathbf{N}_j = y \cdot \mathbf{N}_j = 0$ for some $y \in \Lambda_i(\bar{t}) \cap \Lambda_j(\bar{t})$, and then $s_i(\bar{t}) = y \cdot \mathbf{N}_i = y \cdot \mathbf{N}_j = 0$. For the second and third cases we observe that $O \in \Lambda_i(\bar{t}) \cap \Lambda_j(\bar{t})$, which implies that $s_i(\bar{t}) = s_j(\bar{t}) = 0$. Thus, we observe that $j = i + 1$ with $\bar{t} = T_{i+1}$ by (ii) in Corollary 8, which implies that $\Lambda_j(\bar{t}) = \emptyset$. This is the contradiction. Hence, we obtain (iii). ∎

Proof of Theorem 10. We here mention only the sketch of the proof since the all observation are derived from elemental calculations of inner product and continuity of $y_j(t)$. (See §5.2 for the details.)

Let $\bar{t} \in [T_k, T_{k+1})$ be a first touch time for $\Gamma(t) = \bigcup_{j=0}^{k} \Lambda_j(t)$, and let $\Lambda_i(\bar{t}) \cap \Lambda_j(\bar{t}) \neq \emptyset$. We may assume that

$$\varphi_i \le \varphi_j < \varphi_i + 2\pi \tag{25}$$

without loss of generality by choosing suitable $n \in \mathbb{Z}$ and $\varphi_j + 2\pi n$ instead of φ_j if necessary. We shall prove the followings.

(I) If $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ has facet-vertex type intersection with $\Lambda_i^\circ(\bar{t}) \cap \Lambda_j(\bar{t}) = \{y_j(\bar{t})\}$, then $\Lambda_i(\bar{t})$ and $\Lambda_{j+1}(\bar{t})$ has facet-facet type intersection.

(II) If $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ has vertex-vertex type intersection, then the both pair $(\Lambda_i(\bar{t}), \Lambda_j(\bar{t}))$ and $(\Lambda_{i+1}(\bar{t}), \Lambda_{j+1}(\bar{t}))$ have facet-facet type intersection.

(III) If $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ has facet-facet type intersection, then $\Lambda_i(\bar{t}) = \Lambda_j(\bar{t})$ with $\mathbf{N}_i = \mathbf{N}_j$.

Then, if (I), (II) and (III) are valid, then there exists a pair of facets having facet-facet type intersection, which is still denoted by $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ with $i < j$ for the simplicity. Then, we obtain $\Lambda_i(\bar{t}) = \Lambda_j(\bar{t})$ by (III), which implies that in particular $\Lambda_{i-1}(\bar{t})$ and $\Lambda_{j-1}(\bar{t})$ has facet-facet type intersection. This also implies that $\Lambda_{i-1}(\bar{t}) = \Lambda_{j-1}(\bar{t})$. By iterating the above argument in finite times we obtain $\Lambda_0(\bar{t}) = \Lambda_{j-i}(\bar{t})$. This contradicts to the fact that $\Lambda_0(t)$ is unbounded for $t \ge 0$. According to the above argument, it suffices to verify (I), (II) and (III).

(I) See the proof of Proposition 12 in §5.2 for details.

Note that $\Lambda_{j+1}(\bar{t}) \neq \emptyset$ in this case. In fact, it follows from Lemma 11(iii) if $i < j$. On the other hand, let $i > j$, which implies $i \ge j + 1$, and assume that $\Lambda_{j+1}(\bar{t}) = \emptyset$. Then, we find $i = j + 1$ and then $\bar{t} = T_i$ by (ii) in Corollary 8. This contradicts to $\Lambda_i(\bar{t}) \neq \emptyset$.

We divide this case into two cases; $\Lambda_j^\circ(\bar{t}) \subset \mathcal{O}_i(\bar{t})$, $\Lambda_j^\circ(\bar{t}) \subset \mathcal{I}_i(\bar{t})$, where

$$\mathcal{O}_i(t) := \{x \in \mathbb{R}^2;\ x \cdot \mathbf{N}_i > s_i(t)\}, \tag{26}$$

$$\mathcal{I}_i(t) := \{x \in \mathbb{R}^2;\ x \cdot \mathbf{N}_i < s_i(t)\}. \tag{27}$$

If $\Lambda_j^\circ(\bar{t}) \subset \mathcal{O}_i(\bar{t})$ (see Figure 6), then we observe that

$$\Lambda_{j+1}(\bar{t}) \subset \{x \in \mathbb{R}^2;\ x \cdot \mathbf{N}_j < s_j(\bar{t}),\ \text{and } x \cdot \mathbf{N}_i > s_i(\bar{t})\}$$

(the gray region of the figure (a)). However, the above implies

$$\overline{\mathcal{I}_i(\bar{t})} \cap \overline{\mathcal{I}_j(\bar{t})} \cap \overline{\mathcal{I}_{j+1}(\bar{t})} = \{y_j(\bar{t})\},$$

and then $y_j(\bar{t}) = O$ by (i) in Corollary 8. This implies $s_i(\bar{t}) = s_j(\bar{t}) = s_{j+1}(\bar{t}) = 0$, which contradicts to (ii) in Corollary 8 and (iii) in Lemma 11.

If $\Lambda_j^\circ(\bar{t}) \subset \mathcal{I}_i(\bar{t})$ (see Figure 7), then we observe that

$$\Lambda_{j+1}(\bar{t}) \subset \{x \in \mathbb{R}^2;\ x \cdot \mathbf{N}_j < s_j(\bar{t}),\ \text{and } x \cdot \mathbf{N}_i \le s_i(\bar{t})\}$$

(the gray region of the figure (a)). In this situation $\Lambda_{j+1}(\bar{t})$ only can be located on $\mathcal{L}_i(\bar{t}) := \{x \in \mathbb{R}^2;\ x \cdot \mathbf{N}_i = s_i(\bar{t})\}$ by the property (A2) in Definition 1 of $\Gamma(\bar{t})$, which is the conclusion of (I). In fact, if $\Lambda_{j+1}(\bar{t}) \subset \mathcal{I}_j(\bar{t})$, then we observe that

$$\varphi_j < \varphi_i < \varphi_{j+1}$$

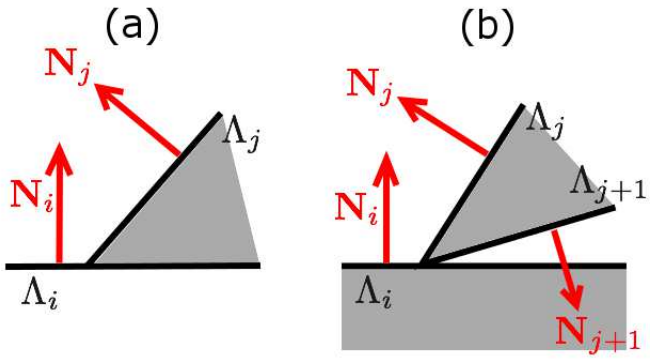

FIG. 6. *The case of facet-vertex intersection with $\Lambda_j(\bar{t}) \subset \mathcal{O}_i(\bar{t})$ (we omit the notation of $\bar{t}$ in the figure). In the situation of the figure (b), the origin should be on the gray regions including the boundary lines by Corollary 8. Thus, the kind of touch like as (b) never occur.*

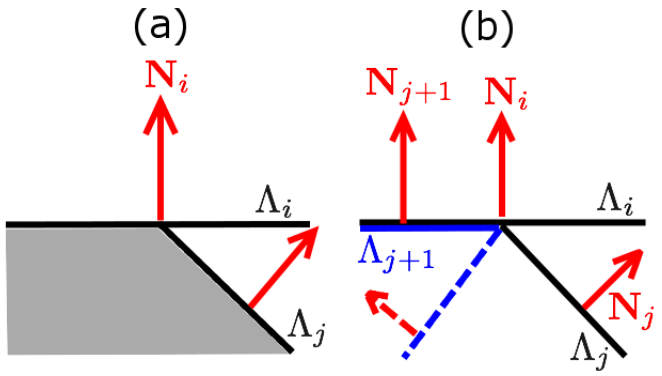

FIG. 7. *The case of facet-vertex intersection with $\Lambda_j \subset \mathcal{I}_i(\bar{t})$. The facet Λ_{j+1} never can be located as the dashed line in (b).*

for $i, j, j+1 \in \mathbb{Z}/(N_\gamma \mathbb{Z})$. This contradicts to the assumption (W1).

(II) Note that both $\Lambda_{i+1}(t)$ and $\Lambda_{j+1}(t)$ are not empty in a neighborhood of $t = \bar{t}$ by Lemma 11(iii). We may assume that $\varphi_i \le \varphi_{j+1} < \varphi_i + 2\pi$, and then it suffices to see $\varphi_i = \varphi_j$. We shall derive a contradiction by assuming that $\varphi_i \neq \varphi_j$. See the proof of Proposition 13 in §5.2 for details.

By (W1) and (A2) one can find that

$$\varphi_i < \varphi_{i+1} \le \varphi_j < \varphi_{j+1} \le \varphi_i + 2\pi. \tag{28}$$

Moreover, one can find $\varphi_j \neq \varphi_i + \pi$ and $\varphi_j \neq \varphi_{i+1} + \pi$ by the parallel argument of the proof of (ii) in Lemma 11. Accordingly, we divide the situation into three situations;

(a) $\varphi_{i+1} \le \varphi_j < \varphi_i + \pi$,
(b) $\varphi_i + \pi < \varphi_j < \varphi_{i+1} + \pi$, and
(c) $\varphi_{i+1} + \pi < \varphi_j < \varphi_i + 2\pi$.

See Figure 8 to find the region where $\Lambda_{j+1}(\bar{t})$ should be there. By (28) $\Lambda_{j+1}(\bar{t})$ should be gray painted and shaded regions. However, $\Lambda_{j+1}(\bar{t})$ never be located in the shaded regions by (W1) and (A2). Then, one can easily find that the case (c) never appear by similar argument of the proof of Lemma 11(i).

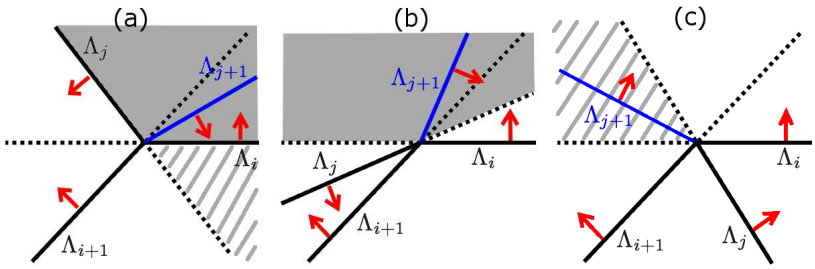

FIG. 8. *Location of $\Lambda_i(\bar{t})$, $\Lambda_{i+1}(\bar{t})$, $\Lambda_j(\bar{t})$ and $\Lambda_{j+1}(\bar{t})$ under the vertex-vertex intersection between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$. The above figures illustrate the case when (a)$\varphi_i < \varphi_j < \varphi_i + \pi$, (b)$\varphi_i + \pi < \varphi_j < \varphi_{i+1} + \pi$, and (c)$\varphi_{i+1} + \pi < \varphi_j < \varphi_i + 2\pi$. The gray regions are where $\Lambda_{j+1}(\bar{t})$ can be located, and gray shaded regions are where the cross type intersection appears between $\Lambda_i(\bar{t}) \cup \Lambda_{i+1}(\bar{t})$ and $\Lambda_j(\bar{t}) \cup \Lambda_{j+1}(\bar{t})$ although $\Lambda_{j+1}(\bar{t})$ seems to be located from the angle condition.*

If the case (a) appears, then we observe that $\varphi_i + \pi \le \varphi_{j+1} < \varphi_j + \pi < \varphi_i + 2\pi$ since cross-type intersection appears between $\Lambda_i(\bar{t}) \cup \Lambda_{i+1}(\bar{t})$ and $\Lambda_j(\bar{t}) \cup \Lambda_{j+1}(\bar{t})$ if $\varphi_j < \varphi_{j+1} < \varphi_i + \pi$. However, $\varphi_i + \pi \le \varphi_{j+1} < \varphi_i + 2\pi$ implies that

$$\overline{\mathcal{I}_j(\bar{t}) \cap \mathcal{I}_{j+1}(\bar{t})} \subset \overline{\mathcal{O}_i(\bar{t})}.$$

Then, we observe that $y_i(\bar{t}) = y_j(\bar{t}) = O$. Also if the case (b) appears, then we observe that $\varphi_{i+1} + \pi < \varphi_{j+1} < \varphi_i + 2\pi$. This implies

$$\overline{\mathcal{I}_i(\bar{t}) \cap \mathcal{I}_{i+1}(\bar{t})} \subset \overline{\mathcal{O}_j(\bar{t})}$$

and then $y_i(\bar{t}) = y_j(\bar{t}) = O$. Hence, we obtain $s_i(\bar{t}) = s_{i+1}(\bar{t}) = s_j(\bar{t}) = s_{j+1}(\bar{t}) = 0$ in the case (a) and (b), which contradicts to (ii) in Corollary 8. Consequently, we obtain the conclusion of (II).

(III) Assume that $\Lambda_i(\bar{t}) \neq \Lambda_j(\bar{t})$. When one is included by the other, we may assume $\Lambda_i(\bar{t}) \subset \Lambda_j(\bar{t})$ without loss of generality. In this case we find $i \neq 0$. In fact, if $i = 0$, then $\Lambda_j(\bar{t})$ is also unbounded, which contradicts to $d_j(t) < \infty$ for $j \ge 1$. Moreover, $\Lambda_i(\bar{t}) \subset \Lambda_j(\bar{t})$ implies that $\dot{s}_i(\bar{t}) = V_i(\bar{t}) < V_j(\bar{t}) = \dot{s}_j(\bar{t})$. Then, there exists $t < \bar{t}$ such that $(\Lambda_{i+1}(t) \cup \Lambda_{i-1}(t)) \cap \Lambda_j(t) \neq \emptyset$. This contradicts to the definition of $\bar{t}$.

It remains the case when $\Lambda_i(\bar{t}) \not\subset \Lambda_j(\bar{t})$ and $\Lambda_i(\bar{t}) \not\supset \Lambda_j(\bar{t})$. If $i \ge 1$ and $j \ge 1$, then we may assume that there exist $\lambda_3 > \lambda_2 > \lambda_1 > 0$ such that

$$y_j(\bar{t}) = y_i(\bar{t}) + \lambda_1 \mathbf{T}_i, \quad y_{i-1}(\bar{t}) = y_i(\bar{t}) + \lambda_2 \mathbf{T}_i, \quad y_{j-1}(\bar{t}) = y_i(\bar{t}) + \lambda_3 \mathbf{T}_i.$$

i.e., $y_j(\bar{t}) \in \Lambda_i^\circ(\bar{t})$ and $y_{i-1}(\bar{t}) \in \Lambda_j^\circ(\bar{t})$. In this case there exists $\mu > 0$ such that either the following (a) or (b) holds;

(a) $s_i < s_j$ on $[\bar{t} - \mu, \bar{t})$,
(b) $s_i > s_j$ on $[\bar{t} - \mu, \bar{t})$.

Then, if $s_i < s_j$ on $[\bar{t} - \mu, \bar{t})$, then there exists $t \in [\bar{t} - \mu, \bar{t})$ such that $\Lambda_i(t) \cap \Lambda_{j+1}(t) \neq \emptyset$. If $s_i > s_j$ on $[\bar{t} - \mu, \bar{t})$, there exists $t \in [\bar{t} - \mu, \bar{t})$ such that $\Lambda_{i-1}(t) \cap \Lambda_j(t) \neq \emptyset$. (See Figure 9.)

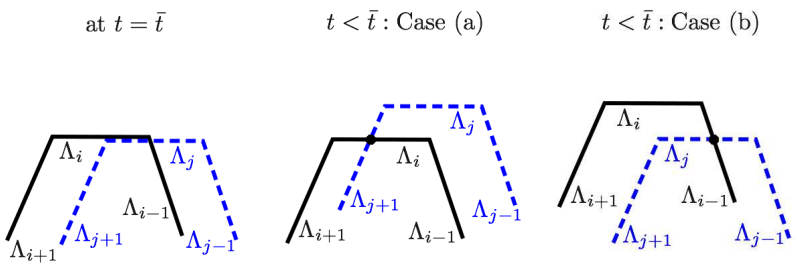

FIG. 9. *The case of facet-facet intersection with* $\Lambda_i(\bar{t}) \not\subset \Lambda_j(\bar{t})$ $\Lambda_i(\bar{t}) \not\supset \Lambda_j(\bar{t})$.

If $j=0$, then $\Lambda_i(\bar{t}) \not\subset \Lambda_j(\bar{t})$ and $\Lambda_i(\bar{t}) \not\supset \Lambda_j(\bar{t})$ implies that

$$y_0(\bar{t}) = y_i(\bar{t}) + \lambda_1 \mathbf{T}_i \in \Lambda_i^\circ(\bar{t}), \quad y_{i-1}(\bar{t}) = y_i(\bar{t}) + \lambda_2 \mathbf{T}_i \in \Lambda_0^\circ(\bar{t})$$

with constants $\lambda_2 > \lambda_1 > 0$. Moreover, we observe that $\dot{s}_j(\bar{t}) = V_0 > \dot{s}_i(\bar{t})$. Thus, we lead a contradiction with the same argument of the case (b) on the above. We thus obtain the conclusion of (III), and accordingly that of Theorem 10. ∎

4. Numerical simulations. In this section we consider the evolution equation

$$\beta(\mathbf{n})V = v_\infty(1 - \rho_c H_\gamma(\mathbf{n}, \nabla\mathbf{n})) \tag{29}$$

instead of (3) for the consistency with [1]; $v_\infty > 0$ and $\rho_c > 0$ are constants denoting the mobility of the evolution and the critical radius of the equilibrium form, respectively. One can easily obtain a solution to (3) (whose vertices are denoted by $y_j(t)$) from that to (29) (whose vertices are denoted by $\eta_j(\tau)$) by the scheme (SP) by rescaling the time and spatial paramters; set

$$y_j(t) = \frac{1}{\rho_c U}\eta_j\left(\frac{v_\infty t}{\rho_c U^2}\right).$$

In this section we only consider the initial curve of spiral as that satisfying (I1), and then the solution should be a positive convex spiral by Theorem 5. Consequently, we calculate the system (16) adapting the coefficients to (29). For example, the third equation of (16), which is typical one, should be revised as

$$\dot{d}_j = v_\infty\left(-b_j\left(1-\rho_c\frac{\ell_j}{d_j}\right) + c_j^+\left(1-\rho_c\frac{\ell_{j+1}}{d_{j+1}}\right) + c_j^-\left(1-\rho_c\frac{\ell_{j-1}}{d_{j-1}}\right)\right),$$

and the other equations should be revised as the same manner. Note that $\rho_c W_\gamma = \{\rho_c x \in \mathbb{R}^2;\ x \in \mathcal{W}_\gamma\}$ is a stationary solution of closed curve to (29), and then the critical length of j-th facet evolving by (29) is $\rho_c \ell_j$. One can easily calculates the system (16) for (29) with an usual explicit finite difference scheme. The figure 10 presents a numerical result of the evolution of a polygonal spiral at time $t = 0, 0.1, 0.5$ and 1.0 with triangle $\mathcal{W}_\gamma$ and isotropic β. We set the parameters of $\mathcal{W}_\gamma$ as

$$N_\gamma = 3,\ \theta_0 = 0,\ \theta_1 = \frac{2\pi}{3},\ \theta_2 = \frac{5\pi}{4},\ \ell_0 = 3,\ \ell_1 = \sqrt{5},\ \ell_2 = 2\sqrt{2}.$$

The parameters of (29) are

$$\beta \equiv 1,\ v_\infty = 1,\ \rho_c = 0.02.$$

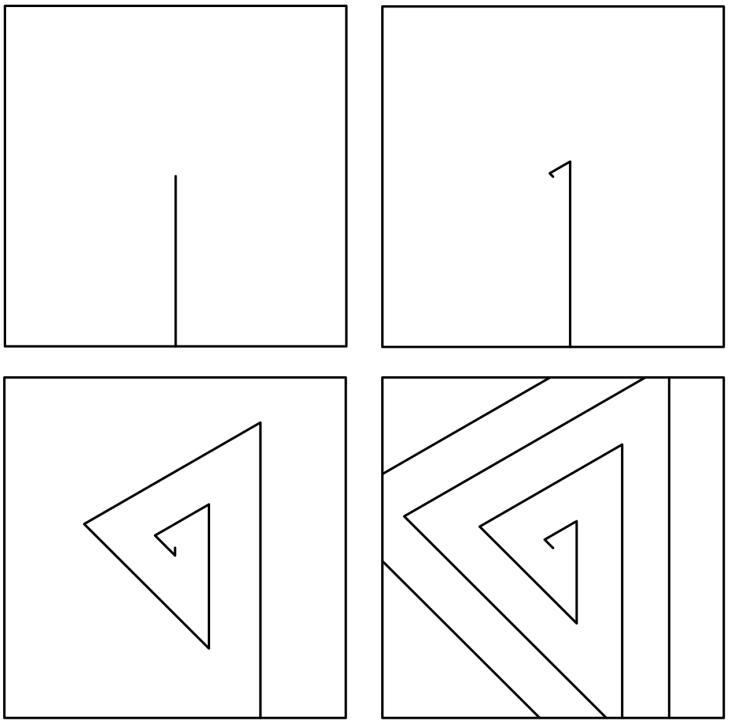

FIG. 10. *Evolution of a polygonal spiral with a triangle Wulff shape. The figures illustrate the shape of a spiral at* $t = 0.0, 0.1, 0.5, 1.0$ *from top left to bottom right.*

The figure 11 presents the numerical results of evolving spirals at $t = 1$ comparing the profiles of spirals with respect to some different anisotropic mobilities under the same $\mathcal{W}_\gamma$. We choose the parameters for $\mathcal{W}_\gamma$ and (29) as

$$N_\gamma = 6,\ \theta_j = \frac{\pi j}{3},\ \ell_j = 1 \quad \text{for } j = 0, 1, 2, 3, 4, 5,$$
$$v_\infty = 1,\ \rho_c = 0.02$$

in these simulations. On the other hand, β_j in these simulation are choosen as

$$\beta_j = \begin{cases} 1 & \text{if } j \text{ is even,} \\ \dfrac{n}{4} & \text{otherwise,} \end{cases}$$

with (a)$n = 4$, (b)$n = 3$, (c)$n = 2$ and (d)$n = 1$. One can find that the shape of spirals on the far away region from the center may reflect the anisotropy of $\beta V = 1$. For evolution of closed curve, Yazaki [23] shows the asymptotic behavior of solution to the Wulff shape for the anisotropy of $\beta V = 1$. In the context of this paper one was afraid that the facets surrounded by other facets having relatively small mobility (and then large velocity) might be vanished. However, a priori estimates in Lemma 7 or Corollary 8 guarantee that the all facets never be vanished and evolve with positive normal velocities. On the other hand, in particular the estimate (iii) in Lemma 7 does not guarantee that $\lim_{t\to\infty} d_j(t) = \infty$; see case (d) in Figure 11.

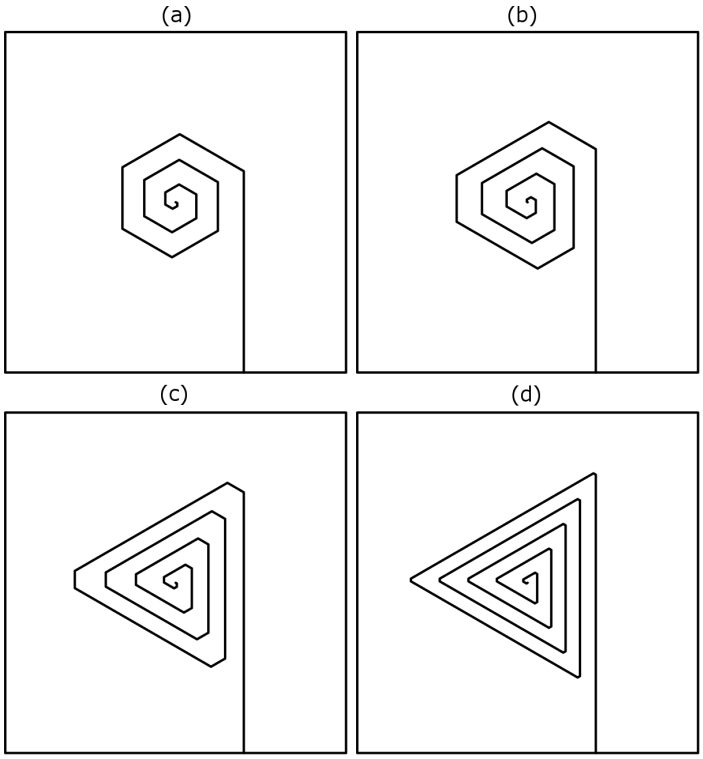

FIG. 11. *Comparison of the profiles of spirals at $t = 1$ with respect to different anisotropic mobilities under the same $\mathcal{W}_\gamma$.*

5. Appendix.

5.1. Remark on the orientation of a new facet. We now remark on the rule of the orientation of generated facet in the generation rule (G) in §2.2. Formally, there exists a possibility setting $\nu(k+1) = \nu(k) - 1$ when $L_{k+1}(t)$ is generated at $t = T_{k+1}$ provided that either $\Gamma(t)$ is positive and $U > 0$ or $\Gamma(t)$ is negative and $U < 0$. We here mention why we can avoid the above situation. Note that the case for negative $\Gamma(t)$ with $U > 0$ (or positive $\Gamma(t)$ with $U < 0$) is considered in parallel.

If we can avoid the assumptions of $\Gamma(t)$ in (I1) or (I2i)–(I2ii), one attempt to consider the evolution of positive and convex $\Gamma(t) = \bigcup_{j=0}^{k+1} L_j(t)$ with an initial data $\Gamma(T_{k+1})$ satisfying

$$d_{k+1}(T_{k+1}) = 0, \quad d_k(T_{k+1}) = \ell_{\nu(k)}/U, \quad d_{k-1}(T_{k+1}) < \ell_{\nu(k)-1}/U.$$

If $L_{k+1}(t)$ is oriented as $\nu(k+1) = \nu(k)+1$, then one can find $d_{k+1} < 0$ in $(T_{k+1}, T_{k+1}+\mu)$ with a small $\mu > 0$. Thus, if we keep setting $y_k(t) = y_{k+1}(t) + d_{k+1}(t)\boldsymbol{\tau}_{k+1}$ in spite of the fact that $d_{k+1}(t) < 0$, the admissibility of $\Gamma(t)$ is broken (see Figure 12(a)). It is natural for the above case that one attempt to set $\nu(k+1) = \nu(k) - 1$ to keep the admissibility of $\Gamma(t)$. (See figure 12(b).)

However, if we set $\nu(k+1) = \nu(k) - 1$, then we have $\sigma_k = 0$, and then the evolution equation for $L_k(t)$ should be $\beta_k V_k = U$, which implies $s_k(t) > 0$ in a very

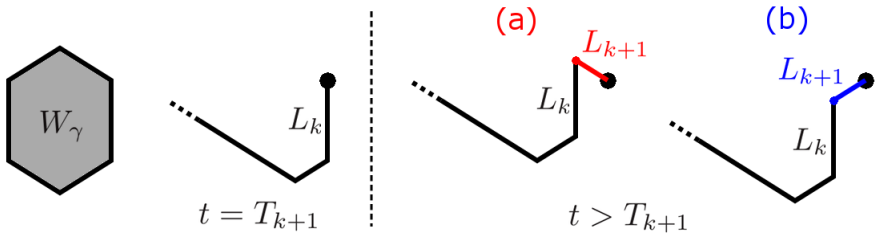

FIG. 12. *The examination of orientation for L_{k+1} provided that $d_k(T_{k+1}) = \ell_k/U$ and $d_{k-1}(T_{k+1}) < \ell_{k-1}/U$ with hexagonal $\mathcal{W}_\gamma$.*

short time from T_{k+1}. Moreover, we observe that

$$\dot{d}_{k+1} = c_{k+1}^- U < 0 \quad \text{since } c_{k+1}^- = \frac{1}{\beta_k \sin(\varphi_{\nu(k)-1} - \varphi_{\nu(k)})} < 0,$$

which implies that $d_{k+1}(t) < 0$ for $t \in (T_{k+1}, T_{k+1} + \mu)$ with a very small $\mu > 0$. Consequently, there is no way to set $\nu(k+1)$ without breaking the admissibility of $\Gamma(t)$ for the above toy case. It is very important to guarantee $d_k \geq \ell_{\nu(k)}/U$ for $t > T_{k+1}$ provided that $\Gamma(t) = \bigcup_{j=0}^{k+1} L_j(t)$ is convex and $U > 0$, and then we can avoid the case $\nu(k+1) = \nu(k) - 1$ under the above guaranty.

5.2. Proof of Theorem 10. In this section we give a proof on the observations (I), (II) and (III) in the proof of Theorem 10. We assume that $\Gamma(t)$ is positive convex, and then (14) holds. Throughout this section we consider $\Gamma(t) = \bigcup_{j=0}^{k} \Lambda_j(t)$ has a first touch time $\bar{t} < \infty$ which is defined as (23), and $\Lambda_i(\bar{t}) \cap \Lambda_j(\bar{t}) \neq \emptyset$. We may assume that (25), i.e.,

$$\varphi_i \leq \varphi_j < \varphi_i + 2\pi$$

without loss of generality. Moreover, we also note that $\Lambda_{i+1}(t)$ and $\Lambda_{j+1}(t)$ exist in a neighborhood of $\bar{t}$ by Lemma 11 (iii).

We first demonstrate the observation (I) on the facet-vertex type intersection; when the facet-vertex type intersection appears, then the adjacent facets have the facet-facet type intersection.

PROPOSITION 12. *If $\Gamma(\bar{t})$ has facet-vertex type intersection between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$ with $\Lambda_i^\circ(\bar{t}) \cap \Lambda_j(\bar{t}) = \{y_j(\bar{t})\}$, then $\Lambda_i(\bar{t})$ and $\Lambda_{j+1}(\bar{t})$ has facet-facet type intersection.*

Proof. Note that $\Lambda_j^\circ(\bar{t}) \cap \Lambda_i(\bar{t}) = \emptyset$, which implies $\varphi_j \neq \varphi_i$ and $\varphi_j \neq \varphi_i + \pi$ provided that (25) holds. Then, the above and assumptions (W1), (W2) and (A2) yield that

$$\varphi_i < \varphi_j < \varphi_{j+1} \leq \varphi_i + 2\pi. \tag{30}$$

We now divide the proof into the following two situations;

(i) $\Lambda_j^\circ(\bar{t}) \subset \mathcal{O}_i(\bar{t})$ (which implies $\varphi_i < \varphi_j < \varphi_i + \pi$),
(ii) $\Lambda_j^\circ(\bar{t}) \subset \mathcal{I}_i(\bar{t})$ (which implies $\varphi_i + \pi < \varphi_j < \varphi_i + 2\pi$),

where $\mathcal{O}_i(\bar{t})$ or $\mathcal{I}_i(\bar{t})$ are define as (26) or (27), respectively.

Case (i)**;** if $\Lambda_j^\circ(\bar{t}) \subset \mathcal{O}_i(\bar{t})$, then we observe that

$$\mathbf{T}_j \cdot \mathbf{N}_i > 0, \quad \mathbf{T}_{j+1} \cdot \mathbf{N}_i < 0. \tag{31}$$

In fact, the first inequality of (31) is derived directly from

$$\mathbf{T}_j \cdot \mathbf{N}_i = \sin(\varphi_j - \varphi_i) > 0.$$

On the other hand, we first deduce a contradiction by assuming $\mathbf{T}_{j+1} \cdot \mathbf{N}_i > 0$ to prove the second inequality of (31). Note that

$$\Lambda_{j+1}(\bar{t}) = \{y_j(\bar{t}) - r\mathbf{T}_{j+1};\ r \in (0, d_{j+1}(\bar{t})]\}$$

by (7), which and $\mathbf{T}_{j+1} \cdot \mathbf{N}_i > 0$ imply

$$x \cdot \mathbf{N}_i - s_i(\bar{t}) = -|x - y_j|\mathbf{T}_{j+1} \cdot \mathbf{N}_i < 0 \quad \text{provided that } x \in \Lambda_{j+1}^\circ(\bar{t}) \tag{32}$$

since $s_i(\bar{t}) = y_j(\bar{t}) \cdot \mathbf{N}_i$ by $y_j \in \Lambda_i(\bar{t})$. We now choose small $\rho > 0$ and a neighborhood of $y_j(\bar{t})$ as

$$B = \{x \in \mathbb{R}^2;\ |x - y_j(\bar{t})| < \rho\}, \tag{33}$$

and $\mu_0 = \mu_0(\rho) > 0$ enough small so that

$$B \cap \partial\mathcal{O}_i(t) = B \cap \partial\mathcal{I}_i(t) = B \cap \Lambda_i^\circ(t) \quad \text{for } t \in [\bar{t} - \mu_0, \bar{t}] \tag{34}$$

by the continuity of every vertex $y_m(t)$ for $m = 0, 1, 2, \ldots, k$. Let us set

$$P_j(t,r) = \begin{cases} y_j(t) + r\mathbf{T}_{j+1} \in \Lambda_{j+1}(t) & \text{if } r \in (-d_{j+1}(t), 0), \\ y_j(t) + r\mathbf{T}_j \in \Lambda_j(t) & \text{if } r \in [0, d_j(t)), \end{cases} \tag{35}$$

$$C_{j,\rho}(t) := \{P_j(t,r);\ r \in [-\rho/2, \rho/2]\}, \tag{36}$$

$$C_j(t) := \Lambda_{j+1}^\circ(t) \cup \Lambda_j(t) \tag{37}$$

for $t \in [\bar{t} - \mu_0, \bar{t}]$ (see Figure 13 for the above notations). Note that $C_{j,\rho}(\bar{t}) \subset B$.

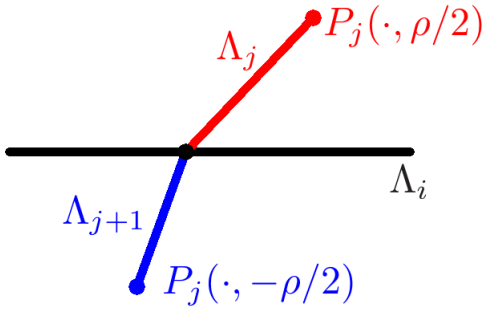

FIG. 13. *The situation when* (31) *does not hold in Case (i).*

Then, there exists $\bar{\mu} < \mu_0$ such that

$$\left.\begin{array}{l} C_{j,\rho}(t) \subset B, \\ C_{j,\rho}(t) \subset (\Lambda_j(t) \cup \Lambda_{j+1}(t)), \\ P_j(t, -\rho/2) \cdot \mathbf{N}_i < s_i(t),\ P_j(t, \rho/2) \cdot \mathbf{N}_i > s_i(t) \end{array}\right\} \quad \text{for } t \in [\bar{t} - \bar{\mu}, \bar{t}]$$

by continuity of every vertex of $\Gamma(t)$, $\Lambda_j^\circ(\bar{t}) \subset \mathcal{O}_i(\bar{t})$ and (32). Then, the above and (34) implies that $C_j(t) \cap \Lambda_i^\circ(t) \neq \emptyset$ for $t \in [\bar{t} - \bar{\mu}, \bar{t}]$, which contradicts to the definition of $\bar{t}$. Hence, we first obtain $\mathbf{T}_{j+1} \cdot \mathbf{N}_i \leq 0$, which implies $\varphi_i + \pi \leq \varphi_{j+1} \leq \varphi_i + 2\pi$. However, $\varphi_{j+1} < \varphi_j + \pi < \varphi_i + 2\pi$. Thus, if $\mathbf{T}_{j+1} \cdot \mathbf{N}_i = 0$ then $\varphi_{j+1} = \varphi_i + \pi$ which contradicts to Lemma 11(ii). Hence, we obtain $\mathbf{T}_{j+1} \cdot \mathbf{N}_i < 0$.

We next deduce a contradiction from (31). By straightforward calculation we observe that

$$\begin{aligned}\overline{\mathcal{I}_j(\bar{t})} \cap \overline{\mathcal{I}_{j+1}(\bar{t})} &= \{x \in \mathbb{R}^2;\ x \cdot \mathbf{N}_j \leq s_j(\bar{t}),\ x \cdot \mathbf{N}_{j+1} \leq s_{j+1}(\bar{t})\} \\ &= \{y_j(\bar{t}) + r\lambda \mathbf{T}_j - r(1-\lambda)\mathbf{T}_{j+1};\ r \geq 0,\ \lambda \in [0,1]\}.\end{aligned}$$

This implies that

$$\begin{aligned}&x \cdot \mathbf{N}_i \geq s_i(\bar{t}) \quad \text{for } x \in \overline{\mathcal{I}_j(\bar{t})} \cap \overline{\mathcal{I}_{j+1}(\bar{t})}, \text{ and} \\ &x \cdot \mathbf{N}_i = s_i(\bar{t}) \quad \text{if and only if } x = y_j(\bar{t}) \text{ provided that } x \in \overline{\mathcal{I}_j(\bar{t})} \cap \overline{\mathcal{I}_{j+1}(\bar{t})}.\end{aligned}$$

Hence, we obtain $\overline{\mathcal{I}_i(\bar{t})} \cap \overline{\mathcal{I}_j(\bar{t})} \cap \overline{\mathcal{I}_{j+1}(\bar{t})} = \{y_j(\bar{t})\}$. Note that $O \in \overline{\mathcal{I}_i(\bar{t})} \cap \overline{\mathcal{I}_j(\bar{t})} \cap \overline{\mathcal{I}_{j+1}(\bar{t})}$ by Corollary 8, which implies $s_i(\bar{t}) = s_j(\bar{t}) = s_{j+1}(\bar{t}) = 0$ and contradicts to Corollary 8 (ii).

Case (ii)**;** if $\Lambda_j^\circ(\bar{t}) \subset \mathcal{I}_i(\bar{t})$, then we observe that

$$\mathbf{T}_j \cdot \mathbf{N}_i < 0, \quad \mathbf{T}_{j+1} \cdot \mathbf{N}_i \geq 0 \tag{38}$$

by the similar argument of the Case (i). By straightforward calculation we have

$$\mathbf{T}_j \cdot \mathbf{N}_i = \sin(\varphi_j - \varphi_i) < 0, \quad \mathbf{T}_{j+1} \cdot \mathbf{N}_i = \sin(\varphi_{j+1} - \varphi_i) \geq 0,$$

which imply $\varphi_i + \pi < \varphi_j < \varphi_i + 2\pi$ and $\varphi_i < \varphi_{j+1} \leq \varphi_i + \pi$ or $\varphi_{j+1} = \varphi_i + 2\pi$. Consequently we observe that $\varphi_{j+1} = \varphi_i + 2\pi$ by (W2), and then $\Lambda_i(\bar{t})$ and $\Lambda_{j+1}(\bar{t})$ has facet-facet type intersection. □

We next demonstrate the observation (II); if the vertex-vertex type intersection appears, then the two pair of facets associated with the touched pair of vertices have facet-facet type intersections.

PROPOSITION 13. *If $\Gamma(\bar{t})$ has a vertex-vertex type self-intersection between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$, then the both pair $(\Lambda_i(\bar{t}), \Lambda_j(\bar{t}))$ and $(\Lambda_{i+1}(\bar{t}), \Lambda_{j+1}(\bar{t}))$ have facet-facet type intersection.*

Proof. It suffices to prove $\mathbf{N}_i = \mathbf{N}_j$, and thus $\varphi_i = \varphi_j$ provided that (25) holds. We now assume that $\varphi_i \neq \varphi_j$ and derive a contradiction.

By (W1), (W2) and (A2) one can find that

$$\varphi_i < \varphi_{i+1} \leq \varphi_j < \varphi_{j+1} \leq \varphi_i + 2\pi. \tag{39}$$

Moreover, we observe that $\varphi_j \neq \varphi_i + \pi$, and $\varphi_j \neq \varphi_{i+1} + \pi$ by the similar argument as in Lemma 11(ii). Then, we divide the proof into three cases;

(a) $\varphi_{i+1} \leq \varphi_j < \varphi_i + \pi$,

(b) $\varphi_i + \pi < \varphi_j < \varphi_{i+1} + \pi$,

(c) $\varphi_{i+1} + \pi < \varphi_j < \varphi_i + 2\pi$.

We first lead a contradiction by assuming (c). We also choose constants $\rho > 0$ and $\mu_0 = \mu_0(\rho)$ small enough so that a neighborhood B of $y_j(\bar{t})$ as (33) satisfies

$$\left.\begin{aligned} &B \cap \partial(\mathcal{I}_i(t) \cap \mathcal{I}_{i+1}(t)) = C_i(t) \cap B, \\ &C_{j,\rho}(t) \subset B \end{aligned}\right\} \quad \text{for } t \in [\bar{t} - \mu_0, \bar{t}] \tag{40}$$

by continuity of every $y_\ell(t)$ for $\ell = 0, 1, 2, \dots, k$, where $C_i(t)$ or $C_{j,\rho}(t)$ are defined as in (37) or (36), respectively. The assumption (c) and (39) imply

$$\begin{aligned} &\mathbf{T}_j \cdot \mathbf{N}_i < 0, \quad \mathbf{T}_j \cdot \mathbf{N}_{i+1} < 0, \\ &\mathbf{T}_{j+1} \cdot \mathbf{N}_i \le 0, \quad \mathbf{T}_{j+1} \cdot \mathbf{N}_{i+1} < 0 \end{aligned}$$

since $\mathbf{T}_j \cdot \mathbf{N}_i = \sin(\varphi_j - \varphi_i)$, and then

$$P_j(\bar{t}, \rho/2) \in \mathcal{I}_i(\bar{t}) \cap \mathcal{I}_{i+1}(\bar{t}), \quad P_j(\bar{t}, -\rho/2) \in \mathcal{O}_i(\bar{t}) \subset \mathrm{Int}(\mathcal{I}_i(\bar{t}) \cap \mathcal{I}_{i+1}(\bar{t}))^c,$$

where P_j is defined as (35). Then, there exists $\bar{\mu} \in (0, \mu_0)$ satisfying

$$P_j(t, \rho/2) \in \mathcal{I}_i(t) \cap \mathcal{I}_{i+1}(t), \quad P_j(t, -\rho/2) \in \mathrm{Int}(\mathcal{I}_i(t) \cap \mathcal{I}_{i+1}(t))^c,$$

for $t \in [\bar{t} - \bar{\mu}, \bar{t}]$ by continuity of $y_j, y_{j\pm1}, y_i$ and $y_{i\pm1}$. The above and (40) yield that $C_{j,\rho}(t) \cap C_i(t) \neq \emptyset$ for $t \in [\bar{t} - \bar{\mu}, \bar{t}]$, which contradicts to the definition of $\bar{t}$.

We next consider the case (a); $\varphi_{i+1} \le \varphi_j < \varphi_i + \pi$. In this case we observe that $\varphi_i + \pi < \varphi_{j+1} < \varphi_j + \pi$. In fact, if $\varphi_{j+1} < \varphi_i + \pi$ (see Figure 14 for this case), then

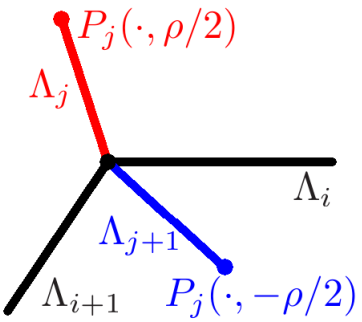

FIG. 14. *The situation case(a) with* $\varphi_{j+1} \in (\varphi_{i+1}, \varphi_i + \pi)$.

we have $\varphi_{i+1} \le \varphi_j < \varphi_{j+1} < \varphi_{i+1} + \pi < \varphi_{i+1} + \pi$ so that

$$\mathbf{T}_j \cdot \mathbf{N}_i > 0, \quad \mathbf{T}_{j+1} \cdot \mathbf{N}_i < 0, \quad \mathbf{T}_{j+1} \cdot \mathbf{N}_{i+1} < 0.$$

This implies

$$P_j(\bar{t}, \rho/2) \in (\mathcal{I}_i(\bar{t}) \cap \mathcal{I}_{i+1}(\bar{t}))^c, \quad P_j(\bar{t}, -\rho/2) \in \mathcal{I}_i(\bar{t}) \cap \mathcal{I}_{i+1}(\bar{t}),$$

which yields the contradiction with the similar argument as the case (c). Then we obtain $\varphi_{i+1} + \pi \le \varphi_{j+1} \le \varphi_j + \pi < \varphi_i + 2\pi$. However, we also have $\varphi_{j+1} \neq \varphi_{i+1} + \pi$ with a similar argument of the proof of Lemma 11(ii). Hence we obtain $\varphi_i + \pi < \varphi_{i+1} + \pi < \varphi_{j+1} < \varphi_j + \pi$. The above and the assumption (a) imply

$$\mathbf{T}_j \cdot \mathbf{N}_i > 0, \quad \mathbf{T}_{j+1} \cdot \mathbf{N}_i < 0, \tag{41}$$

which yields a contradiction by the similar argument as Case (i) in the proof of Proposition 12.

Finally, we consider the case (b); $\varphi_i + \pi < \varphi_j < \varphi_{i+1} + \pi$. Then we observe that

$$-2\pi < \varphi_i - \varphi_j < -\pi, \quad -\pi < \varphi_{i+1} - \varphi_j < 0$$

since $\varphi_{i+1} < \varphi_i + \pi$, which implies that

$$\mathbf{T}_i \cdot \mathbf{N}_j > 0, \quad \mathbf{T}_{i+1} \cdot \mathbf{N}_j < 0.$$

Then, we derive the contradiction with the similar argument as the case (i) in the proof of Proposition 12.

Hence, we obtain the conclusion of Proposition 13. ∎

Finally, we demonstrate the observation (III); if the pair of facets has facet-facet type intersection, then the facets are agree with each other.

PROPOSITION 14. *If $\Gamma(\bar{t})$ has a facet-facet type intersection between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$, then $\Lambda_i(\bar{t}) = \Lambda_j(\bar{t})$ with $\mathbf{N}_i = \mathbf{N}_j$.*

Proof. Note that $\mathbf{N}_i = \mathbf{N}_j$ follows from Lemma 11(ii). Then, we now lead a contradiction with assuming $\Lambda_i(\bar{t}) \neq \Lambda_j(\bar{t})$. By definition of $s_j(t)$ we have $y_j \cdot \mathbf{N}_j = y_{j-1} \cdot \mathbf{N}_j = s_j$, and we also have $y_j \cdot \mathbf{N}_i = y_{j-1} \cdot \mathbf{N}_i = s_j$ since $\mathbf{N}_i = \mathbf{N}_j$. Moreover, by assumption we now have

$$y_j(\bar{t}) \cdot \mathbf{N}_i = y_{j-1}(\bar{t}) \cdot \mathbf{N}_i = s_i(\bar{t}) = s_j(\bar{t}). \tag{42}$$

We also note that each $\Lambda_{i+1}(t)$ or $\Lambda_{j+1}(t)$ is not empty in a neighborhood of $t = \bar{t}$ by Lemma 11(iii).

We first note that $\mathcal{I}_{i+1}(t) \cap \mathcal{I}_{i-1}(t) \cap \mathcal{L}_i(t) = \Lambda_i^\circ(t)$ for $t \geq T_{i+1}$ provided that the set on the left hand is not empty, where

$$\mathcal{L}_i(t) = \{x \in \mathbb{R}^2;\ x \cdot \mathbf{N}_i = s_i(t)\}.$$

In fact, if $x \in \Lambda_i^\circ(t)$, then we have

$$x = y_i(t) + r\mathbf{T}_i = y_{i-1}(t) - (d_i(t) - r)\mathbf{T}_i \tag{43}$$

with some $r \in (0, d_i(t))$. Then, we have

$$\begin{aligned} x \cdot \mathbf{N}_i &= y_i(t) \cdot \mathbf{N}_i = s_i(t), \\ x \cdot \mathbf{N}_{i+1} &= y_i(t) \cdot \mathbf{N}_{i+1} + r\sin(\varphi_i - \varphi_{i+1}) > s_{i+1}(t), \qquad (44) \\ x \cdot \mathbf{N}_{i-1} &= y_{i-1}(t) \cdot \mathbf{N}_{i-1} - (d_i(t) - r)\sin(\varphi_i - \varphi_{i-1}) > s_{i-1}(t). \qquad (45) \end{aligned}$$

Hence, we obtain $x \in \mathcal{I}_{i+1}(t) \cap \mathcal{I}_{i-1}(t) \cap \mathcal{L}_i(t)$, which implies $\Lambda_i^\circ(t) \subset \mathcal{I}_{i+1}(t) \cap \mathcal{I}_{i-1}(t) \cap \mathcal{L}_i(t)$.

To see $\Lambda_i^\circ(t) \supset \mathcal{I}_{i+1}(t) \cap \mathcal{I}_{i-1}(t) \cap \mathcal{L}_i(t)$, we now demonstrate that $x \notin \mathcal{I}_{i+1}(t) \cap \mathcal{I}_{i-1}(t) \cap \mathcal{L}_i(t)$ provided that $x \notin \Lambda_i^\circ(t)$. In fact, $x \notin \Lambda_i^\circ(t)$ implies that $x \notin \mathcal{L}_i(t)$ or $x \in \mathcal{L}_i(t) \setminus \Lambda_i^\circ(t)$. So, it suffices to see $x \notin \mathcal{I}_{i+1}(t) \cup \mathcal{I}_{i-1}$ when $x \in \mathcal{L}_i(t) \setminus \Lambda_i^\circ(t)$.

In fact, we have (43) with $r \le 0$ or $r \ge d_i(t)$ by the assumption. If $r \le 0$, then we obtain $x \cdot \mathbf{N}_{i+1} \le s_{i+1}(t)$ by the first equality of (44), which implies $x \notin \mathcal{I}_{i+1}(t)$. On the other hand, if $r \ge d_i(t)$, then we obtain $x \cdot \mathbf{N}_{i-1} \le s_{i-1}(t)$ by the first equality of (45), which implies $x \notin \mathcal{I}_{i-1}(t)$. Hence, we obtain $\Lambda_i^\circ(t) = \mathcal{I}_{i+1}(t) \cap \mathcal{I}_{i-1}(t) \cap \mathcal{L}_i(t)$.

We now divide the proof of Proposition 14 into two cases.

Step 1. Consider the case $\Lambda_j(\bar t) \subset \Lambda_i(\bar t)$ or $\Lambda_j(\bar t) \supset \Lambda_i(\bar t)$. We may assume that $\Lambda_j(\bar t) \subset \Lambda_i(\bar t)$ by switching the number i and j if necessary. Note that $j \neq 0$. In fact, if $j = 0$ then $\Lambda_i(\bar t)$ and $\Lambda_j(\bar t)$ are unbounded, which contradicts to $d_i < \infty$ for $i \ge 1$. Since $\Lambda_i(\bar t) \neq \Lambda_j(\bar t)$, we observe that $y_j \in \Lambda_i^\circ(\bar t)$ or $y_{j-1} \in \Lambda_i^\circ(\bar t)$. Since the proofs are parallel, we now consider only the case when $y_j \in \Lambda_i^\circ(\bar t)$.

The above assumptions imply $d_j(\bar t) < d_i(\bar t)$. Then, we have $\dot s_j(\bar t) < \dot s_i(\bar t)$ since $\dot s_m = V_m = \beta_m^{-1}(U - \ell_m/d_m)$ for $m \ge 1$ and $\mathbf{N}_i = \mathbf{N}_j$ implies $j = i + nN_\gamma$ with $n \in \mathbb{Z}$. Then, there exists $\mu_0 > 0$ such that

$$s_j(t) < s_i(t) \quad \text{for } t \in [\bar t - \mu_0, \bar t). \tag{46}$$

Let us consider

$$z(t,r) = y_j(t) - r\mathbf{T}_{j+1} \in \overline{\Lambda_{j+1}(t)}$$

for $r \in [0, d_j(t)]$. Since $y_j \in \Lambda_i^\circ(\bar t) = \mathcal{I}_{i+1}(\bar t) \cap \mathcal{I}_{i-1}(\bar t) \cap \mathcal{L}_i(\bar t)$ we have

$$y_j(\bar t) \cdot \mathbf{N}_{i+1} < s_{i+1}(\bar t), \quad y_j(\bar t) \cdot \mathbf{N}_{i-1} < s_{i-1}(\bar t), \quad y_j(\bar t) \cdot \mathbf{N}_i = s_i(\bar t).$$

Then, we first obtain

$$z(\bar t, r) \cdot \mathbf{N}_{i+1} = y_j(\bar t) \cdot \mathbf{N}_{i+1} < s_{i+1}(\bar t) \quad \text{for } r \in [0, d_j(\bar t))]. \tag{47}$$

We next observe that there exists $r_0 \in (0, d_{j+1}(\bar t))$ such that

$$z(\bar t, r) \cdot \mathbf{N}_{i-1} < s_{i-1}(\bar t) \quad \text{for } r \in [0, r_0]. \tag{48}$$

Finally, we observe that

$$z(\bar t, 0) \cdot \mathbf{N}_i = y_j(\bar t) \cdot \mathbf{N}_i = s_i(\bar t), \quad z(\bar t, r_0) \cdot \mathbf{N}_i = s_i(\bar t) - r_0 \sin(\varphi_{j+1} - \varphi_j) < s_i(\bar t)$$

by (42) and (W2). Then, (46) and the continuity of $y_m(t)$ and $d_{j+1}(t)$ imply that there exists $\mu_1 \in (0, \mu_0)$ such that $z(t,r) \in \Lambda_{j+1}^\circ(t)$ for $t \in [\bar t - \mu_1, \bar t]$ and $r \in (0, r_0)$, and

$$\begin{aligned} &z(t,r) \cdot \mathbf{N}_{i+1} < s_{i+1}(t), \quad z(t,r) \cdot \mathbf{N}_{i-1} < s_{i-1}(t) \quad \text{for } (t,r) \in [\bar t - \mu_1, \bar t] \times [0, r_0], \\ &z(t,0) \cdot \mathbf{N}_i > s_i(t), \quad z(t,r_0) \cdot \mathbf{N}_i < s_i(t) \quad \text{for } t \in [\bar t - \mu_1, \bar t). \end{aligned}$$

The above implies that $\Lambda_{j+1}(t) \cap \Lambda_i^\circ(t) \neq \emptyset$ for $t \in [\bar t - \mu_1, \bar t]$, which contradicts to the definition of $\bar t$.

Step 2. We next consider the case $\Lambda_i(\bar t) \not\subset \Lambda_j(\bar t)$ and $\Lambda_i(\bar t) \not\supset \Lambda_j(\bar t)$. We first consider the case when $i \ge 1$ and $j \ge 1$. Then, either the following (A) or (B) holds.

(A) $y_{i-1}(\bar t) \in \Lambda_j^\circ(\bar t)$ and $y_j(\bar t) \in \Lambda_i^\circ(\bar t)$,
(B) $y_{j-1}(\bar t) \in \Lambda_i^\circ(\bar t)$ and $y_i(\bar t) \in \Lambda_j^\circ(\bar t)$.

Since the proofs for the above cases are parallel, we now consider only the case (A).

By definition of $\bar{t}$ there exists $\mu_0 > 0$ such that

$$\text{(a) } s_i(t) < s_j(t) \quad \text{or} \quad \text{(b) } s_i(t) > s_j(t) \quad \text{for } t \in [\bar{t} - \mu_0, \bar{t})$$

even if $\dot{s}_i(\bar{t}) = \dot{s}_j(\bar{t})$ (See Figure 9). In fact, if not, then there exists a sequence $t_n < \bar{t}$ satisfying $s_i(t_n) = s_j(t_n)$ and $\lim_{n\to\infty} t_n = \bar{t}$ by the continuity of s_i and s_j. This implies that $\Lambda_i^\circ(t_n) \cap \Lambda_j^\circ(t_n) \neq \emptyset$ for a enough large n by the continuity of $y_{i-1}(t)$ and $y_j(t)$. Consequently, we obtain the followings with the similar argument as Step 1.

(a) If $s_i < s_j$ on $[\bar{t} - \mu_0, \bar{t})$, then, by $y_j(\bar{t}) \in \Lambda_i^\circ(\bar{t})$, there exists $\mu_1 \in (0, \mu_0)$ such that $\Lambda_{j+1}(t) \cap \Lambda_i(t) \neq \emptyset$ for $t \in [\bar{t} - \mu_1, \bar{t}]$.

(b) If $s_i > s_j$ on $[\bar{t} - \mu_0, \bar{t})$ then, by $y_{i-1}(\bar{t}) \in \Lambda_j^\circ(\bar{t})$, there exists $\mu_1 \in (0, \mu_0)$ such that $\Lambda_{i-1}(t) \cap \Lambda_j(t) \neq \emptyset$ for $t \in [\bar{t} - \mu_1, \bar{t}]$.

We next consider the case $j = 0$. (Switch the notation i and j in the following argument when $i = 0$.) Note that $i \geq 1$ and then $\Lambda_{i-1}(t)$ is not empty in a neighborhood of $t = \bar{t}$. In this case we observe that $y_{i-1}(\bar{t}) \in \Lambda_j^\circ(\bar{t})$, $y_j(\bar{t}) = y_0(\bar{t}) \in \Lambda_i^\circ(\bar{t})$ and $\dot{s}_j(\bar{t}) = V_0 = \beta_0^{-1} U > \dot{s}_i(\bar{t})$. Then, there exists $\mu_0 > 0$ such that $s_i > s_j$ on $[\bar{t} - \mu_0, \bar{t})$, i.e., the case (b) with $y_{i-1}(\bar{t}) \in \Lambda_j^\circ(\bar{t})$ of the above appears. Thus, there exists $\mu_1 \in (0, \mu_0)$ such that $\Lambda_{i-1}(t) \cap \Lambda_j(t) \neq \emptyset$ for $t \in [\bar{t} - \mu_1, \bar{t}]$, which contradicts to the definition of $\bar{t}$.

Hence, we obtain $\Lambda_i(\bar{t}) = \Lambda_j(\bar{t})$. ∎

Proposition 14 implies that the facets with the same direction are agree with each other.

COROLLARY 15. *If $\Gamma(\bar{t})$ has a facet-facet type intersection between $\Lambda_i(\bar{t})$ and $\Lambda_j(\bar{t})$, then $\Lambda_\ell(\bar{t}) = \Lambda_{\ell+i-j+nN_\gamma}(\bar{t})$ as long as all of the above facets exist for $\ell \in \mathbb{Z}/N_\gamma\mathbb{Z}$ and $n \in \mathbb{Z}$.*

Proof. By Proposition 14 we have $\Lambda_i(\bar{t}) = \Lambda_j(\bar{t})$, i.e., $y_i(\bar{t}) = y_j(\bar{t})$ and $y_{i-1}(\bar{t}) = y_{j-1}(\bar{t})$. Then, facet-facet type intersections appear between $\Lambda_{i-1}(\bar{t})$ and $\Lambda_{j-1}(\bar{t})$, and respectively, between $\Lambda_{i+1}(\bar{t})$ and $\Lambda_{j+1}(\bar{t})$ by Proposition 13. Then, we have $\Lambda_{i-1}(\bar{t}) = \Lambda_{j-1}(\bar{t})$, and $\Lambda_{i+1}(\bar{t}) = \Lambda_{j+1}(\bar{t})$. Hence we obtain the conclusion of Corollary 15. ∎

REFERENCES

[1] W. K. BURTON, N. CABRERA, AND F. C. FRANK, *The growth of crystals and the equilibrium structure of their surfaces*, Philosophical Transactions of the Royal Society of London. Series A. Mathematical and Physical Sciences, 243 (1951), pp. 299–358.

[2] R. E. CAFLISCH, *Growth, structure and pattern formation for thin films*, J. Sci. Comput., 37 (2008), pp. 3–17, doi:10.1007/s10915-008-9206-8, http://dx.doi.org/10.1007/s10915-008-9206-8.

[3] N. FORCADEL, C. IMBERT, AND R. MONNEAU, *Uniqueness and existence of spirals moving by forced mean curvature motion*, Interfaces Free Bound., 14 (2012), pp. 365–400, doi:10.4171/IFB/285, http://dx.doi.org/10.4171/IFB/285.

[4] M.-H. GIGA AND Y. GIGA, *Generalized motion by nonlocal curvature in the plane*, Arch. Ration. Mech. Anal., 159 (2001), pp. 295–333, doi:10.1007/s002050100154, http://dx.doi.org/10.1007/s002050100154.

[5] Y. GIGA, *Surface evolution equations: A level set approach*, vol. 99 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 2006.

[6] Y. Giga and N. Požár, *A level set crystalline mean curvature flow of surfaces*, Adv. Differential Equations, 21 (2016), pp. 631–698, http://projecteuclid.org/euclid.ade/1462298654.
[7] M. A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom., 26 (1987), pp. 285–314, http://projecteuclid.org/euclid.jdg/1214441371.
[8] M. E. Gurtin, *Thermomechanics of evolving phase boundaries in the plane*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1993.
[9] H. Imai, N. Ishimura, and T. Ushijima, *Motion of spirals by crystalline curvature*, M2AN Math. Model. Numer. Anal., 33 (1999), pp. 797–806, doi:10.1051/m2an:1999164, http://dx.doi.org/10.1051/m2an:1999164.
[10] T. Ishiwata, *Motion of non-convex polygons by crystalline curvature and almost convexity phenomena*, Japan J. Indust. Appl. Math., 25 (2008), pp. 233–253, http://projecteuclid.org/euclid.jjiam/1215118764.
[11] T. Ishiwata, *Motion of polygonal curved fronts by crystalline motion: V-shaped solutions and eventual monotonicity*, Discrete Contin. Dyn. Syst., (2011), pp. 717–726.
[12] T. Ishiwata, *On the motion of polygonal curves with asymptotic lines by crystalline curvature flow with bulk effect*, Discrete Contin. Dyn. Syst. Ser. S, 4 (2011), pp. 865–873, doi:10.3934/dcdss.2011.4.865, http://dx.doi.org/10.3934/dcdss.2011.4.865.
[13] T. Ishiwata, *Crystalline motion of spiral-shaped polygonal curves with a tip motion*, Discrete Contin. Dyn. Syst. Ser. S, 7 (2014), pp. 53–62, doi:10.3934/dcdss.2014.7.53, http://dx.doi.org/10.3934/dcdss.2014.7.53.
[14] T. Ishiwata, T. K. Ushijima, H. Yagisita, and S. Yazaki, *Two examples of nonconvex self-similar solution curves for a crystalline curvature flow*, Proc. Japan Acad. Ser. A Math. Sci., 80 (2004), pp. 151–154, http://projecteuclid.org/euclid.pja/1116442372.
[15] A. Karma and M. Plapp, *Spiral surface growth without desorption*, Phys. Rev. Lett., 81 (1998), pp. 4444–4447, doi:10.1103/PhysRevLett.81.4444, http://link.aps.org/doi/10.1103/PhysRevLett.81.4444.
[16] R. Kobayashi, *A brief introduction to phase field method*, AIP Conf. Proc., 1270 (2010), pp. 282–291.
[17] H. Miura and R. Kobayashi, *Phase-field modeling of step dynamics on growing crystal surface: Direct integration of growth units to step front*, Crystal Growth & Design, 15 (2015), pp. 2165–2175, doi:10.1021/cg501806d, http://dx.doi.org/10.1021/cg501806d, arXiv:http://dx.doi.org/10.1021/cg501806d.
[18] T. Ohtsuka, *A level set method for spiral crystal growth*, Advances in Mathematical Sciences and Applications, 13 (2003), pp. 225–248.
[19] T. Ohtsuka, Y.-H. Tsai, and Y. Giga, *A level set approach reflecting sheet structure with single auxiliary function for evolving spirals on crystal surfaces*, Journal of Scientific Computing, 62 (2015), pp. 831–874, doi:10.1007/s10915-014-9877-2, http://dx.doi.org/10.1007/s10915-014-9877-2.
[20] P. Smereka, *Spiral crystal growth*, Physica D. Nonlinear Phenomena, 138 (2000), pp. 282–301.
[21] I. Sunagawa and P. Bennema, *Morphology of growth spirals: theoretical and experimental*, Preparation and Properties of Solid State Materials: Vol. 7, Growth Mechanisms and Silicon Nitride, (1982), pp. 1–129.
[22] J. E. Taylor, *Constructions and conjectures in crystalline nondifferential geometry*, in Differential geometry, vol. 52 of Pitman Monogr. Surveys Pure Appl. Math., Longman Sci. Tech., Harlow, 1991, pp. 321–336, doi:10.1111/j.1439-0388.1991.tb00191.x, http://dx.doi.org/10.1111/j.1439-0388.1991.tb00191.x.
[23] S. Yazaki, *Point-extinction and geometric expansion of solutions to a crystalline motion*, Hokkaido Math. J., 30 (2001), pp. 327–357, doi:10.14492/hokmj/1350911957, http://dx.doi.org/10.14492/hokmj/1350911957.