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APPROXIMATION OF GENERAL FACETS BY REGULAR FACETS WITH RESPECT TO ANISOTROPIC TOTAL VARIATION ENERGIES AND ITS APPLICATION TO THE CRYSTALLINE MEAN CURVATURE FLOW

YOSHIKAZU GIGA AND NORBERT POŽÁR

Abstract. We show that every bounded subset of an Euclidean space can be approximated by a set that admits a certain vector field, the so-called Cahn-Hoffman vector field, that is subordinate to a given anisotropic metric and has a square-integrable divergence. More generally, we introduce a concept of facets as a kind of directed sets, and show that they can be approximated in a similar manner.

We use this approximation to construct test functions necessary to prove the comparison principle for viscosity solutions of the level set formulation of the crystalline mean curvature flow that were recently introduced by the authors. As a consequence, we obtain the well-posedness of the viscosity solutions in an arbitrary dimension, which extends the validity of the result in the previous paper.

1. Introduction

In this note we consider the question of approximating general compact subsets of an Euclidean space by sets whose boundary regularity is similar to the regularity of the Wulff shape of a given anisotropic norm. This regularity is related to the existence of the so-called Cahn-Hoffman vector field with an L^2 divergence on the set.

To give a more specific example of what we have in mind, let $n \geq 1$ be a dimension, $A \subset \mathbb{R}^n$ be a compact set and $\rho > 0$. Then there exists a compact set $G \supset A$ with a smooth boundary such that $d_H(\partial G, \partial A) < \rho$, and a Lipschitz continuous vector field $z : G \to \mathbb{R}^n$ such that $|z(x)| \leq 1$, $z \cdot \nu = 1$ on ∂G. Here ν is the unit outer normal vector on ∂G and d_H is the Hausdorff distance with respect to the Euclidean norm. For this result see [GGP2]. Such a vector field is an example of a Cahn-Hoffman vector field for the isotropic metric given by the Euclidean norm $|\cdot|$. The unit ball $\{z : |z| \leq 1\}$ is its Wulff shape. Such vector fields are important for evaluating the anisotropic curvature of a surface or the subdifferential of a total variation energy. Unfortunately, no Cahn-Hoffman vector field might exist for a given set, in which case the anisotropic curvature cannot be evaluated. However, the above result allows us to approximate arbitrary sets by sets for which the anisotropic curvature is well-defined.

The goal of the present note is to extend this result to general anisotropic norm-like functions (non necessarily symmetric) $W : \mathbb{R}^n \to [0, \infty)$. We assume that W is (i) convex, (ii) positively one-homogeneous, $W(tp) = tW(p)$ for all $p \in \mathbb{R}^n$, $t \geq 0$, and (iii) positive definite, $W(p) = 0$ if and only if $p = 0$. We will call such a function W an *anisotropy*. Given W, its

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polar W° is defined as

$$W^\circ(x) := \sup\{x \cdot p : p \in \mathbb{R}^n,\ W(p) \le 1\}. \tag{1.1}$$

W° is also an anisotropy, see [R]. The *Wulff shape* $\mathcal{W}$ of W is given as the set

$$\mathcal{W} := \{x \in \mathbb{R}^n : W^\circ(x) \le 1\}.$$

By the positive definiteness of W°, $\mathcal{W}$ is a compact set containing 0 in its interior. We claim that any compact set A can be approximated by a larger set G with the properties as above but with respect to the anisotropy W. However, we need to relax the notion of boundary values of z. We will show that for every $\rho > 0$ there exists a compact set $G \supset A$, $d_H(\partial G, \partial A) < \rho$, an open set $U \supset G$, a Lipschitz continuous function ψ on U, $\psi \ge 0$, $\psi(x) = 0$ if and only if $x \in G$, and a vector field $z \in L^\infty(U, \mathbb{R}^n)$ with $\operatorname{div} z \in L^2(U)$ such that $z \in \mathcal{W}$, $z \cdot \nabla\psi = W(\nabla\psi)$ a.e. on U. Note that the last condition on the vector field z is equivalent to requiring $z \in \partial W(\nabla\psi)$ a.e. on U, where ∂W is the subdifferential of W with respect to the Euclidean inner product $x \cdot y$, given as

$$\partial W(p) := \{\xi \in \mathbb{R}^n : W(p+h) - W(p) \ge \xi \cdot h,\ h \in \mathbb{R}^n\} = \{\xi \in \mathbb{R}^n : \xi \in \mathcal{W},\ \xi \cdot p = W(p)\}. \tag{1.2}$$

As we will see later, the approximation discussed so far is related to the surfaces of convex or concave solid bodies. To handle general solid bodies, we want to control the direction in which the vector field z flows through the boundary of the set A. Therefore we introduce a concept of *directionality* of a set. In particular, we specify which components of the complement are sources and which are sinks of the vector field. We will call such directed sets *facets*.

Definition 1.1. *Let $n \ge 0$ be the dimension. A compact set $A \subset \mathbb{R}^n$ together with a direction $\chi \in C(A^c, \{-1, 1\})$ is called an* n-dimensional facet*, and we write it as (A, χ). We will set $\chi(x) = 0$ for $x \in A$.*

The direction χ introduces an order on the family of all facets: $(A, \chi) \preceq (G, \tilde\chi)$ if $\chi \le \tilde\chi$. If $\chi, \tilde\chi \ge 0$, we have $(A, \chi) \preceq (G, \tilde\chi)$ if and only if $A \subset G$. We now introduce "regular" facets that admit a certain vector field subordinate to the anisotropy W.

Definition 1.2. *We say that (A, χ) is a W°-(L^2)* Cahn-Hoffman facet *if there exists an open set $U \subset \mathbb{R}^n$, a Lipschitz function ψ on U and a vector field $z \in L^\infty(U, \mathbb{R}^n)$, $\operatorname{div} z \in L^2(U)$, such that $A \subset U$, $\operatorname{sign}\psi = \chi$ on U, $z(x) \in \partial W(\nabla\psi(x))$ for a.e. $x \in U$. Here we define* $\operatorname{sign} s$ *to be $-1, 0, 1$ if $s < 0$, $s = 0$ or $s > 0$, respectively.*

If A has a Lipschitz boundary, we would like to say that (A, χ) is a W°-(L^2) Cahn-Hoffman facet if there exists a vector field $z \in L^\infty(A, \mathbb{R}^n)$, $\operatorname{div} z \in L^2(A)$ such that $z \in \mathcal{W}$ on A and $z(x) \in \partial W(\chi(x + \nu 0+)\nu(x))$ on ∂A, where $\chi(x + \nu 0+)$ is the limit of χ at x from the direction outside of A and $\nu(x)$ is the unit outer normal to A at x. However, we currently do not know if this is equivalent to Definition 1.2.

Such vector fields z are usually referred to as Cahn-Hoffman vector fields. We are mainly interested in the divergence of these vector fields, in particular in the one that is minimal in the L^2-norm on A. We call such an L^2-function the W°-(L^2) *minimal divergence of the facet* (A, χ), Definition 4.2. We will see later, Proposition 4.1, that this minimal divergence is unique and depends only on A and χ.

The main claim of this paper is that an arbitrary n-dimensional facet (A, χ) can be approximated by a Cahn-Hoffman facet $(G, \tilde\chi)$ consistently with the direction, that is, $(A, \chi) \preceq (G, \tilde\chi)$.

Theorem 1.3. *Let (A,χ) be an n-dimensional facet and W an anisotropy. Given $\rho>0$ there exists a W°-(L^2) Cahn-Hoffman facet $(G,\tilde\chi)$ such that $\chi(x)\le\tilde\chi(x)\le\sup_{|x-y|\le\rho}\chi(y)$ for $x\in\mathbb{R}^n$.*

The main application of the approximation result Theorem 1.3 is the recently developed notion of viscosity solutions of the level set formulation of a crystalline curvature flow by the authors in [GP]. In particular, the viscosity solutions of the initial value problem

$$\begin{cases} u_t+F(\nabla u,\operatorname{div}(\nabla W)(\nabla u))=0, & \text{in } \mathbb{R}^n\times(0,\infty),\\ u|_{t=0}=u_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{1.3}$$

were introduced. Here $F:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is assumed to be a continuous function that is nonincreasing in the second variable, and the initial data u_0 is assumed to be continuous and constant outside of a compact set. Finally, W is a *crystalline* anisotropy, that is, a piece-wise linear anisotropy.

This problem can be derived as the level set formulation of the motion of a set $\{E_t\}_{t\ge0}$ by the *crystalline* mean curvature flow

$$V=f(\nu,\kappa_W), \tag{1.4}$$

where V is the normal velocity of the surface ∂E_t, ν is its outer unit normal vector and $\kappa_W=-\operatorname{div}_{\partial E_t}(\nabla W)(\nu)$ is the anisotropic (crystalline) mean curvature of ∂E_t. The function $f:\mathcal{S}^{n-1}\times\mathbb{R}\to\mathbb{R}$ is assumed to be continuous and nondecreasing in the second variable. The idea of the level set method is to introduce an auxiliary function $u:\mathbb{R}^n\times[0,\infty)\to\mathbb{R}$ whose every sub-level set $t\mapsto\{x:u(x,t)<c\}$, $c\in\mathbb{R}$, evolves under the crystalline mean curvature flow. This function then satisfies the initial value problem (1.3) with an appropriate u_0, typically so that $E_0=\{x:u_0(x)<0\}$. Since $V=-\frac{u_t}{|\nabla u|}$, $\nu=\frac{\nabla u}{|\nabla u|}$ and $\operatorname{div}_{\partial E_t}(\nabla W)(\nu)=\operatorname{div}(\nabla W)(\nabla u)$, see [G], we deduce that $F(p,\xi)=|p|f(\frac{p}{|p|},-\xi)$. In [GP], the viscosity solutions of (1.3) were shown to satisfy the comparison principle in three dimensions, and the well-posedness of this problem was established.

Theorem 1.3 allows us to extend the result of [GP] to an arbitrary dimension. For a detailed discussion of the notion of viscosity solutions for (1.3) see Section 4.2. Heuristically, the key idea is to define the viscosity solution using the comparison principle with test functions that have flat parts that are W°-(L^2) Cahn-Hoffman facets, and then interpret the operator $\operatorname{div}(\nabla W)(\nabla u)$ as the W°-(L^2) minimal divergence of the facet. This is consistent with the theory of monotone operators when (1.3) is in a divergence form. The approximation result in Theorem 1.3 implies that the family of these test functions is sufficiently large. We have the following results.

Theorem 1.4 (Comparison principle)**.** *Let $W:\mathbb{R}^n\to\mathbb{R}$ be a crystalline anisotropy, and $F\in C(\mathbb{R}^n\times\mathbb{R})$ be nonincreasing in the second variable, $F(0,0)=0$. Suppose that u is an upper semicontinuous function and v is a lower semicontinuous function, u and v are a viscosity subsolution and a viscosity supersolution, respectively, of (1.3) in $\mathbb{R}^n\times(0,T)$ for some $T>0$, and that there exist a compact set $K\subset\mathbb{R}^n$ and constants $c_u,c_v\in\mathbb{R}$ such that $u\equiv c_u\le c_v\equiv v$ on $(\mathbb{R}^n\setminus K)\times[0,T]$. Then $u(\cdot,0)\le v(\cdot,0)$ on $\mathbb{R}^n$ implies $u\le v$ on $\mathbb{R}^n\times(0,T)$.*

Theorem 1.5 (Stability)**.** *Let $W:\mathbb{R}^n\to\mathbb{R}$ be a crystalline anisotropy, and $F\in C(\mathbb{R}^n\times\mathbb{R})$ be nonincreasing in the second variable. Let u_m, $m\in\mathbb{N}$, be a sequence of viscosity solutions of*

(1.3) *with* $W = W_m$ *and initial data* $u_{0,m}$. *Suppose that the sequence* $u_{0,m}$ *is locally uniformly bounded. If either*

(a) $W_m \in C^2(\mathbb{R}^n)$, $a_m^{-1} \le \nabla^2 W_m \le a_m$ *for some* $a_m > 0$, *and* $W_m \searrow W$, *or*
(b) W_m *are smooth anisotropies, that is,* W_m *is an anisotropy,* $W_m \in C^2(\mathbb{R}^n \setminus \{0\})$ *and* $\{p : W_m(p) \le 1\}$ *is strictly convex, such that* $W_m \to W$ *locally uniformly,*

then

$$\bar{u}(x,t) = \limsup_{(y,s,m)\to(x,t,\infty)} u_m(y,s) \quad \text{and} \quad \underline{u}(x,t) = \liminf_{(y,s,m)\to(x,t,\infty)} u_m(y,s)$$

are respectively a viscosity subsolution and a viscosity supersolution of

$$u_t + F(\nabla u, \operatorname{div}(\nabla W)(\nabla u)) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Theorem 1.6 (Well-posedness)**.** *Let* $W : \mathbb{R}^n \to \mathbb{R}$ *be a crystalline anisotropy, and* $F \in C(\mathbb{R}^n \times \mathbb{R})$ *be nonincreasing in the second variable,* $F(0,0) = 0$. *Suppose that* $u_0 \in C(\mathbb{R}^n)$ *and that there exist a compact set* $K \subset \mathbb{R}^n$ *and a constant* $c \in \mathbb{R}$ *such that* $u_0 = c$ *on* $\mathbb{R}^n \setminus K$. *Then there exists a unique viscosity solution of* (1.3) *such that* $u(x,t) = c$ *on* $\mathbb{R}^n \setminus K_t$ *for some compact set* K_t, $t \ge 0$.

Note that $F(p,\xi) = |p| f(\frac{p}{|p|}, -\xi)$ coming from the level set formulation of the crystalline mean curvature flow (1.4) satisfies automatically $F(0,0) = 0$. In fact, $F(0,\xi) = 0$ for all $\xi \in \mathbb{R}$. Therefore the above theorem implies the global unique existence (up to fattening of the set $\{x : u(x,t) = 0\}$) of the crystalline mean curvature flow for any initial bounded set E_0.

Theorem 1.7. *Let* $W : \mathbb{R}^n \to \mathbb{R}$ *be a crystalline anisotropy, and* $f \in C(\mathcal{S}^{n-1} \times \mathbb{R})$ *be nondecreasing in the second variable. For every bounded open set* $E_0 \subset \mathbb{R}^n$ *there exists a unique evolution* $\{E_t\}_{t\ge 0}$ *of the crystalline mean curvature flow* $V = f(\nu, \kappa_W)$.

1.1. **Relationship to the anisotropic total variation energy.** In this section we relate the above approximation result to the problem of evaluating the subdifferential of a total variation energy functional. Let W be an anisotropy and let Ω be either a smooth domain in $\mathbb{R}^n$ or the torus $\mathbb{T}^n := \mathbb{R}^n \setminus \mathbb{Z}^n$. Consider the anisotropic total variation energy functional $E : L^2(\Omega) \to \mathbb{R}$ defined as

$$E(\psi) := \begin{cases} \int_\Omega W(D\psi)\, dx, & \psi \in BV(\Omega) \cap L^2(\Omega), \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.5}$$

Here $D\psi$ is the distributional gradient of a function of bounded variation, whose space is denoted as $BV(\Omega)$. In general, $D\psi$ is a vector-valued Radon measure. Therefore E is understood as the relaxation (closure or lower semicontinuous envelope) of the energy (1.5) defined for $W^{1,1}(\Omega)$ functions. It is known that this relaxation is equivalent to defining $W(D\psi)$ as the measure

$$W(\nabla\psi)L^n + W\left(\frac{D^s\psi}{|D^s\psi|}\right)|D^s\psi|,$$

where we decompose $D\psi$ into to the absolutely continuous part $\nabla\psi L^n$ and the singular part $D^s\psi$, with respect to the n-dimensional Lebesgue measure L^n. $\frac{D^s\psi}{|D^s\psi|}$ denotes the Radon-Nikodým derivative.

The subdifferential $\partial E : L^2(\Omega) \to 2^{L^2(\Omega)}$ is defined as the set-valued mapping

$$\partial E(\psi) := \big\{ v \in L^2(\Omega) : E(\psi + h) - E(\psi) \geq (v, h),\ h \in L^2(\Omega) \big\}, \tag{1.6}$$

where $(v, h) := \int_\Omega vh\, dx$ is the L^2-inner product. $\partial E(\psi)$ is a closed convex, possibly empty set. The domain of the subdifferential is defined as $\mathcal{D}(\partial E) := \{\psi \in L^2(\Omega) : \partial E(\psi) \neq \emptyset\}$.

The characterization of ∂E is well-known [M, AVCM]. Following [A1], we introduce the set of bounded vector fields with L^2 divergence

$$X_2(\Omega) := \big\{ z \in L^\infty(\Omega; \mathbb{R}^n) : \operatorname{div} z \in L^2(\Omega) \big\}.$$

For $\psi \in Lip(\Omega)$, the set $\partial E(\psi)$ can be characterized as

$$\partial E(\psi) = \{ \operatorname{div} z : z \in X_2(\Omega),\ z \in \partial W(\nabla \psi) \text{ a.e.},\ [z \cdot \nu] = 0 \text{ on } \partial\Omega \},$$

where ∂W is the subdifferential of W with respect to the inner product on $\mathbb{R}^n$, introduced in (1.2). The symbol $[z \cdot \nu]$ denotes the boundary trace of a vector field in $X_2(\Omega)$, see [A1]. The vector fields z are usually called Cahn-Hoffman vector fields, and we shall denote their set as

$$CH(\psi) := \{ z \in X_2(\Omega) :\ z \in \partial W(\nabla \psi) \text{ a.e.} \}, \qquad \psi \in Lip(\Omega). \tag{1.7}$$

Note that $\partial E(\psi)$ might be empty even for smooth ψ. For example, consider the function $\psi(x) = \sin(2\pi x_1)$ on $\mathbb{T}^n$ with anisotropy $W(p) = \|p\|_2 := \left(\sum_{i=1}^n |p_i|^2\right)^{1/2}$, or $W(p) = \|p\|_1 := \sum_{i=1}^n |p_i|$. In this case, every possible candidate vector field z with $z(x) \in \partial W(\nabla \psi)$ a.e. will have a jump discontinuity in z_1 across the sets $\left\{x_1 = \frac{1}{4}\right\}$, where ψ attains its maximum, and $\left\{x_1 = \frac{3}{4}\right\}$, where it attains its minimum. This is a serious difficulty for using the operator ∂E in the theory of viscosity solutions, which usually rely on evaluating the differential operator on a class of sufficiently smooth test functions.

Theorem 1.3 allows us to approximate any function by a function with nonempty ∂E arbitrarily close in the Hausdorff distance of the respective positive and negative sets. Stated in terms of the functions ψ, we get the following theorem.

Theorem 1.8. *Let W be an anisotropy and Ω be either $\mathbb{T}^n$ or a bounded domain with Lipschitz boundary in $\mathbb{R}^n$. If ψ is a Lipschitz function on Ω with a compact zero set, for every $\rho > 0$ there exists a Lipschitz function $\tilde{\psi}$ on Ω such that $\partial E(\tilde{\psi}) \neq \emptyset$ and*

$$\operatorname{sign} \psi(x) \leq \operatorname{sign} \tilde{\psi}(x) \leq \sup_{\substack{y \in \Omega \\ |y-x| \leq \rho}} \operatorname{sign} \psi(y), \qquad x \in \Omega.$$

Note that any Lipschitz function ψ with compact zero set naturally corresponds to a facet

$$(A, \chi) := (\{x : \psi(x) = 0\}, \operatorname{sign} \psi).$$

Intuitively, it is related to the surface facet (flat part of the surface) of the $(n+1)$-dimensional solid body given by the epigraph of ψ, $\operatorname{epi} \psi := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq \psi(x)\}$. If $\chi \geq 0$ or $\chi \leq 0$, the solid body is respectively convex or concave in a neighborhood of the surface facet.

1.2. **Literature overview.** Crystalline mean curvature was introduced independently by Angenent and Gurtin [AG] and Taylor [T] to model the growth of small crystals. It is the first variation of the surface energy given by a *crystalline* anisotropy with respect to the change of volume. A crystalline anisotropy refers to an anisotropic surface energy density σ whose unit ball is a convex polytope. When extended positively one-homogeneously to the full space, the anisotropy σ is a convex piece-wise linear function. Due to the singularity of the surface

energy density, the crystalline mean curvature is a nonlocal quantity on the flat parts, or facets, of the crystal surface. As with smooth anisotropic curvatures, the crystalline mean curvature can be evaluated as the surface divergence of a so-called Cahn-Hoffman vector field on the surface. A Cahn-Hoffman vector field is a selection of the subdifferential of the anisotropy evaluated at the outer normal vector on a given surface whose surface divergence has a certain regularity. However, for crystalline anisotropies, even smooth surfaces might not have a Cahn-Hoffman vector field whose surface divergence is a function. Therefore there has been a lot effort to characterize sets with surfaces that admit reasonable Cahn-Hoffman vector fields. In particular the various notions of ϕ-regular sets were introduced (ϕ being the polar of the surface energy density σ) [BN, BNP1, BNP2, BNP3, B1] and the related notion of rB_ϕ-condition [BCCN2], B_ϕ being the Wulff shape $\mathcal{W}$.

A crystalline mean curvature flow or a motion by the crystalline curvature is an evolution of sets such that the normal velocity of the surface is proportional to the crystalline mean curvature. Already the standard mean curvature flow is known to develop singularities even when starting from smooth initial data, and therefore a weak notion of solutions is necessary. We mention the varifold solutions initiated by K. Brakke [B2, I1, TT], that however apply only to the isotropic mean curvature flow, and the level set method approach [OS, CGG, ES] that can be generalized to the anisotropic mean curvature with a smooth anisotropy as already done in [CGG]. The extension to a crystalline anisotropy is not straightforward even in the case of a curve evolution because of the non-local nature of the crystalline curvature [GG3]. For a more detailed overview of the literature see [GP].

Most of the attempts at defining a reasonable notion of solutions for the crystalline mean curvature flow have required some kind of regularity of the evolution so that the crystalline curvature can be evaluated, such as a ϕ-regular flow [BN, BCCN1] or a rB_ϕ-regular flow [BCCN2].

Recently, A. Chambolle, M. Morini and M. Ponsiglione [CMP] established a unique global solvability of the flow $V = \sigma(\nu)\kappa_\sigma$ for arbitrary convex σ and non necessarily bounded initial data, in an arbitrary dimension. They introduce a notion of solutions of this flow via an anisotropic sign distance function in the spirit of H. M. Soner [S], but in a distributional sense that appeared in [CC], prove a comparison principle, and use the minimizing movements algorithm of Chambolle [C] to construct a solution. This result has been recently improved in [CMNP] to cover $V = \psi(\nu)(\kappa_\sigma + f(x,t))$, where ψ is a convex anisotropy and f is a given Lipschitz function. It is not clear at this moment if their notion of solutions coincide with ours, although it is likely. We mention the result of K. Ishii [I2], who shows that Chambolle's minimizing movements algorithm converges to the viscosity solution of the crystalline mean curvature flow in two dimensions.

We take a different approach using the ideas of the theory of viscosity solutions [GG1, GG3, GGP1, GGP2]. The level set formulation of the crystalline mean curvature flow was introduced by the authors in [GP]. Viscosity solutions are defined via the comparison principle by testing a candidate for a solution by an appropriate class of regular test functions. The main advantage of this approach is that it does not require the solution itself to have any a priori regularity besides continuity. Moreover, testing a solution by a test function is a rather local concept. Since the crystalline mean curvature might be nonlocal on flat parts of the evolving surface, we can localize the construction of test functions to the neighborhood of such flat parts, called (surface) facets. By choosing a local coordinate system so that the surface facet is given as a part of a function graph where the function is equal to zero, we

are at a situation covered by Theorem 1.3. Our notion of facet introduced in Definition 1.1 then corresponds to the set A where the surface facet is located in this coordinate system and directions in which the surface rises above ($\chi = +1$) and falls below ($\chi = -1$) the surface facet. The anisotropy $W^{\mathrm{sl}}_{\hat{p}}$ introduced in (4.2) captures the local lower-dimensional behavior of the full anisotropy W in the direction of the surface facet. A Cahn-Hoffman vector field for (A, χ) then corresponds to a Cahn-Hoffman vector field on the surface in a neighborhood of the surface facet. In general, we consider surface facets to be all the flat features of a surface with various dimensions, including edges ($n = 1$) and (planar) facets ($n = 2$), etc. Their dimension then guides the choice of the ambient space for the facet (A, χ). The approximation result Theorem 1.3 and its corollary in Theorem 1.8 allow us to construct a large family of test functions at any facet of a crystal. This then yields the comparison principle for viscosity solutions and the well-posedness of the level set formulation of the crystalline mean curvature flow [GP]. In [GP], we showed Theorem 1.8 by a direct construction for $n = 1, 2$ and piece-wise linear anisotropies W and therefore we could deduce the well-posedness of the crystalline curvature flow in dimensions $n \leq 3$. The generalization in this paper then automatically extends the results of [GP] to any dimension. We can also slightly simplify the definition of viscosity solutions, see Definition 4.8. Let us mention that in [GP] we described a facet by a pair of open sets (A_-, A_+). They are related to the notion of facet (A, χ) from Definition 1.1 as $A_\pm = \{x : \chi(x) = \pm 1\}$.

Outline. The construction of the facet $(G, \tilde{\chi})$ for Theorem 1.3 will be performed first by reducing the situation to the case of a simple set in Section 2 using the gradient flow of the total variation energy on a torus $\mathbb{T}^n$, and then combining it to produce a facet in Section 3. In Section 4 we outline an important application of Theorem 1.3 to the theory of viscosity solutions for the level set formulation of the crystalline mean curvature problems.

2. Approximability of a single set

We start the proof of Theorem 1.3 with a simple facet (A, χ) with $\chi \leq 0$. Note that we need to construct the Cahn-Hoffman vector field only in a neighborhood of the boundary of the approximating facet since $0 \in \mathcal{W} = \partial W(0)$. We can thus always extend the vector field by 0 in the interior of the facet away from the boundary. To construct the approximating set, we will use the gradient flow of the anisotropic total variation energy (1.5) on the torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$.

Let us thus consider a single open set $D \subset \mathbb{T}^n$ with a *nonempty* boundary. Let $d : \mathbb{T}^n \to \mathbb{R}$ be the signed distance function to ∂D (in the torus topology of $\mathbb{T}^n$) induced by W with $d > 0$ in D. Recall that

$$d(x) := \inf_{y \in D^c} W^\circ(x - y) - \inf_{y \in D} W^\circ(y - x) = \begin{cases} \inf_{y \in D^c} W^\circ(x - y), & x \in D, \\ -\inf_{y \in D} W^\circ(y - x), & x \in D^c, \end{cases} \tag{2.1}$$

where we treat $D \subset \mathbb{R}^n$ as periodic in the sense that if $x \in D$ then $x + \mathbb{Z} \subset D$, and W° is the polar of W defined in (1.1). Recall that W° is again an anisotropy, see [R], and therefore also Lipschitz continuous. In particular, d is a Lipschitz continuous function on $\mathbb{T}^n$.

Fix $\delta > 0$ such that

$$\{x \in \mathbb{R}^n : W^\circ(x) \leq 4\delta\} = 4\delta\mathcal{W} \subset (-\frac{1}{2}, \frac{1}{2})^m. \tag{2.2}$$

We consider the gradient flow (or differential inclusion) on $\mathbb{T}^n$

$$\begin{cases} \dfrac{d\varphi}{dt}(t) \in -\partial E(\varphi(t)), & t > 0, \\ \varphi(0) = \max(-\delta, \min(d, \delta)), \end{cases} \tag{2.3}$$

where the subdifferential ∂E was defined in (1.6) with $\Omega = \mathbb{T}^n$.

It is well-known [B3, B4] that a unique solution $\varphi \in C([0,T]; L^2(\mathbb{T}^n))$ exists (for any $T > 0$) and it is right differentiable for all $t \in (0,T)$. Moreover $\frac{d^+\varphi}{dt}(t) = -\partial^0 E(\varphi(t))$, where $\partial^0 E(\psi)$ is called the minimal section (also the canonical restriction) of $\partial E(\psi)$, and it is the element of $\partial E(\psi)$ with the smallest L^2-norm. In particular, $\varphi(t) \in \mathcal{D}(\partial E)$ for all $t > 0$. Such well-posedness goes back to the work of Y. Komura [K], where initial data is assumed to be in $\mathcal{D}(\partial E)$. Finally, $\varphi(t)$ is Lipschitz for all $t > 0$ since $\varphi(0)$ is Lipschitz and (2.3) has a comparison principle and is translationally invariant. The comparison principle (and Lipschitz continuity of φ in space) can be established, for instance, by approximating the energy E in the Mosco sense by uniformly elliptic energies, for which (2.3) is just a uniformly parabolic PDE. Mosco convergence then implies the convergence of the resolvent problems, see [GP] for details. Alternatively, multiply the difference of equations (2.3) for two solutions u and v by the positive part of the difference, $(u-v)_+ := \max(u-v, 0)$, and integrate by parts.

We want to prove that

$$d_H(\partial\{x : \pm\varphi(x,t) > 0\}, \partial D) \to 0 \qquad \text{as } t \to 0+,$$

where d_H is the Hausdorff distance with respect to the usual Euclidean metric.

To show this, we will compare φ with barriers of the form

$$\psi_c(x) := \min\left\{\min(\max(W^\circ(x-k) - c, -\tfrac{c}{2}), \delta) : k \in \mathbb{Z}^n\right\},$$

for $0 < c < \delta$. These will control the expansion of the set $\{x : \varphi(x,t) = 0\}$, which is equal to ∂D at $t = 0$.

We claim that $\psi_c(x) + Mt$ is a supersolution of (2.3) for $M = M(c) > 0$ large enough. To see this, introduce the vector field for $x \in (-\frac{1}{2}, \frac{1}{2})^n$ as

$$z(x) := \begin{cases} \frac{2x}{c}, & W^\circ(x) < \frac{c}{2}, \\ \frac{x}{W^\circ(x)}, & \frac{c}{2} \le W^\circ(x) < c + 2\delta, \\ \frac{x}{W^\circ(x)} \max(1 + \frac{c+2\delta - W^\circ(x)}{\delta}, 0), & \text{otherwise}, \end{cases}$$

and then extend it periodically to $\mathbb{R}^n$. If $0 < c < \delta$, $z = 0$ in a neighborhood of the boundary of $(-\frac{1}{2}, \frac{1}{2})^n$ by (2.2).

We claim that z is a Cahn-Hoffman vector field for ψ_c, that is, $z \in CH(\psi_c)$. Indeed, working only on the unit cell $x \in (-\frac{1}{2}, \frac{1}{2})^n$, note that $W^\circ(z) \le 1$, with equality if and only if $\frac{c}{2} \le W^\circ(x) \le c + 2\delta$. Therefore $z \in \partial W(0)$ on $\mathbb{R}^n$, and thus $z(x) \in \partial W(\nabla\psi(x)) = \partial W(0)$ for $W^\circ(x) < \frac{c}{2}$ and $W^\circ(x) > c + \delta$. Since $\frac{x}{W^\circ(x)} \in \partial W(p)$ if $p \in \partial W^\circ(x)$, we have that $z \in \partial W(\nabla\psi_c(x))$ whenever W° is differentiable at x and $\frac{c}{2} < W^\circ(x) < c + \delta$. As z is Lipschitz, $\operatorname{div} z \in L^\infty(\mathbb{T}^n) \subset L^2(\mathbb{T}^n)$ and so z is a Cahn-Hoffman vector field for ψ_c.

Let us set $M := \|\operatorname{div} z\|_{L^\infty}$ and $f := M - \operatorname{div} z \in L^2(\mathbb{T}^n)$. Observe that $(x,t) \mapsto \psi_c(x) + Mt$ is the unique solution $\phi \in C([0,\infty); L^2(\mathbb{T}^n))$ of

$$\begin{cases} \dfrac{d\phi}{dt}(t) \in -\partial E(\phi(t)) + f, & t > 0, \\ \phi(0) = \psi_c, \end{cases} \tag{2.4}$$

and that $f \geq 0$, and thus it is a supersolution of (2.3).

Lemma 2.1. *Let φ be the unique solution of (2.3). For every $\rho > 0$ there exists $\tau > 0$ such that for all $x \in D$, $\operatorname{dist}(x, \partial D) > \rho$ we have $\varphi(x,t) > 0$ for all $0 \leq t < \tau$, and for all $x \in D^c$, $\operatorname{dist}(x, \partial D) > \rho$ we have $\varphi(x,t) < 0$ for all $0 \leq t < \tau$. Here* dist *is the usual Euclidean distance.*

Proof. Take $c \in (0, \delta)$ such that $W^\circ(x) > c$ for $|x| \geq \rho$, $M = M(c)$ so that $(x,t) \mapsto \psi_c(x) + Mt$ is a supersolution of (2.3), and $\tau = \frac{c}{2M}$. We claim that by definition of d in (2.1), for any $x_0 \in D^c$ such that $\operatorname{dist}(x_0, \partial D) > \rho$, we have

$$\varphi(\cdot, 0) \leq \psi_c(\cdot - x_0) \qquad \text{in } \mathbb{T}^n.$$

To see this for one such x_0, we only need to show $W^\circ(x - x_0) \geq d(x) + c$ for $x \in \mathbb{R}^n$. If $x \in D$, then $|x - x_0| > \rho$ and hence $W^\circ(x - x_0) > c$. Set $y := \frac{c}{W^\circ(x - x_0)}(x - x_0) + x_0$. Note that $W^\circ(y - x_0) = c$ and therefore $|y - x_0| \leq \rho$. In particular, $y \in D^c$. Then by (2.1)

$$d(x) \leq W^\circ(x - y) = \left(1 - \frac{c}{W^\circ(x - x_0)}\right) W^\circ(x - x_0) = W^\circ(x - x_0) - c.$$

On the other hand, if $x \in D^c$, convexity and positive one-homogeneity yields for all $y \in D$

$$W^\circ(y - x) + W^\circ(x - x_0) \geq W^\circ(y - x_0) > c.$$

Taking the infimum over $y \in D$, we get from (2.1)

$$W^\circ(x - x_0) - c \geq d(x).$$

Fix thus one x_0 as above. By the comparison principle for (2.4),

$$\varphi(x_0, t) \leq \psi_c(0) + Mt < 0, \qquad 0 \leq t < \tau.$$

In particular, $\varphi(x_0) < 0$ for $0 \leq t < \frac{c}{2M}$.

We can argue similarly for $x_0 \in D$ using the subsolution $-\psi_c(-x) - Mt$. □

The lemma above allows us to control the speed of the boundary of $\{\pm\varphi > 0\}$.

Corollary 2.2. *For every $\rho > 0$ there exists $\tau > 0$ such that*

$$\begin{aligned} &d_H(\partial D, \partial\{\varphi(\cdot,t) > 0\}) \leq \rho, \\ &d_H(\partial D, \partial\{\varphi(\cdot,t) < 0\}) \leq \rho, \end{aligned} \qquad \text{for } 0 \leq t < \tau,$$

where d_H is the Hausdorff distance with respect to the usual Euclidean metric.

Proof. Given $\rho > 0$, by compactness there exists $\eta \in (0, \rho)$ such that for all $x \in \partial D$ there exist $y \in D$, $z \in D^c$, such that $\operatorname{dist}(y, x) \leq \rho$, $\operatorname{dist}(z, x) \leq \rho$ while $\operatorname{dist}(y, \partial D) > \eta$, $\operatorname{dist}(z, \partial D) > \eta$. Take $\tau > 0$ from Lemma 2.1 for $\rho = \eta$, and fix $t \in (0, \tau)$. Let us show the inequality for $\Gamma_t := \partial\{\varphi(\cdot, t) > 0\}$, the second one is analogous.

First, let $x \in \Gamma_t$, which by continuity implies $\varphi(x,t) = 0$. By Lemma 2.1 we have $\operatorname{dist}(x, \partial D) \leq \eta < \rho$.

Now suppose that $x \in \partial D$. By the choice of η there exist $y \in D$, $z \in D^c$ with the properties above. By Lemma 2.1 we have $\varphi(y,t) > 0$ and $\varphi(z,t) < 0$ for $0 \leq t < \tau$. The convexity of the norm yields $\operatorname{dist}(x, \partial\Gamma_t) \leq \rho$ since the line segment connecting y with z must contain a point in $\partial\Gamma_t$. □

3. Construction of a Cahn-Hoffman facet

To prove Theorem 1.3, let us fix a facet (A,χ) such that $A \subset (-\frac{1}{4},\frac{1}{4})^n$. Indeed, an arbitrary facet can be reduced to this case by simple scaling since the anisotropy is positively one-homogeneous. We can also assume that $\rho < \frac{1}{4}$. We will use the result of Section 2 to construct the approximating Cahn-Hoffman facet. By continuity of χ on A^c, χ is a constant on $\mathbb{R}^n \setminus (-\frac{1}{4},\frac{1}{4})^n$. Let us assume that $\chi = -1$ on this set. The other case can be handled similarly. Defining the periodic function $\mathring{\chi}(x) := \min\{\chi(y) : y \in x + \mathbb{Z}^n\}$, we see that $\mathring{\chi} = \chi$ on $(-\frac{1}{2},\frac{1}{2})^n$. Set $H_+ = \{x \in \mathbb{R}^n : \sup_{|y-x|<\frac{\rho}{4}} \mathring{\chi}(y) = +1\}$ and $H_- = \{x \in \mathbb{R}^n : \sup_{|y-x|<\frac{3\rho}{4}} \mathring{\chi}(y) \geq 0\}$. Sets H_- and H_+ are open subsets of $\mathbb{T}^n$. Moreover $H_+ \subset H_-$ and $\operatorname{dist}(H_+, H_-^c) = \operatorname{dist}(\partial H_-, \partial H_+) \geq \frac{\rho}{2}$.

Define the open cube $U = (-\frac{1}{2},\frac{1}{2})^n \subset \mathbb{R}^n$. For one fixed $\delta > 0$ as in Section 2, let φ^+ be the solution of (2.3) for set $D = H_+$, and φ^- be the solution $D = H_-$. If ∂H_+ is empty, that is, when H_+ is empty, we set $\varphi^+ \equiv 0$, $z^+ \equiv 0$ below. Similarly if ∂H_- is empty, implying $H_- = \mathbb{T}^n$, we set $\varphi^- \equiv -1$, $z^- \equiv 0$. Applying Lemma 2.1 with $\rho = \frac{\rho}{8}$, we have $\tau > 0$ such that

$$\begin{aligned} &\{\mathring{\chi} = 1\} \subset \{\varphi^+(\cdot,t) > 0\} \subset \{x : \sup_{|y-x|\leq\rho} \mathring{\chi}(y) = 1\}, \\ &\{\mathring{\chi} \geq 0\} \subset \{\varphi^-(\cdot,t) \geq 0\} = \{x : \sup_{|y-x|\leq\rho} \mathring{\chi}(y) \geq 0\}, \end{aligned} \qquad 0 \leq t < \tau, \tag{3.1}$$

and

$$\operatorname{dist}(\{\varphi^+(\cdot,t) > 0\}, \{\varphi^-(\cdot,t) < 0\}) \geq \frac{\rho}{4}, \qquad 0 \leq t < \tau, \tag{3.2}$$

and finally

$$\varphi^-(\cdot,t) < 0 \text{ on } \partial U, \qquad 0 \leq t < \tau. \tag{3.3}$$

Let us fix $t \in (0,\tau)$ and set

$$G_- := \{x \in U : \varphi^-(x,t) < 0\} \cup U^c, \qquad G_+ := \{x \in U : \varphi^+(x,t) > 0\}.$$

By the continuity of $\varphi^\pm$ and (3.3), $G_\pm \subset \mathbb{R}^n$ are open.

We define a facet by setting $G = G_-^c \cap G_+^c \subset U$, $\tilde{\chi} = +1$ on G_+ and $\tilde{\chi} = -1$ on G_-. Then $(G,\tilde{\chi})$ is a facet in the sense of Definition 1.1 and $\chi(x) \leq \tilde{\chi}(x) \leq \sup_{|y-x|\leq\rho} \chi(y)$.

To finish the proof of Theorem 1.3, we need to show that it is actually a W°-(L^2) Cahn-Hoffman facet by finding a Cahn-Hoffman vector field on its neighborhood U. Since $\varphi_\pm$ are solutions of (2.3) and for $t > 0$ we have that $\varphi^\pm(\cdot,t) \in \mathcal{D}(\partial E)$ are Lipschitz, there exist Cahn-Hoffman vector fields $z^\pm \in X_2(\mathbb{T}^n)$ for $\varphi^\pm(\cdot,t)$, respectively. We define the function $\psi : U \to \mathbb{R}$ as

$$\psi(x) := (\varphi^+)_+(x,t) - (\varphi^-)_-(x,t) = \begin{cases} \varphi_+(x,t), & x \in G_+, \\ 0, & x \in G, \\ \varphi_-(x,t), & x \in G_- \cap U, \end{cases}$$

where $(s)_+ := \max(s,0)$ and $(s)_- := \max(-s,0)$ denote the positive and negative parts, respectively. The functions $\varphi^\pm$ are Lipschitz continuous, and so by (3.2) ψ is well-defined and Lipschitz continuous function on U.

Set $\eta = \operatorname{dist}(G_-, G_+)/4$. Let us introduce the vector field

$$z(x) := \xi^+(x)z^+(x) + \xi^-(x)z^-(x),$$

where $\xi^\pm \in C^\infty(\mathbb{R}^n)$ are cut-off functions such that $\xi^\pm(x) = 1$ when $\operatorname{dist}(x, G_\pm) \le \eta$, $\xi^\pm(x) = 0$ when $\operatorname{dist}(x, G_\pm) \ge 2\eta$ and $0 \le \xi^\pm \le 1$ otherwise. Clearly $z \in X_2(U)$. Note that $\xi^- + \xi^+ \le 1$. Since $\psi \equiv 0$ on $G = \{\psi = 0\}$, we see that $z \in \partial W(\nabla\psi)$ on U as $\partial W(p)$ is a convex set. Therefore z is a Cahn-Hoffman vector field on U, and $(G, \tilde{\chi})$ is a W°-(L^2) Cahn-Hoffman facet. The proof of Theorem 1.3 is complete.

4. Level set crystalline mean curvature flow

In this section, we explain the application of the approximation result of Theorem 1.3 to the theory of viscosity solutions of the crystalline mean curvature flow problems. Specifically, let us consider the initial value problem (1.3),

$$\begin{cases} u_t + F(\nabla u, \operatorname{div}(\nabla W)(\nabla u)) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u|_{t=0} = u_0, & \text{in } \mathbb{R}^n. \end{cases} \tag{4.1}$$

The anisotropy $W : \mathbb{R}^n \to \mathbb{R}$ is now assumed to be piece-wise linear. In convex analysis, convex piece-wise linear functions are also known as polyhedral functions [R]. We call such anisotropies *crystalline*. The nonlinearity $F \in C(\mathbb{R}^n \times \mathbb{R})$ is assumed to be nonincreasing in the second variable,

$$F(p, \xi) \le F(p, \eta), \qquad p \in \mathbb{R}^n, \xi \ge \eta.$$

Thanks to this assumption, the problem (4.1) has a comparison principle structure.

Problem (4.1) appears as the level set formulation of an anisotropic mean curvature flow, specifically the crystalline mean curvature flow [GP]. It can be also thought of as an anisotropic total variation flow of non-divergence form. Of particular interest is the singular operator $\operatorname{div}(\nabla W)(\nabla u)$, which is interpreted as the minimal section (also canonical restriction) $-\partial^0 E(u)$ of the subdifferential $\partial E(u)$ defined in (1.6). Since the problem has a comparison principle structure, it falls within the scope of the theory of viscosity solutions. However, the extension of the theory to problems like (4.1) is quite nontrivial. In [GP], the authors succeeded in defining a reasonable notion of viscosity solutions for (4.1). However, due to the difficulty of construction test functions like those in Theorem 1.8 in dimensions $n \ge 3$, the well-posedness was limited to dimensions $n \le 3$. Theorem 1.3 now provides a sufficiently large class of test functions and thus the results of [GP] apply to an arbitrary dimension. We will outline this in the rest of this section.

4.1. Crystalline curvature.

Let us review the notion of the crystalline curvature of a facet. Recall the definition (1.7) of the set $CH(\psi)$ of Cahn-Hoffman vector fields for a given Lipschitz continuous function on an open set $U \subset \mathbb{R}^n$. The set of all divergences of such vector fields $\operatorname{div} CH(\psi) := \{\operatorname{div} z : z \in CH(\psi)\}$ is a closed convex subset of $L^2(U)$. If it is nonempty, there exists a unique element with the minimal L^2 norm. We denote it $\Lambda[\psi]$,

$$\Lambda[\psi] := \operatorname{div} z_{min}, \qquad \|\operatorname{div} z_{min}\|_{L^2(U)} = \inf\Big\{\|\operatorname{div} z\|_{L^2(U)} : z \in CH(\psi)\Big\}.$$

In [GP], we proved the following comparison principle for the quantity Λ.

Proposition 4.1 ([GP, Proposition 4.12]). *If ψ_1, ψ_2 are two Lipschitz functions on an open set U such their zero sets are compact subsets of $U \subset \mathbb{R}^n$, and $\Lambda[\psi_i]$, $i = 1, 2$, are well-defined, then*

$$\operatorname{sign} \psi_1 \leq \operatorname{sign} \psi_2 \quad \text{on } U$$

implies

$$\Lambda[\psi_1] \leq \Lambda[\psi_2], \qquad \text{a. e. on } \{\psi_1 = 0\} \cap \{\psi_2 = 0\}.$$

This allows us to introduce the W°-(L^2) minimal divergence of a W°-(L^2) Cahn-Hoffman facet.

Definition 4.2. *Given a W°-(L^2) Cahn-Hoffman facet (A, χ), that is, a facet for which there exists an open set $U \supset A$, Lipschitz function ψ on U with $\operatorname{sign} \psi = \chi$, and a Cahn-Hoffman vector field $z \in CH(\psi)$, we define the W°-(L^2)* minimal divergence of the facet (A, χ), $\Lambda[(A, \chi)] \in L^2(A)$, *as*

$$\Lambda[(A, \chi)] := \Lambda[\psi] \qquad \text{on } A.$$

By Proposition 4.1, this definition is independent of the choice of U and ψ.

Note that Proposition 4.1 implies a comparison for Λ of facets: if (F_1, χ_1), (F_2, χ_2) are two W°-(L^2) Cahn-Hoffman facets that are ordered, $(F_1, \chi_1) \preceq (F_2, \chi_2)$ in the sense of $\chi_1 \leq \chi_2$, we have

$$\Lambda[(F_1, \chi_1)] \leq \Lambda[(F_2, \chi_2)] \quad \text{a.e. on } F_1 \cap F_2.$$

Example 4.3. Let $n = 1$ and $W(p) = |p|$. Then the facet (A, χ) with $A = \{a\}$, $\chi \in C(\mathbb{R} \setminus \{a\}, \{-1, 1\})$ is always Cahn-Hoffman if and only if $\chi(a-) \neq \chi(a+)$.

Facet (A, χ) with $A = [a, b]$, $a < b$, $\chi \in C(A^c, \{-1, 1\})$ is Cahn-Hoffman. The minimal divergence is constant on the facet, $\Lambda[(A, \chi)] = \frac{\chi(a-)+\chi(b+)}{b-a}$ on A. It is inversely proportional to the length of the facet.

For $n = 0$, $\mathbb{R}^0 = \{0\}$ and the facet $(\{0\}, \chi)$ is always Cahn-Hoffman with $\Lambda[(A, \chi)] = 0$.

Example 4.4. In an arbitrary dimension for any anisotropy W, the rescaled Wulff shape $A := c\mathcal{W} = \{x : W^\circ(x) \leq c\}$, $\chi = 1$ on A^c forms a W°-(L^2) Cahn-Hoffman facet (A, χ) for any $c > 0$ with $\Lambda[(A, \chi)] = \frac{n}{c}$. This can be seen easily by taking $\psi(x) = \max(W^\circ(x) - c, 0)$, $z(x) = \frac{x}{\max(W^\circ(x), c)}$.

Example 4.5. For $n = 2$ and $W(p) = |p_1| + |p_2|$, a rather thorough characterization of $\Lambda[(A, \chi)]$ for axes-aligned polygons A is available in [LMM]. In particular, if $\Lambda[(A, \chi)]$ is a constant, then $\Lambda[(A, \chi)] = \frac{\mathcal{H}^1(\partial A \cap \{\chi=+1\}) - \mathcal{H}^1(\partial A \cap \{\chi=-1\})}{|A|}$, where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure.

Example 4.6. There is an interesting relationship between $\Lambda[(A, \chi)]$ and the Cheeger problem, that is, the problem of finding the subset $\Omega \subset A$ that minimizes the ratio $\frac{P(\Omega)}{|\Omega|}$ among all subsets, where $P(\Omega)$ is the perimeter of Ω. In [BNP4] it was shown that a facet (A, χ) with $A \subset \mathbb{R}^2$ convex and $\chi = +1$ on A^c has $\Lambda[(A, \chi)]$ constant on A if and only if A is a solution of the Cheeger problem on A with the W°-perimeter.

4.2. **Review of the notion of viscosity solutions.** Viscosity solutions are defined as continuous functions that satisfy the comparison (maximum) principle with a class of sufficiently regular test functions. This way we do not need to assume any further regularity about the candidate for a solution, but we have to choose a class of test functions that is sufficiently large. As was pointed out in the introduction, the operator $\operatorname{div}(\nabla W)(\nabla \cdot)$ might not be

well-defined even for smooth functions. We therefore restrict the family of test functions to only the stratified admissible test functions defined below.

Before the definition, let us recall convenient coordinates for the space $\mathbb{R}^n$, introduced in [GP]. For a fixed $\hat{p} \in \mathbb{R}^n$, we define $Z_1 \subset \mathbb{R}^n$ to be the subspace parallel to the *affine hull* $\operatorname{aff} \partial W(\hat{p})$ of $\partial W(\hat{p})$, that is, the smallest subspace such that $Z_1 + \xi \supset \partial W(\hat{p})$ for some $\xi \in \mathbb{R}^n$. Its dimension is then the dimension of the subdifferential $\partial W(\hat{p})$, $k = \dim \partial W(\hat{p}) := \dim Z_1$. Let $Z_2 := Z_1^\perp$ be the orthogonal subspace. Then $\mathbb{R}^n = Z_1 \oplus Z_2$, Z_1 is isometrically-isomorphic to $\mathbb{R}^k$ and Z_2 is isometrically-isomorphic to $\mathbb{R}^{n-k}$. Fixing such isometries $\mathcal{T}_1 : \mathbb{R}^k \to Z_1$, $\mathcal{T}_2 : \mathbb{R}^{n-k} \to Z_2$, we can write every $x \in \mathbb{R}^n$ uniquely in terms of (x', x''), $x = \mathcal{T}_1 x' + \mathcal{T}_2 x''$, where $x' \in \mathbb{R}^k$ and $x'' \in \mathbb{R}^{n-k}$. If $k = 0$ or $k = n$, we simply take $x = x''$ or $x = x'$, respectively.

We also introduce the convex, positively one-homogeneous function $W^{\mathrm{sl}}_{\hat{p}} : \mathbb{R}^k \to \mathbb{R}$ as

$$W^{\mathrm{sl}}_{\hat{p}}(w) := \lim_{\lambda \to 0+} \frac{W(\hat{p} + \lambda \mathcal{T}_1 w) - W(\hat{p})}{\lambda}, \qquad w \in \mathbb{R}^k. \tag{4.2}$$

This function represents the infinitesimal structure of W near $\hat{p}$, sliced in the direction of Z_1. Analogously to (1.5), we define the functional $E^{\mathrm{sl}}_{\hat{p}} : L^2(\Omega) \to \mathbb{R}$.

With $\hat{p}$, k as above, we say that a function $\psi \in Lip(\mathbb{R}^k)$ is a *$\hat{p}$-admissible support function* if $(\{\psi = 0\}, \operatorname{sign} \psi)$ is a $(W^{\mathrm{sl}}_{\hat{p}})^\circ$-$(L^2)$ Cahn-Hoffman facet. For a $\hat{p}$-admissible support function ψ, we define the nonlocal curvature-like operator

$$\Lambda_{\hat{p}}[\psi](x) := \Lambda[(\{\psi = 0\}, \operatorname{sign} \psi)], \qquad x \in \{\psi = 0\},$$

where Λ, defined in Definition 4.2, is used with the anisotropy $W = W^{\mathrm{sl}}_{\hat{p}}$.

Definition 4.7. *Let $\hat{p} \in \mathbb{R}^n$, $(\hat{x}, \hat{t}) \in \mathbb{R}^n \times \mathbb{R}$, $k := \dim \partial W(\hat{p})$. We say that $\varphi(x,t) = \psi(x') + f(x'') + \hat{p} \cdot x + g(t)$ is an* admissible stratified faceted test function at $(\hat{x}, \hat{t})$ with slope $\hat{p}$ *if $f \in C^1(\mathbb{R}^{n-k})$, $\nabla f(\hat{x}'') = 0$, $g \in C^1(\mathbb{R})$, and $\psi \in Lip(\mathbb{R}^k)$ is a $\hat{p}$-admissible support function with $\hat{x}' \in \operatorname{int}\{\psi = 0\}$. Note that if $k = 0$, we have $\varphi(x,t) = f(x) + g(t)$ for some $f \in C^1(\mathbb{R}^n)$, $g \in C^1(\mathbb{R})$.*

Definition 4.8 (Viscosity solution, cf. [GP, Definition 5.2])**.** *We say that an upper semicontinuous function u is a* viscosity subsolution *of $u_t + F(\nabla u, \operatorname{div}(\nabla W)(\nabla u)) = 0$ on $\mathbb{R}^n \times (0,T)$, $T > 0$, if for any $\hat{p} \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^n$, $\hat{t} \in (0,T)$ and any admissible stratified faceted test function φ at $(\hat{x}, \hat{t})$ with slope $\hat{p}$ of the form $\varphi(x,t) = \psi(x') + f(x'') + \hat{p} \cdot x + g(t)$ such that the function $u - \varphi(\cdot - h)$ has a global maximum on $\mathbb{R}^n \times (0,T)$ at $(\hat{x}, \hat{t})$ for all sufficiently small $h' \in \mathbb{R}^k$ and $h'' = 0$, there exists $\delta > 0$ such that*

$$g_t(\hat{t}) + F(\hat{p}, \operatorname*{ess\,inf}_{B_\delta(\hat{x}')} \Lambda_{\hat{p}}[\psi]) \le 0. \tag{4.3}$$

Viscosity supersolutions *are defined analogously as lower semicontinuous functions, replacing a global maximum with a global minimum,* ess inf *with* ess sup*, and reversing the inequality in* (4.3)*.*

A continuous function that is both a viscosity subsolution and a viscosity supersolution is called a viscosity solution*.*

Remark 4.9. Definition 4.8 at $\hat{p} = 0$ is a natural extension of the definition that appeared in an earlier paper [GGP1] for anisotropies W smooth outside of the origin. In that case, if appropriate tests are given at $\hat{p} \neq 0$ where W is smooth, the definition is equivalent to the definition of $\mathcal{F}$-solutions [G]. In fact, this equivalence can be proved along the lines of the proof of [G, Proposition 2.2.8],

where one has to replace $|x-y|^4$ by an appropriate function whose zero set consists of the Wulff shape of W.

Note that we do not need to consider the special test for "curvature-free" directions, $\hat{p}=0$, as in [GP] due to our ability to construct admissible stratified facet functions in an arbitrary dimension.

4.3. **Comparison principle.** One of the main results of [GP] was the comparison principle Theorem 1.4, assuming Theorem 1.8. We shall give a brief sketch of the proof of Theorem 1.4 for the reader's convenience.

Due to the nonlocality of the problem, we require that u and v are constant outside of a compact set to avoid technical issues with unbounded facets. However, it is an interesting question how to handle such solutions as well as boundary conditions. This will be addressed in a future work.

To show Theorem 1.4, we follow a variant of the standard proof by contradiction. The problem (4.1) has features of a second order problem in that similarly to Ishii's lemma that is used to construct test functions with ordered second derivatives even for only semicontinuous solutions in the usual argument, we need to show the existence of ordered Cahn-Hoffman facets at a contact point. In this article we only give the outline of the proof, and show how to apply Theorem 1.3. For full details, see [GP].

We thus suppose that we have a viscosity subsolution u and a viscosity supersolution v satisfying the hypothesis of Theorem 1.4, but for which the conclusion does not hold, that is,

$$m_0 := \sup_Q (u-v) > 0,$$

with $Q := \mathbb{R}^n \times (0,T)$.

Then we follow the standard doubling-of-variables argument with an extra parameter. That is, for $\varepsilon > 0$, $\zeta \in \mathbb{R}^n$, we study the maxima of the functions

$$\Phi_{\zeta,\varepsilon}(x,t,y,s) := u(x,t) - v(y,s) - \frac{|x-y-\zeta|^2}{2\varepsilon} - S_\varepsilon(t,s),$$

over $\overline{Q}\times\overline{Q} = \mathbb{R}^n \times [0,T] \times \mathbb{R}^n \times [0,T]$, where

$$S_\varepsilon(t,s) := \frac{|t-s|^2}{2\varepsilon} + \frac{\varepsilon}{T-t} + \frac{\varepsilon}{T-t}.$$

The introduction of ζ helps "flatten" the profile of u and v near the point of maximum of $\Phi_{\zeta,\varepsilon}$ that we are interested in, see Corollary 4.14 below.

We have the following important observation.

Proposition 4.10 (cf. [GG1]). *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$, $|\zeta| \le \kappa = \kappa(\varepsilon)$, $\Phi_{\zeta,\varepsilon}$ does not attain its maximum on the boundary of $Q\times Q$.*

Therefore, in what follows, we *fix* one such $\varepsilon > 0$ from Proposition 4.10 and $\kappa = \kappa(\varepsilon)$. We then write $\Phi_\zeta = \Phi_{\zeta,\varepsilon}$ and $S = S_\varepsilon$.

Now we follow the notation from [GG1, GP]. We define the maximum of Φ_ζ as

$$\ell(\zeta) := \max_{\overline{Q}\times\overline{Q}} \Phi_\zeta,$$

and the set of the points of maximum

$$\mathcal{A}(\zeta) := \arg\max_{\overline{Q}\times\overline{Q}} \Phi_\zeta := \big\{(x,t,y,s) \in \overline{Q}\times\overline{Q} : \Phi_\zeta(x,t,y,s) = \ell(\zeta)\big\}.$$

Moreover, the set of gradients at maximum will be denoted as

$$\mathcal{B}(\zeta) := \left\{ \frac{x-y-\zeta}{\varepsilon} : (x,t,y,s) \in \mathcal{A}(\zeta) \right\}.$$

We have the following compactness property.

Proposition 4.11 (cf. [GP, Proposition 7.3]). *The graphs of $\mathcal{A}(\zeta)$ and $\mathcal{B}(\zeta)$ over $|\zeta| \le \kappa$ are compact.*

This compactness and the simple structure of ∂W, piece-wise constant on relatively open convex sets, allow us to use the Baire category theorem to find a direction in which u and v have certain flatness.

Proposition 4.12 (cf. [GP, Proposition 7.4]). *There exists a set $\Xi \in \mathbb{R}^n$, a vector $\hat\zeta \in \mathbb{R}^n$ and $\lambda > 0$ such that $|\hat\zeta| + 2\lambda < \kappa$, $\partial W(p)$ is independent of $p \in \Xi$, and*

$$\mathcal{B}(\zeta) \cap \Xi \neq \emptyset \qquad \text{for all } |\zeta - \hat\zeta| < 2\lambda.$$

The set Ξ can be taken as a relatively open convex set in its affine hull aff Ξ*, which is orthogonal to* aff $\partial W(p)$*, $p \in \Xi$.*

In other words, the proposition implies the existence of a set $\Xi \subset \mathbb{R}^n$ and a point $\hat\zeta \in \mathbb{R}^n$ such that for any $\zeta \in \mathbb{R}^n$ close to $\hat\zeta$ there exists a point of maximum $(\hat x, \hat t, \hat y, \hat s) \in \mathcal{A}(\zeta)$ of Φ_ζ such that $\frac{\hat x - \hat y - \zeta}{\varepsilon} \in \Xi$. Noting that $\Phi_\zeta(x,t,\hat y,\hat s) \le \Phi_\zeta(\hat x,\hat t,\hat y,\hat s)$, we see that $\frac{\hat x-\hat y-\zeta}{\varepsilon}$ is the gradient of the smooth test function $(x,t) \mapsto \frac{|x-\hat y-\zeta|^2}{2\varepsilon} + S(t,\hat s)$ touching u at $(\hat x, \hat t)$. A similar reasoning applies for v. Since this gradient always falls into Ξ, we recover some flatness of u and v in the directions orthogonal to aff Ξ, in the sense of the following lemma.

Lemma 4.13 (cf. [GP, Lemma 7.6]). *Suppose that there exist $\hat p, \hat\zeta \in \mathbb{R}^n$, a subspace $Z_2 \subset \mathbb{R}^n$ and $\lambda > 0$ such that $|\hat\zeta| + 2\lambda < \kappa$ and*

$$\mathcal{B}(\zeta) \cap (\hat p + Z_2) \neq \emptyset \qquad \text{for all } |\zeta - \hat\zeta| < 2\lambda.$$

Then

$$\ell(\zeta) - \hat p \cdot \zeta \equiv const \quad \text{for all } \zeta \in \hat\zeta + Z_1,\ |\zeta - \hat\zeta| < 2\lambda,$$

where $Z_1 := Z_2^\perp$.

The following can be derived about the behavior of u and v from the above lemma. Recall that $x' \in \mathbb{R}^k$ is identified with the orthogonal projection of x onto Z_1 and $x'' \in \mathbb{R}^{n-k}$ with the projection onto Z_2.

Corollary 4.14 (cf. [GP, Corollary 7.7]). *Suppose that we have $\hat p$, $\hat\zeta$, λ, Z_1 and Z_2 as in Lemma 4.13. Define*

$$\theta(x,t,y,s) := u(x,t) - v(y,s) - \frac{|x''-y''-\hat\zeta''|^2}{2\varepsilon} - \hat p' \cdot (x'-y'-\hat\zeta') - S(t,s).$$

Then for any $(\hat x, \hat t, \hat y, \hat s) \in \mathcal{A}(\hat\zeta)$ such that $\frac{\hat x' - \hat y' - \hat\zeta'}{\varepsilon} = \hat p'$ we have

$$\theta(x,t,y,s) \le \theta(\hat x,\hat t,\hat y,\hat s) \quad \text{for } (x,t),(y,s) \in \overline{Q},\ |x'-y'-(\hat x'-\hat y')| \le \lambda.$$

Now we want to construct admissible stratified faceted test functions for u and v to reach a contradiction with the definition of a viscosity solution. Therefore for the rest of the proof, we fix Ξ, $\hat\zeta$ and λ from Proposition 4.12, and a point of maximum $(\hat x,\hat t,\hat y,\hat s)\in\mathcal{A}(\hat\zeta)$ such that $\hat p:=\frac{\hat x-\hat y-\hat\zeta}{\varepsilon}\in\Xi$. Given that $\partial W(p)$ is independent of $p\in\Xi$, we set $Z_1\subset\mathbb{R}^n$ parallel to the affine hull $\operatorname{aff}\partial W(p)$, $p\in\Xi$. The convexity of W implies that $\Xi-\hat p\subset Z_2:=Z_1^\perp$, see [GP, Proposition 3.1], and therefore Lemma 4.13 and Corollary 4.14 apply.

Let us set $k:=\dim\partial W(\hat p)=\dim Z_1=n-\dim Z_2$. We need to construct two $\hat p$-admissible support functions on $\mathbb{R}^k$ ordered so that the comparison principle in Proposition 4.1 applies, and which we can use to build the admissible faceted test functions at $(\hat x,\hat t)$ for u and at $(\hat y,\hat s)$ for v, with slope $\hat p$. Since this is trivial if $k=0$, we will from now assume that $k\geq 1$.

4.3.1. *Facet construction.* We first observe that $\hat p\perp Z_1$ since W is positively one-homogeneous and therefore $\hat p'=0$. We need to find two facets to which to apply Theorem 1.3. We follow [GP]. First we define the functions

$$\begin{aligned}\hat u(w)&:=u(\mathcal{T}_1w+\hat x,\hat t)-u(\hat x,\hat t),\\ \hat v(w)&:=v(\mathcal{T}_1w+\hat y,\hat s)-v(\hat y,\hat s),\end{aligned}\qquad\qquad w\in\mathbb{R}^k,$$

and their level sets

$$\hat U:=\big\{w\in\mathbb{R}^k:\hat u(w)\geq 0\big\},\qquad\qquad \hat V:=\big\{w\in\mathbb{R}^k:\hat v(w)\leq 0\big\}.$$

Lemma 4.15. *Sets $\hat U$ and $\hat V$ are closed with compact boundary, and at least one is compact.*

Proof. Closedness follows from semicontinuity of $\hat u$ and $\hat v$. Since $\sup_Q(u-v)>0$ and $u\leq v$ outside of a bounded set, we have $u(\hat x,\hat t)>v(\hat y,\hat s)$ and therefore $\hat u<\hat v$ outside of a bounded set. Since $\hat u$ and $\hat v$ are also both constant outside of a bounded set, $\hat U$ and $\hat V$ cannot be both unbounded, and their boundaries must be compact. The lemma follows. □

For convenience, we set

$$\begin{aligned}\xi_u(x'',t)&:=\frac{|x''-\hat y''-\hat\zeta''|^2}{2\varepsilon}-\frac{|\hat x''-\hat y''-\hat\zeta''|^2}{2\varepsilon}+S(t,\hat s)-S(\hat t,\hat s),\\ \xi_v(y'',s)&:=\frac{|\hat x''-\hat y''-\hat\zeta''|^2}{2\varepsilon}-\frac{|\hat x''-y''-\hat\zeta''|^2}{2\varepsilon}+S(\hat t,\hat s)-S(\hat t,s),\end{aligned}$$

$x'',y''\in\mathbb{R}^{n-k}$, $t,s\in\mathbb{R}$. From Corollary 4.14 we deduce

$$\begin{aligned}&u(x,t)-u(\hat x,\hat t)-\xi_u(x'',t)\leq 0,\quad\text{for }(x,t)\in\overline{Q},\operatorname{dist}(x'-\hat x',\hat V)\leq\lambda,\\ &v(y,s)-v(\hat y,\hat s)-\xi_v(y'',s)\geq 0,\quad\text{for }(y,s)\in\overline{Q},\operatorname{dist}(y'-\hat y',\hat U)\leq\lambda.\end{aligned}\tag{4.4}$$

For reasons that shall become apparent in the proof of Lemma 4.16 below, we set $r:=\frac{\lambda}{5}$. From the definition of $\hat U$, $\hat V$, semicontinuity of u, v, and the fact that u and v are constant outside of a bounded set, there exists $\delta>0$ such that

$$\begin{aligned}&u(x,t)-u(\hat x,\hat t)-\xi_u(x'',t)<0,\quad\operatorname{dist}(x'-\hat x',\hat U)\geq r,|x''-\hat x''|\leq\delta,|t-\hat t|\leq\delta,\\ &v(y,s)-v(\hat y,\hat s)-\xi_v(y'',s)>0,\quad\operatorname{dist}(y'-\hat y',\hat V)\geq r,|y''-\hat y''|\leq\delta,|s-\hat s|\leq\delta.\end{aligned}\tag{4.5}$$

We now introduce k-dimensional facets (F_u, χ_u) and (F_v, χ_v). F_u and $\chi_u \in C(F_u^c, \{-1, 1\})$ are defined as

$$F_u := \{w \in \mathbb{R}^k : \operatorname{dist}(w, \hat{U}) \le 2r \wedge (\operatorname{dist}(w, \hat{U}) \ge 2r \vee \operatorname{dist}(w, \hat{V}) \le \lambda - r)\},$$

$$\chi_u(w) := \begin{cases} 1, & \operatorname{dist}(w, \hat{U}) < 2r \wedge \operatorname{dist}(w, \hat{V}) > \lambda - r, \\ -1, & \operatorname{dist}(w, \hat{U}) > 2r, \\ 0, & \text{otherwise}, \end{cases} \tag{4.6}$$

and F_v, $\chi_v \in C(F_v^c, \{-1, 1\})$ are defined analogously, swapping $\hat{U}$ with $\hat{V}$ and F_u with F_v. Note that F_u and F_v are compact due to Lemma 4.15.

Applying Theorem 1.3 with $\rho = r$, for anisotropy $W_{\hat{p}}^{\mathrm{sl}}$, we obtain a $W_{\hat{p}}^{\mathrm{sl},\circ}$-$(L^2)$ Cahn-Hoffman facet $(G_u, \tilde{\chi}_u)$ for the facet (F_u, χ_u). Similarly, with anisotropy $p \mapsto W_{\hat{p}}^{\mathrm{sl}}(-p)$, we obtain a facet $(G_v, \tilde{\chi}_v)$ for (F_v, χ_v). Note that $(G_v, -\tilde{\chi}_v)$ is a $W_{\hat{p}}^{\mathrm{sl},\circ}$-$(L^2)$ Cahn-Hoffman facet.

To finish the construction of the test functions, we set for $x', y' \in \mathbb{R}^k$

$$\tilde{u}(x' - \hat{x}') := \sup_{|x''-\hat{x}''|\le\delta} \sup_{|t-\hat{t}|\le\delta} \left[u(x,t) - u(\hat{x},\hat{t}) - \xi_u(x'',t)\right],$$

$$\tilde{v}(y' - \hat{y}') := \inf_{|y''-\hat{y}''|\le\delta} \inf_{|s-\hat{s}|\le\delta} \left[v(y,s) - v(\hat{y},\hat{s}) - \xi_v(y'',s)\right].$$

Lemma 4.16. *The facets $(G_u, \tilde{\chi}_u)$ and $(G_v, -\tilde{\chi}_v)$ have the following properties:*

(a) *The facets are ordered, and ordered with respect to the "facets" of $\tilde{u}$ and $\tilde{v}$, namely,*

$$\sup_{|y-x|\le r} \operatorname{sign} \tilde{u}(y) \le \tilde{\chi}_u(x) \le -\tilde{\chi}_v(x) \le \inf_{|y-x|\le r} \operatorname{sign} \tilde{v}(y), \qquad x \in \mathbb{R}^n. \tag{4.7}$$

(b) *The origin 0 lies in the interior of the intersection of the facets, i.e.,*

$$\overline{B}_r(0) \subset G_u \cap G_v.$$

Proof. The lemma was previously proved in [GGP1, Lemma 4.6], using a different notation. For the reader's convenience, we present a self-contained proof using the new notion of facets. To simplify the notation, we write $\sup_{|y-x|\le r} \chi(y)$ as $(\sup^r \chi)(x)$, and analogously for $\inf^r$.

For (a), note that by construction from Theorem 1.3, $\chi_u \le \tilde{\chi}_u \le \sup^r \chi_u$ and $\inf^r(-\chi_v) \le -\tilde{\chi}_v \le -\chi_v$. Therefore the first inequality in (4.7) will follow from $\sup^r \operatorname{sign} \tilde{u} \le \chi_u$, the second will follow from $\sup^r \chi_u \le \inf^r(-\chi_v)$ and the third from $-\chi_v \le \inf^r \operatorname{sign} \tilde{v}$.

To show $\sup^r \chi_u \le \inf^r(-\chi_v)$, we show the equivalent $\chi_u(w) \le -\chi_v(z)$ for all $|z - w| \le 2r$. Fix therefore w, z with $|z - w| \le 2r$. When $\chi_u(w) = -1$, there is nothing to show since $-\chi_v \ge -1$. If $\chi_u(w) = 0$ then by (4.6) $\operatorname{dist}(w, \hat{U}) \le 2r$, and by the triangle inequality $\operatorname{dist}(z, \hat{U}) \le 4r \le \lambda - r$ and therefore $-\chi_v(z) \ge 0 = \chi_u(w)$. Similarly, if $\chi_u(w) = 1$ then by (4.6) $\operatorname{dist}(w, \hat{V}) > \lambda - r$, by the triangle inequality $\operatorname{dist}(z, \hat{V}) > \lambda - 3r \ge 2r$ and therefore $-\chi_v(z) = 1 = \chi_u(w)$. We conclude that $\chi_u(w) \le \chi(z)$ for all w, z, $|z - w| \le 2r$.

To show $\sup^r \operatorname{sign} \tilde{u} \le \chi_u$, fix w, z with $|z - w| \le r$. If $\tilde{u}(w) < 0$, then automatically $\operatorname{sign} \tilde{u}(w) = -1 \le \chi_u(z)$. If $\tilde{u}(w) = 0$, then by (4.5) $\operatorname{dist}(w, \hat{U}) < r$. By the triangle inequality, $\operatorname{dist}(z, \hat{U}) < 2r$ and so by (4.6) $\chi_u(z) \ge 0 = \operatorname{sign} \tilde{u}(w)$. Finally, if $\tilde{u}(w) > 0$, by (4.4) $\operatorname{dist}(w, \hat{V}) > \lambda$ and by (4.5) $\operatorname{dist}(w, \hat{U}) < r$. Therefore the triangle inequality implies $\operatorname{dist}(z, \hat{U}) < 2r$ and $\operatorname{dist}(z, \hat{V}) > \lambda - r$, and so by (4.6) $\chi_u(z) = 1 = \operatorname{sign} \tilde{u}(w)$. The proof of $-\chi_v \le \inf^r \operatorname{sign} \tilde{v}$ is analogous.

For (b), recall that $0 \in \hat{U} \cap \hat{V}$. Therefore by the triangle inequality $\operatorname{dist}(w, \hat{U}) \le r \le 2r$ and $\operatorname{dist}(w, \hat{V}) \le r \le \lambda - r$ for any $|w| \le r$. In particular $\chi_u(w) = 0$ by (4.6). Furthermore, for any z, $|z - w| \le r$, $\operatorname{dist}(z, \hat{U}) \le 2r$, $\operatorname{dist}(z, \hat{V}) \le 2r \le \lambda - r$ and therefore $\chi_u(z) = 0$. We conlude that $0 = \chi_u(w) \le \tilde{\chi}_u(w) \le (\sup^r \chi_u)(w) = 0$, in particular, $w \in G_u$. An analogous argument implies $w \in G_v$. □

Let us now set $m := \sup\{\tilde{u}(w) : \operatorname{dist}(w, \{\tilde{\chi}_u = -1\}) \le \frac{r}{2}\}$ and define

$$\psi_u(x) := \max\left(\frac{2\sup \tilde{u}}{r} \operatorname{dist}(x, G_u)\tilde{\chi}_u(x), m\right).$$

Then due to the ordering in Lemma 4.16(a) and the fact that $\tilde{u}$ is a constant outside of a bounded set, $m < 0$, and $\tilde{u}(\cdot - w) \le \psi_u$ for all $|w| \le \frac{r}{2}$. Note that $\operatorname{sign} \psi_u = \tilde{\chi}_u$. In a similar way we can construct ψ_v such that $\psi_v \le \tilde{v}(\cdot - w)$ for all $|w| \le \frac{r}{2}$.

The functions $\varphi_u(x,t) := \psi_u(x' - \hat{x}') + \xi_u(x'', t)$ and $\varphi_v := \psi_v(x' - \hat{y}') + \xi_v(x'', t)$ are admissible stratified faceted test functions at $(\hat{x}, \hat{t})$ and $(\hat{y}, \hat{s})$, respectively, with slope $\hat{p}$, in the sense of Definition 4.7. Since u is bounded above and ψ_u is bounded, by modifying and smoothly extending ξ_u for $|x'' - \hat{x}''| > \frac{\delta}{2}$, $|t - \hat{t}| > \frac{\delta}{2}$, we can assume that $u - \varphi_u(\cdot - h)$ has a global maximum at $(\hat{x}, \hat{t})$ in Q for all sufficiently small h', $|h'| \le \frac{r}{2}$, and $h'' = 0$. A similar reasoning applies to φ_v. Therefore φ_u and φ_v are test functions in the sense of Definition 4.8.

From the definition of viscosity solutions, Definition 4.8, we infer that for some $\tilde{r} \in (0, r)$,

$$\begin{aligned} (\xi_u)_t(\hat{t}) + F\left(\hat{p}, \operatorname*{ess\,inf}_{B_{\tilde{r}}(0)} [\Lambda_{\hat{p}}[\psi_u]]\right) &\le 0, \\ (\xi_v)_t(\hat{s}) + F\left(\hat{p}, \operatorname*{ess\,sup}_{B_{\tilde{r}}(0)} [\Lambda_{\hat{p}}[\psi_v]]\right) &\ge 0. \end{aligned} \tag{4.8}$$

On the other hand, Lemma 4.16(a–b) and the comparison principle Proposition 4.1 imply

$$\operatorname*{ess\,inf}_{B_{\tilde{r}}(0)} [\Lambda_{\hat{p}}[\psi_u]] \le \operatorname*{ess\,sup}_{B_{\tilde{r}}(0)} [\Lambda_{\hat{p}}[\psi_v]], \tag{4.9}$$

and therefore by subtracting the inequalities in (4.8) and applying the ellipticity of F we obtain

$$0 < \frac{\varepsilon}{(T - \hat{t})^2} + \frac{\varepsilon}{(T - \hat{s})^2} + F\left(\hat{p}, \operatorname*{ess\,inf}_{B_{\tilde{r}}(0)} [\Lambda_{\hat{p}}[\psi_u]]\right) - F\left(\hat{p}, \operatorname*{ess\,sup}_{B_{\tilde{r}}(0)} [\Lambda_{\hat{p}}[\psi_v]]\right) \le 0,$$

a contradiction. This finishes the proof of the comparison principle, Theorem 1.4.

4.4. **Stability and existence of solutions.** With the comparison principle, Theorem 1.4, valid in any dimension, the stability with respect to approximation by regularized problems, Theorem 1.5, and the well-posedness, Theorem 1.6, proved originally in [GP] immediately generalize to all dimensions. Let us give a brief outline of the approach, for all details see [GP]. The argument in the simplified form presented below basically first appeared in [GGP2].

In the standard viscosity theory, the existence of solutions usually follows from Perron's method: the largest subsolution (or the smallest supersolution) turns out to be a solution of the problem, see [CIL, G]. However, it is not clear whether Perron's method can be used for the crystalline curvature problem (1.3) due to the very strong nonlocality of the curvature operator, except in one dimension [GG1, GG3] when the speed of a facet is constant. In case when the speed of the facet is not constant, this approach seems to be difficult except in a

one-dimensional setting, where Perron's method is applied to construct a graph-like solution with non-uniform driving force term by careful classification of speed profile of each facet [GGN]. To be more specific, the standard shift (Ishii's shift) of a faceted test function to create a larger subsolution when the largest subsolution fails to be a supersolution and reach a contradiction cannot be performed unless the crystalline curvature Λ is constant on the facet. Therefore we use the stability of solutions with respect to regularization of W in place of Perron's method.

4.4.1. *Stability.* We consider two modes of regularization of W:

(a) $W_m \in C^2(\mathbb{R}^n)$, $a_m^{-1} \le \nabla^2 W_m \le a_m$ for some $a_m > 0$, and $W_m \searrow W$, or
(b) W_m are smooth anisotropies, that is, W_m is an anisotropy, $W_m \in C^2(\mathbb{R}^n \setminus \{0\})$ and $\{p : W_m(p) \le 1\}$ is strictly convex, such that $W_m \to W$ locally uniformly.

The regularization (a) yields a sequence of degenerate parabolic problems that are within the classical viscosity theory [CIL], while (b) regularizes the crystalline curvature operator by smooth anisotropic curvatures studied in [CGG] when F comes from a level set formulation, or in [GGP2, GGP1] for general F.

Note that both regularizations produce *local* problems, except at $\nabla u_m = 0$ in (b) . The main difficulty in proving the stability property with respect to the approximation is therefore again caused by the nonlocality of the crystalline curvature operator. In the limit $m \to \infty$, the nonlocal information contained in Λ must be recovered. This is achieved with the help of a variant of the perturbed test function method.

Suppose therefore that we approximate W by a sequence of regular W_m as in (a) above and obtain a sequence of solutions u_m of the regularized problems. We want to show that $u(x,t) = \limsup_{(y,s,m)\to(x,t,\infty)} u_m(y,s)$ is a subsolution of (4.1) by verifying Definition 4.8. Let φ be an admissible stratified faceted test function at a point $(\hat{x},\hat{t})$ with slope $\hat{p}$ satisfying the assumptions in Definition 4.8. We need to show (4.3).

To simplify the explanation, let us assume that $k = n$ so that $\hat{p} = 0$, $Z_1 = \mathbb{R}^n$ and $\varphi(x,t) = \psi(x) + g(t)$, where $\psi \in Lip(\mathbb{R}^n)$ is 0-admissible support function. By definition, $(A,\chi) := (\{\psi = 0\}, \operatorname{sign}\psi)$ is a $(W_0^{\mathrm{sl}})^\circ$-$(L^2)$ Cahn-Hoffman facet. For the treatment of the general case, see the details in [GP]. By adding constants and translation, we can assume that $(\hat{x},\hat{t}) = (0,0)$ and $u(\hat{x},\hat{t}) = u(0,0) = 0$, $g(\hat{t}) = g(0) = 0$. By Definition 4.8, we may assume that $u - \varphi(\cdot - h)$ has a global maximum at $(\hat{x},\hat{t}) = (0,0)$ for all $|h| \le \rho$ for some $\rho > 0$, and $\psi(x) = 0 \Leftrightarrow x \in A$ for $|x| \le \rho$.

There are now a few difficulties with trying to follow the standard stability argument for viscosity solutions. The first is the smoothness of φ. We need at least C^2-regularity in space to be able to use φ as a test function for the approximate problems called m-problems. Let us therefore assume that φ is in fact smooth so that this is not an issue. The second problem arises when we try to find a subsequence $m_l \to \infty$ and points of maxima (x_l,t_l) of $u_{m_l} - \varphi$ such that $(x_l,t_l) \to (\hat{x},\hat{t}) = (0,0)$. Such a sequence exists in general only if $u - \varphi$ has a *strict* maximum at $(0,0)$. In the standard argument, this is ensured by a smooth perturbation of φ, for instance by adding a term like $|x|^4 + |t|^2$ to φ. Suppose that we have such a sequence (x_{m_l}, t_{m_l}) of maxima converging to $(0,0)$. Then by the definition of the viscosity solution of the m-problem, we have

$$[\varphi_t + F(\nabla\varphi, \operatorname{div}(\nabla W_{m_l})(\nabla\varphi))](x_{m_l}, t_{m_l}) \le 0, \qquad \text{for all } l.$$

As $\operatorname{div}(\nabla W_m)(\nabla\varphi) = \operatorname{tr}[(\nabla^2 W_m)(\nabla\varphi)\nabla^2\varphi]$ and $\nabla\varphi(0,0) = 0$, there is little hope that this local quantity will converge to anything useful due to the singularity of W at 0, much less to $\Lambda_{\hat{p}}[\psi](\hat{x})$, which is nonlocal.

The key idea is to introduce a uniform perturbation of φ that depends on m, so that it captures the necessary nonlocal information. This basic scheme was introduced with great success in the viscosity theory by Evans [E], where it is called the perturbed test function method. In fact, this idea was actually carried out in the one-dimensional setting, where the test function is taken essentially as W_m° [GG2]. However, in higher dimensional case, one has to test with more functions, which requires a new idea for a choice of test functions depending on m. Such an idea has first appeared in [GGP2]. We shall sketch it below.

As $-\Lambda_{\hat{p}}(\psi)$ coincides on the facet with the minimizing element of the subdifferential of the anisotropic total variation energy at ψ, it can be approximated by its resolvent problem. This problem is equivalent to performing one step of the implicit Euler discretization of the anisotropic total variation flow. To have compactness, we modify ψ far away from them facet and rescale so that it is $\mathbb{Z}^n$-periodic. For given $a > 0$, we find the unique solutions $\psi_a, \psi_{a,m} \in L^2(\mathbb{T}^n)$ of the resolvent problems

$$\frac{\psi_{a,m} - \psi}{a} \in -\partial E_m(\psi_{a,m}), \qquad \frac{\psi_a - \psi}{a} \in -\partial E(\psi_a), \tag{4.10}$$

where E and E_m are the energies defined in (1.5) with $\Omega = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. It is possible to modify ψ away from the facet in such a way that $\psi \in Lip(\mathbb{T}^n)$ and $\partial E(\psi) \neq \emptyset$ since ψ is a 0-admissible support function.

The resolvent problems have a number of very useful properties. Since W_m are smooth, the problem for $\psi_{a,m}$ is a quasilinear elliptic problem. Therefore $\psi_{a,m} \in C^2(\mathbb{T}^n)$ by the elliptic regularity. Moreover, due to the monotone convergence of $W_m \to W$, E_m converges in Mosco sense to E, which implies the resolvent convergence $\psi_{a,m} \to \psi_a$ in $L^2(\mathbb{T}^n)$ as $m \to \infty$ [A2]. By the comparison principle and the translation invariance of the resolvent problems (4.10), $\|\nabla\psi_{a,m}\|_\infty \le \|\nabla\psi\|_\infty$. Therefore $\psi_{a,m} \to \psi_a$ uniformly and $\|\nabla\psi_a\|_\infty \le \|\nabla\psi\|_\infty$. Finally, since $\partial E(\psi) \neq \emptyset$, $\psi_a \to \psi$ uniformly and $\frac{\psi_a - \psi}{a} \to -\partial^0 E(\psi)$ in $L^2(\mathbb{T}^n)$ as $a \to 0+$. Recall that $\Lambda_0[\psi] = -\partial^0 E(\psi)$ on the facet of ψ. This construction yields perturbed test functions $\varphi_{a,m}(x,t) = \psi_{a,m}(x) + g(t)$ and $\varphi_a(x,t) = \psi_a(x) + g(t)$.

We now turn our attention to a compact neighborhood of the facet of ψ,

$$O := \{x : \operatorname{dist}(x, A) \le \rho\}.$$

We assume that we have modified ψ above only far away from the facet (A, χ) so that the value of $\psi(\cdot - w)$ does not change on O for all $|w| \le \rho$. For convenience, we define

$$\bar{u}(x) := \sup_{|t|\le\rho} u(x,t) - g(t).$$

Note that $\bar{u} - \psi(\cdot - h) \le 0$ on O for $|w| \le \rho$, with equality at $x = 0$.

We set $\delta := \frac{\rho}{5}$ and define the critical set

$$N := \{x \in O : \bar{u}(x) \ge 0, \psi(x - w) \le 0 \text{ for some } |w| \le \delta\}.$$

We can deduce that [GP, Corollary 8.3]

$$\bar{u}(x) \le 0, \quad \psi(x - z) \ge 0, \qquad \text{for all } \operatorname{dist}(x, N) \le 3\delta, |z| \le \delta,$$
$$\operatorname{dist}(N, \partial O) \ge 4\delta.$$

In particular, we see that for any $|z| \leq \delta$ and any $\alpha > 0$ all maxima of $\bar{u} - \alpha\psi(\cdot - z)$ in $\{x : \mathrm{dist}(x, N) \leq 3\delta\}$ lie in N. We can therefore make the Lipschitz constant $\|\nabla\psi\|_\infty$ arbitrarily small by multiplying ψ by small $\alpha > 0$ in the sequel. By adding $|t|^2$ to $g(t)$ if necessary, we may assume that all maxima of $u - \varphi(\cdot - z, \cdot)$ are located at $t = 0$.

For $a > 0$ let us fix z_a such that $|z_a| \leq \delta$ and $\psi_a(z_a) = \min_{|w|\leq\delta} \psi_a(w)$. This choice will become important later.

By the above consideration and the uniform convergence of $\psi_a \to \psi$, there exists $a_0 > 0$ such that all the maxima of $u - \varphi_a(\cdot - z_a, \cdot)$ in $M^{3\delta}$ lie in M^δ, where

$$M^s := \{(x, t) : \mathrm{dist}(x, N) \leq s, |t| \leq s\}.$$

Now for every $a \in (0, a_0)$, by the uniform convergence of $\psi_{a,m} \to \psi_a$ and properties of half-relaxed limits, there exist a point $(x_a, t_a) \in M^\delta$ of maximum of $u - \varphi_a(\cdot - z_a, \cdot)$ and sequences $m_l \to \infty$, $(x_l, t_l) \to (x_a, t_a)$ as $l \to \infty$, where (x_l, t_l) is a point of maximum of $u_{m_l} - \varphi_{a,m_l}(x_l - z_a, t_l)$. By the uniform Lipschitz continuity of $\psi_{a,m}$, we can assume that there exists $p_a \in \mathbb{R}^n$, $|p_a| \leq \|\nabla\psi\|_\infty$, such that $\nabla\psi_{a,m_l}(x_l - z_a) \to p_a$ as $l \to \infty$. Since $\varphi_{a,m}$ are smooth, the definition of viscosity solution u_m implies

$$g'(t_l) + F(\nabla\psi_{a,m_l}, \mathrm{div}[(\nabla W_{m_l})(\nabla\psi_{a,m_l})])(x_l - z_a) \leq 0.$$

As $\mathrm{div}[(\nabla W_{m_l})(\nabla\psi_{a,m_l})] = -\partial^0 E_{m_l}(\psi_{a,m_l}) = \frac{\psi_{a,m_l} - \psi}{a}$, the uniform convergence implies in the limit $l \to \infty$

$$g'(t_a) + F\big(p_a, \frac{\psi_a - \psi}{a}(x_a - z_a)\big) \leq 0.$$

The point z_a was chosen above in such a way that a geometric lemma, [GP, Lemma 8.5], implies

$$\frac{\psi_a - \psi}{a}(x_a - z_a) \leq \min_{|w|\leq\delta} \frac{\psi_a - \psi}{a}(w).$$

Monotonicity of F in the second variable thus yields

$$g'(t_a) + F\big(p_a, \min_{|w|\leq\delta} \frac{\psi_a - \psi}{a}(w)\big) \leq 0.$$

Note that $t_a \to 0$ as $a \to 0+$. By finding a sequence of $a_j \to 0+$ such that there exist $p \in \mathbb{R}^n$, $|p| \leq \|\nabla\psi\|_\infty$ with $p_{a_j} \to p$ and $\lim_{j\to\infty} \min_{|w|\leq\delta} \frac{\psi_{a_j} - \psi}{a_j}(w) = \liminf_{a\to 0+} \min_{|w|\leq\delta} \frac{\psi_a - \psi}{a}(w)$, we deduce that

$$g'(0) + F(p, \liminf_{a\to 0+} \min_{|w|\leq\delta} \frac{\psi_a - \psi}{a}(w)) \leq 0.$$

Since $\frac{\psi_a - \psi}{a} \to \Lambda_0[\psi](w)$ in $L^2(B_\delta(0))$, we conclude that

$$\operatorname{ess\,inf}_{|w|\leq\delta} \Lambda_0[\psi](w) \geq \liminf_{a\to 0+} \min_{|w|\leq\delta} \frac{\psi_a - \psi}{a}(w).$$

The monotonicity of F therefore yields

$$g'(0) + F(p, \operatorname{ess\,inf}_{|w|\leq\delta} \Lambda_0[\psi](w)) \leq 0.$$

Finally, we recall that we can assume that $|p| \leq \|\nabla\psi\|_\infty$ is arbitrarily small. Continuity of F in the first variable yield the final conclusion that (4.3) is satisfied.

The full argument is more technically involved. In particular, we need to decompose the space $\mathbb{R}^n$ into the direct sum of Z_1 and Z_2 as explained in Section 4.2, and treat them independently. In essence, the resolvents $\frac{\psi_a - \psi}{a}$ do not depend on the directions parallel to Z_2, [GP, Lemma 3.9], and therefore we can treat these directions as we treat time in the above simplified argument. A symmetric argument implies that $u(x,t) = \liminf_{(y,s,m)\to(x,t,\infty)} u_m$ is a viscosity supersolution.

To address the stability with respect to positively one-homogeneous approximations W_m in (b) above, we approximate W_m by a sequence W_m^δ as in (a) and modify the previous stability argument. In particular, we have solutions u_m^δ besides u_m, and $u_m^\delta \rightrightarrows u_m$ locally uniformly as $\delta \to 0+$ due to the stability results in [GGP2, GGP1].

4.4.2. *Existence.* Existence of solutions of the equation (4.1) is now a standard consequence of the stability property. Fixing initial data u_0 and an approximating sequence of one-homogeneous W_m so that the stability result Theorem 1.5 holds, we get a sequence of viscosity solutions of (4.1) with the anisotropy W_m and initial data u_0. The existence of such solutions follows from the standard theory of viscosity solutions [CGG, G] if F comes from the level set formulation of (1.4), or from [GGP1] in the general case. By the stability result Theorem 1.5, the half-relaxed limits $\bar{u}(x,t) = \limsup_{(y,s,m)\to(x,t,\infty)} u_m(y,s)$ and $\underline{u}(x,t) = \liminf_{(y,s,m)\to(x,t,\infty)} u_m(y,s)$ are respectively a subsolution and a supersolution of (4.1), without the initial data. Clearly $\underline{u} \le \bar{u}$. To show the other inequality, we use the comparison theorem Theorem 1.4 after we show that $\bar{u}$ and $\underline{u}$ attain the correct initial data u_0 and that they are equal to a constant outside of a compact set at each time. This can be done via the comparison principle for the regularized problems in a rather standard way using translations of the barriers for the solutions u_m of the type $(x,t) \mapsto \min(\max(W_m^\circ(x), c_1), c_2) + ct$ for appropriate constants c_1, c_2, c, where W_m° is the polar of W_m. Since $W_m \to W$ locally uniformly, $W_m^\circ \to W^\circ$ locally uniformly as well. The existence part of the well-posedness theorem, Theorem 1.6, is established. The uniqueness is a direct consequence of the comparison principle, Theorem 1.4.

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