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THE NAVIER-STOKES EQUATIONS FOR LINEARLY GROWING VELOCITY WITH NONDECAYING INITIAL DISTURBANCE

OKIHIRO SAWADA AND TOSHIOMI USUI

Abstract. The locally-in-time solvability of the Cauchy problem of the incompressible Navier-Stokes equations is established with initial velocity U_0 of the form $U_0(x) := u_0(x) - Mx$, where M is a real-valued matrix and u_0 is a bounded function. It is also shown that in 2-dimensional case the Navier-Stokes equations admit a unique globally-in-time smooth solution, due to the uniform bound for vorticity. Although the semigroup is not analytic, our mild solution satisfies the Navier-Stokes equations in the classical sense, provided the pressure term is suitably chosen. The form of the pressure is uniquely determined, provided the disturbance of velocity is bounded and the modified pressure is in a certain function space.

1. Introduction and Main Results.

We consider the flow of an incompressible viscous fluid in the whole space $\mathbb{R}^n$, $n \geq 2$:

$$\left\{ \begin{array}{rl} U_t - \Delta U + (U, \nabla)U + \nabla P = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ \nabla \cdot U = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ U|_{t=0} = U_0 & \text{in } \mathbb{R}^n. \end{array} \right. \tag{1.1}$$

Here, $U = (U^1(x,t), \ldots, U^n(x,t))$ represents the unknown velocity vector field of the fluid at location $x \in \mathbb{R}^n$ and time $t \in (0,T)$, and $P = P(x,t)$ is its pressure of scalar field; U_0 is a given initial velocity satisfying $\nabla \cdot U_0 = 0$ in tempered distribution sense (compatibility condition). We used the standard notations $U_t := \partial U/\partial t$, $\partial_j := \partial/\partial x_j$ for $j = 1, \ldots, n$, $\Delta := \sum_{j=1}^n \partial_j^2$, $\nabla := (\partial_1, \ldots, \partial_n)$, and $(a,b) := a \cdot b := \sum_{j=1}^n a_j b_j$ for $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. For the shake of simplicity, we assume that the external force terms is conservative, i.e., given by a potential.

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There are huge literatures on (1.1). Particularly, on the existence of solutions of (1.1) in $\mathbb{R}^n$ with decaying initial velocity, we refer to e.g. [20, 28, 34, 46, 47, 48, 53, 54, 60]. In 1930's J. Leray [53, 54] proved that in 2-dimensional case there exists a globally-in-time smooth solution, and in 3-dimensional case there exist weak solutions, when U_0 is square-integrable. His proofs are based on the energy method, dealing with the kinematic energy $\frac{1}{2}\|U(t)\|_2^2$.

In 1960's H. Fujita and T. Kato introduced the notion of a mild solution (solution to the integral equation), and they proved the locally-in-time solvability of (1.1) if the initial velocity U_0 is in a certain fractional power Sobolev space. It is also well-known that there exists a unique locally-in-time smooth solution to (1.1) provided U_0 belongs to $L^p_\sigma(\mathbb{R}^n)$ for $p \geq n$; see [28, 34, 46]. Moreover, T. Kato [46] showed that if $U_0 \in L^n_\sigma(\mathbb{R}^n)$ is small enough, then the smooth solution exists globally-in-time. This argument is only applicable when the function space of the initial data is invariant under the scaling of self-similar solutions, for instance, $\dot{H}^{n/2-1}(\mathbb{R}^n)$ and $\dot{B}^{-1+n/p}_{p,q}(\mathbb{R}^n)$; see [28, 73].

On the other hand, there are some interests in the equations (1.1) with initial data which do not decay at spatial infinity. When U_0 is merely bounded, we refer to [8, 9, 14, 31, 50, 51, 58, 67]. In 2-dimensional case, even though the kinematic energy is not finite, we can also obtain the globally-in-time smooth solutions by deriving a priori estimates from a uniform bound for vorticity; see [33, 69, 70]. J. Kato [45] established the uniqueness of weak solutions to (1.1), when U is bounded and P is BMO-valued locally-in-time integrable.

We also have strong interests the case when the velocity grows at $|x| \to \infty$, since there are many exact solutions in this framework; see [21]. H. Okamoto [65] and D. Chae and N. Kim [49] showed the uniqueness of classical solutions to (1.1) which enjoy the property that grows linearly at spatial infinity, besides, the pressure has a suitable decay. More precisely, they discuss the solutions in the class

$$|U(x,t)| = O(|x|),\ |\nabla U(x,t)| = O(1),\ |P(x,t)| = O(|x|^{-\frac{1}{2}}) \text{ as } |x| \to \infty.$$

In this paper, we consider initial velocity of the form

$$U_0(x) = u_0(x) - Mx, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where the initial disturbance u_0 is a function nondecaying at space infinity satisfying $\nabla \cdot u_0 = 0$ in the tempered distribution sense, and

$M = (m_{jk})$ is an $n \times n$ real-valued matrix fulfilling the two conditions:

$$\mathrm{tr}M = 0, \quad M^2 \text{ is symmetric.} \tag{1.3}$$

It was mentioned by e.g. [21, 59, 66] that the pair $U = -Mx$ and $P = \Pi := -\frac{1}{2}(M^2x, x)$ is a stationary solution to (1.1). For the case $M = M_r$ where M_r is skew-symmetric, we investigate a rotation phenomena of fluid (or, rigid body). Besides, when $M = M_j$ where M_j is diagonal, this illustrates an axisymmetric straining flow.

We consider the substitutions

$$u := U + Mx \quad \text{and} \quad \tilde{P} := P - \Pi. \tag{1.4}$$

Hence, (U, P) is a classical solution to (1.1) if and only if $(u, \tilde{P})$ fulfills

$$\begin{cases} u_t + Au = -(u, \nabla)u + 2Mu - \nabla\tilde{P}, \\ \nabla \cdot u = 0 \quad \text{in } \mathbb{R}^n \times (0, T), \\ u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^n. \end{cases} \tag{1.5}$$

Here we denote the linear operator $A := -\Delta - (Mx, \nabla) + M$. We call the terms $(Mx, \nabla)u$ drift terms. If M is skew-symmetric, then the terms Mu express a Coriolis force. Throughout of this paper it is assumed that the initial disturbance u_0 belongs to

$$\dot{\mathcal{B}}^0_{\infty,1,\sigma} := \{v \in \dot{B}^0_{\infty,1}; \sum_{j\in\mathbb{Z}} \phi_j * v = v, \ \nabla \cdot v = 0 \text{ in } \mathcal{S}'\},$$

where $\{\phi_j\}_{j\in\mathbb{Z}}$ is the Littlewood-Paley decomposition. Although $\dot{\mathcal{B}}^0_{\infty,1,\sigma}$ is strictly smaller than L^∞_σ, that still contains nondecaying functions. See Section 2 for the details on this space.

It is known that $-A$ with suitable domain $D(A)$ generates a semigroup $\{e^{-tA}\}_{t\geq 0}$ in L^p_σ for $p \in [1, \infty]$. In Section 3 we see that $\{e^{-tA}\}_{t\geq 0}$ acts onto $\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^n)$ for $n \geq 2$. Applying the projection $\mathbb{P}$ onto solenoidal subspace into the first equations of (1.5) to annihilate the gradient terms $\nabla\tilde{P}$, we deduce the integral equation of (1.5):

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathbb{P}\{(u(s), \nabla)u(s) - 2Mu(s)\}ds \tag{1.6}$$

by Duhamel's principle. The solution u of (1.6) is often called a mild solution, we also use this terminology in this paper. We rather discuss (1.5) or (1.6) than (1.1).

We now state the main results of this paper:

1.1. **Theorem.** *Let $n \geq 2$, M satisfy (1.3), and let $u_0 \in \dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^n)$. Then there exist a $T > 0$ and a unique smooth mild solution u on $(0, T)$*

in the class

$$[t \mapsto t^{k/2}\nabla^k u(t)] \in C([0,T);\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n)) \quad \text{for } k=0,1.$$

The mild solution u satisfies (1.5) *in the classical sense provided*

$$\partial_l \tilde{P} = \sum_{j,k=1}^{n} \Big\{ \partial_l R_j R_k u^j u^k - 2R_j R_l m_{jk} u^k \Big\}. \tag{1.7}$$

A weak solution $(u,\tilde{P})$ for bounded u and $\tilde{P} \in L^1_{loc}(0,T;BMO(\mathbb{R}^n))$, is a classical solution. If $n=2$, one can take $T=\infty$.

Note that $(u+Mx, \tilde{P}-\Pi)$ solves (1.1). Our solutions (U,P) are out of the setting in [49, 65], unfortunately, since P grows quadratically at spatial infinity. The existence of mild solutions is based on the fixed-point theorem. In 2-dimensional case we focus on the uniformly boundedness of vorticity $\omega := \operatorname{rot} u$ of scalar-field from the maximal principle. Using the notion of a weak solution we show that the mild solution is a classical solution when $\tilde{P}$ is given suitably.

We are now position to explain the known results. We first recall the works related to (1.1) with the initial data of (1.2). Consider the case when $n=3$ and $M=M_r$ is skew-symmetric, which demonstrates the rotational fluid mechanics. Employing the rotational coordinate $(x \mapsto e^{tM_r}x)$, we derive the Navier-Stokes equations with Coriolis force terms. P. Constantin and C. Fefferman [12] proved the global regularity on this problem, that is, the weak solution is unique for all time, provided the speed of rotation $|M_r|$ is fast enough and u_0 has decaying property and C^2 regularity. A. Babin, A. Mahalov and B. Nicolaenko [2, 3] also showed the global regularity with less-smooth periodic initial data. Under this setting in [68] the first author of this paper proved the existence and uniqueness of locally-in-time solutions when $u_0 \in \dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^3)$. Theorem 1.1 is an improvement of [68] in the sense that we can treat a general matrix M under (1.3). Recently, it was obtained by [29, 30] that the existence time of the smooth solution is estimated by below uniformly in the rotating speed $|M_r|$, when the initial disturbance u_0 is periodic and smooth.

The case where $M=M_j$ is diagonal (describing an axisymmetric straining flow) was investigated by Y. Giga and T. Kambe [32]. More precisely, the solution U is of the form

$$U(x,t) = (-\frac{1+\lambda}{2}x_1, -\frac{1-\lambda}{2}x_2, x_3) + (u^1(x',t), u^2(x',t), 0)$$

for $x = (x', x_3) \in \mathbb{R}^3$ and $\lambda \in [0, 1)$. Under this setting the vorticity $\operatorname{rot} u = (0, 0, \omega)$ can be regarded as a scalar function $\omega(x_1, x_2, t)$. For $\lambda = 0$ they obtained the stability of the vortex (so-called Burgers vortex), when $\omega_0 := \operatorname{rot} u_0$ is a finite Radon measure on $\mathbb{R}^2$, and the convergence to the Burgers vortex as $t \to \infty$ under the smallness of $\omega_0 \in L^1(\mathbb{R}^2)$. See also [10, 27]. Th. Gallay and C. E. Wayne [23] also investigated the convergence of the Burgers vortex without smallness of $\|\omega_0\|_1$. Their analysis is essentially based on the spectral theory of the linear operator A in Gaussian weighted L^2 space, when $M = I$ is the identity matrix. For the case when $\lambda > 0$ they also observed the behavior of the Burgers vortex in various situations, for example, the case when λ is sufficiently small in [22, 24, 25]. Recently, Y. Maekawa improved their results in [56, 57], he discussed the stability of Burgers vortex for $\lambda < 1$ and the convergence for $\lambda < 1/2$ in the modified Gaussian weighted L^2, when $\|\omega_0\|_1$ is sufficiently large.

Secondly, we shall refer to the known results related to (1.5). In 1990's W. Borchers started to study the rotating obstacle problem. He considered the initial-boundary value problem of the incompressible Navier-Stokes flows in time-depending exterior domains $\Omega_t := \{y = e^{tM_r}x; x \in \Omega\}$, where M_r is skew-symmetric and $\Omega \subset \mathbb{R}^3$ is an exterior domain. In order to fix the domain we employ the transformation (1.4) and the rotational coordinate, then it leads us the similar equations to (1.5) in Ω with non-slip boundary conditions. On this problem W. Borchers [6] established the existence theorem of weak solutions by Galerkin method. Using Yosida approximation, Z.-M. Chen and T. Miyakawa [11] also constructed the globally-in-time weak solutions.

In 1999 T. Hishida [40] constructed a unique locally-in-time solution of the rotating obstacle problem in L^2 context, provided u_0 is in the Sobolev space $H^{1/2}(\Omega)$, He established the contraction property of $\{e^{-tA}\}_{t\geq 0}$ in $L^2(\Omega)$, where $M = M_r$ skew-symmetric. After T. Hishida's work, there are many researching on this topic, e.g. [15, 16, 17, 18, 19]. M. Geissert, H. Heck and M. Hieber [26] proved the existence of a unique locally-in-time solution when $u_0 \in L^p_\sigma(\Omega)$ for $p \geq 3$, using the cut-off technique and the direct expression of the resolvent $(\lambda - \mathbb{P}A)^{-1}$ from iterated convolutions. Recently, a deep result is obtained by T. Hishida and Y. Shibata [41, 42]. They established precise $L^p - L^q$ estimates of the semigroup, and showed the globally-in-time solvability

and stability of the rotating obstacle problem in L^3-framework when the initial disturbance and the rotating speed are small enough, due to T. Kato's arguments.

Thirdly, we shall mention the results directly related to Theorem 1.1. It is known by [39] that (1.5) admits a unique locally-in-time smooth solution, when the initial disturbance $u_0 \in L^p_\sigma(\mathbb{R}^n)$ for $p \in [n, \infty)$. Our results can be regarded as an improvement of [39] in the sense that the spatial decay condition for the initial disturbance is relaxed. These solutions obtained by [39] are analytic in the spatial variables provided M is skew-symmetric as well as the solutions of the heat equation or, as the solutions to (1.1) proved by [35]. Spatial analyticity implies that the propagation speed of solutions is infinite. In fact, even if the support of the initial data u_0 is compact, then the support of the solution $u(t)$ coincides $\mathbb{R}^n$ for any small $t > 0$. It was also shown by [38] that when $u_0 \in L^p_\sigma(\mathbb{R}^n)$, there exists a unique locally-in-time solution to the problem (1.5) replaced to Mx by $f(x)$, where f is a globally Lipschitz continuous function satisfying some conditions. It is not clear that the solution obtained by [38] fulfills (1.5) in the classical sense. When we consider $u_0 \in \dot{\mathcal{B}}^0_{\infty,1,\sigma}$ and general f instead of Mx, it is also not clear that one can get the locally-in-time existence of solutions, since we need the explicit form of the Ornstein-Uhlenbeck semigroup (3.4) in our proof.

This paper is organized as follows. In Section 2 we define the function spaces, and verify several properties of $\dot{\mathcal{B}}^0_{\infty,1}$. We recall the theory of Ornstein-Uhlenbeck semigroup in Section 3. In Section 4 we divide the assertions of Theorem 1.1 into four propositions, and give the proofs of propositions.

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2. Function Spaces.

In this section we shall give the definitions of function spaces. We also verify the properties of homogeneous Besov space, in particular, $\dot{\mathcal{B}}^0_{\infty,1}$.

We denote $L^p(\mathbb{R}^n)$ the standard Lebesgue space on $\mathbb{R}^n$ for $p \in [1,\infty]$, and its norm is denoted by $\|\cdot\|_p$. It is a standard way that $L^p_\sigma(\mathbb{R}^n)$ stands for the solenoidal subspace of $L^p(\mathbb{R}^n)^n$ even for $p = 1$ and $p = \infty$. In order to simplify our notation, we do not distinguish in the function spaces of between scalar and vector-valued as well as functions. We sometimes suppress the notation of the domain $(\mathbb{R}^n)$, if no confusion occurs likely.

For $p \in [1,\infty]$ and $s \in \mathbb{R}$ we define $H^s_p := (I-\Delta)^{-s/2}L^p$ the (inhomogeneous) Sobolev spaces; its homogeneous version stands for $\dot{H}^s_p := (-\Delta)^{-s/2}L^p$. Here $(I-\Delta)^{-s/2}\varphi := \mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\hat{\varphi}$ and $(-\Delta)^{-s/2}\varphi := \mathcal{F}^{-1}|\xi|^s\hat{\varphi}$, where

$$\mathcal{F}\varphi(\xi) := \hat{\varphi}(\xi) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\varphi(x)e^{-ix\cdot\xi}dx$$

denotes the Fourier transform of φ, and $\mathcal{F}^{-1}$ is its inverse; $i := \sqrt{-1}$. We sometimes suppress $H^s := H^s_2$.

Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity, i.e., $\hat{\phi}_0 \in C^\infty_0(\mathbb{R}^n)$, $\operatorname{supp}\hat{\phi}_0 \subset \{\xi \in \mathbb{R}^n; 1/2 \le |\xi| \le 2\}$, $\hat{\phi}_j(\xi) := \hat{\phi}_0(2^{-j}\xi)$, and $\sum_{j=-\infty}^{\infty}\hat{\phi}_j(\xi) = 1$ except $\xi = 0$. Let $\mathcal{S}'(\mathbb{R}^n)$ be the space of all tempered distributions, i.e., the topological dual of $\mathcal{S}(\mathbb{R}^n)$ which is the space of rapidly decreasing functions in the sense of L. Schwartz.

2.1. **Definition** (homogeneous Besov spaces)**.** *Let $s \in \mathbb{R}$ and $p,q \in [1,\infty]$. A homogeneous Besov space is defined by*

$$\dot{B}^s_{p,q} := \{v \in \mathcal{Z}'; \|v\|_{\dot{B}^s_{p,q}} < \infty\},$$

$$\|v\|_{\dot{B}^s_{p,q}} := \begin{cases} \Big[\sum_{j\in\mathbb{Z}} 2^{jsq}\|\phi_j * v\|_p^q\Big]^{1/q} & \text{if} \quad q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\phi_j * v\|_p & \text{if} \quad q = \infty. \end{cases}$$

Here $\mathcal{Z}'$ stands for the topological dual space of

$$\mathcal{Z} := \{\varphi \in \mathcal{S}; \partial^\alpha\hat{\varphi}(0) = 0,\ \forall\alpha \in \mathbb{N}^n_0\}.$$

Here we denote $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, where $\mathbb{N}$ is the set of positive integers.

Following J. Johnsen [44], we call (s,p,q) differentiability-, integral- and sum-exponent, respectively. It is well-known that the homogeneous

Besov space can be regarded as a subspace of $\mathcal{S}'$ if one of the following two conditions of exponents holds

$$\text{(i)}\ \ s < n/p, \qquad \text{(ii)}\ \ s = n/p \text{ and } q = 1; \tag{2.1}$$

see e.g. [7, 52]. We only treat the functions which belong to these homogeneous Besov spaces with exponents satisfying one of (2.1).

Note that polynomials are identically zero in $\|\cdot\|_{\dot{B}^s_{p,q}}$-norm. Particularly, $\|v\|_{\dot{B}^0_{\infty,1}} = 0$ with constant function $v \equiv C$ for all constant C. To avoid the ambiguity of polynomials we now introduce a new function space of homogeneous type:

$$\dot{\mathcal{B}}^s_{p,q} := \{v \in \dot{B}^s_{p,q}; \sum_{j\in\mathbb{Z}} \phi_j * v = v \text{ in } \mathcal{S}'\}$$

with the exponents satisfying one of (2.1). Evidently, $\dot{\mathcal{B}}^s_{p,q}$ is a Banach space, since that is a closed subspace of $\dot{B}^s_{p,q}$. We treat $\dot{\mathcal{B}}^0_{\infty,1}$ mainly. The virtue of this definition is to get the natural inequality

$$\|v\|_\infty \le \sum_{j\in\mathbb{Z}} \|\phi_j * v\|_\infty = \|v\|_{\dot{B}^0_{\infty,1}} \tag{2.2}$$

for all $v \in \dot{\mathcal{B}}^0_{\infty,1}$. Besides, the inequality (2.2) does not hold for $v \in \dot{B}^0_{\infty,1}$, in general.

We now recall the properties of $\dot{\mathcal{B}}^0_{\infty,1}$. We can see easily that

$$\dot{\mathcal{B}}^0_{\infty,1} \subset BUC \subset L^\infty \subset BMO. \tag{2.3}$$

Here BUC is the space of bounded and uniformly continuous functions (see e.g. [1]), and BMO is the space of bounded mean oscillation functions (see e.g. [71]). All of these embeddings is continuous, clearly. Moreover, three inclusions (2.3) are strict. Indeed, the non-zero constant function does not belong to $\dot{\mathcal{B}}^0_{\infty,1}$, however, belongs to BUC; the Heavyside function is bounded , however, not uniformly continuous. For the last inclusion consider the Logarithmic function $\log x$. Nevertheless, $\dot{\mathcal{B}}^0_{\infty,1}$ contains non-decaying functions. By interpolation it follows that $v \in \dot{\mathcal{B}}^0_{\infty,1}$, if the derivative and the primitive of v are bounded. A typical example is $v(x) = \sin x$. Notice that $\dot{\mathcal{B}}^0_{\infty,1}$ contains not only periodic functions but also several nondecaying functions at space infinity, for example, the following almost periodic function (in the sense of H. Bohr [5]):

$$v(x) := \sum_{j=1}^\infty \alpha_j e^{i\lambda_j\cdot x} \quad \text{for} \quad \{\alpha_j\}_{j\in\mathbb{N}} \in l^1 \ \text{ and } \ \lambda_j \in \mathbb{R}^n\backslash\{0\}.$$

Such a representation of v is often said to be a generalized trigonometric series. This comes from the fact that $\|v_\lambda\|_{\dot{B}^0_{\infty,1}}$ is independent of $\lambda \in \mathbb{R}^n\backslash\{0\}$, where $v_\lambda(x) := \sin(\lambda\cdot x)$. In general, not every almost periodic function does have such a representation.

The advantage to use $\dot{\mathcal{B}}^0_{\infty,1}$ consists of that the Riesz transforms $R_k := \partial_k(-\Delta)^{-1/2}$ for $k = 1, \dots, n$ are bounded in $\dot{\mathcal{B}}^0_{\infty,1}$, besides, not bounded in L^∞. Observing at support of $\hat{\phi}_j$, for $v \in \dot{\mathcal{B}}^0_{\infty,1}$ we have

$$v = \sum_{j\in\mathbb{Z}} \phi_j * v = \sum_{j\in\mathbb{Z}} \tilde{\phi}_j * \phi_j * v,$$

where $\tilde{\phi}_j := \phi_{j-1} + \phi_j + \phi_{j+1}$. Therefore, Young's inequality yields

$$\|R_k v\|_{\dot{B}^0_{\infty,1}} = \sum_{j\in\mathbb{Z}} \|(R_k\tilde{\phi}_j) * \phi_j * v\|_\infty \le 3 \sup_{l\in\mathbb{Z}} \|R_k\phi_l\|_1 \|v\|_{\dot{B}^0_{\infty,1}}. \tag{2.4}$$

By dilation argument we easily verify that $\|R_k\phi_l\|_1$ is independent of $l \in \mathbb{Z}$ as well as $\|\phi_l\|_1$. Therefore, R_k is bounded in $\dot{\mathcal{B}}^0_{\infty,1}$.

Notice that $\dot{\mathcal{B}}^0_{\infty,1}$ is not a Banach algebra with respect to the pointwise multiplication. Indeed, let $n = 1$ and $v(x) := \sin x$, then $v^2 \notin \dot{\mathcal{B}}^0_{\infty,1}$. To estimate the bilinear terms we appeal to the next lemma of Leibnitz role type associated with the positivity of the differential exponent. Recall that $\|\nabla\cdot\|_{\dot{B}^0_{\infty,1}} \cong \|\cdot\|_{\dot{B}^1_{\infty,1}}$, see e.g. [4]. Here $\|\cdot\|_X \cong \|\cdot\|_Y$ stands for the equivalent of norms, i.e., there exists a constant $C > 1$ such that $C^{-1}\|v\|_X \le \|v\|_Y \le C\|v\|_X$ for all v.

2.2. **Lemma.** *There exists a positive constant C such that*

$$\|fg\|_{\dot{B}^1_{\infty,1}} \le C(\|f\|_{\dot{B}^1_{\infty,1}}\|g\|_{\dot{B}^0_{\infty,1}} + \|f\|_{\dot{B}^0_{\infty,1}}\|g\|_{\dot{B}^1_{\infty,1}}) \tag{2.5}$$

for all $f, g \in \dot{\mathcal{B}}^0_{\infty,1} \cap \dot{B}^1_{\infty,1}$.

Proof. Although the similar lemma can be found in [36], we shall give a complete proof for readers convenience. Apply the equivalent norm (see [13, 72]):

$$\|v\|_{\dot{B}^s_{p,q}} \cong \Big[\int_0^\infty t^{-1-sq} \sup_{|y|\le t} \|\tau_y v - v\|_p^q dt\Big]^{1/q}, \tag{2.6}$$

which is valid for $1 \le p, q \le \infty$ and $0 < s < 1$, and

$$\|v\|_{\dot{B}^s_{p,q}} \cong \Big[\int_0^\infty t^{-1-sq} \sup_{|y|\le t} \|\tau_y v + \tau_{-y} v - 2v\|_p^q dt\Big]^{1/q}, \tag{2.7}$$

which is valid for $1 \le p, q \le \infty$ and $0 < s < 2$, where $\tau_y f(x) := f(x-y)$.

We now calculate

$$|\tau_y(fg)+\tau_{-y}(fg)-2fg| \le |\tau_y f(\tau_y g+\tau_{-y}g-2g)|+ \\ +|(\tau_y f+\tau_{-y}f-2f)\tau_{-y}|+2|(\tau_y f-f)(\tau_{-y}g-g)|.$$

Hence, we plug-in $v=fg$ into (2.7) to get

$$\|fg\|_{\dot{B}^1_{\infty,1}} \le C\int_0^\infty t^{-2}\sup_{|y|\le t}\big\{\|\tau_y f\|_\infty\|\tau_y g+\tau_{-y}g-2g\|_\infty+ \\ +\|\tau_y f+\tau_{-y}f-2f\|_\infty\|\tau_{-y}g\|_\infty+2\|\tau_y f-f\|_\infty\|\tau_{-y}g-g\|_\infty\big\}dt \\ \le C\big\{\|f\|_\infty\|g\|_{\dot{B}^1_{\infty,1}}+\|f\|_{\dot{B}^1_{\infty,1}}\|g\|_\infty+\|f\|_{\dot{B}^{1/2}_{\infty,1}}\|g\|_{\dot{B}^{1/2}_{\infty,1}}\big\}$$

by Hölder's inequality and (2.6). Using (2.2) and the interpolation inequality (see e.g. [37])

$$\|v\|_{\dot{B}^{1/2}_{\infty,1}} \le C\|v\|^{1/2}_{\dot{B}^0_{\infty,1}}\|v\|^{1/2}_{\dot{B}^1_{\infty,1}},$$

we obtain (2.5) by the Cauchy-Schwarz inequality. □

This lemma implies that the first terms in right hand side in (1.7) are well-defined as functions of valued in $\dot{\mathcal{B}}^0_{\infty,1}$.

3. Ornstein-Uhlenbeck Semigroup.

In this section we recall the theory of Ornstein-Uhlenbeck semigroup $\{e^{-tA}\}_{t\ge0}$ for $A=-\Delta-(Mx,\nabla)+M$. We also study the relationship between semigroup and the homogeneous Besov norm, that is, making sense of $\|e^{-tA}u\|_{\dot{B}^0_{\infty,1}}$ for $u\in\dot{\mathcal{B}}^0_{\infty,1}$, and deriving benefit estimates. At the end of this section we see that $\{e^{-tA}\}_{t\ge0}$ is a C_0-semigroup in $\dot{\mathcal{B}}^0_{\infty,1,\sigma}$.

Let M be an $n\times n$ constant real-valued matrix, we do not impose (1.3) in this section. We define L as a realization of the operator

$$\mathcal{L}u := -\Delta u-(Mx,\nabla)u$$

in $L^p(\mathbb{R}^n)$ by

$$\begin{cases} Lu &:= \mathcal{L}u, \\ D(L) &:= \{u\in H^2_p(\mathbb{R}^n);(Mx,\nabla)u\in L^p(\mathbb{R}^n)\}. \end{cases}$$

The following results on the Ornstein-Uhlenbeck semigroup were proved by G. Metafune and his collaborators [55, 61, 62, 63, 64].

3.1. **Proposition.** *Let $1\le p\le\infty$. Then the operator $-L$ generates a semigroup $\{e^{-tL}\}_{t\ge0}$ on $L^p(\mathbb{R}^n)$. Moreover, the semigroup $\{e^{-tL}\}_{t\ge0}$ is given by*

$$e^{-tL}\varphi(x) := \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}}\int_{\mathbb{R}^n}\varphi(e^{tM}x-y)e^{-\frac{(Q_t^{-1}y,y)}{4}}dy \tag{3.1}$$

for $x \in \mathbb{R}^n$ and $t > 0$, where $Q_t := \int_0^t e^{sM} e^{sM^T} ds$ for $t > 0$. Here M^T is the transposed matrix of M.

3.2. **Remark.** (i) *When $M = 0$, e^{-tL} coincides the heat semigroup $e^{t\Delta}$. The semigroup $\{e^{-tL}\}_{t\geq 0}$ does not commute the spatial differentiation ∇, in general. Indeed, we see that*

$$\nabla e^{-tL}\varphi = e^{tM} e^{-tL} \nabla \varphi. \tag{3.2}$$

(ii) *For $p \in [1, \infty)$ it turns out that $\{e^{-tL}\}_{t\geq 0}$ is a C_0-semigroup in L^p. For $p \in (1, \infty)$ we have $\|e^{-tL}\|_{\mathcal{L}(L^p)} \leq e^{-\frac{t}{p}\operatorname{tr} M}$ for all $t \geq 0$. Here $\|\cdot\|_{\mathcal{L}(L^p, L^q)}$ is the operator norm from L^p to L^q, and $\|\cdot\|_{\mathcal{L}(L^p)} := \|\cdot\|_{\mathcal{L}(L^p, L^p)}$.*
(iii) *By* [18, 19, 40] *it is well-known that the semigroup $\{e^{-tL}\}_{t\geq 0}$ is not analytic.*
(iv) *Let $\tilde{L} := -\Delta - (f, \nabla)$, where $f \in C^{1+\alpha}$ is a globally Lipschitz continuous function. Then $-\tilde{L}$ also generates a semigroup $\{e^{-t\tilde{L}}\}_{t\geq 0}$ in L^p for $p \in [1, \infty]$. However, $\{e^{-t\tilde{L}}\}_{t\geq 0}$ has neither an expression* (3.1) *nor the commutate relationship* (3.2)*, in general.*

To solve (1.5) by an approach via semigroup theory we require an additional property on the semigroup. Obviously, solutions must satisfy the divergence free condition. So, we modify the semigroup which maps onto L^p_σ. To this end, we introduce the matrix operator $\mathcal{A}$ by

$$\mathcal{A}u := \mathcal{L}Iu + Mu$$

for $u = (u^1, \ldots, u^n) \in L^p(\mathbb{R}^n)^n$. Here I is the identity matrix. It is clear that

$$\nabla \cdot \{-(Mx, \nabla)u + Mu\} = 0, \qquad \text{provided} \quad \nabla \cdot u = 0. \tag{3.3}$$

We thus define A as a realization of $\mathcal{A}$ in $L^p_\sigma(\mathbb{R}^n)$ by

$$\begin{cases} Au &:= \mathcal{A}u, \\ D(A) &:= D(L)^n \cap L^p_\sigma. \end{cases}$$

By a standard perturbation theory we see that

3.3. **Lemma.** *The operator $-A$ generates a semigroup on $L^p_\sigma(\mathbb{R}^n)$ for $p \in [1, \infty]$, which is given by*

$$e^{-tA}\varphi(x) := \frac{e^{-tM}}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} \varphi(e^{tM}x - y) e^{-\frac{(Q_t^{-1}y, y)}{4}} dy \tag{3.4}$$

for $x \in \mathbb{R}^n$, $t > 0$ and $\varphi \in L^p_\sigma$.

The same assertions of Remark 3.2 hold for $\{e^{-tA}\}_{t\ge 0}$, for example, the semigroup e^{-tA} is not analytic due to the fact that e^{-tL} is not analytic. Thanks to (3.3), e^{-tA} commutes $\mathbb{P}$.

We now turn to the $L^p - L^q$-smoothing properties for the semigroup $\|e^{-tA}\|_{\mathcal{L}(L^p,L^q)}$ as well as the gradient estimates. Due to the non-analyticity of e^{-tA}, gradient estimates do not follow from the general theory of analytic semigroup.

3.4. **Lemma.** *Let $n \ge 2$ and $1 \le p \le q \le \infty$.*
(a) *Then there exist constants $C > 0$ and $\rho \ge 0$ such that*

$$\|e^{-tA}\varphi\|_q \le Ce^{\rho t}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\varphi\|_p, \tag{3.5}$$

$$\|\nabla e^{-tA}\varphi\|_p \le Ce^{\rho t}t^{-\frac{1}{2}}\|\varphi\|_p \tag{3.6}$$

for $t > 0$ and $\varphi \in L^p(\mathbb{R}^n)$.
(b) *Then there exist constants $C_j > 0$ and $\rho_j \ge 0$ such that*

$$\|\nabla^m e^{-tA}\varphi\|_q \le C_1 e^{(\rho_1+\rho_2 m)t}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|\nabla^m\varphi\|_p \tag{3.7}$$

for $t > 0$, $m \in \mathbb{N}$ and $\varphi \in H^m_p(\mathbb{R}^n)$, and

$$\|\nabla^m e^{-tA}\varphi\|_q \le C_2(C_3 m)^{m/2}e^{(\rho_3+\rho_4 m)t}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}}\|\varphi\|_p \tag{3.8}$$

for $t > 0$, $m \in \mathbb{N}$ and $\varphi \in L^p(\mathbb{R}^n)$.

The proofs are given by the direct calculation for the kernel of the expression (3.4). Precise proofs can be found in [39], so we skip them. When $M = M_r$ is skew-symmetric, we are allowed to take $\rho = \rho_j = 0$, since $\operatorname{tr} M_r = 0$ and e^{tM_r} is unitary, i.e., $\|e^{tM_r}\| = 1$ for all $t \ge 0$. Considering $\tilde{A} := \tilde{L} + \nabla f$ with a globally Lipschitz continuous function f, the same $L^p - L^q$-smoothing estimates for $\{e^{-t\tilde{A}}\}_{t\ge 0}$ can be obtained, except (3.7) and (3.8) for $m \ge 3$; see [38].

In this paper we assume that $u_0 \in \dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^n)$. Here we denote by

$$\dot{\mathcal{B}}^0_{\infty,1,\sigma} := \{\varphi \in \dot{\mathcal{B}}^0_{\infty,1}; \nabla\cdot u_0 = 0 \text{ in } \mathcal{S}'\}.$$

Since $\dot{\mathcal{B}}^0_{\infty,1,\sigma} \subset L^\infty_\sigma$, $e^{-tA}u_0$ can be regarded as a bounded function. We now verify that e^{-tA} maps onto $\dot{\mathcal{B}}^0_{\infty,1,\sigma}$ for each $t > 0$. Because ∇ does not commute e^{-tA}, it is not clear that $-A$ generates a semigroup in $\dot{H}^1_\infty$ or $\dot{H}^{-1}_\infty$. So, usual interpolation arguments are not applicable in our situation.

3.5. **Lemma.** *Let $n \geq 2$, and let M be a real-valued matrix. Then there exist constants $C > 0$ and $\rho \geq 0$ such that*

$$\|e^{-tA}\varphi\|_{\dot{B}^0_{\infty,1}} \leq C(1+t)e^{\rho t}\|\varphi\|_{\dot{B}^0_{\infty,1}} \tag{3.9}$$

for all $t > 0$ and $\varphi \in \dot{\mathcal{B}}^0_{\infty,1}$.

Proof. Let $j \in \mathbb{Z}$, $0 < t < t'$, and let $\varphi \in \dot{\mathcal{B}}^0_{\infty,1}$. Although $\phi_j *$ does not commute e^{-tA} as well as ∇, we have the following identity. There exists a constant $K \geq 0$ (depending only on M) such that

$$\phi_j * e^{-tA}\varphi = \phi_j * e^{-tA}(\tilde{\phi}_{j,[Kt']} * \varphi) \tag{3.10}$$

for $t \in [0, t']$, where $\tilde{\phi}_{j,a} := \sum_{k=j-1-a}^{j+1+a} \phi_k$. Here $[a]$ is the Gaussian integer. Once we obtain (3.10), by (3.5) it follows that

$$\begin{aligned}\|e^{-tA}\varphi\|_{\dot{B}^0_{\infty,1}} &\leq \sum_{j\in\mathbb{Z}} \|\phi_j\|_1 \|e^{-tA}\|_{\mathcal{L}(L^\infty)} \|\tilde{\phi}_{j,[Kt']} * \varphi\|_\infty \\ &\leq \|\phi_0\|_1 C e^{\rho t}(3 + 2[Kt'])\|\varphi\|_{\dot{B}^0_{\infty,1}}\end{aligned}$$

for $t \in [0, t']$. By $\rho \geq 0$ we take $\sup_{t\in[0,t']}$ to have (3.9).

By (3.4) we compute

$$\begin{aligned}&\phi_j * e^{-tA}\varphi(z) \\ &= \int_{\mathbb{R}^n} \phi_j(z-x)\left(\frac{e^{-tM}}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} \varphi(e^{tM}x - y)e^{-\frac{(Q_t^{-1}y,y)}{4}} dy\right) dx \\ &= \int_{\mathbb{R}^n} \frac{(-1)^n e^{-tM} e^{-\frac{(Q_t^{-1}y,y)}{4}}}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \left(\int_{\mathbb{R}^n} \phi_j(w)\varphi(e^{tM}z - e^{tM}w - y)dw\right) dy.\end{aligned}$$

Since $\varphi = \sum_{k=-\infty}^{\infty} \phi_k * \varphi$, it is enough to see that

$$\begin{aligned}\Phi_{jk}(y) := \int_{\mathbb{R}^n} \phi_j(w)\phi_k(y - e^{tM}w)dw = 0 \quad &\text{for all} \quad y \in \mathbb{R}^n \\ &\text{if either} \quad k \leq j - 2 - [Kt'] \quad \text{or} \quad k \geq j + 2 + [Kt'].\end{aligned}$$

Compute the Fourier transform of Φ_{jk}:

$$\begin{aligned}(2\pi)^{n/2}\hat{\Phi}_{jk}(\xi) &= \int_{\mathbb{R}^n} e^{-iy\cdot\xi}\left(\int_{\mathbb{R}^n} \phi_j(w)\phi_k(y - e^{tM}w)dw\right) dy \\ &= \int_{\mathbb{R}^n} \phi_j(w)\left(\int_{\mathbb{R}^n} e^{-iy\cdot\xi}\phi_k(y - e^{tM}w)dy\right) dw \\ &= \int_{\mathbb{R}^n} \phi_j(w)e^{-ie^{tM}w\cdot\xi}dw\hat{\phi}_k(\xi) \\ &= \hat{\phi}_j(e^{tM^T}\xi)\hat{\phi}_k(\xi).\end{aligned}$$

Notice that

$$\begin{aligned}\operatorname{supp}\hat{\phi}_j(e^{tM^T}\cdot) &\subset \{\xi; 2^{-j-1} \le |e^{tM^T}\xi| \le 2^{-j+1}\}\\ &= \{\xi; \|e^{tM^T}\|^{-1}2^{-j-1} \le |\xi| \le \|e^{-tM^T}\|2^{-j+1}\}.\end{aligned}$$

We thus calculate that $\Phi_{jk} \equiv 0$, if either $-k-1 \ge \log_2 \|e^{-tM^T}\| - j + 1$ or $-k+1 \le -\log_2 \|e^{tM^T}\| - j - 1$. Therefore, we suitably choose K by the eigenvalue of M^T to see (3.10). □

3.6. **Corollary.** *Since $\phi_j *$ commutes ∇, the identify* (3.10) *is still valid if ∇ appears in front of e^{-tA}. Therefore,* (3.6) *and* (3.8) *we thus obtain*

$$\|\nabla^m e^{-tA}\varphi\|_{\dot{B}^0_{\infty,1}} \le C_1(C_2 m)^{m/2} t^{-m/2}(1+t)e^{(\rho_1+\rho_2 m)t}\|\varphi\|_{\dot{B}^0_{\infty,1}} \tag{3.11}$$

for $t > 0$, $m \in \mathbb{N}$ and $\varphi \in \dot{\mathcal{B}}^0_{\infty,1}$ with some constants C_j and ρ_j.

We have seen that e^{-tA} acts on $\dot{\mathcal{B}}^0_{\infty,1,\sigma}$. In order to get the continuity with respect to time including the initial time of mild solutions with valued in $\dot{\mathcal{B}}^0_{\infty,1,\sigma}$, we should check that $\{e^{-tA}\}_{t\ge 0}$ is a C_0-semigroup (strong continuous) in $\dot{\mathcal{B}}^0_{\infty,1,\sigma}$.

3.7. **Lemma.** *For $\varphi \in \dot{\mathcal{B}}^0_{\infty,1}$ we have*

$$\|e^{-tA}\varphi - \varphi\|_{\dot{B}^0_{\infty,1}} \to 0 \quad \text{as } t \to 0.$$

Proof. Let us recall that the heat semigroup $\{e^{t\Delta}\}_{t\ge 0}$ is not a C_0-semigroup in L^∞. Indeed, it was proved by e.g. [31] that for $\varphi \in L^\infty$

$$\varphi \in BUC \ \text{ if and only if } \ \|e^{t\Delta}\varphi - \varphi\|_\infty \to 0 \text{ as } t \to 0.$$

Recall also that $\dot{\mathcal{B}}^0_{\infty,1} \subset BUC$; see [67].

For $\varphi \in \dot{\mathcal{B}}^0_{\infty,1}$ we see that

$$\begin{aligned}&e^{-tA}\varphi(x) - \varphi(x)\\ &= \frac{e^{-tM}}{(4\pi)^{n/2}(\det Q_t)^{1/2}}\int_{\mathbb{R}^n}\left\{\varphi(e^{tM}x - y) - e^{tM}\varphi(x)\right\}e^{-\frac{(Q_t^{-1}y,y)}{4}}dy\end{aligned}$$

from (3.4). We now divide $\{\cdots\}$ into three parts:

$$\begin{aligned}\varphi(e^{tM}x - y) - e^{tM}\varphi(x) &= \left\{\varphi(e^{tM}x - y) - \varphi(x - y)\right\} +\\ &+ \left\{\varphi(x - y) - e^{tM}\varphi(x - y)\right\} + \left\{e^{tM}\varphi(x - y) - e^{tM}\varphi(x)\right\}.\end{aligned}$$

Note that the first two terms of right-hand-side tend to zero as $t \to 0$, since $e^{tM} \to I$. Splitting the integral $\int_{\mathbb{R}^n} = \int_{|y-x|<\delta} + \int_{|y-x|\ge\delta}$ for some $\delta > 0$ to the same arguments of [31, Lemma 5], the last terms also tend to zero as $y \to x$ and $t \to 0$, since φ is uniformly continuous. This completes the assertion. □

4. Proof of Theorem 1.1.

We give a full proof of Theorem 1.1. We split it into four parts;

(I) locally-in-time existence of mild solutions,
(II) mild solutions satisfy (1.5) in the classical sense,
(III) globally-in-time solvability in 2-dimension,
(IV) uniqueness under a certain class of pressure.

4.1. **Existence of Mild Solutions.** We recall the notion of a mild solution. Applying the Helmholtz projection $\mathbb{P}$ onto solenoidal subspace to the first equations of (1.5), we derive the abstract equation

$$u_t + Au = -\mathbb{P}\nabla\cdot(u\otimes u) + 2\mathbb{P}Mu, \quad u(0) = u_0. \tag{4.1}$$

Here $Au := -\Delta u - (Mx,\nabla)u + Mu$. In the whole space the projection $\mathbb{P}$ can be expressed explicitly by $\mathbb{P} := (\delta_{jk} + R_jR_k)_{j,k}$, where δ_{jk} stands for Kronecker's delta, and R_j is the Riesz transform defined by $R_j := \partial_j(-\Delta)^{-1/2}$ for $j = 1,\dots,n$. Observe that A commutes $\mathbb{P}$, and that $\mathbb{P}u = u$ if $\nabla\cdot u = 0$. Note that $\mathbb{P}$ is bounded in $\dot{\mathcal{B}}^0_{\infty,1}$ as well as the Riesz transform by (2.4).

In Section 3 we showed that $-A$ generates a C_0-semigroup $\{e^{-tA}\}_{t\ge0}$ in $\dot{\mathcal{B}}^0_{\infty,1,\sigma}$. By Duhamel's principle the integral equation (1.6) is deduced. For $T > 0$ we call a mild solution as a function u which belongs to $C([0,T);\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^n))$, if u satisfies (1.6) for $t \in (0,T)$ with $u(0) = u_0$. We now state the locally-in-time existence and uniqueness results for mild solutions.

4.1. **Proposition.** *Let $n \ge 2$, M satisfy* (1.3)*, and let $u_0 \in \dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^n)$. Then there exist $T_1 > 0$ and a unique mild solution u on $(0,T_1)$ in the class*

$$[t \mapsto t^{k/2}\nabla^k u(t)] \in C([0,T_1);\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n)) \quad \text{for } k = 1,2.$$

Proof. We use the iteration arguments, that is, successive approximation. Let n, M and u_0 satisfy the hypothesis of the Proposition 4.1. For $j \ge 1$ and $0 < t < T \le 1$ we define the sequence of functions $\{u_j\}_{j\ge1}$ successively by $u_1(t) := e^{-tA}u_0$ and

$$u_{j+1}(t) := u_1(t) - \int_0^t e^{-(t-s)A}\mathbb{P}\{(u_j(s),\nabla)u_j(s) - 2Mu_j(s)\}ds.$$

Clearly, u_j is divergence-free for all $t > 0$ and all j.

Put $K_0 := \|u_0\|_{\dot{B}^0_{\infty,1}}$, and

$$K_j := \sup_{0<t\le T} \|u_j(t)\|_{\dot{B}^0_{\infty,1}} \quad \text{and} \quad K_j' := \sup_{0<t\le T} t^{\frac{1}{2}} \|\nabla u_j(t)\|_{\dot{B}^0_{\infty,1}}.$$

By (3.9), (3.11) and $t < T \le 1$ we easily see that

$$K_1 \le C_1 K_0 \quad \text{and} \quad K_1' \le C_1' K_0$$

with some positive constants C_1 and C_1'.

Next, it follows from the approximation that

$$\begin{aligned}
&\|u_{j+1}(t)\|_{\dot{B}^0_{\infty,1}} \\
&\le C_1 K_0 + \int_0^t \|e^{-(t-s)A}\mathbb{P}\{(u_j(s),\nabla)u_j(s) - 2Mu_j(s)\}\|_{\dot{B}^0_{\infty,1}} ds \\
&\le C_1 K_0 + C\int_0^t \|u_j \otimes u_j\|_{\dot{B}^1_{\infty,1}} ds + C\int_0^t \|u_j\|_{\dot{B}^0_{\infty,1}} ds \\
&\le C_1 K_0 + C_2\sqrt{t} K_j K_j' + C_3 t K_j
\end{aligned}$$

with some positive constants C_2 and C_3. We have used (2.4), (2.5) and the identity $(u,\nabla)u = \nabla\cdot(u\otimes u)$, which is valid for $\nabla\cdot u = 0$; $u\otimes u := (u^j u^k)_{j,k}$ is the tensor matrix. Taking $\sup_{0<t\le T}$ on both sides, we obtain

$$K_{j+1} \le C_1 K_0 + C_2\sqrt{T} K_j K_j' + C_3 T K_j.$$

Similarly, estimating ∇u_j, it follows that

$$K_{j+1}' \le C_1' K_0 + C_2'\sqrt{T} K_j K_j' + C_3' T K_j$$

for some constants C_2' and C_3'.

Hence, K_j and K_j' are bounded uniformly in $j \in \mathbb{N}$ if T is chosen small enough. Indeed, $\sup_j (K_j + K_j') \le 3(C_1 + C_1')K_0$ for $T \le T_0$ if

$$T_0 := \min\left(1, \frac{4}{9(C_2+C_2')^2(C_1+C_1')K_0}, \frac{1}{3(C_3+C_3')}\right).$$

Similarly, we derive estimates for the differences $u_{j+1} - u_j$. Let

$$L_j := \sup_{0<t\le T} \|u_{j+1}(t) - u_j(t)\|_{\dot{B}^0_{\infty,1}}, \quad L_j' := \sup_{0<t\le T} t^{\frac{1}{2}} \|u_{j+1}(t) - u_j(t)\|_{\dot{B}^1_{\infty,1}}$$

for $j \in \mathbb{N}$. Similarly as before, we have for all j

$$L_{j+1} + L_{j+1}' \le C_4\sqrt{T}(L_j + L_j') + C_5 T L_j$$

with some positive constants C_4 and C_5. We choose $T_1 \le T_0$ small enough so that $(L_{j+1}+L_{j+1}')/(L_j+L_j') \le 1/2$ for all $j \in \mathbb{N}$ and $T \le T_1$. This implies that L_j and L_j' tend to zero as $j \to \infty$. It follows that the $\{t^{k/2}\nabla^k u_j(t)\}_{j\in\mathbb{N}}$ are Cauchy sequences in $\dot{B}^0_{\infty,1}$ for $k = 0, 1$. We can

also show $\|u_j(t) - u_j(s)\|_{\dot{B}^0_{\infty,1}} \to 0$ as $s \to t$, this implies the continuity with respect to time including the initial time by Lemma 3.7.

We thus conclude that there are unique limit functions

$$[t \mapsto u(t)],\ [t \mapsto v(t)] \in C([0,T_0];\dot{\mathcal{B}}^0_{\infty,1})$$

of the sequences $\{t^{k/2}\nabla^k u_j(t)\}_{j\in\mathbb{N}}$ for $k = 0, 1$. Obviously, $v(t) = t^{1/2}\nabla u(t)$, and u is a mild solution. Uniqueness of mild solutions follows from standard Gronwall's inequality. This completes the proof of Proposition 4.1. □

From the construction of mild solutions, we may estimate the existence time of mild solutions by below; $T_1 \geq C/\|u_0\|_{\dot{B}^0_{\infty,1}}$ with some numerical constant C for large initial disturbance.

One can also involve the $t^{k/2}\|\nabla^k u(t)\|_{\dot{B}^0_{\infty,1}}$ into the iteration scheme for all $k \in \mathbb{N}$. We thus get the mild solutions in the class

$$[t \mapsto t^{k/2}\nabla^k u(t)] \in C([0,T_k];\dot{\mathcal{B}}^0_{\infty,1})$$

if T_k is chosen small enough. The mild solution is unique, and does not blow-up until T_1. These imply that the mild solution is smooth with respect to spatial variables, namely, $u(t) \in C^\infty(\mathbb{R}^n)$ for all $t \in (0,T_1)$. Also, following from the construction of mild solutions, $u \in C(0,T_1;C^\infty(\mathbb{R}^n))$.

Furthermore, we can prove that $u(t) \in C^\omega(\mathbb{R}^n)$ real analytic in spatial variables, provided M is skew-symmetric. To show it we appeal to (3.7) and (3.8) with $\rho = \rho_j = 0$. Especially, the fact that $\rho_2 = 0$ is essential. The strategy of the proof of spatial analyticity is established by [39], and we can adjust it in our situation.

4.2. **Classical Solutions.** It is a natural question whether the mild solutions obtained by Proposition 4.1 satisfies (1.5) in the classical sense. As Remark 3.2-(i), the semigroup $\{e^{-tA}\}_{t\geq 0}$ is not analytic. So, it is difficult to handle the time-derivative of u with valued in $\dot{\mathcal{B}}^0_{\infty,1}$. Actually, one can not expect that the mild solution is a strong solution. Here we call u a strong solution if u solves (1.5) in the class $C^1(0,T;\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^n))$. Nevertheless, our mild solution u satisfies (1.5) in the classical sense under suitable setting of gradient terms $\nabla\tilde{P}$.

Before stating the proposition, we introduce the notion of a weak solution. We call $(u,\tilde{P})$ a weak solution of the modified Navier-Stokes

equations (1.5) on $\mathbb{R}^n \times (0,T)$ with initial data u_0 if $(u, \tilde{P})$ satisfies $\nabla \cdot u = 0$ in $\mathcal{S}'$ for almost every t and

$$\int_0^T \Big\{ \langle u(t), \partial_t \Phi(t) \rangle + \langle u(t), \Delta \Phi(t) \rangle + \langle (u \otimes u)(t), \nabla \Phi(t) \rangle + \langle Mu(t), \Phi(t) \rangle + \langle \tilde{P}(t), \nabla \cdot \Phi(t) \rangle \Big\} dt = -\langle u_0, \Phi(0) \rangle$$

for $\Phi \in C^1(\mathbb{R}^n \times [0,T])^n$ with $\Phi(\cdot, t) \in \mathcal{S}(\mathbb{R}^n)^n$ for all $t \in [0,T]$ and $\Phi(\cdot, T) \equiv 0$. Here $\langle u, \Phi \rangle$ is the canonical pairing of $u \in \mathcal{S}'$ and $\Phi \in \mathcal{S}$, and $\langle u \otimes u, \nabla \Phi \rangle := \sum_{j,k=1}^n \langle u^j u^k, \partial_j \Phi^k \rangle$.

4.2. **Proposition.** *Let $n \geq 2$, $T > 0$, and let $M = (m_{jk})_{1 \leq j,k \leq n}$ satisfy* (1.3). *Assume that $u \in C(0,T; C^2(\mathbb{R}^n))$ is a mild solution. Then $(u, \tilde{P})$ satisfies* (1.5) *in the classical sense provided ∇P is given by* (1.7).

Proof. The idea of proof arises from the notion of a weak solution. Firstly, we verify that u satisfies the abstract equation (4.1) in the classical sense. Let $\Phi := h\eta$, where $\eta \in \mathcal{S}(\mathbb{R}^n)$, $t \in (0,T]$ and $h \in C^1([0,T])$ assuming $h(0) = h(T) = 0$. We denote by A^* the dual operator of A, that is,

$$\begin{cases} A^* & := & -\Delta + (Mx, \nabla) + \operatorname{tr} M + M^T, \\ D(A^*) & := & D(A). \end{cases}$$

By bounded perturbation theory we see that $-A^*$ generates a semigroup $\{e^{-tA^*}\}_{t \geq 0}$ in L^p for $p \in [1, \infty]$. Using the integral equation (1.6), it holds that

$$\begin{aligned}
&\int_0^T \langle u(t), \eta \rangle h'(t) dt \\
&= \int_0^T \langle e^{-tA} u_0 - \int_0^t e^{-(t-s)A} \mathbb{P} \big\{ \nabla \cdot (u \otimes u)(s) - 2Mu(s) \big\} ds, \eta \rangle h'(t) dt \\
&= \int_0^T \langle u_0, e^{-tA^*} \eta \rangle h'(t) dt \\
&\quad - \int_0^T \left(\int_0^t \langle \mathbb{P} \nabla \cdot (u \otimes u)(s) - 2\mathbb{P} Mu(s), e^{-(t-s)A^*} \eta \rangle ds \right) h'(t) dt \\
&= - \int_0^T \langle u_0, \partial_t e^{-tA^*} \eta \rangle h(t) dt \\
&\quad + \int_0^T \partial_t \left(\int_0^t \langle \mathbb{P} \nabla \cdot (u \otimes u)(s) - 2\mathbb{P} Mu(s), e^{-(t-s)A^*} \eta \rangle ds \right) h(t) dt \\
&= \int_0^T \langle u_0, A^* e^{-tA^*} \eta \rangle h(t) dt
\end{aligned}$$

$$+\int_0^T \langle \mathbb{P}\nabla\cdot(u\otimes u)(t) - 2\mathbb{P}Mu(t),\eta\rangle h(t)dt$$
$$-\int_0^T \left(\int_0^t \langle \mathbb{P}\nabla\cdot(u\otimes u)(s) - 2\mathbb{P}Mu(s), A^*e^{-(t-s)A^*}\eta\rangle ds\right) h(t)dt$$
$$=\int_0^T \langle Au(t)+\mathbb{P}\nabla\cdot(u\otimes u)(t) - 2\mathbb{P}Mu(t),\eta\rangle h(t)dt.$$

We have used the fact that $\eta\in\mathcal{S}\subset D(A^*)$, and then $\partial_t e^{-tA^*}\eta = -A^*e^{-tA^*}\eta$. One can make sense $Au(x,t)$ pointwisely for each $x\in\mathbb{R}^n$ and $t\in(0,T)$, and $Au(x,\cdot)$ is continuous in time, since $u\in C(0,T;C^2(\mathbb{R}^n))$. From the above and the arbitrariness of h

$$\langle D_t u(t),\eta\rangle = -\langle Au(t)+\mathbb{P}\nabla\cdot(u\otimes u)(t) - 2\mathbb{P}Mu(t),\eta\rangle$$

is continuous in t. Here D_t is the time-differentiation in the distribution sense. This implies that $\langle u(\cdot),\eta\rangle\in C^1(0,T)$. Since the test function η in spatial variables is arbitrary, we conclude that u fulfills the abstract equation (4.1) in the classical sense for each t and x.

Secondary, it turns to (1.5). Once we assume that $\nabla\tilde{P}$ is given by (1.7), for $t\in(0,T)$

$$\langle u_t(t)+Au(t)+(u(t),\nabla)u(t)-2Mu(t)+\nabla\tilde{P}(t),\eta\rangle = 0.$$

This holds true for all $\eta\in\mathcal{S}(\mathbb{R}^n)$, therefore, $(u,\tilde{P})$ satisfies (1.5) in the classical sense for each t and x. This completes the proof of Proposition 4.2. □

We employ the inverse of the substitution (1.4); $U(x,t)=u(x,t)-Mx$ and $P(x,t):=\tilde{P}(x,t)-\frac{1}{2}(M^2x,x)$. By Proposition 4.2 we see that (U,P) solves (1.1) in the classical sense.

4.3. **2-D Globally-in-time Existence.** This subsection is devoted to discuss the existence of the globally-in-time unique smooth solutions when $n=2$. If it is supposed that the initial disturbance $u_0\in L^2_\sigma(\mathbb{R}^2)$, then one can also construct the mild solution locally-in-time; see [39]. We thus obtain the globally-in-time unique smooth solution by the following a priori estimate of energy type:

$$\frac{1}{2}\|u(t)\|_2^2 \le C\exp\{|M|t\},\quad t\ge 0 \tag{4.2}$$

with some positive constants C and $|M|:=\max_{j,k}|m_{jk}|$. The estimate (4.2) is yielded by the usual technique that we multiply u into the first equations of (1.5), and integrate in $x\in\mathbb{R}^2$. Once we have (4.2),

it turns out that the mild solution can be extended globally-in-time. This follows from the fact that the mild solution is unique as long as it exists, and the existence time is estimated by below. This argument is still valid for the case when $u_0 \in L^p_\sigma(\mathbb{R}^2)$ for $p \in (2, \infty)$. However, the approach to derive (4.2) is not applicable in our situation, because u is not integrable.

We now state the proposition for an a priori estimate, which implies that our mild solutions can be extended globally-in-time.

4.3. **Proposition.** *Let $n = 2$, M satisfy (1.3), and let $u_0 \in \dot{B}^0_{\infty,1,\sigma}(\mathbb{R}^2)$ with $\omega_0 := \operatorname{rot} u_0 \in L^\infty(\mathbb{R}^2)$. Assume that u is a mild solution. Then there exist positive constants K_1, K_2 and K_3 such that for all $t > 0$*

$$\|u(t)\|_{\dot{B}^0_{\infty,1}} \le K_1(1+t)\|u_0\|_{\dot{B}^0_{\infty,1}} \exp\Big[K_2 t\Big(1+t+\|\omega_0\|_\infty+\exp\{K_3 t\}\Big)\Big].$$

Proof. The proof follows from the arguments by [70], that is, we use the Littlewood-Paley decomposition and uniform bound for vorticity. Firstly, we are focusing on the uniformly boundedness of vorticity $\omega := \operatorname{rot} u := \partial_1 u^2 - \partial_2 u^1$. Taking rot into the first equations of (1.5), we derive the vorticity equation:

$$\omega_t - \Delta\omega + (u, \nabla)\omega - (Mx, \nabla)\omega = 0, \quad \omega(0) = \omega_0. \tag{4.3}$$

We have used that $\operatorname{rot}(Mx, \nabla)u = (Mx, \nabla)\omega - \operatorname{rot} Mu$, since $\operatorname{tr} M = 0$ and $\nabla \cdot u = 0$. It is well-known by [27, 43] that one can apply the maximal principle for (4.3), even though the drift terms appear. Assume that u and ω are bounded for $x \in \mathbb{R}^2$ and $t \in (0, T)$, and that $\omega_0 \in L^\infty(\mathbb{R}^2)$, then we obtain $\|\omega(t)\|_\infty \le \|\omega_0\|_\infty$ for all $t \in [0, T)$.

For $N \in \mathbb{Z}$

$$\|u(t)\|_{\dot{B}^0_{\infty,1}} = \sum_{j=-\infty}^{N} \|\phi_j * u(t)\|_\infty + \sum_{j=N+1}^{\infty} \|\phi_j * u(t)\|_\infty =: I + II.$$

We shall establish the estimates for I and II. Since u is a mild solution, we estimate low frequencies I by

$$\begin{aligned} I \le \sum_{j=-\infty}^{N} \Big(&\|\phi_j * e^{-tA}u_0\|_\infty + \int_0^t \|\phi_j * e^{-(t-s)A}\mathbb{P}\nabla(u\otimes u)(s)\|_\infty ds \\ &+ 2\int_0^t \|\phi_j * e^{-(t-s)A}\mathbb{P}Mu(s)\|_\infty ds\Big) \\ =: I_a + I_b + I_c. \end{aligned}$$

We see by (3.10) that

$$I_a = \sum_{j=-\infty}^{N} \|\phi_j * e^{-tA}\tilde{\phi}_{j,[Kt]} * u_0\|_\infty \le C(1+t)e^{\rho t}\|u_0\|_{\dot{B}^0_{\infty,1}}.$$

Similarly, by (3.10) we show that

$$\begin{aligned} I_c &= 2\sum_{j=-\infty}^{N}\int_0^t \|\mathbb{P}\phi_j * e^{-(t-s)A}\tilde{\phi}_{j,[K(t-s)]} * Mu(s)\|_\infty ds \\ &\le C(1+t)e^{\rho t}\int_0^t \|u(s)\|_{\dot{B}^0_{\infty,1}} ds. \end{aligned}$$

Using (3.2) and (3.5), we derive

$$\begin{aligned} I_b &= \sum_{j=-\infty}^{N}\int_0^t \|\mathbb{P}\phi_j * \nabla\cdot e^{-(t-s)M}e^{-(t-s)A}(u\otimes u)(s)\|_\infty ds \\ &\le \sum_{j=-\infty}^{N}\int_0^t C2^j e^{\rho(t-s)}\|u(s)\|_\infty^2 ds \\ &\le Ce^{\rho t}2^N\Big(\int_0^t \|u(s)\|_{\dot{B}^0_{\infty,1}} ds\Big)^2. \end{aligned}$$

To estimate high frequencies II we use uniform bound for ω:

$$II \le \sum_{j=N+1}^{\infty} \|(-\Delta)^{-1}\mathrm{rot}\,\phi_j\|_1 \|\mathrm{rot}\,u(t)\|_\infty \le C2^{-N}\|\omega_0\|_\infty.$$

We now choose $N\in\mathbb{Z}$ so that $2^{-N} \le \int_0^t \|u(s)\|_{\dot{B}^0_{\infty,1}} ds \le 2^{-N+1}$. Hence,

$$\|u(t)\|_{\dot{B}^0_{\infty,1}} \le C(1+t)e^{\rho t}\|u_0\|_{\dot{B}^0_{\infty,1}} + C(1+t+\|\omega_0\|_\infty+e^{\rho t})\int_0^t \|u(s)\|_{\dot{B}^0_{\infty,1}} ds.$$

Finally, we apply the Gronwall inequality to obtain the desired a priori estimate. □

To get the globally-in-time solvability it is not necessary to assume that $\omega_0 \in L^\infty(\mathbb{R}^2)$, in general. Indeed, by smoothing effect the mild solution $u(t_0) \in C^1(\mathbb{R}^2)$ for any small $t_0 \in (0, T_1)$. Taking t_0 as an initial time, we may derive the a priori estimate.

4.4. **Uniqueness of Weak Solutions.** We argue the uniqueness of weak solutions to (1.5) for a bounded disturbance u and a modified pressure $\tilde{P}$ in a certain class. J. Kato [45] showed that the weak solution (U, P) to (1.1) is uniquely determined by U_0, when

$$U \in L^\infty(\mathbb{R}^n\times[0,T]) \quad\text{and}\quad P \in L^1_{loc}(0,T;BMO(\mathbb{R}^n)).$$

One can obtain the similar results for $(u, \tilde{P})$ in our situation.

4.4. **Proposition.** *Let $n \geq 2$, $T > 0$, and $u_0 \in \dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^n)$. Assume that $u \in L^\infty(0,T;\dot{\mathcal{B}}^0_{\infty,1,\sigma})$, $\tilde{P} \in L^1_{loc}(0,T;BMO)$, and that $(u,\tilde{P})$ is a weak solution to (1.5). Then u is a mild solution, and $\tilde{P}$ is given by (1.7). Furthermore, $(u,\tilde{P})$ is uniquely determined by u_0.*

Proof. We give the proof by J. Kato's strategy [45]. For $N \in \mathbb{N}$ and for $l, j = 1, \dots, n$ we put the Riesz transform type operator R^N_{lj} by

$$R^N_{lj}\varphi := \partial_l \partial_j k_N * \varphi,$$

where $k_N(x) := \sum_{j=-N}^N \hat{\phi}_j(x) k(x)$, and k is the fundamental solution of $-\Delta$, i.e.,

$$k(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & \text{if } \ n = 2, \\ \frac{1}{(n-2)|S^{n-1}|}|x|^{2-n} & \text{if } \ n \geq 3. \end{cases}$$

It is known by [45, Theorem 2] that

$$\lim_{N\to\infty} \langle R^N_{lj}\varphi, \eta\rangle = \langle R_l R_j \varphi, \eta\rangle, \quad \varphi \in L^\infty, \ \ \eta \in \mathcal{S}, \ \ \hat{\eta}(0) = 0.$$
$$\lim_{N\to\infty} R^N_{lj}\partial_k \varphi = \partial_l R_j R_k \varphi \quad \text{in } \ \mathcal{S}', \quad \varphi \in L^\infty.$$
$$\sum\nolimits_j R^N_{lj}\varphi^j = 0 \quad \text{in } \ \mathcal{S}', \quad \varphi \in \mathcal{S}', \ \ \nabla\cdot\varphi = 0, \ \ N \in \mathbb{N}.$$
$$\lim_{N\to\infty} \sum\nolimits_j R^N_{lj}\partial_j \psi = -\partial_l \psi \quad \text{in } \ \mathcal{S}', \quad \psi \in BMO.$$

We now show that $\nabla\tilde{P}$ is given by (1.7) automatically under the hypothesis of Proposition 4.4. Fix $N \in \mathbb{N}$ and $l \in \{1,\dots,n\}$. Put the test function $\Phi = R^N_{lj}\tilde{\Phi}$, where $\tilde{\Phi} \in C^1(\mathbb{R}^n \times [0,T])^n$ satisfying $\tilde{\Phi}(\cdot,s) \in \mathcal{S}(\mathbb{R}^n)^n$ for all $s \in [0,T]$ and $\tilde{\Phi}(\cdot,T) \equiv 0$. Then

$$\sum_{j,k=1}^n \int_0^T \Big\{ \langle R^N_{lj}u^j, \partial_t\tilde{\Phi}\rangle + \langle R^N_{lj}u^j, \Delta\tilde{\Phi}\rangle + \langle \partial_k R^N_{lj}u^j u^k, \tilde{\Phi}\rangle$$
$$+ \langle R^N_{lj} m_{jk} u^k, \tilde{\Phi}\rangle + \langle \partial_j R^N_{lj}\tilde{P}, \tilde{\Phi}\rangle \Big\} dt = \langle R^N_{lj}u^j_0, \tilde{\Phi}(0)\rangle.$$

Notice that the first and second terms of left-hand-side are zero, since $\nabla\cdot u = 0$. The right-hand-side also vanishes. We thus take a limit as $N \to \infty$ to see

$$\int_0^T \langle \partial_l \tilde{P}, \tilde{\Phi}\rangle dt = \int_0^T \langle \sum_{j,k=1}^n \partial_l R_j R_k u^j u^k - 2R_l R_j m_{jk} u^k, \tilde{\Phi}\rangle dt.$$

Therefore, the gradient terms are given by (1.7).

Next, we verify that u is a mild solution. Let $t' \in (0,T)$, and fix $\delta > 0$ small so that $t' + \delta < T$. Put the test function Φ as

$$\Phi(x,t) := \begin{cases} h(t)(e^{-(t'-t+\delta)A^*}\eta)(x) & \text{if } \ 0 < t < t' + \delta, \\ 0 \quad \text{if } \ t' + \delta \le t \le T. \end{cases}$$

Here $\eta \in \mathcal{S}(\mathbb{R}^n)$ and $h \in C^1(\mathbb{R})$ assuming $h(0) = 1$ and $\operatorname{supp} h \subset (-\infty, t' + \delta)$. Let us see

$$\partial_t(he^{-(t'-t+\delta)A^*}\eta) = h'e^{-(t'-t+\delta)A^*}\eta + he^{-(t'-t+\delta)A^*}A^*\eta,$$

which is valid since $\eta \in D(A^*)$. So, by (1.7) we have

$$\int_0^T \Big\{ \langle u, h'e^{-(t'-t+\delta)A^*}\eta\rangle - \langle \mathbb{P}\nabla\cdot(u\otimes u) - 2\mathbb{P}Mu, he^{-(t'-t+\delta)A^*}\eta\rangle \Big\} dt$$
$$= -\langle u_0, e^{-(t'+\delta)A^*}\eta\rangle.$$

We now set $h(t) := \int_t^\infty \tilde{h}_\varepsilon(\tau - t')d\tau$, where $\tilde{h}_\varepsilon(t) := \varepsilon^{-1}\tilde{h}(t/\varepsilon)$ for $\varepsilon \in (0,\delta)$ and $\tilde{h} \in C(\mathbb{R})$, $\tilde{h} \ge 0$, $\operatorname{supp}\tilde{h} \subset (-1,1)$. Note that $h'(t) = -\tilde{h}_\varepsilon(t-t')$ and $lim_{\varepsilon\to 0}\int_t^\infty \tilde{h}_\varepsilon(\tau - t')d\tau = \chi_{(-\infty,t']}(t)$. Tending $\varepsilon \to 0$,

$$\int_0^T \Big\{ \langle u(t), -\tilde{h}_\varepsilon(t-t')e^{-(t'-t+\delta)A^*}\eta\rangle dt \to -\langle u(t'), e^{-\delta A^*}\eta\rangle,$$
$$\int_0^T \langle \mathbb{P}\nabla\cdot(u\otimes u)(t) - 2\mathbb{P}Mu(t), he^{-(t'-t+\delta)A^*}\eta\rangle dt$$
$$\to \int_0^{t'} \langle \mathbb{P}\nabla\cdot(u\otimes u)(t) - 2\mathbb{P}Mu(t), e^{-(t'-t+\delta)A^*}\eta\rangle dt.$$

Therefore, sending $\delta \to 0$, we obtain

$$\langle u(t') - e^{t'A}u_0 + \int_0^{t'} e^{-(t'-t)A}\mathbb{P}\{\nabla\cdot(u\otimes u)(t) - 2\mathbb{P}Mu(t)\}dt, \eta\rangle = 0.$$

It follows from arbitrariness of η that u satisfies the integral equation (1.6). Since a mild solution is unique as long as it exists, the above u is uniquely determined by u_0. □

Gathering Proposition 4.1-4.4, we complete the proof of Theorem 1.1.

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