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RESIDUES OF HOLOMORPHIC VECTOR FIELDS ON SINGULAR VARIETIES

DANIEL LEHMANN AND TATSUO SUWA

In this article we discuss various residue problems of holomorphic vector fields on (possibly singular) complex analytic varieties from a unified view point and summarize mainly the results in [LS] and [LSS].

The basic idea is to generalize the residues of R. Bott for holomorphic vector bundles with an action of a holomorphic vector field on a complex manifold ([B1]) in two ways: on one hand to the case of virtual bundles (with an action of a holomorphic vector field, which is a quite different situation than the case of the virtual normal bundle for a singular foliation used by Baum-Bott in [BB1]), and on the other hand with basis a singular variety. The method is a generalization of the one initiated in [Lh2] for a single vector bundle with a non-singular basis.

Let W be a complex manifold of dimension n and V an analytic subvariety of pure dimension p. Suppose we have a holomorphic vector field X on W leaving V invariant and define the singular set Σ to be the set of singular points of X on V and singular points of V; $\Sigma = (\mathrm{Sing}(X) \cap V) \cup \mathrm{Sing}(V)$. In this situation, the above generalization can be done with the assumption that each of the vector bundles involved has a C^∞ extension on a neighborhood of V in W. Let (E, E') be a pair of X-bundles on V (cf. section 2). With the vanishing theorem away from Σ, we can define the residue for each compact component of Σ and a pair of Chern polynomials (φ, φ') with the sum of degrees equal to p and, when V itself is compact, the sum of the residues over the components of Σ is equal to the characteristic number of (E, E') for (φ, φ') (Theorem (2.2)). This naturally includes the single vector bundle case. When the singularity is isolated, there is a "good coordinate system" to compute the residues (Lemma (2.9)) and they are expressed as "Grothendieck residues relative to a subvariety" (Theorem (2.11)).

As applications, we first discuss the residues for the restriction $TW|_V$ to V of the holomorphic tangent bundle of W. If V is now a local complete intersection in W, the usual normal bundle to the regular part of V in W has a natural extension N_V on the whole V and we discuss also the residues for N_V and for the virtual tangent bundle $\check{T}V = TW|_V - N_V$ of V. Each of these bundles is equipped with a natural X-action. The first bundle has an obvious extension, however for the second and the third we need to further assume that N_V admits a C^∞ extension on a neighborhood of V. Besides the case V is non-singular, this condition is

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satisfied, for example, when V is a hypersurface, a complete intersection or a (global) complete intersection in the projective space. The residues for N_V generalize the index introduced by C. Camacho and P. Sad in [CS] for a holomorphic vector field on a complex surface leaving a (non-singular) curve invariant and are interpreted as the residues of the vector field X relative to V. The global invariants in this case are the characteristic numbers of the bundle N_V (Theorem (4.2)). The theory can also be developed in the case of higher dimensional singular foliations, generalizing the previous works of several authors (see Theorem (4.6) and the subsequent Particular cases). The residues for the virtual tangent bundle $\check{T}V$ (see Theorem (5.3) and Proposition (5.6)) include as a special case the topological index of X. Gómez-Mont, J. Seade and A. Verjovsky ([GSV], see also [BG] [Gz] and [SS]) defined for a holomorphic vector field on a singular variety and they reduce to the residues of P. Baum and R. Bott ([BB1] and [BB2]) when V is non-singular. We give an explicit formula for the residues in the "non-degenerate case" (Proposition (6.2)) and compute various examples in section 7.

We would like to thank X. Gómez-Mont for useful discussions.

1. Local complete intersections

Let W be a complex manifold of complex dimension $n = p + q$ and V an analytic subvariety of pure complex dimension p. A "system of reduced local defining functions" for V is a holomorphic map $f : U \longrightarrow \mathbb{C}^q$, defined on an open set U of W, such that $V \cap U = f^{-1}(0)$ and that the q components of f generate the ideal $I(V \cap U)$ of holomorphic functions which vanish on $V \cap U$. The subvariety V is said to be a "local complete intersection" (briefly, LCI), if there exists a family $\{f_\alpha : U_\alpha \longrightarrow \mathbb{C}^q\}_\alpha$ of systems of reduced local defining functions for V such that $\bigcup_\alpha U_\alpha \supset V$. We denote by TW the holomorphic tangent bundle of W and by $N_{V'}$ the normal bundle of the regular part $V' = V - \mathrm{Sing}(V)$ of V. Recall that (e.g., [LS] Proposition 1), if V is an LCI, there exist a natural holomorphic vector bundle N_V of rank q over V and a bundle map $\pi : TW|_V \longrightarrow N_V$ such that $N_V|_{V'} = N_{V'}$ and that we have a commutative diagram with an exact row

$$
(1.1)\qquad
\begin{array}{ccccccccc}
 & & & & TW|_V & \xrightarrow{\ \pi\ } & N_V & & \\
 & & & & \uparrow{\scriptstyle \text{incl.}} & & \uparrow{\scriptstyle \text{incl.}} & & \\
0 & \longrightarrow & TV' & \longrightarrow & TW|_{V'} & \longrightarrow & N_{V'} & \longrightarrow & 0.
\end{array}
$$

An LCI subvariety V of W will be said a "strong local complete intersection" (briefly, SLCI), if there exists a C^∞ vector bundle $\tilde{N}$, defined over some neighborhood of V in W, whose restriction to V is equal to N_V. The last condition implies that in a neighborhood of every point of V, $\tilde{N}$ admits a C^∞ trivialization whose restriction to V is holomorphic. If V is an LCI and if $f : U \longrightarrow \mathbb{C}^q$ is a system of reduced local defining functions, the holomorphic bundle N_V is trivial on

$V \cap U$ and there is a trivialization which, on the regular part of $V \cap U$, is given by $\pi(\frac{\partial}{\partial f_1}), \dots, \pi(\frac{\partial}{\partial f_q})$ taking the components of f as a part of a local coordinate system on W. We call it the trivialization associated with f. If, moreover, V is an SLCI with a C^∞ extension $\tilde{N}$ of N_V, choosing a smaller U if necessary, there is a C^∞ trivialization of $\tilde{N}$ on U extending the trivialization associated with f. Here are some examples of SLCI:

(1) If V is a non-singular subvariety, i.e., a submanifold, of W, then clearly it is an LCI. It is moreover an SLCI. In fact let U be a tubular neighborhood of V with C^∞ projection $\rho : U \to V$. Then $\tilde{N} = \rho^* N_V$ is an extension of ν with the desired properties. In this case, however, we do not need to consider the extension $\tilde{N}$ in the following argument.

(2) Any hypersurface V of W ($q = 1$) is an SLCI. In fact, if we set $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$, where $\{f_\alpha\}$ denotes a family of reduced local defining functions, then the system $\{g_{\alpha\beta}\}$ of non-vanishing holomorphic functions satisfies the cocycle condition and it defines a holomorphic extension $\tilde{N}$ of N_V on the union of the domains U_α of the f_α's (which may be assumed to be W). Note that the collection $\{f_\alpha\}$ defines a global section of $\tilde{N}$ non-vanishing away from V.

(3) A complete intersection, i.e., a subvariety V which is covered by the domain U of a single system defining functions $f : U \longrightarrow \mathbb{C}^q$, is an SLCI. In fact, since N_V is trivial in this case, it admits a trivial extension $\tilde{N}$ on U.

(4) Let $W = \mathbb{CP}^n$, the n-dimensional complex projective space with homogeneous coordinates $[X_0, X_1, ..., X_n]$. Any algebraic set V in $\mathbb{CP}^n$ which is a (global) complete intersection is also an SLCI. In fact, by definition, the ideal $I(V)$ of homogeneous polynomials in $(X_0, \dots, X_n)$ vanishing on V can be generated by q elements $F_1, F_2, ..., F_q$ ($q = \operatorname{codim} V$). In the affine open subset U_i of $\mathbb{CP}^n$ defined by $X_i \neq 0$, $V \cap U_i$ has for defining equations $\frac{F_k}{X_i^{d_k}} = 0$, $d_k = \deg F_k$, $1 \leq k \leq q$. Therefore, on $U_i \cap U_j$ the change of defining functions is given by the $q \times q$ diagonal matrix with entries $\left(\frac{X_j}{X_i}\right)^{d_1}, ..., \left(\frac{X_j}{X_i}\right)^{d_q}$. Thus, denoting by $L \longrightarrow \mathbb{CP}^n$ the hyperplane bundle (the dual to the tautological line bundle), $\tilde{N}$ is defined on the whole $\mathbb{CP}^n$ by

$$\tilde{N} = \oplus_{k=1}^q L^{\otimes d_k}.$$

Hence we have $c(\tilde{N}) = 1 + c_1(\tilde{N}) + \cdots + c_q(\tilde{N}) = \prod_{k=1}^q (1 + d_k c)$, where $c = c_1(L)$.

(5) In general, let $\tilde{N}$ be a holomorphic vector bundle of rank q over W and V the subvariety of W defined by a holomorphic section σ of $\tilde{N}$. Suppose σ is a regular section, i.e., a section such that, at each point of V, the germs of its components $(f_1, \dots, f_q)$ with respect to a local (holomorphic) trivialization of $\tilde{N}$ near the point form a regular sequence (in fact, this is the case if and only if the codimension of V is q). Then V is an LCI, locally defined by $f_1 = \cdots = f_q = 0$. Moreover it is an SLCI with $\tilde{N}$ itself a holomorphic extension of N_V. (We assume that V is reduced, to be consistent with the definition in the beginning of this section.)

2. Bott residues on singular varieties

Let W be a complex manifold of complex dimension n and V an analytic subvariety of W of pure complex dimension p, as before. Also, let X be a holomorphic vector field on the regular part of V and write $\Sigma = \mathrm{Sing}(X) \cup \mathrm{Sing}(V)$. (Recall that a singular point of X is either a point where X vanishes or a point where it is not defined.)

A holomorphic vector bundle $E \to V$ will be said an "X-bundle", if
(i) $E|_{V-\Sigma}$ is equipped with a holomorphic action of X in the sense of Bott [B1], i.e., there exists a $\mathbb{C}$-linear endomorphism

$$\theta_X : C^\infty(E|_{V-\Sigma}) \to C^\infty(E|_{V-\Sigma})$$

of the space of C^∞ sections of E over $V - \Sigma$ with the properties

$$\begin{cases} \theta_X(u\sigma) = (Xu)\sigma + u\theta_X(\sigma) & \text{for a } C^\infty \text{ function } u, \\ \theta_X(\sigma) \text{ is holomorphic if } \sigma \text{ is,} \end{cases}$$

(ii) either V is non-singular or there exists a C^∞ extension $\tilde{E}$ of the bundle E to some neighborhood of V in W.

Let (E, θ_X) be an X-bundle. A connexion ω on $E|_{V-\Sigma}$ will be said *special* relative to θ_X, if its derivation law ∇ satisfies:

$$\begin{cases} \nabla_X(\sigma) = \theta_X(\sigma) & \text{for all } \sigma \in C^\infty(E|_{V-\Sigma}), \\ \nabla_Z(\sigma) = 0 & \text{for a } C^\infty \text{ section } Z \text{ of } \overline{T}(V-\Sigma) \text{ and a holomorphic section } \sigma. \end{cases}$$

It is known that such a connection always exists ([B1]).

We denote by Δ_ω the Chern-Weil homomorphism defined by a connection ω and by $\Delta_{\omega_0\omega_1\ldots\omega_k}$ the Bott's operator for iterated differences so that $d \circ \Delta_{\omega_0\omega_1\ldots\omega_k} = \sum_{i=0}^k (-1)^i \Delta_{\omega_0\ldots\hat{\omega}_i\ldots\omega_k}$ ([B2]). In particular, we have $d \circ \Delta_{\omega\omega'} = \Delta_{\omega'} - \Delta_\omega$. In general, let E be a complex vector bundle of rank r and let $\mathbb{C}[c_1, \ldots, c_r]$ denote the ring of Chern polynomials with complex coefficients. The ring is graded by $\deg(c_i) = i$ and its homogeneous component of degree d is denoted by $\mathbb{C}[c_1, \ldots, c_r]_d$. Thus, if φ is a polynomial in $\mathbb{C}[c_1, \ldots, c_r]_d$, then $\Delta_{\omega_0\omega_1\ldots\omega_k}(\varphi)$ is a $(2d-k)$-form. In particular, for a connection ω on E, $\Delta_\omega(\varphi)$ is a closed $2d$-form and its cohomology class is independent of the choice of the connection and gives the characteristic class $\varphi(E)$ of E for the polynomial φ. Note that if the degree of φ is 0, i.e., if $\varphi = c$ is a constant, $\Delta_\omega(\varphi) = c$ and $\Delta_{\omega_0\omega_1\ldots\omega_k}(\varphi) = 0$ for $k \geq 1$.

(2.1) Lemma (Vanishing theorem). *Let (E, θ_X) and (E', θ'_X) be two X-bundles on V of respective ranks r and r' and let $\varphi \in \mathbb{C}[c_1, \ldots, c_r]_d$ and $\varphi' \in \mathbb{C}[c_1, \ldots, c_{r'}]_{d'}$ be homogeneous Chern polynomials, defining a characteristic class of dimension $2d$ for E and one of dimension $2d'$ for E'. Suppose $\omega_1, \ldots, \omega_k$ are connections on $E|_{V-\Sigma}$*

all special relative to θ_X and $\omega'_1, \dots, \omega'_{k'}$ connections on $E'|_{V-\Sigma}$ all special relative to θ'_X. If $d + d'$ is equal to the complex dimension p of V, then we have

$$\Delta_{\omega_1 \dots \omega_k}(\varphi) \wedge \Delta_{\omega'_1 \dots \omega'_{k'}}(\varphi') = 0.$$

Note that if $k = p$ or $k' = p$, by letting $\varphi' = 1$ or $\varphi = 1$, respectively, we have the vanishing theorem in the single vector bundle case:

$$\Delta_{\omega_1 \dots \omega_k}(\varphi) = 0 \quad \text{or} \quad \Delta_{\omega'_1 \dots \omega'_{k'}}(\varphi') = 0.$$

The proof of the lemma is an obvious generalization of the single vector bundle case (see [B1]).

This lemma in particular implies that, for $d + d' = p$, the cup product $\varphi(E) \smile \varphi'(E')$ of characteristic classes vanishes over $V - \Sigma$. Thus this product "localizes" near Σ, in the sense that it has a natural lift to $H^{2p}(V, V - \Sigma)$ giving rise to residues in $H_0(\Sigma)$ by duality when Σ is compact. In fact this is done as follows.

Let Σ_α be a compact connected component of Σ and U_α an open neighborhood of Σ_α in W such that $V_\alpha - \Sigma_\alpha$ is in the regular part of V, $V_\alpha = U_\alpha \cap V$. Also, let $\tilde{\mathcal{T}}_\alpha$ be a compact (real) manifold of dimension $2n$ with boundary in U_α such that Σ_α is in the interior of $\tilde{\mathcal{T}}_\alpha$ and that the boundary $\partial\tilde{\mathcal{T}}_\alpha$ is transverse to V. We write $\mathcal{T}_\alpha = \tilde{\mathcal{T}}_\alpha \cap V$ and $\partial\mathcal{T}_\alpha = \partial\tilde{\mathcal{T}}_\alpha \cap V$.

Now suppose we have two X-bundles (E, θ_X) and (E', θ'_X). We take an arbitrary connection ω_α on $\tilde{E}|_{U_\alpha}$ (or on $E|_{V_\alpha}$ if V is non-singular) and a connection ω on $E|_{V_\alpha - \Sigma_\alpha}$ special relative to θ_X. Take also ω'_α and ω' similarly for (E', θ'_X). Let $\varphi \in \mathbb{C}[c_1, \dots, c_r]_d$ and $\varphi' \in \mathbb{C}[c_1, \dots, c_{r'}]_{d'}$ be as in Lemma (2.1) with $d + d' = p$. Then, using the integration theory on the Čech-de Rham cohomology ([Lh1], [Lh2]) and the vanishing theorem (2.1), we get the following theorem ([LS] Proposition 2, [LSS] Lemma 2).

(2.2) Theorem. *If we define the residue $\mathrm{Res}_{\Sigma_\alpha}(\varphi(E, \theta_X) \cdot \varphi'(E', \theta'_X))$ by*

$$\begin{aligned} &\mathrm{Res}_{\Sigma_\alpha}(\varphi(E, \theta_X) \cdot \varphi'(E', \theta'_X)) \\ &\quad = \int_{\mathcal{T}_\alpha} \Delta_{\omega_\alpha}(\varphi) \wedge \Delta_{\omega'_\alpha}(\varphi') - \int_{\partial\mathcal{T}_\alpha} (\Delta_\omega(\varphi) \wedge \Delta_{\omega'\omega'_\alpha}(\varphi') + \Delta_{\omega\omega_\alpha}(\varphi) \wedge \Delta_{\omega'_\alpha}(\varphi')), \end{aligned}$$

then

(i) *This number does not depend on the choices of $\tilde{\mathcal{T}}_\alpha$, ω, ω_α, ω', and ω'_α.*

(ii) *Assume V to be compact and let $(\Sigma_\alpha)_\alpha$ be the partition of Σ into connected components. Then, we have*

$$\sum_\alpha \mathrm{Res}_{\Sigma_\alpha}(\varphi(E, \theta_X) \cdot \varphi'(E', \theta'_X)) = (\varphi(E) \smile \varphi'(E')) \frown [V],$$

where $[V]$ denotes the homology class of V. In particular, the sum depends only on V, E, E', φ and φ', but not on X, θ_X and θ'_X.

If $d = p$ and $\varphi' = 1$, we write the residue simply $\mathrm{Res}_{\Sigma_\alpha}(\varphi(E, \theta_X))$ instead of $\mathrm{Res}_{\Sigma_\alpha}(\varphi(E, \theta_X) \cdot 1(E', \theta'_X))$. Thus we have

$$\mathrm{Res}_{\Sigma_\alpha}(\varphi(E, \theta_X)) = \int_{T_\alpha} \Delta_{\omega_\alpha}(\varphi) - \int_{\partial T_\alpha} \Delta_{\omega\omega_\alpha}(\varphi) \tag{2.3}$$

and the residue formula in Theorem (2.2) (ii) becomes

$$\sum_\alpha \mathrm{Res}_{\Sigma_\alpha}(\varphi(E, \theta_X)) = \varphi(E) \frown [V]. \tag{2.4}$$

Also, if $d' = p$ and $\varphi = 1$, we use similar notation and get similar identities.

As usual, denoting by $c(E)$ and $c(E')$ the total Chern classes of E and E', we define the i-th Chern classe $c_i(E - E')$ of the virtual bundle $E - E'$ to be equal to the homogeneous component of degree i in the formal expansion of $c(E) \cdot c(E')^{-1}$. So for a polynomial φ'' in $\mathbb{C}[c''_1, \cdots, c''_{r''}]$, $r'' = r - r'$, we may define the characteristic class $\varphi''(E - E')$ as the sum of classes of the form $\varphi_i(E) \smile \varphi'_i(E')$. More precisely, letting

$$\rho : \mathbb{C}[c''_1, \ldots, c''_{r''}] \to \mathbb{C}[c_1, \ldots, c_r] \otimes \mathbb{C}[c'_1, \ldots, c'_{r'}] \tag{2.5}$$

be the morphism generated by the correspondence

$$1 + c''_1 + \ldots + c''_{r''} \mapsto (1 + c_1 + \ldots + c_r) \cdot (1 + c'_1 + \ldots + c'_{r'})^{-1}$$

with the terms of sufficiently large degree truncated, we may write $\rho(\varphi'') = \sum_i \varphi_i \cdot \varphi'_i$ with $\varphi_i \in \mathbb{C}[c_1, \ldots, c_r]$ and $\varphi'_i \in \mathbb{C}[c'_1, \ldots, c'_{r'}]$. Then we define $\varphi''(E - E') = \sum_i \varphi_i(E) \smile \varphi'_i(E')$ and, moreover, if the degree d'' of φ'' is equal to p, we may define

$$\mathrm{Res}_{\Sigma_\alpha}(\varphi''((E, \theta_X) - (E', \theta'_X))) = \sum_i \mathrm{Res}_{\Sigma_\alpha}(\varphi_i(E, \theta_X) \cdot \varphi'_i(E', \theta'_X)). \tag{2.6}$$

Thus, from the residue formula in Theorem (2.2) (ii), we have

$$\sum_\alpha \mathrm{Res}_{\Sigma_\alpha}(\varphi''((E, \theta_X) - (E', \theta'_X))) = \varphi''(E - E') \frown [V]. \tag{2.7}$$

Now if V is non-singular and if there exists an exact sequence of holomorphic vector bundles

$$0 \to E'' \to E \to E' \to 0$$

on V, then the characteristc classes of E'' and those of $E - E'$ are obviously the same. Assume furthermore that E'' is also an X-bundle with action θ''_X. If θ''_X is equal to the restriction of θ_X to the sections of E'' and if θ'_X is defined from θ_X by passing to the quotient, it is then less obvious, but true, that the residues of E'' and those of $E - E'$ are also the same ([LSS] Lemma 3):

(2.8) Lemma (Compatibility of the residues). *We have*

$$\mathrm{Res}_{\Sigma_\alpha}(\varphi''(E'',\theta''_X)) = \mathrm{Res}_{\Sigma_\alpha}(\varphi''((E,\theta_X)-(E',\theta'_X))).$$

We will now give a formula for the residue in the case Σ_α consists of an isolated point P in V and, near P, X is the restriction of a holomorphic vector field $\tilde{X}$ defined on a neighborhood of P in W. We also assume that V is a complete intersecion near P. Then we have the following lemma ([LS] Theorem 2).

(2.9) Lemma (Existence of a good coordinate system). *There exists a local coordinate system $(z_1,\ldots,z_n)$ near P in W such that, if we express $\tilde{X}$ as*

$$\tilde{X} = \sum_{i=1}^{n} a_i(z_1,\ldots,z_n)\frac{\partial}{\partial z_i}$$

and if we denote by $(f_1,\ldots,f_q)$ a system of defining functions of V near P, the sequence $(a_1,\ldots,a_p,f_1,\ldots,f_q)$ is regular, i.e., the set of common zeros of the holomorphic functions $a_1,\ldots,a_p,f_1,\ldots,f_q$ consists only of P.

Let $\varphi \in \mathbb{C}[c_1,\ldots,c_r]$ and $\varphi' \in \mathbb{C}[c_1,\ldots,c_{r'}]$ be homogeneous polynomials of respective degrees d and d' with $d+d'=p$. Taking a holomorphic trivialization $(\sigma_1,\ldots,\sigma_r)$ of E on a neighborhood V_P of P, we define the matrix $M=(m_{ij})$ whose entries are holomorphic functions m_{ij} on $V_P-\{P\}$ determined by

$$\theta_X(\sigma_i) = \sum_{j=1}^{r} m_{ij}\sigma_j. \tag{2.10}$$

We define the matrix M' similarly for E'.

In general, for an $r\times r$ matrix A, we define $c_i(A)$, $i=1,\ldots,r$, by

$$\det\left(I + t\frac{\sqrt{-1}}{2\pi}A\right) = 1 + tc_1(A) + \cdots + t^r c_r(A).$$

Thus, for a polynomial φ in $\mathbb{C}[c_1,\ldots,c_r]$, we may also define $\varphi(A)$, which is a holomorphic function, if A is a matrix with holomorphic entries. Similarly we define $\varphi'(A')$ for an $r'\times r'$ matrix A'.

With these, we have the following theorem ([LS] Remark after Theorem 2, [LSS] Corollary to Theorem 2 and Lemma 4).

(2.11) Theorem. *In the above situation, if we take a coordinate system as in Lemma (2.9), we have*

$$\mathrm{Res}_P(\varphi(E,\theta_X)\cdot\varphi'(E',\theta'_X)) = \int_\Gamma \frac{\varphi(M)\varphi'(M')dz_1\wedge dz_2\wedge\cdots\wedge dz_p}{a_1a_2\cdots a_p},$$

where Γ *denotes the* p*-cycle in* V *given by*

$$\Gamma = \{\, z \mid |a_1(z)| = \cdots = |a_p(z)| = \varepsilon,\ f_1(z) = \cdots = f_q(z) = 0\,\},$$

for a small positive number ε*, which is oriented so that* $d\theta_1 \wedge \cdots \wedge d\theta_p$ *is positive,* $\theta_i = \arg a_i$.

For the formula in terms of general coordinate system, see [LS] Theorem 1' and [LSS] Theorem 2.

3. Residues for $TW|_V$

Let W and V be as before and let $\tilde{X}$ be a holomorphic vector field on a neighborhood of V which leaves V invariant. Letting X be the vector field on (the non-singular part of) V induced by $\tilde{X}$, we define the singular set Σ as before. Then $TW|_V$ is an X-bundle with the action θ_X given as follows. Let Y be a C^∞ section of $TW|_V$ on $V - \Sigma$. Taking its extension $\tilde{Y}$ to some neighborhood of $V - \Sigma$, we set

$$\theta_X(Y) = [\tilde{X}, \tilde{Y}]|_{V-\Sigma}. \tag{3.1}$$

Then it does not depend on the extension and defines a holomorphic action. From Theorem (2.2), we have the followingtheorem.

(3.2) Theorem. *For a compact connected component* Σ_α *of* Σ *and a polynomial* φ *in* $\mathbb{C}[c_1, \ldots, c_n]_p$*, we have the residue* $\mathrm{Res}_{\Sigma_\alpha}(\varphi(TW|_V, \theta_X))$ *and, if* V *is compact, we have*

$$\sum_\alpha \mathrm{Res}_{\Sigma_\alpha}(\varphi(TW|_V, \theta_X)) = \varphi(TW) \frown [V].$$

We denote $\mathrm{Res}_{\Sigma_\alpha}(\varphi(TW|_V, \theta_X))$ by $\mathrm{Res}_{\Sigma_\alpha}(\varphi(TW|_V, X))$ hereafter.

Now we assume that Σ_α consists of an isolated point P, $\tilde{X}$ is defined near P and that V is a complete intersection near P. If $(z_1, \ldots, z_n)$ is a local coordinate system on W near P, $\left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right)$ gives a local holomorphic trivialization of $TW|_V$. Expressing $\tilde{X} = \sum_{i=1}^n a_i \frac{\partial}{\partial z_i}$, we compute

$$\theta_X\left(\frac{\partial}{\partial z_i}\right) = -\sum_{j=1}^n \frac{\partial a_j}{\partial z_i}\frac{\partial}{\partial z_j}$$

Hence the matrix M defined by (2.10) is given by $M = -J$, $J = \frac{\partial(a_1, \ldots, a_n)}{\partial(z_1, \ldots, z_n)}$, the Jacobian matrix of $(a_1, \ldots, a_n)$ with respect to $(z_1, \ldots, z_n)$. Let φ be a polynomial in $\mathbb{C}[c_1, \ldots, c_n]_p$. Then, from Theorem (2.11), we have the following theorem.

(3.3) Theorem. *In the above situation, if* $(z_1, \ldots, z_n)$ *denotes a coordinate system as in Lemma (2.9), we have*

$$\mathrm{Res}_P(\varphi(TW|_V, X)) = \int_\Gamma \frac{\varphi(-J)dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{a_1 a_2 \cdots a_p},$$

where Γ *denotes a* p*-cycle as in (2.11).*

4. Residues for N_V (residues of a holomorphic vector field relative to an invariant subvariety, [LS])

Let W, V, $\tilde{X}$, X and Σ be as in the previous section. In this section, we assume that V is an SLCI so that the bundle N_V admits a C^∞ extension $\tilde{N}$. Then N_V is an X-bundle with the action θ_X given as follows. First, note that any C^∞ section σ of N_V over $V - \Sigma$ may be written as $\sigma = \pi(Y)$ for some section Y of $TW|_{V-\Sigma}$. Let Y be a C^∞ section of $TW|_V$ on $V - \Sigma$. Taking an extension $\tilde{Y}$ of Y to some neighborhood of $V - \Sigma$, we set

$$\theta_X(\sigma) = \pi\left([\tilde{X}, \tilde{Y}]|_{V-\Sigma}\right). \tag{4.1}$$

Then this action is well defined and N_V becomes an X-bundle (cf. Remark (4.3) 2° below). From Theorem (2.2), we have the following theorem.

(4.2) Theorem. *For a compact connected component Σ_α of Σ and a polynomial φ in $\mathbb{C}[c_1, \ldots, c_q]_p$, we have the residue $\mathrm{Res}_{\Sigma_\alpha}(\varphi(N_V, \theta_X))$ and, if V is compact, we have*

$$\sum_\alpha \mathrm{Res}_{\Sigma_\alpha}(\varphi(N_V, \theta_X)) = \varphi(N_V) \frown [V].$$

(4.3) *Remarks.* 1°. We call $\mathrm{Res}_{\Sigma_\alpha}(\varphi(N_V, \theta_X))$ the residue of the vector field X (or of $\tilde{X}$) relative to the invariant subvariety V at Σ_α for φ. It will be sometimes denoted by $\mathrm{Res}_{\Sigma_\alpha}(\varphi(N_V, X))$.

2°. Theorem (4.2) is valid when there is no global vector fields. More precisely, a dimension one singular foliation $\mathcal{F}$ on W is defined locally by a holomorphic vector field so that two vector fields defining $\mathcal{F}$ differ by multiplication by a non-vanishing holomorphic function in the common domain of definition. Thus we may patch together the singular sets of the vector fields defining $\mathcal{F}$ to get the singular set $S(\mathcal{F})$ of $\mathcal{F}$. If V is a subvariety of W invariant by $\mathcal{F}$, i.e., invariant by every vector field in $\mathcal{F}$, $\mathcal{F}$ induces a dimension one foliation $\mathcal{F}_V$ on the non-singular part of V. We set $\Sigma = (S(\mathcal{F}) \cap V) \cup \mathrm{Sing}(V)$. Then, on $V - \Sigma$, there is an action $\theta_{\mathcal{F}_V}$ of $\mathcal{F}_V$ on N_V similar to (4.1) ([Lh2] Lemme Fondamental 2.1) and there is also a vanishing theorem as (2.1). Thus Theorem (4.2) remains true if we replace θ_X by $\theta_{\mathcal{F}_V}$. We denote $\mathrm{Res}_{\Sigma_\alpha}(\varphi(N_V, \theta_{\mathcal{F}_V}))$ by $\mathrm{Res}_{\Sigma_\alpha}(\varphi(N_V, \mathcal{F}))$ hereafter.

3°. Theorem (4.2) is also readily generalized to the case of higher dimensional singular foliations on W (see Theorem (4.6) below).

Now we assume that Σ_α consists of an isolated point P and that $\tilde{X}$ is defined near P. If we let $f = (f_1, \ldots, f_q)$ be a system of local reduced defining functions of V near P, the condition that V is invariant by $\tilde{X}$ is expresed as

$$\tilde{X}(f_i) = \sum_{i=1}^{q} c_{ij} f_j,$$

for some holomorphic functions c_{ij} ([St], [BR]). Denoting by $(\sigma_1, \ldots, \sigma_q)$ the trivialization of N_V associated with f (see section 1), we have ([LS] Lemma 1 (iii))

$$\theta_X(\sigma_i) = -\sum_{j=1}^{q} c_{ji}\sigma_j.$$

Hence the matrix M defined by (2.10) is given by $M = -{}^tC$, $C = (c_{ij})$. Let φ be a polynomial in $\mathbb{C}[c_1, \ldots, c_q]_p$. Then, denoting by $\mathcal{F}$ the foliation defined by $\tilde{X}$ (Remark (4.3) 2°), from Theorem (2.11), we have the following theorem.

(4.4) Theorem. *In the above situation, if $(z_1, \ldots, z_n)$ denotes a coordinate system as in Lemma (2.9), we have*

$$\mathrm{Res}_P(\varphi(N_V, \mathcal{F})) = \int_\Gamma \frac{\varphi(-C)dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{a_1 a_2 \cdots a_p},$$

where Γ denotes a p-cycle as in (2.11).

(4.5) *Remark.* The right hand side above depends only on $\mathcal{F}$ and not on $\tilde{X}$, in accordance with Remark (4.3) 2°. Namely, if we take $u\tilde{X}$ instead of $\tilde{X}$ for a non-vanishing holomorphic function (in fact, any function) u, each a_i is multiplied by u, however, since the matrix C is also multiplied by u, the integrant remains the same. Note that, on the contrary, if we replace $\tilde{X}$ by $u\tilde{X}$, the right hand side of the formula in Theorem (3.3) changes in general.

The case of higher dimensional foliations

In the case of higher dimensional foliations, Theorem (4.2) is generalized as follows. For details we refer to [Sw2]. Let W and V be as above with V an SLCI. Also, let $\mathcal{F}$ be an s-dimensional singular foliation on W leaving V invariant and set $\Sigma = (S(\mathcal{F}) \cap V) \cup \mathrm{Sing}(V)$.

(4.6) Theorem. *Let φ be a polynomial in $\mathbb{C}[c_1, \ldots, c_q]$ of degree $d > p - s$.*
(i) *For a compact connected component Σ_α of Σ we have the residue*

$$\mathrm{Res}_{\Sigma_\alpha}(\varphi(N_V, \mathcal{F})) \in H_{2p-2d}(\Sigma_\alpha; \mathbb{C}),$$

which depends only on the local behavior of $\mathcal{F}$ and V near Σ_α.
(ii) *If V is compact, we have*

$$\sum_\alpha (i_\alpha)_* \mathrm{Res}_{\Sigma_\alpha}(\varphi(N_V, \mathcal{F})) = \varphi(N_V) \frown [V] \quad \text{in} \quad H_{2p-2d}(V; \mathbb{C}),$$

where i_α denotes the embedding $\Sigma_\alpha \hookrightarrow V$.

Particular cases

(I) Let W be a (non-singular) complex surface, V a complex curve in W and $\mathcal{F}$ a dimension one singular foliation on W leaving V invariant ($p = q = s = 1$). In this case, we have the residue essentially only for $\varphi = c_1$ and Σ consists of isolated points. If V is non-singular at P, the residue $\mathrm{Res}_P(c_1(N_V, \mathcal{F}))$ coincides with the index defined in [CS] by C. Camacho and P. Sad (see the case (III) below) and, in general, it coinsides with the one defined in [Sw1], which is also equal to the one in [Ln1] by A. Lins Neto, provided V is locally irreducible at P. Also, if V is compact, we have, by Theorem (4.2),

$$\sum_{P\in\Sigma} \mathrm{Res}_P(c_1(N_V, \mathcal{F})) = c_1(N_V) \frown [V] = V^2,$$

where V^2 denotes the self-intersection number of V in W. This is the formula proved in [CS] when V is non-singular and is the one in [Sw1], in general. Equivalent formulas had been obtained for $W = \mathbb{CP}^2$ in [Ln1] and for W a complete intersection in $\mathbb{CP}^n$ in [Sr] by M. Soares.

(II) Let V be a hypersurface in W and $\mathcal{F}$ a codimension one singular foliation on W leaving V invariant ($q = 1$, $s = p$). In this case, we have the residue essentially only for $\varphi = c_1^k$, $k = 1, \dots, p$. When V is non-singular and $\dim \Sigma = p - 1$, the residue $\mathrm{Res}_{\Sigma_\alpha}(\varphi(N_V, \mathcal{F}))$ is defined and the residue formula as in Theorem (4.6) is proved by J.-P. Brasselet, B. Gmira [Gr] and A. Lins Neto [Ln2].

(III) If V is non-singular, Theorems (4.2), (4.4) and (4.6) reduce to the ones in [Lh2]. In particular, the formula in (4.4) becomes as follows. When P is a non-singular point of V, we may take a local coordinate system $(z_1, \dots, z_n)$ so that $f_1 = z_{p+1}, \dots, f_q = z_n$ define V near P. Then the invariance condition for V by $\tilde{X}$ is that $a_{p+1}, \dots, a_n$ vanish on V. Thus $(z_1, \dots, z_n)$ is a good coordinate system in the sense of Lemma (2.9). On the other hand, c_{ij} and $\frac{\partial a_{p+i}}{\partial z_{p+j}}$ are equal on V. Hence denoting by J' the Jacobian matrix $\frac{\partial(a_{p+1}, \dots, a_n)}{\partial(z_{p+1}, \dots, z_n)}$, we recover the formula

$$\mathrm{Res}_P((\varphi(N_V, \mathcal{F})) = \int_\Gamma \frac{\varphi(-J')dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{a_1 a_2 \cdots a_p},$$

in [Lh2] Théorème 1. Note that the right hand side is expressed as a Grothendieck residue.

5. Residues for $TW|_V - N_V$ (residues of holomorphic vector fields on varieties, [LSS])

Again, let W, V, $\tilde{X}$, X and Σ be as in section 3 and assume that V is an SLCI as in section 4. Taking as E the bundle $TW|_V$ and as E' the bundle N_V with the X-actions θ_X and θ'_X given by (3.1) and (4.1), from Theorem (2.2), we have the following theorem (see also (2.6) and (2.7)).

(5.1) Theorem. *For a compact connected component Σ_α of Σ and a polynomial φ in $\mathbb{C}[c_1, \dots, c_p]_p$, we have the residue $\mathrm{Res}_{\Sigma_\alpha}(\varphi((TW|_V, \theta_X) - (N_V, \theta'_X)))$ and if V is compact we have*

$$\sum_\alpha \mathrm{Res}_{\Sigma_\alpha}(\varphi((TW|_V, \theta_X) - (N_V, \theta'_X))) = \varphi(TW|_V - N_V) \frown [V].$$

We denote $\mathrm{Res}_{\Sigma_\alpha}(\varphi((TW|_V, \theta_X) - (N_V, \theta'_X)))$ by $\mathrm{Res}_{\Sigma_\alpha}(\varphi(TW|_V - N_V, X))$ hereafter. We employ further abbreviation and denote the virtual bundle $TW|_V - N_V$ simply by $\check{T}V$ and call it the virtual tangent bundle of V (in W):

$$\check{T}V = TW|_V - N_V.$$

(5.2) *Remark.* If Σ_α is in the regular part of V, then, by Lemma (2.8), the residue $\mathrm{Res}_{\Sigma_\alpha}(\varphi(\check{T}V, X))$ coincides with that of P. Baum and R. Bott ([BB1], [BB2]) of X for φ at Σ_α.

Now we assume that Σ_α consists of an isolated point P and that $\tilde{X}$ is defined near P. Let φ be a polynomial in $\mathbb{C}[c_1, \dots, c_p]_p$ and write $\rho(\varphi) = \sum_i \varphi_i \cdot \varphi'_i$ with $\varphi_i \in \mathbb{C}[c_1, \dots, c_n]$ and $\varphi'_i \in \mathbb{C}[c_1, \dots, c_q]$ (see (2.5)). Then, from Theorem (2.11), we have the following theorem.

(5.3) Theorem. *In the above situation, if $(z_1, \dots, z_n)$ denotes a coordinate system as in Lemma (2.9), we have*

$$\mathrm{Res}_P(\varphi(\check{T}V, X)) = \sum_i \int_\Gamma \frac{\varphi_i(-J)\varphi'_i(-C) dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{a_1 a_2 \cdots a_p},$$

where Γ denotes a p-cycle as in (2.11).

In particular, if $\varphi = c_p$, denoting by $[c(-J) \cdot c(-C)^{-1}]_k$ the holomorphic function given as the coefficient of t^k in the formal power series expansion of $\det\left(I - t\frac{\sqrt{-1}}{2\pi}J\right) \cdot \det\left(I - t\frac{\sqrt{-1}}{2\pi}C\right)^{-1}$ in t, we have

$$\mathrm{Res}_P(c_p(\check{T}V, X)) = \int_\Gamma \frac{[c(-J) \cdot c(-C)^{-1}]_p dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{a_1 a_2 \cdots a_p}. \tag{5.4}$$

(5.5) *Remark.* If P is a regular point of V, we may take a coordinate system $(z_1, \dots, z_n)$ about P as in Particular case (III) in section 4. Then, on V, $C = J'$ and $\frac{\partial a_{p+i}}{\partial z_j} = 0$ for $i = 1, \dots, q$ and $j = 1, \dots, p$. Hence denoting by J'' the Jacobian matrix $\frac{\partial(a_1, \dots, a_p)}{\partial(z_1, \dots, z_p)}$, the formula in (5.3) reduces to

$$\mathrm{Res}_P((\varphi(\check{T}V, X)) = \int_\Gamma \frac{\varphi(-J'') dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{a_1 a_2 \cdots a_p},$$

which coincides with the ones in [BB1] and [BB2] expressing the residue as a Grothendieck residue.

Now we can show that the residue in (5.4) coincides with the topological index considered in [GSV] and [SS], recalling that the latter is equal to the total index of a vector field obtained from X by perturbation on a "canonical" smoothing of V.

(5.6) Proposition. *The residue* $\mathrm{Res}_P(c_p(\check{T}V, X))$ *coincides with the topological index of* [GSV] *and* [SS]. *In particular, if P is a regular point of V, then it is equal to the usual Poincaré-Hopf index at P of the vector field X on V.*

The proof is done by first noting that the class $c_p(\check{T}V)$ is localized by two different reasons; (1) because of the existence of (natural) X-actions on $TW|_V$ and N_V, away from the singularity, and (2) because of the existence of a non-vanishing section X of the tangent bundle of V away from the singularity, and then showing that these two define in fact the same residue. Since the first gives $\mathrm{Res}_P(c_p(\check{T}V, X))$, the proof is completed by showing that the residue given by the second is equal to the index of [GSV] and [SS]. For details, see [LSS] Lemma 5.

6. Grothendieck residues relative to subvarieties

As we have seen in the previous sections, we encounter integrals of the form

$$\int_\Gamma \frac{h(z)\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_p}{a_1 a_2 \cdots a_p},$$

where Γ denotes a p-cycle as in (2.11). We give a formula for this integral in the case V is a hypersurface and the system $(a_1, \ldots, a_p)$ is "non-degenerate" in the following sense. We refer to, e.g., [GH] for integral representations of usual Grothendieck residues and their properties.

We denote by $\mathcal{O}_n$ the ring of germs of holomorphic functions at the origin 0 in $\mathbb{C}^n$ and let $(z_1, \ldots, z_n)$ be a coordinate system near 0 in $\mathbb{C}^n$. Also, let $a_1, \ldots, a_{n-1}$ be germs in $\mathcal{O}_n$ vanishing at 0 and V a germ of hypersurface with isolated singularity at 0 in $\mathbb{C}^n$ with defining function f. We further assume:

(i) $\det\left(\frac{\partial(a_1,\ldots,a_{n-1})}{\partial(z_1,\ldots,z_{n-1})}\right)(0) \neq 0$, thus $(a_1, \ldots, a_{n-1}, z_n)$ form a coordinate system.

(ii) The linear part of a_i, $i = 1, \ldots, n-1$, depends only on $z_1, \ldots, z_{n-1}$.

(iii) In the coordinate system $(a_1, \ldots, a_{n-1}, z_n)$, f is regular in z_n. Thus, if we denote by ℓ the order of f in z_n, we may assume that f is equal to a Weierstrass polynomial $P(z_n)$ of degree ℓ in z_n.

(iv) When we write

$$P(z_n) = z_n^\ell + g_1(a')z_n^{\ell-1} + \cdots + g_\ell(a')$$

with g_i holomorphic functions in $a' = (a_1, \ldots, a_{n-1})$ vanishing at $a' = 0$, the order of g_i is strictly greater than 1 for $i \geq 2$.

(6.1) *Remarks.* 1°. The condition (iii) implies that $(a_1, \ldots, a_{n-1}, f)$ is a regular sequence.
2°. If the (total) order of f is ℓ, then the condition (iv) is automatically satisfied.
3°. If we impose the following condition, in place of (ii), then (iv) is not necessary:
(ii)′ Each a_i, $i = 1, \ldots, n-1$, depends only on $z_1, \ldots, z_{n-1}$.
In fact, we gave a formula as in (6.2) below under the condition (ii)′ in [LSS]. We are indebted to X. Gómez-Mont for suggesting us to replace (ii)′ by (ii).

Denoting by Γ the $(n-1)$-cycle in V given by

$$\Gamma = \{ z \in V \mid |a_1(z)| = \cdots = |a_{n-1}(z)| = \varepsilon \},$$

for a small positive number ε, which is oriented so that $d\theta_1 \wedge \cdots \wedge d\theta_{n-1}$ is positive, $\theta_i = \arg\, a_i$, we have the following formula.

(6.2) Proposition. *In the above situation, we have, for a holomorphic function h near* 0,

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^{n-1} \int_\Gamma \frac{h(z)\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n-1}}{a_1 a_2 \cdots a_{n-1}} = \frac{\ell \cdot h(0)}{\det\left(\frac{\partial(a_1, \ldots, a_{n-1})}{\partial(z_1, \ldots, z_{n-1})}\right)(0)}.$$

Proof. First, denote by z_n' the function z_n seen as a coordinate belonging to the system $(a_1, \ldots, a_{n-1}, z_n)$. We have

$$dz_1 \wedge \cdots \wedge dz_{n-1} = \delta\, da_1 \wedge \cdots \wedge da_{n-1} + \sum_{i=1}^{n-1} \delta_i\, da_1 \wedge \cdots \wedge \widehat{da_i} \wedge \cdots \wedge da_{n-1} \wedge dz_n,$$

where δ denotes the partial Jacobian of $(z_1, \ldots, z_{n-1})$ with respect to $(a_1, \ldots, a_{n-1})$ and δ_i that with respect to $(a_1, \ldots, \hat{a}_i, \ldots, a_{n-1}, z_n')$ in the coordinate system $(a_1, \ldots, a_i, \ldots, a_{n-1}, z_n')$. Note that each term in δ_i involves $\frac{\partial z_j}{\partial z_n'}$ for some $j = 1, \ldots, n-1$. We have $\frac{\partial z_j}{\partial z_n'} = -\sum_{k=1}^{n-1} \frac{\partial z_j}{\partial a_k} \frac{\partial a_k}{\partial z_n}$. By the condition (ii), $\frac{\partial a_k}{\partial z_n}(0) = 0$ and thus $\frac{\partial z_j}{\partial z_n'}(0) = 0$. Hence the proposition is proved if we prove the identity

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^{n-1} \int_\Gamma \frac{g\, da_1 \wedge da_2 \wedge \cdots \wedge da_{n-1}}{a_1 a_2 \cdots a_{n-1}} = \ell \cdot g(0) \tag{6.3}$$

for an arbitrary holomorphic function g and the identity

$$\int_\Gamma \frac{g\, da_1 \wedge \cdots \wedge \widehat{da_i} \wedge \cdots \wedge da_{n-1} \wedge dz_n}{a_1 a_2 \cdots a_{n-1}} = 0 \tag{6.4}$$

for a holomorphic function g with $g(0) = 0$. To prove these, we consider the decomposition

$$P(z_n) = \prod_{k=1}^{\ell} (z_n - \varphi_k(a')) .$$

Notice that the functions φ_k themselves are not well defined but that the symmetric functions of the φ_k's are well defined, holomorphic in a' and vanishing at $a' = 0$. In particular, $s_p = \sum_{k=1}^{\ell} \varphi_k^p$ has the properties for each $p \geq 1$. Then, denoting by Γ_0 the $(n-1)$-cycle $\{ a' \mid |a_1| = \cdots = |a_{n-1}| = \varepsilon \}$ in the a'- space, the integral in (6.3) is given by

$$\int_{\Gamma_0} \frac{\sum_{k=1}^{\ell} g(a', \varphi_k)\, da_1 \wedge da_2 \wedge \cdots \wedge da_{n-1}}{a_1 a_2 \cdots a_{n-1}} .$$

If we write $g(a', z_n) = \sum_{p \geq 0} g_p(a') z_n^p$ with g_p holomorphic functions in a', we have $\sum_{k=1}^{\ell} g(a', \varphi_k) = \sum_{p \geq 0} g_p(a') s_p$, which is holomorphic in a' and takes the value $\ell \cdot g(0)$ at 0. Hence we get (6.3). To show (6.4), note that, on V, we may write $dz_n = \sum_{i=1}^{n-1} \frac{\partial z_n}{\partial a_i} da_i$. Hence the integral in (6.4) is given (up to sign) by

$$\int_{\Gamma_0} \frac{\sum_{k=1}^{\ell} g(a', \varphi_k) \frac{\partial \varphi_k}{\partial a_i}\, da_1 \wedge da_2 \wedge \cdots \wedge da_{n-1}}{a_1 a_2 \cdots a_{n-1}} .$$

If we express g as before, we have $\sum_{k=1}^{\ell} g(a', \varphi_k) \frac{\partial \varphi_k}{\partial a_i} = \sum_{p \geq 0} \frac{1}{p+1} g_p(a') \frac{\partial s_{p+1}}{\partial a_i}$, which is holomorphic in a' and vanishes at 0, because of the assumptions (iv) and $g(0) = 0$. □

(6.5) *Remarks.* 1°. If P is a non-singular point of V and if the coordinate system $(z_1, \ldots, z_n)$ is taken so that V is defined by $z_n = 0$, then, only under the condition (i), we have the formula in (6.2) with $\ell = 1$. In fact, in this case, we have the identity (6.4) without the condition (ii), since $dz_n = 0$ on V.

2°. For the formula in (6.2), it is not necessary to assume that V is invariant by the vector field $\sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}$.

Applications

Let W, V, $\tilde{X}$, X and Σ be as in sections 3 - 5. Here we assume that V is a hypersurface. For an isolated point P in Σ, under the additional conditions above, we may compute the residues in Theorems (3.3), (4.4) and (5.3) by the formula in Proposition (6.2).

Let V be defined by f near P and $(z_1, \ldots, z_n)$ a coordinate system about P as in Lemma (2.9). We write $\tilde{X} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}$ and assume that the conditions (i) - (iv) (or (i), (ii)$'$ and (iii)) are satisfied. Note that under the condition (ii), the eigenvalues of $\frac{\partial(a_1, \ldots, a_{n-1})}{\partial(z_1, \ldots, z_{n-1})}(0)$ are part of those of $J(0) = \frac{\partial(a_1, \ldots, a_n)}{\partial(z_1, \ldots, z_n)}(0)$.

So let $\lambda_1, \ldots, \lambda_{n-1}$ and $\lambda_1, \ldots, \lambda_{n-1}, \lambda_n$ be the ones for these matrices. By (i), $\lambda_1, \ldots, \lambda_{n-1}$ are all non-zero, while λ_n may be zero. Since $q = 1$ in this case, C is a 1×1 matrix. We set $\gamma = C(0)$.

In what follows, for complex numbers $\lambda_1, \ldots, \lambda_r$, we define $c_i(\lambda_1, \ldots, \lambda_r)$, $i = 1, \ldots, r$, by

$$\prod_{i=1}^{r}(1 + t\lambda_i) = 1 + tc_1(\lambda_1, \ldots, \lambda_r) + \cdots + t^r c_r(\lambda_1, \ldots, \lambda_r).$$

Thus for a polynomial φ in $\mathbb{C}[c_1, \ldots, c_r]$, we may define $\varphi(\lambda_1, \ldots, \lambda_r)$.
(I) Residues for $TW|_V$: For a polynomial φ in $\mathbb{C}[c_1, \ldots, c_n]_{n-1}$ the residue in Theorem (3.3) is given by

$$\mathrm{Res}_P(\varphi(TW|_V, X)) = \ell \cdot \frac{\varphi(\lambda_1, \ldots, \lambda_n)}{\lambda_1 \cdots \lambda_{n-1}}. \tag{6.6}$$

(II) Residue for N_V: In this case, we have the residue essentially only for $\varphi = c_1^{n-1}$ and for this polynomial the residue in Theorem (4.4) is given by

$$\mathrm{Res}_P(c_1^{n-1}(N_V, X)) = \ell \cdot \frac{\gamma^{n-1}}{\lambda_1 \cdots \lambda_{n-1}}. \tag{6.7}$$

(III) Residues for $\check{T}V$: For a polynomial φ in $\mathbb{C}[c_1, \ldots, c_{n-1}]_{n-1}$ the residue in Theorem (5.3) is given by

$$\mathrm{Res}_P(\varphi(\check{T}V, X)) = \ell \cdot \sum_{i=0}^{n-1} \frac{\varphi_i(\lambda_1, \ldots, \lambda_n)\gamma^i}{\lambda_1 \cdots \lambda_{n-1}}, \tag{6.8}$$

where, for each $i = 0, \ldots, n-1$, φ_i is a polynomial in $\mathbb{C}[c_1, \ldots, c_n]_{n-i-1}$ determined by $\rho(\varphi) = \sum_{i=0}^{n-1} \varphi_i \cdot (c_1')^i$ (see (2.5)). In particular, for $\varphi = c_{n-1}$, we have, from (5.4),

$$\mathrm{Res}_P(c_{n-1}(\check{T}V, X)) = \ell \cdot \frac{\lambda_1 \cdots \lambda_n - (\lambda_1 - \gamma) \cdots (\lambda_n - \gamma)}{\lambda_1 \cdots \lambda_{n-1}\, \gamma}. \tag{6.9}$$

If $\gamma = 0$, the right hand side in the above is understood to be the limit as γ approaches 0.

As a special case, if P is a non-singular point of V and if $(z_1, \ldots, z_n)$ is taken so that V is defined by $z_n = 0$, then $\gamma = \lambda_n$ and (6.6) - (6.8) become (see Particular case (III) in section 4, Remarks (5.5) and (6.5) 1°)

$$\begin{aligned}
&\mathrm{Res}_P(\varphi(TW|_V, X)) = \frac{\varphi(\lambda_1, \ldots, \lambda_n)}{\lambda_1 \cdots \lambda_{n-1}} \quad \text{for} \quad \varphi \in \mathbb{C}[c_1, \ldots, c_n]_{n-1}, \\
&\mathrm{Res}_P(c_1^{n-1}(N_V, X)) = \frac{\lambda_n^{n-1}}{\lambda_1 \cdots \lambda_{n-1}}, \\
&\mathrm{Res}_P(\varphi(\check{T}V, X)) = \frac{\varphi(\lambda_1, \ldots, \lambda_{n-1})}{\lambda_1 \cdots \lambda_{n-1}} \quad \text{for} \quad \varphi \in \mathbb{C}[c_1, \ldots, c_{n-1}]_{n-1}.
\end{aligned} \tag{6.10}$$

We have, in particular, $\mathrm{Res}_P(c_{n-1}(\check{T}V, X)) = 1$.

7. Examples

(7.1) Example. Let V be a hypersurface in $\mathbb{C}^n = \{(z_1, \dots, z_n)\}$ defined by a weighted homogeneous polynomial f of type $(d_1, ..., d_n)$ with isolated singularity at the origin 0. For the holomorphic vector field $X = \sum_{i=1}^n \frac{z_i}{d_i}\frac{\partial}{\partial z_i}$, we have $X(f) = f$ and thus V is invariant by X. We asssume that f is regular in z_n. This implies that d_n is a positive integer and f is regular in z_n of order d_n. If we let $a_i = \frac{z_i}{d_i}$, $i = 1, \dots, n$, $(a_1, \dots, a_{n-1}, f)$ is a regular sequence and the conditions (i), (ii)$'$ and (iii) in section 6 are satisfied. We have $\ell = d_n$, $\lambda_i = \frac{1}{d_i}$ and $\gamma = 1$. Hence, for a polynomial φ in $\mathbb{C}[c_1, \dots, c_n]_{n-1}$, from (6.6), we have

$$\mathrm{Res}_0(\varphi(TW|_V, X)) = \varphi\left(\frac{1}{d_1}, \dots, \frac{1}{d_n}\right) d_1 \cdots d_n.$$

Also from (6.7), we have

$$\mathrm{Res}_0(c_1^{n-1}(N_V, X)) = d_1 \cdots d_n.$$

Finally, for a polynomial φ in $\mathbb{C}[c_1, \dots, c_{n-1}]_{n-1}$, from (6.8), we have

$$\mathrm{Res}_0(\varphi(\check{T}V, X)) = \sum_{i=0}^{n-1} \varphi_i \left(\frac{1}{d_1}, \dots, \frac{1}{d_n}\right) d_1 \cdots d_n,$$

where, for each $i = 0, \dots, n-1$, φ_i is a polynomial in $\mathbb{C}[c_1, \dots, c_n]_{n-i-1}$ determined by $\rho(\varphi) = \sum_{i=0}^{n-1} \varphi_i \cdot (c_1')^i$ (see (2.5)). In particular, for $\varphi = c_{n-1}$, we have, from (6.9),

$$\mathrm{Res}_0(c_{n-1}(\check{T}V, X)) = 1 + (-1)^{n-1}(d_1 - 1)(d_2 - 1) \cdots (d_n - 1).$$

Note that, since X is transversal to the boundary of the Milnor fiber of f, $\mathrm{Res}_0(c_{n-1}(\check{T}V, X))$ is also equal to the Euler number $1 + (-1)^{n-1}\mu$ of the Milnor fiber, where μ denotes the Milnor number of f at 0 (see Proposition (5.6) and the paragraph right before it). Thus we reprove the formula

$$\mu = (d_1 - 1)(d_2 - 1) \cdots (d_n - 1)$$

for the Milnor number ([MO] Theorem 1).

In what follows, let W be the three dimensional complex projective space $\mathbb{CP}^3$ with homogeneous coordinates $[X, Y, Z, T]$ and V the cone in $\mathbb{CP}^3$ defined by

$$X^m + Y^m + Z^m = 0$$

for a positive integer m. The singular set of V consists of a single point $P_1 = [0, 0, 0, 1]$. Denote by U_T, U_Z and U_Y the affine spaces $T \neq 0$, $Z \neq 0$ and $Y \neq$

0 with respective coordinates $(x,y,z) = (\frac{X}{T},\frac{Y}{T},\frac{Z}{T})$, $(x',y',t') = (\frac{X}{Z},\frac{Y}{Z},\frac{T}{Z})$ and $(x'',z'',t'') = (\frac{X}{Y},\frac{Z}{Y},\frac{T}{Y})$. These three open sets U_T, U_Z, U_Y cover V since the point $[1,0,0,0]$ does not belong to V. The defining functions of V are given, respectively, by $f_T(x,y,z) = x^m + y^m + z^m$, $f_Z(x',y',t') = (x')^m + (y')^m + 1$ and $f_Y(x'',z'',t'') = (x'')^m + (z'')^m + 1$. If we denote by $L \longrightarrow \mathbb{CP}^3$ the hyperplane bundle, we have $T(\mathbb{CP}^3) \oplus 1 = 4L$ and $\tilde{N} = L^{\otimes m}$ (see section 1). Hence we have

$$(7.2)\qquad \begin{aligned} &c_1^2(TW) \frown [V] = 16m, && c_2(TW) \frown [V] = 6m,\\ &c_1^2(N_V) \frown [V] = m^3, && \\ &c_1^2(\check{T}V) \frown [V] = m(m-4)^2, && c_2(\check{T}V) \frown [V] = m(m^2-4m+6). \end{aligned}$$

(7.3) Example. We take for $\tilde{X}$, the extension H to the whole $\mathbb{CP}^3$ of the vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ in U_T. In U_Z and U_Y, H is equal, respectively, to $-t'\frac{\partial}{\partial t'}$ and $-t''\frac{\partial}{\partial t''}$. This vector field has for singular set the union of $\{P_1\}$ and the hyperplane $T = 0$, thus Σ has two connected components: $\Sigma_1 = \{P_1\}$ and Σ_2 is the curve $\{X^m + Y^m + Z^m = 0,\ T = 0\}$. Let $\mathcal{F}$ be the dimension one foliation defined by H on U_T, by $\frac{\partial}{\partial t'}$ on U_Z and by $\frac{\partial}{\partial t''}$ on U_Y. Then Σ_2 does not contain any singularity for the foliation $\mathcal{F}$, so that we can already assert that (see Remarks (4.3) 2° and (4.5))

$$\mathrm{Res}_{\Sigma_2}(c_1^2(N_V,H)) = \mathrm{Res}_{\Sigma_2}(c_1^2(N_V,\mathcal{F})) = 0.$$

Now we compute the residues at P_1. If we take the coordinate system $(z_1,z_2,z_3) = (x,y,z)$, $f_T = z_1^2 + z_2^2 + z_3^2$ and the coefficients of the vector field H are given by $(a_1,a_2,a_3) = (z_1,z_2,z_3)$. Thus (a_1,a_2,f_T) is a regular sequence and the conditions (i), (ii)$'$ and (iii) in section 6 are satisfied. We have $\ell = m$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\gamma = m$. Hence we get, from (6.6) - (6.9),

$$\begin{aligned} &\mathrm{Res}_{P_1}(c_1^2(TW|_V,H)) = 9m, && \mathrm{Res}_{P_1}(c_2(TW|_V,H)) = 3m,\\ &\mathrm{Res}_{P_1}(c_1^2(N_V,H)) = \mathrm{Res}_{P_1}(c_1^2(N_V,\mathcal{F})) = m^3, && \\ &\mathrm{Res}_{P_1}(c_1^2(\check{T}V,H)) = m(m-3)^2, && \mathrm{Res}_{P_1}(c_2(\check{T}V,H)) = m(m^2-3m+3). \end{aligned}$$

Note that these are also obtained by setting $n = 3$ and $d_1 = d_2 = d_3 = m$ in Example (7.1). (Multiplying the vector field by a constant, in this case m, does not affect the residues.) We have, from (7.2), Theorems (3.2) and (5.1),

$$\begin{aligned} &\mathrm{Res}_{\Sigma_2}(c_1^2(TW|_V,H)) = 7m, && \mathrm{Res}_{\Sigma_2}(c_2(TW|_V,H)) = 3m,\\ &\mathrm{Res}_{\Sigma_2}(c_1^2(\check{T}V,H)) = m(7-2m), && \mathrm{Res}_{\Sigma_2}(c_2(\check{T}V,H)) = m(3-m). \end{aligned}$$

Note that the values of the residues for $TW|_V$ coincide with the ones computed using Theorem 2 in [B1] (see [LS]).

(7.4) Example. We let $m = 2$ and take for $\tilde{X}$, the extension R to the whole $\mathbb{CP}^3$ of the vector field $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ in U_T. In U_Z and U_Y, R are given, respectively, by $y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}$ and $((x'')^2+1)\frac{\partial}{\partial x''} + x''z''\frac{\partial}{\partial z''} + x''t''\frac{\partial}{\partial t''}$. Now Σ consists of three isolated points: $P_1 = [0,0,0,1]$, $P_2 = [\sqrt{-1},1,0,0]$ and $P_3 = [-\sqrt{-1},1,0,0]$. Note that V is non-singular at P_2 and P_3. We have $R(f_T) = 0$, $R(f_Z) = 0$, and $R(f_Y) = 2x''f_Y$ and see that V is invariant by R.
1) Residues at P_1: If we take the coordinate system $(z_1,z_2,z_3) = (x,y,z)$, $f_T = z_1^2 + z_2^2 + z_3^2$ and the coefficients of the vector field R are given by $(a_1,a_2,a_3) = (z_2,z_1,0)$. Thus (a_1,a_2,f_T) is a regular sequence and the conditions (i), (ii)$'$ and (iii) in section 6 are satisfied. We have $\ell = 2$, $\lambda_1 = \sqrt{-1}$, $\lambda_2 = -\sqrt{-1}$, $\lambda_3 = 0$ and $\gamma = 0$. Hence we get, from (6.6) - (6.9),

$$\begin{aligned}
&\mathrm{Res}_{P_1}(c_1^2(TW|_V,R)) = 0, && \mathrm{Res}_{P_1}(c_2(TW|_V,R)) = 2,\\
&\mathrm{Res}_{P_1}(c_1^2(N_V,R)) = 0, && \\
&\mathrm{Res}_{P_1}(c_1^2(\check{T}V,R)) = 0, && \mathrm{Res}_{P_1}(c_2(\check{T}V,R)) = 2.
\end{aligned}$$

2) Residues at P_2 and P_3: We set $\varepsilon = \sqrt{-1}$ at P_2 and $\varepsilon = -\sqrt{-1}$ at P_3. If we take the coordinate system $(z_1,z_2,z_3) = (z'',t'',f_Y)$, the coefficients of the vector field R are given by $(a_1,a_2,a_3) = (x''z_1, x''z_2, 2x''z_3)$. Since x'' is a unit at P_i, $i = 2,3$, we have $\lambda_1 = \lambda_2 = \varepsilon$, $\lambda_3 = 2\varepsilon$. Hence we get, from (6.10), for $i = 2,3$,

$$\begin{aligned}
&\mathrm{Res}_{P_i}(c_1^2(TW|_V,R)) = 16, && \mathrm{Res}_{P_i}(c_2(TW|_V,R)) = 5,\\
&\mathrm{Res}_{P_i}(c_1^2(N_V,R)) = 4, && \\
&\mathrm{Res}_{P_i}(c_1^2(\check{T}V,R)) = 4, && \mathrm{Res}_{P_i}(c_2(\check{T}V,R)) = 1.
\end{aligned}$$

Note that the sums of the residues give the values of (7.2) for $m = 2$, verifying Theorems (3.2), (4.2) and (5.1).

(7.5) Example. We again let $m = 2$ and take for $\tilde{X}$, the linear combination $X_\omega = aH + bR$ of the vector fields in Examples (7.3) and (7.4), where ω is a real number with $0 < \omega \le \frac{\pi}{2}$, $a = \cos\omega$, $b = \sin\omega$. The singular set of X_ω is the same as that of R and again Σ consists of P_1, P_2 and P_3 as in Example (7.4).
1) Residues at P_1: If we take the coordinate system $(z_1,z_2,z_3) = (x,y,z)$, $f_T = z_1^2 + z_2^2 + z_3^2$ and the coefficients of the vector field X_ω are given by $(a_1,a_2,a_3) = (az_1 + bz_2, az_2 - bz_1, az_3)$. Thus (a_1,a_2,f_T) is a regular sequence and the conditions (i), (ii)$'$ and (iii) in section 6 are satisfied. We have $\ell = 2$, $c_1(\lambda_1,\lambda_2,\lambda_3) = 3a$, $c_2(\lambda_1,\lambda_2,\lambda_3) = 2a^2+1$, $\gamma = 2a$ and $\lambda_1\lambda_2 = 1$. Hence we get, from (6.6) - (6.9),

$$\begin{aligned}
&\mathrm{Res}_{P_1}(c_1^2(TW|_V,X_\omega)) = 18a^2, && \mathrm{Res}_{P_1}(c_2(TW|_V,X_\omega)) = 2(2a^2+1),\\
&\mathrm{Res}_{P_1}(c_1^2(N_V,X_\omega)) = 8a^2, && \\
&\mathrm{Res}_{P_1}(c_1^2(\check{T}V,X_\omega)) = 2a^2, && \mathrm{Res}_{P_1}(c_2(\check{T}V,X_\omega)) = 2.
\end{aligned}$$

2) Residues at P_2 and P_3: Let ε be as in Example (7.4). If we take the coordinate system $(z_1, z_2, z_3) = (z'', t'', f_Y)$, the coefficients of the vector field R are given by $(a_1, a_2, a_3) = (bx''z_1, (bx'' - a)z_2, 2bx''z_3)$. Since x'' is a unit at P_i, $i = 2, 3$, we have $\lambda_1 = b\varepsilon$, $\lambda_2 = -a + b\varepsilon$ and $\lambda_3 = 2b\varepsilon$. Hence we get, from (6.10), for $i = 2, 3$,

$$\mathrm{Res}_{P_i}(c_1^2(TW|_V, X_\omega)) = 9b^2 + 7 - c\varepsilon, \quad \mathrm{Res}_{P_i}(c_2(TW|_V, X_\omega)) = 2b^2 + 3 - 2ab\varepsilon,$$

$$\mathrm{Res}_{P_i}(c_1^2(N_V, X_\omega)) = 4b(b - a\varepsilon),$$

$$\mathrm{Res}_{P_i}(c_1^2(\check{T}V, X_\omega)) = b^2 + 3 + \frac{a^3}{b}\varepsilon, \qquad \mathrm{Res}_{P_i}(c_2(\check{T}V, X_\omega)) = 1,$$

$c = \frac{a(8b^2 - a^2)}{b}$.

Note that we recover the values of Example (7.2) for $\omega = \frac{\pi}{2}$. Note also that the sums of the residues give the values of (7.2) for $m = 2$, verifying Theorems (3.2), (4.2) and (5.1).

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