21st Century COE Program:
Mathematics of Nonlinear Structure via Singularity

COE Symposium
Nonlinear Dispersive Equations

Edited by
T. Ozawa and Y. Tsutsumi

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Dedicated to Professor Masayoshi Tsutsumi
on the occasion of his sixtieth birthday

PREFACE

This volume is intended as the proceedings of COE Symposium "Nonlinear Dispersive Equations," held on the 23rd and 24th of September in 2004 at Sapporo Convention Center.

COE Symposium "Nonlinear Dispersive Equations" is meant for a forum for exchanging views and ideas on the latest developments of mathematical analysis on nonlinear evolution equations related to wave propagation in various nonlinear media with emphasis on dispersive effects.

We would like to thank

- J. Ginibre and J. Shatah for excellent invited lectures.
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We wish to dedicade this volume to Professor Masayoshi Tsutsumi in celebration of his sixtieth birthday.

T. Ozawa and Y. Tsutsumi

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Mathematics of Nonlinear Structure via Singularity

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TSUTSUMI, Yoshio (堤 誉志雄)(Kyoto U.)

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Venue (場所): Small Hall, Sapporo Convention Center
(札幌コンベンションセンター小ホール)
Higashi-Sapporo 6-jo 1-chome, Shiroishi-ku, Sapporo
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Long range scattering for
some Schrödinger related non linear systems

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This lecture is devoted to the theory of scattering in long range cases for some non linear equations and systems based on the Schrödinger equation, and principally for the Wave-Schrödinger system $(\mathrm{WS})_3$ and for the Maxwell-Schrödinger system $(\mathrm{MS})_3$ in space dimension 3. In order to put the subject in perspective, I first list a few equations and systems which can be considered along similar lines.

Linear Schrödinger equation :

$$i\partial_t u = -(1/2)\Delta u + Vu \qquad (\mathrm{LS})_\mathrm{n}$$

Nonlinear Schrödinger equation :

$$i\partial_t u = -(1/2)\Delta u + \kappa |u|^{p-1} u \qquad (\mathrm{NLS})_\mathrm{n}$$

Hartree equation :

$$i\partial_t u = -(1/2)\Delta u + (V \ * \ |u|^2) u \qquad (\mathrm{R3})_\mathrm{n}$$

Klein-Gordon-Schrödinger system :

$$\begin{cases} i\partial_t u = -(1/2)\Delta u + Au \\ (\square + 1)A = -|u|^2 \end{cases} \qquad (\mathrm{KGS})_\mathrm{n}$$

Wave-Schrödinger system :

$$\begin{cases} i\partial_t u = -(1/2)\Delta u + Au \\ \square A = -|u|^2 \end{cases} \qquad (\mathrm{WS})_\mathrm{n}$$

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Zakharov system :

$$\begin{cases} i\partial_t u = -(1/2)\Delta u + Au \\ \\ \Box A = \Delta |u|^2 \end{cases} \qquad (\mathrm{Z})_{\mathrm{n}}$$

Maxwell-Schrödinger system (in space dimension $n = 3$ and in the Coulomb gauge) :

$$\begin{cases} i\partial_t u = -(1/2)\Delta_A u + g(|u|^2)u \\ \\ \Box A = P \ \mathrm{Im} \ \overline{u}\nabla_A u \ , \quad \nabla \cdot A = 0 \ , \end{cases} \qquad (\mathrm{MS})_3$$

with $g(|u|^2) = (4\pi |x|)^{-1} \ * \ |u|^2$.

Here space time is $I\!R^{n+1}$, with n the space dimension, $u : I\!R^{n+1} \to \mathbb{C}$, $V : I\!R^n \to I\!R$ is a given potential, $\kappa \in I\!R$, $p > 1$, $A : I\!R^{n+1} \to I\!R$ except for $(\mathrm{MS})_3$ where $A : I\!R^{3+1} \to I\!R^3$, $\Box = \partial_t^2 - \Delta$ and (for $(\mathrm{MS})_3$) $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$ (covariant Laplacian) and $P = 1\!\!1 - \nabla\Delta^{-1}\nabla$ (projector on divergence free vector fields).

We shall regard scattering theory as a method to study the asymptotic behaviour in time of the global solutions of the above systems and hopefully to classify those solutions by their asymptotic behaviour. The first step is the construction of the wave operators and for that purpose one has to solve the local Cauchy problem at infinity, namely construct solutions with prescribed asymptotic behaviour u_a or (u_a, A_a) parametrized by asymptotic data u_+ or $(u_+, A_+, \dot{A}_+)$. In this lecture I concentrate on that problem. One has to distinguish the short range case, where u_a can be taken as a solution of the free Schrödinger equation

$$u_a = U(t) \ u_+ \equiv \exp(i(t/2)\Delta)u_+ \ , \qquad (1)$$

where "ordinary" wave operators do exist, from the long range case where (1) is inadequate and has to be modified by a suitable phase, thereby leading to "modified" wave operators. In that respect $(\mathrm{LS})_{\mathrm{n}}$ and $(\mathrm{R3})_{\mathrm{n}}$ with $V(x) = |x|^{-\gamma}$ are short (resp. long) range for $\gamma > 1$ (resp. $\gamma \leq 1$), $(\mathrm{NLS})_{\mathrm{n}}$ is short (resp. long) range for $p - 1 > 2/n$ (resp. $p - 1 \leq 2/n$), $(\mathrm{WS})_3$ and $(\mathrm{MS})_3$ are borderline long range (corresponding to $\gamma = 1$), $(\mathrm{Z})_3$ is short range, and $(\mathrm{Z})_2$ and $(\mathrm{KGS})_2$, although short range (no phase needed), exhibit technical difficulties typical of the long range cases. The Cauchy problem at infinity for the previous equations and systems has been treated in the long range cases by two methods, of which I shall consider only the first one, initiated in [O] for $(\mathrm{NLS})_1$ and subsequently applied to $(\mathrm{NLS})_{2,3}$, $(\mathrm{R3})_{\mathrm{n}}$, $(\mathrm{KGS})_2$, $(\mathrm{WS})_3$, $(\mathrm{Z})_3$ and $(\mathrm{MS})_3$ (see the references). That method is intrinsically restricted to small Schrödinger data and to the borderline long range case $\gamma = 1$.

Early applications to the $(KGS)_2$ and $(MS)_3$ systems required in addition a support condition on the asymptotic Schrödinger data to take into account the difference in propagation properties of the Schrödinger equation and of the wave or KG equation, and a smallness condition of the A-field [OT] [T]. The latter restrictions were subsequently removed in recent works [GV] [Sh]. In this lecture (based on [GV]) I shall explain the method on the example of $(WS)_3$ and quote some of the results for other systems, in particular for $(MS)_3$.

We first consider $(WS)_3$. The method proceeds in two steps.

Step 1. One looks for (u, A) in the form $(u, A) = (u_a + v, A_a + B)$. The system satisfied by (v, B) is

$$\begin{cases} i\partial_t v = -(1/2)\Delta v + Av + Bu_a - R_1 \\ \square B = -(|v|^2 + 2 \text{ Re } \overline{u}_a v) - R_2 \end{cases} \tag{2}$$

where the remainders R_1, R_2 are defined by

$$\begin{cases} R_1 = i\partial_t u_a + (1/2)\Delta u_a - A_a u_a \\ R_2 = \square A_a + |u_a|^2 \ . \end{cases} \tag{3}$$

The first step consists in solving the auxiliary system (2) for (v, B) tending to zero at infinity under assumptions on (u_a, A_a) of a general nature, the most important of which being decay assumptions of the remainders. That can be done by a partial linearization of the system (2) followed by a contraction argument in a suitable space $X(I)$ of functions defined in a time interval $I = [T, \infty)$ with T sufficiently large and with sufficient time decay at infinity.

Step 2. The second step consists in constructing approximate solutions (u_a, A_a) of the given system (here $(WS)_3$) satisfying the assumptions needed for Step 1.

We come back to Step 1. The main problem is to choose $X(I)$. We look for $X(I)$ as large as possible, in order to accomodate the largest possible set of asymptotic data, and in particular with the weakest possible time decay, in order to accomodate the simplest possible (and therefore not accurate) asymptotic forms (u_a, A_a). The choice of the regularity of the functions is dictated by the available estimates, namely

Energy (L^2) estimates. From (2) it follows formally that

$$\partial_t \parallel v \parallel_2 \le \parallel Bu_a \parallel_2 \ + \ \parallel R_1 \parallel_2 \ . \qquad (4)$$

Strichartz estimates for the Schrödinger equation ($n = 3$).

Let $0 \le 2/q_i = 3/2 - 3/r_i \le 1$, $i = 1, 2$ and let u satisfy

$$i\partial_t u = -(1/2)\Delta u + f$$

in some interval I with $u(t_0) = u_+$ for some $t_0 \in I$. Then

$$\parallel u; L^{q_1}(I, L^{r_1}) \parallel \le \ C \left(\parallel u_+ \parallel_2 \ + \ \parallel f; L^{\overline{q}_2}(I, L^{\overline{r}_2}) \parallel \right) \qquad (5)$$

with C a constant independent of I, and with $1/p + 1/\overline{p} = 1$.

Strichartz estimates for the Wave equation ($n = 3$, special cases).

Let $\omega = (-\Delta)^{1/2}$ and let A satisfy

$$\square A = f$$

in some interval I with $(A, \partial_t A)(t_0) = (A_+, \dot{A}_+)$ for some $t_0 \in I$. Then

$$\parallel A; L^4(I, L^4) \parallel \le \ C\Big(\parallel \omega^{1/2} A_+ \parallel_2 \ + \ \parallel \omega^{-1/2} \dot{A}_+ \parallel_2 \ + \ \parallel f; L^{4/3}(I, L^{4/3}) \parallel \ , \qquad (6)$$

$$\parallel \nabla A; L^\infty(I, L^2) \parallel \ + \ \parallel \partial_t A; L^\infty(I, L^2) \parallel \le \ C\Big(\parallel \nabla A_+ \parallel_2 \ + \ \parallel \dot{A}_+ \parallel_2$$
$$+ \ \parallel f; L^1(I, L^2) \parallel \Big) \ . \qquad (7)$$

We shall say that solutions of the Schrödinger (resp. Wave) equation with initial data u_+ (resp. $(A_+, \dot{A}_+)$) in H^k (resp. in $\dot{H}^\ell \oplus \dot{H}^{\ell-1}$) are of level k (resp. ℓ). Thus (5) is of level 0 while (6) (7) are of level 1/2 and 1 respectively. The joint level $(k, \ell) = (1, 1)$ is the energy level.

The time decay included in the choice of $X(\cdot)$ will be characterized by a function $h \in \mathcal{C}([1, \infty), I\!R^+)$ such that $\overline{h}(t) \equiv t^\lambda h(t)$ be nonincreasing and tend to zero as $t \to \infty$, for some $\lambda > 0$ to be chosen appropriately.

The simplest (largest simple) choice of $X(I)$ where Step 1 can be performed for $(WS)_3$ is then

$$X(I) = \Big\{ (v, B) : v \in \mathcal{C}(I, L^2), \parallel (v, B); X(I) \parallel \equiv \mathop{\mathrm{Sup}}_{t \in I} \ h(t)^{-1}$$
$$\Big(\parallel v(t) \parallel_2 \ + \ \parallel v; L^{8/3}([t, \infty), L^4) \parallel \ + \ \parallel B; L^4([t, \infty), L^4) \parallel \Big) < \infty \Big\} \qquad (8)$$

with $\lambda = 3/8$ and the result is as follows.

Proposition 1. *Let $X(\cdot)$ be defined by (8) with $\lambda = 3/8$. Let u_a, A_a, R_1, R_2 satisfy the estimates*

$$\| u_a(t) \|_4 \leq \ c_4 \ t^{-3/4}$$

$$\| A_a(t) \|_\infty \leq \ a \ t^{-1}$$

$$\| R_1; L^1([t,\infty), L^2) \| \leq \ r_1 \ h(t)$$

$$\| R_2; L^{4/3}([t,\infty), L^{4/3}) \| \leq \ r_2 \ h(t)$$

for some constants c_4, a, r_1, r_2 with c_4 sufficiently small and for all $t \geq 1$. Then there exists T, $1 \leq T < \infty$, and there exists a unique solution (v, B) of the system (2) in $X([T,\infty))$.

Note that there is an absolute smallness condition on u_a (through c_4), but none other. In particular A_a can be arbitrarily large.

We now turn to Step 2. With the weak ($\lambda = 3/8$) decay allowed by Proposition 1, the simplest choice of (u_a, A_a) will suffice. We recall that

$$U(t) = \exp(i(t/2)\Delta) = M \ D \ F \ M \ ,$$

where

$$M = \exp(ix^2/2t) \ , \ \ D \equiv D(t) = (it)^{-3/2} D_0(t) \ , \ \ (D_0(t)f)(x) = f(x/t) \ ,$$

and F is the Fourier transform. We choose

$$u_a = M \ D \exp(-i\varphi) w_+ \qquad , \quad w_+ = F u_+ \ , \tag{9}$$

$$A_a = A_0 + A_1 \qquad \text{with} \quad \Box A_a = \Box A_1 = -|u_a|^2 \tag{10}$$

namely

$$A_0(t) = \cos \omega t \ A_+ + \omega^{-1} \sin \omega t \ \dot{A}_+ \tag{11}$$

$$A_1(t) = t^{-1} D_0(t) \widetilde{A}_1 \tag{12}$$

$$\widetilde{A}_1 = - \int_1^\infty d\nu \ \nu^{-3} \ \omega^{-1} \sin(\omega(\nu - 1)) D_0(\nu) |w_+|^2 \tag{13}$$

and

$$\varphi = (\ell n \ t) \ \widetilde{A}_1 \ . \tag{14}$$

With that choice, $R_2 = 0$ and

$$R_1 = (2t^2)^{-1} M \; D \, \Delta \exp(-i\varphi) w_+ - A_0 \; u_a \; . \tag{15}$$

One can then state the final result as follows.

Proposition 2. *Let $h(t) = t^{-1/2}$. Let (u_a, A_a) be defined as above. Let $w_+ = Fu_+ \in H^2$ with $c_4 = \| w_+ \|_4$ sufficiently small. Let A_+, $\omega^{-1}\dot{A}_+ \in L^2$ and $\nabla^2 A_+$, $\nabla \dot{A}_+ \in L^1$. Then there exists T, $1 \leq T < \infty$ and there exists a unique solution (u, A) of the system (WS)$_3$ such that $(v, B) = (u - u_a, A - A_a) \in X([T, \infty))$.*

We have treated the (WS)$_3$ system at the level $(k, \ell) = (0, 1/2)$ for (v, B). It is easy to treat the same system at higher levels of regularity, for instance at the energy level $(k, \ell) = (1, 1)$. As a slightly more regular example, we now state the result at the level $(k, \ell) = (2, 1)$. It seems to be a general feature of the method that no additional smallness condition is needed beyond that of Propositions 1 and 2, and that the time decay (namely $\lambda = 3/8$ in Proposition 1) remains the same. We use the notation W_r^k for the usual Sobolev spaces (derivatives of order up to k in L^r) and $H^{k,s}$ for the weighted spaces

$$H^{k,s} = \left\{ v : \| v; H^{k,s} \| = \| < x >^s < \omega >^k v \|_2 < \infty \right\}$$

where $< \cdot >= (1 + |\cdot|^2)^{1/2}$. The relevant function space can be taken as

$$\begin{aligned} & X_2(I) = \Big\{ (v, B) : v \in \mathcal{C}(I, H^2) \cap \mathcal{C}^1(I, L^2), \nabla B, \partial_t B \in \mathcal{C}(I, L^2), \\ & \| (v, B); X_2(I) \| = \mathop{\rm Sup}_{t \in I} \ h(t)^{-1} \Big(\| v(t); H^2 \| \ + \ \| \partial_t v(t) \|_2 \\ & + \ \| v; L^{8/3}([t, \infty), W_4^2) \| \ + \ \| \partial_t v; L^{8/3}([t, \infty), L^4) \| \ + \ \| B; L^4([t, \infty), L^4) \| \\ & + \ \| \nabla B(t) \|_2 \ + \ \| \partial_t B(t) \|_2 \Big) < \infty \Big\} \ . \end{aligned} \tag{16}$$

The result of Step 1 takes the following form :

Proposition 3. *Let $X_2(\cdot)$ be defined by (16) with $\lambda = 3/8$. Let u_a, A_a, R_1, R_2 satisfy the assumptions of Proposition 1 and in addition*

$$\| u_a(t) \|_\infty \leq \ c \ t^{-3/2} \quad , \quad \| \partial_t u_a(t) \|_4 \leq \ c \ t^{-3/4}$$

$$\| \partial_t A_a \|_\infty \leq \ a \ t^{-1}$$

$$\| \partial_t R_1 ; L^1([t,\infty), L^2) \| \le \ r_1 \ h(t)$$
$$\| R_1 ; L^{8/3}([t,\infty), L^4) \| \le \ r_1 \ h(t)$$
$$\| R_2 ; L^1([t,\infty), L^2) \| \le \ r_2 \ t^{-1/2} \ h(t)$$

for some constants c_4, c, a, r_1, r_2 with c_4 sufficiently small and for all $t \ge 1$. Then there exists T, $1 \le T < \infty$ and there exists a unique solution (v, B) of the system (2) in $X_2([T,\infty))$. Furthermore B satisfies the estimate

$$\| \nabla B(t) \|_2 \ \vee \ \| \partial_t B(t) \|_2 \le \ C \left(t^{-1/2} + t^{1/4} \ h(t) \right) h(t)$$

for all $t \ge T$.

Using the same (u_a, A_a) as before, we obtain the following final result.

Proposition 4. *Let $h(t) = t^{-1/2}$. Let (u_a, A_a) be defined by (9)-(14). Let $u_+ \in H^{1,3} \cap H^{2,2}$ with $c_4 = \| Fu_+ \|_4$ sufficiently small. Let A_+, $\omega^{-1}A_+ \in H^1$ and $\nabla^2 A_+$, $\nabla \dot{A}_+ \in W_1^1$. Then there exists T, $1 \le T < \infty$, and there exists a unique solution (u, A) of the system $(WS)_3$ such that $(v, B) = (u - u_a, A - A_a) \in X_2([T,\infty))$. Furthermore B satisfies the estimate*

$$\| \nabla B(t) \|_2 \ \vee \ \| \partial_t B(t) \|_2 \le \ C \ t^{-3/4}$$

for all $t \ge T$.

Remark. By using the correction of the asymptotic u_a given in [Sh 1], one can obtain a faster rate of convergence, namely $h(t) = t^{-1}(2 + \ell n \ t)^2$ on a subset of more regular and decaying asymptotic data.

We now turn to the $(MS)_3$ system. The method is exactly the same and the results are similar. In particular Step 1 requires the same time decay, namely $\lambda = 3/8$. We only state the results. The auxiliary system satisfied by (v, B) is now

$$\begin{cases} i\partial_t v = -(1/2)\Delta_A v + g(|u|^2)v + G_1 - R_1 \\ \square B = G_2 - R_2 \end{cases} \tag{17}$$

where G_1 and G_2 are defined by

$$\begin{cases} G_1 = iB \cdot \nabla_{A_a} u_a + (1/2)B^2 u_a + g\left(|v|^2 + 2 \ \mathrm{Re} \ \overline{u}_a v\right) u_a \\ G_2 = P \ \mathrm{Im}\left(\overline{v}\nabla_A v + 2\overline{v}\nabla_A u_a\right) - P \ B|u_a|^2 \end{cases} \tag{18}$$

and the remainders are defined by

$$\left\{ \begin{array}{l} R_1 = i\partial_t u_a + (1/2)\Delta_{A_a} u_a - g\left(|u_a|^2\right) u_a \\ \\ R_2 = \square A_a - P \ \mathrm{Im} \ \overline{u}_a \nabla_{A_a} u_a \ . \end{array} \right. \tag{19}$$

The relevant function space can be taken as

$$\begin{aligned} & X(I) = \Big\{ (v, B) : v \in \mathcal{C}(I, H^2) \cap \mathcal{C}^1(I, L^2), \\ & \| \ (v, B); X(I) \ \| \equiv \mathop{\mathrm{Sup}}_{t \in I} \ h(t)^{-1} \Big(\ \| \ v(t); H^2 \ \| \ + \ \| \ \partial_t v(t) \ \|_2 \\ & + \ \| \ v; L^{8/3}([t, \infty), W_4^1) \ \| \ + \ \| \ B; L^4([t, \infty), W_4^1) \ \| \\ & + \ \| \ \partial_t B; L^4([t, \infty), L^4) \ \| \ \Big) < \infty \Big\} \ . \end{aligned} \tag{20}$$

The result of Step 1 takes the following form.

Proposition 5. *Let $X(\cdot)$ be defined by (20) with $\lambda = 3/8$. Let u_a, A_a, R_1 and R_2 satisfy the estimates*

$$\| \ \partial_t^j \nabla^k u_a(t) \ \|_r \leq \ c \ t^{-(3/2 - 3/r)} \qquad \textit{for } 2 \leq r \leq \infty$$

and in particular

$$\| \ u_a \ \|_3 \leq c_3 \ t^{-1/2} \quad , \quad \| \ \nabla u_a \ \|_4 \leq c_4 \ t^{-3/4} \ ,$$

$$\| \ \nabla^2 u_a(t) \ \|_4 \ \vee \ \| \ \partial_t \nabla u_a(t) \ \|_4 \leq c \ t^{-3/4} \ ,$$

$$\| \ \partial_t^j \nabla^k A_a(t) \ \|_\infty \leq a \ t^{-1} \ ,$$

$$\| \ \partial_t^j \nabla^k R_1; L^1([t, \infty), L^2) \ \| \leq r_1 \ h(t) \ ,$$

$$\| \ R_2; L^{4/3}([t, \infty), W_{4/3}^1) \ \| \leq r_2 \ h(t) \ ,$$

for $0 \leq j + k \leq 1$, for some constants c, c_3, c_4, a, r_1 and r_2 with c_3 and c_4 sufficiently small and for all $t \geq 1$. Then there exists T, $1 \leq T < \infty$ and there exists a unique solution (v, B) of the system (17) in $X([T, \infty))$. If in addition

$$\| \ R_2; L^1([T, \infty), L^2) \ \| \leq r_2 \ t^{-1/2} \ h(t) \ ,$$

then ∇B, $\partial_t B \in \mathcal{C}([T, \infty), L^2)$ and B satisfies the estimate

$$\| \ \nabla B(t) \ \|_2 \ \vee \ \| \ \partial_t B(t) \ \|_2 \leq C \left(t^{-1/2} + t^{1/4} \ h(t) \right) h(t)$$

for some constant C and for all $t \geq T$.

With the weak time decay ($\lambda = 3/8$) allowed by Proposition 5, the simplest choice of (u_a, A_a) is again sufficient. Thus we define again u_a by (9) and $A_a = A_0 + A_1$ with A_0 and A_1 defined by (11) (12) where now however $\widetilde{A}_1$ is given by

$$\widetilde{A}_1 = \int_1^\infty d\nu \ \nu^{-3} \ \omega^{-1} \sin(\omega(\nu - 1)) D_0(\nu) P \ x|w_+|^2 \ , \tag{21}$$

and we take

$$\varphi = (\ell n \ t) \left(g(|w_+|^2) - x \cdot \widetilde{A}_1 \right) \ . \tag{22}$$

The final result can then be stated as follows.

Proposition 6. *Let $h(t) = t^{-1}(2 + \ell n \ t)^2$. Let (u_a, A_a) be defined as above. Let $u_+ \in H^{3,1} \cap H^{1,3}$ with $\| xw_+ \|_4$ and $\| w_+ \|_3$ sufficiently small, where $w_+ = Fu_+$. Let $\nabla^2 A_+$, $\nabla \dot{A}_+$, $\nabla^2(x \cdot A_+)$ and $\nabla(x \cdot \dot{A}_+) \in W_1^1$ with A_+, $x \cdot A_+ \in L^3$ and $\dot{A}_+$, $x \cdot \dot{A}_+ \in L^{3/2}$ and let $\nabla \cdot A_+ = \nabla \cdot \dot{A}_+ = 0$.*

Then there exists T, $1 \leq T < \infty$ and there exists a unique solution (u, A) of the system (MS)$_3$ such that $(v, B) = (u - u_a, A - A_a) \in X([T, \infty))$. Furthermore ∇B, $\partial_t B \in \mathcal{C}([T, \infty), L^2)$ and B satisfies the estimate

$$\| \nabla B(t) \|_2 \ \vee \ \| \partial_t B(t) \|_2 \leq C \ t^{-3/2} (2 + \ell n \ t)^2$$

for some constant C depending on $(u_+, A_+, \dot{A}_+)$ and for all $t \geq T$.

Remark. The only smallness conditions bear on $\| xw_+ \|_4$ and on $\| w_+ \|_3$ and are required by the magnetic interaction and the Hartree interaction respectively. In particular there is no smallness condition on $(A_+, \dot{A}_+)$.

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MODIFIED WAVE OPERATORS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH STARK EFFECTS

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1. Introduction

We study the global existence and large time behavior of solutions for the nonlinear Schrödinger equation with the Stark effect in one or two space dimensions:

$$i\partial_t u = -\frac{1}{2}\Delta u + (E\cdot x)u + \widetilde{F}_n(u), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n, \tag{1.1}$$

where $n = 1, 2$ and u is a complex valued unknown function of (t, x). Here $\widetilde{F}_n(u)$ and $E\cdot x$ are a nonlinearity and a linear potential, respectively. The nonlinearity is

$$\begin{aligned} \widetilde{F}_n(u) &= G_n(u) + \widetilde{N}_n(u),\\ G_n(u) &= \lambda_0|u|^{2/n}u,\\ \widetilde{N}_1(u) &= \lambda_1 u^3 + \lambda_2\bar{u}^3, \quad \text{when } n=1,\\ \widetilde{N}_2(u) &= \lambda_1 u^2 + \lambda_2\bar{u}^2 + \lambda_3 u\bar{u}, \quad \text{when } n=2, \end{aligned} \tag{1.2}$$

where $\lambda_0\in\mathbb{R}$, $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$ and $E\in\mathbb{R}^n\setminus\{0\}$. We remark that the cubic nonlinearity $u\bar{u}^2$ is excluded in one dimensional case. $\widetilde{F}_n$ is a summation of the gauge invariant nonlinearity $G_n(u)$ and the non-gauge invariant one $\widetilde{N}_n(u)$, and it is a critical power nonlinearity between the short range case and the long range one in n space dimensions ($n = 1, 2$). The above potential $E\cdot x$ is called the Stark potential with a constant electric field E. In this talk, we prove the existence of modified wave operators to the equation (1.1) for small final states.

Let $U(t)$ be the free Schrödinger group, that is,

$$U(t) = e^{it\Delta/2}.$$

The Schrödinger operator $-(1/2)\Delta + E\cdot x$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. H_E denotes the self-adjoint realization of that operator

This is a joint work with Satoshi Tonegawa (Nihon University).

defined on $C_0^\infty(\mathbb{R}^n)$ and we define the unitary group U_E generated by H_E:

$$U_E(t) = e^{-itH_E}.$$

$\widetilde{F}_n(u)$ is a critical power nonlinearity between the short range scattering and the long range one. The modified wave operator $\widetilde{W}_+$ for the equation (1.1) is defined as follows. Let ϕ be a final state. Modifying the solution $U_E(t)\phi$ for the linear Schrödinger equation with the Stark potential, we construct a suitable modified free dynamics A, which depends on ϕ, and we show the existence of a unique solution u for the equation (1.1) which approaches A in L^2 as $t \to \infty$. The mapping

$$\widetilde{W}_+ : \phi \mapsto u(0)$$

is called a modified wave operator. In this talk, we prove the existence of modified wave operators for the equation (1.1).

The theory of scattering for the ordinary nonlinear Schrödinger equations with critical power nonlinearities was studied, e.g., in [3, 4, 5, 6, 7, 8].

Before stating our main results, we introduce several notations.

Notation. We denote the Schwartz space on $\mathbb{R}^n$ by $\mathcal{S}$. Let $\mathcal{S}'$ be the set of tempered distributions on $\mathbb{R}^n$. For $w \in \mathcal{S}'$, we denote the Fourier transform of w by $\hat{w}$. For $w \in L^1(\mathbb{R}^n)$, $\hat{w}$ is represented as

$$\hat{w}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(x) e^{-ix\cdot\xi}\, dx.$$

For $s, m \in \mathbb{R}$, we introduce the weighted Sobolev spaces $H^{s,m}$ corresponding to the Lebesgue space L^2 as follows:

$$H^{s,m} \equiv \{\psi \in \mathcal{S}' \colon \|\psi\|_{H^{s,m}} \equiv \|(1+|x|^2)^{m/2}(1-\Delta)^{s/2}\psi\|_{L^2} < \infty\}.$$

Our result is as follows.

Theorem 1.1. *Assume that $\phi \in H^2 \cap H^{0,2}$ and that $\|\phi\|_{H^2\cap H^{0,2}}$ is sufficiently small. Then the equation* (1.1) *has a unique solution u for satisfying*

$$u \in C([0,\infty); L^2),$$

$$\sup_{t\geq 1}(t^d \|u(t) - U_E(t) e^{-i|\cdot|^2/2t} e^{-iS(t,-i\nabla)}\phi\|_{L^2}) < \infty,$$

$$\sup_{t\geq 1}\left[t^d \left(\int_t^\infty \|U(s)(U_E(-s)u(s) - e^{-i|\cdot|^2/2s} e^{-iS(s,-i\nabla)}\phi)\|_{Y_n}^4 \, ds \right)^{1/4} \right] < \infty,$$

where

$$S(t,x) = \lambda_0 |\hat{\phi}(x)|^{2/n} \log t,$$

d is a constant satisfying $n/4 < d < 1$, $Y_1 = L^\infty_x$ and $Y_2 = L^4_x$. Furthermore the modified wave operator

$$\widetilde{W}_+ \colon \phi \mapsto u(0)$$

is well-defined.

A similar result holds for negative time.

Remark 1.1. Since the multiplication operator $e^{-i|\cdot|^2/2t}$ appraoches the identity in L^2 as $t \to \infty$, the solution obtained in Theorem 1.1 approaches $U_E(t)e^{-iS(t,-i\nabla)}\phi$ in L^2. Noting the phase correction S depends only on the gauge invariant nonlinearity $G_n(u)$, we see that the contribution of the non-gauge invariant term $\widetilde{N}_n(u)$ is a short range interaction. that is, it is negligible as $t \to \infty$, under our assumptions. We also note that the assumption $\phi \in H^2$ is needed only if $\widetilde{N}_n(u) \neq 0$.

Remark 1.2. If we consider the asymptotic behavior of solutions to the Cauchy problem for equation (1.1) with initial data $u(0, x) = \phi_0(x)$, $x \in \mathbb{R}^n$, then we see from Theorem 1.1 that for any initial data ϕ_0 belonging to the range of the modified wave operator $\widetilde{W}_+$, there exists a unique global solution $u \in C([0,\infty); L^2)$ of the Cauchy problem for equation (1.1) which has the modified free profile $U_E(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi$. More precisely, u satisfies the asymptotic formula of Theorem 1.1. However it is not clear how to describe the initial data belonging to the range of the operator $\widetilde{W}_+$.

2. The strategy of the proof

The idea of the proof of Theorem 1.1 is as follows. We reduce our problem to the following non-autonomous nonlinear Schrödinger equation without a potential

$$i\partial_t v = -\frac{1}{2}\Delta v + F_n(t, v), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{2.1}$$

where $n = 1, 2$,

$$\begin{aligned} F_n(t, v) &= G_n(v) + N_n(t, v), \\ N_1(t, v) &= \lambda_1 v^3 e^{-2i(tE\cdot x - t^3|E|^2/3)} + \lambda_3 \bar{v}^3 e^{4i(tE\cdot x - t^3|E|^2/3)}, \\ N_2(t, v) &= \lambda_1 v^2 e^{-i(tE\cdot x - t^3|E|^2/3)} + \lambda_2 \bar{v}^2 e^{3i(tE\cdot x - t^3|E|^2/3)} \\ &\quad + \lambda_3 v\bar{v} e^{i(tE\cdot x - t^3|E|^2/3)}, \end{aligned}$$

$G_n(v)$ is defined by (1.2). By the change of variables

$$v(t, x) = u\left(t, x - \frac{t^2}{2}E\right) e^{i(tE\cdot x - t^3|E|^2/3)},$$

our problem is equivalent to constructing modified wave operators for the equation (2.1) (see, e.g., Cycon-Froese-Kirsch-Simon [2]). In order to overcome difficulty caused by the gauge invariant nonlinearity

$G_n(v)$ which is a long range interaction (see Barab [1]), we introduce a modified free dynamics of the form

$$\begin{aligned} v_a(t,x) =& (U(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\phi)(x) \\ =& \frac{1}{(it)^{n/2}}\hat{\phi}\left(\frac{x}{t}\right)e^{i|x|^2/2t-iS(t,x/t)}, \end{aligned}$$

with the phase shift $S(t,x) = \lambda_0|\hat{\phi}(x)|^{2/n}\log t$ so that $\mathcal{L}v_a - G_n(v_a)$ decays faster than $G(v_a)$, where $\mathcal{L} = i\partial_t + (1/2)\Delta$. This modified free dynamics v_a was introduced by Ozawa [7] for the ordinary nonlinear Schrödinger equation with a nonlinearity $\lambda|u|^2u$ in one space dimension. In order to treat the non-gauge invariant nonlinearity $N_n(t,v)$, we show that

$$\left\| \int_t^\infty U(t-s)N_n(s,v_a(s))\,ds \right\|_{L^2_x},$$

which appears in the associate integral equation, is integrable over the interval $[1,\infty)$. More precisely, it decays suitably in time. Hence we see that

$$\left\| \int_t^\infty U(t-s)(\mathcal{L}v_a(s) - F_n(s,v_a(s)))\,ds \right\|_{L^2_x}$$

decays suitably in time and we can directly construct a unique solution u which approaches the asymptotic profile v_a.

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1

GLOBAL SOLUTIONS FOR QUASILINEAR WAVE EQUATIONS IN THREE SPACE DIMENSIONS

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1. **Introduction.**

The goal of this paper is to prove global existence of solutions to quadratic quasilinear Dirichlet-wave equations exterior to a class of compact obstacles. As in Metcalfe-Sogge [23], the main condition that we require for our class of obstacles is exponential local energy decay. Our result improves upon the earlier one of Metcalfe-Sogge [23] by allowing a more general null condition which only puts restrictions on the self-interaction of each wave family. In Minkowski space, such equations were studied and shown to have global solutions by Sideris-Tu [30], Agemi-Yokoyama [1], and Kubota-Yokoyama [18].

We use Klainerman's commuting vector fields method [16]:

$$\partial_0 = \partial_t, \quad \Omega_{ij} = x_i\partial_j - x_j\partial_i, \quad 1 \le i \ne j \le 3, \quad L = t\partial_t + \sum_{1\le j\le 3} x_j\partial_j.$$

L is called the scaling operator. We denote $\{\partial_j\}_{0\le j\le 3}$ by ∂, $\{\Omega_{ij}\}_{1\le i\ne j\le 3}$ by Ω, $\{\partial, \Omega\}$ by Z, and $\{L, Z\}$ by Γ. For functions u, u' denotes ∂u. These operators have the commuting relations with d'Alembertian $\Box$:

$$\Box\Omega_{ij} = \Omega_{ij}\Box, \qquad \Box L = (L+2)\Box, \qquad L\Omega_{ij} = \Omega_{ij}L, \qquad \partial_j L = (L+1)\partial_j. \tag{1.1}$$

Using Z, we can earn one weight by Klainerman-Sobolev inequality :

Lemma 1.1. [16] [13, Lemma 2.4] [28, Lemma 3.3] *Suppose that $h \in C^\infty(\mathbb{R}^3)$. Then, for $R > 2$,*

$$\|h\|_{L^\infty(R<|x|<R+1)} \le CR^{-1} \sum_{|\alpha|+|\beta|\le 2} \|\Omega^\alpha \partial_x^\beta h\|_{L^2(R-1<|x|<R+2)}. \tag{1.2}$$

We describe our assumptions on our obstacles $\mathcal{K} \subset \mathbb{R}^3$. We shall assume that $\mathcal{K}$ is smooth and compact, but not necessarily connected. By scaling, without loss of generality, we may assume

$$\mathcal{K} \subset \{\, x \in \mathbb{R}^3 \,:\, |x| < 1\},\ 0 \in \mathcal{K}\backslash\partial\mathcal{K}.$$

The only additional assumption states that there is exponential local energy decay with a possible loss of regularity. That is, if u is a solution to

$$\begin{cases} \Box u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3\backslash\mathcal{K} \\ u(t,\cdot)|_{\partial\mathcal{K}} = 0 \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g, & \operatorname{supp} f \cup \operatorname{supp} g \subset \{\mathbb{R}^3\backslash\mathcal{K}, |x| \le 4\}, \end{cases} \tag{1.3}$$

[1] A personal note on the joint work with Jason Metcalfe and Christopher D. Sogge [22]

then there must be constants $c, C > 0$ so that

$$\|u'(t,\cdot)\|_{L^2(x\in\mathbb{R}^3\backslash\mathcal{K},|x|\le 4)} \le Ce^{-ct}\sum_{|\alpha|\le 1}\|\partial_x^\alpha u'(0,\cdot)\|_2. \tag{1.4}$$

Throughout this paper, we assume this local energy decay estimate for $\mathcal{K}$.

Lax, Morawetz and Phillips have shown (1.4) without a loss of regularity, namely $|\alpha| = 0$ in the RHS, when $\mathcal{K}$ is star-shaped in [19] (see also [20, Theorem 3.2]).

Morawetz, Ralston and Strauss have shown (1.4) without a loss of regularity ($|\alpha| = 0$) when $\mathcal{K}$ is bounded connected and nontrapping in [25, (3.1)]. Here if the lengths of all rays in $B_1(0)\backslash\mathcal{K}$ are bounded, then waves are not trapped and (1.4) holds without a loss of regularity. They also treat the multi-dimensional cases. See Melrose [21] for further results. Ralston [26] has shown that (1.4) could not hold without a loss of regularity when there are trapped rays..

Ikawa has shown (1.4) with an additional loss of regularity, namely $|\alpha| \le \ell$ with $\ell \ge 1$ in the RHS, when $\mathcal{K}$ is trapping. He has shown (1.4) with $\ell = 6$ when $\mathcal{K}$ consists of two disjoint strictly convex bodies in [9], and (1.4) with $\ell = 2$ when $\mathcal{K}$ consists of sufficiently separated several disjoint strictly convex bodies in [10]. Since we have the standard energy preservation

$$\|u'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} = \|u'(0,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}$$

(see (3.3) with $\gamma = 0$), we can reduce the estimate (1.4) with an additional regularity, $\ell \ge 1$, to the estimate for $\ell = 1$ with different constants c and C by the interpolation. Therefore we can treat the above obstacles by the condition (1.4).

We note that we do not require exponential decay; in fact, $O((1+t)^{-1-\delta-m})$ with $\delta > 0$ and $m \ge 0$ may be sufficient with a tighter argument, where we need $1+\delta$ for the integral ability and m is the number of L we need in our argument (see the argument below (4.4) to bound $t^\mu e^{-ct/2}$). Currently, the authors are not aware of any 3-dimensional example that involves polynomial decay, but does not have exponential decay.

We consider quadratic, quasilinear systems of the form

$$\begin{cases} \Box u = F(\partial u, \partial^2 u),\ (t,x)\in\mathbb{R}_+\times\mathbb{R}^3\backslash\mathcal{K} \\ u(t,\,\cdot\,)|_{\partial\mathcal{K}} = 0 \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g. \end{cases} \tag{1.5}$$

Here $\Box$ denotes a vector-valued multiple speed d'Alembertian :

$$\Box u = (\Box_{c_1}u^1, \Box_{c_2}u^2, \dots, \Box_{c_D}u^D), \quad F = (F^1, \cdots, F^D), \quad D \ge 1, \tag{1.6}$$

where

$$\Box_{c_I} = \partial_t^2 - c_I^2\Delta, \quad 1 \le I \le D.$$

We assume that the wave speeds c_I are positive and distinct:

$$0 < c_1 < \cdots < c_D.$$

Straightforward modifications of the argument give the more general case where the various components are allowed to have the same speed.

We shall assume that $F(\partial u, \partial^2 u)$ is of the form

$$F^I(\partial u, \partial^2 u) = \sum_{\substack{1\le J,K\le D\\ 0\le j,k\le 3}} A^{IJK}_{jk}\partial_j u^J \partial_k u^K + \sum_{\substack{0\le j,k,l\le 3\\ 1\le J,K\le D}} B^{IJK}_{jkl}\partial_j u^J \partial_k\partial_l u^K, \quad 1\le I\le D. \tag{1.7}$$

For the energy estimates, we require the symmetry condition:

$$B^{IJK}_{jkl} = B^{KJI}_{jkl} = B^{IJK}_{jlk}.$$

To obtain global existence, we also require that the equations satisfy the following null condition which only involves the self-interactions of each wave family :

$$\sum_{0\le j,k\le 3} A^{II}_{jk}\xi_j\xi_k = 0 \quad \text{whenever} \quad \xi_0^2 = c_I^2(\xi_1^2+\xi_2^2+\xi_3^2), \quad I = 1,\dots,D, \tag{1.8}$$

$$\sum_{0\le j,k,l\le 3} B^{III}_{jkl}\xi_j\xi_k\xi_l = 0 \quad \text{whenever} \quad \xi_0^2 = c_I^2(\xi_1^2+\xi_2^2+\xi_3^2), \quad I = 1,\dots,D. \tag{1.9}$$

The terms which satisfy the above null conditions are treated by the following estimates :

Lemma 1.2. [30, 33] *If the semilinear null condition* (1.8) *holds, then*

$$\Big|\sum_{0\le j,k\le 3} A^{II}_{jk}\partial_j u\partial_k v\Big| \le C\frac{|\Gamma u||\partial v| + |\partial u||\Gamma v|}{\langle r\rangle} + C\frac{\langle c_I t - r\rangle}{\langle t+r\rangle}|\partial u||\partial v|. \tag{1.10}$$

Suppose that the quasilinear null condition (1.9) *holds. Then,*

$$\Big|\sum_{0\le j,k,l\le 3} B^{III}_{jkl}\partial_l u\partial_j\partial_k v\Big| \le C\frac{|\Gamma u||\partial^2 v| + |\partial u||\partial\Gamma v|}{\langle r\rangle} + C\frac{\langle c_I t - r\rangle}{\langle t+r\rangle}|\partial u||\partial^2 v|. \tag{1.11}$$

We refer to compatibility conditions. For the solution u of (1.5), the functions $\{\partial_t^j u(0,x)\}_{j\ge 0}$ are called compatible functions. The compatible functions are functions of spatial variables and $\partial_t^j u(0,x)$ are expressed by $\{\partial_x^\alpha f\}_{|\alpha|\le j}$ and $\{\partial_x^\alpha g\}_{|\alpha|\le j-1}$. We say that the compatibility conditions of order s are satisfied if $\partial_t^j u(0,x)|_{\partial\mathcal{K}} = 0$ for all $0\le j\le s$ (See [12, Definition 9.2]). Additionally, we say that $(f,g)\in C^\infty$ satisfies the compatibility conditions to infinite order if the compatibility conditions are satisfied to any order $s\ge 0$.

We can now state our main result:

Theorem 1.3. *Let $\mathcal{K}$ be a fixed compact obstacle with smooth boundary that satisfies* (1.4). *Assume that $F(\partial u, \partial^2 u)$ and $\Box$ are as above and that $(f,g)\in C^\infty(\mathbb{R}^3\backslash\mathcal{K})$ satisfy the compatibility conditions to infinite order. Then there is a constant $\varepsilon_0 > 0$, and an integer $N > 0$ so that for all $\varepsilon < \varepsilon_0$, if*

$$\sum_{|\alpha|\le N} \|\langle x\rangle^{|\alpha|}\partial_x^\alpha f\|_2 + \sum_{|\alpha|\le N-1} \|\langle x\rangle^{1+|\alpha|}\partial_x^\alpha g\|_2 \le \varepsilon \tag{1.12}$$

then (1.5) *has a unique solution $u\in C^\infty([0,\infty)\times\mathbb{R}^3\backslash\mathcal{K})$.*

This paper is organized as follows. In the next section, we will collect some preliminary results which are frequently used in this paper. We put several sections for energy estimates, L^2 estimates in space and time, and Sobolev embeddings, respectively.

2. Preliminaries.

We use the following Poincaré inequalities to bound u by u' near the obstacle:

$$\|u\|_{L^2(\mathbb{R}^3\backslash\mathcal{K},|x|<R)} \le C_R \|\nabla u\|_{L^2(\mathbb{R}^3\backslash\mathcal{K},|x|<R)} \quad \text{if} \quad u|_{\partial\mathcal{K}} = 0, \tag{2.1}$$

where C_R is a constant dependent on $R \ge 1$ (cf. [4, (7.44)]).

We also use the following elliptic regularity : for any fixed $M \ge 0$

$$\begin{aligned} \sum_{2\le|\alpha|\le M+2} \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^3\backslash\mathcal{K},|x|<R)} \le C_R \Big(&\sum_{|\alpha|\le M} \|\partial_x^\alpha \nabla u\|_{L^2(\mathbb{R}^3\backslash\mathcal{K},|x|<R+1)} \\ &+ \sum_{|\alpha|\le M} \|\partial_x^\alpha \Delta u\|_{L^2(\mathbb{R}^3\backslash\mathcal{K},|x|<R+1)} \Big) \end{aligned} \tag{2.2}$$

if $u|_{\partial\mathcal{K}} = 0$ (cf. [4, Theorem 8.13]).

Here we briefly sketch the elementary method to treat the nonlinearity.

Lemma 2.1. *Let $u \in C^\infty((0,\infty)\times\mathbb{R}^3\backslash\mathcal{K})$. Suppose u has the bound*

$$\sum_{|\alpha|\le M_0} \|Z^\alpha u'(t,x)\|_{L^\infty_x} \le \frac{C_0\varepsilon}{1+t} \tag{2.3}$$

for some constants $M_0 \ge 0$ and $C_0 \ge 0$. Then for any $M \ge 0$ and $\mu_0 \ge 0$, there exists a constant C such that we have

$$\begin{aligned} \sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} \|L^\mu \partial^\alpha (u')^2(t)\|_{L^2_x} \le{}& \frac{C_0\varepsilon}{1+t} \sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} \|L^\mu \partial^\alpha u'(t)\|_{L^2_x} \\ &+ C \sum_{M_0+1\le|\alpha|\le M-M_0+1} \|\langle x\rangle^{-1/2} Z^\alpha u'(t)\|_{L^2_x} \sum_{M_0+1\le|\alpha|\le M-M_0-1} \|\langle x\rangle^{-1/2} \partial^\alpha u'(t)\|_{L^2_x} \\ &+ C \sum_{\substack{\mu+|\alpha|\le M-M_0+1\\ 1\le\mu\le\mu_0}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u'(t)\|_{L^2_x} \sum_{|\alpha|\le M-1} \|\langle x\rangle^{-1/2} \partial^\beta u'(t)\|_{L^2_x} \\ &+ C \sum_{\substack{\mu+|\alpha|\le M/2+2\\ 1\le\mu\le\mu_0-1}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u'(t)\|_{L^2_x} \sum_{\substack{\mu+|\alpha|\le M-1\\ 1\le\mu\le\mu_0-1}} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u'(t)\|_{L^2_x}. \end{aligned} \tag{2.4}$$

Here ∂ can be replaced by Z in the above inequality.

Proof of Lemma 2.1 : We use the following estimates:

$$
\begin{aligned}
\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} \|L^\mu\partial^\alpha(u')^2\|_2 &\lesssim \sum_{\substack{\mu+|\alpha|+\nu+|\beta|\le M\\ \mu+\nu\le\mu_0}} \|L^\mu\partial^\alpha u' L^\nu\partial^\beta u'\|_2 \\
&\lesssim \sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} \|L^\mu\partial^\alpha u'\|_2 \sum_{|\beta|\le M_0} \|\partial^\beta u'\|_\infty + \sum_{\substack{M_0+1\le|\alpha|\le M-M_0-1\\ M_0+1\le|\beta|\le M-M_0-1}} \|\partial^\alpha u'\partial^\beta u'\|_2 \qquad (2.5)\\
&\quad + \sum_{\substack{\mu+|\alpha|\le M-M_0-1\\ 1\le\mu\le\mu_0}} \sum_{M_0+1\le|\beta|\le M-1} \|L^\mu\partial^\alpha u'\partial^\beta u'\|_2 \\
&\quad + \sum_{\substack{\mu+|\alpha|\le M/2\\ 1\le\mu\le\mu_0-1}} \sum_{\substack{\nu+|\beta|\le M-1\\ 1\le\nu\le\mu_0-1}} \|L^\mu\partial^\alpha u' L^\nu\partial^\beta u'\|_2.
\end{aligned}
$$

Since we have by (1.2)

$$
\begin{aligned}
|L^\mu\partial^\alpha u'(t,x)| &\lesssim \langle x\rangle^{-1}\sum_{|\beta|\le 2}\|Z^\beta L^\mu\partial^\alpha u'(t,x)\|_{L^2(|x|-1\le|y|\le|x|+1)} \\
&\lesssim \langle x\rangle^{-1/2}\sum_{\nu+|\beta|\le\mu+|\alpha|+2}\|\langle x\rangle^{-1/2}L^\mu Z^\beta u'\|_2,
\end{aligned}
$$

we obtain the required result using (2.3). □

3. **Energy Estimates.**

Since we are considering the quasilinear wave equation, we need associated energy estimates as follows. Let $\gamma = \{\gamma^{IJ,jk}\}_{1\le I,J\le D, 0\le j,k\le 3}$ be any smooth functions on $[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K}$. We consider $\Box_\gamma$ which is defined by

$$(\Box_\gamma u)^I(t,x) = (\partial_t^2 - c_I^2\Delta)u^I(t,x) + \sum_{J=1}^{D}\sum_{j,k=0}^{3}\gamma^{IJ,jk}(t,x)\partial_j\partial_k u^J(t,x), \quad 1\le I\le D.$$

And we define the energy form associated with $\Box_\gamma$ as follows :

$$e_0^I(u) = (\partial_0 u^I)^2 + \sum_{k=1}^{3} c_I^2(\partial_k u^I)^2 + 2\sum_{J=1}^{D}\sum_{k=0}^{3}\gamma^{IJ,0k}\partial_0 u^I\partial_k u^J - \sum_{J=1}^{D}\sum_{j,k=0}^{3}\gamma^{IJ,jk}\partial_j u^I\partial_k u^J \tag{3.1}$$

$$e_0 = e_0(u) = \sum_{I=1}^{D} e_0^I(u).$$

We define the other components of the energy-momentum vector. For $I = 1, 2, \cdots, D$, and $k = 1, 2, 3$, let

$$e_k^I = e_k^I(u) = -2c_I^2\partial_0 u^I\partial_k u^I + 2\sum_{J=1}^{D}\sum_{j=0}^{3}\gamma^{IJ,jk}\partial_0 u^I\partial_j u^J$$

$$e_j = e_j(u) = \sum_{I=1}^{D} e_j^I, \quad j = 1, 2, 3$$

$$R_0^I(u) = 2\sum_{J=1}^{D}\sum_{k=0}^{3}(\partial_0\gamma^{IJ,0k})\partial_0 u^I\partial_k u^J - \sum_{J=1}^{D}\sum_{j,k=0}^{3}(\partial_0\gamma^{IJ,jk})\partial_j u^I\partial_k u^J$$

$$R_k^I(u) = 2\sum_{J=1}^{D}\sum_{j=0}^{3}(\partial_k\gamma^{IJ,jk})\partial_0 u^I \partial_j u^J$$

$$R(u) = \sum_{I=1}^{D}\sum_{k=0}^{3} R_k^I(u).$$

Then we have the following most fundamental energy estimates :

Lemma 3.1. *Suppose that the functions $\gamma^{IJ,jk}$ satisfy the symmetry conditions*

$$\gamma^{IJ,jk} = \gamma^{JI,jk} = \gamma^{IJ,kj} \quad \text{for} \quad 1 \le I, J \le D, \quad 0 \le j,k \le 3. \tag{3.2}$$

For any function u in $C^2((0,\infty)\times\mathbb{R}^3\backslash\mathcal{K})$, the following equation holds:

$$\partial_t e_0 + \mathrm{div}(e_1, e_2, e_3) = 2\partial_t u \cdot \Box_\gamma u + R(u). \tag{3.3}$$

Proof of Lemma 3.1: By direct computation, we have

$$\begin{aligned}\partial_0 e_0^I = {}& 2\partial_0 u^I \partial_0^2 u^I + 2\sum_{k=1}^{3} c_I^2 \partial_k u^I \partial_0\partial_k u^I + 2\partial_0 u^I \sum_{J=1}^{D}\sum_{k=0}^{3}\gamma^{IJ,0k}\partial_0\partial_k u^J \\ & + 2\sum_{J=1}^{D}\sum_{k=0}^{3}\gamma^{IJ,0k}\partial_0^2 u^I \partial_k u^J - \sum_{J=1}^{D}\sum_{j,k=0}^{3}\gamma^{IJ,jk}(\partial_0\partial_j u^I \partial_k u^J + \partial_j u^I \partial_0\partial_k u^J) + R_0^I\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\sum_{k=1}^{3}\partial_k e_k^I = {}& -2\partial_0 u^I c_I^2 \Delta u^I - 2\sum_{k=1}^{3} c_I^2 \partial_k u^I \partial_0 \partial_k u^I \\ & + 2\partial_0 u^I \sum_{J=1}^{D}\sum_{j=0}^{3}\sum_{k=1}^{3}\gamma^{IJ,jk}\partial_j\partial_k u^J + 2\sum_{J=1}^{D}\sum_{j=0}^{3}\sum_{k=1}^{3}\gamma^{IJ,jk}\partial_0\partial_k u^I \partial_j u^J + \sum_{k=1}^{3} R_k^I.\end{aligned} \tag{3.5}$$

We obtain the required result using the symmetry condition (3.2). □

We use (3.3) to show the energy estimates for $L^\mu Z^\alpha u$. However, direct application causes derivative losses from $\mathrm{div}(e_1, e_2, e_3)$ since L, Ω, ∂_x don't preserve the Dirichlet condition. To avoid it, we cut L near the obstacle and construct the energy estimates for $\partial_t^j u$. Let $\eta \in C^\infty(\mathbb{R}^3)$ be a smooth function with $\eta(x) = 0$ for $|x| \le 1$ and $\eta(x) = 1$ for $|x| \ge 2$. We define $\tilde{L}$ by $\tilde{L} = t\partial_t + \eta r \partial_r$. By simple calculation, we have for any $\mu \ge 0$

$$\tilde{L}^\mu = L^\mu + \sum_{j+|\alpha|\le\mu-1} C_{\mu,j,\alpha}\chi_{\mu,j,\alpha}(x) L^j \partial_x^\alpha \partial_x, \quad \chi_{\mu,j,\alpha} \in C_0^\infty(\mathbb{R}^3), \quad \mathrm{supp}\,\chi_{\mu,j,\alpha} \subset B_2(0), \tag{3.6}$$

where $\{C_{\mu,j,\alpha}\}$ are constants dependent on lower indices.

Our first task is to show the energy estimates for $\tilde{L}^\mu \partial_t^j u$. We put

$$E_{M,\mu_0}(t) = E_{M,\mu_0}(u)(t) = \int \sum_{\substack{\mu+j\le M \\ \mu\le\mu_0}} e_0(\tilde{L}^\mu \partial_t^j u)(t,x)\,dx.$$

The estimate for $E_{M,\mu_0}(t)$ is given by the following lemma. And the energy estimates for $L^\mu \partial^\alpha u$ follows from it due to the elliptic regularity :

Lemma 3.2. *Assume that the perturbation terms* $\gamma^{IJ,jk}$ *satisfy* (3.2) *and the size condition*

$$\sum_{I,J=1}^{D}\sum_{j,k=0}^{3}\|\gamma^{IJ,jk}(t,x)\|_{L^\infty_{t,x\in\mathbb{R}^3\backslash\mathcal{K}}}\le\delta \tag{3.7}$$

for δ *sufficiently small. Then for any* $M\ge 0$ *and* $\mu_0\ge 0$*, there exists a constant* $C=C(M,\mu_0,\mathcal{K})$ *so that for any smooth function* u *in* $[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K}$ *with* $u(t,x)|_{x\in\partial\mathcal{K}}=0$*, and* $u(t,x)=0$ *for large* x *for every* t*, the following estimate holds.*

$$\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha u'(t,\cdot)\|_2\le CE^{1/2}_{M,\mu_0}+C\sum_{\substack{\mu+|\alpha|\le M-1\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha\Box u(t,\cdot)\|_2. \tag{3.8}$$

$$\begin{aligned}\partial_t E^{1/2}_{M,\mu_0}(t) &\le C\sum_{\substack{\mu+j\le M\\ \mu\le\mu_0}}\|\Box_\gamma\tilde{L}^\mu\partial_t^j u(t,\,\cdot)\|_2+C\|\gamma'(t,\,\cdot)\|_\infty E^{1/2}_{M,\mu_0}(t)\\ &\le C\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha\Box_\gamma u(t,\,\cdot)\|_2+C\|\gamma'(t,\,\cdot)\|_\infty E^{1/2}_{M,\mu_0}(t)\\ &\quad+C\sum_{\substack{\mu_1+|\alpha_1|+\mu_2+|\alpha_2|\le M\\ \mu_1+\mu_2\le\mu_0\\ \mu_2+|\alpha_2|\le M-1}}\|(L^{\mu_1}\partial^{\alpha_1}\gamma(t,\cdot))(L^{\mu_2}Z^{\alpha_2}u'(t,\cdot))\|_2\\ &\quad+C\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0-1}}\|L^\mu\partial^\alpha u'(t,x)\|_{L^2(|x|<2)}.\end{aligned} \tag{3.9}$$

When we apply Gronwall's inequality to (3.9), we need the following lemma to bound the last term in (3.9).

Lemma 3.3. *For any* $M\ge 0$ *and* μ_0*, there exists a constant* $C=C(M,\mu_0,\mathcal{K})$ *such that for any smooth function* u *in* $[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K}$ *with the Dirichlet condition* $u(t,x)|_{x\in\partial\mathcal{K}}=0$ *the following estimate holds.*

$$\begin{aligned}\sum_{\substack{\mu+j\le M\\ \mu\le\mu_0}}\int_0^t\|L^\mu\partial^\alpha u'(s,x)\|_{L^2(|x|<2)}ds &\le C\sum_{\substack{\mu+j\le M+2\\ \mu\le\mu_0}}\|\langle x\rangle(L^\mu\partial^\alpha u)(0,\cdot)\|_2\\ &+\sum_{\substack{\mu+|\alpha|\le M+1\\ \mu\le\mu_0}}\int_0^t\int_0^s\|L^\mu\partial^\alpha G(\tau,y)\|_{L^2(||y|-(s-\tau)|<10)}d\tau ds\\ &+\sum_{\substack{\mu+|\alpha|\le M+1\\ \mu\le\mu_0}}\int_0^t\|L^\mu\partial^\alpha\Box u(s,y)\|_{L^2(|y|<4)}ds.\end{aligned} \tag{3.10}$$

For the energy estimates for $L^\mu Z^\alpha u$, we need the following estimates. Begin by setting

$$Y_{M,\mu_0}(t)=\int\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}}e_0(L^\mu Z^\alpha u)(t,x)\,dx. \tag{3.11}$$

We, then, have the following lemma which shows how the energy estimates for $L^\mu Z^\alpha u$ can be obtained from the ones involving $L^\mu\partial^\alpha u$.

Lemma 3.4. *Assume* (3.2), (3.7) *and*

$$\|\gamma'(t,\cdot)\|_\infty \equiv \sum_{I,J=1}^{D}\sum_{j,k,l=0}^{3}\|\partial_l\gamma^{IJ,jk}(t,\cdot)\|_\infty \le \delta \tag{3.12}$$

for sufficiently small δ. *Then,*

$$\begin{aligned}\partial_t Y_{M,\mu_0} \quad &\le \quad CY_{M,\mu_0}^{1/2}\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}}\|\Box_\gamma L^\mu Z^\alpha u(t,\cdot)\|_2 \\ &\quad +C\|\gamma'(t,\cdot)\|_\infty Y_{M,\mu_0}+C\sum_{\substack{|\alpha|+\mu\le M+1\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha u'(s,\cdot)\|^2_{L^2(|x|<2)} \\ &\le \quad CY_{M,\mu_0}^{1/2}\{\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}}\|L^\mu Z^\alpha\Box_\gamma u(t,\cdot)\|_2 \\ &\qquad + \sum_{\substack{\mu_1+|\alpha_1|+\mu_2+|\alpha_2|\le M\\ \mu_1+\mu_2\le\mu_0\\ \mu_2+|\alpha_2|\le M-1}}\|(L^{\mu_1}Z^{\alpha_1}\gamma)(L^{\mu_2}Z^{\alpha_2}\partial^2 u)\|_2\} \\ &\quad +C\|\gamma'(t,\cdot)\|_\infty Y_{M,\mu_0}+C\sum_{\substack{|\alpha|+\mu\le M+1\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha u'(s,\cdot)\|^2_{L^2(|x|<2)}\end{aligned} \tag{3.13}$$

4. Local energy estimates and L^2 estimates in space and time.

First we derive local energy estimates for inhomogeneous wave equations near the obstacle.

Lemma 4.1. *Let* $\mathcal{K}$ *satisfy the local energy decay* (1.4). *Let* u *be the solution of*

$$\begin{cases}\Box u = F, \qquad \mathrm{supp}_x F(t,x)\subset B_4(0)\\ u|_{\partial\mathcal{K}}=0\\ u(0)=f,\ \partial_t u(0)=g, \qquad \mathrm{supp}\ f\cup\mathrm{supp}\ g\subset B_4(0).\end{cases} \tag{4.1}$$

Then for any $M\ge 0$ *and* $\mu_0\ge 0$, *the following estimates holds :*

$$\begin{aligned}\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha u'(t,x)\|_{L^2(|x|<4)} &\le Ce^{-ct/2}\sum_{|\alpha|\le M+1}\|\partial^\alpha u'(0,x)\|_{L^2(|x|<4)} \\ &+C\int_0^t e^{-c(t-s)/2}\sum_{\substack{\mu+|\alpha|\le M+1\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha F(s,\cdot)\|_2 ds+\sum_{\substack{\mu+|\alpha|\le M-1\\ \mu\le\mu_0}}\|L^\mu\partial^\alpha F(t,\cdot)\|_2.\end{aligned} \tag{4.2}$$

Proof of Lemma 4.1 : First we show (4.2) for $\mu_0=0$ using induction. The estimate for $M=0$ follows from (1.4) and the Duhamel priciple. Let's assume that the estimate for $M\ge 0$, and we consider the case $M+1$. We have

$$\begin{aligned}\sum_{|\alpha|\le M+1}\|\partial^\alpha u'\|_{L^2(|x|<4)} \lesssim \sum_{|\alpha|\le M}\|\partial^\alpha u'\|_{L^2(|x|<4)}+\sum_{\substack{j+|\alpha|\le M+2\\ j\ge 1}}\|\partial_t^j\partial_x^\alpha u\|_{L^2(|x|<4)} \\ +\sum_{|\alpha|=M+2}\|\partial_x^\alpha u\|_{L^2(|x|<4)}.\end{aligned} \tag{4.3}$$

Ther first two terms in the RHS are treated by induction since $\partial_t u$ satisfies the Dirichlet condition. Applying (2.1) and (2.2) to the last term, we have

$$\sum_{|\alpha|=M+2} \|\partial_x^\alpha u(t)\|_{L^2(|x|<4)} \lesssim \|u'\|_{L^2(|x|<5)} + \sum_{|\alpha|\le M} \|\partial_x^\alpha \partial_t^2 u\|_{L^2(|x|<5)} + \sum_{|\alpha|\le M} \|\partial_x^\alpha \Box u\|_{L^2(|x|<5)}.$$

Again by induction, we obtain the required estimate for $M+1$. Here we can replace $c/2$ with c in (4.2) when $\mu_0 = 0$.

Next we show (4.2) for $\mu_0 \ge 1$ by induction. Let's assume that (4.2) holds for M and μ_0. We consider the case $\mu_0 + 1$. Since we have

$$\sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0+1}} \|L^\mu \partial^\alpha u'\|_{L^2(|x|<4)} \lesssim \sum_{\substack{\mu+|\alpha|\le M\\ \mu\le\mu_0}} \|L^\mu \partial^\alpha u'\|_{L^2(|x|<4)} + \sum_{\substack{\mu+|\alpha|\le M\\ 1\le\mu\le\mu_0+1}} t^\mu \|\partial_t^\mu \partial^\alpha u'\|_{L^2(|x|<4)}, \tag{4.4}$$

it suffices by induction to show the last term in the RHS is bounded by the RHS in (4.2). If we use (4.2) for $\mu_0 = 0$ for $\partial_t^j u$ which satisfies the Dirichlet condition, and we use that $t^\mu e^{-ct/2}$ is bounded, then we obtain the required estimate. □

We need weighted L^2 estimates. Put

$$S_T = \{[0,T] \times \mathbb{R}^3 \backslash \mathcal{K}\}$$

to denote the time strip of height T in $\mathbb{R}_+ \times \mathbb{R}^3\backslash\mathcal{K}$.

Lemma 4.2. *(1) (Boundaryless case* [13, Proposition 2.1]*) There exists a constant $C > 0$ so that for any function u in $[0,\infty)\times\mathbb{R}^3$, the following estimate holds.*

$$(\log(2+T))^{-1/2} \|\langle x\rangle^{-1/2} u'\|_{L^2([0,T]\times\mathbb{R}^3)} \le C \sum_{|\alpha|\le 1} \|\partial^\alpha u(0,\,\cdot\,)\|_2 + C\int_0^T \|\Box u(t,\,\cdot\,)\|_2\, dt. \tag{4.5}$$

(2) (Exterior domain case [14, (6.8), (6.9)]*) There exists a constant C so that for any function u in $[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K}$ with the Dirichlet condition $u(t,x)|_{x\in\partial\mathcal{K}} = 0$, the following estimate holds. For any $M \ge 0$ and $\mu_0 \ge 0$*

$$\begin{aligned}(\log(2+T))^{-1/2} \sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}} \|\langle x\rangle^{-1/2} L^\mu \partial^\alpha u'\|_{L^2(S_T)} &\le C \sum_{\substack{|\alpha|+\mu\le M+2\\ \mu\le\mu_0}} \|(L^\mu\partial^\alpha u)(0,\cdot)\|_2 \\ &+ C\int_0^T \sum_{\substack{|\alpha|+\mu\le M+1\\ \mu\le\mu_0}} \|L^\mu \partial^\alpha \Box u(t,\,\cdot\,)\|_2\, dt + C \sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}} \|L^\mu\partial^\alpha \Box u\|_{L^2(S_T)}\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}(\log(2+T))^{-1/2} \sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}} \|\langle x\rangle^{-1/2} L^\mu Z^\alpha u'\|_{L^2(S_T)} &\le C \sum_{\substack{|\alpha|+\mu\le M+2\\ \mu\le\mu_0}} \|L^\mu Z^\alpha u(0,x)\|_{L^2_x} \\ &+ C\int_0^T \sum_{\substack{|\alpha|+\mu\le M+1\\ \mu\le\mu_0}} \|\Box L^\mu Z^\alpha u(t,\,\cdot\,)\|_2\, dt + C \sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}} \|\Box L^\mu Z^\alpha u\|_{L^2(S_T)}\end{aligned} \tag{4.7}$$

5. **Pointwise Estimates.**

We consider pointwise estimates in this section.

Lemma 5.1. *Let F, f and g be any functions.*

(1) (Boundaryless case) Let u be a solution to

$$\begin{cases} (\partial_t^2-\Delta)u(t,x)=F(t,x), & (t,x)\in[0,\infty)\times\mathbb{R}^3 \\ u(0,x)=f(x), \quad \partial_t u(0,x)=g(x). \end{cases}$$

Then

$$(1+t+|x|)|u(t,x)| \le C\sum_{\substack{\mu+|\alpha|\le 3\\ \mu\le 1, j\le 1}} \|(\langle x\rangle^j\partial_{t,x}^j L^\mu Z^\alpha u)(0,x)\|_{L_x^2} + C\int_0^t\int_{\mathbb{R}^3}\sum_{\substack{\mu+|\alpha|\le 3\\ \mu\le 1}} |L^\mu Z^\alpha F(s,y)|\,\frac{dy\,ds}{\langle y\rangle}. \tag{5.1}$$

(2) (Exterior domain case) Let u be a solution to

$$\begin{cases} (\partial_t^2-\Delta)u(t,x)=F(t,x), & (t,x)\in[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K} \\ u(t,x)|_{x\in\partial\mathcal{K}}=0 \\ u(0,x)=f(x), \quad \partial_t u(0,x)=g(x). \end{cases}$$

Then for any $M\ge 0$ and $\mu_0\ge 0$

$$\begin{aligned}(1+t+|x|)\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le\mu_0}}|L^\mu Z^\alpha u(t,x)| &\le C\sum_{\substack{j+\mu+|\alpha|\le M+8\\ \mu\le\mu_0+2,\ j\le 1}} \|(\langle x\rangle^j\partial_{t,x}^j L^\mu Z^\alpha u)(0,x)\|_{L_x^2} \\ &\quad + C\int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}}\sum_{\substack{|\alpha|+\mu\le M+7\\ \mu\le\mu_0+1}}|L^\mu Z^\alpha F(s,y)|\,\frac{dy\,ds}{|y|} \\ &\quad + C\int_0^t\sum_{\substack{|\alpha|+\mu\le M+4\\ \mu\le\mu_0+1}}\|L^\mu\partial^\alpha F(s,y)\|_{L^2(|y|<4)}\,ds.\end{aligned} \tag{5.2}$$

Here and throughout $\{|y|<4\}$ is understood to mean $\{y\in\mathbb{R}^3\backslash\mathcal{K} : |y|<4\}$.

The proof of the above lemma for vanishing Cauchy data has been shown by Keel-Smith-Sogge in [14, (2.3), (2.4) and (4.2)] and Metcalfe-Sogge in [23, (3.2)].

The following estimates are the special version to treat the inhomogeneity F near the light cones, which follows from the Huygens principle.

Lemma 5.2. *Let F be any function.*

(1) (Boundaryless case) Let u be a solution to

$$\begin{cases} (\partial_t^2-c_I^2\Delta)u(t,x)=F(t,x), & (t,x)\in[0,\infty)\times\mathbb{R}^3 \\ u(0,\cdot)=0, \quad \partial_t u(0,\cdot)=0. \end{cases}$$

Assume

$$\mathrm{supp}F \subset \{(t,x); t \geq 1,\ \frac{c_1 t}{10} \leq |x| \leq 10c_D t\}.$$

Then

$$\sup_{|x|\leq c_1 t/2}(1+t)|u(t,x)| \leq C \sup_{0\leq s\leq t}\int_{\mathbb{R}^3}\sum_{\substack{\mu+|\alpha|\leq 3\\ \mu\leq 1}}|L^\mu Z^\alpha F(s,y)|dy. \tag{5.3}$$

(2) (Exterior domain case) Let u be a solution to

$$\begin{cases} (\partial_t^2 - c_I^2\Delta)u(t,x) = F(t,x), \quad (t,x)\in[0,\infty)\times\mathbb{R}^3\backslash\mathcal{K} \\ u(t,x)|_{x\in\partial\mathcal{K}} = 0 \\ u(t,\cdot) = 0 \quad for \quad t\leq 0. \end{cases}$$

Assume

$$\mathrm{supp}F \subset \{(t,x); t \geq 1\vee\frac{6}{c_1},\ \ \frac{c_1 t}{10} \leq |x| \leq 10c_D t\}.$$

Then for any $M\geq 0$ and $\mu_0 \geq 0$

$$\sup_{|x|\leq c_1 t/2}(1+t)\sum_{\substack{\mu+|\alpha|\leq M\\ \mu\leq\mu_0}}|L^\mu Z^\alpha u(t,x)| \leq C \sup_{0\leq s\leq t}\int_{\mathbb{R}^3\backslash\mathcal{K}}\sum_{\substack{|\alpha|+\mu\leq M+7\\ \mu\leq\mu_0+1}}|L^\mu Z^\alpha F(s,y)|dy \\ + \sup_{0\leq s\leq t}(1+s)\sum_{\substack{|\alpha|+\mu\leq M+3\\ \mu\leq\mu_0}}\|L^\mu\partial^\alpha F(s,y)\|_{L^2(|y|<4)}. \tag{5.4}$$

We also need the following $L^\infty - L^\infty$ estimates to treat the inhomogeneity away from the light cones, which are special (more elementary) version of Kubota-Yokoyama estimates (see Kubota-Yokoyama [18, Theorem 3.4] for the boundaryless case).

Lemma 5.3. *Let F, f and g be any functions.*

(1) (Boundaryless case) Let u be a solution to

$$\begin{cases} (\partial_t^2 - c_I^2\Delta)u(t,x) = F(t,x), \quad (t,x)\in[0,\infty)\times\mathbb{R}^3 \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x). \end{cases}$$

Assume

$$\mathrm{supp}F \subset \{(t,x); 0\leq t\leq 2,\ |x|\leq 2\}\cup\{(t,x); |x| \leq \frac{c_I t}{5} \ \ or \ \ |x|\geq 5c_I t\ \}. \tag{5.5}$$

Then for any $\theta > 0$, there exists a constant $C = C(\theta)$ such that

$$\sup_{|x|\leq c_I t/2}(1+t)|u(t,x)| \leq C\sum_{\substack{\mu+|\alpha|\leq 3\\ \mu\leq 1, j\leq 1}}\|(\langle x\rangle^j\partial_{t,x}^j L^\mu Z^\alpha u)(0,x)\|_{L^2_x} \\ + C\sup_{\substack{s\geq 0\\ y\in\mathbb{R}^3}}\langle y\rangle^{2-\theta}(1+s+|y|)^{1+\theta}|F(s,y)|. \tag{5.6}$$

(2) (Exterior domain case) Let u be a solution to

$$\begin{cases} (\partial_t^2 - c_I^2\Delta)u(t,x) = F(t,x), & (t,x) \in [0,\infty)\times\mathbb{R}^3\backslash\mathcal{K} \\ u(t,x)|_{x\in\partial\mathcal{K}} = 0 \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x). \end{cases}$$

Assume (5.5). *Then for any $\theta > 0$, $M \geq 0$ and $\mu_0 \geq 0$, there exists a constant $C = C(\theta, M, \mu_0, \mathcal{K})$ such that*

$$\begin{aligned} \sup_{|x|\leq c_I t/2} (1+t) \sum_{\substack{\mu+|\alpha|\leq M \\ \mu\leq\mu_0}} |L^\mu Z^\alpha u(t,x)| &\leq C \sum_{\substack{j+\mu+|\alpha|\leq M+8 \\ \mu\leq\mu_0+2,\ j\leq 1}} \|(\langle x\rangle^j \partial_{t,x}^j L^\mu Z^\alpha u)(0,x)\|_{L^2_x} \\ &\quad + C \sup_{\substack{s\geq 0 \\ y\in\mathbb{R}^3\backslash\mathcal{K}}} \langle y\rangle^{2-\theta}(1+s+|y|)^{1+\theta} \sum_{\substack{|\alpha|+\mu\leq M \\ \mu\leq\mu_0}} |L^\mu Z^\alpha F(s,y)| \\ &\quad + C \sup_{\substack{s\geq 0 \\ y\in\mathbb{R}^3\backslash\mathcal{K}}} \langle y\rangle^{2-\theta}(1+s+|y|)^{1+\theta} \sum_{\substack{|\alpha|+\mu\leq M+4 \\ \mu\leq\mu_0}} |L^\mu \partial^\alpha F(s,y)|. \end{aligned} \tag{5.7}$$

6. **Sobolev-type Estimates.**

We need the following Sobolev inequalities. The first inequality is due to Klainerman-Sideris [17], Sideris [28], and Hidano-Yokoyama [6]. The second one is the exterior domain analog of the first one.

Lemma 6.1. *Let $c > 0$, $0 \leq \theta \leq 1/2$ be any constants.*

(1) (Boundaryless case) For any function $u \in C_0^\infty((0,\infty)\times\mathbb{R}^3)$

$$\langle x\rangle^{1/2+\theta}\langle ct-|x|\rangle^{1-\theta}|u'(t,x)| \leq C \sum_{\substack{\mu+|\alpha|\leq 2 \\ \mu\leq 1}} \|L^\mu Z^\alpha u'(t,x)\|_{L^2_x} + C\sum_{|\alpha|\leq 1} \|\langle t+|x|\rangle Z^\alpha \Box_c u(t,x)\|_{L^2_x}. \tag{6.1}$$

(2) (Exterior domain case) For any function $u \in C_0^\infty((0,\infty)\times\mathbb{R}^3\backslash\mathcal{K})$ with the Dirichlet condition $u|_{\partial\mathcal{K}} = 0$, and any $M \geq 0$, $\mu_0 \geq 0$

$$\begin{aligned} \langle x\rangle^{1/2+\theta}\langle ct-|x|\rangle^{1-\theta} \sum_{\substack{\mu+|\alpha|\leq M \\ \mu\leq\mu_0}} |L^\mu Z^\alpha u'(t,x)| &\leq C \sum_{\substack{\mu+|\alpha|\leq M+2 \\ \mu\leq\mu_0+1}} \|L^\mu Z^\alpha u'(t,x)\|_{L^2_x} \\ &\quad + C \sum_{\substack{\mu+|\alpha|\leq M+1 \\ \mu\leq\mu_0}} \|\langle t+|x|\rangle Z^\alpha \Box_c u(t,x)\|_{L^2_x} \\ &\quad + C(1+t)\sum_{\mu\leq\mu_0} \|L^\mu u'(t,x)\|_{L^\infty(|x|<2)}. \end{aligned} \tag{6.2}$$

Proof of Lemma 6.1 : By (3.14c) in [28], and (4.2) in [18], we have

$$\langle x\rangle^{1/2+\theta}\langle ct-|x|\rangle^{1-\theta}|u'(t,x)| \leq C\sum_{|\alpha|\leq 2} \|Z^\alpha u'(t,x)\|_{L^2_x} + C\sum_{|\alpha|\leq 1} \|\langle ct-|x|\rangle Z^\alpha \partial^2 u(t,x)\|_{L^2_x}$$

for any θ with $0 \leq \theta \leq 1/2$. By (2.10) and (3.1) in [17], we have

$$\|\langle ct-|x|\rangle \partial^2 u(t,x)\|_{L^2_x} \leq C \sum_{\substack{\mu+|\alpha|\leq 1 \\ \mu\leq 1}} \|L^\mu Z^\alpha u'(t,x)\|_{L^2_x} + C\|\langle t+|x|\rangle \Box_c u(t,x)\|_{L^2_x}.$$

Combining the above two estimates, we obtain (6.1). The proof of (2) can be found as (4.7) in [22]. □

7. Proof of Theorem 1.3.

To prove our global existence theorem, we shall need a standard local existence theorem (See [7, Theorem 6.4.11] for the local existence theorem for the boundaryless case).

Theorem 7.1. [12, Theorem 9.4] *Let $s \geq 7$. Let $(f, g) \in H^s \oplus H^{s-1}$ satisfy the compatibility conditions of order $s-1$. Then (1.5) has a local solution $u \in C([0,T); H^s)$, where T depends on s and the norms of f and g. Moreover if $\|f\|_{H^s} + \|g\|_{H^{s-1}}$ is sufficiently small, then there exists C and T independent of f and g so that the solution of (1.5) exists for $0 \leq t \leq T$ and satisfies*

$$\sup_{0\le t\le T} \sum_{j=0}^{s} \|\partial_t^j u(t,\cdot)\|_{H^{s-j}} \leq C(\|f\|_{H^s} + \|g\|_{H^{s-1}}).$$

Based on this local exsistence theorem, we can show the global solutions by the continuity argument.

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Mode generating property of solutions to nonlinear Schrödinger equations

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Abstract

We consider the initial value problem of the nonlinear Schrödinger equation with superposed δ-functions as initial data. The speaker will treat this problem case by case, i.e., the cases in which the initial data consists of single and double δ-functions, respectively. In particular, when the initial data consists of double δ-functions, the solution receives the generation of new modes which is visible only in the nonlinear problem (see section 3).

1 Introduction

In this talk, we present several results on the initial value problem of the nonlinear Schrödinger equation like

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u = -\partial_x^2 u + \lambda \mathcal{N}(u), \\ u(0,x) = (\text{superposition of } \delta\text{-functions}), \end{cases}$$

where $(t,x) \in \mathbf{R} \times \mathbf{R}$ and the unknown function $u = u(t,x)$ takes complex values. The nonlinearity $\mathcal{N}(u)$ is given by

$$\mathcal{N}(u) = |u|^{p-1}u \quad \text{with } 1 < p < 3.$$

The nonlinear coefficient λ takes arbitrary complex number. The functional δ_a denotes the well-known point mass measure supported at $x = a \in \mathbf{R}$.

From the physical point of view, the cubic nonlinearity (i.e. $p = 3$ which is excluded in our assumption for mathematical reason) frequently appears. For example, (NLS) with $\lambda \in \mathbf{R}$ and $p = 3$ is said to govern the motion of vortex filament in the ideal fluid. In fact, letting $\kappa(t,x)$ be the curvature of the filament and $\tau(t,x)$ the tortion, we observe that $u(t,x) = \kappa(t,x)\exp(i\int_0^x \tau(t,y)\ dy)$ (which is called "Hasimoto transform" [3]) satisfies (NLS), where x stands for the position parameter along the filament.

To our regret, our argument does not contain the cubic nonlinearity. However, if one allows us to treat the solution as a fine approximation of the physically important case, we can imagine the time evolution of vortex filament with the locally bended initial state (which is described as $\kappa(0,x) = \delta_a$).

The nonlinear evolution equations with measures as initial data are extensively sutudied for various kinds of initial value problem. As for the nonlinear parabolic equations like $\partial_t u - \partial_x^2 u + |u|^{p-1}u = 0$ with $u(0,x) = \delta_0$, Brezis-Friedman [2] give the critical power of nonlinearity concerning the solvability and unsolvability of the equation. They prove that, if $3 \le p$, there exists no solution continuous at $t = 0$ in the distribution sense and that, if $1 < p < 3$, it is posibble to construct a solution with a general measure as the initial data. For the KdV equation, Tsutsumi [5] constructs a solution by making use of Miura transformation which deforms the original KdV equation into the modified one. Recently, Abe-Okazawa [1] have studied this kind of problem for the complex Ginzburg-Landau equation. The ideas of the proof for these known results are based on the strong smoothing effect of linear part or the nonlinear transformation of unknown functions into the suitably handled equation. In the present case, however, the nonlinear Schrödinger equation does not have the useful smoothing properties and the transformation into easily handeled equation. Therefore, it is still open whether we can construct a solution when the initial data is arbitrary measure.

We remark that Kenig-Ponce-Vega [4] studied the ill-posedness aspect of the nonlinear Schrödinger equation with $u(0,x) = \delta_0$ and $3 \le p$. The situation is very similar to the nonlinear heat case introduced above. They proved that (NLS) possesses either no solution or more than one in $C([0,T];\mathcal{S}'(\mathbf{R}))$, where $\mathcal{S}'(\mathbf{R})$ denotes the tempered distribution. In this talk, we consider the construction of the solution to (NLS) for the subcritical nonlinearity. We prove that the solution is explicitly obtained when the initial data consists of single δ-function (see section 2). Furthermore, we observe that, when the initial data consists of double (or more) δ-functions, the superposition of infinitely many linear solutions immediately appers (see section 3). This aspect is called "the generalization of new modes". Throughout this note, the Lebesgue space L^q_θ denotes

$$L^q_\theta = \{f(\theta);\ \|f\|^q_{L^q_\theta} = \int_0^{2\pi} |f(\theta)|^q \, d\theta < \infty\}.$$

Let us state our main theorems case by case.

2 The case $u(0,x) = \mu_0\delta_0$

This case simply gives an explicit solution. Namely, the solution to (NLS) is given by

$$u(t,x) = A(t)\exp(it\partial_x^2)\delta_0, \tag{2.1}$$

where $\exp(it\partial_x^2)\delta_0 = (4\pi it)^{-1/2}\exp(ix^2/4t)$ and the modified amplitude $A(t)$ is

$$(2.2)\ A(t) = \begin{cases} \mu_0 \exp\left(\dfrac{2\lambda|\mu_0|^{p-1}}{i(3-p)}|4\pi t|^{-(p-1)/2}t\right) & \text{if } \mathrm{Im}\lambda = 0, \\ \mu_0\left(1 - \dfrac{2(p-1)\mathrm{Im}\lambda|\mu_0|^{p-1}}{3-p}|4\pi t|^{-(p-1)/2}t\right)^{\frac{i\lambda}{(p-1)\mathrm{Im}\lambda}} & \text{if } \mathrm{Im}\lambda \neq 0. \end{cases}$$

In fact, by substituting (2.1) into (NLS), we have the ordinary differential equation (ODE) of $A(t)$:

$$\begin{cases} i\dfrac{dA}{dt} = \lambda|4\pi t|^{-(p-1)/2}\mathcal{N}(A), \\ A(0) = \mu_0. \end{cases}$$

This is easily solved and yields (2.2). Note that $\mathrm{Im}\lambda > 0$ implies blowing-up of $A(t)$ in positive finite time.

3 The case $u(0,x) = \mu_0\delta_0 + \mu_1\delta_a$

The superposition of δ-functions causes "the mode generation" for $t \neq 0$. Before stating our results, let ℓ^2_α be the weighted sequence space defined by

$$\ell^2_\alpha = \{\{A_k\}_{k\in\mathbf{Z}};\ \|\{A_k\}_{k\in\mathbf{Z}}\|^2_{\ell^2_\alpha} = \sum_{k\in\mathbf{Z}}(1+|k|^2)^\alpha|A_k|^2 < \infty\}.$$

For the simplicity of description, we often use the notation $\{A_k\}$ in place of $\{A_k\}_{k\in\mathbf{Z}}$. Then, our results are

Theorem 3.1 (local result) *For some $T > 0$, there exists a unique solution to (NLS) discribed as*

$$(3.1)\qquad u(t,x) = \sum_{k\in\mathbf{Z}} A_k(t)\exp(it\partial_x^2)\delta_{ka},$$

where $\{A_k(t)\} \in C([0,T];\ell^2_1) \cap C^1((0,T];\ell^2_1)$ with $A_0(0) = \mu_0$, $A_1(0) = \mu_1$ and $\mu_k = 0$ $(k \neq 0,1)$.

Remark 3.1. Let us call $A_k(t)\exp(it\partial_x^2)\delta_{ka}$ the k-th mode. Then, (3.1) suggests that new modes away from 0-th and first ones appear in the solution while the initial data contains only the two modes. This special property is visible only in the nonlinear case.

Remark 3.2. Reading the proof of Theorem 3.1, we see that it is possible to generalize the initial data. Namely, we can construct a solution even when point masses are distributed on a line at equal intervals – more precisely, the initial data is given like

$$u(0,x) = \sum_{k\in\mathbf{Z}} \mu_k \delta_{ka}(x),$$

where $\{\mu_k\}_{k\in\mathbf{Z}} \in \ell_1^2$. In this case, the solution is described similarly to (3.1) but $\{A_k(0)\} = \{\mu_k\}$. The decay condition on the coefficients described in terms of ℓ_1^2 is required to estimate the nonlinearity. This is because we will use the inequality like $\|\mathcal{N}(g)\|_{L^2_\theta} \leq C\|g\|_{L^\infty_\theta}^{p-1}\|g\|_{L^2_\theta}$ where $g = g(t,\theta) = \sum_k A_k e^{-ik\theta} e^{i(ka)^2/4t}$ and $\theta \in [0, 2\pi]$. Accordingly, to estimate $\|g\|_{L^\infty_\theta}$, we require the decay condition of $\{A_k\}$.

The sign of Imλ determines the global solvability of (NLS).

Theorem 3.2 (blowing up or global result) *(1) Let Im$\lambda > 0$. Then, the solution as in Theorem 3.1 blows up in positive finite time. Precisely speaking, the ℓ_0^2-norm of $\{A_k(t)\}$ tends to infinity at some positive time.*

(2) Let Im$\lambda \leq 0$. Then, there exists a unique global solution to (NLS) discribed as in Theorem 3.1 with $\{A_k(t)\} \in C([0,\infty);\ell_1^2) \cap C^1((0,\infty);\ell_1^2)$.

In what follows, we present the rough sketch to prove Theorem 3.1 and 3.2. The idea is based on the reduction of (NLS) into the ODE system of $\{A_k\}_{k\in\mathbf{Z}}$. The next key lemma gives the representation formula of $\mathcal{N}(\sum_k A_k \exp(it\partial_x^2)\delta_{ka})$.

Lemma 3.3 *Let $\{A_k\} \in C([-T,T];\ell_1^2)$. Then, we have*

$$\mathcal{N}\Big(\sum_{k\in\mathbf{Z}} A_k(t)\exp(it\partial)\delta_{ka}\Big) = |4\pi t|^{-n(p-1)/2} \sum_{k\in\mathbf{Z}} \tilde{A}_k(t)\exp(it\partial)\delta_{ka}, \tag{3.2}$$

where

$$\tilde{A}_k(t) = (2\pi)^{-1} e^{i(ka)^2/4t} \langle \mathcal{N}\Big(\sum_j A_j e^{-ij\theta} e^{-i(ja)^2/4t}\Big), e^{-ik\theta}\rangle_\theta,$$

with $\langle f,g\rangle_\theta = \int_0^{2\pi} f(\theta)\overline{g(\theta)}d\theta$.

Proof of Lemma 3.3. Note that the linear Schrödinger group is factorized as follows.

$$\begin{aligned} \exp(it\partial_x^2)f &= (4\pi i t)^{-1/2}\int \exp(i|x-y|^2/4t) f(y)dy \\ &= MD\mathcal{F}Mf, \end{aligned}$$

where

$$\begin{aligned}
Mg(t,x) &= e^{ix^2/4t}g(x),\\
Dg(t,x) &= (2it)^{-1/2}g(x/2t),\\
\mathcal{F}g(\xi) &= (2\pi)^{-1/2}\int e^{-i\xi x}g(x)dx \quad \text{(Fourier transform of } g).
\end{aligned}$$

Then, we see that

$$\begin{aligned}
(3.3)\qquad & \mathcal{N}(\sum_k A_j(t)\exp(it\partial_x^2)\delta_{ja})\\
&= \mathcal{N}((2\pi)^{-1/2}MD\sum_j A_j(t)e^{-ija\cdot x-i(ja)^2/4t})\\
&= |4\pi t|^{-(p-1)/2}(2\pi)^{-1/2}MD\mathcal{N}(\sum_j A_j(t)e^{-ija\cdot x-i(ja)^2/4t}).
\end{aligned}$$

Note that, to show the last equality in (3.3), we make use of the gauge invariance of the nonlinearity. Replacing $a\cdot x$ by θ, we can regard $\mathcal{N}(\sum_j A_j(t)e^{-ij\theta-i(ja)^2/4t})$ as the 2π-periodic function of θ. Therefore, by the Fourier series expansion,

$$\begin{aligned}
\mathcal{N}(\sum_j A_j(t)e^{-ij\theta-i(ja)^2/4t}) &= \sum_k \tilde{A}_k(t)e^{-i(ka)^2/4t}e^{-ik\theta}\\
&= (2\pi)^{n/2}\sum_k \tilde{A}_k(t)\mathcal{F}M\delta_{ka}.
\end{aligned}$$

Plugging this into (3.3), we obtain Lemma 3.3. ❑

Our idea to solve the nonlinear equation is based on the reduction of (NLS) into the system of ODE's. By substituting $u=\sum_k A_k(t)\exp(it\partial_x^2)\delta_{ka}$ into (NLS) and noting that $i\partial_t\exp(it\partial_x^2)\delta_{ka}=-\partial_x^2\exp(it\partial_x^2)\delta_{ka}$, Lemma 3.3 yields

$$\sum_k i\frac{dA_k}{dt}\exp(it\partial_x^2)\delta_{ka} = |4\pi t|^{-(p-1)/2}\sum_k \tilde{A}_k\exp(it\partial_x^2)\delta_{ka}.$$

Equating the terms on both hand sides, we arrive at the desired ODE system:

$$(3.4)\qquad i\frac{dA_k}{dt}=|4\pi t|^{-(p-1)/2}\tilde{A}_k$$

with the initial condition $A_k(0)=\mu_k$. Now, showing the existence and uniqueness of (NLS) is equivalent to showing those of (3.4). To solve (3.4), let us consider the following integral equation.

$$\begin{aligned}
A_k(t) &= \Phi_k(\{A_k(t)\}_{k\in\mathbf{Z}})\\
(3.5)\qquad &\equiv \mu_k - i\int_0^t |4\pi\tau|^{-(p-1)/2}\tilde{A}_k(\tau)\,d\tau.
\end{aligned}$$

Then, we want to see the contraction mapping property of $\{\Phi_k\}_{k\in\mathbf{Z}}$. The simple aplication of Parseval's identity derives the following.

Lemma 3.4 *Let $I = [0,T]$ and $\{A_k\} = \{A_k\}_{k\in\mathbf{Z}}$. Then, we have*

$$\|\{\tilde{A}_k\}\|_{L^\infty(I;\ell_1^2)} \le C\|\{A_k\}\|^p_{L^\infty(I:\ell_1^2)}, \tag{3.6}$$

$$\begin{aligned} &\|\{\tilde{A}_k^{(1)}\} - \{\tilde{A}_k^{(2)}\}\|_{L^\infty(I;\ell_0^2)} \\ &\le C(\max_{j=1,2}\|\{A_k^{(j)}\}\|_{L^\infty(I;\ell_1^2)})^{p-1}\|\{A_k^{(1)}\} - \{A_k^{(2)}\}\|_{L^\infty(I;\ell_0^2)}. \end{aligned} \tag{3.7}$$

Proof of Lemma 3.4. According to the description of $\tilde{A}_k$ as in Lemma 3.3 and the integration by parts, we see that

$$k\tilde{A}_k \ = \ (2\pi)^{-1} i e^{-i(ka)^2/4t}\langle \partial_\theta \mathcal{N}(\sum_j A_j e^{-ij\theta}e^{i(ja)^2/4t}), e^{-ik\theta}\rangle_\theta.$$

Then, Parseval's equality yields

$$\begin{aligned} \|\{k\tilde{A}_k\}\|_{\ell_0^2} &= (2\pi)^{-1/2}\|\partial_\theta\mathcal{N}(\sum_j A_j e^{-ij\theta}e^{i(ja)^2/4t})\|_{L^2_\theta} \\ &\le C\|\sum_j A_j e^{-ij\theta}e^{i(ja)^2/4t}\|^{p-1}_{L^\infty_\theta}\|\sum_j jA_j e^{-ij\theta}e^{i(ja)^2/4t}\|_{L^2_\theta} \\ &\le C\|\{A_j\}\|^p_{\ell_1^2}. \end{aligned}$$

Thus, we obtain (3.6). The proof for (3.7) follows similarly. Since there is a singularity at $u = 0$ of the nonlinearity $\mathcal{N}(u)$, we do not employ ℓ_1^2-norm to measure $\{A_k^{(1)}\} - \{A_k^{(2)}\}$. ❑

We are now in the position to prove Theorem 3.1.

Proof of Theorem 3.1. The proof relies on the contraction mapping principle of $\{\Phi_k(\{A_j\})\}$. Let $\|\{\mu_k\}\|_{\ell_1^2} \le \rho_0$ and

$$\overline{B}_{2\rho_0} = \{\{A_k\} \in L^\infty([0,T];\ell_1^2);\ \|\{A_k\}\|_{L^\infty([0,T];\ell_1^2)} \le 2\rho_0\}$$

endowed with the metric in $L^\infty([0,T];\ell_0^2)$. Then, in virture of Lemma 3.4, we see that $\{\Phi_k(\{A_j\})\}$ is the contraction map on $\overline{B}_{2\rho_0}$ if T is sufficiently small. Thus, Theorem 3.1 is obtained. ❑

To prove Theorem 3.2, we apply the a priori estimates described in the following.

Lemma 3.5 *Let $\{A_k(t)\}$ be the solution to (3.4) in $C([0,T];\ell_1^2) \cap C^1((0,T];\ell_1^2)$.*

(1) Then, we have

$$\frac{d\|\{A_k(t)\}\|_{\ell_0^2}}{dt} = \frac{Im\lambda}{\pi}(4\pi t)^{-(p-1)/2}\|v(t)\|_{L_\theta^{p+1}}^{p+1}, \tag{3.8}$$

where $v(t,\theta) = \sum_k A_k(t)e^{-k\theta}e^{i(ka)^2/4t}$.

(2) In addition, if $Im\lambda < 0$*, then we have*

$$\|\{kA_k(t)\}\|_{\ell_0^2} \le Ce^{t/2}, \tag{3.9}$$

where the positive constant C *does not depend on* T.

Remark 3.3 The a priori bound in (3.9) may be refined by sophisticating the estimates in the proof.

Formal Proof of Lemma 3.5. Note that $v(t,\theta)(=v)$ satisfies the nonlinear equation like

$$i\partial_t v = -\frac{a^2}{4t^2}\partial_\theta^2 v + \lambda|4\pi t|^{-(p-1)/2}\mathcal{N}(v). \tag{3.10}$$

Also, let us remark that $\|\{A_k(t)\}\|_{\ell_0^2} = \|v(t)\|_{L_\theta^2}$ and $\|\{kA_k(t)\}\|_{\ell_0^2} = \|\partial_\theta v(t)\|_{L_\theta^2}$. Then, multiplying $\overline{v}$ and taking the imaginary part of integration, we obtain (3.8). On the other hand, multiplying $\overline{\partial_t v}$ and taking the real part of integration, we have

$$\begin{aligned} 0 \;=\; & -\frac{a^2}{4t^2}\frac{d}{dt}\|\partial_\theta v\|_{L_\theta^2}^2 + \frac{2\mathrm{Re}\lambda}{p+1}|4\pi t|^{-(p-1)/2}\frac{d}{dt}\|v\|_{L_\theta^{p+1}}^{p+1} \\ & -2(\mathrm{Im}\lambda)|4\pi t|^{-(p-1)/2}\mathrm{Im}\langle\mathcal{N}(v),\partial_t v\rangle_\theta. \end{aligned} \tag{3.11}$$

To estimate $\mathrm{Im}\langle\mathcal{N}(v),\partial_t v\rangle_\theta$ in (3.11), let us multiply $\overline{\mathcal{N}(v)}$ on both hand sides of (3.10). Then, we see that

$$\begin{aligned} \mathrm{Im}\langle\mathcal{N}(v),\partial_t v\rangle_\theta \;&=\; -\frac{a^2}{4t^2}\mathrm{Re}\langle\partial_\theta^2 v,\mathcal{N}(v)\rangle_\theta + (\mathrm{Re}\lambda)|4\pi t|^{-(p-1)/2}\|v\|_{L_\theta^{2p}}^{2p} \\ &\ge\; (\mathrm{Re}\lambda)|4\pi t|^{-(p-1)/2}\|v\|_{L_\theta^{2p}}^{2p}, \end{aligned} \tag{3.12}$$

since $\mathrm{Re}\langle\partial_\theta^2 v,\mathcal{N}(v)\rangle_\theta \le 0$. Combining (3.11) and (3.12), we have

$$\frac{d}{dt}\|\partial_\theta v\|_{L_\theta^2}^2 + K_1(\mathrm{Re}\lambda)t^{(5-p)/2}\frac{d}{dt}\|v\|_{L_\theta^{p+1}}^{p+1} - K_2(\mathrm{Im}\lambda)(\mathrm{Re}\lambda)t^{3-p}\|v\|_{L_\theta^{2p}}^{2p} \le 0, \tag{3.13}$$

where $K_1 = \dfrac{8}{(p+1)a^2(4\pi)^{(p-1)/2}}$ and $K_2 = \dfrac{8}{a^2(4\pi)^{p-1}}$. This is equivalent to

$$\frac{d}{dt}E(t) \le \frac{(5-p)K_1\mathrm{Re}\lambda}{2}t^{(3-p)/2}\|v\|_{L_\theta^{p+1}}^{p+1}, \tag{3.14}$$

where

$$E(t) = \|\partial_\theta v\|_{L^2_\theta}^2 + K_1(\mathrm{Re}\lambda)t^{(5-p)/2}\|v\|_{L^{p+1}_\theta}^{p+1} - K_2(\mathrm{Im}\lambda)(\mathrm{Re}\lambda)\int_{t_0}^t \tau^{3-p}\|v(\tau)\|_{L^{2p}_\theta}^{2p}\,d\tau.$$

In this proof, we only consider the most complicated case that $\mathrm{Im}\lambda$ and $\mathrm{Re}\lambda < 0$. The other case follows more easily. By (3.14), we have $E(t) \le$ (const.) for $t > t_0$, i.e.,

$$\|\partial_\theta v\|_{L^2_\theta}^2 \le C_1 + C_2 t^{(5-p)/2}\|v\|_{L^{p+1}_\theta}^{p+1} + C_3\int_{t_0}^t \tau^{3-p}\|v(\tau)\|_{L^{2p}_\theta}^{2p}\,d\tau \tag{3.15}$$

for some positive constants C_1, C_2 and C_3. Applying the Gagliardo-Nirenberg inequalities:

$$\begin{aligned}\|v\|_{L^{p+1}_\theta}^{p+1} &\le C\|v\|_{H^1_\theta}^{(p+1)\beta}\|v\|_{L^2_\theta}^{(p+1)(1-\beta)},\\ \|v\|_{L^{2p}_\theta}^{2p} &\le C\|v\|_{H^1_\theta}^{2p\gamma}\|v\|_{L^2_\theta}^{2p(1-\gamma)},\end{aligned}$$

where $1/(p+1) = \beta(1/2-1) + (1-\beta)2$ and $1/2p = \gamma(1/2-1) + (1-\gamma)/2$, and using Young's inequality, we have

$$\|v(t)\|_{H^1_\theta}^2 \le C\langle t\rangle^3 + \int_{t_0}^t \|v(\tau)\|_{H^1_\theta}^2\,d\tau. \tag{3.16}$$

We here note that, since $\|v(t)\|_{L^2}$ has a finite bound in virture of (3.8), it is included in the positive constant C. Applying Gronwall's inequality to (3.16), we obtain (3.9). ❑

Proof of Theorem 3.2. If $\mathrm{Im}\lambda > 0$, then, Lemma 3.5 (3.8) and Hölder's inequality $\|v\|_{L^{p+1}_\theta}^{p+1} \ge (2\pi)^{-(p-1)/2}\|v\|_{L^2_\theta}^{p+1}$ give

$$\frac{d}{dt}\|v\|_{L^2_\theta}^2 \ge C\|v\|_{L^2_\theta}^{p+1}.$$

This implies that $\|v(t)\|_{L^2_\theta} = \|\{A_k(t)\}\|_{\ell^2_0}$ blows up in positive finite time. On the other hand, if $\mathrm{Im}\lambda \le 0$, then, Lemma 3.5 gives the a priori bound of $\|\{A_k(t)\}\|_{\ell^2_1}$ for any positive t. Hence, the local solution to (3.4) is continuated to the global one. ❑

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Time local well-posedness for Benjamin-Ono equation with large initial data

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We consider the initial value problem for the Benjamin-Ono equation:

$$\begin{cases} \partial_t u + \mathcal{H}_x \partial_x^2 u + u\partial_x u = 0, & x, t \in \mathbf{R}, \\ u(x,0) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{0.1}$$

where $\mathcal{H}_x$ denotes the Hilbert transform, i.e., $\mathcal{H}_x = \mathcal{F}^{-1}(-i\xi/|\xi|)\mathcal{F}$. The equation (0.1) arises in the study of long internal gravity waves in deep stratified fluid.

We present the time local well-posedness of (0.1). Namely, we prove the existence, uniqueness of the solution and the continuous dependence on the initial data. There are several known results about this problem. One of their concern is to overcome the regularity loss arising from the nonlinearity. Because of this difficulty, the contraction mapping principle via the associated integral equation does not work as long as we consider the estimates only in the Sobolev space $H_x^{s,0}$, where $H_x^{s,\alpha}$ is defined by

$$H_x^{s,\alpha} = \{f \in \mathcal{S}'(\mathbf{R}); \|f\|_{H_x^{s,\alpha}} < \infty\}$$

with $\|f\|_{H_x^{s,\alpha}} = \|\langle x\rangle^\alpha \langle D_x\rangle^s f\|_{L_x^2}$, $\langle x\rangle^\alpha = (1+x^2)^{\alpha/2}$ and $\langle D_x\rangle^s = \mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}$. Indeed, Molinet-Saut-Tzvetkov [6] negatively proved the solvability of the integral equation in $H_x^{s,0}$ for any $s \in \mathbf{R}$.

Recently, Koch-Tzvetkov [4] (see also Ponce [7]) have studied the local well-posedness with $s > 5/4$ due to the cut off technique of $\mathcal{F}u(\xi)$. Furthermore, Kenig-Koenig [2] proved the local well-posedness with $s > 9/8$. We remark here that it is possible to minimize the regularity of u_0 by inducing another kind of function space. In fact, Kenig-Ponce-Vega [3] construct a time local solution via the integral equation by applying the smoothing property like

$$\|D_x \int_0^t V(t-t')F(t')dt'\|_{L_x^\infty(L_T^2)} \le C\|F\|_{L_x^1(L_T^2)},$$

where $\|u\|_{L_x^p(L_T^r)} = \|(\|u\|_{L^r[0,T]})\|_{L_x^p(\mathbf{R})}$, $D_x = \mathcal{F}^{-1}|\xi|\mathcal{F}$ and $V(t) = \exp(-t\mathcal{H}_x\partial_x^2)$. They obtained the time local well-posedness in $H_x^{s,0}$ $(s > 1)$ for the cubic nonlinearity (Their argument is also applicable to the quadratic case if u_0 satisfies $u_0 \in H_x^{s,0}$ $(s > 1)$ and the additional weight condition). In their result, however, the smallness of the initial data is required. This is because the inclusion $L_x^1(L_T^\infty) \cdot L_x^\infty(L_T^2) \subset L_x^1(L_T^2)$ yields $\|u\|_{L_x^1(L_T^\infty)}$ in the nonlinearity and we can not expect that $\|u\|_{L_x^1(L_T^\infty)} \to 0$ even when $T \to 0$.

Our concern in this talk is to remove this smallness condition of u_0. Before presenting the rough sketch of our idea, we introduce the function space Y_T in which the solution is constructed:

$$Y_T = \{u : [0,T] \times \mathbf{R} \to \mathbf{R}; |||u|||_{Y_T} < \infty\},$$

where $|||u|||_{Y_T} = \|u\|_{L_T^\infty(H_x^{s,0} \cap H_x^{s_1,\alpha_1})} + \|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u\|_{L_x^{1/\varepsilon}(L_T^2)} + \|\langle D_x\rangle^{\mu}\langle x\rangle^{\alpha_1}u\|_{L_x^2(L_T^\infty)}$ with $\rho, \mu > 0$ sufficiently small and $0 < \varepsilon < \rho$. We first consider the modified equation such that

$$\begin{cases} \partial_t u_\nu + \mathcal{H}_x \partial_x^2 u_\nu + u_\nu \partial_x \eta_\nu * u_\nu = 0, \\ u_\nu(0,x) = u_0(x), \end{cases} \tag{0.2}$$

where $\eta_\nu(x) = \nu^{-1}\eta(x/\nu)$ with $\eta \in C_0^\infty, \int \eta(x)dx = 1$ and $\nu \in (0,1]$. Then, the existence of u_ν in Y_T easily follows and it is continuated as long as $\|u_\nu(t)\|_{H_x^{s,0} \cap H_x^{s_1,\alpha_1}} < \infty$. Note that $|||u_\nu|||_{Y_T}$ is continuous with respect to T. To seek for the a priori estimate of $|||u_\nu|||_{Y_T}$, we deform (0.2). Let $\varphi \in C_0^\infty(\mathbf{R})$ and write $u_\nu \partial_x \eta_\nu * u_\nu = \varphi \partial_x \eta_\nu * u_\nu + (u_\nu - \varphi)\partial_x \eta_\nu * u_\nu$. Note here that, if φ is close to u_0, one can make $u_\nu - \varphi$ sufficiently small when $t \to 0$. To control $\varphi(\partial_x \eta_\nu * u_\nu)$, we employ the gauge transform so that this quantity is, roughly speaking, absorbed in the linear operator. Then, our desired a priori estimate follows via the integral equation. As for the convergence of nonlinearity $u_\nu \partial_x \eta_\nu * u_\nu \to u\partial_x u$, we also consider the estimate of $u_\nu - u_{\nu'}$. Let us now state our main theorem.

Theorem 0.1 *(i) Let $u_0 \in H_x^{s,0} \cap H_x^{s_1,\alpha_1} \equiv X^s$ with $s_1 + \alpha_1 < s$, $1/2 < s_1$ and $1/2 < \alpha_1 < 1$. Then, for some $T = T(u_0) > 0$, there exists a unique solution to (0.1) such that $u \in C([0,T]; X^s) \cap Y_T$.*
(ii) Let $u'(t)$ be the solution to (0.1) with the initial data u_0' satisfying $\|u_0' - u_0\|_{X^s} < \delta$. If $\delta > 0$ is sufficiently small, then there exist some $T' \in (0,T)$ and $C > 0$ such that

$$\|u' - u\|_{L_{T'}^\infty(X^s)} \le C\|u_0' - u_0\|_{X^s},$$
$$\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}(u' - u)\|_{L_x^{1/\varepsilon}(L_{T'}^2)} \le C\|u_0' - u_0\|_{X^s}.$$

In Theorem 0.1, the conditions on the initial data are determined by the estimate of maximal function, where, we call $\|f(\cdot,x)\|_{L_T^\infty}$ the maximal function of $f(t,x)$. Concretely speaking, the quantity $\|u\|_{L_x^1(L_T^\infty)}$ is bounded by $C(\|u_0\|_{H_x^{s,0}} + \|u_0\|_{H_x^{s_1,\alpha_1}})$.

Remark. Recently, Tao [8] has studied the global well-posedness in $H_x^{1,0}$ but the L^2-stability of the data-to-solution map holds while the initial data belongs to $H_x^{1,0}$, i.e., $\|u'(t) - u(t)\|_{L^2} \le C\|u_0' - u_0\|_{H_x^{1,0}}$.

We also remark that Koch-Tzvetkov [6] negatively proved the strong stability like

$$\|u'(t) - u(t)\|_{H_x^{s,0}} \leq C\|u_0' - u_0\|_{H_x^{s,0}} \quad \text{for } s > 0,$$

if there is no weight condition on u_0 and u_0'. Though our result requires slightly large regularity in comparison with Tao's work, it suggests that the additional weight condition yields the strong stability of the data-to-solution map in the sense that its target space coincides with that of initial data. Recently, K. Kato [1] obtained the similar result via the Fourier restriction method.

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On the nonlinear Schrödinger limit of the Klein-Gordon-Zakharov system[1]

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1. Introduction

The nonlinear Schrödinger equation (NLS) describes nonlinear dispersive wave propagation in various phenomena. One of such contexts is the plasma physics, where one derives the nonlinear Schrödinger equation via the Klein-Gordon-Zakharov system (KGZ) or the simpler Zakharov system (Z), starting from the fluid equations of the ions and the electrons, coupled to the Maxwell equations (see e.g., [7]). In this talk, we consider convergence of solutions in the limit from (KGZ) to (NLS).

This problem has two difficulties, which seem typical to this kind of singular limits. One is that the bilinear interactions have certain resonant frequency which tends to infinity in the limit. Bilinear estimates can not control this part, because of non-oscillatory interaction at this frequency. The other is that the conserved energy tends to infinity in the limit. Hence the energy itself can not bound the solution along the limit.

To overcome these difficulties, we decompose the solution into the resonant part and the remaining non-resonant part in the frequency, and apply modified nonlinear energy and bilinear estimates, so that they can compliment each other.

In the following, $\widetilde{u}$ and $\widehat{u}$ denote the Fourier transforms in the space and in the space-time respectively. H^s denotes the Sobolev space on $\mathbb{R}^3$. We denote

$$\langle a\rangle := \sqrt{|a|^2+1}, \quad \widetilde{f(\nabla)u} := f(i\xi)\widetilde{u}. \tag{1}$$

2. Formal limit

First we formally derive (NLS) from (KGZ). The Klein-Gordon-Zakharov system (KGZ) reads

$$\begin{cases} c^{-2}\ddot{E} - \Delta E + c^2 E = nE, \\ \alpha^{-2}\ddot{n} - \Delta n = -\Delta|E|^2, \end{cases} \tag{2}$$

where $E : \mathbb{R}^{1+3} \to \mathbb{R}^3$ and $n : \mathbb{R}^{1+3} \to \mathbb{R}$ are unknown functions, and c, α are given parameters. In the context of the plasma physics, E and n approximately describe the electric field and the ion density fluctuation respectively, c^2 is the plasma frequency[2], α is the ion sound speed and we usually have $c \gg \alpha \gg 1$, which is the physical ground for the following approximation. Replacing $E = \Re(e^{ic^2 t}E)$ and neglecting the complex conjugate[3], we get the usual Zakharov system (Z) in the limit $c \to \infty$:

$$\begin{cases} 2i\dot{E} - \Delta E = nE, \\ \alpha^{-2}\ddot{n} - \Delta n = -\Delta|E|^2. \end{cases} \tag{3}$$

[1] Joint work with Nader Masmoudi.

[2] In the real physical model, the solenoidal part of E obeys faster propagation speed.

[3] We can not really ignore the complex conjugate and their interactions; see our theorem for a correct treatment.

Substitution $\alpha \to \infty$ gives the nonlinear Schrödinger equation:

$$2i\dot{E} - \Delta E = |E|^2 E, \quad n = |E|^2. \tag{4}$$

3. Known results

The above equations have the following conserved energy respectively:

$$\begin{aligned}
\mathcal{E} &= \int_{\mathbb{R}^3} |c^{-1}\dot{E}|^2 + |\nabla E|^2 + |cE|^2 + \frac{|\alpha^{-1}|\nabla|^{-1}\dot{n}|^2 + |n|^2}{2} - n|E|^2 dx, \quad (KGZ),\\
\mathcal{E} &= \int_{\mathbb{R}^3} |\nabla E|^2 + \frac{|\alpha^{-1}|\nabla|^{-1}\dot{n}|^2 + |n|^2}{2} - n|E|^2 dx, \quad (Z),\\
\mathcal{E} &= \int_{\mathbb{R}^3} |\nabla E|^2 - \frac{|E|^4}{2} dx, \quad (NLS),
\end{aligned} \tag{5}$$

so the energy class of solutions are defined by the following:

$$\begin{aligned}
(KGZ) &: (E(t), \dot{E}(t), n(t), |\nabla|^{-1}\dot{n}(t)) \in H^1 \times L^2 \times L^2 \times L^2,\\
(Z) &: (E(t), n(t), |\nabla|^{-1}\dot{n}(t)) \in H^1 \times L^2 \times L^2\\
(NLS) &: E(t) \in H^1.
\end{aligned} \tag{6}$$

Local wellposedness is known to hold in the above class for each equation[4]. Indeed, (Z) and (NLS) is wellposed even in larger spaces, such as $H^{1/2} \times L^2$. See [5, 4, 3].

On the other hand, the convergence of solutions in the limit has been proved only in Sobolev spaces with much higher regularity. The convergence from (Z) to (NLS) was proved by [6] in H^5, assuming that $n = |E|^2$ at $t = 0$. [1] proved the convergence in H^6, assuming smallness of the energy, but allowing the initial layer. The convergence from (KGZ) to (Z) was proved in [2] in H^s for $s > 7/2$. We are not aware of any result in the literature for the convergence from (KGZ) to (NLS).

4. Main difficulties

There is an essential obstacle which prevents those wellposedness results from deducing the convergence in the same spaces. Local solutions are usually constructed by using the iteration scheme such as

$$\begin{cases} 2i\dot{E}_k - \Delta E_k = n_{k-1}E_{k-1},\\ \alpha^{-2}\ddot{n}_k - \Delta n_k = -\Delta|E_{k-1}|^2. \end{cases} \tag{7}$$

This iteration sequence (E_k, n_k) in general does not converge uniformly for large α. More precisely, we have the following.

Theorem 1. *For any $s \in \mathbb{R}$, there exist $\varphi \in H^s$ such that the second iteration E_2 defined by (7) together with the initial condition*

$$E_k(0) = \varphi, \quad n_k(0) = \dot{n}_k(0) = 0, \tag{8}$$

satisfies

$$\limsup_{\alpha\to\infty} \|E_2(t)\|_{H^{s-1+\varepsilon}} = \infty, \tag{9}$$

[4]The local wellposedness of (KGZ) in the energy class is known in the case $c \neq \alpha$. The case $c = \alpha$ seems to be open.

for any $0 < t \ll 1$ *and* $\varepsilon > 0$.

We have a similar result for (KGZ). Thus we need something else to bound the nonlinearity in the limit. The above divergence occurs only at a certain resonant frequency, where the bilinear estimate can not derive any cancellation from oscillatory integrals. The resonant frequency is given by the intersection in the space-time Fourier space of the characteristic surfaces of the Schrödinger equation and of the wave equation, where both the linear equations behave the same. This portion does not matter at all if the parameter α is fixed, because the resonant frequency then remains bounded.

Another standard way to control the nonlinearity is to use the conserved energy. In fact, the above convergence results for (Z) rely mostly on nonlinear energy argument, or the conservation structure. In the case of (KGZ), however, we encounter another difficulty in using the energy. Namely, the conserved energy $\mathcal{E}$ is not bounded uniformly for $c \to \infty$, because of the term $|cE|^2$. The energy for $e^{ic^2t}E$ (or $e^{-ic^2t}E$), which appears formally to be bounded, is useless in our context: forcing a uniform bound on that energy makes the limit solution vanish identically, as long as the original E is real-valued.

5. Main result

Theorem 2. *Let* $s > 3/2$ *and* $0 < \gamma < 1$. *Let* E *and* n *be the solution of (KGZ), and assume that* $(E(0), c^{-1}\langle\nabla/c\rangle^{-1}\dot{E}(0))$ *converges in* H^s *and* $n(0)$ *is bounded in* H^{s-1}, *as* $(c, \alpha) \to \infty$ *with* $\alpha/c < \gamma$. *Let* $\mathbb{E}^\infty = (E_+^\infty, E_-^\infty)$ *be the solution of the nonlinear Schrödinger equation*

$$\begin{cases} 2i\mathbb{E}^\infty - \Delta\mathbb{E}^\infty - |\mathbb{E}^\infty|^2\mathbb{E}^\infty = 0, \\ E_+^\infty(0) = \lim \frac{1}{2}(E(0) - ic^{-1}\langle\nabla/c\rangle^{-1}\dot{E}(0)), \\ E_-^\infty(0) = \lim \frac{1}{2}(E(0) - \overline{ic^{-1}\langle\nabla/c\rangle^{-1}\dot{E}(0)}), \end{cases} \tag{10}$$

and denote its maximal existence time by T^∞. *Then the maximal existence time* T *of* (E, n) *satisfies* $\liminf T \geq T^\infty$, *and* (E, n) *have the following asymptotic behavior:*

$$\begin{aligned} &E - (\mathbb{E}_+^\infty e^{ic^2t} + \overline{\mathbb{E}_-^\infty} e^{-ic^2t}) \to 0 \text{ in } C([0, T^\infty); H^s), \\ &n - n_f - |\mathbb{E}^\infty|^2 \to 0 \text{ in } C([0, T^\infty); H^{s-1}), \end{aligned} \tag{11}$$

where n_f *is the free wave solution* $\alpha^{-2}\ddot{n}_f - \Delta n_f = 0$ *satisfying the initial condition*

$$n(0) - n_f(0) = |\mathbb{E}^\infty(0)|^2, \quad \dot{n}(0) - \dot{n}_f(0) = -\Im[\overline{\mathbb{E}^\infty(0)} \cdot \Delta\mathbb{E}^\infty(0)]. \tag{12}$$

We have a similar result for the limit from (Z) to (NLS). The convergence from (KGZ) to (Z) is much easier and can be proved in the energy class. On the other hand, the limit equation changes if the plasma frequency c^2 is comparable to, or smaller than the ion sound speed α. For example, if we take the limit $\alpha \to \infty$ first and $c \to \infty$ later, then we get a different limit system

$$\begin{cases} 2i\dot{E}_+^\infty - \Delta E_+^\infty - (|E_+^\infty|^2 + 2|E_-^\infty|^2)E_+^\infty = 0, \\ 2i\dot{E}_-^\infty - \Delta E_-^\infty - (2|E_+^\infty|^2 + |E_-^\infty|^2)E_-^\infty = 0. \end{cases} \tag{13}$$

6. Outline of proof

The main part of proof is deriving estimates which are uniform in (c,α). We first rewrite the equation into the first order system by putting

$$\begin{aligned} &e^{ic^2t}E_+ := \frac{1}{2}(E - ic^{-1}\langle\nabla/c\rangle^{-1}\partial_t E), \quad e^{ic^2t}E_- = \frac{1}{2}(\overline{E} - ic^{-1}\langle\nabla/c\rangle^{-1}\partial_t\overline{E}), \\ &N := n - i\alpha^{-1}|\nabla|^{-1}\partial_t n. \end{aligned} \tag{14}$$

Let $\mathbb{E} := (E_+, E_-)$ and $\mathbb{E}^* := e^{-2ic^2t}(\overline{E_-}, \overline{E_+})$. Then (KGZ) is reduced to

$$\begin{cases} 2i\dot{\mathbb{E}} - \Delta_c\mathbb{E} = \langle\nabla/c\rangle^{-1}n(\mathbb{E}+\mathbb{E}^*), \\ i\dot{N} + \alpha|\nabla|N = \alpha|\nabla|\langle\mathbb{E}, \mathbb{E}+\mathbb{E}^*\rangle, \end{cases} \tag{15}$$

where we denote $\Delta_c = 2c^2(1-\langle\nabla/c\rangle)$. The original function is given by $E = e^{ic^2t}E_+ + e^{-ic^2t}\overline{E_-}$, $n = \Re N$. The initial condition implies that

$$\mathbb{E}(0) \to \mathbb{E}^\infty(0) \text{ in } H^s, \quad N(0) \text{ bounded in } H^{s-1}, \tag{16}$$

and the convergence result is equivalent to

$$\begin{aligned} &\mathbb{E} \to \mathbb{E}^\infty \text{ in } C([0,T);H^s), \\ &N - N_f - |\mathbb{E}^\infty|^2 \to 0 \text{ in } C([0,T);H^{s-1}), \end{aligned} \tag{17}$$

where $N_f = e^{i\alpha|\nabla|t}(N(0) - |\mathbb{E}^\infty(0)|^2)$.

Next we decompose the solution $(\mathbb{E}, N)$ into the resonant part and non-resonant part. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function satisfying $\varphi(\xi) = 1$ for $2/3 < |\xi| < 3/2$ and $\varphi(\xi) = 0$ for $|\xi| < 1/2$ and $|\xi| > 2$. We define the resonant frequency by

$$M := 2\alpha c^2/(c^2-\alpha^2), \quad \widetilde{\mathbb{E}_M} := \varphi(\xi/M)\widetilde{\mathbb{E}}, \quad \mathbb{E}_X := \mathbb{E} - \mathbb{E}_M. \tag{18}$$

We define N_M and N_X in the same way.

The main uniform estimates are given in the following spaces. Fix a small $\kappa > 0$ such that $s - 3\kappa > 3/2$ and denote $2+ := 1/(1/2-\kappa)$. We introduce the Strichartz space-time norms by

$$\begin{aligned} &\mathrm{Str}^E(0,T) := L^\infty(0,T;H^s) \cap \langle\nabla/c\rangle^{1/2-\kappa}L^{2+}(0,T;B^{s-1/2+\kappa}_{1/\kappa}), \\ &\mathrm{Str}^n(0,T) := L^\infty(0,T;H^{s-1}) \cap \alpha^{-1/2+\kappa}L^{2+}(0,T;B^{s-2+2\kappa}_{1/\kappa}), \end{aligned} \tag{19}$$

where $B^s_p := B^s_{p,2}$ denotes the Besov space on $\mathbb{R}^3$. The latter space on each line is close to the endpoint Strichartz norm. We define the Fourier restriction norms by

$$\begin{aligned} &X^{s,1}(0,T) := e^{i\Delta_c t/2}H^1(0,T;H^s_x), \\ &Y^{s,1}(0,T) := e^{i\alpha|\nabla|t}H^1(0,T;H^s_x). \end{aligned} \tag{20}$$

We prove that the following norms are bounded uniformly for (c,α).

$$\begin{aligned} &\|\mathbb{E}\|_{(0,T)} := \|\mathbb{E}\|_{L^\infty(0,T;H^s)\cap X^{s-1,1}(0,T)} + \|\mathbb{E}_X\|_{\mathrm{Str}^E(0,T)}, \\ &\|N\|_{(0,T)} := \|N\|_{L^\infty(0,T;H^{s-1})\cap\alpha Y^{s-1,1}(0,T)} + \|N_X\|_{\mathrm{Str}^n(0,T)}. \end{aligned} \tag{21}$$

We derive the uniform estimates by the bootstrap argument: we estimate the above norms by themselves with a small factor depending on the time interval T. The outline is as follows. The standard energy estimate yields bounds on the $X^{s-1,1}$ and $\alpha Y^{s-1,1}$ norms in terms of $\|\mathbb{E}\|_{L^\infty H^s}$ and $\|N\|_{L^\infty H^{s-1}}$. We can use the Strichartz estimate to control the high-high interactions in the space-time norms $\mathrm{Str}^E, \mathrm{Str}^n$. For the low-high interactions, we use the bilinear estimate to recover the derivative loss, where we have to restrict the frequency to the non-resonant part. Hence we get bounds for the non-resonant part in the space-time norms. For the resonant frequency part, we use a modified energy, which is localized in the frequency and does not contain the divergence factor $|cE|^2$. Thus we obtain a bound in $L^\infty H^s \times L^\infty H^{s-1}$, which closes our bootstrap argument.

7. Bilinear estimate

The bilinear estimate uses the following geometric property of the non-resonant interaction. We consider the trilinear form

$$\langle \Re(N)E \mid F\rangle_{t,x} = \int_{\mathbb{R}^{1+3}} \Re(N)E\overline{F}dtdx, \tag{22}$$

for general functions N, E, F. We decompose each function at a distance $\delta > 0$ from each characteristic surface:

$$\begin{aligned} N &= N^C + N^F, \quad \widehat{N^C}(\tau,\xi) = \chi((\tau - |\xi|)/\delta)\widehat{N}(\tau,\xi), \\ E &= E^C + E^F, \quad \widehat{E^C}(\tau,\xi) = \chi((\tau - \omega(\xi))/\delta)\widehat{E}(\tau,\xi), \\ F &= F^C + F^F, \quad \widehat{F^C}(\tau,\xi) = \chi((\tau - \omega(\xi))/\delta)\widehat{F}(\tau,\xi), \end{aligned} \tag{23}$$

where $\omega(\xi) := 2c^2(\langle \xi/c\rangle - 1)$ and $\chi \in C_0^\infty(\mathbb{R})$ is a cut-off function satisfying $\chi(t) = 1$ for $|t| < 1$ and $\chi(t) = 0$ for $|t| > 2$. Furthermore, we restrict each function into spatial frequency j, k, l, respectively, using the following notation $\widetilde{u_j} := \varphi(\xi/j)\widetilde{u}$. The non-resonant property can be formulated as follows. Assume that

$$k \ll j \sim l \not\sim M, \quad \delta \ll (\alpha + \min(c,l))l. \tag{24}$$

Then we have $\langle \Re(n_j^C)E_k^C \mid F_l^C\rangle_{t,x} = 0$. In other words, one of the three functions at least has to be away from the characteristic. For that function, we use the $X^{s-1,1}$ type norm. For instance, we have

$$\|E^F\|_{L^2H^{s-1}} \lesssim \delta^{-1}\|E\|_{X^{s-1,1}}, \tag{25}$$

which gives us, roughly speaking, two derivative gain, compensating the derivative loss.

There is no resonance in the terms including $\mathbb{E}^*$: it holds $\langle \Re(n_j^C)E_k^{C*} \mid E_l^C\rangle_{t,x} = 0$ whenever $\delta \ll c(c + j + k + l)$, without any condition on j, k, l.

8. Modified energy for resonant part

We define the following modified energy to control the resonant part.

$$\mathcal{E}_M := 2(\Delta_c\mathbb{E} \mid \langle\nabla/c\rangle\mathbb{E}_M)_x + (N \mid N_M)_x/2 - (n_M\mathbb{E}_M \mid \mathbb{E}_M)_x, \tag{26}$$

where $(f \mid g)_x$ denotes the $L^2(\mathbb{R}^3)$ inner product. Using the equation, we have

$$\begin{aligned} \partial_t\mathcal{E}_M &= -2(n\mathbb{E} - n_M\mathbb{E}_M + n\mathbb{E}^* \mid i\Delta_c\mathbb{E}_M)_x \\ &\quad + (n_M\mathbb{E}_M \mid iP_M\langle\nabla/c\rangle^{-1}n(\mathbb{E} + \mathbb{E}^*))_x \\ &\quad + (i\alpha|\nabla|N_M \mid \langle\mathbb{E},\mathbb{E}\rangle - \langle\mathbb{E}_M,\mathbb{E}_M\rangle + \langle\mathbb{E},\mathbb{E}^*\rangle)_x. \end{aligned} \tag{27}$$

Notice that the error terms do not contain any triple interaction of the resonant part $(\mathbb{E}_M, N_M)$. They can be estimated by using the Strichartz estimate for the high-high interaction and the bilinear estimate for the low-high interaction, just in the same spirit as in estimating the non-resonant part. The lower bound $3/2$ of the regularity is the most crucial in this step, where we are forced to bound n in L^∞_x.

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Schrödinger Maps:
Local regularity and singularity formation

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Abstract

Schrödinger Maps are a generalization of the Landau Lifschitz equations that arise in ferromagnets. We will discuss the physical meaning of these equations and the equivalence between Schrödinger Maps and nonlinear Schrödinger equations. We will show how this equivalence leads to local and global well posedness results. We will also discuss how singularities might form for data with large energy.

Existence and uniqueness of the solution to the modified Schrödinger maps

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We consider the local well-posedness of the initial value problem for the system of the nonlinear Schrödinger equations on $(0,T)\times\mathbf{R}^2$,

$$(\mathrm{MSM})\begin{cases} i\,\partial_t u_1 + \Delta u_1 = -2i\,A\cdot\nabla u_1 + Bu_1 + |A|^2 u_1 + i\,\Im(u_2\bar{u}_1)u_2, \\ i\,\partial_t u_2 + \Delta u_2 = -2i\,A\cdot\nabla u_2 + Bu_2 + |A|^2 u_2 + i\,\Im(u_1\bar{u}_2)u_1, \\ u_1(0,x) = u_0^1(x), \quad u_2(0,x) = u_0^2(x), \quad x\in\mathbf{R}^2, \end{cases}$$

under the low regularity assumption on the initial data, where u_1, u_2 are complex valued functions (we set $u=(u_1,u_2)$ in the following), and $A=(A_1[u],A_2[u])$, $B=B[u]$ are defined by

$$A_j[u] = 2\,G_j * \Im(u_1\bar{u}_2), \quad j=1,2, \tag{1}$$

$$G_1(x) = \frac{1}{2\pi}\frac{x_2}{|x|^2}, \quad G_2(x) = -\frac{1}{2\pi}\frac{x_1}{|x|^2}, \tag{2}$$

$$B[u] = -\sum_{j,k=1}^{2} 2\big(R_jR_k\Re(u_j\bar{u}_k) + |u|^2\big). \tag{3}$$

Here, for a complex number z, $\Re z$ and $\Im z$ denotes the real part of z and the imaginary part respectively, and R_j denotes the Riesz transform. We notice that

$$\operatorname{div} A = 0 \quad \text{and} \quad \operatorname{rot} A = \partial_1 A_2 - \partial_2 A_1 = 2\Im(u_1\bar{u}_2)$$

hold from the definition of A. These properties are useful to construct the solution to (MSM) for the low regularity initial data.

The system (MSM) above is called the modified Schrödinger map which is derived by Nahmod-Stefanov-Uhlenbeck [3] from Schrödinger map from $\mathbf{R}\times\mathbf{R}^2$ to the unit sphere S^2 choosing an appropriate gauge change so that the first order derivatives of the Schrödinger map satisfy (MSM). Roughly speaking, well-posedness of (MSM) in H^s corresponds to the well-posedness of the Schrödinger map in H^{s+1}. As for the modified Schrödinger map, Nahmod-Stefanov-Uhlenbeck [4] showed the existence and uniqueness of the solution for the data $u_0\in H^s(\mathbf{R}^2)$ with $s>1$ by using the energy method. In this talk, we show the improvement of their result.

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Theorem 1. *Let $u_0 \in H^s(\mathbf{R}^2)$ for $s > 1/2$. Then, there exists $T > 0$ satisfying*

$$\min\{1, C/\big((1 + \|u_0\|_{L^2}^q)\|u_0\|_{H^s}^q\big)\} \le T \le 1,$$

and at least one solution $u \in L^\infty(0, T;\, H^s(\mathbf{R}^2))$ to (MSM) such that

$$J^\delta u \in L^p(0, T;\, L^q(\mathbf{R}^2)), \tag{4}$$

where $s - 1/2 > \delta > 2/q > 0$, $1/p = 1/2 - 1/q$, and $J^\delta = (I - \Delta)^{\delta/2}$.

Remark 2. (1) The modified Schrödinger map is invariant under the scale transformation

$$u(t, x) \mapsto \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Thus, the critical space for the well-posedness of the Cauchy problem (MSM) is considered to be $L^2(\mathbf{R}^2)$, which corresponds to the energy class for the original Schrödinger map.

(2) As is pointed out in [3, §3], it is not possible to go back directly from solutions of the (MSM) to the original Schrödinger map. However, a priori estimate and the estimate on the time of existence on the smooth solution to (MSM) are made use of in order to construct the low regularity solution to the Schrödinger map. See [3, §3] for details.

In the theorem above, the uniqueness of the solution is not known. However, we have the following result by using the Vladimirov's argument [6] (see also [5]).

Theorem 3. *Let $u_0 \in H^1(\mathbf{R}^2)$. We assume that u and v are solutions to (MSM) on $(0, T) \times \mathbf{R}^2$ in the distribution sense with the same data u_0 satisfying*

$$u,\, v \in C([0, T];\, L^2(\mathbf{R}^2)),$$
$$\|u\|_{L_T^\infty H_x^1} \le M, \quad \|v\|_{L_T^\infty H_x^1} \le M.$$

Then, we have $u(t) = v(t)$ in $L^2(\mathbf{R}^2)$ for $0 \le t \le T$.

Corollary 4. *If we assume $u_0 \in H^1(\mathbf{R}^2)$, then the solution in the class of Theorem 1 is uniquely determined.*

For the proof of Theorem 1 we use the compactness argument. Because the local well-posedness for smooth data is already known, our task is to show a priori estimate for the solution to (MSM). To recover the loss of the derivatives caused by the nonlinearity, the following type of estimate

$$\|J^s w\|_{L_T^p L_x^q} \lesssim \|w\|_{L_T^\infty H_x^{s+1/2+\varepsilon'}} + \|F\|_{L_T^2 H_x^{s-1/2}} \tag{5}$$

for the solution to $i\partial_t w + \Delta w = F$ is crucial in our argument, where p, q are the admissible exponent for Strichartz estimates, i.e. $1/p = 1/2 - 1/q$, $2 < q < \infty$, and $s \in \mathbf{R}$. Compared with the usual Strichartz estimate

$$\|J^s w\|_{L^p_T L^q_x} \lesssim \|w(0)\|_{H^s} + \|F\|_{L^1_T H^s_x},$$

estimate (5) says that we have a gain of regularity $1/2$ on the inhomogeneous term at the cost of a loss of regularity $1/2+\varepsilon'$ on the homogeneous term. This type of estimate is first appeared in Koch-Tzvetkov [2] and refined by Kenig-Koenig [1] in the context of the Benjamin-Ono equation.

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Conservation Laws with Vanishing Diffusion and Dispersion

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1 Introduction

We study scalar conservation laws with diffusion and dispersion terms:

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u - \delta \partial_x^3 u, \quad (x,t) \in \mathbf{R} \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0^{\varepsilon,\delta}(x), \quad x \in \mathbf{R}, \tag{1.2}$$

where $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ tend to zero and the initial data $u_0^{\varepsilon,\delta}$ is an approximation of a given initial condition $u_0 : \mathbf{R} \to \mathbf{R}$. We show that the sequence $u^{\varepsilon,\delta}$ of solutions for (1.1) converges to the unique entropy solution of the hyperbolic conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}, \tag{1.4}$$

under the assumption that the dispersion parameter δ is small compared with the diffusion parameter ε.

We recall that the existence and uniqueness of an entropy solution were first proved by S.N. Kružkov [11] for the Cauchy problem (1.3)-(1.4). Furthermore we recall that J.L. Bona-R. Smith [4] showed that there exist the solutions $u^{\varepsilon,\delta} \in L^\infty(0,T;H_L^k)$ $(T > 0)$ when the initial data $u_0^{\varepsilon,\delta} \in H_L^k$ for a positive integer k to the Korteweg-de Vries equation (1.1)-(1.2). Here $H^s = H^s(\mathbf{R})$ is the Sbolev space for integers $s \geq 0$.

There are many previous results for Eqs. (1.1)-(1.2): M.E. Schonbek, P.G. LeFloch-R. Natalini, etc. At first, M.E. Schonbek [20] gave a convergence

*This note is a joint work with Mitsuru Yamazaki (University of Tsukuba).

result under the assumption that either $\delta = O(\varepsilon^2)$ for $f(u) = u^2/2$, or $\delta = O(\varepsilon^3)$ for abitrary subquadratic flux functions f. Next P.G. LeFloch-R. Natalini [17] studied the equation with nonlinear diffusion and showed that the sequence $u^{\varepsilon,\delta}$ converges to the entropy weak solution under the assumption $\delta = o(\varepsilon^{1/r})$ $(r \geq 1)$. See also a convergence result for systems in B.T. Hayes-P.G. LeFloch [7].

On the other hand, when the flux is a smooth function with linear growth at infinity: $|f'(u)| \leq M$, for $u \in \mathbf{R}$, some $M > 0$, C.I. Kondo-P.G. LeFloch [10] proved that the sequence $u^{\varepsilon,\delta}$ for the linear diffusion and dispersion terms converge in $L^s(\mathbf{R}_+; L^p(\mathbf{R}))$ $(1 < s < \infty$ and $1 < p < 2)$ to a weak solution under the assumption $\delta = O(\varepsilon^2)$ and that the limit is the unique entropy solution in the sence of Kružkov under the stronger condition on $\delta = o(\varepsilon^2)$.

In this note, we restrict ourselves to assume that the flux function is a special form $f(u) = u^2/2$ i.e. Burgers equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \varepsilon\frac{\partial^2 u}{\partial x^2} - \delta\frac{\partial^3 u}{\partial x^3}.$$

Here we assume the diffusion term to be a natural form $\varepsilon\partial_x^2 u$ which did not apparently treated in [17]. For this equation, we obtain a similar result to P.G. LeFloch-R. Natalini, weakening a vanishing order between diffusion and dispersion terms without the condition: $|f'(u)| \leq M$, for $u \in \mathbf{R}$, some $M > 0$. More explicitly, P.G. LeFloch-R. Natalini showed that a sequence $u^{\varepsilon,\delta}$ converges strongly to $u \in L^\infty(0, T_*; L^q(\mathbf{R}))$ in $L^s(0, T_*; L^p(\mathbf{R}))$ $(s < \infty$ and $p < q)$ using a priori estimate, compensated compactness and Young measure. We improve a relation between ε and δ, a priori estimate being carried out for our conservation law. Then we describe that the sequence $u^{\varepsilon,\delta}$ of the smooth approximate solution converges to the unique entropy solution $u \in L^\infty(0, T_*; L^q(\mathbf{R}))$ in $L^s(0, T_*; L^p(\mathbf{R}))$ $(s < \infty$ and $p < q)$.

In the case of $f(u) = u^2/2$, by making a consideration for the travelling wave solution, which are of the form

$$u(x, t) = u\left(\frac{x - ct}{\varepsilon}\right), \quad c > 0,$$

we are led to the differential equation:

$$-cu' + uu' = u'' - \frac{\delta}{\varepsilon^2}u'''.$$

This structure suggests that for $\delta \leq k\varepsilon^2$ with a some appropriate k, there is convergence to the solution of Eq. (1.3), that is, if $\delta = O(\varepsilon^2)$, Eq. (1.1) could be reduced to a conservation law with diffusion:

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u,$$

it is rather trivial that the sequence $u^{\varepsilon,\delta}$ of solutions to Eq. (1.1) converge to the solution of the hyperbolic conservation law (1.3). On the other hand, if $\delta > k\varepsilon^2$, Eq. (1.1) could be reduced to a conservation law with dispersion:

$$\partial_t u + \partial_x f(u) + \delta \partial_x^3 u = 0,$$

it is considered that the solutions of Eq. (1.1) do not converge in general to the solution of Eq. (1.3) (cf. P.D. Lax-C.D Levermore [13, 14]) (See also Lax [12]). Contrary to the above suggestion, we will show that for $\delta = o(\varepsilon)$, even if $\delta > k\varepsilon^2$, the sequence $u^{\varepsilon,\delta}$ of solutions to Eq. (1.1) converge to the entropy weak solution of the hyperbolic conservation law (1.3).

In the section 2, we recall some important tools: Young measures, entropy measure-valued (m.-v.) solutions. In the section 3, we apply a priori estimates to a scalar conservation law (1.1). In the last section, we show that the sequence $u^{\varepsilon,\delta}$ of the solutions to Eqs. (1.1)-(1.2) converges to the unique entropy solution of Eq. (1.3).

2 Young measures and entropy mesure-valued solutions

In this section, we recall a generalization of the Young measures associated to sequences and entropy measure-valued solutions.

Lemma 2.1 *Let* $\{u_j\}$ *be a uniformly bounded sequence in* $L^\infty(\mathbf{R}_+; L^q(\mathbf{R}))$. *Then there exists a subsequence* $\{u_{j'}\}$ *and a weakly-*$\star$ *measurable mapping* $\nu : \mathbf{R} \times \mathbf{R}_+ \to \mathrm{Prob}(\mathbf{R})$ *such that, for all functions* $g \in \mathcal{C}(\mathbf{R})$ *satisfying*

$$g(u) \leq c(1 + |u|^r) \quad \text{for some } 0 < r < q, \tag{2.1}$$

the following limit holds

$$g(u_{j'}) \to \int_{\mathbf{R}} g(u) d\nu(u) \quad \text{as } j \to \infty$$

in $L^s(\mathbf{R}_+)$ *for some* $1 < s < q/r$, *i.e.*

$$\lim_{j'\to\infty} \iint_{\mathbf{R}\times\mathbf{R}} g(u_{j'}(x,t))\phi(x,t)dxdt = \iint_{\mathbf{R}\times\mathbf{R}} \int_{\mathbf{R}} g(\lambda)d\nu_{(x,t)}(\lambda)\phi(x,t)dxdt \tag{2.2}$$

for all $\phi \in C_0^\infty(\mathbf{R}\times\mathbf{R})$.

Here $\mathrm{Prob}(\mathbf{R})$ is the space of nonnegative Borel measures with unit total mass and the measure-valued function $\nu_{(x,t)}$ is a Young measure associated with the sequence $\{u_{j'}\}$ and

$$\langle \nu_y, g(\lambda)\rangle := \int_{\mathbf{R}} g(\lambda)d\nu_y.$$

Lemma 2.2 *Suppose that* ν *is a Young measure associated with a sequence* $\{u_j\}$, *uniformly bounded in* $L^\infty(\mathbf{R}_+; L^q(\mathbf{R}))$. *Then, for* $u \in L^\infty(\mathbf{R}_+; L^q(\mathbf{R}))$,

$$\lim_{j\to\infty} u_j = u \quad in \;\; L^\infty(\mathbf{R}_+; L^r_{loc}(\mathbf{R})) \quad for \;\; some \;\; 1 \le r < q$$

if and only if

$$\nu_{(x,t)}(\lambda) = \delta_{u(x,t)}(\lambda) \quad for \;\; a.e. \;\; (x,t) \in \mathbf{R}\times\mathbf{R}_+.$$

In the above, the notation $\delta_{u(x,t)}(\lambda) = \delta(\lambda - u(x,t))$ is used for the Dirac measure defined by

$$\iint_{\mathbf{R}\times\mathbf{R}_+} \langle \delta_{u(x,t)}, g(\cdot)\rangle \phi(x,t)dxdt = \iint_{\mathbf{R}\times\mathbf{R}_+} g(u(x,t))\phi(x,t)dxdt$$

for all $g \in \mathcal{C}(\mathbf{R})$ satisfying Eq. (2.1) and all $\phi \in C_0^\infty(\mathbf{R}\times\mathbf{R})$.

Next, following DiPerna [5], LeFloch-Natalini [17] and Szepessy [21], we define the measure-valued (m.-v.) solutions to the Cauchy problem (1.3)-(1.4).

Definition 2.1 *Let* $f \in C(\mathbf{R})$ *satisfy the growth condition (2.1) and* $u_0 \in L^1(\mathbf{R})\cap L^q(\mathbf{R})$. *A Young measure* ν *associated with the sequence* $\{u_j\}$, *which is assumed to be uniformly bounded in* $L^\infty(\mathbf{R}_+; L^q(\mathbf{R}))$, *is then called an entropy measure-valued (m.-v.) solution of Cauchy problem (1.3)-(1.4) if*

$$\partial_t\langle \nu_{(\cdot)}, |\lambda - k|\rangle + \partial_x\langle \nu_{(\cdot)}, \mathrm{sgn}(\lambda - k)(f(\lambda) - f(k))\rangle \le 0, \tag{2.3}$$

in $\mathcal{D}'(\mathbf{R}\times\mathbf{R}_+)$ *for all* $k \in \mathbf{R}$ *(i.e. in the sence of distribution), and for all compact sets* $K \subset \mathbf{R}$,

$$\lim_{T\to 0^+} \frac{1}{T}\int_0^T \int_K \langle \nu_{(x,t)}, |\lambda - u_0(x)|\rangle dxdt = 0. \tag{2.4}$$

The following theorems were proved in Szepessy [21], Theorem 2.1 states that entropy m.-v. solutions are solutions of Krŭzkov. Theorem 2.2 states that there exists a unique entropy solution to the Cauchy probrem (1.3)-(1.4).

Theorem 2.1 *Let f satisfy Eq. (2.1) and $u_0 \in L^1(\mathbf{R}) \cap L^q(\mathbf{R})$. Suppose that ν is an entropy m.-v. solution of Eqs. (1.3)-(1.4). Then there exists a function $w \in L^\infty(\mathbf{R}; L^1(\mathbf{R}) \cap L^q(\mathbf{R}))$ such that*

$$\nu_{(x,t)} = \delta_{w(x,t)} \quad for \ \ a.a. \ (x,t) \in \mathbf{R} \times \mathbf{R}_+. \tag{2.5}$$

Theorem 2.2 *Let f satisfy Eq. (2.1) and $u_0 \in L^1(\mathbf{R}) \cap L^q(\mathbf{R})$. Then there exists a unique entropy solution $u \in L^\infty(\mathbf{R}; L^1(\mathbf{R}) \cap L^q(\mathbf{R}))$ of Eqs. (1.3)-(1.4) which satisfies*

$$\| u(\cdot,t) \|_{L^r(\mathbf{R})} \leq \| u_0 \|_{L^r(\mathbf{R})} \quad for \ \ a.a \ t \in \mathbf{R}_+ \ \ and \ \ 1 \leq r \leq q. \tag{2.6}$$

Moreover the measure-valued mapping $\nu_{(x,t)} = \delta_{u(x,t)}$ is the unique entropy m.-v. solution of Eqs. (1.3)-(1.4).

Combining the above results, we obtain the following main convergence tool which was proved in P.G. LeFloch- R. Natalini [17].

Corollary 2.1 *Let f satisfy Eq. (2.1) and $u_0 \in L^1(\mathbf{R}) \cap L^q(\mathbf{R})$. Suppose that ν is a Young measure associated with a sequence $\{u_j\}$, unifomly bounded in $L^\infty(\mathbf{R}_+; L^q(\mathbf{R}))$ for $q \geq 1$. If ν is an entropy m.-v. solution of Eqs. (1.3)-(1.4), then*

$$\lim_{j\to\infty} u_j = u \quad in \ \ L^\infty(\mathbf{R}_+; L^r_{loc}(\mathbf{R})) \quad for \ \ all \ \ 1 \leq r < q,$$

where $u \in L^\infty(\mathbf{R}_+; L^q(\mathbf{R}))$ is the unique entropy solution of Eqs. (1.3)-(1.4).

In the framework of the above strategy, we establish several a priori estimates in the followoing section.

3 A priori estimates

In this section, we study a sequence $\{u^{\varepsilon,\delta}\}$ of smooth solutions to Eqs. (1.1)-(1.2) vanishing at infinity. We assume that the initial data $\{u_0^{\varepsilon,\delta}\}$ are smooth functions with compact support, uniformly bounded in $L^1(\mathbf{R}) \cap L^q(\mathbf{R})$

for $q > 1$. Moreover we assume that the flux function is a special form $f(u) = u^2/2$ i.e. Burgers equation. Here we recall again that there exist the solutions $u^{\varepsilon,\delta} \in L^\infty(0,T;H_L^k)$ when the initial data $u_0^{\varepsilon,\delta} \in H_L^k$ for a positive integer k to the Korteweg-de Vries equation (1.1)-(1.2).

There is a time $T_* \in (0,\infty]$ such that the initial problem (1.1)-(1.2) is well-posed in the strip $\mathbf{R} \times (0,T_*)$. Then we have

Lemma 3.1 *For $T \in (0,T_*)$, we have*

$$\int_{\mathbf{R}} u^2(x,T)dx + 2\varepsilon \int_0^T \int_{\mathbf{R}} u_x^2(x,t)dxdt = \int_{\mathbf{R}} u_0^2(x)dx \ . \tag{3.1}$$

Let be $F(u) = u^3/3$ then $F'(u) = f(u)$. We obtain the following

Lemma 3.2 *For every $T \in (0,T_*)$, we have*

$$\begin{aligned}&\frac{\delta}{2}\int_{\mathbf{R}} u_x^2(x,T)dx + \varepsilon\delta \int_0^T \int_{\mathbf{R}} u_{xx}^2 dxdt \\ &\quad = \frac{\delta}{2}\int_{\mathbf{R}} u_{0,x}^2(x)dx + \int_{\mathbf{R}} F(u(x,T))dx - \int_{\mathbf{R}} F(u_0(x))dx + \varepsilon \int_0^T \int_{\mathbf{R}} u u_x^2 dxdt.\end{aligned}$$

Using Lemma 3.1, we have obtained a norm of u in $L^\infty(0,T_*;L^2(\mathbf{R}))$ and of εu_x^2 in $L^1((0,T_*) \times \mathbf{R})$ which are both uniformly bounded with respect to $\varepsilon \in (0,1]$.

Now we introduce a condition for a general flux $f(u)$:

$$(A) \qquad \exists c_1 > 0,\ m > 1 \quad s.t. \quad |f'(u)| \le c_1(1 + |u|^{m-1}) \qquad for\ all\ \ u \in \mathbf{R},$$

we get an estimate of u in the L^∞ norm.

Lemma 3.3 *If $m < 5$ in condition (A), then for the solution to Eq. (1.1), there exist $c > 0$ such that*

$$|u(x,t)| \le c\delta^{-1/(5-m)} \qquad for\ \ all\ \ (x,t) \in \mathbf{R} \times (0,T_*).$$

In the case of $f(u) = u^2/2$, we can take $m = 2 \in (1,5)$. Using the same arguments as in the proof of Lemma 3.3, we also get:

Lemma 3.4 *For any $T \in (0,T_*)$, we have*

$$\frac{1}{2}\int_{\mathbf{R}} u_x^2(x,T)dx + \varepsilon \int_0^T \int_{\mathbf{R}} u_{xx}^2 dxdt \le c\delta^{-4/(5-m)}.$$

Moreover we get the following result by differentiating Eq. (1.1).

Lemma 3.5 *For any $T \in (0, T_*)$, we have*

$$\int_{\mathbf{R}} u_x^2(x,T)dx + \int_0^T \int_{\mathbf{R}} u_x^3(x,t)dxdt + 2\varepsilon \int_0^T \int_{\mathbf{R}} u_{xx}^2(x,t)dxdt = \int_{\mathbf{R}} u_{0,x}^2(x)dx. \tag{3.2}$$

From the above Lemmas, we are led to uniform bound in $L^\infty(0,T_*; L^q(\mathbf{R}))$ with $q < 6$ which is an improvement of the L^2 bound in Lemma 3.1.

Proposition 3.1 *Assume that $q + m < 8$ $(m < q)$ in (A). There exists a constant $C > 0$ (depending only on the initial data), such that, for all small enough δ and ε,*

$$\sup_{t\in(0,T_*)} ||u(\cdot,t)||_{L^q(\mathbf{R})}^q \leq C(1 + \delta^{(8-q-m)/(5-m)}). \tag{3.3}$$

When $\delta = 0$ in an estimate (3.3), Eqs. (1.1)-(1.2) are reduced to the conservation law with viscosity but no dispersion.

4 Convergence results

In this section, we show that the sequence $\{u^{\varepsilon,\delta}\}$ of solutions to Eqs. (1.1)-(1.2) converge to the unique entropy solution to Eqs. (1.3)-(1.4).

Assume again that the flux function $f(u) = u^2/2$ and that the initial data $\{u_0^{\varepsilon,\delta}\}$ are smooth functions with compact support, uniformly bounded in $L^1(\mathbf{R}) \cap L^q(\mathbf{R})$ for $q > 1$ and there exists a function $u_0 \in L^1(\mathbf{R}) \cap L^q(\mathbf{R})$ for $q > 1$ such that, if $\delta = O(\varepsilon)$,

$$\lim_{\varepsilon\to 0} u_0^{\varepsilon,\delta} = u_0 \qquad \text{in} \quad L^1(\mathbf{R}) \cap L^q(\mathbf{R}). \tag{4.1}$$

In the case of $f(u) = u^2/2$, the sequence $u^{\varepsilon,\delta}$ is uniformly bounded in $L^\infty(0,T_*; L^q(\mathbf{R}))$ from Proposition 3.1 for $q \in (2,6)$.

Now, we state main results of this note.

Theorem 4.1 *Let $u^{\varepsilon,\delta}$ be a sequence of the smooth solutions to Eqs. (1.1)-(1.2) defined on $\mathbf{R} \times (0,T_*)$, vanishing at infinity and associated with initial data $\{u_0^{\varepsilon,\delta}\}$ satisfying Eq. (4.1) with $q \in (2,6)$. If $\delta = o(\varepsilon)$, the sequence $u^{\varepsilon,\delta}$ of solutions converges to the unique entropy solution $u \in L^\infty(0,T_*; L^q(\mathbf{R}))$ to Eqs. (1.3)-(1.4) in $L^s(0,T_*; L^p(\mathbf{R}))$ $(s < \infty$ and $p < q)$.*

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Weak solutions for the Falk model system of shape memory alloys in energy class

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1 Summary

We study the following initial boundary value problem of the following Boussinesq-heat system:

$$u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x, \qquad (t,x) \in \mathbb{R}^+ \times (0,1), \tag{1.1}$$
$$\theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt}, \tag{1.2}$$
$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x), \tag{1.3}$$
$$u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = \theta_x(t,0) = \theta_x(t,1) = 0. \tag{1.4}$$

where $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t > 0\}$.

This system describes the dynamics of first order martensitic phase transitions occurring in a sufficiently thin rod of a shape memory alloys, where u denotes the longitudinal displacement of the rod, and θ is the temperature. In [7], Falk proposes a Landau-Ginzburg theory using the shear strain $v = u_x$ as an order parameter in order to explain the occurrence of the martensitic phase transitions in shape memory alloys such as *Nitinol* (Ni-Ti alloy). In this paper we assume that the Helmholtz free energy density F is a potential of the following simple form that accounts quite well for the experimentally observed behavior, i.e.

$$\begin{aligned} F &= F(v, v_x, \theta) \\ &= F_0(\theta) - \theta F_1(v) + F_2(v) + \frac{1}{2}v_x^2, \end{aligned}$$

where $F_1(r)$ and $F_2(r)$ are the primitives of $f_1(r)$ and $f_2(r)$, respectively. For more details of the Falk model system, we refer the reader to Chapter 5 in the literature [5].

Before stating our results, let us first recall some results related to this article. Sprekels and Zheng [14] proved the unique global existence of smooth solution for (1.1)-(1.4). In [6], Bubner and Sprekels established unique global

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existence results of (1.1)-(1.4) with the moving boundary condition for data $(u_0, u_1, \theta_0) \in H^3 \times H^1 \times H^1$, and discussed the optimal control problem in the case

$$f_1(r) = -r \text{ and } f_2(r) = r^5 - r^3 + r. \tag{A0}$$

T. Aiki [1] proved unique global existence of solution with $(u_0, u_1, \theta_0) \in H^3 \times H^1 \times H^1$ for more general nonlinearity, that is,

$$f_1, f_2 \in C^2(\mathbb{R}), \tag{A1}$$

and

$$F_2(r) \geq -C \text{ for } r \in \mathbb{R}. \tag{A2}$$

We note that the condition (A0) implies the conditions (A1) and (A2). Systems related to (1.1)-(1.4) have been studied for the case of viscous materials, that is, the stress σ contains additional viscous component of the following form,

$$\sigma = \frac{\partial F}{\partial v} + u_{xt}.$$

Correspondingly, the equations (1.1) and (1.2) are modified as follows:

$$u_{tt} + u_{xxxx} - u_{xxt} = (f_1(u_x)\theta + f_2(u_x))_x, \tag{1.5}$$

$$\theta_t - \theta_{xx} - |u_{xt}|^2 = f_1(u_x)\theta u_{xt}. \tag{1.6}$$

The viscosity term simplifies the analysis because this term has smoothing property. In fact, K.-H. Hoffman and Zochowski establish the existence result decomposing (1.5) into a system of two parabolic equations in [10]. Sprekels, Zheng and Zhu [15] prove the asymptotic behavior of the solution for (1.5)-(1.6) as $t \to \infty$. However, the literature [5] says that there is no interior friction from the experimental evidence. Moreover, it seems that for (1.1)-(1.4) has not been determined the asymptotic behavior of the solution as $t \to \infty$. Another interesting property of shape memory alloys is *hysteresis*. There are a lot of models and results from this point of view. For related results to hysteresis, we refer to e.g. [2].

System (1.1)-(1.2) conserves the energy, namely, the integral

$$E(u(t), u_t(t), \theta(t)) = \frac{1}{2}(\|u_t\|_{L_x^2}^2 + \|u_{xx}\|_{L_x^2}^2) + \int_0^1 \theta dx + \int_0^1 F_2(u_x)dx \tag{1.7}$$

does not depend on the time t. Therefore, the energy class of this system is $H^2 \times L^2 \times L^1$. In the author's master thesis [18], the unique global existence theorem in $H^2 \times L^2 \times L^2$ is proved, which is slightly smaller than the energy space. When we consider the solvability of (1.1)-(1.4), the energy class seems most natural. Nevertheless, there have been no papers on the solvability of (1.1)-(1.4) in the energy class up to the present. The aim of this paper is to prove the unique global existence of solution for (1.1)-(1.4) in this space. Here the spaces $W^{m,p}$ and H^m are the standard Sobolev spaces, that is, $W^{m,p}$ is equipped with the norm

$$\|f\|_{W^{m,p}} = \sum_{m \geq k \geq 0} \|\partial_x^k f\|_{L^p},$$

and $H^m = W^{m,2}$.

Our main results in this paper are stated as follows:

Theorem 1.1 (Local existence and uniqueness). *Assume that f_1, f_2 satisfy the condition (A1). Let any $\varepsilon \in (0, 1/6)$ be fixed. Then for any $(u_0, u_1, \theta_0) \in H^2 \times L^2 \times L^1$ with $u_0(0) = u_0(1) = 0$, there exists $T = T(\|u_0\|_{H^2}, \|u_1\|_{L^2}, \|\theta_0\|_{L^1}) > 0$ such that the problem (1.1)-(1.4) has a unique solution (u, θ) on the time interval $[0, T]$, satisfying*

$$\begin{aligned} u &\in C([0,T]; H^2(0,1)) \cap L^4(0,T; W^{2,4}(0,1)),\\ u_t &\in L^\infty(0,T; L^2(0,1)) \cap L^4(0,T; L^4(0,1)),\\ \theta &\in C([0,T]; L^1(0,1)),\\ \theta_x &\in L^{\frac{4}{3}+\varepsilon}(0,T; L^{\frac{4}{3}+\varepsilon}(0,1)). \end{aligned}$$

Our main tools of the proof of this theorem are the maximal regularity estimate and the Strichartz estimate. In general, the derivative of a solution for most of the equations is less regular than the right-hand side of the corresponding equations. However for parabolic equations such a loss of regularity does not occur, as in the case of elliptic equations. The estimate ensuring this regularity is called the maximal regularity. For this estimate, we refer to [3], [11] and [12]. The Strichartz estimate established in [16] is closely related to the restriction theory of the Fourier transform to surfaces and used often in various areas of the study of nonlinear wave equations. For the application of this estimate, we refer to [9], [13] and [17]. Corresponding results in the spatially periodic setting are established by J. Bourgain [4], and more transparent version is given by Fang and Grillakis in [8]. Therefore we first consider the following initial value problem with periodic boundary conditions, which is closely related to (1.1)-(1.4).

$$u_{tt} + u_{xxxx} = (f_1(u_x)\theta + f_2(u_x))_x, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{T}, \tag{1.8}$$

$$\theta_t - \theta_{xx} = f_1(u_x)\theta u_{xt}, \tag{1.9}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x), \tag{1.10}$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Theorem 1.2. *For the problem (1.8)-(1.10), the same conclusion as in Theorem 1.1 holds.*

From a physical point of view, the problem (1.8)-(1.10) describes the dynamics of the ring made of shape memory alloys. So it is a very interesting problem. Moreover Theorem 1.1 can be proved in the same way as Theorem 1.2. Roughly speaking, extending the solutions u and θ of (1.1)-(1.4) as odd and even periodic function respectively, we regard the initial boundary value problem as the problem with periodic boundary condition.

In order to regard the third term of the right hand side of (1.7) as L^1-norm of θ, we give the following lemma related to a sign property for the temperature θ

Lemma 1.1 (Maximum principle). *If $\theta_0 \geq 0$ on $\mathbb{T}$ (resp. $[0,1]$) then the solution θ of (1.8)-(1.10) (resp. (1.1)-(1.4)) satisfies $\theta \geq 0$ a.e. on $\mathbb{T}$ (resp. $[0,1]$) $\times [0,T]$.*

Combining these results with the energy conservation law, we can easily obtain the following global result.

Theorem 1.3 (Global existence). *In addition to the assumptions of Theorems 1.1 and 1.2, suppose that* (*A*2) *and* $\theta_0 \geq 0$. *Then, the solution given by Theorems 1.1 and 1.2 can be extended globally in time.*

In the end we state the Strichartz estimate and the maximal regularity. For a 1-parameter (semi-)group $V(t)$, we write

$$\Gamma(V)f := \int_0^t V(t-s)f(s)ds.$$

Lemma 1.2 (Strichartz type estimate [4], [8]). *The following estimates holds,*

$$\|V_\pm(\cdot)g; L_T^4 L_x^4\| \leq C\|g; L_x^2\|, \tag{1.11}$$

$$\left\|\Gamma(V_\pm)f; L_T^4 L_x^4\right\| \leq C\|f; L_T^{\frac{4}{3}} L_x^{\frac{4}{3}}\|, \tag{1.12}$$

and

$$\left\|\Gamma(V_\pm)f; L_T^\infty L_x^2\right\| \leq C\left\|f; L_T^{\frac{4}{3}} L_x^{\frac{4}{3}}\right\|, \tag{1.13}$$

where $V_\pm := e^{\pm it\partial_x^2}$.

Lemma 1.3 (Maximal regularity). *For any* $q \in (1, \infty)$, *we have*

$$\|\partial_x^2 \Gamma(U)f; L_T^q L_x^q\| \leq C(1+T)\|f; L_T^q L_x^q\|, \tag{1.14}$$

where $U(t) := e^{t\partial_x^2}$.

Remark. We note that the nonlinear term of (1.2) and (1.9) is rewritten as the following form:

$$f_1(u_x)\theta u_{tx} = (f_1(u_x)\theta u_t)_x - f_1'(u_x)u_{xx}\theta u_t - f(u_x)\theta_x u_t,$$

which makes sense in the distribution class.

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Strong instability of standing waves for nonlinear Klein-Gordon equations

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This note is based on a joint work with Grozdena Todorova (University of Tennessee).

We consider nonlinear Klein-Gordon (KG) equation of the form

$$\partial_t^2 u - \Delta u + u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1}$$

where $N \geq 2$, $1 < p < 1 + 4/(N-2)$. We study instability of standing wave solutions $u(t,x) = e^{i\omega t}\varphi(x)$ for (1), where $-1 < \omega < 1$, and $\varphi \in H^1(\mathbb{R}^N)$ is a nontrivial solution of

$$-\Delta\varphi + (1-\omega^2)\varphi - |\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}^N. \tag{2}$$

Recall that the Cauchy problem for (1) is locally well-posed in the energy space $X := H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ (see [4]): For any $(u_0, u_1) \in X$ there exists a unique solution $\vec{u} := (u, \partial_t u) \in C([0, T_{\max}); X)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t \to T_{\max}} \|\vec{u}(t)\|_X = \infty$. Moreover, the solution $u(t)$ satisfies $E(\vec{u}(t)) = E(u_0, u_1)$ and $Q(\vec{u}(t)) = Q(u_0, u_1)$ for all $t \in [0, T_{\max})$, where

$$E(u,v) = \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{3}$$

$$Q(u,v) = \operatorname{Im} \int_{\mathbb{R}^n} \overline{u} v \, dx. \tag{4}$$

Let ϕ_ω be the ground state (unique positive radially symmetric solution) of (2). The stability of standing waves $u_\omega(t) = e^{i\omega t}\phi_\omega$ for (1) has been studied by many authors. First, we consider the orbital stability of $u_\omega(t)$. Shatah [8] proved that $u_\omega(t)$ is orbitally stable if $p < 1 + 4/N$ and $\omega_c < |\omega| < 1$, where

$$\omega_c = \sqrt{\frac{p-1}{4-(N-1)(p-1)}}. \tag{5}$$

Moreover, Shatah and Strauss [10] proved that $u_\omega(t)$ is orbitally unstable when $p < 1 + 4/N$ and $|\omega| < \omega_c$ or when $p \geq 1 + 4/N$ and $\omega \in (-1, 1)$. Here, we say that a standing wave solution $e^{i\omega t}\varphi$ is orbitally stable for KG (1) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u_0, u_1) \in X$ satisfies $\|(u_0, u_1) - (\varphi, i\omega\varphi)\|_X < \delta$, then the solution $u(t)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ exists globally in time and satisfies

$$\sup_{t\geq 0} \inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^N} \|\vec{u}(t) - e^{i\theta}(\varphi(\cdot + y), i\omega\varphi(\cdot + y))\|_X < \varepsilon.$$

Otherwise, $e^{i\omega t}\varphi$ is said to be orbitally unstable.

Next, we consider instability of $u_\omega(t)$ in stronger senses. Berestycki and Cazenave [1] proved that $u_\omega(t)$ is very strongly unstable in the sense of Definition 1 when $\omega = 0$ and $1 < p < 1+4/(N-2)$. Moreover, Shatah [9] studied nonlinear Klein-Gordon equation with more general nonlinearity, and proved that $u_\omega(t)$ is strongly unstable in the sense of Definition 2 when $\omega = 0$. Recently, the authors [7] proved that $u_\omega(t)$ is very strongly unstable for (1) when $|\omega| \leq \sqrt{(p-1)/(p+3)}$, $1 < p < 1 + 4/(N-2)$ and $N \geq 3$. Here, we give the definitions of very strong instability and strong instability.

Definition 1 (very strong instability) We say that $e^{i\omega t}\varphi$ is *very strongly unstable* for KG (1) if for any $\varepsilon > 0$ there exists $(u_0, u_1) \in X$ such that $\|(u_0, u_1) - (\varphi, i\omega\varphi)\|_X < \varepsilon$ and the solution $u(t)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ blows up in a finite time.

Definition 2 (strong instability) We say that $e^{i\omega t}\varphi$ is *strongly unstable* for (1) if for any $\varepsilon > 0$ there exists $(u_0, u_1) \in X$ such that $\|(u_0, u_1) - (\varphi, i\omega\varphi)\|_X < \varepsilon$ and the solution $u(t)$ of (1) with $\vec{u}(0) = (u_0, u_1)$ either blows up in a finite time or exists globally in time and satisfies $\limsup_{t\to\infty} \|\vec{u}(t)\|_X = \infty$.

Note that, by the definitions, if $e^{i\omega t}\varphi$ is very strongly unstable, then it is strongly unstable, and if $e^{i\omega t}\varphi$ is strongly unstable, then it is orbitally unstable.

For the nonlinear Schrödinger (NLS) equation

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \tag{6}$$

it is known that for any $\omega > 0$ the standing wave solution $e^{i\omega t}\phi_\omega$ for (6) is orbitally stable when $1 < p < 1+4/N$, and it is very strongly unstable when $1+4/N < p < 1+4/(N-2)$, where $\phi_\omega \in H^1(\mathbb{R}^N)$ is the ground state of

$$-\Delta\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^N \tag{7}$$

(see [1, 3]). For the critical case $p = 1+4/N$, for any $\omega > 0$ and any nontrivial solution $\varphi \in H^1(\mathbb{R}^N)$ of (7), it is known that the standing wave $e^{i\omega t}\varphi$ is very strongly unstable for (6) (see [12]).

The main results in this note are as follows.

Theorem 1 *Let $N \geq 2$, $1 < p < 1+4/(N-2)$, $\omega \in (-1,1)$ and ϕ_ω be the ground state of* (2). *Assume that $|\omega| < \omega_c$ if $p < 1+4/N$. Then, $e^{i\omega t}\phi_\omega$ is strongly unstable for KG* (1) *in the sense of Definition 2.*

In Theorem 1, one may expect that $u_\omega(t)$ is very strongly unstable for (1) in the sense of Definition 1. In this direction, Cazenave [2] proves that any global solution $u(t)$ of (1) is uniformly bounded in X, i.e., $\sup_{t\geq 0} \|\vec{u}(t)\|_X < \infty$, if $1 < p \leq 5$ and $N = 2$, and if $1 < p \leq N/(N-2)$ and $N \geq 3$. Moreover, using arguments in Merle and Zaag [5], we can prove the uniform boundedness of global solutions of (1) in X when $1 < p < 1+4/(N-1)$ and $N \geq 2$. Note that $1+4/(N-1) = 5$ if $N = 2$, $1+4/(N-1) = N/(N-2) = 3$ if $N = 3$, and $N/(N-2) < 1+4/(N-1)$ if $N \geq 4$. Therefore, as a corollary of Theorem 1, we have the following.

Corollary 2 *In addition to the assumptions in Theorem 1, assume that $p \leq 1+4/(N-1)$ if $N = 2, 3$, and that $p < 1+4/(N-1)$ if $N \geq 4$. Then, $e^{i\omega t}\phi_\omega$ is very strongly unstable for* (1) *in the sense of Definition 1.*

For the critical frequencies $\omega = \pm\omega_c$ in the case $p < 1+4/N$, we have the following.

Theorem 3 *Let $N \geq 2$, $1 < p < 1+4/N$ and $\varphi \in H^1(\mathbb{R}^N)$ be a nontrivial radially symmetric solution of* (2) *with $\omega = \omega_c$. Then, the standing wave solution $e^{i\omega_c t}\varphi$ is very strongly unstable for* (1).

As mentioned above, similar result is known for NLS (6) in the critical case $p = 1+4/N$ without assuming the radial symmetry of solution of (7).

The proofs of Theorems 1 and 3 are based on the argument in Shatah [9] and on local versions of the virial type identities. In [9], Shatah considers a local version of the following identity

$$\begin{aligned}(8)\qquad &\frac{d}{dt}\operatorname{Re}\int_{\mathbb{R}^N} x\cdot\nabla u\partial_t\bar{u}\,dx = NK_1(\vec{u}(t)),\\ &K_1(u,v) = -\frac{1}{2}\|v\|_2^2+\left(\frac{1}{2}-\frac{1}{N}\right)\|\nabla u\|_2^2+\frac{1}{2}\|u\|_2^2-\frac{1}{p+1}\|u\|_{p+1}^{p+1}.\end{aligned}$$

Since the integral in the left-hand side of (8) is not well-defined on the energy space X, we need to approximate the weight function x in (8) by suitable bounded functions. To control error terms by the approximation, we assume that the initial perturbations are radially symmetric, and use the decay estimate for radially symmetric functions in $H^1(\mathbb{R}^N)$:

$$(9)\qquad \|w\|_{L^\infty(|x|\ge m)} \le Cm^{-(N-1)/2}\|w\|_{H^1}$$

(see [11]). The assumption $N\ge 2$ is needed here. This kind of approach has been also used for blowup problem of NLS (6) (see, e.g., [6]).

For the case $p\ge 1+4/N$ in the proof of Theorem 1, we use a local version of the identity

$$\begin{aligned}(10)\qquad &-\frac{d}{dt}\operatorname{Re}\int_{\mathbb{R}^N}\{2x\cdot\nabla u+Nu\}\partial_t\bar{u}\,dx = 4P(u(t)),\\ &P(u)=\frac{1}{2}\|\nabla u\|_2^2-\frac{N(p-1)}{4(p+1)}\|u\|_{p+1}^{p+1}.\end{aligned}$$

Note that the functional P in (10) appears in the virial identity for NLS (6):

$$(11)\qquad \frac{d^2}{dt^2}\|xu(t)\|_2^2 = 16P(u(t)).$$

For the case $p<1+4/N$, we use a local version of the identity

$$\begin{aligned}(12)\qquad &-\frac{d}{dt}\operatorname{Re}\int_{\mathbb{R}^N}\{2x\cdot\nabla u+(N+\alpha)u\}\partial_t\bar{u}\,dx = K(\vec{u}(t)),\\ &K(u,v) = -\alpha\|v\|_2^2+\alpha\|u\|_2^2+(\alpha+2)\{\|\nabla u\|_2^2-\frac{2}{p+1}\|u\|_{p+1}^{p+1}\},\end{aligned}$$

where $\alpha := 4/(p-1)-N>0$ (cf. [10, page 185]). Note that

$$\begin{aligned}K(u,v) = &-2(\alpha+1)\|v-i\omega u\|_2^2+2(\alpha+2)(E-\omega Q)(u,v)\\ &-2\alpha\omega Q(u,v)-2\{1-(\alpha+1)\omega^2\}\|u\|_2^2,\end{aligned}$$

and that if $|\omega|=\omega_c$ then $(\alpha+1)\omega^2=1$.

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Fast Singular Oscillating Limits, Restricted Convolutions and Global Regularity of the 3D Navier-Stokes Equations of Gephysics

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Abstract

We prove existence on infinite time intervals of regular solutions to the 3D Navier-Stokes Equations for fully three-dimensional initial data characterized by uniformly large vorticity and for the full 3D Navier-Stokes Equations of Gephysics in the regime of strong stratification and rotation; smoothness assumptions for initial data are the same as in local existence theorems. There are no conditional assumptions on the properties of solutions at later times, nor are the global solutions close to any 2D manifold. The global existence is proven using techniques of fast singular oscillating limits, lemmas on restricted convolutions and the Littlewood-Paley dyadic decomposition. The approach is based on the computation of singular limits of rapidly oscillating operators and cancellation of oscillations in the nonlinear interactions for the vorticity field. With nonlinear averaging methods in the context of almost periodic functions, we obtain fully 3D limit resonant Navier-Stokes equations. We establish the global regularity of the latter without any restriction on the size of 3D initial data. With strong convergence theorems, we bootstrap this into the global regularity of the 3D Navier-Stokes Equations for above classes of fully 3D initial data.