# Surface Evolution Equations
# — a level set method —

Yoshikazu Giga
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

March, 2002

# Preface

This book is intended to be a self-contained introduction to analytitc foundation of a level set method for various surface evolution equations including curvature flow equations. These equations are important in various fields including material sciences, image processing and differential geometry. The goal of this book is to introduce a generalized notion of solutions allowing singularities and solve the initial-value problem globally-in-time in generalized sense. Various equivalent definitions of solutions are studied. Several new results on equivalence are also presented.

This book contains rather complete introduction to the theory of viscosity solutions which is a key tool for the level set method. Also a self-contained explanation is given for general surface evolution equations of the second order. Although most of results in this book is more or less known, they are scattered in several references sometimes without proof. This book presents these results in a synthetic way with full proofs. However, the references are not exhaustive at all.

This book is suitable for applied researchers who would like to know the detail of the theory as well as its flavour. No familiarity of differential geometry and the theory of viscosity solutions is required. Only prerequisites are calculus, linear algebra and some familiarity of semicontinuous functions. This book is also suitable for upper level of under graduate students who are interested in the field.

This book is based on my Lipschitz lectures in Bonn 1997. The author is grateful to its audience for their interest. The author is also grateful to Professor Naoyuki Ishimura, Professor Reiner Schätzle, Professor Katsuyuki Ishii and Professor Masaki Ohnuma for their critical remarks on earlier versions of this book. The financial support of the Japan Society for the Promotion of Science (no. 10304010, 11894003, 12874024, 13894003) is gratefully acknowledged. Finally, the author is grateful to Ms. Hisako Iwai for careful typing of the manuscripts in latex style.

Y. Giga

Sapporo
March 2002

# Contents

# Chapter 0

# Introduction

In various fields of science there often arise phenomena in which phases (of materials) can coexist without mixing. A surface bounding the two phases is called a *phase boundary*, an *interface* or *front* dependeing upon situation. In the process of phase transition a phase boundary is moving by thermodynamical driving forces. Since evolution of a phase boundary is unknown and it should be determined as a part of solutions, the problem including such a phase boundary is called in general a free boundary problem. The motion of a phase boundary between ice and water is a typical example, and it has been well studied– the Stefan problem. For classical Stefan problems the reader is referred to the books of L. I. Rubinstein (1971) and of A. M. Meirmanov (1992). The reader is referred to the recent book of A. Visintin (1996) for free boundary problems related to phase transition. In the Stefan problem evolution of a phase boundary is affected by the physical situation of the exterior of the surface. However, there is a special but important class of problems where evolution of a phase boundary does not depend on the physical situation outside the phase boundary, but only on its geometry. The equation that describes such motion of the phase boundary is called a *surface evolution equation* or *geometric evolution equations*. There are several examples in material sciences and the equation is also called an *interface controlled model*. Examples are not limited in material sciences. Some of those comes from geometry, crystal growth problems and image processing. An important subclass of surface evolution equations consists of equations that arise when the normal velocity of the surface depends locally on its normal and the second fundamental form as well as on position and time. In this book we describe analytic foundation of the level set method which is useful to analyse such surface evolution equations including the mean curvature flow equation as a typical example. We intend to give a symmetric and synthetic approach since the results are scattered in the literature. This book also includes several new results on barrier solutions (Chapter V).

We consider a family $\{\Gamma_t\}_{t\geq 0}$ of hypersurfaces embedded in $N$-dimensional Euclidean space $\mathbf{R}^N$ parametrized by time $t$. We assume that $\Gamma_t$ is a compact hypersurface so that $\Gamma_t$ is given as a boundary of a bounded open set $D_t$ in $\mathbf{R}^N$ by Jordan–Brouwer's decomposition theorem. Physically, we regard $\Gamma_t$ as a phase boundary bounding $D_t$ and $\mathbf{R}^N \backslash D_t$ each of which is occupied by different phases. To write down a surface evolution equation we assume that $\Gamma_t$ is smooth and changes its shape smoothly in time. Let $\mathbf{n}$

be a unit normal vector field of $\Gamma_t$ outward from $D_t$. Let $V = V(x,t)$ be the *normal velocity* in the direction of $\mathbf{n}$ at a point of $\Gamma_t$. If $V$ depends locally on normal $\mathbf{n}$ and the second fundamental form $-\nabla\mathbf{n}$ of $\Gamma_t$, as well as on position $x$ and time $t$, a general form of surface evolution equation is

$$V = f(x, t, \mathbf{n}, \nabla\mathbf{n}) \quad \text{on } \Gamma_t, \tag{0.0.1}$$

where $f$ is a given function. We list several examples of (0.0.1).

(1) Mean curvature flow equation : $V = H$, where $H$ is the sum of all principal curvatures in the direction of $\mathbf{n}$ and is called the mean curvature throughout in this book (although many authors since Gauss call the average of principal curvatures the mean curvature). The mean curvature is expressed as $H = -\mathrm{div}\ \mathbf{n}$, where div is the surface divergence on $\Gamma_t$. This equation was first proposed by W. W. Mullins (1956) to describe motion of grain boundaries in annealing metals.

(2) Gaussian curvature flow equation : $V = K$, where $K$ is the Gaussian curvature of $\Gamma_t$, that is, the product of all principal curvatures in the direction of $\mathbf{n}$. For this problem we take $\mathbf{n}$ inward so that a sphere shrinks to a point in a finite time if it evolves by $V = K$. This equation was proposed by W. Fiery (1974) to describe shapes of rocks in sea shore.

(3) General evolutions of isothermal interface:

$$\beta(\mathbf{n})V = -a \ \mathrm{div}\ \xi(\mathbf{n}) - c(x,t), \tag{0.0.2}$$

where $\beta$ is a given positive function on a unit sphere $S^{N-1}$ and $a$ is nonnegative constant and $c$ is a given function. The quantity $\xi$ is the *Cahn-Hoffman vector* defined by the gradient of a given nonnegative positively homogeneous function $\gamma$ of degree one, i.e., $\xi = \nabla\gamma$ in $\mathbf{R}^N$. In problems of crystal growth we should often consider the anisotropic property of the surface structure of phase boundaries; in one direction the surface is easy to grow, but in the other direction it is difficult to grow. This kind of thing often happens. The equation (0.0.2) includes this effect and was derived by M. E. Gurtin (1988a), (1988b) and by S. Angenent and M. E. Gurtin (1989) from the fundamental laws of thermodynamics and the balance of forces. Note that if $\gamma(p) = |p|$ and $\beta(p) \equiv 1$ with $c \equiv 0$, $a \equiv 1$, then (0.0.2) becomes $V = H$. If $a = 0$, the equation (0.0.2) becomes simpler:

$$V = -c/\beta(\mathbf{n}) \quad \text{on } \Gamma_t \tag{0.0.3}$$

This equation is a kind of Hamilton-Jacobi equation. If $\beta \equiv 1$ and $c < 0$ is a constant, this equation describes the wave front propagation based on Huygens' principle.

(4) Affine curvature flow equation : $V = K^{1/(N+1)}$ or $V = (tK)^{1/(N+1)}$ which were axiomatically derived by L. Alvarez, F. Guichard, P.-L. Lions and J.-M. Morel (1993). The feature of these equations is that it is invariant by affine transform of coordinates. For this problem we take $\mathbf{n}$ inward as for the Gaussian curvature flow equation.

Examples of surface evolution equations are provided by the singular limit of reaction-diffusion equations as many authors studied. See for example papers of X.-Y. Chen (1991) and X. Chen (1992).

A fundamental question of analysis is to construct a unique family $\{\Gamma_t\}_{t\geq 0}$ satisfying (0.0.1) for given initial hypersurface $\Gamma_0$ in $\mathbf{R}^N$. In other words it is the question whether there exists a unique solution $\{\Gamma_t\}_{t\geq 0}$ of the initial value problem for (0.0.1) with $\Gamma_t|_{t=0} = \Gamma_0$. This problem is classified as unique existence of a local solution or of a global solution depending on whether one can construct a solution of (0.0.1) in a short time interval or for infinite time. If the equation (0.0.1) is strictly parabolic in a neighborhood of initial hypersurface $\Gamma_0$, then there exists a unique local smooth solution $\{\Gamma_t\}$ for given initial data provided that the dependence of variables in $f$ is smooth. It applies to the mean curvature flow equation and its generalization (0.0.2) with $a > 0$ and smooth $\beta$ and $c$ for general initial data $\Gamma_0$ provided that the Frank diagram

$$\text{Frank } \gamma = \{p \in \mathbf{R}^N;\ \gamma(p) \leq 1\} \tag{0.0.4}$$

has a smooth, strictly convex boundary in the sense that all inward principal curvature are positive. For the Gaussian curvature flow equations and the affine curvature flow equation the equation may not be parabolic for general initial data. It resembles to solve the heat equation backward in time, so for general initial data it is not solvable. However, if we restrict ourselves to strictly convex initial surfaces, the problems are strictly parabolic around the initial surfaces and locally uniquely solvable. A standard method to construct a unique local solution is to analyse an equation of a "height" function, where the evolving surface is parametrized by the height (or distance) from the initial surface. See for example a paper by X.-Y. Chen (1991), where he discussed (0.0.2) with $\gamma(p) = |p|,\ \beta \equiv 1,\ a = 1$. Major machinary is the classical parabolic theory in a book of O. A. Ladyžehnskaja, V. A. Solonnikov and N. N. Ural'ceva (1968) since the equation of a height function is a strictly parabolic equation of second order (around zero height) although it is nonlinear. For (0.0.3) the equation of a height function is of first order so a local smooth solution can be constructed by a method of characteristics. However, as we see later, such a local smooth solution may cease to be smooth in a finite time and singularities may develop even for the mean curvature flow equation where a lot of regularizing effects exist. Since the phenomena may continue after solution cease to be smooth, it is natural to continue the solution by generalizing the notion of solution. The *level set method* is a powerful tool in constructing global solutions allowing singularities. It also provides a correct notion of solutions, if (0.0.1) is degenerate parabolic but not of first order so that a 'solution' may lose smoothness even instanteneously and that smooth local solution may not exist for smooth initial data. The analytic foundation of this theory was established independently by Y.-G. Chen, Y. Giga and S. Goto (1989), (1991a) and by L. C. Evans and J. Spruck (1991), where the latter work concentrated on the mean curvature flow equation while the former work handled a more general equation of form (0.0.1). These works were preceded by a numerical computation by S. Osher and J. A. Sethian (1988) using a level set method. (We point out that there are two proceeding works implicitly related to level set method for first order problems. One is by G. Barles (1985) and the other one is by

L. C. Evans and P. E. Souganidis (1984).) After the analytic foundation was established, a huge number of articles on the level set method has been published. One of purposes of this book is to explain the analytic foundation of the level set method in a systematic and synthetic way so that the reader can access the field without scratching references. For the development of numerical aspects of the theory the reader is referred to a recent book of J. A. Sethian (1996). Before explaining the level set method we discuss phenomena of formation of singularities for solutions of surface evolution equations.

**Formation of singularities.** We consider the mean curvature flow equation $V = H$. If the initial surface $\Gamma_0$ is a sphere of radius $R_0$, a direct calculation shows that the sphere with radius $R(t) = (R_0^2 - 2(N-1)t)^{1/2}$ is the exact solution. At time $t_* = R_0^2/(2(N-1))$ the radius of the sphere is zero so it is natural to interpret $\Gamma_t$ as the empty set after $t_*$. If $\Gamma_0$ is a smooth, compact, and convex hypersurface, G. Huisken (1984) showed that the solution $\Gamma_t$ with initial data $\Gamma_0$ remains smooth, compact and convex until it shrinks to a "round point" in a finite time; the asymptotic shape of $\Gamma_t$ just before it disappears is a sphere. He proved this result for hypersurfaces of $\mathbf{R}^N$ with $N \geq 3$ but his method does not apply to the case $N = 2$. Later M. Gage and R. Hamilton (1986) showed that it still holds when $N = 2$, for simple convex curves in the plane. The methods used by G. Huisken (1984), and M. Gage and R. Hamilton (1986) do not resemble each other. M. Gage and R. Hamilton (1986) also observed that any smooth family of plane-immersed curves that moved by its curvature which is initially embedded curves remains embedded. M. Grayson (1987) proved the remarkable fact that such a family must become convex before it becomes singular. Thus, in the plane $\mathbf{R}^2$ if the initial data $\Gamma_0$ is a smooth, compact, (and embedded) curve, then the solution $\Gamma_t$ remains smooth (and embedded), becomes convex in a finite time and remains convex until it shrinks to a "round point". The situation is quite different for higher dimensions. While it is still true that smooth immersed solutions remain embedded if their initial data is embedded, M. Grayson (1989) also showed that there exists smooth solutions $\Gamma_t$ that becomes singular before they shrink to a point. His example consisted of a barbell: two spherical surfaces connected by a sufficiently thin "neck". In this example the inward curvature of the neck is so large that it will force the neck to pinch before the two spherical ends can shrink appreciably. In this example the surface has genus zero but there also is an example the surface with genus one the becomes singular in a finite time. For example if the initial data $\Gamma_0$ is a fat doughnut, then $\Gamma_t$ becomes singular before they shrink to a point. Since the outward curvature of the hole so large that it will force the hole to pinch. Of course if a doughnut is thin and axisymmetric it is known that it shrinks to a ring as shown by K. Ahara and N. Ishimura (1993). See also a paper of K. Smoczyk (1994) for symmetric surfaces under rotation fixing $\mathbf{R}^m$ with $m \geq 2$.

We consider (0.0.3) with $\beta \equiv 1$ and $c \equiv -1$ so that (0.0.3) is $V = 1$. For this equation it is natual to think that the "solution" $\Gamma_t$ with initial data $\Gamma_t$ equals the set of all points $x$ with $\mathrm{dist}(x, \Gamma_0) = t$ that lie in the positive direction $\mathbf{n}$ from $\Gamma_0$. If initial $\Gamma_0$ has a "hollow" that $\Gamma_t$ becomes singular in a finite time even if $\Gamma_0$ is smooth. The solution $\Gamma_t$ given by the distance function provides a natural candidate of an extension of smooth solution after singularities develop. For the second order equation including the mean curvature flow, it is not expected that $\Gamma_t$ is expressed by a simple explicit function like a distance function. So one introduce a generalized notion of solutions after singularities develop.

For the Gaussian curvature and the affine curvature flow equations as for the mean curvature flow equation if $\Gamma_0$ is a strictly convex smooth hypersurface, then $\Gamma_t$ is a strictly convex smooth hypersurface and stays convex until it shrinks to a point. This was proved by K. Tso (1985) and B. Andrews (1994). (For the affine curvature flow equation the shrinking pattern may not be a "round point" since the equation is affine invariant so that there is a solution of shrinking ellipse.) If $\Gamma_0$ is convex but not strictly convex, these equations are degenerate parabolic, so one should be afraid that $\Gamma_t$ may lose smoothness even instanteneously. For (0.0.2) if Frank $\gamma$ is merely convex, then it actually happens that $\Gamma_t$ ceases to be $C^1$ instanteneously even if initial data $\Gamma_t$ is smooth as shown in the work of Y. Giga (1994). We need a generalized notion of solution to track these phenomena.

**Tracking solutions after formation of singularities.** Suppose that singularities develop when a hypersurface evolves by a surface evolution equation. In what way should one construct the solution after that? From the geometric point of view, one method may be to classify all possible sigularities which may appear and then construct the solution restarting with the shape having such singularities as initial data. This method has a drawback when it is difficult to classify singularities. In fact, even for the mean curvature flow equation, the classification of singularities is not completed. (We do not pursue this topic in the present version of the book. The reader is referred to the review by Y. Giga (1995a).) On the other hand, in analysis it is standard to introduce a suitable generalized notion of solutions so that it allows singularities. Such a solution is called a weak solution or generalized solution.

The notion of generalized solution for the mean curvature flow equation was first introduced by K. A. Brakke (1978). He showed, how, using geomtrtic measure theory, one can construct a theory of generalized solutions for variables in $\mathbf{R}^N$ of arbitrary dimension and co-dimendion. His generalized solution is a kind of family of measure supported in $\Gamma_t$ that satisfies transport inequality in a generalized sense. He proved the existence of at least one global solution for any initial data. In fact, he showed in an example that one initial datum may have many different solution. This feature of his theory is related to the fattening phenomena in the level set method.

The second way to try to construct weak solutions is to study the singular limit of reaction-diffusion equations approximating the surface evolution equations. For the mean curvature flow equation, we consider the semilinear heat equation called the Allen-Cahn equation of the form

$$\frac{\partial u}{\partial t} - \Delta u + \frac{W'(u)}{\varepsilon^2} = 0 \quad \text{in } \mathbf{R}^N \times (0, \infty) \tag{0.0.5}$$

which was introduced by S. M. Allen and J. W. Cahn (1979) to describe motion of anti-phase boundary in material sciences. Here $W$ is a function that has only two equal minima; its typical form is $W(v) = \frac{1}{2}(v^2 - 1)^2$ and $W'$ denotes the derivative. In the Allen-Cahn equation (0.0.5) $\varepsilon$ is a positive parameter. For given initial data we solve the Allen-Cahn equation globally in time and denote its solution by $u^\varepsilon$. The equation (0.0.5) is semilinear parabolic and there is no growing effect for the nonlinear term so it is globally solvable. We guess that if $\varepsilon$ tends to 0, $u^\varepsilon$ tends to the minimal values of $W$, i.e., $-1$ or 1. In fact it is easy to check this if there is no Laplace term $\Delta u$ (unless its value is unstable stationary point of $W$). The boundary between regions where the limit equals $-1$ and 1 is known to evolve by the mean curvature flow equation up to constant multiplication. This is well-known, at least formally, by asymptotic analysis. In fact, if the mean curvature flow equation has a smooth solution, the convergence was proved rigorously in a time interval where the smooth solution exists by several authors including L. Bronsard and R. Kohn (1991), P. de Mottoni and M. Schatzman (1995) and X. Chen (1992) in various setting. It is expected that a weak solution is obtained as a limit of $u^\varepsilon$ (as $\varepsilon \to 0$) since $u^\varepsilon$ itself exists globally in time. This idea turns to produce Brakke's generalized solution as shown by T. Ilmanen (1993b) and improved by H. M. Soner (1997). Extending these idea a Brakke's type generalized solution was also constructed for the Mullins-Sekerka problem with or without kinetic undercooling as a singular limit of phase-field model and of the Cahn-Hilliard equation by H. M. Soner (1995) and X. Chen (1996), respectively. We won't touch these problems in this book but we note that this method is based on variational structure of problems and does not appeal to *comparison principle* which says that the solution $\Gamma_t$ is always enclosed by another solution $\Gamma'_t$ if initially $\Gamma_0$ is enclosed by $\Gamma'_0$. This is why this method works for higher order nonlocal surafce evolution equation like the Mullins-Sekerka problem.

A level set method, which is the main topic of this book, is another analytic method to construct weak solutions. It is based on comparison principle (or order-preserving structure of the solutions). It does not depend on variational surface evolution equation (0.0.1) even if other methods fails to apply. For the mean curvature flow equation the relation between above three methods were clarified by T. Ilmanen (1993) preceded by the work of L. C. Evans, H. M. Soner and P. Souganidis (1992). We shall explain these works after we explain the idea of the level set method.

**Level set method.** To describe a hypersurface, one can represent it as the zero set of a function, i.e., the zero level set. Compared with the method of representation by local coordinates, there is an advantage since the zero level set is allowed to have singularities even if the function is smooth. In other words, a hypersurface with singularities may be represented by a smooth function. The idea of the level set method is to regard a hypersurface $\Gamma_t$ as the zero level set of some auxiliary function $u : \mathbf{R}^N \times [0, \infty) \to \mathbf{R}$ and to derive an equation which guarantees that its zero level set will evolve by a surface evolution equation (0.0.1). Many equations will have this property, but if one requires that not only the zero level set, but *also all* level sets of the function $u$ evolves by the same surface evolution equation (0.0.1), then, as in Giga and Goto (1992a), there exists a unique partial differential equations of form

$$\frac{\partial u}{\partial t} + F(x, t, \nabla u, \nabla^2 u) = 0; \qquad (0.0.6)$$

such an equation is called the *level set equation* of (0.0.1). Here $\nabla u$ denotes the spatial gradient and $\nabla^2 u$ dennotes the Hessian; $F$ is a function determined by the values of $(x, t, \nabla u(x,t), \nabla^2 u(x,t))$.

For example for the surface evolution equation $V = 1$ its level set equation is

$$\frac{\partial u}{\partial t} - |\nabla u| = 0 \tag{0.0.7}$$

once we take the orientation $\mathbf{n} = -\nabla u/|\nabla u|$ of each level set so that $V = \partial u/\partial t/|\nabla u|$. For the mean curvature flow equations its level set equation is

$$\frac{\partial u}{\partial t} - |\nabla u| \operatorname{div}(\frac{\nabla u}{|\nabla u|}) = 0 \tag{0.0.8}$$

The idea to represent hypersurfaces as level sets is of course common in differential geometry. In the present context it goes back to T. Ohta, D. Jasnow and K. Kawasaki (1982) who used the level set equation (0.0.8) to derive a scaling law for "dynamic structure functions" from a physical point of view. There were some article on combustion theory that (0.0.8) was implicit before their paper but is seems that their paper is the first one to use (0.0.8) effectively. The idea to use (0.0.8) to study motion by mean curvature numerically was used by S. Osher and J. Sethian (1988).

A level set method for initial value problem of (0.0.1) is summarized as follows.

1°. For a given initial hypersurface $\Gamma_0$ which is the boundary of a bounded open set $D_0$ we take an auxiliary function $u_0$ which is at least continuous such that

$$\Gamma_0 = \{x \in \mathbf{R}^N;\ u_0(x) = 0\},\ D_0 = \{x \in \mathbf{R}^N;\ u_0(x) > 0\}.$$

(For convenience we often arrange so that $u_0$ equals a negative constant $\alpha$ outside some big ball.)
2°. We solve the initial value problem globally-in-time for the level set equation (0.0.6) with initial condition $u(x, 0) = u_0(x)$.
3°. We then set

$$\begin{aligned} \Gamma_t &= \{x \in \mathbf{R}^N;\ u(x,t) = 0\}, \\ D_t &= \{x \in \mathbf{R}^N;\ u(x,t) > 0\} \end{aligned} \tag{0.0.9}$$

and expect that $\Gamma_t$ is a kind of generalized solution.

The first step is easy. The second step is not easy since the level set equation is not very nice from the point of analysis. If (0.0.1) is parabolic, it is a parabolic equation but it is very degenerate. There is no diffusion effect normal to its level set since each level set of $u$ moves independently of the others. Thus classical techniques and results in the theory of parabolic equations cannot be expected to apply. We do not expect to have global smooth solution for (0.0.6) even if initial data is smooth. It is necessary to introduce the notion of weak solutions to (0.0.6). As known by Y.-G. Chen, Y. Giga and S. Goto (1989), (1991a) for (0.0.6) and L. C. Evans and J. Spruck (1991) for (0.0.8) the concept that fits this situation perfectly is a notion of viscosity solutions initiated by M. G. Crandall and P.-L. Lions (1981), (1983). The reader is referred to the review article by M. G. Crandall, H. Ishii and P.-L. Lions (1992) for development of the theory of viscosity solutions. The theory of viscosity solutions applies nonlinear degenerate elliptic and parabolic single equation including equations of first order where comparison principle is expected. The key step of the theory is to establish a comparison principle for viscosity solutions. For (0.0.7) the theory for first order equations applies. For (0.0.8) the equation is singular at $\nabla u = 0$ which is a new aspect of problems in the theory of viscosity solutions. Since the mean curvature flow equation has a comparison principle or order-preserving properties for smooth solutions, comparison principle for its level set equation is expected. It turns out that the extended theory of viscosity solutions yields a unique global continuous solution $u$ of (0.0.6) with $u(x,0) = u_0(x)$ (with the property $u(x,t) - \alpha$ is compatly supported as a function of space variables for all $t \geq 0$) provided that (0.0.1) is degenerate parabolic and $f$ in (0.0.1) does not grow superlinearly in $\nabla \mathbf{n}$. To apply this theory for the Gaussian curvature flow equation we need to extend the theory so that $f$ is allowed to grow superlinearly in $\nabla \mathbf{n}$. This extension is done by S. Goto (1994) and independently by H. Ishii and P. Souganidis (1995). We note that order-preserving structure of (0.0.1) is essential to get a global continuous solution to (0.6).

The method to construct $\Gamma_t$ by $1^\circ - 3^\circ$ is extrinsic. There are huge freedom to choose $u_0$ for given $\Gamma_0$. Although the solution $u$ of (0.0.6) for given initial data $u_0$ is unique, we wonder whether $\Gamma_t$ and $D_t$ in (0.0.9) are determined by $\Gamma_0$ and $D_0$ respectively independent of the choice of $u_0$. This problem is the uniqueness of level set of the initial value problem for (0.0.6). Since $F$ in (0.0.6) has a scaling property (called *geometricity*):

$$F(x, t, \lambda p, \lambda X + \sigma p \otimes p) = \lambda F(x, t, p, X)$$

for all $\lambda > 0, \sigma \in \mathbf{R}$, real symmetric matrix $X, p \in \mathbf{R}^N \setminus \{0\}, x \in R^N, t \in [0, \infty)$, the equation (0.0.6) has the invariance property: $u$ solves (0.0.6) so does $\theta(u)$ for every non-decreasing continuous function $\theta$ in viscosity sense. Using the invariance and the comparison principle, we get the uniqueness of level sets. In other words $\Gamma_t$ and $D_t$ in (0.0.9) is uniquely determined by $\Gamma_0$ and $D_0$ respectively. It is also possible to prove that $\Gamma_t$ is an extended notion of smooth solution.

**Fattening.** One disturbing aspect of the solution $\Gamma_t$ defined by (0.0.9) is that for $t > 0$ $\Gamma_t$ may have a nonempty interior even if the initial hypersurface is smooth, except for a few isolated singularities. An example is provided by L. C. Evans and J. Spruck (1991) for the mean curvature flow equation, where it is argued that the solution in $\mathbf{R}^2$ whose initial shape is a "figure eight" has nonempty interior. Such phenomena was studied by many authors in various setting and several sufficient conditions of nonfattening was provided. For the mean curvature flow problem it is observed that $\Gamma_t$ may fatten (i.e. have empty interior) even if initial hypersurface is smooth as observed by S. Angenent, D. L. Chopp and T. Ilmanen (1995) for $N = 3$, S. Angenent, T. Ilmanen and J. J. L. Velazquez for $4 \leq N \leq 8$. We do not pursue this problem in this book. If we introduce the notion of set-theoretic solutions, the fattening phenomena can be interpreted as nonuniqueness of the set-theoretic solution. Such a notion was first introduced by H. M. Soner (1993) for (0.0.1) when $f$ is independent of $x$ by using distance function from a set. It turns out to be more natural that a family of set $\Omega_t$ is a set-theoretic solution of (0.0.1) if the characteristic function $\chi_{\Omega_t}$ is a solution of its level set solution in viscosity sense. $D_t$ and $D_t \cup \Gamma_t$ defined by (0.0.9) are typical examples of set-theoretic solutions. If fattening occurs, $\overline{D}_t$ and $D_t \cup \Gamma_t$ are essentially different and we have at least two solutions for given initial data $D_0$; here $\overline{D}_t$ denotes the cross section at $t$ of the closure of the set where $u$ is positive in $\mathbf{R}^N \times [0,\infty)$ and not the closure of $D_t$ in $\mathbf{R}^N$. Intuitively, one can understand this fattening phenomena as follows. If we approximate $\Gamma_0$ from inside by a smooth hypersurface $\Gamma_i \to \Gamma_0$ $(i \to \infty)$ then taking the limit of corresponding solution $\Gamma_{it}$ as $i \to \infty$. In particular, one can approximate $\Gamma_0$ from the inside and obtain one solution, and one can approximate $\Gamma_0$ from the outside to obtain another solution. If these "inner" and "outer" solutions coincide, then it follows from the comparison principle that any sequence will have the same limiting solution. This corresponds the case of nonfattening. If they are different, there is no preferred smooth solution, and $\Gamma_t$ in (0.0.9) will consists of the entire region between the inner and outer solutions. Thus $D_t$ is a minimal set-theoretic solution while $D_t \cup \Gamma_t$ is a maximal set theoretic solutions.

There is another notion of set-theoreic solution called barrier solution. Our solution $D_t$ in (0.0.9) is a kind of barrier solution since by comparison principle all smooth evolving hypersurface $\Sigma_t$ solving (0.0.1) remains to be contained in $D_t$ or $\mathbf{R}^N \backslash \overline{D}_t$ for $t \geq t_0$ if $\Sigma_{t_0}$ is contained in $D_{t_0}$ or $\mathbf{R}^N \backslash \overline{D}_{t_0}$. In other words $(\partial D)_t$ avoids all smooth evolutions. It turns out that a barrier solution is an equivalent notion of set-theoretic solutions even without comparison principle for (0.0.6). The notion of barrier solutions was first introduced by De Giorgi (1990) and T. Ilmanen (1993a) (see also T. Ilmanen (1992)) for the mean curvature flow equations and many authors develop the theory. However, the above equivalence has not been observed in the literature so the theory developed here (Chapter V) is new. This characterization provides an alternative way to prove the comparison principle for the level set equation in a set theoretic way at least for (0.0.1) with $f$ independent of $x$ including the mean curvature flow equation. Actually, the idea of the proof is also useful fo establish the comparison theorem for the level set equation for the crystalline curvature flow equation in the plane; the equation is formally written as (0.0.2) but the Frank disgram of $\gamma$ in (0.0.4) is a convex polygon so that (0.0.2) is no longer a partial differential equation. See M.-H. Giga and Y. Giga (1998a). For the background of motion by crystalline curvature the reader is referred to a book of M. E. Gurtin (1993) or a review paper by J. Taylor (1992). We won't touch this problem in this book except in the end of §3.8. For the level set method for crystalline curvature flow equations the reader is referred to papers by M.-H. Giga and Y. Giga (1998a), (1998b), (1999), (2000), (2001), a review paper by Y. Giga (2000) and references cited there. The idea of barrier solution provides an alternate proof for the convergence of solutions of the Allen-Cahn equation to our generalized solution $\Gamma_t$ and $D_t$ which was originally proved by L. C. Evans, H. M. Soner and P. Souganidis (1992) by using distance functions. This was remarked by G. Barles and P. Souganidis (1998). The convergence results are global. For example it reads: $u^\varepsilon(x,0) = 2\chi_{D_0}(x) - 1$ then $u^\varepsilon$ converges to one (as $\varepsilon \to 0$) on $D_t$ and to minus one outside $\Gamma_t \cup D_t$, where $\Gamma_t$ and $D_t$ are our generalized solution defined in (0.0.9). We do not know the behaviour of $u^\varepsilon$ on $\Gamma_t$ if $\Gamma_t$ fattens. The Brakke type solution is always a kind of set-theoretic solution as proved by T. Ilmanen (1993b) and L. Ambrosio and H. M. Soner (1996) so it is contained in $\Gamma_t$. Moreover, T. Ilmanen (1993b) proved that as a limit Brakke solution is obtained. Thus his method also recovers the above convergence results on $D_t$ and outside of $D_t \cup \Gamma_t$.

The idea of level set method is fundamentally important to study behaviour of solutions. We do not mention any application of the method in the present version of our book. The reader is referred to a review paper of Y. Giga (1995a).

This book is organized as follows. In Chapter I we formulate surface evolution equations regorously by defining several relevant quantities. We pay attention to modify the Gaussian curvature flow equation and related equations so that the equation becomes parabolic. We also derive level set equations and study their structual properties. We conclude Chapter I by giving several explicit solutions for typical surface evolution equation having curvature effects. In Chapter II we prepare the theory of viscosity solution which is necessary to analyze level set equations. In Chapter II we mainly discuss stability and Perron's method. A comparison principle which is always fundamental in the theory of viscosity solutions is discussed in Chapter III. In Chapter II and III we did not use geometricity of the equation so that the theory applies other equations including $p$-Laplace heat equation. Some of comparison theorems presented in Chapter III are standard but there are several versions for the equation depending on the space variables. A coordinate free version seems to be new. In Chapter III we also establish convexity and Lipschitz preserving properties for spatially homogeneous equations. In Chapter IV we apply the theory of viscosity solutions to get a generalized solution by a level set method. In Chapter V we consider set-theoretic approach of the level set method. In particular we introduce notion of set-theoretic solutions and barrier solutions. We give an alternate approach to establish comparison results for level set equations. Most of the contents in Chapter V is new at least in this generality. In this book the level set method is adjusted so that it applies to evolutions of noncompact hypersurfaces. Also the evolution with boundary conditions is discussed.

This book is written so that no knowledge of differential geometry is required although such a knowledge is helpful to understand. No knowledge of the theory of viscosity solutions is required except standard maximum principle for semicontinuous functions.

Finally, we note that since the level set method of the present version depends on comparison principle it is impossible to extend directly to higher order surface evolution equation for example the surface diffusion equation:

$$V = -\Delta H \quad \text{on } \Gamma_t,$$

where $\Delta$ denotes the Laplace-Beltrami operator of $\Gamma_t$. This equation was analyzed by C. Elliott and H. Garcke (1997) and J. Escher, U. Mayer and G. Simonett (1997) where local existence and global existence near equilibrium are established. Even for curves in plane, solution behaves differently from that of the curvature flow equation. Embedded curve may lose embeddedness in a finite time as proved in Y. Giga and K. Ito (1998).

# Chapter 1

# Surface evolution equations

There are several interesting examples of equations governing motion of hypersurfaces bounding two phases of materials in various sciences. Such a hypersurface is called an interface or a phase boundary. When the motion depends only on geometry of the hypersurface as well as position and time, the governing equation is often called a *surface evolution equation* or a *geometric evolution equation*. In material sciences it is also called an interface controlled model. Although there are several types of surface evolution equations, we focus on equations of evolving hypersurfaces whose speed depend on its shape through its local geometric quantities such as normals and curvatures. In this chapter we formulate such equations. We derive various useful expression of these quantities especially when the hypersurface is given as a level set of a function. The main goal of this chapter is to study structural properties of level set equations obtained by level set formulation of surface evolution equations. We introduce the notion of geometric equations which is fundamental in a level set method. We also give a few self-similar shrinking solutions as examples of exact solutions.

## 1.1 Representation of hypersurface

There are at least three ways in representing (locally) a hypersurface embedded in $\mathbf{R}^N$. These are representation by local coordinates, by zero level set of a function and by a graph of a function. To fix the idea a set $\Gamma$ in $\mathbf{R}^N$ is called a $C^m$ hypersurface around a point $x_0$ of $\Gamma$ if there is a $C^m$ $(m \geq 1)$ function $u(x)$ defined in a neighborhood $U$ of $x_0$ such that

$$\Gamma \cap U = \{x \in U; u(x) = 0\} \tag{1.1.1}$$

and that the gradient

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_N}\right) = (u_{x_1}, \cdots u_{x_N})$$

of $u$ does not vanish on $\Gamma$. We say this representation by a *level set representation.* If $u$ can be taken $C^\infty$ in $U$ i.e., $u$ is $C^m$ in $U$ for all $m \geq 1$, $\Gamma$ is called smooth around $x_0$. If $\Gamma$ is a $(C^m)$ hypersurface around every point $x_0$ of $\Gamma$, $\Gamma$ is simply called a $(C^m)$ hypersurface.

Of course, by the implicit function theorem, one may assume that $\Gamma$ is locally represented by a graph of a function. By rotating coordinates and shrinking $U$ if necessary, there is a $C^m$ function of $N-1$ variables defined in a neighborhood $U'$ of $x_0'$ that satisfies

$$\Gamma \cap U = \{x_N = g(x'); x' \in U' \subset \mathbf{R}^{N-1}\}$$

with $g(x_0') = x_{0N}$, where $x = (x', x_N)$. This representation is called a *graph representation*. If $\Gamma$ is represented by a graph of $g$, it is represented by the zero level of $u = -x_N + g(x')$.

Another representation is by local cordinates and it includes a graph representation as a special case. It represents a hypersurface $\Gamma$ around $x_0$ by the image $\varphi(U')$ of some $C^m$ mapping $\varphi$ (of full rank) from some open set $U'$ in $\mathbf{R}^{N-1}$ to $\mathbf{R}^N$. By full rank we mean that the Jacobi matrix $\nabla\varphi$ of $\varphi$ has the maximal rank (i.e., in this case the rank of $\nabla\varphi$ equals $N-1$) at each point of $U'$. This representation is called a *parametric representation* and $U'$ is called a space of parameters. The equivalence of a level set representation and a parametric representation is guaranteed by the implicit function theorem.

**Tangents and normals**. Let $\Gamma$ be a hypersurface around $x$. A vector $\tau$ in $\mathbf{R}^N$ is called a *tangent vector* of $\Gamma$ at $x$ if there is a ($C^1$) curve $\zeta$ on $\Gamma$ that satisfies $\zeta = x$, $d\zeta/dt = \tau$ at $t = 0$ , where $\zeta$ is defined at least in a neighborhood of 0. The space of tangent vectors at $x$ is called the *tangent space* of $\Gamma$ at $x$ and is denoted $T_x\Gamma$.

We shall calculate $T_x\Gamma$ when $\Gamma$ is represented by a graph of a function. We may assume that $x = 0$ and that $\Gamma$ is represented by a graph $x_N = g(x')$ around a point $x = 0$. A curve $\zeta$ on $\Gamma$ through 0 is of the form

$$\zeta(t) = (\sigma(t), g(\sigma(t)))$$

with a curve $\sigma(t)$ in $\mathbf{R}^{N-1}$ through 0 of $\mathbf{R}^{N-1}$.Note that

$$\frac{d\zeta}{dt} = \left(\frac{d\sigma}{dt}, \left\langle \nabla' g(\sigma(t)), \frac{d\sigma}{dt} \right\rangle\right) \in \mathbf{R}^N,$$

where $\nabla'$ denotes the gradient in $x'$ and $\langle\ ,\ \rangle$ denotes the standard inner product in the Euclidean space. For given $\tau' \in \mathbf{R}^{N-1}$ there is a curve $\sigma$ that satisfies $d\sigma/dt = \tau'$ at $t = 0$ with $\sigma(0) = 0$ so $\tau$ is a tangent vector (of $\Gamma$ at 0) if and only if

$$\tau = (\tau', \langle \nabla' g(0), \tau' \rangle).$$

In other words

$$T_x\Gamma = \{(\tau', \langle \nabla' g(x), \tau' \rangle), \tau' \in \mathbf{R}^{N-1}\},$$

which in particular implies that $T_x\Gamma$ is an $N-1$ dimensional vector subspace of $\mathbf{R}^N$.

A *unit normal vector* $\mathbf{n}(x)$ at $x$ of $\Gamma$ is a unit vector of $\mathbf{R}^N$ orthogonal to $T_x\Gamma$ with respect to the standard innerproduct $\langle\ ,\ \rangle$ of $\mathbf{R}^N$. It is unique up to multiplier $\pm 1$ since $T_x\Gamma$ is an $N-1$ dimensional space.

Suppose that $\Gamma$ is a hypersurface around $x_0 \in \Gamma$. If $\mathbf{n}(x)$ is a unit normal vector at $x$ of $\Gamma$ near $x_0$ and $\mathbf{n}$ depends on $x$ at least continuously, we say that $\mathbf{n}$ is a *unit normal vector field* of *orientation* (around $x_0$) of $\Gamma$. Such a field $\mathbf{n}$ is of course exists around $x_0$.

To see this we use a level set representation (1.1.1) of $\Gamma$. For $x \in \Gamma \cap U$ let $\zeta(t)$ be a curve on $\Gamma$ through $x$ at $t = 0$, i.e., $\zeta(0) = x$. Differentiate $u(\zeta(t)) = 0$ in $t$ and evaluate at $t = 0$ to get

$$\left\langle \nabla u(x), \frac{d\zeta}{dt}(0) \right\rangle = 0.$$

This implies that $\nabla u(x)$ is orthogonal to $T_x\Gamma$. Since $u$ is $C^m$ $(m \geq 1)$, $\nabla u$ is $C^{m-1}$ and at least continuous. Since $\nabla u$ does not vanish around $x_0$,

$$\mathbf{n}(x) = -\frac{\nabla u(x)}{|\nabla u(x)|}$$

is a unit normal vector field around $x_0$. Here $|p|$ denotes the Euclidean length of vector $p$, i.e., $|p| = \langle p, p\rangle^{1/2}$. There are exactly two unit normal vector fields around $x_0$. The other field is of course $-\mathbf{n}(x)$ . We take $\mathbf{n}$ as above just to fix the idea. If $\Gamma$ is given as a graph $x_N = g(x')$, setting $u(x) = -x_N + g(x')$ yields the upward unit normal:

$$\mathbf{n}(x') = \frac{(-\nabla' g(x'), 1)}{(1 + |\nabla' g(x)|^2)^{1/2}}. \tag{1.1.2}$$

In any case if $\mathbf{n}$ is a unit normal vector field around $x_0$, then

$$T_x\Gamma = \{\tau \in \mathbf{R}^N; \langle \tau, \mathbf{n}(x)\rangle = 0\}$$

for $x \in \Gamma$ around $x_0$. We shall often suppress the words 'vector field'. If a hypersurface $\Gamma$ is a topological boundary $\partial D$ of a domain $D$, the unit normal (vector field) pointing outward from $D$ is called the *outward* unit normal (vector field).

**Evolving hypersurface**. Suppose that $\Gamma_t$ is a set in $\mathbf{R}^N$ depending on the time variable $t$. We say that a family $\{\Gamma_t\}$ or simply $\Gamma_t$ is a ($C^{2m,m}$) *evolving hypersurface* around $(x_0, t_0)$ (with $x_0 \in \Gamma_{t_0}$) if there is a $C^{2m,m}$ $(m \geq 1)$ function $u(x,t)$ defined for $t_0 - \delta < t < t_0 + \delta$, $x \in U$ for some $\delta > 0$ and some neighborhood $U$ of $x_0$ in $\mathbf{R}^N$ such that

$$\Gamma_t \cap U = \{x \in U; u(x,t) = 0\} \tag{1.1.3}$$

and that the spatial gradient $\nabla u$ of $u$ does not vanish on $\Gamma_t$. (This is a level set representation of $\Gamma_t$.) Here by a $C^{2m,m}$ function we mean that derivatives $\nabla^{(k)}\partial_t^h u$ is continuous for $k + 2h \leq 2m$, where $\partial_t^h$ denotes the $h$-th differentiation in the time variable and $\nabla^{(k)}$ denotes the $k$-th differentiation in the space variables. If $u$ can be taken $C^\infty$ i.e., $C^{2m,m}$ for all $m \geq 1$, we just say that $\Gamma_t$ is a smoothly evolving hypersurface around $(x_0, t_0)$. If $\Gamma_t$ is a $C^{2m,m}$ (resp. smoothly) evolving hypersurface around all $(x,t)$ with $x \in \Gamma_t$ and $t$ belonging to an interval $I$, we say that $\Gamma_t$ is a $C^{2m,m}$ (resp. smoothly) evolving hypersurface on $I$. In this Chapter we always assume that $\Gamma_t$ is a smoothly evolving hypersurface in some time interval unless otherwise claimed.

## 1.2 Normal velocity

Let $\mathbf{n}$ be a unit normal vector field of $\Gamma_t$ so that it depends on time $t$ smoothly. The reasonable quantity which describes the motion of $\Gamma_t$ is a normal velocity, that is the speed in the direction of $\mathbf{n}$. Note that there is a chance that each point of $\Gamma_t$ moves but the set $\Gamma_t$ is independent of time like a rotating sphere.

**Definition 1.2.1.** Let $x_0$ be a point of $\Gamma_{t_0}$. Let $x(t)$ be a ($C^1$) curve defined on $(t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$ such that $x(t)$ is a point on $\Gamma_t$ and $x(t_0) = x_0$. The quantity

$$V = \left\langle \frac{dx}{dt}(t_0), \mathbf{n} \right\rangle$$

is called the *normal velocity* at $x_0$ of $\Gamma_t$ at the time $t_0$ in the direction of $\mathbf{n}$.

As we will see later $V$ is independent of the choice of curve $x(t)$. We shall give various expression of $V$.

**Level set representation.** Suppose that $\Gamma_t$ is represented by (1.1.3). We shall calculate the normal velocity $V$ at $(x_0, t_0)$ in the direction of $\mathbf{n}$ defined by

$$\mathbf{n}(x_0, t_0) = -\frac{\nabla u(x_0, t_0)}{|\nabla u(x_0, t_0)|}.$$

Let $x(t)$ be a curve in $\mathbf{R}^N$ on $\Gamma_t$ with $x(t_0) = x_0$. Since $\Gamma_t$ is the zero level set of $u(\cdot, t)$, $u(x(t), t) = 0$ for $t$ near $t_0$. Differentiate $u(x(t), t)$ in $t$ and evaluate at $(x_0, t_0)$ to get

$$u_t(x_0, t_0) + \left\langle \frac{dx}{dt}(t_0), \nabla u(x_0, t_0) \right\rangle = 0,$$

where $u_t = \partial_t u$. Recalling $\mathbf{n} = -\nabla u / |\nabla u|$, we obtain

$$V = \frac{u_t(x_0, t_0)}{|\nabla u(x_0, t_0)|}. \tag{1.2.1}$$

Clearly, this shows that $V$ is independent of the choice of $x(t)$.

**Graph.** We shall give a formula for $V$ when $\Gamma_t$ is represented by a graph of a function. By rotating coodinates we may assume that $\Gamma_t$ is expressed as

$$\Gamma_t = \{x_N = g(x', t); x' \in \mathbf{R}^{N-1}\}$$

around $(x_0, t_0)$, where $g(x_0') = x_{0N}$ and $x' = (x_1, \ldots, x_{N-1})$. If $\mathbf{n}$ is taken upward, $\Gamma_t$ is given as a zero level set of

$$u(x, t) = -x_N + g(x', t)$$

with $\mathbf{n} = -\nabla u / |\nabla u|$ as in (1.1.2). Then by (1.2.1) we see

$$V = \frac{g_t(x_0', t_0)}{(1 + |\nabla' g(x_0', t_0)|^2)^{1/2}}. \tag{1.2.2}$$

**Axisymmetric surface**. Suppose now that $\Gamma_t$ is obtained by rotating the graph of a function $\varphi(x_1, t)$ around $x_1$-axis and that $x_0$ is not on the axis. In other words, around $(x_0, t_0)$ $\Gamma_t$ is of the form

$$\Gamma_t = \left\{ r = \varphi(x_1, t); r = \left( \sum_{j=2}^{N} x_j^2 \right)^{1/2} \right\}.$$

Since $\Gamma_t$ is regarded as the zero level set of

$$u(x, t) = -r + \varphi(x_1, t),$$

the normal velocity $V$ in the direction of

$$\mathbf{n} = \frac{(-\varphi_{x_1}(x_{01}, t_0), x_0''/|x_0''|)}{(1 + (\varphi_{x_1}(x_{01}, t_0))^2)^{1/2}} = -\frac{\nabla u(x_0, t_0)}{|\nabla u(x_0, t_0)|} \tag{1.2.3}$$

at $(x_0, t_0)$ is of the form

$$V = \frac{\varphi_t(x_{01}, t_0)}{(1 + (\varphi_{x_1}(x_{01}, t_0))^2)^{1/2}} \tag{1.2.4}$$

where $x_0'' = (x_{02}, \cdots, x_{0N})$ and $\varphi_{x_1} = \partial\varphi/\partial x_1$.

**Rescaled motion**. If we would like to study the behavior of $\Gamma_t$ near $x_* \in \mathbf{R}^N$ as $t$ tends to $t_*$ with $t < t_*$, we often magnify $\Gamma_t$ near $(x_*, t_*)$ by rescaling. There are of course several ways to rescale but here we only give a typical example. Let $(y, s)$ be defined by

$$y = (t_* - t)^{-1/2}(x - x_*), \ s = -\log(t_* - t) \tag{1.2.5}$$

for $t < t_*$. Let $\hat{V}$ denote the normal velocity of

$$\hat{\Gamma}_s = \{y \in \mathbf{R}^N; \ y = (t_* - t)^{-1/2}(x - x_*), \ x \in \Gamma_t\}$$

at $(y_0, s_0)$, where $s$ is regarded as the new time variable. Here the unit normal vector $\hat{\mathbf{n}}(y_0, s_0)$ of $\hat{\Gamma}_{s_0}$ is taken so that its direction is the same as the unit normal $\mathbf{n}(x_0, t_0)$ of $\Gamma_{t_0}$ at $x_0$, where $(x_0, t_0)$ is determined by (1.2.5) by setting $(y, s) = (y_0, s_0)$. Let $V$ be the normal velocity of $\Gamma_{t_0}$ at $x_0$ in the direction of $\mathbf{n}(x_0, t_0)$. Then

$$\hat{V} = e^{-s_0/2} \ V + \frac{1}{2} \ \langle \ y_0, \ \hat{\mathbf{n}}(y_0, s_0) \ \rangle. \tag{1.2.6}$$

Note that the behavior of $\Gamma_t$ as $t$ tends to $t_*$ with $t < t_*$ corresponds to the large time behavior of $\hat{\Gamma}_s$ by (1.2.5).

To see this formula we use level set representation of $\Gamma_t$ near $(x_0, t_0)$. Suppose that $\Gamma_t$ is represented by (1.1.3). We may assume $\mathbf{n}(x_0, t_0)$ is of form

$$\mathbf{n}(x_0, t_0) = -\frac{\nabla u(x_0, t_0)}{|\nabla u(x_0, t_0)|}.$$

We set

$$w(y,s) = u((t_* - t)^{1/2}\ y + x_*,\ t),\ s = -\log(t_* - t)$$

so that $\hat{\Gamma}_s$ is represented as zero level set of $w$ near $(y_0, s_0)$. Since

$$\hat{\mathbf{n}}(y_0, s_0) = -\frac{\nabla w(y_0, s_0)}{|\nabla w(y_0, s_0)|},$$

the formula (1.2.1) yields

$$\hat{V} = \frac{w_s(y_0, s_0)}{|\nabla w(y_0, s_0)|}.$$

A direct calculation shows that

$$\left(w_s + \frac{1}{2} \langle\ y,\ \nabla w\ \rangle\right)(y_0, s_0) = e^{-s_0}\ u_t(x_0, t_0),\ \nabla w(y_0, s_0) = e^{-s_0/2}\ \nabla u(x_0, t_0).$$

These two identities together with representation of $V$, $\hat{\mathbf{n}}$, $\hat{V}$ yields (1.2.6).

If we consider a little bit general rescaling than (1.2.5) of form

$$y = (t_* - t)^{-\alpha}\ (x - x_*),\ s = -\log(t_* - t) \quad \text{with} \quad \alpha > 0,$$

then

$$\hat{V} = e^{-(1-\alpha)s_0}\ V + \alpha\ \langle\ y_0,\ \hat{\mathbf{n}}(y_0,\ s_0)\ \rangle$$

instead of (1.2.6).

## 1.3 Curvatures

Let $\Gamma$ be a hypersurface in $\mathbf{R}^N$. For a point $x_0$ let $\tau$ be a tangent vector of $\Gamma$ at $x_0$. Let $X$ be a ($C^1$) vector field on $\Gamma$ around $x_0$, i.e., $X$ be a $C^1$ function from $\Gamma$ to $\mathbf{R}^N$ around $x_0$. Here $C^1$ means that $X$ can be extended to a $C^1$ function in a neighborhood of $x_0$ in $\mathbf{R}^N$. The vector field $X$ needs not be tangential to $\Gamma$. Let $\zeta$ be a curve on $\Gamma$ that satisfies

$$\zeta(0) = x_0, \quad \frac{d\zeta}{dt}(0) = \tau.$$

A tangential derivative in the direction of $\tau$ is defined by

$$(D_\tau X)(x_0) = \frac{d}{dt}(X(\zeta(t))|_{t=0}.$$

If $X$ is extended to a neighborhood of $x_0$ in $\mathbf{R}^N$, we observe that

$$(D_\tau X)(x_0) = (\tau \cdot \nabla)X = \sum_{j=1}^{N} \tau_j \frac{\partial}{\partial x_j} X, \qquad \tau = (\tau_1, \ldots .\tau_N).$$

This shows that the operator $D_\tau$ is independent of the choice of the curve $\zeta$. By the definition of $D_\tau$ the quantity $(\tau \cdot \nabla)X$ is independent of the extension of $X$ outside $\Gamma$. Thus the operator $D_\tau$ is well-defined for any vector field on $\Gamma$.

**Second fundamental form**. Suppose now that $\Gamma$ is a ($C^2$) hypersurface around $x$ on $\Gamma$. Let $\mathbf{n}$ be a unit normal vector field around $x$. From the level set representation it is clear that $\mathbf{n}$ is $C^1$ on $\Gamma$. For each $\tau \in T_x\Gamma$ we set

$$\mathbf{A}\tau = -D_\tau \mathbf{n} \in \mathbf{R}^N.$$

Since $|\mathbf{n}| = 1$,

$$\langle \mathbf{A}\tau, \mathbf{n} \rangle = -\frac{1}{2}\, D_\tau(|\mathbf{n}|^2) = 0$$

so that $\mathbf{A}\tau \in T_x\Gamma$. The linear operator $\mathbf{A} = \mathbf{A}_x$ from $T_x\Gamma$ into itself is called the *Weingarten map* (in the direction of $\mathbf{n}(x)$). The bilinear form on $T_x\Gamma \times T_x\Gamma$ associated with $\mathbf{A}$ defined by

$$B_x(\tau, \eta) = \langle\ \mathbf{A}\tau,\ \eta\ \rangle \tag{1.3.1}$$

is called the *second fundamental form* (in the direction of $\mathbf{n}(x)$) at $x \in \Gamma$.

To see the geometric meaning of $B_x$ for $\tau,\ \eta \in T_x\Gamma$ let $\phi$ be a function from a neighborhood of the origin of $\mathbf{R}^2$ to $\Gamma \subset \mathbf{R}^N$ that satisfies

$$\frac{\partial \phi}{\partial x_1}(0,0) = \tau,\ \frac{\partial \phi}{\partial x_2}(0,0) = \eta,\ \phi(0,0) = x.$$

Since

$$\left\langle\ \mathbf{n}(\phi(x_1,x_2)),\ \frac{\partial \phi}{\partial x_1}(x_1,x_2)\ \right\rangle\ = 0 \tag{1.3.2}$$

near $(x_1, x_2) = (0,0) \in \mathbf{R}^2$, differentiating in $x_2$ and evaluating at zero yields

$$\left\langle\ \left(\frac{\partial \phi}{\partial x_2}(0,0)\ \cdot\ \nabla\right)\mathbf{n},\ \frac{\partial \phi}{\partial x_1}(0,0)\ \right\rangle + \left\langle\ \mathbf{n},\ \frac{\partial^2 \phi}{\partial x_1\, \partial x_2}(0,0)\ \right\rangle\ = 0,$$

or

$$\langle\ (\eta\ \cdot\ \nabla)\mathbf{n},\ \tau\ \rangle\ + \left\langle\ \mathbf{n},\ \frac{\partial^2 \phi}{\partial x_1\, \partial x_2}(0,0)\ \right\rangle\ = 0.$$

By definition this yields

$$B_x(\eta,\ \tau) = \left\langle\ \mathbf{n},\ \frac{\partial^2 \phi}{\partial x_1\, \partial x_2}(0,0)\ \right\rangle. \tag{1.3.3}$$

In particular $B_x$ is a symmetric bilinear form and the Weingarten map $\mathbf{A}$ is a symmetric linear operator. Thus its eigenvalues are all real and called *principal curvatures* of $\Gamma$ at $x$ (in the direction of $\mathbf{n}(x)$). The principal curvatures are denoted $\kappa_1, \cdots, \kappa_{N-1}$. Differentiating (1.3.2) in $x_1$ and evaluating at zero, we get

$$B_x(\tau,\ \tau) = \left\langle\ \mathbf{n},\ \frac{\partial^2 \phi}{\partial x_1^2}(0,0)\ \right\rangle$$

instead of (1.3.3). If $|\tau| = 1$, this quantity is called the *normal curvature* of $\Gamma$ in the direction of $\tau$.

**Surfaces of higher codimension.** As well known the second fundamental form is defined even for any embedded manifold $M$ in $\mathbf{R}^N$ whose *dimension* $k$ is strictly less than $N-1$ (i.e., *codimension* $N-k$ is strictly greater than 1). We briefly review the definition since it is almost the same as (1.3.1). For the tangent space $T_xM$ of $M$ at $x$ let $N_xM$ denote its orthogonal complement in $\mathbf{R}^N$. The space $N_xM$ is called the *normal space* of $M$ at $x$. Since $M$ has dimension $k$, the dimension of $T_xM$ and $N_xM$ equal $k$ and $N-k$, respectively. Let $\mathbf{n}^1, \cdots, \mathbf{n}^{N-k}$ be ($C^1$) vector fields near $x \in M$ such that $\{\mathbf{n}^i(z);\ 1 \le i \le N-k\}$ is an orthonormal basis of $N_zM$ for every $z$ near $x$. The *second fundamental form* of $M$ at $x$ is defined by

$$\mathbf{B}_x(\tau,\ \eta) = -\sum_{j=1}^{N-k} \langle\ (D_\tau \mathbf{n}^i)(x),\ \eta\ \rangle\ \mathbf{n}^i(x),\ \tau,\ \eta \in T_xM$$

as a mapping

$$\mathbf{B}_x\ :\ T_xM \times T_xM \to N_xM.$$

As in the same way to derive (1.3.3) we have

$$\mathbf{B}_x(\eta,\ \tau) = \pi\left(\frac{\partial^2 \phi}{\partial x_1\, \partial x_2}\,(0,0)\right) \ \text{ with } \ \frac{\partial \phi}{\partial x_1}(0,0) = \tau,\ \frac{\partial \phi}{\partial x_2}(0,0) = \eta, \phi(0,0) = x,$$

where $\pi$ denotes the orthogonal projection from $\mathbf{R}^N$ onto $N_xM$. In particular $\mathbf{B}_x$ is symmetric and $\mathbf{B}_x$ is independent of the choice of $\mathbf{n}^1, \cdots, \mathbf{n}^{N-k}$ forming an orthonormal basis of $N_zM$ for $z$ near $x$. Note that the definition does not require $k < N-1$. If $k = N-1$, then, as expected

$$\langle\ \mathbf{B}_x(\tau, \eta),\ \mathbf{n}\ \rangle\ = B_x(\tau, \eta),$$

where $B_x$ is the second fundamental form in the direction of $\mathbf{n}$. This shows that $\mathbf{B}_x$ is a natural generalization of $B_x$.

**Surface divergence.** Let $X$ be a $C^1$ vectorfield on a hypersurface $\Gamma$. For $x \in \Gamma$ let $\{\tau^\ell;\ 1 \le \ell \le N-1\}$ be an orthonormal basis of $T_x\Gamma$. The *surface divergence* of $X$ is denoted by $\mathrm{div}_\Gamma\ X$ and is defined by

$$(\mathrm{div}_\Gamma\ X)(x) = \sum_{\ell=1}^{N-1} \langle\ (D_{\tau^\ell}\ X)(x),\ \tau^\ell\ \rangle. \tag{1.3.4}$$

If we extend $X$ around $\Gamma$ so that $\nabla X$ is well-defined as an $N \times N$ matrix, then (1.3.4) yields

$$(\mathrm{div}_\Gamma\ X)(x) = \mathrm{trace}\left(\sum_{\ell=1}^{N-1}\ \tau^\ell \otimes\ \tau^\ell\right)(\nabla X)(x) \tag{1.3.5}$$

since $D_\tau X = (\tau\ \cdot\ \nabla)X$, where $\otimes$ denotes the tensor product of $N$-vectors. The definition of $\mathrm{div}_\Gamma\ X$ is independent of the extension of $X$, so the right hand side of (1.3.5) is independent of the extension of $X$. Since

$$I_N - \sum_{\ell=1}^{N-1}\ \tau^\ell \otimes\ \tau^\ell = \mathbf{n}(x) \otimes \mathbf{n}(x)$$

as matrices, the identity (1.3.5) is rewritten as

$$(\mathrm{div}_\Gamma\ X)(x) = \mathrm{trace}\{(I - \mathbf{n}(x) \otimes\ \mathbf{n}(x))(\nabla X(x))\}, \tag{1.3.6}$$

where $I$ denotes the $N \times N$ unit matrix. From (1.3.6) it follows that the surface divergence is defined independent of the choice of orthonormal basis of $T_x\Gamma$ and the orientation $\mathbf{n}$.

The surface divergence $\mathrm{div}_M\ X$ is also defined for a vector field $X$ on an embedded manifold $M$ of dimension $k$ in $\mathbf{R}^N$ whose codimension $N - k > 1$. It is defined as (1.3.4) i.e.,

$$(\mathrm{div}_M\ X)(x) = \sum_{\ell=1}^{k} \langle (D_{\tau^\ell}\ X)(x),\ \tau^\ell\ \rangle,$$

where $\{\tau^\ell;\ 1 \le \ell \le k\}$ is an orthonormal basis of $T_xM$. As for a hypersurface, $\mathrm{div}_M\ X$ is independent of the choice of orthonormal basis of $T_xM$ which can be proved directly from (1.3.5) with $N - 1$ replaced by $k$.

**Mean curvature.** Let $\mathbf{n}$ be a unit normal vector field around $x$ on a ($C^2$) hypersurface $\Gamma$. Let $H$ be the sum of all principal curvatures $\kappa_1(x), \cdots, \kappa_{N-1}(x)$ at $x$ in the direction of $\mathbf{n}$, i.e.,

$$H = \kappa_1(x) + \cdots + \kappa_{N-1}(x).$$

We say that $H$ is the *mean curvature* (of $\Gamma$) at $x$ (in the direction of $\mathbf{n}$). We do *not* take the average of principal curvatures although many authors have taken the average to define the mean curvature since the time of Gauss. Since $H$ is the trace of the Weigarten map,

$$H = \sum_{\ell=1}^{N-1} \mathbf{B}_x(\tau^\ell,\ \tau^\ell) = -\sum_{\ell=1}^{N-1} (D_{\tau^\ell}\ \mathbf{n}(x),\ \tau^\ell) = -(\mathrm{div}_\Gamma\ \mathbf{n})(x),$$

where $\tau^\ell$ ($\ell = 1, \cdots, m-1$) is an orthonormal basis of $T_x\Gamma$. If $N = 2$, $H$ is simply called the *curvature* at $x \in \Gamma$ (in the direction of $\mathbf{n}$) and is denoted $\kappa$.

Even if a manifold $M$ in $\mathbf{R}^N$ has higher codimension, or its dimension $k < N - 1$, the mean curvature is defined as a vector. We say that

$$\mathbf{H} = \sum_{\ell=1}^{N-1} \mathbf{B}_x(\tau^\ell,\ \tau^\ell) \in N_xM$$

is the *mean curvature vector* at $x \in M$. By definition of the second fundamental form and the surface divergence

$$\mathbf{H} = -\sum_{i=1}^{N-k} (\mathrm{div}_M\ \mathbf{n}^i)\mathbf{n}^i,$$

where $\mathbf{n}^i (1 \le i \le N - k)$ is the same as in the definition of $\mathbf{B}_x$. If $k = N - 1$, then by definition

$$H = \langle\ \mathbf{H},\ \mathbf{n}\ \rangle,$$

where $H$ is the mean curvature in the direction of $\mathbf{n}$. Since $N_x\Gamma$ is one dimensional, the mean curvature $H$ has all information of the mean curvature vector if we take the sign of $H$ into account.

**Symmetric curvatures.** Let $\kappa_1(x)\cdots\kappa_{N-1}(x)$ be principal curvatures at $x \in \Gamma$ (in the direction of $\mathbf{n}$), where $\Gamma$ is a ($C^2$) hypersurface in $\mathbf{R}^N$. We shall consider elementary symmetric polynomials of principal curvatures. Let $e_m(1 \le m \le N-1)$ denote the $m$-th *elementary symmetric polynomial* of $\lambda_1, \cdots, \lambda_{N-1}$, i.e.,

$$e_m(\lambda_1, \cdots, \lambda_{N-1}) = \sum \lambda_{i_1} \cdots \lambda_{i_m}$$

where sum is taken over all integers $i_1, \cdots i_m$ that satisfies $1 \le i_1 < i_i < \cdots < i_m \le N-1$. In particular

$$e_1(\lambda_1, \cdots, \lambda_{N-1}) = \lambda_1 + \cdots + \lambda_{N-1},$$
$$e_{N-1}(\lambda_1, \cdots, \lambda_{N-1}) = \lambda_1 \cdots \lambda_{N-1}.$$

Clearly, the mean curvature is of form

$$H = e_1(\kappa_1, \cdots, \kappa_{N-1}).$$

The quantity

$$K = e_{N-1}(\kappa_1, \cdots, \kappa_{N-1})$$

is called the *Gaussian curvature*. In general we say that

$$H_m = e_m(\kappa_1, \cdots, \kappa_{N-1})$$

is the $m$-th *symmetric curvature*. Sometimes we consider a little more complicated curvature defined by the ratio $H_m/H_\ell$ with $m > \ell$. The quantity $H_{N-1}/H_{N-2}$ for $N \ge 3$ is called the *harmonic curvature* since

$$H_{N-1}/H_{N-2} = \left( \sum_{i=1}^{N-1} \frac{1}{\kappa_i} \right)^{-1}.$$

**Anisotropic curvatures**. It is well-known that the mean curvature $H$ is the change ratio of surface area of $\Gamma$ per change of volume of $D$ enclosed by $\Gamma$, where $\Gamma$ is a ($C^2$) hypersurface in $\mathbf{R}^N$. If the hypersurface $\Gamma$ has anisotropic structure depending on its normal direction, it is natural to consider surface energy instead of area. Let $\gamma_0$ be a positive function defined on the unit sphere

$$S^{N-1} = \left\{ p \in \mathbf{R}^N;\ |p| = 1 \right\}.$$

We extend $\gamma_0$ on $\mathbf{R}^n$ so that

$$\gamma(p) = \gamma_0(p/|p|)\ |p|. \tag{1.3.7}$$

Clearly, $\gamma(p)$ is positively homogeneous of degree one, i.e.,

$$\gamma(\lambda p) = \lambda \gamma(p) \quad \text{for all} \quad \lambda > 0,\ p \in \mathbf{R}^N. \tag{1.3.8}$$

Conversely, $\gamma$ satisfying (1.3.8) with $\gamma|_{S^{N-1}} = \gamma_0$ is always expressed by (1.3.7). The *surface energy* of $\Gamma$ with *surface energy density* $\gamma_0$ is defined by

$$\int_\Gamma \gamma_0(\mathbf{n})d\sigma$$

where $d\sigma$ denotes the surface element. Of course, this is nothing but a surface area if $\gamma_0 \equiv 1$. We shall define a quantity (called the anisotropic mean curvature) which describes the change ratio of surface energy of $\Gamma$ per change of volume of $D$ enclosed by $\Gamma$ as a generalization of the mean curvature.

For a given surface energy density $\gamma_0$ let $\xi$ be the gradient of the homogenization $\gamma$ of $\gamma_0$ given by (1.3.7), i.e. $\xi = \nabla\gamma$. The vector $\xi$ is called the *Cahn-Hoffman vector* of $\gamma_0$. We say

$$h(x) = -(\mathrm{div}_\Gamma\ \xi(\mathbf{n}))(x)$$

is the *anisotropic* (or *weighted*) *mean curvature* of $\Gamma$ at $x$ (in the direction of $\mathbf{n}$) with respect to surface energy density $\gamma_0$. To define $h$ as a continuous function we need to assume that $\gamma$ is $C^2$ outside the origin. Of course, if $\gamma_0 \equiv 1$, then $\xi(p) = p/|p|$ so that $\xi(\mathbf{n}) = \mathbf{n}$. Thus the anisotropic mean curvature agrees with usual mean curvature when $\gamma_0 \equiv 1$ as expected.

## 1.4 Expression of curvature tensors

Let $\Gamma \subset \mathbf{R}^N$ be a ($C^2$) hypersurface around $x_0 \in \Gamma$. Let $\mathbf{A} = \mathbf{A}_{x_0}$ denote the Weingarten map in the direction of $\mathbf{n}(x_0)$, where $\mathbf{n}$ is a unit normal vector field on $\Gamma$ around $x_0$. We shall give a various expression of $\mathbf{A}$ and curvatures.

**Level set representation.** Suppose that $\Gamma$ is represented by (1.1.1) with

$$\mathbf{n}(x_0) = -\frac{\nabla u(x_0)}{|\nabla u(x_0)|}. \tag{1.4.1}$$

Then for $\tau \in T_{x_0}\Gamma$

$$\begin{aligned} \mathbf{A}_{x_0}\tau &= -(D_\tau\mathbf{n})(x_0) = (\tau\cdot\nabla)\frac{\nabla u}{|\nabla u|} \\ &= \frac{1}{|\nabla u|}\left\{(\tau\cdot\nabla)\nabla u - \langle\ (\tau\cdot\nabla)\nabla u, \nabla u\ \rangle\frac{\nabla u}{|\nabla u|^2}\right\} \quad \text{at} \quad x = x_0. \end{aligned} \tag{1.4.2}$$

It is convenient to introduce the orthogonal projection $\Pi_{x_0}$ from $T_{x_0}\mathbf{R}^n$ to $T_{x_0}\Gamma$. Its matrix expression is

$$R_{\mathbf{n}(x_0)} = I - \mathbf{n}(x_0)\otimes\mathbf{n}(x_0)$$

so that

$$\Pi_{x_0}\zeta = R_{\mathbf{n}(x_0)}\zeta, \quad \zeta \in T_{x_0}\mathbf{R}^N = \mathbf{R}^N,$$

where $\zeta$ is regarded as a column vector. Using the notation

$$R_p = I - p\otimes p/|p|^2 \quad \text{for} \quad p \in \mathbf{R}^N,\ p \neq 0, \tag{1.4.3}$$

we observe from (1.4.2) that

$$\mathbf{A}_{x_0}\tau = \frac{1}{|\nabla u|} R_p\ (\nabla^2 u)\tau \quad \text{at } x_0 \text{ with} \quad p = \nabla u(x_0),$$

where $\tau$ is regarded as a column vector. Since

$$R_p\tau = \tau,\ p = \nabla u(x_0)$$

by definition, we see

$$\mathbf{A}_{x_0}\tau = \frac{1}{|\nabla u(x_0)|} Q_p(\nabla^2 u(x_0))\tau \quad \text{at} \quad x_0$$

with

$$Q_p(X) = R_p X R_p, \tag{1.4.4}$$

where $X$ is an $n \times n$ real symmetric matrix. Although $R_p \nabla^2 u$ may not be symmetric, $Q_p(\nabla^2 u)$ is now symmetric. It is not difficult to see that the symmetric operator $\tilde{\mathbf{A}}_{x_0}$ from $T_{x_0}\mathbf{R}^N$ to $T_{x_0}\mathbf{R}^N$ defined by

$$\tilde{\mathbf{A}}_{x_0}\zeta = \frac{1}{|\nabla u(x_0)|} Q_{\nabla u(x_0)}(\nabla^2 u(x_0))\zeta,\ \zeta \in T_{x_0}\mathbf{R}^N = \mathbf{R}^N \tag{1.4.5}$$

is a unique linear operator with the property that

$$\tilde{\mathbf{A}}_{x_0}\zeta = \mathbf{A}_{x_0}\Pi_{x_0}\zeta.$$

We often identify the Weingarten map $\mathbf{A}_{x_0}$ by $\tilde{\mathbf{A}}_{x_0}$. By definition $\tilde{\mathbf{A}}_{x_0}$ is given by a direct sum of operators

$$\tilde{\mathbf{A}}_{x_0} = \mathbf{A}_{x_0} \oplus 0$$

corresponding to the decomposition of the tangent space $\mathbf{R}^N = T_{x_0}\mathbf{R}^N$ of form

$$\mathbf{R}^N = T_{x_0}\Gamma \oplus N_{x_0}\Gamma,$$

where $N_{x_0}\Gamma$ denotes the normal vector space at $x_0$. Thus the eigenvalues of $\tilde{\mathbf{A}}_{x_0}$ consist of principal curvatures $\kappa_1, \cdots, \kappa_{N-1}$ and 0.

We shall derive the level set representation of various curvatures from (1.4.5). The mean curvature $H$ at $x_0 \in \Gamma$ in the direction of $\mathbf{n}(x_0)$ (defined by (1.4.1)) is

$$\begin{aligned} H &= \kappa_1 + \cdots + \kappa_{N-1} + 0 = \text{trace } \tilde{\mathbf{A}}_{x_0} \\ &= \frac{1}{|p|} \text{ trace} Q_p(X) \quad \text{with} \quad p = \nabla u(x_0),\ X = \nabla^2 u(x_0) \end{aligned}$$

Since $R_p^2 = R_p$ by (1.4.3) and trace $(R_p X R_p) = \text{trace}(R_p^2 X)$, $H$ is of form

$$\begin{aligned} H &= \frac{1}{|\nabla u(x_0)|} \text{ trace} \left( \left( I - \frac{\nabla u(x_0) \otimes \nabla u(x_0)}{|\nabla u(x_0)|} \right) \nabla^2 u(x_0) \right) \\ &= \frac{1}{|\nabla u|} \left( \left( \Delta u - \sum_{1 \le 1,\ j \le N} \frac{u_{x_i} u_{x_j}}{|\nabla u|^2} \right) u_{x_i x_j} \right) \quad \text{at} \quad x = x_0, \end{aligned} \tag{1.4.6}$$

where $u_{x_i} = \partial u/\partial x_i$, $u_{x_ix_j} = \partial^2 u/\partial_{x_i}\partial_{x_j}$ and $\Delta u = \sum_{j=1}^N u_{jj}$. For $m$-th symmetric curvature with $m \le N-1$ by the form of $\tilde{\mathbf{A}}_x$ we see

$$H_m = e_m(\kappa_1, \cdots, \kappa_{N-1}, 0)$$

where $\kappa_1, \cdots, \kappa_{N-1}, 0$ are eigenvalues of $Q_p(X)/|p|$ with $p = \nabla u(x_0)$, $X = \nabla^2 u(x_0)$, since

$$e_m(\kappa_1, \cdots, \kappa_{N-1}, 0) = e_m(\kappa_1, \cdots, \kappa_{N-1}).$$

If $m = N-1$, then the Gaussian curvature $K$ is of form

$$K = e_{N-1}(\kappa_1, \cdots, \kappa_{N-1}, 0) = e_N(\kappa_1, \cdots, \kappa_{N-1}, 1).$$

This observation gives a simple representation of the Gaussian curvature

$$K = \det\left(\frac{Q_p(X)}{|p|} + \frac{p \otimes p}{|p|^2}\right),\ p = \nabla u(x_0),\ X = \nabla^2 u(x_0) \tag{1.4.7}$$

since the eigenvalues of $Q_p(X)/|p| + p \otimes p/|p|^2$ are $\kappa_1, \cdots, \kappa_{N-1}, 1$. If we arrange $\kappa_1 \le \cdots \le \kappa_{N-1}$, $\kappa_i (1 \le i \le N-1)$ is written as

$$\kappa_i = k_i(p, X),\ p = \nabla u(x_0) \neq 0,\ X = \nabla^2 u(x_0)$$

where $k_i(p, X) \le k_2(p, X) \le \cdots \le k_{N-1}(p, X)$ are the eigenvalues of the linear operator $Q_p(X)/|p|$ the orthogonal complement of the vector $p$. By this expression it is possible to express the $m$-th symmetric curvature by $u$.

There is another way to derive the level set representation of the mean curvature and anisotropic mean curvature without using the representation (1.4.5) but using surface divergence. We set $\mathbf{m} = -\nabla u/|\nabla u|$ and note that the vector field $\mathbf{m}$ is defined not only on $\Gamma$ but also some neighborhood of $\Gamma$ near $x_0$. By (1.3.6) we observe that

$$\begin{aligned} H &= -\mathrm{div}_\Gamma \mathbf{n} = -\mathrm{trace}((I - \mathbf{n}(x_0) \otimes \mathbf{n}(x_0))(\nabla \mathbf{m})(x_0)) \\ &= -\mathrm{trace}(\nabla \mathbf{m}(x_0)) + \mathrm{trace}(\mathbf{n}(x_0) \otimes \mathbf{n}(x_0)(\nabla \mathbf{m})(x_0)) \\ &= \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)(x_0) + \sum_{1 \le i,\ j \le N} n_i(x_0) n_j(x_0) (\frac{\partial}{\partial x_j} m_i)(x_0), \end{aligned} \tag{1.4.8}$$

where $\mathbf{n} = (n_1, \cdots, n_N)$, $\mathbf{m} = (m_1, \cdots, m_N)$. Since $\mathbf{m}$ is a *unit* vector field near $x_0$,

$$\sum_{i=1}^N n_i(x_0)\left(\frac{\partial}{\partial x_j} m_i\right)(x_0) = \frac{1}{2}\left(\frac{\partial}{\partial x_j} \sum_{i=1}^N m_i^2\right)(x_0) = \frac{1}{2}\frac{\partial}{\partial x_j} 1 = 0.$$

Thus the last term of (1.4.8) disappears and we obtain

$$H = \left(\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right)(x_0) \tag{1.4.9}$$

which is the same as (1.4.6).

We shall give a level set representation of anisotropic mean curvature. We recall (1.3.6) to calculate

$$\begin{aligned} h &= -\mathrm{div}_\Gamma \xi(\mathbf{n}) = -\mathrm{trace}\{(I - \mathbf{n}(x_0) \otimes \mathbf{n}(x_0))(\nabla \xi(\mathbf{m}))(x_0)\} \\ &= -\mathrm{trace}(\nabla \xi(\mathbf{m}))(x_0) + \mathrm{trace}\{(\mathbf{n}(x_0) \otimes \mathbf{n}(x_0))(\nabla \xi(\mathbf{m}))(x_0)\}, \end{aligned} \tag{1.4.10}$$

The second term is of form

$$\sum_{1 \le i,k,\ell \le N} n_i(x_0) n_j(x_0) \frac{\partial \xi^i}{\partial p_\ell}(\mathbf{n}(x_0)) \frac{\partial m_\ell}{\partial x_j}(x_0)$$

with $\xi = (\xi^1, \cdots, \xi^N)$. Since $\xi = \nabla \gamma$ and $\gamma$ is posively homogeneous of degree one so that

$$\gamma(\lambda p) = \lambda \gamma(p), \ \lambda > 0,$$

differentiating in $p_\ell$ yields

$$\lambda \xi^\ell(\lambda p) = \lambda \xi^\ell(p) \quad \text{or} \quad \xi^\ell(\lambda p) = \xi^\ell(p) \quad \text{for} \ \ 1 \le \ell \le N.$$

In other words $\xi^\ell$ is positively homogeneous of degree zero. Differentiating in $\lambda$ and setting $\lambda = 1$ yields the Euler equation

$$\sum_{i=1}^N p_i \frac{\partial \xi^\ell}{\partial p_i}(p) = 0, \ 1 \le \ell \le N \tag{1.4.11}$$

Since $\partial \xi^\ell / \partial p_i = \partial \xi^i / \partial p_\ell$, the second term of (1.4.10) can be rewritten as

$$\begin{aligned} &\sum_{1 \le i,j,\ell \le N} n_i(x_0) n_j(x_0) \frac{\partial \xi^\ell}{\partial p_i}(\mathbf{n}(x_0)) \frac{\partial m_\ell}{\partial x_j}(x_0) \\ = &\sum_{1 \le i,j,\ell \le N} \left( \sum_{i=1}^N n_i(x_0) \frac{\partial \xi^\ell}{\partial p_i}(\mathbf{n}(x_0)) \right) \frac{\partial m_\ell}{\partial x_j}(x_0) n_j(x_0) = 0 \quad \text{by (1.4.11).} \end{aligned}$$

Thus we have

$$h = -(\mathrm{div}\ \xi(\mathbf{m}))(x_0); \tag{1.4.12}$$

note that this formula holds for any extension $\mathbf{m}$ of $\mathbf{n}$ in a tubular neighborhood of $\Gamma$ near $x_0$ since we do not use the property $|\mathbf{m}| = 1$. Instead, $\xi$ should be a Cahn-Hoffman vector. If $\gamma(p) = |p|$ so that $\xi = p/|p|$, we recover the formula (1.4.9) with $\mathbf{m} = -\nabla u / |\nabla u|$. Since $\xi$ is homogeneous of degree zero, (1.4.12) yields

$$\begin{aligned} h &= -\mathrm{div}(\xi(-\nabla u))(x_0) \\ &= \sum_{1 \le i,j \le N} \frac{\partial^2 \gamma}{\partial p_i \partial p_j}(-\nabla u(x_0)) \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0). \end{aligned} \tag{1.4.13}$$

(Of course if $\gamma(p) = |p|$, this again yields (1.4.9).) By (1.4.11) we see $R_p \nabla^2 \gamma(p) = \nabla^2 \gamma(p) = \nabla^2 \gamma(p) R_p$. From (1.4.13) it now follows that

$$h = \frac{1}{|\nabla u(x_0)|} \mathrm{trace} \left( \nabla^2 \gamma(\mathbf{n}(x_0)) Q_{\mathbf{n}(x_0)} (\nabla^2 u)(x_0) \right) \quad \text{with } \mathbf{n}(x_0) = -\nabla u(x_0) / |\nabla u(x_0)| \tag{1.4.14}$$

since $\partial^2\gamma/\partial p_i \partial p_j$ is homogeneous of degree minus one so that

$$\frac{\partial^2\gamma}{\partial p_i \partial p_j}(-\nabla u) = \frac{1}{|\nabla u|}\frac{\partial^2\gamma}{\partial p_i \partial p_j}\left(-\frac{\nabla u}{|\nabla u|}\right),\ 1 \le i,\ j \le N.$$

Using the second fundamental form in (1.4.14) we have

$$h = \mathrm{trace}(\nabla^2\gamma(\mathbf{n}(x_0))\tilde{\mathbf{A}}_{x_0}). \tag{1.4.15}$$

The formula (1.4.15) may explain a reason why $h$ is sometimes called a weighted mean curvature.

**Graph representation.** It is easy to derive formula for the second fundamental form from (1.4.5) when $\Gamma$ is given as the graph of a function. By rotating coordinates we may assume that $\Gamma$ is expressed as

$$\Gamma = \{x_N = g(x'),\ x' \in \mathbf{R}^{N-1}\}$$

around $x_0 \in \Gamma$, where $g(x_0') = x_{0N}$. If $\mathbf{n}$ is taken upward, then $\Gamma$ is given as the zero level set of

$$u(x) = -x_N + g(x')$$

with $\mathbf{n} = -\nabla u/|\nabla u|$ which is the same as (1.1.2). Plugging in (1.4.5) we obtain a formula of $\tilde{\mathbf{A}}_{x_0}$ written by $g$. A general formula is complicated so we do not give it here. However if $\nabla g'(x_0') = 0$ so that $\mathbf{n}(x_0) = (0, 0, \cdots, 0, 1)$ then the expression of $\tilde{\mathbf{A}}_{x_0}$ is simple. Indeed since $\nabla u(x_0) = -\mathbf{n}(x_0)$, we see

$$Q_{\nabla u(x_0)}(\nabla^2 u(x_0)) = \begin{pmatrix} \nabla'^2 g(x_0) & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus we obtain

$$\tilde{\mathbf{A}}_{x_0}\zeta = \nabla'^2 g(x_0)\zeta',\ \zeta' \in \mathbf{R}^{N-1} \quad \text{with } \zeta = (\zeta', \zeta_n) \in \mathbf{R}^N. \tag{1.4.16}$$

We shall calculate the mean curvature $H$ at $x_0$ in the direction of $\mathbf{n}$. We plug $u$ in (1.4.9) to get

$$\begin{aligned} H &= \left(\mathrm{div}'\left(\frac{\nabla g'}{\sqrt{1+|\nabla g'|^2}}\right)\right)(x_0') + \frac{\partial}{\partial x_N}\left(\frac{-1}{\sqrt{1+|\nabla g'|^2}}(x_0')\right) \\ &= \left(\mathrm{div}'\left(\frac{\nabla g'}{\sqrt{1+|\nabla g'|^2}}\right)\right)(x_0'), \end{aligned} \tag{1.4.17}$$

where $\mathrm{div}'$ denotes the divergence in $x'$ variables.

**Axisymmetric surface.** Suppose that $\Gamma$ is obtained by rotating the graph of a function $\varphi(x_1)$ around $x_1$-axis and that $x_0 \in \Gamma$ is not on the axis. Around $x_0 \in \Gamma$ the hypersurface $\Gamma$ is of form

$$\Gamma = \left\{ r = \varphi(x_1);\ r = \left( \sum_{j=2}^{N} x_j^2 \right)^{1/2} \right\}.$$

If $\mathbf{n}$ is taken outward from $x_1$-axis, i.e. $\mathbf{n}$ is given by (1.2.3), then $\Gamma$ is given as the zero level set of

$$u(x_1, \cdots, x_N) = -r + \varphi(x_1)$$

with $\mathbf{n} = -\nabla u/|\nabla u|$ (around $x_0$). We obtain a formula of $\tilde{\mathbf{A}}_{x_0}$ written by $\varphi$ by plugging above $u$ into (1.4.5). However, we do not give its explicit formula. Here we only calculate the mean curvature $H$ at $x_0$ in the direction of $\mathbf{n}$. We plug $u$ in (1.4.9) and using the formula (1.2.3) to get

$$\begin{aligned} H &= \frac{\partial}{\partial x_1} \left( \frac{\varphi_{x_1}}{(1+\varphi_{x_1}^2)^{1/2}} \right) - \frac{1}{(1+\varphi_{x_1}^2)^{1/2}} \sum_{j=2}^{N} \frac{\partial}{\partial x_j} \frac{x_j}{r} \\ &= \frac{\varphi_{x_1 x_1}}{(1+\varphi_{x_1}^2)^{3/2}} - \frac{1}{(1+\varphi_{x_1}^2)^{1/2}} \frac{N-2}{r} \quad \text{at } x = x_0 = (x_x, x_0'') \end{aligned} \tag{1.4.18}$$

where $r = |x''|$ with $x'' = (x_2, \cdots, x_N)$.

**Gradient of normal vector fields** Let $\mathbf{n}$ be a unit normal vector field of $\Gamma$ around $x_0$. Let $\mathbf{m}$ be a $(C^1)$ extension of $\mathbf{n}$ to a tubular neighborhood of $\Gamma$ around $x_0$. Since

$$\mathbf{A}_{x_0}\ \tau = -((\tau \cdot \nabla)\mathbf{m})(x_0),$$

for $\tau \in T_{x_0}\Gamma$,

$$\begin{aligned} \tilde{\mathbf{A}}_{x_0}\zeta &= -((\Pi_{x_0}\zeta \cdot \nabla)\mathbf{m})(x_0) \\ &= -(\zeta \cdot \nabla)\mathbf{m})(x_0) + \langle\, \zeta, \mathbf{n}(x_0)\, \rangle(\mathbf{n}(x_0) \cdot \nabla)\mathbf{m}(x_0). \end{aligned}$$

This implies that the matrix expression of $\tilde{\mathbf{A}}_{x_0}$ (with respect to the standard Euclidean basis) equals

$$-\frac{\partial}{\partial x_i} m_j(x_0) + n_i(x)(\mathbf{n}(x_0) \cdot \nabla) m_j(x_0),\ 1 \le i,\ j \le N$$

(which should be symmetric since $\tilde{\mathbf{A}}_{x_0}$ is symmetric). If the extension $\mathbf{m}$ has the property that

$$((\mathbf{n}(x_0) \cdot \nabla)\mathbf{m})(x_0) = 0, \tag{1.4.19}$$

then one may identify $-\nabla\mathbf{m}(x_0)$ by $\tilde{\mathbf{A}}_{x_0}$. We shall use the notation $\nabla\mathbf{n}$ by extending $\mathbf{n}$ to a tublar neighborhood of $\Gamma$ around $x_0$ such that (1.4.19) holds. By this interpretation $-\nabla\mathbf{n} = \tilde{\mathbf{A}}_{x_0}$, so we shall often identify second fundamental form $\mathbf{A}_{x_0}$ with $-\nabla\mathbf{n}$.

We conclude this section by studying the range of $\nabla\mathbf{n}$. Let $\mathbf{S}^N$ denote the space of all $N \times N$ real symmetric matrices.

**Lemma 1.4.1.** *Let $x_0$ be a point in $\mathbf{R}^N$. For each $p \in S^{N-1}$ and $X \in \mathbf{S}^N$ there is a smooth hypersurface $\Gamma$ around $x_0$ with the property that $\mathbf{n}(x_0) = p$, $\nabla\mathbf{n}(x_0) = Q_p(X)$ where $\mathbf{n}$ is a unit normal vector field on $\Gamma$ around $x_0$ and is extended to a tubular neighborhood of $\Gamma$ satisfying $((\mathbf{n}(x_0)\cdot\nabla)\mathbf{n})(x_0) = 0$.*

*Proof.* We may assume $p = (0, 0, \cdots, 1)$ by rotation of coordinates and $x_0 = 0$ by translation. The matrix $Q_p(X)$ is of form

$$\begin{pmatrix} & & 0 \\ & Y & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

with $Y \in \mathbf{S}^{N-1}$. We set

$$g(x') = -\frac{1}{2}\,\langle\, Y\ x', x'\,\rangle \quad \text{for } x' \in \mathbf{R}^{N-1}$$

to get $g(0) = 0$, $\nabla' g(0) = 0$, $\nabla'^2 g(0) = Y$. Since $\nabla n(x_0) = -\tilde{\mathbf{A}}_{x_0}$, the formula (1.4.16) yields

$$\nabla\mathbf{n}(x_0) = -\nabla'^2 g(0) = Q_p(X). \quad \square$$

## 1.5 Examples of surface evolution equations

We give general examples of equations of an evolving hypersurface whose normal velocity $V$ is determined by its normals and second fundamental forms. In general such an evolution equation is of form

$$V = f(x,\ t,\ \mathbf{n},\ \nabla\mathbf{n}) \tag{1.5.1}$$

on an evolving hypersurface $\Gamma_t$, where $f$ is a given function and $\mathbf{n}$ is a unit normal vector field of $\Gamma_t$. For cosistency with the literature we take the minus of the second fundamental form $\mathbf{A}$ of $\Gamma_t$ as an independent variable of $f$ and denote it by $\nabla\mathbf{n}$. Here $V$ is the velocity in the direction of $\mathbf{n}$.

### 1.5.1 General evolutions of isothermal interfaces

For an evolving hypersurface $\Gamma_t$ we consider

$$\beta(\mathbf{n})V = -a\ \mathrm{div}_{\Gamma_t}\xi(\mathbf{n}) - c(x,t). \tag{1.5.2}$$

Here $\xi$ is the Cahn-Hoffman vector of a surface energy density $\gamma_0 : S^{n-1} \to (0,\infty)$ and $\beta$ is a given positive function on $S^{n-1}$, $a$ is a nonnegative constant and $c$ is a given function. We always assume that $N \geq 2$ if $a \neq 0$ since the curvature term equals zero when $N = 1$. As we observed in (1.4.15), this equation is an example of (1.5.1). If $c$ is independent of $x$ and $a = 1$, (1.5.2) is often used to describe motion of isothermal interface; there $c$

is regarded as bulk free energy difference such as temperature difference in both phases. The function $1/\beta$ is called the *mobility*. It may be again anisotropic in the sense that it depends on the direction of normals. The mobility is determined by microstructure of the hypersurface. Sometimes it is proprotional to $\gamma_0$, i.e., $\beta\gamma_0$ is constant independent of $\mathbf{n}$ but in general $1/\beta$ is not necessarily propotional to $\gamma_0$. The equation (1.5.2) has an energy structure. Indeed, we set

$$G(\Gamma) = \int_\Gamma a\ \gamma_0(\mathbf{n})d\sigma + \int_D c\ dx \tag{1.5.3}$$

for a hypersurface $\Gamma$ surrounding $D$, then (1.5.2) is of the form

$$\beta(\mathbf{n})V = -\delta G/\delta\Gamma_t$$

where $\delta G/\delta\Gamma_t$ denotes the change ratio of (free energy) $G$ per change of volume of $D$ in the direction of $\mathbf{n}$. Here $\mathbf{n}$ is taken outward from $D$. In other words (1.5.2) is a gradient flow of $G$.

**Mean curvature flow equation.** The equation (1.5.2) includes several interesting important examples as a special case. If the mobility and the surface energy density is isotropic with no driving force $c$ and $a = 1$, then (1.5.2) becomes

$$V = H \tag{1.5.4}$$

by rescaling time if necessary (or taking $\beta \equiv \gamma_0 \equiv 1$, $a = 1$, $c = 0$). This equation is called the *mean curvature flow equation.* If $N = 2$ so that $\Gamma_t$ is a curve, (1.5.4) is called the *curve shortening equation.* If $\kappa$ denotes the curvature in the direction of $\mathbf{n}$, the curve shortening equation is of form

$$V = \kappa. \tag{1.5.5}$$

For the mean curvature flow equation the energy $G$ in (1.5.3) is the surface area of $\Gamma$. Thus (1.5.4) gives a deformation so that decrease ratio of area is steepest. This is why (1.5.5) is called the curve shortening equation. Note that the equation (1.5.4) is invariant under the change of orientation $\mathbf{n}$. In other words the evolution law (1.5.4) is the same even if we replace $\mathbf{n}$ by $-\mathbf{n}$.

**Hamilton Jacobi equations.** If $a = 0$, then (1.5.2) becomes

$$\beta(\mathbf{n})V = c(x,t). \tag{1.5.6}$$

This equation is regarded as a special form of the Hamilton-Jacobi equation. Indeed, if $\Gamma_t$ is represented by the graph of a function i.e., $x_N = g(x', t)$, then by (1.1.2) and (1.2.2) the equation (1.5.6) is of the form

$$g_t + H(x',\ t,\ g,\ \nabla' g) = 0 \tag{1.5.7}$$

with

$$H(x',\ t,\ r,\ p') = -c(x',\ r,\ t)\mu(p')/\beta(-p'/\mu(p'),\ \mu(p')^{-1}),\ \mu(p') = (1 + |p'|^2)^{1/2}.$$

The equation (1.5.7) is a first order equation and the curvature plays no role in (1.5.6).

### 1.5.2 Evolution by principal curvatures

In the mean curvature flow equation the normal velocity depends only on principal curvatures of hypersurfaces. There are several other examples of form

$$V = g(\kappa_1, \cdots \kappa_{N-1};\ \mathbf{n}), \tag{1.5.8}$$

where $g$ is a given function of principal curvatures of $\kappa_1, \cdots, \kappa_{N-1}$ and $\mathbf{n}$. In the mean curvature flow equation, $g$ is independent of $\mathbf{n}$ and is taken as the first elementary symmetric polynomial $e_1$.

If $g$ is taken $e_{N-1}$ so that $e_{N-1}(\kappa_1, \cdots, \kappa_{N-1})$ equals the Gaussian curvature $K$, (1.5.8) becomes the *Gaussian curvature flow equation*

$$V = K. \tag{1.5.9}$$

Of course if $N = 2$, then this equation becomes the curve shortening equation $V = \kappa$. More generally, for each $1 \leq m \leq N-1$

$$V = H_m \tag{1.5.10}$$

is called the *m-th symmetric curvature flow* equation, where $H_m$ is the $m$-th symmetric curvature $e_m(\kappa_1, \cdots \kappa_{N-1})$. The equation

$$V = H_{N-1}/H_{N-2} \quad (N \geq 3) \tag{1.5.11}$$

is called the *harmonic curvature flow equation*, which is a special case of

$$V = H_m/H_\ell. \tag{1.5.12}$$

All equations (1.5.9)-(1.5.12) are examples of (1.5.8). As we see later, we impose the restriction $\ell < m$ in (1.5.12) so that the equation is parabolic at least for convex surfaces. Note that in general the evolution law (1.5.9) or (1.5.10) (with even $m$) may depends on the choice of orientation $\mathbf{n}$. For a closed evolving hypersurface $\Gamma_t$ we take the inward normal vector field as $\mathbf{n}$ so that sphere shrinks as time develops. The same remark applies to (1.5.12). We use this convention when we consider the evolution by principle curvatures (1.5.10) with even $m$ and (1.5.12) with even $m - \ell$.

### 1.5.3 Other examples

In general the right hand side of (1.5.2) is not propotional to the velocity. A natural generalization is

$$V = h(-a \ \mathrm{div}_{\Gamma_t}\ \xi(\mathbf{n}) + c,\ \mathbf{n}) \tag{1.5.13}$$

where $h$ is nondecreasing in the first variable and $h(0, \mathbf{n}) = 0$. If $N = 2$ so that $\Gamma_t$ is a curve in the plane and $h(\sigma, \mathbf{n}) = \sigma_+^{1/3}$ with $c = 0$, $a = 1$, $\gamma_0 \equiv 1$, then (1.5.13) becomes

$$V = (\kappa_+)^{1/3} \tag{1.5.14}$$

where $\sigma_+ = \max(\sigma, 0)$. The equation (1.5.14) is called the *affine curvature flow* equation since the equation is invariant under affine transformation.

### 1.5.4 Boundary conditions

It often happens that a hypersurface $\Gamma_t$ moves in a domain $\Omega$ in $\mathbf{R}^N$ and the geometric boundary of $\Gamma_t$ intersects the boundary $\partial\Omega$ of $\Omega$. In this case in addition to the equation (1.5.1) in $\Omega$ we have to impose the boundary condition so that evolution is determined by equations. We give here typical examples of them. We assume that the boundary $\partial\Omega$ is at least $C^1$ hypersurface in $\mathbf{R}^N$. Let $\nu$ be the unit normal vector field of $\partial\Omega$ outward from $\Omega$. Let $\mathbf{n}$ be the unit normal vector field of a smooth hypersurface $\{\Gamma_t\}$ in $\overline{\Omega}$.

**Neumann boundary condition.** This condition imposes

$$\langle\ \nu,\ \mathbf{n}\ \rangle\ = 0$$

on the intersection of $\Gamma_t$ and $\partial\Omega$. Geometrically speaking, $\Gamma_t$ intersects $\partial\Omega$ orthogonally, i.e., $\Gamma_t \perp \partial\Omega$.

**Prescribed contact angle boundary condition.** Let $z$ be a given real-valued continuous function on $\partial\Omega$ that satisfies $|z| < 1$ on $\partial\Omega$. The prescribed contact boundary condition imposes

$$\langle\ \nu,\ \mathbf{n}\ \rangle\ = z$$

on the intersection of $\Gamma_t$ and $\partial\Omega$. Of course if $z = 0$, this condition is exactly the Neumann boundary condition. Although the orientation of $\Gamma_t$ is irrelvant to describe the Neumann boundary condition, the presdribed angle condition depends on the orientation of $\Gamma_t$. In the literature the prescribed contact angle condition is often referred to as the Neumann boundary condition.

**Dirichlet boundary condition.** Let $S$ be a given codimension two closed surfaces in $\mathbf{R}^N$. The Dirichlet condition imposes that the geometric boundary of $\Gamma_t$ always equals $S$. This condition is so far not easy to treat in a level set method so we do not discuss this problem much in this book.

## 1.6 Level set equations

For a given surface evolution equation we shall introduce its level set equation. We shall study various properties of level set equations.

### 1.6.1 Examples

We consider a surface evolution equation

$$V = f(x,\ t,\ \mathbf{n},\ \nabla\mathbf{n}) \tag{1.6.1}$$

on an evolving hypersurface $\Gamma_t$ in a domain $\Omega$ in $\mathbf{R}^N$. Here $f(x,\ t,\ \cdot,\ \cdot)$ for $(x,t) \in \Omega\times[0,T]$ is a given real-valued function defined in

$$\mathbf{E} = \{(p, Q_p(X));\ p \in S^{N-1},\ X \in \mathbf{S}^N\},$$

By Lemma 1.4.1 the set $\mathbf{E}$ is a natural space so that $(\mathbf{n}, \nabla\mathbf{n})$ lives. We say an equation

$$u_t(x,t) + F(x,\ t,\ \nabla u(x,t),\ \nabla^2 u(x,t)) = 0 \quad \text{for all } (x,t) \in \Omega \times (0,T) \tag{1.6.2}$$

is a *level set equation* of (1.6.1) if for *each* level set

$$\Gamma_t = \{x;\ u(x,t) = c\}$$

of a $C^{2,1}$ solution $u$ of (1.6.2) near $(x_0,t_0) \in \Omega$ satisfies (1.6.1) at $(x_0,t_0)$ provided that $\nabla u(x_0,t_0) \neq 0$ and that the orientation $\mathbf{n}$ is chosen so that $\mathbf{n}(x_0,t_0) = -\nabla u(x_0,t_0) / |\nabla u(x_0,t_0)|$, where $x_0 \in \Gamma_{t_0}$. Here $F$ is a real-valued function defined in $\Omega \times [0,T] \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$.

Such an equation is uniquely determined by (1.6.1). Indeed, using the level set representation of $V, \mathbf{n}$ and $\nabla\mathbf{n}$, (1.6.1) is of form

$$\frac{u_t}{|\nabla u|} = f\left(x, t, -\frac{\nabla u}{|\nabla u|},\ -\frac{1}{|\nabla u|}\ Q_{\nabla u}(\nabla^2 u)\right)$$

on the evolving hypersurface. The representation of $V, \mathbf{n}$ and $\nabla\mathbf{n}$ is the same if the hypersurface is a $c$-level set instead of the zero level set of $u$. Thus, the equation

$$u_t - |\nabla u| f\left(x, t, -\frac{\nabla u}{|\nabla u|},\ -\frac{1}{|\nabla u|}\ Q_{\nabla u}(\nabla^2 u)\right) = 0$$

is the unique level set equation. It may be rewritten as

$$u_t + F_f(x, t, \nabla u, \nabla^2 u) = 0 \tag{1.6.3}$$

with

$$F_f(x,t,p,X) = -|p| f\left(x, t, -\frac{p}{|p|},\ -\frac{1}{|p|}\ Q_p(X)\right) \tag{1.6.4}$$

for $p \in \mathbf{R}^N \backslash \{0\}$, $X \in \mathbf{S}^N$, $(x,t) \in \Omega \times [0,T]$. Note that the function $F_f$ is not defined for $p = 0$ in general as we will see in following examples.

**Level set mean curvature flow equation.** If (1.6.1) is the mean curvature flow equation (1.5.4) : $V = H$, then by (1.2.1) and (1.4.6), the level set equation is

$$u_t - \Delta u + \sum_{1 \le i,\ j \le N} \frac{u_{x_i} u_{x_j}}{|\nabla u|^2}\ u_{x_i x_j} = 0 \tag{1.6.5}$$

or (1.6.3) with

$$F_f(x,t,p,X) = F_f(p,X) = -\text{trace}\ \left(I - \frac{p \otimes p}{|p|^2}\right) X,\ p \neq 0 \tag{1.6.6}$$

which is independent of $(x,t)$ and all $X \in \mathbf{S}^N$. Note that $F$ is not defined for $p = 0$. Using (1.4.9) we often write the level set equation of (1.5.4) as

$$u_t - |\nabla u| \text{div}\ \left(\frac{\nabla u}{|\nabla u|}\right) = 0 \tag{1.6.7}$$

which is of course the same as (1.6.5). The equation (1.6.5) (and its equivalent form (1.6.7)) is called the *level set mean curvature* flow equation.

**Level set Hamilton-Jacobi equation.** We consider the level set equation of the Hamilton-Jacobi equation (1.5.6)

$$\beta(\mathbf{n})V = c(x,t)$$

By (1.2.1) and (1.4.1) its level set equation is of form

$$u_t - c(x,t)|\nabla u|\beta(-\nabla u/|\nabla u|) = 0 \tag{1.6.8}$$

or (1.6.3) with

$$F(x,t,p,X) = -|p|\beta(-p/|p|)c(x,t), \tag{1.6.9}$$

which is independent of $X$. The equation (1.6.8) is again the first order Hamilton-Jacobi equation

$$u_t + H(x,t,\nabla u) = 0$$

with the Hamiltonian $H(x,t,p) = F_f(x,t,p,X)$ which is not necessarily convex in $p$. We also note that $H(x,t,p)$ can be extended continuously to $p = 0$.

**Anisotropic version.** We consider the anisotropic version of the mean curvature flow given by (1.5.2). By (1.2.1), (1.4.1) and (1.4.13) its level set equation is

$$u_t - |\nabla u| \left( a \sum_{1\le i,\ j\le N} \frac{\partial^2\gamma}{\partial p_i\,\partial p_j}(-\nabla u)\frac{\partial^2 u}{\partial x_i\,\partial x_j} + c \right) \frac{1}{\beta(-\nabla u/|\nabla u|)} = 0 \tag{1.6.10}$$

or (1.6.3) with

$$\begin{aligned} F_f(x,t,p,X) &= -\{a\ \mathrm{trace}(\nabla^2\gamma(-p)X) + c(x,t)\}\ \frac{|p|}{\beta(-p/|p|)} \\ &= -\{a\ \mathrm{trace}(\nabla^2\gamma(-p)R_p\ X\ R_p) + c(x,t)\}\ \frac{|p|}{\beta(-p/|p|)} \end{aligned} \tag{1.6.11}$$

by (1.4.14), where $R_p$ is given by (1.4.3). This $F_f$ depends on $(x,t)$ through $c$ and it is defined for all $p \in \mathbf{R}^N\backslash\{0\}$ and $X \in \mathbf{S}^N$.

**Level set Gaussian curvaure flow equation.** We consider the Gaussian curvature flow equation (1.5.9) : $V = K$. Its level set equation is of form

$$u_t - |\nabla u|\ \det\left( \left( I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right) \frac{\nabla^2 u}{|\nabla u|} \left( I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right) + \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right) = 0 \tag{1.6.12}$$

since we have (1.4.7) for the Gaussian curvature. If we write it in the form (1.6.3),

$$F_f(x,t,p,X) = -|p|\ \det\left( R_p X P_p + \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \right). \tag{1.6.13}$$

In above examples the level set equation is directly computable by (1.6.4) once the surface equation is given by (1.6.1) with explicit $f$. For example, the mean curvature flow equation is of form

$$V = -\text{trace } \nabla \mathbf{n}$$

so that $f(\mathbf{n}\ \nabla\mathbf{n}) = \text{trace } \nabla\mathbf{n}$. Then by (1.6.4)

$$F_f(x,t,p,X) = -|p| \text{ trace } Q_p(X)/|p| = -\text{trace } Q_p(X),$$

which is the same as in (1.6.6).

**Boundary conditions.** If the boundary condition on $\partial\Omega$ is imposed for an evolving hypersurface $\Gamma_t$ in a domain $\Omega$, it should be included in the level set equation. In the level set representation the Neumann boundary condition

$$\langle\ \nu,\ \mathbf{n}\ \rangle = 0$$

is written as

$$\left\langle\ \nu,\ -\frac{\nabla u}{|\nabla u|}\ \right\rangle = 0$$

if we take $\mathbf{n}$ as in (1.4.1). More generally, the prescribed contact angle condition can be written as

$$\left\langle\ \nu,\ -\frac{\nabla u}{|\nabla u|}\ \right\rangle = z$$

or

$$\frac{\partial u}{\partial \nu} + z\ |\nabla u| = 0, \tag{1.6.14}$$

where $\nu$ is the outward unit normal vector field of $\partial\Omega$ and $|z| < 1$; $\partial u/\partial\nu$ denotes the directional derivative of $u$ in the direction of $\nu$, i.e. $\partial u/\partial\nu = (\nu\cdot\nabla)u$. If boundary condition is imposed for (1.6.1) on $\Omega$, its level set equation should include the boundary condition. For prescribed contact angle condition it is easy to include. However, for the Dirichlet problem, it is not clear in what way we include it. The level set equation of the boundary problem for (1.6.1) requires that each level set of solutions must satisfy the boundary condition. For example the level set equation of $V = H$ with the prescribed contact angle condition $\langle\ \nu,\ \mathbf{n}\ \rangle = z$ is

$$\begin{cases} u_t - \Delta u + \Sigma\ \dfrac{u_{x_i}u_{x_j}}{|\nabla u|^2}\ u_{x_ix_j} = 0, & \text{in } \Omega\times(0,T) \\ \qquad\qquad\quad \dfrac{\partial u}{\partial \nu} + z\ |\nabla u| = 0 & \text{on } \partial\Omega\times(0,T). \end{cases}$$

## 1.6.2 General scaling invariance

The function $F_f$ defined by (1.6.4) has special scaling properties in $p$ and $X$. To see this we suppress the dependence in $(x,t)$. Let $f$ be a real-valued function defined in

$$\mathbf{E} = \{(p, Q_p(X));\ p\in S^{N-1},\ X\in\mathbf{S}^N\}. \tag{1.6.15}$$

We set

$$F_f(p,X) = -|p|\ f(-\frac{p}{|p|},\ -\frac{1}{|p|}\ Q_p(X)),\ p \in \mathbf{R}^N\backslash\{0\},\ X \in \mathbf{S}^N \tag{1.6.16}$$

Then $F_f$ fulfills

(G1) $F_f(\lambda p,\ \lambda X) = \lambda F_f(p,X)$, for all $\lambda > 0,\ p \in \mathbf{R}^N\backslash\{0\},\ X \in \mathbf{S}^N$,
(G2) $F_f(p, X + p \otimes y + y \otimes p) = F_f(p,X)$, for all $p \in \mathbf{R}^N\backslash\{0\},\ X \in \mathbf{S}^N$.
Indeed, (G1) follows from definition of $F_f$ since $Q_p$ is a linear operator. To show (G2) we note the identity

$$(x \otimes p)(p \otimes y) = x \otimes y|p|^2.$$

Using this identity we see that

$$R_p\ p \otimes y = \left(I - \frac{p \otimes p}{|p|^2}\right)\ p \otimes y = p \otimes y - p \otimes y = 0$$

and similarly $y \otimes p\ R_p = 0$, where $R_p$ is given in (1.4.3). Thus

$$Q_p(p \otimes y + y \otimes p) = R_p(p \otimes y + y \otimes p)\ R_p = 0. \tag{1.6.17}$$

From this identity (G2) follows.

**Definition 1.6.1.** Let $F$ be a real-valued function in $(\mathbf{R}^N\backslash\{0\}) \times \mathbf{S}^N$. We say that $F$ is *strongly geometric* if $F$ fulfills (G1) and (G2).

We have seen that $F_f$ is always strongly geometric. We shall prove its converse: if $F$ is strongly geometric, then there is (unique) $f$ with $F = F_f$ (Theorem 1.6.4). To see this we study structure of $\mathbf{E}$ as a bundle over a unit sphere $S^{N-1}$. The vector space $\mathbf{S}^N$ is equipped with an inner product

$$\langle\ A \mid B\ \rangle = \text{trace}\ AB,\ A,\ B \in \mathbf{S}^N.$$

For a given $p \in S^{N-1}$ we consider the linear operator from $\mathbf{S}^N$ into itself defined by

$$Q_p(X) = R_p\ X\ R_p,\ X \in \mathbf{S}^N$$

as in (1.4.3), (1.4.4). Let $Q_p^*$ denote the adjoint operator of $Q_p$ with respect to the inner product $\langle\ \cdot|\cdot\ \rangle$.

**Lemma 1.6.2.** *(i) The operator $Q_p$ is an orthogonal projection on $\mathbf{S}^N$, i.e. $Q_p^2 = Q_p$ and $Q_p^* = Q_p$.*
*(ii) The kernel of $Q_p$ in $\mathbf{S}^N$ equals one-dimensional space*

$$L_p = \{p \otimes y + y \otimes p;\ y \in \mathbf{R}^N\}$$

*Proof.* (i) The property $Q_p^2 = Q_p$ follows from $R_p^2 = R_p$. Since

$$\langle\ Q_p(X) \mid Y\ \rangle = \text{trace}(R_p X R_p\ Y) = \text{trace}(X\ R_p\ Y\ R_p) = \langle\ X \mid Q_p(Y)\ \rangle$$

for all $X, Y \in \mathbf{S}^N$, the operator $Q_p$ is self-adjoint, i.e. $Q_p^* = Q_p$.
(ii) By (1.6.17) $L_p$ is contained in the kernel of $Q_p$. It remains to prove that $Q_p(X) = 0$ for $X \in \mathbf{S}^N$ implies $X \in L_p$. By definition of $Q_p$ we see

$$U^{-1}\ Q_p(X)U = Q_q(Y),\ q = pU,\ Y = U^{-1}XU,\ X \in \mathbf{S}^N$$

for any orthogonal matrix $U$, where $p, q$ are regarded as row vectors. We take $U$ so that $q = (1, 0, \cdots, 0)$ and observe that $Q_q(Y) = 0$ implies

$$Y = \begin{pmatrix} 2y_1 & y_2 & \dots & y_N \\ y_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ y_N & 0 & \dots & 0 \end{pmatrix},$$

since $X \in \mathbf{S}^N$ implies $Y \in \mathbf{S}^N$. Thus $Q_p(X) = 0$ implies $Y = q \otimes y + y \otimes q$ with $y = (y_1, \cdots, y_N)$, which is the same as $X \in L_p$. $\square$

The set $\mathbf{E}$ defined by (1.6.15) is regarded as a (smooth) vector subbundle of a trivial bundle $S^{N-1} \times \mathbf{S}^N$. The fibre dimension of $\mathbf{E}$ equals $N(N-1)/2$. Let $Q$ be a bundle map

$$Q : S^{N-1} \times \mathbf{S}^N \to \mathbf{E}$$

defined by

$$Q(p, X) = (p, Q_p(X)).$$

Let $\mathbf{L}$ be a bundle over $S^{N-1}$ of form

$$\mathbf{L} = \{(p, X);\ p \in S^{N-1},\ X \in L_p\}.$$

Since $Q$ is surjective to $\mathbf{E}$, Lemma 1.6.3 provides a direct sum decomposition of $S^{N-1} \times \mathbf{S}^N$.

**Lemma 1.6.3.** *The bundle $S^{N-1} \times \mathbf{S}^N$ is expressed as a orthogonal sum of form $\mathbf{L} \oplus \mathbf{E}$ as bundles over $S^{N-1}$. The operator $Q$ gives a projection to $\mathbf{E}$ on fibres.*

Let $\mathcal{G}$ be the set of all strongly geometric real-valued function $F$ defined in $(\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$. Let $\mathcal{I}$ be the set of all real-valued function $f$ defined in $\mathbf{E}$. Let $\mathbf{F}$ denote the mapping corresponds $F_f$ to $f$, where $F_f$ is defined by (1.6.16).

**Theorem 1.6.4.** The mapping $\mathbf{F}$ is a bijection from $\mathcal{I}$ to $\mathcal{G}$.

*Proof.* Let $\mathcal{G}'$ be the set of all real-valued function $F'$ on $S^{N-1} \times \mathbf{S}^N$ satisfying (G2). Then the mapping $F' \mapsto F$ defined by

$$F(p, X) = |p|\ F'\left(\frac{p}{|p|},\ \frac{X}{|p|}\right)$$

gives a bijection from $\mathcal{G}'$ to $\mathcal{G}$. By dedinition of $\mathbf{L}$ and $L_p$ one may identity $F' \in \mathcal{G}'$ with a function on the quotient bundle $S^{N-1} \times \mathbf{S}^N/\mathbf{L}$. By Lemma 1.6.3 $S^{N-1} \times \mathbf{S}^N/\mathbf{L}$ is identified with $\mathbf{E}$ and the mapping $f \mapsto F'$ defined by

$$F'(p, X) = -f(-p, -Q_p(X)),\ p \in S^{N-1},\ X \in \mathbf{S}^N/L_p$$

gives a bijection from $\mathcal{G}$ to $\mathcal{G}'$ since $Q_p = Q_{-p}$. Since $\mathbf{F}:\ f \mapsto F_f$ is a composition of $f \mapsto F'$ and $F' \mapsto F$, $\mathbf{F}$ is a bijection from $\mathcal{I}$ to $\mathcal{G}$. $\Box$

**Remark 1.6.5.** The bijective property of $\mathbf{F}$ is still valid for function defined in a subset of $\mathbf{E}$ if $\mathcal{I}$ and $\mathcal{G}$ are appropriately modified. For a subset $\Sigma$ of $S^{N-1}$ let $\mathbf{E}_\Sigma$ be of form

$$\mathbf{E}_\Sigma = \{(p, Q_p(X));\ p \in \Sigma,\ X \in \mathbf{S}^N\}.$$

Let $\mathcal{I}$ be the set of all real-valued function defined in $\mathbf{E}_\Sigma$. Let $\mathcal{G}$ be the set of all strongly geometric real-valued function defined in $\tilde{\Sigma} \times \mathbf{S}^N$ with the cone $\tilde{\Sigma} = \{\lambda p;\ p \in \Sigma,\ \lambda > 0\}$; there we understand that (G1), (G2) holds for $p \in \tilde{\Sigma}$. Then $\mathbf{F}$ gives a bijection from $\mathcal{I}$ to $\mathcal{G}$ as before. The proof is the same.

### 1.6.3 Ellipticity

As we know the backward heat equation cannot be solvable for general smooth initial data even locally in time. We need some structual conditions for $f$ to find solution $\Gamma_t$ with initial data $\Gamma_0$. We recall the notion of degenerate ellipticity and parabolicity for this purpose.

**Definition 1.6.6.** Let $F$ be a real-valued function defined in $(\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$ (or in its subset $\tilde{\Sigma} \times \mathbf{S}^N$, where $\tilde{\Sigma} = \{\lambda p;\ p \in \Sigma \subset S^{N-1},\ \lambda > 0\}$). We say $F$ is *degenerate elliptic* if

$$F(p, X) \le F(p, Y),\ p \in (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N \text{ (or } \tilde{\Sigma} \times \mathbf{S}^N) \tag{1.6.18}$$

for all $X, Y \in S^N$ with $X \ge Y$. Here $X \ge Y$ means that $X - Y$ is a nonnegative matrix, i.e., $\langle\ (X-Y)\xi, \xi\ \rangle\ \ge 0$ for all $\xi \in \mathbf{R}^N$.

This condition is a kind of monotonicity of $F$ in $X$. Fortunately, many examples $F_f$ of level set equations fulfill this property as we see below.

**Level set mean curvature flow equation.** The function $F_f(p, X)$ defined by (1.6.6) is degenerate elliptic. Indeed, by definition

$$F_f(p, X) = -\mathrm{trace}(R_p\, Y) - \mathrm{trace}\ R_p(X - Y).$$

Since trace $AB \ge 0$ for $A \ge O,\ B \ge O$ and $R_p \ge O$, the last term is nonpositive if $X \ge Y$ so (1.6.18) follows.

**Level set Hamilton-Jacobi equation.** The function $F_f(x, t, \cdot, \cdot)$ defined by (1.6.9) is independent of $X$ so $F_f(x, t, \cdot, \cdot)$ is degenerate elliptic for all $x, t$. If a level set equation is of first order, $F_f$ is always degenerate elliptic.

**Anisotropic version.** The function $F_f(x,t,\cdot,\cdot)$ defined by (1.6.11) is degenerate elliptic if $\nabla^2\gamma \geq O$ as well as $a \geq 0$ and $\beta > 0$; as before we here assume that $\gamma$ is $C^2$ outside the origin. The idea to prove (1.6.18) is the same as the proof for $-\mathrm{trace}\,(R_p X)$.

**Surface evolution equation by principal curvatures.** The function $F_f$ defined by (1.6.13) is no longer degenerate elliptic unless $N = 2$. We shall modify $e_m$ so that $F_f$ is degenerate elliptic. If we consider the equation (1.5.8), $F_f$ in the level set equation is of form

$$F_f(p,X) = -|p|\; g\left(k_1(p,X), \cdots, k_{N-1}(p,X),\ -\frac{p}{|p|}\right), \tag{1.6.19}$$

where $k_i$'s are eigenvalues of $Q_p(X)/|p|$ as defined in in the paragraph on the Gaussian curvature in §1.4. There is a sufficient condition on $g$ so that $F_f$ is degenerate elliptic.

**Proposition 1.6.7.** *For each* $i = 1, \cdots N-1$, $p \in S^{N-1}$ *and* $(\lambda_1, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_{N-1}) \in \mathbf{R}^{N-2}$ *the function* $\lambda_i \mapsto g(\lambda_1, \cdots, \lambda_{N-1};\ p)$ *is nondecreasing in* $\mathbf{R}$. *Then* $F_f$ *given by (1.6.19) is degenerate elliptic.*

*Proof.* It suffices to prove that $X \geq Y$ implies $k_i(p,X) \leq k_i(p,Y)$ for $i = 1, 2, \cdots N-1$. By rotation of $p$ as in the proof of Lemma 1.6.2 we may assume that $p = (1, 0, \cdots, 0)$ and

$$Q_p(X) = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & X' & \\ 0 & & \end{pmatrix},\ Q_p(Y) = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & Y' & \\ 0 & & \end{pmatrix}.$$

with $X', Y' \in \mathbf{S}^{N-1}$. Then $k_i(p,X)$ equals the $i$-th eigenvalue $\mu_i$ of $X'/|p|$ denoted $\mu_i(X'/|p|)$ where $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N-1}$. If $X \geq Y$, then evidently $X' \geq Y'$.

A minimax characterization (e.g. R. Courant and D. Hilbert (1962)) of eigenvalues implies that $\mu_i(X'/|p|) \geq \mu_i(Y'/|p|)$, $1 \leq i \leq N-1$ for $X' \geq Y'$, so $k_i(p,X) \geq k_i(p,Y)$ for $i = 1, 2, \cdots, N-1$. □

**Remark 1.6.8.** A minimax characterization of eigenvalues also implies that $k_i(p,X)$ is continuous on $(\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$; see for example a book of T. Kato (1982). By the observation we consider

$$g(\lambda_1, \cdots, \lambda_{N-1};\ p) = \lambda_1^+ \cdots \lambda_{N-1}^+$$

instead of $\lambda_1 \cdots \lambda_{N-1}$ so that $F_f$ in (1.6.19) is degenerate elliptic, where $\lambda_i^+ = \max(\lambda_i, 0)$. Moreover, $F_f$ is continuous in $(\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$ since $k_i$'s are continuous there.

The operation of the plus part depends upon the orientation $\mathbf{n}$. When we consider convex surface, taking $\mathbf{n}$ inward is a way not to trivialize the problem.

More generally, when

$$g(\lambda_1, \cdots, \lambda_{N-1};\ \mathbf{n}) = e_m(\lambda_1, \cdots, \lambda_{N-1})$$

there is a way to modify $g$ so that it satisfies the assumption on $g$ in Proposition 1.6.7.

**Lemma 1.6.9.** *Let $N \geq 2$ and $1 \leq m \leq N-1$. There is a closed convex cone $K_m$ in $\mathbf{R}^{N-1}$ with vertex at the origin such that*

*(i) $e_m(\lambda_1, \cdots, \lambda_{N-1})$ satisfies the monotonicity assumption of Proposition 1.6.7 as long as $(\lambda_1, \cdots, \lambda_{N-1}) \in K_m$.*

*(ii) $[0, \infty)^{N-1} \subset K_m$.*

*(iii) $e_m(\lambda_1, \cdots, \lambda_{N-1}) > 0$ for $(\lambda_1, \cdots, \lambda_{N-1})$ belonging to the interior int $K_m$ of $K_m$.*

*(iv) $e_m(\lambda_1, \cdots, \lambda_{N-1}) = 0$ for $(\lambda_1, \cdots, \lambda_{N-1}) \in \partial K_m$.*

*(v) $K_m \backslash \{0\} \subset$ int $K_\ell$ for $\ell < m$.*

For the proof the reader is referred to a book of D. S. Mintrinovic (1970) [p. 102, Theorem 1] and the article of N. S. Trudinger (1990). When $m = N-1$, it is easy to see that $K_m = [0, \infty)^{N-1}$ so that (i)-(iv) holds.

By this consideration we set

$$\hat{e}_m(\lambda_1, \cdots, \lambda_{N-1}) = \begin{cases} e_m(\lambda_1, \cdots, \lambda_{N-1}), & (\lambda_1, \cdots, \lambda_{N-1}) \in K_m \\ 0 & \text{otherwise} \end{cases} \tag{1.6.20}$$

and observe by Lemma 1.6.9 that $\hat{e}_m$ fulfills the monotonicity condition of Proposition 1.6.7 as well as continuity on $\mathbf{R}^{N-1}$.

**Theorem 1.6.10.** *The function $F_f$ defined by (1.6.19) with*

$$g(\lambda_1, \cdots, \lambda_{N-1};\ p) = \hat{e}_m(\lambda_1, \cdots, \lambda_{N-1}) \quad (2 \leq m \leq N-1)$$

*is degenerate elliptic. Moreover, $F_f$ is continuous in $(\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$.*

When we consider a quotient $e_m/e_\ell$, we note that $\hat{e}_m/e_\ell$ for $1 \leq \ell < m \leq N-1$ satisfies the monotonicity condition of Proposition 1.6.7 for

$$g(\lambda_1, \cdots, \lambda_{N-1};\ p) = (\hat{e}_m/e_\ell)(\lambda_1, \cdots, \lambda_{N-1});$$

see the book of D. S. Mintrinovic (1970) [p. 102, Theorem 1] for the proof. Note that by Lemma 1.6.9 (v) $\hat{e}_m/e_\ell$ is a well-defined continuous function in $\mathbf{R}^{N-1}$ by assigning zero as the value at $(0, \cdots, 0)$. Thus the function $F_f$ defined by (1.6.19) with this $g$ is again degenerate elliptic and continuous in $(\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$.

**Other examples.** We finally consider the level set equations of (1.5.13) and (1.5.14). For (1.5.13) the function $F_f$ is degenerate elliptic of $h$ is nondecreasing and $\nabla^2 \gamma \geq 0$ as well as $a \geq 0$. For (1.5.14) the function $F_f$ is always degenerate elliptic.

The condition (1.6.18) for $F_f$ defined by (1.6.16) is equivalent to say

$$f(p, Q_p(X)) \leq f(p, Q_p(Y)) \quad \text{whenever} \quad Q_p(X) \geq Q_p(Y). \tag{1.6.21}$$

We say that $f$ defined in $\mathbf{E}$ (or $\mathbf{E}_\Sigma$) is *degenerate elliptic* if (1.6.21) is fulfilled for all $(p, Q_p, (X)),\ (p, Q_p(Y)) \in \mathbf{E}$ (or $\mathbf{E}_\Sigma$). We say that (1.6.1) is *degenerate parabolic* if $f(x, t, \cdot, \cdot)$ is degenerate elliptic for all $(x, t) \in \Omega \times [0, T]$. As we study in this section the mean curvature flow equation and the Hamilton-Jacobi equation (1.5.6) are of course degenerate parabolic. The anisotropic version (1.5.13) as well as (1.5.2) is degenerate parabolic when $h$ is nondecreasing and $\nabla^2\gamma \geq 0$ and $a \geq 0$. The Gaussian curvature equation for $N \geq 3$ is not degenerate parabolic. More generally, $m$-th symmetric curvature flow equation (1.5.10) with $2 \leq m \leq N-1$ is not degenerate elliptic. We modify these equations (1.5.10), (1.5.12) by replacing $e_m$ by $\hat{e}_m$ as defined in (1.6.20). Then the modified equations

$$V = \hat{e}_m(\kappa_1, \cdots, \kappa_{N-1}), \tag{1.6.22}$$

$$V = (\hat{e}_m/e_\ell)(\kappa_1, \cdots, \kappa_{N-1}), \quad 1 \leq \ell < m \leq N-1 \tag{1.6.23}$$

are degenerate parabolic.

If (1.6.18) is replaced by the strict monotonicity

$$F(p, X) < F(p, Y) \quad \text{for } X \geq Y \quad \text{with trace } (X - Y) > 0$$

is fulfilled, $F$ is called *strictly elliptic*. A typical example of such $F$ is

$$F(p, X) = -\text{trace } X$$

so that $F(\nabla u, \nabla^2 u) = -\Delta u$. We note that a strongly geometric $F$ is not strictly elliptic because of condition (G2). Similarly we say $f$ is *strictly elliptic* if (1.6.21) is replaced by the strict monotonicity

$$f(p, Q_p(X)) < f(p, Q_p(Y))$$

for $Q_p(X) \geq Q_p(Y)$ with trace $Q_p(X - Y) > 0$. The equation (1.6.1) is called *strictly parabolic* if $f(x, t, \cdot, \cdot)$ is strictly elliptic. The mean curvature flow equation is strictly parabolic while the Hamilton-Jacobi equation is not strictly parabolic. The anisotropic version (1.5.13) as well as (1.5.2) is strictly parabolic when $h$ is strictly increasing and $R_p \nabla^2\gamma(p) R_p > O$ for $p \neq 0$. Here by $X > O$ we mean $X \geq 0$ and det $X > O$. The equation (1.6.23) with $1 \leq \ell < m \leq N-1$ is not strictly parabolic because $K_m \neq \mathbf{R}^{N-1}$.

### 1.6.4 Geometric equations

A familiar condition on scaling invariance of $F_f$ defined by (1.6.16) appears to be a little bit weaker than (G1), (G2).

**Definition 1.6.11.** Let $F$ be a real-valued function on $(\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$. We say that $F$ is *geometric* if $F$ fulfills (G1) and

$$\text{(G2')} \qquad F(p, X + \sigma\, p \otimes p) = F(p, X) \quad \text{for all } p \in \mathbf{R}^N \backslash \{0\},\ X \in \mathbf{S}^N,\ \sigma \in \mathbf{R}.$$

Clearly (G2) implies (G2'). The condition (G2') is certainly weaker than (G2). For example if we set

$$F_O(p, X) = ||R_p X||_2,\ R_p = I - \frac{p \otimes p}{|p|^2}$$

then $F_O$ is geometric but not strongly geometric. Here $||Y||_2$ denotes the Hilbert-Schmidt norm of $N \times N$ matrix $Y$, i.e.

$$||Y||_2 = (\sum_{1 \le i,\ j \le N} |Y_{ij}|^2)^{1/2}$$

where $Y_{ij}$ denotes the $ij$ component of the matrix $Y$. Since

$$F_O(\lambda p,\ \lambda X) = ||R_p(\lambda X)||_2 = \lambda F_O(p, X),$$
$$F_O(p, X + \sigma\ p \otimes p) = ||R_p X + R_p \sigma\ p \otimes p||_2 = ||R_p X||_2 = F_O(p, X),$$

$F_O$ fulfills (G1) and (G2'). However, $F_O$ does not fulfill (G2). Indeed, if we take $p = (0, \cdots 0, 1)$, then

$$F_O(p, X) = (\sum_{i=1}^{N-1} \sum_{j=1}^{N} |X_{ij}|^2)^{1/2}.$$

We take $y = (\alpha, 0, \cdots, 0)(\neq 0)$ and observe that

$$F_O(p, X + y \otimes p + p \otimes y)^2 - F_O(p, X)^2 = F_O(p, X + y \otimes p)^2 - F_O(p, X)^2 = (X_{1N} + \alpha)^2 - X_{1N}^2 \neq 0$$

since $X$ is symmetric. Thus (G2) is not fulfilled. However, if we assume that $F$ is degenerate elliptic, then geometricity and strong geometricity are equivalent conditions.

**Theorem 1.6.12.** *For $p \in (\mathbf{R}^n \backslash \{0\})$ let $X \mapsto F(p, X)$ be a continuous function in $\mathbf{S}^N$. Assume that*

$$F(p, X + \sigma\ p \otimes p) = F(p, X)$$

*for all $X \in \mathbf{S}^N,\ \sigma \in \mathbf{R}$ and that*

$$F(p, X) \le F(p, Y)$$

*for all $X, Y \in \mathbf{S}^N$ with $X \ge Y$. Then*

$$F(p, X + y \otimes p + p \otimes y) = F(p, X)$$

*for all $X \in \mathbf{S}^N,\ y \in \mathbf{R}^N$. In particular, if a real-valued function $F = F(p, X)$ on $(\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$ is continuous in $X$ and degenerate elliptic then $F$ is strongly geometric if and only if $F$ is geometric. (The set $\mathbf{R}^N \backslash \{0\}$ may be replaced by a cone $\tilde{\Sigma}$ with vertex and $0 \in \tilde{\Sigma}$.)*

*Proof.* An elementary calculation shows that

$$\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \le \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \le \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

provided that $ab \ge 1,\ cd \ge 1,\ a > 0,\ c < 0$. This estimate yields

$$c\ p \otimes p + d\ y \otimes y \le p \otimes y + y \otimes p \le a\ p \otimes p + b\ y \otimes y \quad \text{in } \mathbf{S}^N.$$

By the invariance and monotonicity assumption on $F$ we see

$$\begin{aligned}F(p, X + b\ y \otimes y) &= F(p, X + a\ p \otimes p + b\ y \otimes y)\\ &\le F(p, X + p \otimes y + y \otimes p)\\ &\le F(p, X + c\ p \otimes p + d\ y \otimes y)\\ &\le F(p, X + d\ y \otimes y).\end{aligned}$$

Keeping the relation $ab \ge 1,\ cd \ge 1,\ a > 0,\ c < 0$, we send $b, d$ to zero to get

$$F(p, X) = F(p, X + p \otimes y + y \otimes p)$$

by continuity of $F$ in $X$. □

In this book we consider degenerate parabolic equations so we do not need to distinguish strongly geometricity and geometricity. We shall mainly use the geometricity to describe the scaling property of level set equations.

It is sometimes convenient to extend the notion of the geometricity to general second order operators to handle boundary value problems.

**Definition 1.6.13.** Let $E$ be a real-valued function defined in a dense subset $W$ of $\mathbf{R}^d \times \mathbf{S}^d$. Assume that $(q, Y) \in W$ implies $(\lambda q, \lambda Y + \sigma\ q \otimes q) \in W$ for $\lambda > 0,\ \sigma \in \mathbf{R}$. We say that $E$ is *geometric* on $E = 0$ if $E(q, Y) \le 0$ (resp. $E(q, Y) \ge 0$) implies $E(\lambda q, \lambda Y + \sigma\ q \otimes q) \le 0$ (resp. $E(\lambda q, \lambda Y + \sigma\ q \otimes q) \ge 0$) for all $\lambda > 0$, $\sigma \in \mathbf{R}$.

Let $E(z, \cdot, \cdot)$ be a real-valued function defined in $W$ with $z \in \mathcal{O}$, where $\mathcal{O}$ is a locally compact subset of $\mathbf{R}^d$. We say an equation

$$E(z,\ Du,\ D^2u) = 0,\ z \in \mathcal{O}$$

is *geometric* in $\mathcal{O}$ if $E(z, \cdot, \cdot)$ is geometric on $E = 0$ for all $z \in \mathcal{O}$. Here $Du = (\partial u/\partial z)_{i=1}^d$, $D^2u = (\partial^2 u/\partial z_i \partial z_j)_{1 \le i,j \le d}$.

It is strightforward to see that

$$E(q, Y) = \tau + F(p, X),\ q = (\tau, p) \in \mathbf{R} \times (\mathbf{R}^N \backslash \{0\}),\ Y = \{0\} \oplus X,\ X \in \mathbf{S}^N$$

with $W = \mathbf{R} \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$, $d = N + 1$ is geometric on $E = 0$ if and only if $F$ is geometric. In particular a level set equation (1.6.2) is always geometric equation in the sense of Definition 1.6.13. Thanks to Theorems 1.6.4 and 1.6.12 if an equation

$$u_t + F(x, t, \nabla u, \nabla^2 u) = 0,\ (x, t) \in \Omega \times (0, T)$$

is geometric in $\Omega \times (0, T)$, then it is a level set equation of some surface evolution equation (1.6.1) provided that $F(x, t, \cdot, \cdot)$ is degenerate elliptic for all $(x, t) \in \Omega \times (0, T)$, where $\Omega$ is a domain in $\mathbf{R}^N$. (We say that the preceeding equation is *degenerate parabolic* if $F$ is degenerate elliptic.)

**Remark 1.6.14.** We have introduced the notion of geometricity for $E$ since it is convenient to handle boundary value problems. We consider

$$u_t + F(x, t, \nabla u, \nabla^2 u) = 0, \quad \text{in } \Omega \times (0, T)$$

with, for example, the boundary condition (1.6.14):

$$\frac{\partial u}{\partial \nu} + k|\nabla u| = 0 \quad \text{on } \partial\Omega \times (0,T),$$

where $|k| < 1$, $k \in \mathbf{R}$. As we see in §2.3, to study this problem it is natural to indroduce

$$E(x,\tau,p,X) = \begin{cases} \tau + F(p,X), & x \in \Omega \\ (\tau + F(p,X)) \wedge (\langle\ \nu,\ p\ \rangle + k|p|), & x \in \partial\Omega \end{cases}$$

where $a \wedge b = \min(a,b)$ and regard the boundary value problem as

$$E(x,u_t,\nabla u,\nabla^2 u) = 0 \quad \text{in } \overline{\Omega} \times (0,T).$$

For this operator, it is easy to see that $E$ is geometric at $E = 0$ so that the equation is geometric. We may replace (1.6.14) by a more general first order boundary condition

$$B(x,t,\nabla u) = 0$$

with $B(x,t,\lambda p) = \lambda B(x,t,p),\ \lambda > 0$ so that $E$ is geometric at $E = 0$; here $E$ is defined by

$$E(x,t,\tau,p,X) = (\tau + F(p,X)) \wedge B(x,t,p).$$

In Chapter 4 we shall see how important the geometric property of equations. The main observation is that if $u$ solves a geometric equation so does $\theta \circ u = \theta(u)$ with nondecreasing $\theta$, where $\theta \circ u$ denotes the composition of functions. This indicates that a geometric equation is invariant under coordinate change of dependent variables. Although we postpone a rigorous proof for general solutions, we here formally indicate how geometricity yields such a property. Assume that $u$ solves a geometric equation

$$u_t + F(\nabla u, \nabla^2 u) = 0$$

and $\theta' \geq 0$. For $v = \theta \circ u$ we calculate

$$\begin{aligned} & v_t + F(\nabla v, \nabla^2 v) \\ = \ & \theta'(u)u_t + F(\theta'(u)\nabla u,\ \theta'(u)\nabla^2 u + \theta''(u)\nabla u \otimes \nabla u) \\ = \ & \theta'(u)(u_t + F(\nabla u,\ \nabla^2 u)) = 0. \end{aligned}$$

by geometricity of $F$. This invariance property is natural if we recall that a geometric equation is the level set equation of some surface evolution equation so that motion of each level set of solutions is independent of other levels and the value of levels.

### 1.6.5 Singularities in level set equations

Regularity of $f$ in (1.6.1) is of course reflects to $F_f$ in (1.6.4). Here is a trivial observation.

**Proposition 1.6.15.** *Let $f$ be a real-valued function defined in $\mathbf{E}$ given by (1.6.15). The associate function $F_f$ defined by (1.6.16) is continuous in $(\mathbf{R}^N\backslash\{0\}) \times \mathbf{S}^N$ if and only if $f$ is continuous in $\mathbf{E}$.*

Examples of equations (1.6.1) with $f$ continuous in its variables includes (1.5.2), (1.5.13), (1.5.14) with continuous $h, \beta > 0, c$ with $C^2$ $\gamma$ (outside the origin) as well as the mean curvature flow equation. Examples also include (1.5.9)-(1.5.12) but these equations may not degenerate parabolic so we rather consider modified equations (1.6.22), (1.6.23) instead of them. By Theorem 1.6.10 (and its following paragrah) these equations (1.6.22), (1.6.23) can be written in the form of (1.6.1) with continuous $f$. So for such a $f$ the associate function $F_f$ is continuous in $(\mathbf{R}^N\backslash\{0\}) \times \mathbf{S}^N$.

We next study the magnitude of singularity of $F_f(p, X)$ near $p = 0$. For this purpose we introduce the upper semicontinuous envelope

$$F^* : \mathbf{R}^N \times \mathbf{S}^N \to \mathbf{R} \cup \{\infty\}$$

of $F$ defined on $(\mathbf{R}^N\backslash\{0\}) \times \mathbf{S}^N$ by setting

$$F^*(p, X) = \lim_{\varepsilon\downarrow 0} \sup\{F(q, Y);\ |q - p| \leq \varepsilon,\ ||X - Y||_2 \leq \varepsilon\}.$$

The lower semicontinuous envelope $F_*$ is defined by $F_* = -(-F)^*$ with valued in $\mathbf{R} \cup \{-\infty\}$. If $F$ is continuous in $(\mathbf{R}^N\backslash\{0\}) \times \mathbf{S}^N$ then

$$F(p, X) = F^*(p, X) = F_*(p, X) \quad \text{for} \quad (p, X) \in (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N.$$

If $F$ is geometric and degenerate elliptic, so is $F^*$ and $F_*$.

**Lemma 1.6.16.** *Assume that $F$ is continuous $(\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N$ and that $F$ is geometric and degenerate elliptic. Let $M$ and $m$ denote*

$$M = \sup\{F(p, -I);\ |p| \leq 1,\ p \neq 0\},$$
$$m = \inf\{F(p, I);\ |p| \leq 1,\ p \neq 0\}.$$

*Then the following three conditions are equivalent*
- (a) $F^*(0, O) < \infty$ *(resp.* $F_*(0, O) > -\infty$*),*
- (b) $M < \infty$ *(resp.* $m > -\infty$*),*
- (c) $F^*(0, O) = 0$ *(resp.* $F_*(0, O) = 0$*).*

*Proof.* Let $|X|$ denote the operator norm of $X$ as a self-adjoint operator. In other words $|X|$ equals the largest modulus of eigenvalues of $X$. Since the Hilbert-Schmidt norm $||X||_2$ is equivalent to the operator norm $|X|$ for finite dimensional $\mathbf{S}^N$, we may replace $||X - Y||_2$ by $|X - Y|$ in the definition of $F^*$. We first note that $|X| \leq \varepsilon$ implies

$$-\varepsilon I \leq X \leq \varepsilon I$$

for $\varepsilon > 0$. Since $F$ is degenerate elliptic, we observe that

$$\sup_{|X|\leq\varepsilon} F(p, X) \leq F(p, -\varepsilon I),\ p \neq 0.$$

The converse inequality is trivial since $|-\varepsilon I| = \varepsilon$. We thus observe that

$$\sup_{\substack{|p|\le\varepsilon \\ p\neq 0}} \sup_{|X|\le\varepsilon} F(p,X) = \sup_{\substack{|p|\le\varepsilon \\ p\neq 0}} F(p,-\varepsilon I) = \varepsilon \sup_{\substack{|p|\le\varepsilon \\ p\neq 0}} F(p/\varepsilon,-I) = \varepsilon M$$

since $F$ is geometric. Thus the equivalence of (a), (b), (c) is clear for $F^*$. The proof for $F_*$ is symmetric. □

**Remark 1.6.17.** Even if $F$ depends on $(x,t) \in \Omega \times (0,T)$ the same proof shows that

$$F^*(x,t,0,O) = \lim_{\varepsilon\downarrow 0}(\varepsilon\ M^*(x,t))$$
$$\text{with } M^*(x,t) = \sup\{F(x,t,p,-I);\ |p| \le 1,\ p \neq 0\},$$

provided that $F$ is continuous in its variables and that $F(x,t,\cdot,\cdot)$ is geometric and degenerate elliptic for all $(x,t)$. Here $F^*$ denotes the upper semicontinuous envelope as a function of $(x,t,p,Y)$ (see for definition §2.1.1). Again if $M^*(x,t) < \infty$ then $F^*(x,t,0,O) = 0$ and of course if $m_*(x,t) > -\infty$, then $F_*(x,t,0,O) = 0$, where $m$ is defined in the same way of $M$ by replacing sup by inf.

Notice that the condition

$$-\infty < F_*(x,t,0,O) = F^*(x,t,0,O) < \infty$$

is equivalent to say that $F$ can be continuously extended to $(x,t,0,O)$.

**Proposition 1.6.18.** *Let $f$ be a real-valued continuous function defined in $\mathbf{E}$ by (1.6.15). Assume that $f$ is degenerate elliptic. Then the associate function $F_f$ defined by (1.6.16) can be continuously extended to $(0,O)$ with value zero if and only if*

$$\inf_{0<\rho\le 1} \rho \inf_{|p|=1} f(-p,\ -R_p I/\rho) > -\infty, \quad \sup_{0<\rho\le 1} \rho \sup_{|p|=1} f(-p,\ R_p I/\rho) < +\infty. \tag{1.6.24}$$

*(The first (second) quantity equals $-M$ (resp. $-m$) defined in Lemma 1.6.16 with $F = F_f$.)*

This follows from Lemma 1.6.16 since geometricity of $F_f$ implies

$$\begin{aligned} M &= \sup\{F(p,-I);\ |p| \le 1,\ p \neq 0\} \\ &= \sup|p|F(p/|p|,-I/|p|);\ |p| \le 1,\ p \neq 0\} \\ &= -\inf_{0<\rho<1} \rho \inf_{|p|=1}\ f(-p,\ -R_p I/\rho). \end{aligned}$$

and similar expression is valid for $m$.

The condition (1.6.24) is a growth restriction of $f = f(\mathbf{n}, \nabla\mathbf{n})$ in $\nabla\mathbf{n}$. It roughly says that $f$ grows either linearly or sublinearly in $\nabla\mathbf{n}$ as $|\nabla\mathbf{n}| \to \infty$. For example if $f$ is positively homogeneously of degree one in the second variable, i.e. $f(p,\lambda Z) = \lambda f(p,Z)$ for $\lambda > 0, (p,Z) \in \mathbf{E}$ then (1.6.24) is fulfilled. If we write (1.5.2) in the form of (1.6.1), then evidently $f$ satisfies (1.6.24) (with constant $c$) since $f$ is linear in $\nabla\mathbf{n}$. In particular, $f$

corresponding to the mean curvature flow equation fulfills (1.6.24). The condition (1.6.24) is also fulfilled for (1.5.13) and (1.5.14) provided that $\overline{\lim}_{z\to\infty} |h(z)|/|z| < \infty$.

Both estimates of (1.6.24) are violated for $f$ corresponding to (1.6.22) (for $2 \leq m \leq N-1$) and (1.6.23) (for $1 \leq \ell < m \leq N-1$, $\ell - 1 < m$); if $m = \ell - 1$, the growth condition (1.6.24) is fulfilled for (1.6.23). Note that if both inequalities in (1.6.24) are, violated, then

$$F_*(0, O) = -\infty, \ F^*(0, O) = +\infty$$

by Lemma 1.6.16.

## 1.7 Exact solutions

We give here some explicit solutions mainly for the mean curvature flow equation (1.5.4) and its anisotropic version (1.5.2).

### 1.7.1 Mean curvature flow equation

**Shrinking sphere.** As expected there is a solution $\Gamma_t$ of (1.5.4) that is a family of spheres of radius $R(t)$ centered at the origin. The equation (1.5.4) is now of form

$$-dR/dt = (N-1)/R \tag{1.7.1}$$

since the left hand side is the inward velocity and the right hand side is the inward mean curvature; note that for the sphere of radius $R$ all principal curvatures are $1/R$. Integrating (1.7.1), we see

$$R(t) = (R_0^2 - 2(N-1)t)^{1/2} \tag{1.7.2}$$

with $R(0) = R_0 > 0$. Thus an evolving sphere

$$\Gamma_t = \{x \in \mathbf{R}^N; |x| = R(t)\} \tag{1.7.3}$$

solves (1.5.4) if (and only if) $R(t)$ is of form (1.7.2). Note that if $t > t_* = R_0^2/(2(N-1))$, then $R(t)$ is not well-defined as a real number. It is natural to interpret that $\Gamma_t$ becomes empty after the time $t_*$ when $\Gamma_t$ shrinks to a point.

Similarly for the Gaussian curvature flow equation (1.5.9) there is a shrinking sphere solution $\Gamma_t$ of form (1.7.3) provided that $R(t)$ solves

$$-dR/dt = 1/R^{N-1} \tag{1.7.4}$$

instead of (1.7.1). Integrating (1.7.4) yields

$$R(t) = (R_0^N - Nt)^{1/N}$$

instead of (1.7.2). For more general equation (1.5.12) it is still easy to find a shrinking sphere solution although we do not its explicit form here.

**Shrinking cylinders.** We consider a little bit general evolving hypersurface called a cylinder:

$$\Gamma_t = \{(x_1, \dots, x_j, x_{j+1}, \dots, x_N); (x_{j+1}^2 + \dots + x_N^2)^{1/2} = R(t)\}; \tag{1.7.5}$$

of course $\Gamma_t$ is a sphere if $j = 0$. The equation (1.5.4) is interpreted as

$$-dR/dt = (N-1-j)/R \tag{1.7.6}$$

which generalizes (1.7.1); note that $\kappa_1 = \dots = \kappa_{N-1-j} = 1/R$, $\kappa_{N-j} = \kappa_{N-j+1} = \dots = \kappa_{N-1} = 0$. Solving (1.7.6) to get

$$R(t) = (R_0^2 - 2(N-1-j)t)^{1/2}. \tag{1.7.7}$$

Thus an evolving cylinder $\Gamma_t$ of (1.7.5) solves the mean curvature flow equation if and only if $R(t)$ is given by (1.7.7).

Since the Gaussian curvature of cylinder of form (1.7.5) with $j \geq 1$ is always zero, any cylinder is a stationary solution of the Gaussian curvature flow equation (1.5.9).

In these exact solutions we notice that the shape of an evolving hypersurface is independent of time up to dilation. Such a solution is often called *self-similar* solution. We here give a rigorous definition of self similarity.

**Definition 1.7.1.** $\Gamma_t$ be an evolving hypersurface $t \in I$ where $I$ is a time interval. We say that $\Gamma_t$ is *self-similar* if there is $x_0 \in \mathbf{R}^N$ and a hypersurface $\Gamma$ (independent of time) in $\mathbf{R}^N$ such that for some $\lambda = \lambda(t)$, $\Gamma_t$ is of form

$$\Gamma_t = \{\lambda(t)(x - x_0) + x_0;\ x - x_0 \in \Gamma\}$$

For example the evolving cylinder $\Gamma_t$ given (1.7.5) is self-similar. If (1.7.7) is fulfilled, then it is a self-similar solution of (1.5.4).

**Level set approach.** It is sometimes convenient to find self-similar solution by using level set equations. We seek a solution of the level set equation (1.6.7) of form

$$u(x,t) = -(t + \zeta(r)),\ r = |x|$$

with nondecreasing $\zeta$. Then $\zeta(r(x))$ must solve

$$1 = |\nabla_x \zeta(r)| \mathrm{div} \left( \frac{\nabla \zeta(r)}{|\nabla \zeta(r)|} \right) = \zeta(r) |\nabla_x r| \mathrm{div}(\frac{\nabla r}{|\nabla r|}).$$

Since $\partial r / \partial x_i = x_i / r$ and $\partial / \partial x_j (x_i / r) = (\delta_{ij} - x_i x_j / r^2) r^{-1}$, this implies

$$1 = \zeta'(r)(N-1)/r.$$

By normalizing $\zeta(0) = 0$ we get $\zeta(r) = r^2/(2(N-1))$. Thus

$$u(x,t) = -(t + |x|^2/2(N-1))$$

solves (1.6.7) at least outside the place where $\nabla u = 0$. Each level set gives a shirinking sphere solution of (1.5.4). Similarly, we see

$$u(x,t) = -(t + \sum_{\ell=j+1}^{N} x_\ell^2/(2(N-1)))$$

solves (1.6.7) at least formally. Each level set of $u$ gives a shirinking cylinder solution of (1.5.4). In these problems $-u$ also solves (1.6.7). This reflects the fact that the evolution law (1.5.4) is independent of the choice of the orientation $\mathbf{n}$.

### 1.7.2 Anisotropic version

We shall consider a class of anisotropic curvature flow equation (1.5.2) with constant $c$ which includes the mean curvature flow equation as a special case. We try to find a self-similar shrinking solution similar to a sphere. A typical property is a sphere is that its mean curvature is constant on the sphere. (If an (embedded) closed sphere has constant mean curvature, then it must be a sphere by a result of A. D. Alexandorv(1956).) We shall seek a hypersurface whose anisotropic mean curvature is constant.

**Wulff shape.** Let $\gamma_0$ be a surface energy. We say that

$$\mathcal{W} = \bigcap_{|q|=1} \{x \in \mathbf{R}^N; \langle x, q\rangle \le \gamma_0(q)\}$$

is the Wulff shape associated with $\gamma_0$. If $\gamma_0 \equiv 1$, $\mathcal{W}$ is a unit sphere.

According to Wulff's theorem, $\mathcal{W}$ minimizes the surface energy

$$\int_{\partial D} \gamma_0(\mathbf{n}) d\sigma$$

among all set $D$ with the same volume as $\mathcal{W}$. Such a minimizer is unique up to translation. In other words $\mathcal{W}$ is a unique solution of anisotropic isoperimetric problem.

The Wulff shape is also characterized by zero set of the conjugate convex function $\Gamma$ of $\gamma$ defined as

$$\gamma^\natural(x) = \sup\{\langle x, q\rangle - \gamma(q); q \in \mathbf{R}^N\}$$

where $\gamma$ is given by (1.3.7). The function $\gamma^\natural$ is convex even if $\gamma$ is not convex. Since $\gamma$ is positively homogeneous of degree one, we see that $\gamma^\natural$ is the indicator function of $\mathcal{W}$, i.e., $\gamma^\natural = 0$ on $\mathcal{W}$ and $\gamma^\natural = +\infty$ outside $\mathcal{W}$. Indeed, if $x \in \mathcal{W}$, then $\langle x, q\rangle - \gamma(q)$ attains zero so $\gamma^\natural(x) = 0$. If $x \notin \mathcal{W}$, then $c = \langle x, q\rangle - \gamma > 0$ for some $q$ satisfying $|q| = 1$. Since $\gamma$ is positively homogeneous of degree one,

$$\langle x, q\rangle - \gamma(\lambda q) = \lambda c$$

for all $\lambda > 0$. This implies $\gamma^\natural(x) = \infty$.

Since $\gamma^\natural$ is convex and lower semicontinuous, $\mathcal{W}$ is convex and closed. Since the surface energy density $\gamma_0$ is always assumed to be positive, $\mathcal{W}$ contains the origin as an interior point. The next lemma shows that the boundary of a Wulff shape substitutes the role of a sphere for anisotropic mean curvature. Let $P$ be the *Minkowski function* of $\mathcal{W}$ defined by

$$P(x) = \inf\{\lambda \in (0,\infty);\ x/\lambda \in \mathcal{W}\}.$$

Clearly, $P$ is positively homogeneous of degree one and convex. Since $\mathcal{W}$ contains the origin as an interior point, $P$ is defined in whole $\mathbf{R}^N$. Clearly

$$\mathcal{W} = \{x \in \mathbf{R}^N;\ P(x) \le 1\}.$$

**Lemma 1.7.2.** (Anisotropic mean curvature of the Wulff shape) *Assume that surface energy density $\gamma_0 > 0$ is $C^m (m \ge 2)$ in the sense that $\gamma$ is $C^m$ outside the origin. Assume that $\gamma$ satisfies a strict convexity assumption: $R_p \nabla^2\gamma(p) R_p > 0$ for $p \ne 0$ or equivalently,*

$$\sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2\gamma}{\partial p_i \partial p_j}(p)\eta_i\eta_j > 0 \quad \textit{for all } \eta, p \in \mathbf{R}^N\backslash\{0\} \quad \textit{with } \langle \eta, p\rangle = 0,$$

*where $\eta = (\eta_1, \ldots, \eta_n)$. Then the boundary $\Gamma = \partial\mathcal{W}$ of the Wulff shape associate to $\mathcal{W}$ is $C^m$ (so that $P$ is $C^m$ outside the origin). Moreover*

$$\gamma(\nabla P(x)) = 1 \quad \textit{in } \mathbf{R}^N\backslash\{0\}, \tag{1.7.8}$$

$$\xi(\nabla P(x)/|\nabla P(x)|) = \xi(\nabla P(x)) = x/P(x),\ x \in \mathbf{R}^N,\ x \ne 0, \tag{1.7.9}$$

$$\langle \nabla \left(\frac{|x|}{P(x)}\right), \frac{x}{|x|}\rangle = 0,\ x \in R^N,\ x \ne 0, \tag{1.7.10}$$

*where $\xi$ is the Cahn-Hoffman vector of $\gamma$ i.e. $\xi = \nabla\gamma$. In particular $\xi(\mathbf{n}(x)) = x, x \in \Gamma$ so that the anisotropic mean curvature $h$ in the direction of $\mathbf{n}$ equals $-(N-1)$, where $\mathbf{n}$ is the outward unit normal vector field of $\Gamma$.*

We postpone the proof in the next subsection. We seek a self-similar solution of (1.5.2) with $a \ge 0$ when $c$ is a constant by the level set approach as for (1.5.4). We set

$$u(x,t) = -(t + \zeta(P)) \tag{1.7.11}$$

with nondecreasing $\zeta$ defined on $[0,\infty)$, where $P$ is the Minkowski functional of $\mathcal{W}$. The level set equation of (1.5.2) is (1.6.10) or equivalently

$$u_t - |\nabla u|(-a \ \mathrm{div}\ \xi(-\nabla u) + c)/\beta(-\nabla u/|\nabla u|) = 0$$

by the identity (1.4.13). For the special form of $u$ in (1.7.11), this equation is equivalent to

$$1 = \zeta'(P)|\nabla P|(a \ \mathrm{div}\ \xi(\nabla P) - c)/\beta(\nabla P/|\nabla P|). \tag{1.7.12}$$

Using (1.7.9) and (1.7.10), we have

$$\mathrm{div}\ \xi(\nabla P) = \mathrm{div}\ (x/P) = (|x|/P)\mathrm{div}(x/|x|) + \langle \nabla\frac{|x|}{P},\ \frac{x}{|x|}\rangle = (N-1)/P + 0.$$

Thus if $\beta\gamma_0$ is a constant function with $\beta\gamma_0 = 1/\sigma_0$ on $S^{N-1}$, then, by (1.7.8), the identity (1.7.12) is equivalent to

$$1 = \zeta'(P)\sigma_0(a(N-1)/P - c). \tag{1.7.13}$$

Regarding $P$ as an independent variable, (1.7.13) is an ordinary differential equation for $\zeta$. It is easy to integrate (1.7.13). If $c \leq 0$, then solution of (1.7.13) is a strictly increasing function of form

$$\zeta(\rho) = \int_0^\rho \frac{\tau}{\sigma_0(a(N-1) - c\tau)} d\tau \tag{1.7.14}$$

by normalizing $\zeta(0) = 0$. For example, if $c = 0$ with $a > 0$ then $\zeta(\rho) = \rho^2/(2\sigma_0 a(N-1))$. If $a = 0$ and $c < 0$, $\zeta(p) = -\rho/\sigma_0 c$. We thus observe that $u$ of (1.7.11) with $\zeta$ of form (1.7.14) solves the level set equation of (1.5.2) at least where $\nabla u \neq 0$. Since each level set of $u$ solves (1.5.2) at least where $\nabla u \neq 0$,

$$\Gamma_t = \{x \in \mathbf{R}^N; P(x) = R(t)\} \tag{1.7.15}$$

solves (1.5.2) provided that

$$R(t) = \zeta^{-1}(\zeta(R(0)) - t), \tag{1.7.16}$$

where $\zeta^{-1}$ denotes the inverse function of $\zeta$. This generalizes (1.7.2). If $c = 0$, $\Gamma_t$ disappears in a finite time; it disappears at $t = \zeta(R(0))$. If $c > 0$, then we cannot find an increasing function $\zeta$ solving (1.7.13). This corresponds to the phenomena that there is a growing solution of form (1.7.15) to (1.5.2) with $c > 0$ when $\mathbf{n}$ is taken outward.

We give another way to find self-similar solution to see these properties depending on $c$. We argue as in the same way to derive (1.7.1) from (1.5.3). We note that $dR/dt$ may not equal the outward formal velocity $V$ of $\Gamma_t$ given by (1.7.15) but

$$dR/dt = V/\langle \mathbf{n}, x/R(t)\rangle.$$

By homogeneity of $\gamma$ we see

$$\langle \xi(\mathbf{n}), \mathbf{n}\rangle = \gamma(\mathbf{n});$$

indeed differentiating $\gamma(\lambda p) = \lambda\gamma(p)$ in $\lambda$ and setting $\lambda = 1$ yields the Euler equation

$$\langle \xi(p), p\rangle = \gamma(p). \tag{1.7.17}$$

By (1.7.9) this yields

$$\gamma(\mathbf{n}) = \langle \frac{x}{R(t)},\ \mathbf{n}\rangle$$

so we obtain

$$dR/dt = V/\gamma_0(\mathbf{n}).$$

If $\beta\gamma_0 = 1/\sigma_0$, the equation (1.5.2) becomes

$$V = \sigma_0\gamma_0(\mathbf{n})(-a\ \mathrm{div}_{\Gamma_t}\xi(\mathbf{n}) + c). \tag{1.7.18}$$

This yields

$$dR/dt = \sigma_0(-a(N-1)/R + c) \tag{1.7.19}$$

since $\mathrm{div}_{\Gamma_t}\xi(\mathbf{n}) = (N-1)/R$ by Lemma 1.7.2. If $c \le 0, a \ge 0$, $R(t)$ given by (1.7.16) is the unique solution of (1.7.18) with given $R(0)$. If $c > 0$, then if initial data $R(0) = R_0$ is small say

$$a(N-1)/R_0 > c,$$

then the solution disappear in a finite time. If $a(N-1)/R_0 = c$, then $R(t) \equiv R_0$. If $a(N-1)/R_0 < c$, the solution $R(t)$ exists globally in time and its asymptotics as $t \to \infty$ is $\sigma_0 ct$. Thus in any case we have self-similar solution since

$$\Gamma_t = \{R(t)x;\ x \in \Gamma\}.$$

**Theorem 1.7.3.** *Assume the same hypothesis of Lemma 1.7.2 concerning $\gamma_0$ and $\gamma$. Assume that $\beta\gamma_0$ is a constant function on $S^{N-1}$ with value $1/\sigma_0 > 0$ and that $c$ is a constant. Let $R$ be a solution of (1.7.19). Then $\Gamma_t$ given by (1.7.15) or*

$$\Gamma_t = \{R(t)x;\ x \in \Gamma = \partial\mathcal{W}\} \tag{1.7.20}$$

*is a self-similar solution of (1.7.18) (or (1.5.2) with $\beta\gamma_0 = 1/\sigma_0, a \ge 0,\ c \in \mathbf{R}$), where $\mathcal{W}$ denotes the Wulff shape of $\gamma_0$.*

*For $a \ge 0, c \le 0$ with $(c,a) \ne (0,0)$ all solution of form (1.7.20) disappear in a finite time while for $c > 0$, solution with $R(0) < a(N-1)/c$ disappears in a finite time while $R(0) > a(N-1)/c$ the solution exists globally in time and grows with $\lim_{t\to\infty} R(t)/t = \sigma_0 c$. If $R(0) = a(N-1)/c$, $\Gamma_t$ with $R(t) = R(0)$ is a stationary solution.*

### 1.7.3 Anisotropic mean curvature of the Wulff shape

We shall prove Lemma 1.7.2. For the Minkowski functional $P$ of the Wulff function we set

$$P(x) = |x|/w(x)$$

so that $w$ is positively homogeneous of degree zero. We use the convention $\hat{x} = x/|x|$ for $x \in \mathbf{R}^N$, $x \ne 0$.

**Proposition 1.7.4.** *Assume the same hypothesis of Lemma 1.7.2 concerning $\gamma_0$ and $\gamma$. Let $\xi$ be the Cahn-Hoffman vector of $\gamma$, i.e., $\xi = \nabla\gamma$.*

*(i) For $x \in \mathbf{R}^N$ with $x \ne 0$*

$$w(x) = \min\left\{\gamma(q)/\langle q, \hat{x}\rangle; q \in S^{N-1} \quad \text{and } \langle \hat{x}, q\rangle > 0\right\} > 0. \tag{1.7.21}$$

*(ii) For $x \ne 0$ let $\Theta(x) \subset S^{N-1}$ be the set of minimizers of the right hand side of (1.7.21):*

$$\Theta(x) = \left\{q \in S^{N-1};\ w(x) = \gamma(q)/\langle q, \hat{x}\rangle \quad \text{and } \langle \hat{x}, q\rangle > 0\right\}. \tag{1.7.22}$$

*Then $\Theta(x)$ is a singleton $\{q(x)\}$. The mapping $q : (\mathbf{R}^n\backslash\{0\}) \to S^{N-1}$ is $C^{m-1}$. Moreover,*

$$\xi(q(x))\langle q(x), \hat{x}\ \rangle - \gamma(q(x))\hat{x} = 0 \tag{1.7.23}$$

*(iii) For* $x \neq 0$

$$\gamma(q(x)) = w(x)\langle q(x), \hat{x}\rangle = \max\{w(y)\langle q(x), \hat{y}\rangle;\ y \neq 0\} \tag{1.7.24}$$

*In particular differentiating* $w(y)\langle q(x), \hat{y}\rangle$ *in* $y$ *yields*

$$\nabla w(x)\langle q(x), \hat{x}\rangle + w(x)\frac{q(x)}{|x|} - w(x)\langle q(x), \hat{x}\rangle\frac{x}{|x|^2} = 0 \tag{1.7.25}$$

*(iv) For* $x \neq 0$,

$$\nabla P(x) = q(x)/\gamma(q(x)), \tag{1.7.26}$$
$$\xi(q(x)) = x/P(x). \tag{1.7.27}$$

Lemma 1.7.2 easily follows Proposition 1.7.4. Indeed, $\partial\mathcal{W}$ is $C^1$ by (1.7.26) and the implicit function theorem. The $C^m$ regularity follows from $C^m$ regularity of $w$ which follows from $C^{m-1}$ regularity of $q$ and (1.7.26). The formula (1.7.26) yields (1.7.8) since $\gamma$ is positively homogeneous of degree one. The formula (1.7.27) yields (1.7.9) by (1.7.26) and the homogeneity of $\xi$. The formula (1.7.10) follows from (1.7.25) since $|x|/P = w$. The anisotropic curvature $h$ on $\mathcal{W}$ equals

$$\begin{aligned} h &= -\mathrm{div}_\Gamma \xi(\mathbf{n}(x)) = -\mathrm{div}_\Gamma x \\ &= -\mathrm{trace}((I - \mathbf{n}(x) \otimes \mathbf{n}(x))\nabla x) = -(N-1). \end{aligned}$$

by (1.3.6).

*Proof of Proposition 1.7.4.* (i) Let $w_0$ denote the right hand side of (1.7.21). Since $|x| \leq w_0(x)$ is equivalent to

$$\langle q, x\rangle - \gamma(q) \leq 0 \quad \text{for all } q \in S^{N-1},\ \langle q, x\rangle > 0,$$

we see, by interpreting $|x|/w_0(x) = 0$ for $x = 0$,

$$\mathcal{W} = \{x \in \mathbf{R}^N;\ |x|/w_0(x) \leq 1\}$$

by definition of $\mathcal{W}$; apparent extra condition $\langle q, x\rangle > 0$ does not play a role at all. By definition of the Minkowski functional $w_0$ must equal $w$.
(ii) We put

$$\begin{aligned} G(p,x) &= (G_1(p,x), \cdots, G_N(p,x)) \\ G_i(p,x) &= \frac{\partial}{\partial p_i}\left(\frac{\gamma(p)}{\langle p, \hat{x}\rangle}\right)\langle\ p, \hat{x}\ \rangle^2 \\ &= \langle p, \hat{x}\rangle\frac{\partial\gamma}{\partial p_i}(p) - \gamma(p)\hat{x}_i,\ 1 \leq i \leq N. \end{aligned}$$

Since $\gamma(p)/\langle p, \hat{x}\rangle$ is invariant under positive muptiplication of $p$ and $q \in \Theta$ is a mimimizer, we see

$$G(q,x) = 0 \quad \text{for } q \in \Theta(x).$$

We differentiate $G_i$ in $p_j$ to get

$$\frac{\partial G_i}{\partial p_j}(p,x) = \langle p, \hat{x}\rangle \frac{\partial^2 \gamma}{\partial p_i \partial p_j}(p) + \hat{x}_j \frac{\partial \gamma}{\partial p_i}(p) - \hat{x}_i \frac{\partial \gamma}{\partial p_j}(p).$$

We then obtain

$$\sum_{1\le i,j\le N} \frac{\partial G_i}{\partial p_j}(p,x)\eta_i\eta_j = \langle p, \hat{x}\rangle \sum_{1\le i,j\le N} \frac{\partial^2 \gamma}{\partial p_i \partial p_j}(p)\eta_i\eta_j + 0$$

for $\eta = (\eta_1, \cdots, \eta_N) \in \mathbf{R}^N$. The strictly convexity assumption of $\gamma$ now yields

$$\sum_{1\le i,j\le N} \frac{\partial G_i}{\partial p_j}(p,x)\eta_i\eta_j > 0 \quad \text{for all } \eta,\ p \in \mathbf{R}^N \backslash \{0\} \quad \text{with } \langle p, \hat{x}\rangle > 0,\ \langle \eta, p\rangle = 0$$

Thus for every $x$, there is a unique solution $p = q$ of $G(p,x) = 0$ so that $\Theta(x)$ is a singleton. The implicit function theorem implies that $x \mapsto q(x)$ is $C^{m-1}$ since $G$ is $C^{m-1}$. The identity (1.7.23) is the same as $G(q(x),x) = 0$.
(iii) By definition of $w(y)$ we see

$$w(y)\langle q(x), \hat{y}\rangle \le \gamma(q(x)) = w(x)\langle q(x), \hat{x}\rangle,\ y \ne 0,\ x \ne 0.$$

This yields (1.7.24).
(iv) For $x \ne 0$ we differntiate $P$ to get

$$\nabla P(x) = \frac{x}{|x|w(x)} - \frac{|x|}{(w(x))^2}\nabla w(x)$$

Since

$$\nabla w(x) = w(x)\frac{x}{|x|^2} - w(x)\frac{q(x)}{|x|\langle q(x), \hat{x}\rangle}$$

by (1.7.25) we obtain

$$\nabla P(x) = q(x)/(w(x)\langle q(x), \hat{x}\rangle) = q(x)/\gamma(q(x)).$$

It remains to prove (1.7.27). Since

$$\gamma(q(x)) = \langle q(x), \xi(q(x))\rangle$$

by homogeneity (1.7.17), from (1.7.24) it follows that $\xi(q(x)) = \lambda\hat{x}$ with some $\lambda \in \mathbf{R}$. Plugging this into (1.7.23) yields

$$\lambda\hat{x}\langle q(x), \hat{x}\rangle = \gamma(q(x))\hat{x}.$$

Thus $\lambda = \gamma(q(x))/\langle q(x), \hat{x}\rangle = w(x)$ and $\xi(q(x)) = w(x)\hat{x} = x/P(x)$.

### 1.7.4 Affine curvature flow equation

We consider the affine curvature flow equation (1.5.14) and its generalization

$$V = k_+^\alpha \tag{1.7.28}$$

for $\alpha > 0$ in the plane. As expected, this equation admits a shrinking circle as a self-similar solution. However, for $\alpha = 1/3$ (corresponding to the affine curvature flow equaion), it also admits a shrinking ellipse as a self-similar solution while for other $\alpha$ the only self-similar shrinking ellipse is a circle. We study this aspect below.

We interpret $V$ and $k$ in (1.5.14) as inward velocity and inward curvature, respectively when $\Gamma_t$ is a closed curve. For level set representation we take $\nabla u/|\nabla u|$ as inward normal and obtain the level set equation of (1.7.28) of form

$$u_t = -|\nabla u| \left( -\mathrm{div}\ \frac{\nabla u}{|\nabla u|} \right)_+^\alpha \tag{1.7.29}$$

As in §1.7.1 we set

$$u(x,t) = -(t + \zeta(S)),\ S(x) = \left( \frac{x_1}{a_1} \right)^2 + x_2^2,\ a_1 > 0,\ a_1 \neq 0 \tag{1.7.30}$$

with nondecreasing $\zeta : [0,\infty) \to [0,\infty)$. Pluging (1.7.30) in (1.7.29) yields

$$1 = \zeta'(S)|\nabla S|(\mathrm{div}\ (\nabla S/|\nabla S|))^\alpha. \tag{1.7.31}$$

We calculate

$$\nabla S = 2(x_1/a_1^2,\ x_2)$$

to get

$$|\nabla S| = 2\rho,\ \rho = ((x_1/a_1^2)^2 + x_2^2)^{1/2}.$$

We further calculate

$$\mathrm{div}\ \frac{\nabla S}{|\nabla S|} = \frac{x_2^2}{a_1^2\rho^3} + \frac{(x_1/a_1^2)^2}{\rho^3} = \frac{S}{a_1^2\rho^3}$$

and observe that (1.7.31) is equivalent to

$$1 = \zeta'(S)2\rho[S/(a_1^2\rho^3)]^\alpha.$$

The dependence of $\rho$ disappears if and only if $\alpha = 1/3$. If $\alpha = 1/3$ we proceed to get

$$1 = \zeta'(S)2a_1^{-2/3}S^{1/3}.$$

Integrating the last equation to get

$$\zeta(S) = 3a_1^{2/3}S^{2/3}/4 \tag{1.7.32}$$

by normalizing $\zeta(0) = 0$. We now conclude

$$u(x,t) = -(t + \zeta(S))$$

with (1.7.32) solves (1.7.29) when $\alpha = 1/3$. To see its level set we conclude:

**Theorem 1.7.5.** *Assume that $\alpha = 1/3$. Then for each ellipse $\Gamma$ there is a self-similar evolving curve $\Gamma_t$ defined as in Definition 1.7.1 which solves (1.7.28). If*

$$\Gamma_t|_{t=1} = \left\{(x_1, x_2);\ \frac{x_1^2}{a_1^2} + x_1^2 = s_0\right\}$$

*then it extincts at time $\zeta(s_0)$. If $\alpha \neq 1/3$ there is no self-similar solution of ellipse unless ellipse is a circle.*

## 1.8 Notes and comments

In the first four sections we review several notions of geometric quantities such as curvature and give their various representation. Except normal velocity and anisotropic curvature these notions are standard in differential geometry. For further background the reader is referred to classical books by S. Kobayashi and K. Nomizu (1963), (1969). Explanation of curvature and the second fundamental form follows that of a book of L. Simon (1983). Surface energy has been popular in material sciences. The Cahn–Hoffman vector has been introduced by J. Cahn and D. W. Hoffman (1974). For further background the reader is referred to a nice review article on anisotropic curvature by J. Taylor (1992).

The mean curvature flow equation was first introduced by W. W. Mullins (1956) to model motion of grain boundaries in material sciences. The Eikonal equation $V = 1$ in geometric optics is a typical example of the Hamilton-Jacobi equation. Even for general $\beta$ the equation is well studied in material sciences to describe growth of crystals. For the development of this topic the reader is referred to a review by J. W. Cahn, J. E. Taylor and C. A. Handwerker (1991). The anisotropic version was derived by S. Angenent and M. Gurtin (1989) form balance of forces and the second law of thermomecanics. However, even before it was used to describe a crystal growth phenomena by H. Müller-Krumbhaar, T. W. Burkhardt and D. M. Kroll (1977) (see also a book by A. A. Chernov (1984)). There are nice review articles on anitotropic curvature flow equations by M. Gurtin (1993) and J. E. Taylor, J. W. Cahn and C. A. Handwerker (1992). The Gaussian curvature flow was first introduced by W. Firey (1974) to describe motion of surface of stones worn in a seashore. The affine curvature flow was axiomatically derived by L. Alvarez, F. Guichand, P.-L. Lions (1993) to propose a way of deformation of image in image processing.

The level set mean curvature flow equation was first effectively used to derive scaling law for "dynamic structure functions" of motion by mean curvature flow equation in T. Ohta, D. Jasnow and K. Kawasaki (1982). S. Osher and J. Sethian (1988) used the level set equations to track the evolution numerically. Except §1.6.3 most of contents of §1.6 is taken from the paper by Y. Giga and S. Goto (1992a). The definition of geometricity is due to Y.-G. Chen, Y. Giga and S. Goto (1991a). The property in Lemma 1.6.9 is well-studied to solve the Dirichlet problem for Monge–Ampère type equations. It is known that $\partial e_m/\partial\lambda_j > 0$ $(1 \leq j \leq N-1, 1 \leq m \leq N-1)$ and $e_m^{1/m}$ is concave in $K_m$; see

L. Caffarelli, L. Nirenberg and J. Spruck (1985). Using the concaving properties, these authors proved a necessary and sufficient condition on domain $\Omega \subset \mathbf{R}^{N-1}$ such that the Dirichlet problem $\ell_m(\lambda_1, \cdots, \lambda_{N-1}) = \psi$ in $\Omega$ with $u = $ const. on $\partial\Omega$ is solvable, where $\lambda_j$ denotes the eigenvalues of the Hessian $\nabla^2 u$ of $u$ in $\Omega$. Later L. Caffarelli, L. Nirenberg and J. Spruck (1988) also studied the problem where $\lambda_j$ is a principal curvature of the graph of $u$. N. Trudinger (1990) also studied this problem by using viscosity solutions. For further generalization the reader is referred to the paper of N. M. Ivochkina, S. I. Prokof'eva and G. V. Yakunina (1995).

For surfaces of higher codimension it is possible to consider the mean curvature flow equation by assigning its velocity vector by its mean curvature vector defined in §1.3. A level set method is proposed for such an equation by L. Ambrosio and H. M. Soner (1996).

**Wulff's theorem and Wulff shape (§1.7.2).** G. Wulff (1901) formulated the generalized isoperimetric problem "Find a set minimizing the surface energy with fixed volume" and conjecture that the answer is a dilation of the Wulff shape $\mathcal{W}$. A. Dinghas (1944) gave a formal proof. J. Taylor (1978) gave a precise proof for very general surface energies and a very general class of set for which the surface energy is defined by using geometric measure theory. B. Dacorogna and C. E. Pfister (1991) gave an analytic proof when $N = 2$. I. Fonseca (1991) and I. Fonseca and S. Müller (1991) gave a simpler proof for arbitrary dimensions. The minimizer of the generalized isoperimetric problem, or Wulff's problem is also unique up to translation and it is a dilation of $\mathcal{W}$. We do not discuss Wulff's problem further. See also the book of Morgan (1993) [Chapter 10] for elementary proof when $N = 2$. For more information of convex bodies $\mathcal{W}$ see a book of R. Schneider (1993).

The self-similar solution in Theorem 1.7.3 for (1.5.2) is constructed by H. M. Soner (1993) by proving Lemma 1.7.2. Lemma 1.7.2 says that anisotropic mean curvature is constant if the surface is the boundary of the Wulff shape. The converse problem seems to be open unless $\gamma_0$ is a constant. The problem is of form: if an embedded compact hypersurface has a constant anisotropic mean curvature, is the hypersurface a boundary of the Wulff shape up to translation and dilation?

The fact that ellipse gives a self-similar solution for the affine curvature flow equation is easy if we admit that the equation is affine invariant. The higher dimensional version of affine curvature flow equations is of form $V = K^{1/(N+1)}$.

**Existence of self-similar solution for anisotropic curvature flow equation.** If $c = 0$ in (1.5.2) and $a > 0$, is there still a self-fimilar solution $\Gamma_t$ of form (1.7.20) even if $\beta\gamma_0$ is not a constant? The answer is affirmative if $N = 2$. In fact if $\gamma_0$ is $C^2$ and $R_p \nabla^2 \gamma(p) R_p > 0$ for $p \neq 0$ and $\beta$ is continuous, then there is $\tilde{\gamma}$ such that (1.5.2) can be rewritten as

$$V = \tilde{\gamma} \operatorname{div}_\Gamma \tilde{\xi}(\mathbf{n}), \tag{1.8.1}$$

where $\tilde{\xi} = \nabla \tilde{\gamma}$; $\tilde{\gamma}_0$ another surface energy which satisfies the same property as $\gamma_0$. The existence of a self-similar solution follows from that for (1.8.1). This is proved by M. E.

Gage and Y. Li (1994) when $\gamma_0$ and $\beta$ is $C^4$. Later, a direct proof for general $\gamma_0$ and $\beta$ is given by C. Dohmen, Y. Giga and N. Mizoguchi (1996). It is also proved that such self-similar solution is unique (up to translation in time and space) if $\gamma$ and $\beta$ are even, i.e., $\gamma(p) = \gamma(-p)$, $\beta(p) = \beta(-p)$. This is proved by M. E. Gage (1993); see also C. Dohmen and Y. Giga (1994) and Y. Giga (2000). There are several researches on existence of self-similar solutions of $V = \kappa^\alpha$ and its anisotropic version $N = 2$. We do not intend to explain the detail. The reader is referred to a recent review article by Y. Giga (2000) and a book by K. S. Chou and X. P. Zhu (2001) with references cited there on this topics as well as the article by B. Andrews (1998). For self-similar solutions moved by the power of the Gaussian curvature the reader is referred to a review article by J. Urbas (1999) and references cited there. For the harmonic curvature flow see a paper by K. Anada (2001).

**Asymptotic self-similarity.** Although self-similar solutions are special solutions, they are important since they often represent a typical asymptotic behaviour of solutions. For example for the mean curvature flow equation (1.5.4) with $N \geq 3$ G. Huisken (1984) proved that a convex hypersurface shrinks to a point in finite time and the way of shrinking is asymptotically equal to the sphere shrinking. For the curve shortening equation (1.5.5) the corresponding result has been proved by M. E. Gage and R. S. Hamilton (1986). For its anisotropic version (1.8.1) M. E. Gage (1993) proved that a convex curve shrinks to a point and the way of shrinking is asymptotically like the shrinking Wulff shape provided that the equation is orientation free. For further extensions of this results the reader is referred to papers of M. E. Gage and Y. Li (1994), K.-S. Chou and X.-P. Zhu (1999a), X.-P. Zhu (1998) and a recent book by K.-S. Chou and X.-P. Zhu (2001). If one starts from a non convex curve, it becomes convex in finite time for (1.5.5) as Grayson (1987) proved. Such convexity formulation is also generalized by J. Oaks (1994) for anisotropic orientation free equation including (1.8.1). For more development of the theory the reader is referred to papers of K.-S. Chou and X.-P. Zhu (1999b), X.-P. Zhu (1998) and a book by K.-S. Chou and X.-P. Zhu (2001).

There are several related results for the Gaussian curvature flow equation (1.5.9) and its modification $V = K^\alpha (\alpha > 0)$. K. Tso (1985) proved that solution of (1.5.9) remains smooth and strictly convex and shrinks to a point if initial hypersurface is strictly convex. For $V = K^{1/(n-1)}$ B. Chow (1985) proved that a strict convex hypersurface shrinks to a point in finite time and the way of shrinking is asymptotically equal to the sphere shrinking which corresponds to the results of G. Huisken (1984) for the mean curvature flow equation. B. Andrews (1994) extended the theory so that it includes both (1.5.4) and $V = K^{1/(n-1)}$. Note that the homogeneous degree with respect to principal curvatures are the same both for $K^{1/(n-1)}$ and $H$ so it can be treated simultaneously. For the affine curvature flow equation $V = K^{1/(n+1)}$ the way of shrinking is asymptotically equal to an ellipsoid shrinking. This is proved by G. Sapiro and A. Tannenbaum (1994) for strict convex curves moved by (1.5.14) and later by B. Andrews (1996) for strict convex hypersurfaces including curves. The situation for $V = K^\alpha$ with $\alpha > 1/(n + 1)$ is similar to the case $\alpha = 1/(n - 1)$ according to forthcoming papers of B. Andrews. In fact, B. Andrews (2000) confirmed it for $\alpha \in (1/(n + 1), 1/(n - 1)]$. On the other hand if

$\alpha < 1/(n+1)$, then there seems to be no general asymptotic shrinking shapes. This conjecture was verified for $N = 2$ by B. Andrews (2002?).

**Self-similar solutions for the mean curvature flow equation.** Classification of self-similar solutions is rather difficult topic even for the mean curvature flow equation with $N \geq 3$. There exists a torus type self-similar solution as proved by S. Angenent (1992). The existence of a self-similar solution whose geners is more than one is conjectured by D. L. Chopp (1994). If self-similar solution is monotone shrinking and diffeomorphic to sphere, it has been proved that it must be a shrinking sphere by G. Huisken (1990). However, without monotonicity it is not known whether there is another self-similar solution diffeomorphic to sphere.

**Singularities for the mean curvature flow equation.** The blow up rate of curvatures near singularity may be higher than the self-similar rate. Such a singularity is called type II otherwise it is called Type I. Shrinking sphere is of course type I. There exists a type II singularity as proved in S. Altschuler, S. Angenent and Y. Giga (1995), where they constructs a smooth surface shrinking to a point without becoming convex. They applied a level set method with topological argument; see also Y. Giga (1995a). The existence of other type II singularity is constructed for a higher dimensional surface by J. J. L. Velásquez (1994). Later S. B. Angenent and J. J. L. Velásquez (1997) give more explicit examples.

If the evolution is monotone in time, the asymptotic shape of singularity is always convex (irrelvant of types of singularities). This statement has been proved by G. Huisken and C. Sinestrari (1999) and independently by B. White (1998) by a completely different method.

**Other important equation.** The equation $V = -1/H$ is used to prove the Riemannian Penrose inequality in cosmology by G. Huisken and T. Ilmanen (1997).

**Local solvability.** In Chapter 4 we consider initial value problem for (1.5.1). The first question would be whether there is a unique solution $\{\Gamma_t\}$ satisfying (1.5.1) with some time duration $(0, T)$ for a given initial data $\Gamma_0$. If (1.5.1) is strictly parabolic near initial data, the unique local existence of smooth solutions can be proved. One way is to analyze an equation of a height function, where the evolving surface is parametrized by the height from the initial surface. This idea has been carried out by X.-Y. Chen (1991) for a class of equations including the mean curvature flow equation. Another way is to solve the equation whose solution is the (signed) distance function of $\Gamma_t$. This idea is introduced by L. C. Evans and J. Spruck (1992a) for the mean curvature flow equation and generalized by Y. Giga and S. Goto (1992b) for general strictly parabolic equation. In this method one has to solve a fully nonlinear strict parabolic equation even if the original equation is quasilinear. However, the theory of local solvability for fully nonlinear parabolic equations (including higher order equations) is well developed in a book by A. Lunardi (1995) improving the theory of O. A. Ladyžehnskaja, V. Solonnikov and N. Ural'ceva (1968).

A general result of Y. Giga and S. Goto (1992b) yields the unique local existence of smooth solution for (1.5.4) for arbitrary smooth initial data but for (1.5.9), (1.5.10) with $m > 1$, (1.5.12), (1.5.14), $V = K^\alpha$ for strict convex smooth initial data so that the equation is strictly parabolic. If (1.5.2) is strictly parabolic and $\gamma$ and $\beta$ are smooth enough, the same result for (1.5.4) holds. However, if there is degeneraty of parabolicity in (1.5.2) with $a > 0$, we do not expect to have a smooth solution even locally-in-time for arbitrary smooth initial data as observed by Y. Giga (1994). For the first order equation like (1.5.6) one can prove the unique local existence of smooth solution for arbitrary smooth initial data by a method of characteristics.

There is a general theory of local solvability by A. Polden (See an article by G. Huisken and A. Polden (1999)) for strictly parabolic equation (1.5.1) including higher order equations.

# Chapter 2

# Viscosity solutions

Level set equations we would like to study are parabolic, but they are very degenerate. The classical techniques and results cannot be expected to apply. For example the initital value problem for degenerate parabolic equations may not admit a smooth solution for smooth initial data even locally in time. Moreover, level set equations may be singular at place where the spatial gradient of solutions vanishes as noted for the level set mean curvature flow equations. Even for such singular degenerate equations it turns out that the theory of viscosity solutions provides a right notion of generalized solutions. In this chapter we shall present the basic aspect of the theory especially for singular degenerate parabolic equations.

## 2.1 Definitions and main expected properties

To motivate definition of viscosity solution we consider a $C^2$ function $u = u(z)$ which satisfies a differential inequality:

$$u_t(z) + F(z, u(z), \nabla u(z), \nabla^2 u(z)) \leq 0 \tag{2.1.1}$$

for all $z = (x, t)$ in $\Omega \times (0, T)$, where $\Omega$ is a domain in $\mathbf{R}^\mathbf{N}$ and $T > 0$, and $F$ is a function in $\Omega \times \mathbf{R} \times \mathbf{R}^N \times S^N$. We assume that $F$ is (degenerate) elliptic i.e.

$$F(z, r, p, X) \leq F(z, r, p, Y) \qquad \text{for } X \geq Y,$$

so that the equation

$$u_t + F(z, u, \nabla u, \nabla^2 u) = 0 \tag{2.1.2}$$

is a parabolic equation. Suppose that $\varphi$ is also $C^2$ and $u - \varphi$ takes its local maximum at $\hat{z} = (\hat{x}, \hat{t})$ with $\hat{z} \in \Omega \times (0, T)$. Then the classical maximum principle which is just calculus implies that

$$u_t(\hat{z}) = \varphi_t(\hat{z}),\ \nabla u(\hat{z}) = \nabla\varphi(\hat{z}),\ \nabla^2 u(\hat{z}) \leq \nabla^2\varphi(\hat{z}).$$

The degenerate ellipticity together with (2.1.1) implies that

$$\begin{aligned}&\varphi_t(\hat{z}) + F(\hat{z}, u(\hat{z}), \nabla\varphi(\hat{z}), \nabla^2\varphi(\hat{z}))\\ &\leq u_t(\hat{z}) + F(\hat{z}, u(\hat{z}), \nabla u(\hat{z}), \nabla^2 u(\hat{z})) \leq 0.\end{aligned}$$

The same inequality for $\varphi$ is obtained even if $\Omega \times (0,T]$ so that $\hat{t}$ may equal $T$, since we still have

$$u_t(\hat{z}) \le \varphi_t(\hat{z})$$

although $u_t(\hat{z})$ may not equal $\varphi_t(\hat{z})$. The inequality

$$\varphi_t(\hat{z}) + F(\hat{z}, u(\hat{z}), \nabla\varphi(\hat{z}), \nabla^2\varphi(\hat{z})) \le 0 \tag{2.1.3}$$

does not include derivatives of $u$ so we are tempting to define an arbitrary function $u$ to be a generalized subsolution of (2.1.2) in $\Omega \times (0,T)$ if for each $\hat{z} \in \Omega \times (0,T)$ an upper test function $\varphi$ of $u$ around $\hat{z}$ always solves (2.1.3). There are a couple of freedom to define viscosity solutions when $F$ is not continuous. One should take a good definition of solutions which enjoys major expected properties including comparison principle, stability principle and Perron's construction of solutions. We shall give such definitions for a typical class of equations. For technical convenience we define a notion of solutions for arbitrary discontinuous functions.

### 2.1.1 Definition for arbitrary functions

To handle arbitrary functions we recall upper and lower semicontinuous envelope. Let $h$ be a function defined in a set $L$ of a metric space $X$ with values in $\mathbf{R}^{\cup}\{\pm\infty\}$. The *upper semicontinuous envelope* $h^*$ and the *lower semicontinuous envelope* $h_*$ of $h$ are defined by

$$\begin{aligned} h^*(z) &= \lim\nolimits_{r\downarrow 0} \sup\{h(\zeta); \zeta \in B_r(z) \cap L\} \\ h_*(z) &= \lim\nolimits_{r\downarrow 0} \inf\{h(\zeta); \zeta \in B_r(z) \cap L\} \end{aligned} \qquad z \in \overline{L} \tag{2.1.4}$$

respectively, as functions defined on the closure $\overline{L}$ of $L$ with values in $\mathbf{R}^{\cup}\{\pm\infty\}$, where $B_r(z)$ is a closed ball of radius $r$ centered at $z$ in $X$. In other words $h_*$ is the greatest lower semicontinuous function on $\overline{L}$ which is smaller than $h$ on $L$ and similarly, $h^*$ is the smallest upper semicontinuous function on $\overline{L}$ which is greater than $h$ on $L$. Clearly $h_* = h$ if lower semicontinuous in $\overline{L}$ and $h^* = h$ if $h$ is upper semicontinuous in $\overline{L}$. For continuous $\varphi$ on $\overline{L}$ it is clear that $(h-\varphi)_* = h_* - \varphi$. Also by definition $h_* = -(-h)^*$. For $h^*$ the maximum over compact set $K$ is always attained provided that $h^*(z) < \infty$ for $z \in K$. This property is often used in the sequel.

We first give a rigorous definition of solution of (2.1.2) when $F$ has no singularities.

**Definition 2.1.1.** Let $\Omega$ be an open set in $\mathbf{R}^N$ and $T > 0$. Let $F$ be a continuous on $\overline{\Omega} \times [0,T] \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ with values in $\mathbf{R}$. Let $\mathcal{O}$ be an open set in $\Omega \times (0,T)$.

(i) A function $u : \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ is a (viscosity) *subsolution* of

$$u_t + F(z, u, \nabla u, \nabla^2 u) = 0 \tag{2.1.5}$$

in $\mathcal{O}$ (or equivalently, solution of $u_t + F(z,u,\nabla u, \nabla^2 u) \le 0$) if

(a) $u^*(z) < \infty$ for $z \in \mathcal{O}$;

(b) If $(\varphi, \hat{z}) \in C^2(\mathcal{O}) \times \mathcal{O}$ satisfies

$$\max_{\mathcal{O}}(u^* - \varphi) = (u^* - \varphi)(\hat{z}), \tag{2.1.6}$$

then

$$\varphi_t(\hat{z}) + F(\hat{z}, u^*(\hat{z}), \nabla\varphi(\hat{z}), \nabla^2\varphi(\hat{z})) \leq 0. \tag{2.1.7}$$

(ii) A function $u : \mathcal{O} \to \mathbf{R} \cup \{+\infty\}$ is a (viscosity) *supersolution* of (2.1.5) in $\mathcal{O}$ (or equivalently, solution of $u_t + F(z, u, \nabla u, \nabla^2 u) \geq 0$) if

(a) $u_*(z) > -\infty$ for $z \in \mathcal{O}$;

(b) If $(\varphi, \hat{z}) \in C^2(\mathcal{O}) \times \mathcal{O}$ satisfies

$$\min_{\mathcal{O}}(u_* - \varphi) = (u_* - \varphi)(\hat{z}) \tag{2.1.8}$$

then

$$\varphi_t(\hat{z}) + F(\hat{z}, u^*(\hat{z}), \nabla\varphi(\hat{z}), \nabla^2\varphi(\hat{z})) \geq 0. \tag{2.1.9}$$

By definition this is equivalent to say that $v = -u$ is a solution of

$$v_t - F(z, -v, -\nabla v, -\nabla^2 v) \leq 0.$$

We shall often supress the word "viscosity". In every circumstance when sub- and supersolutions are defined, we say a function is a (viscosity) *solution* of the equation in a given set if it is both a sub- and supersolution of the equation in the set. A function $\varphi$ satisfying (2.1.6) is called a *upper test function of* $u$ at $\hat{z}$ over $\mathcal{O}$ and $\varphi$ satisfying (2.1.8) is called a *lower test function of* $u$ at $\hat{z}$ over $\mathcal{O}$. As observed from (2.1.3) the notion of viscosity solutions are a natural extension of notion of solutions if $F$ is degenerate elliptic.

**Proposition 2.1.2.** *Assume that $u \in C^2(\mathcal{O})$ (i.e. $u$ is $C^2$ on $\mathcal{O}$). Assume that $F$ is degenerate elliptic. Then $u$ is a subsolution (resp. supersolution) of (2.1.5) in $\mathcal{O}$ if and only if $u$ satisfies*

$$u_t + F(z, u, \nabla u, \nabla^2 u) \leq 0 \quad (\text{resp. } \geq 0)$$

*in $\mathcal{O}$.*

**Remark 2.1.3.** It is sometimes convenient to define solutions in an open set $\mathcal{O}$ in $\Omega \times (0, T]$ instead of $\Omega \times (0, T)$. It is defined in the same way as in Definition 2.1.1. As remarked before (2.1.3), the statement of Proposition 2.1.2 is still valid even if $\mathcal{O}$ intersects $t = T$.

The notation of viscosity solutions can be localized.

**Localization property.** *(i) If $u$ is a subsolution of an equation in a neighborhood of each point of $\Omega \times (0, T)$, then it is a subsolution (of the same equation) in $\Omega \times (0, T)$.*
*(ii) If $u$ is a subsolution (of an equation) in $\Omega \times (0, T)$, then it is a subsolution (of the same equation) in every open set $\mathcal{O}$ in $\Omega \times (0, T)$.*
*(iii) The properties (i), (ii) are still valid if a supersolution replaces a subsolution.*

This is easily proved for equation (2.1.5) by restricting and extending test functions.

Unfortunately, if (2.1.5) is a level set equation, the value of $F$ at $\nabla u = 0$ may not be defined. We have to extend definition of viscosity solutions for (2.1.5) with discontinuous $F$ to handle such equations.

**Definition 2.1.4.** (i) Assume that $F$ is lower semicontinuous in $W = \overline{\Omega} \times [0,T] \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ with values in $\mathbf{R} \cup \{-\infty\}$. A subsolution of (2.1.5) is defined as in Definition 2.1.1 (i).
(ii) Assume that $F$ is upper semicontinuous in $W$ with values in $\mathbf{R} \cup \{+\infty\}$. A supersolution of (2.1.5) is defined as in Definition 2.1.1 (ii). Here we use the convention that $a + \infty = \infty > 0$ and $a - \infty = -\infty < 0$ for any real number $a$.
(iii) Assume that $F$ is defined only in a dense subset of $W$ and that $F_* < \infty, F^* > -\infty$ in $W$. If $u$ is a subsolution of

$$u_t + F_*(z, u, \nabla u, \nabla^2 u) = 0$$

in $\mathcal{O}$, then $u$ is called a subsolution of (2.1.5) in $\mathcal{O}$. If $u$ is a supersolution of

$$u_t + F^*(z, u, \nabla u, \nabla^2 u) = 0$$

in $\mathcal{O}$, then $u$ is called a supersolution of (2.1.5). (In applications the case when $F$ is continuous in $\overline{\Omega} \times [0,T] \times \mathbf{R} \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$ has a particular importance.)

As stated for continuous $F$, the statement of Proposition 2.1.2 and localization property are still valid for such an $F$.

### 2.1.2 Expected properties of solutions

We shall explain in an informal way three major properties which viscosity solutions are expected to satisfy.

**Comparison principle.** *If $u$ is a subsolution of (2.1.5) in $Q = \Omega \times [0,T)$ and $v$ is a supersolution of (2.1.5) in $Q$, then $u^* \le v_*$ in $Q$ if $u^* \le v_*$ on the parabolic boundary $\partial_p Q$ of $Q$, i.e.*

$$\partial_p Q = \Omega \times \{0\} \cup \partial\Omega \times [0,T).$$

This is a comparison principle for semicontinuous functions so it is sometimes called a strong comparison principle to distinguish the comparison principle for continuous functions.

The major structual assumptions to guarantee comparison principle are degenerate ellipticity of $F$ and monotonicity of $F = F(z,r,p,X)$ in $r$ in the sense that

$$r \mapsto F(z,r,p,X)$$

is nondecreasing. The last property can be weaken to the nondecreasing of

$$r \mapsto F(z,r,p,X) + c_0\, r$$

for some $c_0$ independent of $(z,p,X)$. (Indeed, by the change of variable $u$ by $e^{c_0 t}v$, this condition deduces to the stronger version of monotonicity condition for equation for $v$.) We need other regularity assumptions for $F$. A simplest set of assumptions for continuous $F$ is that $F$ is independent of $z, r$ and degenerate elliptic. In this case as we will see later the comparison principle holds at least for bounded $\Omega$. (For unbounded $\Omega$ one should modify the interpretation of the inequality $u^* \le v_*$.)

If $F = F(z,r,p,X)$ has a singularity (discontinuity) at $p = 0$ (but is continuous elsewhere), we assume that

$$-\infty < F_*(z,r,0,O) = F^*(z,r,0,O) < \infty \tag{2.1.10}$$

to get the comparison principle. This assumption cannot be removed completely.

**Counterexample.** We set $F(p,X) = -X/|p|^\alpha$ with $N = 1, \alpha > 0$. Then both $u \equiv 0$ and

$$v(x,t) = \begin{cases} 1 & t \ge T \\ 0 & t < T \end{cases}$$

are solutions of (2.1.5) in $\mathbf{R} \times (0,\infty)$. Of course $F$ is degenerate elliptic but it violates (2.1.10); indeed

$$F_*(0,O) = -\infty,\ F^*(0,O) = +\infty.$$

Since both $u$ and $v$ are initially zero, the comparison principle is violated. One complaint lies in the fact that $\mathbf{R}$ is unbounded. But it still gives a counterexample for the comparison principle which is expected to hold for an unbounded domain. We do not pursue to find a counterexample for a bounded domain.

**Stability principle.** *We begin with its naive form. Assume that $F_\varepsilon$ converges to $F$ locally uniform (as $\varepsilon \to 0$ with $\varepsilon > 0$) in the domain of definition of $F$. Assume that $u_\varepsilon$ solves*

$$u_{\varepsilon t} + F_\varepsilon(z, u_\varepsilon, \nabla u_\varepsilon, \nabla^2 u_\varepsilon) \le 0 \quad (\textit{resp.} \ge 0) \quad \textit{in} \quad \mathcal{O}$$

*and that $u_\varepsilon$ converges to $u$ locally uniform (as $\varepsilon \to 0$) in $\mathcal{O}$. Then $u$ solves*

$$u_t + F(z,u,\nabla u,\nabla^2 u) \le 0 \quad (\text{resp.} \ge 0) \quad \text{in} \quad \mathcal{O}.$$

This property is useful when we want to contruct a solution $u$ of (2.1.5) by approximating the equation. There are several options to weaken the assumption of convergences $F_\varepsilon \to F$, $u_\varepsilon \to u$. We may replace these convergence by 'semi-convergence'. Let $h_\varepsilon$ be a function defined in a set $L$ of a metric space $X$ with values in $\mathbf{R} \cup \{\pm\infty\}$, where $\varepsilon > 0$. The *upper relaxed limit* $\overline{h} = \limsup^* h_\varepsilon$ and the *lower relaxed limit* $\underline{h} = \liminf_* h_\varepsilon$ are defined by

$$\begin{aligned} \overline{h}(z) = (\limsup_{\varepsilon\to 0}{}^* h_\varepsilon)(z) &= \limsup_{\substack{\zeta\to z\\ \varepsilon\to 0}}\ h_\varepsilon(\zeta) \\ &= \lim_{\varepsilon\to 0}\ \sup\{h_\delta(\zeta); \zeta \in L \cap B_\varepsilon(z),\ 0<\delta<\varepsilon\}, \\ \underline{h}(z) = (\liminf_{\varepsilon\to 0}{}_* h_\varepsilon)(z) &= \liminf_{\substack{\zeta\to z\\ \varepsilon\to 0}}\ h_\varepsilon(\zeta) \\ &= \lim_{\varepsilon\to 0}\ \inf\{h_\delta(\zeta); \zeta \in L \cap B_\varepsilon(z),\ 0<\delta<\varepsilon\}, \end{aligned}$$

respectively. These limits are a kind of 'lim sup' and 'lim inf' but take such an operation in *both* $\varepsilon$ and $\zeta$. The functions $\limsup^* h_\varepsilon$ and $\liminf_* h_\varepsilon$ are defined for all $z$ in $\overline{L}$ allowing the values $\pm\infty$. Evidently, if $h_\varepsilon = h$, then

$$h^* = \limsup{}^* h_\varepsilon \quad \text{and} \quad h_* = \liminf{}_* h_\varepsilon.$$

By definition $\overline{h} = \limsup^*(h_\varepsilon^*)$ and $\underline{h} = \liminf_*(h_\varepsilon)_*$. If we set $k(z,\varepsilon) = h_\varepsilon(z)$ as a function of $z \in L$ and $\varepsilon > 0$, then by definition

$$\overline{h}(z) = k^*(z,0), \ \underline{h}(z) = k_*(z,0) \quad \text{for} \quad z \in \overline{L}.$$

If $\overline{h} = \underline{h}$ on $\overline{L}$, then $k^*$ and $k_*$ are continuous at $\varepsilon = 0$ as functions in $\overline{L} \times (0,1)$. This implies that $h_\varepsilon$ converges to $\overline{h} = \underline{h}$ locally uniformly in $\overline{L}$ as $\varepsilon \to 0$.

The convergence $u_\varepsilon \to u$ may be replaced by semi-convergence $u = \overline{u}$ if we consider sequences of subsolutions. For sequence of supersolutions we may replace $u_\varepsilon \to u$ by $u = \underline{u}$. The convergence $F_\varepsilon \to F$ is replaced by $F \le \underline{F}$ and $F \ge \overline{F}$ for sub and supersolutions respectively. If convergence is replaced by semi-convergence, in stability principle, it is sometimes called strong stability principle. The major advantage lies in the fact that $\overline{u}$ and $\underline{u}$ are always exists allowing the values $\pm\infty$. So to apply strong stability principle we need not to estimate $\{u_\varepsilon\}$ to get locally uniform convergence of $u_\varepsilon$. Once we prove $(-\infty <)\ \overline{u} \le \underline{u}\ (< \infty)$ for example by comparison principle, we have $\overline{u} = \underline{u}$. This implies *a posteori* locally uniform convergence $u_\varepsilon \to \overline{u} = \underline{u} = u$ and the limit $u$ is continuous.

**Perron's method.** (Construction of solutions) *If $u_+$ and $u_-$ are super- and subsolution of (2.1.5) in $\Omega \times [0,T)$, respectively, then there is a solution $u$ of (2.1.5) in $\Omega \times [0,T)$ that satisfies $u_- \le u \le u_+$ on $\Omega \times (0,T)$ provided that $u_- \le u_+$ on $\Omega \times (0,T)$.*

This method reduces the business of construction of solutions to finding appropriate sub- and supersolutions satisfying the initial and boundary conditions. For example consider initial value problem for (2.1.5) with continuous initial data $u_0$, where $\Omega = \mathbf{R}^N$. If there are sub- and supersolution $u^\pm$ of (2.1.5) in $\mathbf{R}^N \times (0,T)$ that satisfies

$$u^{+*}(x,0) = u_0(x) = u^-_*(x,0),$$

then by Perron's method we have a solution $u$. Moreover, if the comparison principle applies, then $u^* \le u_*$ in $\mathbf{R}^N \times [0,T)$. This implies that $u$ is continuous $\mathbf{R}^N \times [0,T)$ with $u(0,x) = u_0(x)$. The uniqueness of such a solution is clear from the comparison principle. Since $\mathbf{R}^N$ is unbounded, the comparison principle is not exactly as stated above, this discussion is a little bit informal but it explains the basic idea to get solution.

Apparently, Definition 2.1.4 is a natural extension for discontinuous $F$. Indeed the definition for continuous $F$ is included. Moreover, stability principle and Perron's method is available. If (2.1.10) is fulfilled, as we see later, the comparison principle hold under reasonable regularity assumptions on $F$ at least when $F(z,r,p,X)$ is continuous outside $p = 0$. This class of equations includes many important examples of level set equations satisfying (1.6.24) so for example it applies to the level set mean curvature flow equation (1.6.5). However, for the level set equation of the Gaussian curvature flow (1.6.12), (2.1.10) is not fulfilled (see §1.6.5). If $F$ fails to satisfy (2.1.10), the comparison principle may not hold. One should modify the notion of viscosity solutions for such problems as discussed in the next subsection.

In this Chapter we only discuss the stability principle and Perron's method. We discuss the comparison principle in the next Chapter.

### 2.1.3 Very singular equations

Let $F$ be continuous on $W_0 = \overline{\Omega} \times [0,T] \times \mathbf{R} \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$ with values in $\mathbf{R}$. We restrict test functions. Let $\mathcal{F}_\Omega = \mathcal{F}_\Omega(F)$ be the set of functions $f \in C^2[0,\infty)$ such that

$$f(0) = f'(0) = f''(0) = 0 \quad \text{and} \quad f''(r) > 0 \quad \text{for} \quad r > 0$$

and that for $s = \pm 1$

$$\lim_{p\to 0} \sup_{z\in\overline{\mathcal{O}}} \sup_{r\in\mathbf{R}} |F(z, r, \nabla_p(f(|p|)), s\nabla_p^2(f(|p|)))| = 0. \tag{2.1.11}$$

The formula (2.1.11) is often written as

$$\lim_{p\to 0} \sup_{z\in\overline{\mathcal{O}}} \sup_{r\in\mathbf{R}} |F(z, r, \nabla(f|p|)), \pm\nabla^2(f(|p|))| = 0.$$

We suppress $\Omega$ dependence of $\mathcal{F}$ in this Chapter since we fix $\Omega$. We say that $\varphi \in C^2(\mathcal{O})$ is *compatible* with $F$ if for any $\hat{z} = (\hat{x}, \hat{t})$ in $\mathcal{O}$ with $\nabla\varphi(\hat{z}) = 0$, there is a constant $\delta > 0$ and functions $f \in \mathcal{F}$ and $\omega_1 \in C[0,\infty)$ with $\lim_{\sigma\to 0} \omega_1(\sigma)/\sigma = 0$ that satisfies

$$|\varphi(x,t) - \varphi(\hat{z}) - \varphi_t(\hat{z})(t - \hat{t})| \le f(|x - \hat{x}|) + \omega_1(|t - \hat{t}|) \tag{2.1.12}$$

for all $(x,t) \in \mathcal{O}$ with $|x - \hat{x}| \le \delta$, $|t - \hat{t}| \le \delta$. The set of $\varphi \in C^2(\mathcal{O})$ compatible with $F$ denotes $C_F^2(\mathcal{O})$.

Note that the set $\mathcal{F}$ can be empty. Indeed, for $N = 1$ let

$$F = F(p, X) = -X/|p|^\alpha. \tag{2.1.13}$$

If $\alpha \ge 1$, then, by the Gronwall inequality for $f'$, a function $f \in C^2[0,\infty)$ that satisfies

$$\lim_{x\to 0} \frac{f''}{f'^\alpha} = 0 \quad \text{with} \quad f' \ge 0,\ f'(0) = 0$$

is a constant near $x = 0$. This violates $f'(\sigma) > 0$ for $\sigma > 0$ so the set $\mathcal{F}$ is empty for $\alpha \ge 1$. If $0 < \alpha < 1$, $f(\sigma) = \sigma^\gamma$ belongs to $\mathcal{F}$ provided that $\gamma > (2-\alpha)/(1-\alpha)$. The case $\alpha \ge 1$ the initial value problem for (2.1.5) with (2.1.13) cannot be expected solvable except for monotone initial data. The case $0 < \alpha < 1$ the equation (2.1.5) with (2.1.13) is the $q$-Laplace diffusion equation (with $1 < q < 2$)

$$u_t - c\ \mathrm{div}(|\nabla u|^{q-2}\nabla u) = 0, \tag{2.1.14}$$

where $q = 2 - \alpha,\ c = 1/(q-1),\ N = 1$. If $F = F(p, X)$ is geometric, degenerate elliptic and continuous outside $p = 0$, then it turns out that $\mathcal{F}$ is not empty. We shall discuss this point in the next Chapter. It is easy to see that $\mathcal{F}$ is invariant under a positive multiplication i.e., $af \in \mathcal{F}$ if $f \in \mathcal{F}$ and $a > 0$ for (2.1.14) with $p > 1$ and also for geometric $F$.

**Definition 2.1.5.** Assume that $F$ is continuous on $W_0 = \overline{\Omega} \times [0,T] \times \mathbf{R} \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$ with values in $\mathbf{R}$. Assume that $\mathcal{F}$ is nonempty. An $\mathcal{F}$-subsolution of (2.1.5) with $F$ is defined as in Definition 2.1.1 by replacing (i)(b) by the condition:

$$\begin{cases} \varphi_t(\hat{z}) + F(\hat{z},\ u^*(\hat{z}),\ \nabla\varphi(\hat{z}),\ \nabla^2\varphi(\hat{z})) & \le 0 \quad \text{if} \quad \nabla\varphi(\hat{z}) \ne 0 \\ \varphi_t(\hat{z}) & \le 0 \quad \text{otherwise} \end{cases}$$

for all $(\varphi, \hat{z}) \in C^2_F(\mathcal{O}) \times \mathcal{O}$ satisfying (2.1.6). An $\mathcal{F}$-supersolution of (2.1.5) with $F$ is defined as in Definition 2.1.1 by replacing (ii)(b) by the condition:

$$\begin{cases} \varphi_t(\hat{z}) + F(\hat{z},\ u^*(\hat{z}),\ \nabla\varphi(\hat{z}),\ \nabla^2\varphi(\hat{z})) & \ge 0 \quad \text{if} \quad \nabla\varphi(\hat{z}) \ne 0 \\ \varphi_t(\hat{z}) & \ge 0 \quad \text{otherwise} \end{cases}$$

for all $(\varphi, \hat{z}) \in C^2_F(\mathcal{O}) \times \mathcal{O}$ satisfying (2.1.8). One may define $\mathcal{F}$ subsolution is an open set $\mathcal{O}$ in $\Omega \times (0, T]$ as for usual subsolutions.

**Remark 2.1.6.** If

$$\lim_{\substack{p \to 0 \\ X \to O}} \sup_{z \in \overline{\mathcal{O}}} \sup_{r \in \mathbf{R}} |F(z, r, p, X)| = 0, \tag{2.1.15}$$

(which in particular implies (2.1.10)), then clearly

$$\mathcal{F} = \{f \in C^2(0,\infty); f(0) = f'(0) = f''(0) = 0 \quad \text{and} \quad f''(r) > 0 \quad \text{for} \quad r > 0\}.$$

It turns out that $C^2_F(\mathcal{O})$ equals

$$\mathcal{A}_0 = \{\varphi \in C^2(\mathcal{O});\ \nabla\varphi(\hat{z}) = 0 \quad \text{implies} \quad \nabla^2\varphi(\hat{z}) = O\}.$$

We postpone its proof. By this observation a subsolution is an $\mathcal{F}$-subsolution since $F_*(z, r, 0, O) = 0$ by (2.1.15). It is curious whether or not an $\mathcal{F}$-subsolution argrees with a subsolution in Definirion 2.1.4. It turns out that the converse is true if (2.1.15) is fulfilled and $F$ is degenerate elliptic as proved in Proposition 2.2.8. Here we just give a simple equivalent definition of $\mathcal{F}$- subsolutions.

**Proposition 2.1.7.** *Assume that (2.1.15) holds for $F$. A function $u : \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ is an $\mathcal{F}$-subsolution of (2.1.5) if and only if $u$ is a solution of*

$$u_t + F_\#(z, u, \nabla u, \nabla^2 u) \le 0 \quad \textit{in} \quad \mathcal{O}$$

*with*

$$F_\#(z, r, p, X) = \begin{cases} F(z, r, p, X) & \textit{if } p \ne 0 \\ 0 & \textit{if } (p, X) = (0, O) \\ -\infty & \textit{otherwise.} \end{cases}$$

*The similar assertion holds for an $\mathcal{F}$-supersolution by replacing the above inequality by*

$$u_t + F^{\#}(z, u, \nabla u, \nabla^2 u) \geq 0 \quad \text{in} \quad \mathcal{O}$$

*with*

$$F^{\#}(z, r, p, X) = \begin{cases} F(z, r, p, X) & \text{if } p \neq 0 \\ 0 & \text{if } (p, X) = (0, O) \\ +\infty & \text{otherwise.} \end{cases}$$

This follows from definition of $\mathcal{F}$-solution if we admit $\mathcal{A}_0 = C_F^2(\mathcal{O})$, which is proved in Proposition 2.1.8. Note that $F_{\#}(\leq F_*)$ and $F^{\#}(\geq F^*)$ are, respectively, still lower and upper semicontinuous function on $\overline{\Omega} \times [0, T] \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ by (2.1.15). Here is a trivial remark. One may weaken (2.1.15) by

$$\lim_{\substack{p \to 0 \\ X \to O}} \sup_{z \in \overline{\mathcal{O}}} \sup_{r \in \mathbf{R}} |F(z, r, p, X) - a_0| = 0$$

for a fixed constant $a_0$ so that $F_*(z, r, 0, O) = a_0 = F^*(z, r, 0, O)$. To define an $\mathcal{F}$-subsolution in this situation we should replace (2.1.11) by

$$\lim_{\substack{p \to 0 \\ X \to O}} \sup_{z \in \overline{\mathcal{O}}} \sup_{r \in \mathbf{R}} |F(z, r, \nabla(f(|p|)), \pm\nabla^2(f(|p|))) - a_0| = 0$$

and replace $\varphi_t(\hat{z}) \leq 0$ by $\varphi_t(\hat{z}) + a_0 \leq 0$ and $\varphi_t(\hat{z}) \geq 0$ by $\varphi_t(\hat{z}) + a_0 \geq 0$ in Definition 2.1.5. The statement of Proposition 2.1.7 is still valid by replacing 0 by $a_0$ in the definition of $F_{\#}$ and $F^{\#}$. One may even replace $a_0$ by a continuous function $a_0(z, r)$ to handle the equation having external force term.

For the level set equation (1.6.12) of the Gaussian curvature flow equation it is not difficult to see that $F_*(0, X) = -\infty$ and $F^*(0, X) = +\infty$ (see §1.6.5) so a subsolution in Definition 2.1.4 may not be an $\mathcal{F}$-subsolution while an $\mathcal{F}$-subsolution is always a subsolution. In this case Definition 2.1.5 is more restrictive than Definition 2.1.4 which is indeed important to get the comparison principle.

**Proposition 2.1.8.** *Assume that (2.1.15) holds for $F$. Then $C_F^2(\mathcal{O}) = \mathcal{A}_0$.*

*Proof.* It is easy to see that $\mathcal{A}_0$ includes $C_F^2(\mathcal{O})$. Indeed, the condition (2.1.12) implies that $\nabla^2 \varphi(\hat{z}) = O$ for $\hat{z}$ with $\nabla \varphi(\hat{z}) = 0$.

It remains to prove that $C_F^2(\mathcal{O})$ includes $\mathcal{A}_0$. If $\nabla \varphi(\hat{z}) = 0$ and $\nabla^2 \varphi(\hat{z}) = O$, then

$$|\varphi(z) - \varphi(\hat{z}) - \varphi_t(\hat{z})(t - \hat{t})| \leq \omega_1(|t - \hat{t}|) + \omega_2(|x - \hat{x}|)$$

for all $(x, t)$ with $|x - \hat{x}| \leq \delta$, $|t - \hat{t}| \leq \delta$ for some $\delta > 0$ and $\omega_1, \omega_2 \geq 0$ with

$$\lim_{\sigma \to 0} \omega_k(\sigma)/\sigma^k = 0 \quad (k = 1, 2). \tag{2.1.16}$$

We would like to find $f_0 \in C^2[0, \infty)$ with $f_0(0) = f_0'(0) = f_0''(0) = 0$ and $f_0'' \geq 0$ that satisfies

$$\omega_2(\sigma) \leq f_0(\sigma) \quad \text{for} \quad 0 \leq \sigma \leq \delta. \tag{2.1.17}$$

If such an $f_0$ exists, $f(\sigma) = f_0(\sigma) + \sigma^4$ belongs to $\mathcal{F}$ since (2.1.11) is always fulfilled for $f \in C^2[0,\infty)$ with $f'(0) = f''(0) = 0$ under the condition (2.1.15).

We shall prove the existence of $f_0$. If we set

$$\omega_0(\sigma) = \sup\{\omega_2(\eta)/\eta^2;\ 0 \le \eta \le \sigma,\ \eta \le \delta\},$$

then $\omega_0$ is a nondecreasing function on $[0,\infty)$ with $\omega_0(0) = 0$ which is continuous at zero by (2.1.16). Since

$$\omega_2(\sigma) \le \omega_0(\sigma)\sigma^2 \quad \text{for} \quad 0 \le \sigma \le \delta,$$

the existence of $f_0$ that satisfy (2.1.17) follows from the next elementary lemma by taking $f_0 = \theta$ with $k = 2$. □

**Lemma 2.1.9.** *(i) Let $\omega_0$ be a nondecreasing function on $[0,\infty)$ with $\omega_0(0) = 0$. Assume that $\omega_0$ is continuous at zero. Then there is a modulus $\omega$ (i.e., $\omega$ is a nondecreasing continuous function on $[0,\infty)$ with $\omega(0) = 0$) with $\omega \in C^\infty(0,\infty)$ that satisfies $\omega_0 \le \omega$ on $[0,\infty)$.*
*(ii) Let $\omega$ be a modulus and let $k$ be a positive integer. Then there is $\theta \in C^k[0,\infty)$ such that $\theta^{(j)}(0) = 0$ and $\theta^{(j)}(\sigma) \ge 0$ for $\sigma \ge 0$ with $0 \le j \le k$ and $\omega(\sigma)\sigma^k \le \theta(\sigma)$ for all $\sigma \ge 0$.*

*Proof.* (i) We set $\omega(m) = \omega_0(m+1)$, $m = 1,2,\cdots$ and $\omega(1/2^j) = \omega_0(1/2^{j-1})$, $j = 1,2,\cdots$. Interpolating the value of $\omega(\sigma)$ for $m \le \sigma \le m+1$ and $1/2^j \le \sigma \le 1/2^{j-1}$ by an affine function, we get a continuous function $\omega$ on $(0,\infty)$. It is possible to mollify $\omega$ without changing values at $1/2^j$, $j = 1,2,\cdots$, so that $\omega \in C^\infty(0,\infty)$ and $\omega_0 \le \omega$. Since $\omega_0$ is nondecreasing so is $\omega$ and $\omega_0 \le \omega$ on $(0,\infty)$. Since $\omega_0$ is continuous at zero, we have

$$\lim_{\sigma\to 0} \omega(\sigma) = \lim_{j\to\infty} \omega_0(1/2^{j-1}) = 0.$$

Thus $\omega$ extends a continuous function on $[0,\infty)$ (still denoted $\omega$) by setting $\omega(0) = 0$.
(ii) We set

$$\theta_j(\sigma) = \int_\sigma^{2\sigma} \theta_{j-1}(s)ds,\ j \ge 1,\ \theta_0 = \omega \quad \text{for} \quad \sigma \ge 0,$$

so that $\theta_j \in C^j[0,\infty)$ with $\theta_j^{(i)}(0) = 0$ for $0 \le i \le j$. Since $\theta_j$ is nondecreasing, we have

$$\theta_j(\sigma) \ge \sigma\theta_{j-1}(\sigma) \quad \text{for} \quad \sigma \ge 0$$

so that $\theta_j(\sigma) \ge \sigma^j\omega(\sigma)$. We thus observe that $\theta = \theta_k$ has all desired properties. □

**Remark 2.1.10.** For $\mathcal{F}$-subsolutions the statement of Proposition 2.1.2 should be altered. Assume that $u \in C^2(\mathcal{O})$ satisfies

$$u_t(\hat{z}) + F(\hat{z}, u(\hat{z}), \nabla u(\hat{z}), \nabla^2 u(\hat{z})) \le 0 \quad (\text{resp. } \ge 0.)$$

for $\hat{z} \in \mathcal{O}$ with $\nabla u(\hat{z}) \ne 0$ and that

$$u_t(\hat{z}) \le 0 \quad (\text{resp. } \ge 0). \tag{2.1.18}$$

for $\hat{z} \in \mathcal{O}$ with $\nabla u(\hat{z}) = 0$. Then $u$ is an $\mathcal{F}$-subsolution (resp. $\mathcal{F}$-supersolution) of (2.1.5) in $\mathcal{O}$ provided that $F$ is degenerate elliptic. For $\hat{z}$ with $\nabla u(\hat{z}) = 0$, (2.1.18) is unnecessary if there are no $f \in \mathcal{F}$ and $\omega_1$ (with $\omega_1(\sigma)/\sigma \to 0$ as $\sigma \to 0$) that satisfies

$$u(x,t) - u(\hat{z}) - u_t(\hat{z})(t - \hat{t}) \leq (\text{resp. } \geq) f(|x - \hat{x}|) + \omega_1(t - \hat{t})$$

for all $z = (x,t)$ sufficiently close to $\hat{z} = (\hat{x}, \hat{t})$. These statements immediately follow from definition. We also note that the localization property is still valid for $\mathcal{F}$-solutions. Although the proof for (i) (in localization property) is immediate, the proof for (ii) needs an extension property as follows. Suppose that

$$\max_{\mathcal{O}'}(u^* - \varphi) = (u^* - \psi)(\hat{z})$$

in a neighborhood $\mathcal{O}'$ of $\hat{z}$ in $\mathcal{O}$ with $\varphi \in C^2_F(\mathcal{O}')$. If $\nabla\varphi(\hat{z}) \neq 0$, then there is $\psi \in C^2_F(\mathcal{O})$ that satisfies $\varphi = \psi$ in some neighborhood of $\hat{z}$ in $\mathcal{O}'$, $\nabla\psi(z) \neq 0$ for all $z \in \mathcal{O}$ and

$$\max_{\mathcal{O}}(u^* - \psi) = (u^* - \psi)(\hat{z}).$$

If $\nabla\varphi(\hat{z}) = 0$, then there is $\psi \in C^2_F(\mathcal{O})$ that satisfies $\varphi \leq \psi$ in some neighborhood of $\hat{z}$ in $\mathcal{O}'$ and $\varphi(\hat{z}) = \psi(\hat{z})$. The proof is easy and left to the reader.

## 2.2 Stability results

We shall give typical results on stability principle in its strong form.

**Theorem 2.2.1.** *Let $\Omega$ be a domain in $\mathbf{R}^N$ and $T > 0$. Let $\mathcal{O}$ be an open set in $\Omega \times (0,T)$.*
*(i) Assume that $F_\varepsilon$ and $F$ are lower (resp. upper) semicontinuous in $W = \overline{\Omega} \times [0,T] \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ with values in $\mathbf{R} \cup \{-\infty\}$ (resp. $\mathbf{R} \cup \{+\infty\}$) for $\varepsilon > 0$. Assume that*

$$F \leq \liminf_{\varepsilon\to 0}{}_* F_\varepsilon \quad \text{in} \quad W \quad (\text{resp. } F \geq \limsup_{\varepsilon \to 0}{}^* F_\varepsilon \text{ in } W). \tag{2.2.1}$$

*Assume that $u_\varepsilon$ is a subsolution (resp. supersolution) of*

$$u_t + F_\varepsilon(z, u, \nabla u, \nabla^2 u) = 0 \tag{2.2.2}$$

*in $\mathcal{O}$. Then $\overline{u} = \limsup_{\varepsilon\to 0}{}^* u_\varepsilon$ (resp. $\liminf_{\varepsilon\to 0}{}_* u_\varepsilon$) is a subsolution of*

$$u_t + F(z, u, \nabla u, \nabla^2 u) = 0 \tag{2.2.3}$$

*in $\mathcal{O}$ provided that $\overline{u}(z) < \infty$ (resp. $\underline{u}(z) > -\infty$) for each $z$ in $\mathcal{O}$.*
*(ii) Assume that $F_\varepsilon$ and $F$ are continuous in $W_0 = \overline{\Omega} \times [0,T] \times \mathbf{R} \times (\mathbf{R}^N \backslash \{0\}) \times \mathcal{S}^N$ with values in $\mathbf{R}$. Assume that $\mathcal{F}(F)$ is included in $\mathcal{F}_\varepsilon = \mathcal{F}(F_\varepsilon)$ for all (sufficiently small) $\varepsilon$ and that $F_\varepsilon$ converges to $F$ locally uniformly in $W_0$ as $\varepsilon \to 0$. Assume that $\mathcal{F}$ is invariant*

*under positive multiplication, i.e., $f \in \mathcal{F} = \mathcal{F}(F)$ implies $af \in \mathcal{F}$ for all $a > 0$. Assume that for any $f \in \mathcal{F}(F)$,*

$$\begin{array}{ll} & \liminf\ F_\varepsilon(\zeta, \rho, \nabla(f(|p|)), \nabla^2(f(|p|))) \geq 0 \\ (resp. & \limsup\ F_\varepsilon(\zeta, \rho, \nabla(f(|p|)), \nabla^2(f(|p|))) \leq 0) \end{array} \tag{2.2.4}$$

*as $\varepsilon \to 0$, $\zeta \to z$, $\rho \to r$, $p \to 0$ for all $z \in \mathcal{O}, r \in \mathbf{R}$. Assume that $u_\varepsilon$ is an $\mathcal{F}_\varepsilon$-subsolution (resp. supersolution) of (2.2.2) in $\mathcal{O}$. Then $\overline{u}$ is an $\mathcal{F}$-subsolution of (2.2.3) in $\mathcal{O}$ provided that $\overline{u}(z) < \infty$ (resp. $\underline{u}(z) > -\infty$) for each $z \in \mathcal{O}$.*

Clearly, the condition (2.2.1) is fulfilled if $F_\varepsilon$ converges to $F$ locally uniformly in $W$ as $\varepsilon \to 0$. However, the uniform convergence in $W$ is not expected if $F$ has singularities. This is a reason why we assume (2.2.1) instead of uniform continuity in $W$. If $F$ and $F_\varepsilon(\varepsilon > 0)$ satisfy (2.1.15), then $\mathcal{F} = \mathcal{F}_\varepsilon = \mathcal{A}_0$ by Proposition 2.1.8. In this case if $F_\varepsilon \to F_\#$ locally uniform in $W_0$ and

$$F_\# \leq \liminf_{\varepsilon \to 0} F_\varepsilon \quad \text{in} \quad W \tag{2.2.5}$$

(which is a special form of $(2.2.1)_1$), then it is easy to see that $(2.2.4)_1$ holds, since $F_\#(z, r, 0, O) = 0$ by Proposition 2.1.8. However, $(2.2.4)_1$ does not seem to imply (2.2.5) under the uniform convergence assumption in $W_0$ even if $F = F(z, r, p, X)$ does not depend on $z$ and $r$, although both conditions are closely related.

The proof of Theorem 2.2.1 is not difficult if we are familiar with various equivalent definition of viscosity solutions and convergence of maximum points explained in the next few subsections.

### 2.2.1 Remarks on a class of test functions

We give several observation on test functions which are practically important to prove expected properties for viscosity solutions.

**Proposition 2.2.2.** *In (2.1.6) of Definition 2.1.1, 2.1.4, 2.1.5 the maximum in the sense that (2.1.6) holds with $\mathcal{O}$ replaced by some neighborhood of $\hat{z}$. It is also replaced by a strict maximum in the sense that*

$$(u^* - \varphi)(z) < (u^* - \varphi)(\hat{z}),\ z \neq \hat{z},\ z \in \mathcal{O} \tag{2.2.6}$$

*or even by a local strict maximum in the sense that (2.2.6) holds with $\mathcal{O}$ replaced by some neighborhood of $\hat{z}$ provided that $\mathcal{F}$ is invariant under positive multiplication. Similarly, in (2.1.8) the minimum may be replaced by a strict minimum or by a local strict minimum.*

*Proof.* It is easy to see that a global maximum may be replaced by a local maximum in (2.1.6) and (2.2.6) in Definitions similar to the proof of the localization property in §2.1.1. For $\mathcal{F}$-subsolution see also Remark 2.1.10.

If $\varphi \in C^2(\mathcal{O})$ satisfies (2.1.6), then $\psi(z) = \varphi(z) + |z - \hat{z}|^4$ satisfies (2.2.6). If (2.1.7) holds for $\psi$, so does $\varphi$. This shows that global maximum in (2.1.6) may be reduced by global strict maximum in Definitions 2.1.1 and 2.1.5. For $\mathcal{F}$-subsolutions one should be a little bit careful. If $\varphi \in C^2_F(\mathcal{O})$ satisfies (2.1.6), then $\psi(z) = \varphi(z) + f(|z-\hat{z}|) + (t-\hat{t})^2$ with $f \in \mathcal{F}$ satisfies (2.2.6). However, $\psi$ may not be in $C^2_F(\mathcal{O})$. So we choose $\psi$ in another way. We may assume that $\nabla\varphi(\hat{z}) = 0$ and recall (2.1.12). By Lemma 2.1.9 (ii) there is $\theta_1 \in C^1[0,\infty)$ with $\theta_1(0) = \theta_1'(0) = 0$ that satisfies

$$\omega_1(|t-\hat{t}|) \le \theta_1(|t-\hat{t}|).$$

If we set

$$\begin{aligned}\psi(x,t) = \; & \varphi(\hat{z}) + \varphi_t(\hat{z})(t-\hat{t}) + f(|x-\hat{x}|) + \theta_1(|t-\hat{t}|) \\ & + f(|x-\hat{x}|) + (t-\hat{t})^2\end{aligned} \tag{2.2.7}$$

then $\varphi(\hat{z}) = \psi(\hat{z})$ and

$$\varphi(z) < \psi(z), \quad \text{for} \quad z \ne \hat{z},\ z = (x,t) \in \mathcal{O}'$$

with

$$\mathcal{O}' = \{(x,t); |x-\hat{x}| < \delta,\ |t-\hat{t}| < \delta\}.$$

The last two terms $f(|x-\hat{x}|)$ and $|t-\hat{t}|^2$ and added to get strict inequality. Since $2f \in \mathcal{F}, \psi \in C^2_F(\mathcal{O}')$ and (2.2.6) holds with $\mathcal{O}$ replaced by $\mathcal{O}'$. Here the property that $\mathcal{F}$ is invariant under positive multiplication is invoked. If the second differential inequality in Definition 2.1.5 holds for $\psi$, i.e., $\psi_t(\hat{z}) \le 0$, so does $\varphi_t(\hat{z}) \le 0$. This explain the reason why we may replace maximum by strict maximum even for $\mathcal{F}$-subsolutions. The proof for supersolutions parallels that for subsolutions. □

**Proposition 2.2.3.** *(i) In Definitions 2.1.1, 2.1.4 the class of test functions may be replaced by $C^k(\mathcal{O}), C^\infty(\mathcal{O})$ or*

$$A^k(\mathcal{O}) = \{\varphi(x,t) = b(x) + g(t) \in C^k(\mathcal{O})\}$$

*where $k \ge 2$. In Definition 2.1.5 $C^2_F(\mathcal{O})$ may be replaced by*

$$A^k_F(\mathcal{O}) = \{\varphi(x,t) = b(x) + g(t) \in C^2_F(\mathcal{O}), g \in C^k(\mathbf{R})\}$$

*where $k \ge 1$ provided that $\mathcal{F}$ is invariant under positive multiplication.*
*(ii) The classes of test functions $C^2(\mathcal{O})$ and $C^2_F(\mathcal{O})$ may be replaced also by $C^{2,1}(\mathcal{O})$ and*

$$C^{2,1}_F(\mathcal{O}) = \text{the set of } \varphi \in C^{2,1}(\mathcal{O}) \text{ that satisfies (2.1.12),}$$

*where $C^{2,1}(\mathcal{O})$ is the space of $\varphi$ whose derivatives $\varphi_t, \nabla^2\varphi, \nabla\varphi$ are continuous in $\mathcal{O}$. Of course for $C^{2,1}_F$ the class $\mathcal{F}$ should be invariant under a positive multiplication.*

**Remark 2.2.4.** It is sometimes convenient to extend a class of test functions other than $C^2$ for continuous $F$ in Definition 2.1.1. For example one would like to consider Sobolev spaces $W^{2,1}_p(\mathcal{O})$ as a class of test functions. Here $W^{2,1}_p(\mathcal{O})$ denotes the space of $\varphi \in L^p(\mathcal{O})$ that satisfies $\nabla^2\varphi \in L^p(\mathcal{O})$ and $\varphi_t \in L^p(\mathcal{O})$. By the Sobolev embedding for large $p$, say $p > N+1$, $W^{2,1}_p(\mathcal{O}) \subset C(\mathcal{O})$ so that the value of $\varphi$ at each point of $\mathcal{O}$ is meaningful. However, $\nabla^2\varphi$ may not be continuous, it is merely $p$-th integrable measurable function and the value of $\nabla^2\varphi$ at each point of $\mathcal{O}$ is only determined up to measure zero set in $\mathcal{O}$. The condition (i)(b) in Definition 2.1.1 should be interpreted as follows. If $(\varphi, \hat{z}) \in W_p^{2,1}(\mathcal{O}) \times \mathcal{O}$ satisfies

$$\varphi_t(z) + F(z, u^*(z),\ \nabla\varphi(z),\ \nabla^2\varphi(z)) \geq \varepsilon > 0$$

for some $\varepsilon > 0$ in some neighborhood of $\hat{z}$, then $u^* - \varphi$ does not attain its maximum over $\mathcal{O}$ at $\hat{z}$. It is immediate that for $\varphi \in C^2(\mathcal{O})$ this is equivalent to (i)(b). Such type of definition is important to study regularity theory for fully nonlinear equations.

To prove (i) we approximate a test function $\varphi$ by a smoother function $\varphi_\varepsilon$. However, the maximum point $z_\varepsilon$ of $u^* - \varphi_\varepsilon$ may be different from the maximum point $\hat{z}$ of $u^* - \varphi$ so we need to study the behavior of $z_\varepsilon$ as $\varepsilon \to 0$. The next general lemma is useful not only to prove Proposition 2.2.3 but also to prove stability results.

## 2.2.2 Convergence of maximum points

**Lemma 2.2.5.** *Let $X$ be a metric space. For $\varepsilon > 0$ let $U_\varepsilon$ be an upper semicontinuous function on $X$ with values in $\mathbf{R} \cup \{-\infty\}$. Let $\overline{U}$ be a function on $X$ defined by $\overline{U} = \limsup^*_{\varepsilon\to 0} U_\varepsilon$. Let $B$ and $S$ be compact sets in $X$. Assume that $S$ is included in the interior of $B$, i.e., $S \subset$ int $B$ in the topology of $X$. Assume that $\overline{U}$ equals a constant $M$ on $S$ and that $\overline{U}(z) < M$ for $z \in B\backslash S$. (In other words $\overline{U}$ takes 'strict' maximum over $B$ modulo points of $S$.) Let $S_\varepsilon$ be the set of maximum points of $U_\varepsilon$ on $B$, i.e.,*

$$S_\varepsilon = \{z \in B;\ U_\varepsilon(z) = \max_B U_\varepsilon\}.$$

*(The set $S_\varepsilon$ is nonempty and $U_\varepsilon(z_\varepsilon) < \infty$ for $z_\varepsilon \in S_\varepsilon$ since $U_\varepsilon$ is upper semicontinuous and $B$ is compact.) Then there exists subsequence $\varepsilon(j) \to \infty$ (as $j \to \infty$) such that*

$$\mathit{Limsup}_{j\to\infty} S_{\varepsilon(j)} \subset S$$

*in the sense that for each $r > 0$ there exists $j_1$ satisfying*

$$S_{\varepsilon(j)} \subset B_r(S) = \{z \in B; d(z, S) \leq r\} \quad \textit{for all} \quad j \geq j_1.$$

*(In particular, $S_{\varepsilon(j)} \subset intB$ for sufficiently large $j$ since $S \subset intB$.) Moreover, for any $z_{\varepsilon(j)} \in S_{\varepsilon(j)}$*

$$\lim_{j\to\infty} U_{\varepsilon(j)}(z_{\varepsilon(j)}) = M.$$

*Proof.* 1. We may assume that $U_\varepsilon \not\equiv -\infty$ on $B$ by taking a subsequence if necessary. By definition of $\overline{U}$ for each $\zeta \in S$ there exist a subsequence $\{U_{\varepsilon(j)}\}_{j=1}^\infty$ and a sequence $\zeta_j \in X$ converging to $\zeta$ as $j \to \infty$ that satisfies

$$M = \overline{U}(\zeta) = \lim_{j\to\infty} U_{\varepsilon(j)}(\zeta_j).$$

2. For a sequence $\{z_{\varepsilon(j)}\}(z_{\varepsilon(j)} \in S_{\varepsilon(j)})$ let $A$ denote the set of its accumulation points, i.e.,

$$A = \{z \in B; \text{ there is a subsequence } \{z_{\varepsilon(j(k))}\} \text{ of } \{z_{\varepsilon(j)}\} \text{ that satisfies } z_{\varepsilon(j(k))} \to z \text{ as } k \to \infty\}.$$

The set $A$ is a non-empty compact set since $B$ is compact.

If $z_{\varepsilon(j(k))} \to z$, then by definition of $\overline{U}$, $\limsup_{k\to\infty} U_{\varepsilon(j(k))}(z_{\varepsilon(j(k))}) \le \overline{U}(z) \le \sup_A \overline{U}$; the last equality is trivial since $z \in A$ by definition of $A$. Since any subsequence limit of $\{z_{\varepsilon(j)}\}$ belongs to $A$, we have

$$\limsup_{j\to\infty} U_{\varepsilon(j)}(z_{\varepsilon(j)}) \le \sup_A \overline{U}.$$

3. Since $\zeta_j$ converges to a maximum point $\zeta \in S$ of $\overline{U}$ over $B$ and $S$ is contained in int $B$, we observe $\zeta_j \in B$ for sufficiently large $j$, say $j \ge j_0$. Since

$$U_{\varepsilon(j)}(\zeta_j) \le U_{\varepsilon(j)}(z_{\varepsilon(j)})$$

for $j \ge j_0$, Steps 1 and 2 yield

$$\begin{aligned} M = \lim_{j\to\infty} U_{\varepsilon(j)}(\zeta_j) &\le \liminf_{j\to\infty} U_{\varepsilon(j)}(z_{\varepsilon(j)}) \\ &\le \limsup_{j\to\infty} U_{\varepsilon(j)}(z_{\varepsilon(j)}) \le \sup_A \overline{U}. \end{aligned}$$

Since $M \ge \sup_A \overline{U}$ is trivial, so we get

$$M = \sup_A \overline{U} = \lim_{j\to\infty} U_{\varepsilon(j)}(z_{\varepsilon(j)}).$$

Since $S$ is the set of all maximum points of $\overline{U}$ in $B$, $S$ includes $A$. This inclusion implies that

$$\text{Limsup}_{j\to\infty} S_{\varepsilon(j)} \subset S. \quad \square$$

**Remark 2.2.6.** (i) $U_\varepsilon$ converges to $\overline{U}$ locally uniformly in a neighborhood of $S$ then we need not take a subsequence $\varepsilon(j)$. If, moreover, $S$ consists of only one point $\overline{z}$, so that $\overline{U}$ attains a strict local maximum at $\overline{z}$, then $z_\varepsilon$ converges to $\overline{z}$ and $U_\varepsilon(z_\varepsilon) \to U(\overline{z})$ *without* taking subsequences.

Indeed, if $U_\varepsilon$ converges to $\overline{U}$ locally uniformly in $B_r(\overline{z})$, for any sequence $\zeta_\varepsilon \to \overline{z}$ the formula

$$M = \overline{U}(\overline{z}) = \lim_{\varepsilon\to 0} U_\varepsilon(\zeta_\varepsilon)$$

is valid without taking any subsequence of $U_\varepsilon$. This is stronger than Step 1. We argue as in Step 2, 3 but $z_\varepsilon, U_\varepsilon$ replaces $z_{\varepsilon(j)}, U_{\varepsilon(j)}$ respectively. Since $A$ is contained in $S = \{\overline{z}\}$, $z_\varepsilon$ converges to $\overline{z}$ without taking a subsequence.

This simple version is also important to develop the theory of viscosity solutions, for example, in proving Proposition 2.2.3.

(ii) Lemma 2.2.5 is a trivial modification of [Barles (1994), lemma 4.2] where $S$ is assumed to be a singleton. An essentially same proof is presented there. However, it seems that [p.90, line 6] needs a further explanation, which is included in our proof. The inequality in [p.90, line 6]

$$\overline{v}(y) \leq \limsup v_{\varepsilon'}(x_{\varepsilon'}) \text{ for } y \in \overline{\Omega} \cap B_r(x)$$

may not be true even for $y = x$ unless one first choose subsequence of $v_\varepsilon$ such that $\overline{v}(x) = \lim_{j\to\infty} v_{\varepsilon(j)}(z_j)$ for some $z_j \to x$ and then take further subsequences $v_{\varepsilon'}$ and $x_{\varepsilon'}$. In our terminology, we should first take subsequence $U_{\varepsilon(j)}$ as in Step 1 of our proof.
(iii) Similar statements for lower semicontinuous $U_\varepsilon$ (with values in $\mathbf{R}\cup\{+\infty\}$) is obtained by replacing $\overline{U}$ by $\underline{U} = \liminf_* U_\varepsilon$, $\overline{U}(z) < M$ by $\underline{U}(z) > M$ and a maximum point by a minimum point. This assertion follows by Lemma 2.2.5 by replacing $U_\varepsilon$ by $-U_\varepsilon$. Also, we may replace a maximum point by a minimum point in Remark 2.2.6 (i).

### 2.2.3 Applications

**Proof of Proposition 2.2.3.** (i) As in Proposition 2.2.2 we may interpret the maximum in (2.1.6) as a local maximum for each class of test functions. To show the first part it suffices to prove (2.1.7) (with lower semicontinuous $F$) for $\varphi \in C^2(\mathcal{O})$ satisfying (2.1.6) by assuming that (2.1.7) holds for all $z \in \mathcal{O}$ and all $\psi \in \cap_{k=2}^{\infty} A^k(\mathcal{O})$ such that $u^* - \psi$ takes a local maximum at $z$. Since $\varphi$ is $C^2$ there is a modulus $\omega_0$ that satisfies

$$\begin{aligned}\Phi = \varphi(x,t) \quad & -\varphi(\hat{x},\hat{t}) - \varphi_t(\hat{x},\hat{t})(t-\hat{t}) - \langle \nabla\varphi(\hat{x},\hat{t}), x-\hat{x}\rangle \\ & -\frac{1}{2}\langle \nabla^2\varphi(\hat{x},\hat{t})(x-\hat{x}), x-\hat{x}\rangle \\ & \leq \omega_0(|x-\hat{x}|^2 + |t-\hat{t}|)(|x-\hat{x}|^2 + |t-\hat{t}|)\end{aligned}$$

for $(x,t)$ sufficiently close to $(\hat{x},\hat{t})$. The rightest hand side is dominated from above by

$$\begin{array}{ll} 2\omega_0(2|x-\hat{x}|^2)|x-\hat{x}|^2 & \text{for } |x-\hat{x}|^2 \geq |t-\hat{t}|, \\ 2\omega_0(2|t-\hat{t}|^2)|t-\hat{t}|^2 & \text{for } |x-\hat{x}|^2 \leq |t-\hat{t}|. \end{array}$$

The leftest hand side $\Phi$ is now dominated by

$$\Phi \leq \overline{\omega_0}(|x-x|^2)|x-\hat{x}|^2 + \overline{\omega_0}(|t-\hat{t}|)|t-\hat{t}|$$

with $\overline{\omega_0}(\sigma) = 2\omega_0(2\sigma)$. Applying Lemma 2.1.9 (ii) yields the existence of $\theta_1 \in C^1[0,\infty)$ and $\theta_2 \in C^2[0,\infty)$ with $\theta_k^{(j)}(0) = 0$ for $0 \leq j \leq k$ that satisfies

$$\Phi \leq \theta_2(|x-\hat{x}|) + \theta_1(|t-\hat{t}|)$$

for $(x,t)$ sufficiently close to $(\hat{x},\hat{t})$. If we set

$$\psi_0(x,t) = \varphi - \Phi + \theta_2(|x-\hat{x}|) + \theta_1(|t-\hat{t}|) + |x-\hat{x}|^4 + \ |t-\hat{t}|^2$$

then $\psi_0$ is a separable function

$$\begin{aligned}\psi_0(x,t) = &\ b(x) + g(t),\\ b(x) = &\ \langle \nabla\varphi(\hat{x},\hat{t}),\ x-\hat{x}\rangle + \tfrac{1}{2}\langle \nabla^2\varphi(\hat{x},\hat{t}),\ x-\hat{x}\rangle\\ &+\theta_2(|x-\hat{x}|) + |x-\hat{x}|^4,\\ g(t) = &\ \varphi_t(\hat{x},\hat{t})(t-\hat{t}) + \theta_1(|t-\hat{t}|) + |t-\hat{t}|^2,\end{aligned}$$

and $b \in C^2(\mathbf{R}^N)$, $g \in C^1(\mathbf{R})$. Since $\varphi < \psi_0$ near $(\hat{x},\hat{t})$ (except $(\hat{x},\hat{t})$) and $\varphi(\hat{x},\hat{t}) = \psi_0(\hat{x},\hat{t})$, $u^* - \psi_0$ takes its local strict maximum at $\hat{z} = (\hat{x},\hat{t})$ with $\nabla\varphi(\hat{x},\hat{t}) = \nabla b(\hat{x})$, $\nabla^2\varphi(\hat{x},\hat{t}) = \nabla^2 b(\hat{x})$, $\varphi_t(\hat{x},\hat{t}) = g'(\hat{t})$.

We approximate $b$ by a smooth function $b_\varepsilon$ so that $b_\varepsilon$ converges to $b$ locally uniformly together with its derivatives up to second order (as $\varepsilon \to 0$). We approximate $g$ by a smooth function so that $g_\varepsilon \to g$ locally uniformly with its derivative. We now apply Lemma 2.2.5 or its simpler version Remark 2.2.6 (i) to $u^* - b_\varepsilon - g_\varepsilon$ with $X = \mathcal{O}, S = \{\hat{z}\}$ to get there is a sequence $z_\varepsilon = (x_\varepsilon, t_\varepsilon)$ converging to $\hat{z}$ such that

$$u^* - b_\varepsilon - g_\varepsilon \le (u^* - b_\varepsilon - g_\varepsilon)(z_\varepsilon) \quad \text{near} \quad z_\varepsilon, \tag{2.2.8}$$

$$\lim_{n\to\infty} (u^* - b_\varepsilon - g_\varepsilon)(z_\varepsilon) = (u^* - b - g)(\hat{z}). \tag{2.2.9}$$

Since $b_\varepsilon \to b$, $g_\varepsilon \to g$ uniformly (together with its derivatives), we get

$$\left\{\begin{array}{rl} b_\varepsilon(x_\varepsilon) \to b(\hat{x}), & g_\varepsilon(t_\varepsilon) \to g(\hat{t}),\\ \nabla b_\varepsilon(x_\varepsilon) \to \nabla b(\hat{x}), & g'_\varepsilon(t_\varepsilon) \to g'(\hat{t}),\\ \nabla^2 b_\varepsilon(x_\varepsilon) \to \nabla^2 b(\hat{x}) & \text{with} \quad z_\varepsilon = (x_\varepsilon, t_\varepsilon).\end{array}\right.$$

By (2.2.9) this implies

$$u^*(z_\varepsilon) \to u^*(\hat{z}). \tag{2.2.11}$$

Since (2.2.8) holds, we apply (2.1.7) at $z_\varepsilon$ to get

$$g'_\varepsilon(t_\varepsilon) + F(z_\varepsilon,\ u^*(z_\varepsilon),\ \nabla b_\varepsilon(z_\varepsilon),\ \nabla^2 b_\varepsilon(z_\varepsilon)) \le 0. \tag{2.2.12}$$

Since $F$ is lower semicontinuous, sending $\varepsilon$ to 0 yields, by (2.2.10) and (2.2.11),

$$g'(\hat{t}) + F(\hat{z},\ u^*(\hat{z}),\ \nabla b(\hat{x}),\ \nabla^2 b(\hat{x})) \le 0,$$

which is

$$\varphi_t(\hat{t}) + F(\hat{z},\ u^*(\hat{z}),\ \nabla\varphi(\hat{z}),\ \nabla^2\varphi(\hat{z})) \le 0.$$

This is what we want to prove. The proof for supersolutions is similar so is omitted.

It remains to study whether test functions in Definition 2.1.5 may be replaced by $A_F^k(\mathcal{O})$. It suffices to prove $\varphi_t(\hat{z}) \le 0$ for $\varphi \in C_F^2(\mathcal{O})$ satisfying (2.1.6) when $\nabla\varphi(\hat{z}) = 0$ by assuming that differential inequalities in Definition 2.1.5 holds for all $z \in \mathcal{O}$ and all $\psi \in \cap_{k=z}^{\infty} A_F^k(\mathcal{O})$ such that $u^* - \psi$ takes a local maximum at $z$. The basic idea of the proof is the same as above. We take $\psi_0 = \psi$ as defined in (2.2.7) and observe that $u^* - \psi_0$ takes a strict local maximum at $\hat{z}$ and that $\psi_0$ is of form

$$\psi_0(x,t) = f(|x - \hat{x}|) + g(t)$$

with some $f \in \mathcal{F}$ and $g \in C^1(\mathbf{R})$ (Note that $af \in \mathcal{F}, a > 0$). We approximate $g$ by $g_\varepsilon \in C^\infty(\mathbf{R})$ as before. (Approximation of $f \in \mathcal{F}$ by $f_\varepsilon \in C^\infty(0,\infty) \cap \mathcal{F}$ may be impossible in this generality so we only approximate $g$.) As before we get (2.2.8)-(2.2.11), where we set $f_\varepsilon(x) = f(|x - \hat{x}|) = f(x)$ by abuse of notation. By differential inequalities in Definition 2.1.5 at $z_\varepsilon$ we get (2.2.12) if $z_\varepsilon \neq \hat{z}$, and $g'_\varepsilon(t_\varepsilon) \leq 0$ if $z_\varepsilon = \hat{z}$. Sending $m$ to $\infty$ we obtain $g'(\hat{t}) = \varphi_t(\hat{t}, \hat{x}) \leq 0$ by using (2.1.11) and convergence results (2.2.10), (2.2.11). This is what we want to prove.
(ii) It suffices to prove that $\varphi \in C^{2,1}(\mathcal{O})$ (resp. $\varphi \in C^{2,1}_F(\mathcal{O})$) satisfy (2.1.7) (resp. differential inequalities in Definition 2.1.5) if $\varphi$ is a test function at $\hat{z}$ of a (resp. $\mathcal{F}$-) subsolution $u$ of (2.1.5). Let $u$ be a subsolution in Definitions 2.1.1 and 2.1.4. As in (i) we may assume that max in (2.1.6) is a local strict maximum and that $\varphi$ is a separable type : $\varphi(x,t) = b(x) + g(t)$ with $b \in C^2(\mathbf{R}^N)$ and $g \in C^1(\mathbf{R})$. We approximate $g$ be $C^2$ function $g_\varepsilon$ so that $g_\varepsilon \to g$ locally uniformly with its first derivative. The rest of the proof is the same as in (i). For $\mathcal{F}$-subsolution we may assume $\nabla\varphi(\hat{z}) = 0$ for $\varphi \in C^{2,1}_F(\mathcal{O})$. By (2.2.7) we may assume that $\varphi$ is a separable type.

$$\varphi(x,t) = f(|x - \hat{x}|) + g(t)$$

with $f \in \mathcal{F},\ g \in C^1(\mathbf{R})$. We do not need to approximate $f$; we only approximate $g$ by $C^2$ function. The rest of the proof is the same as in (i) so it is safely left to the reader. □

**Remark 2.2.7.** From the proof of Proposition 2.2.3 we obtain the following equivalent definition of an $\mathcal{F}$-subsolution of (2.1.5) if $\mathcal{F}$ is invariant under positive multiplication. Let $u : \mathcal{O} \to \mathbf{R}^{\cup}\{-\infty\}$ fulfill $u^*(z) < \infty$ for all $z \in \mathcal{O}$. Then $u$ is an $\mathcal{F}$-subsolution of (2.1.5) in $\mathcal{O}$ if and only if the next two conditions are fulfilled.
(A) $g'(\hat{t}) + F(\hat{z}, u^*(\hat{z}), \nabla b(\hat{z}), \nabla^2 b(\hat{z})) \leq 0$ holds for all $\hat{z} = (\hat{x}, \hat{t}) \in \mathcal{O}$ and for all $\varphi = \varphi(x,t) := b(x) + g(t)$ which is $C^{2,1}$ near $\hat{z}$ provided that $u^* - \varphi$ attains its local strict maximum at $\hat{z}$ and that $\nabla b(\hat{x}) \neq 0$.
(B) $g'(\hat{t}) \leq 0$ holds for all $\hat{z} = (\hat{x}, \hat{t}) \in \mathcal{O}$ and for all

$$\varphi = \varphi(x,t) := f(|x - \hat{x}|) + g(t)$$

with $f \in \mathcal{F}$ and with $g$ which is $C^1$ near $\hat{t}$ provided that $u^* - \varphi$ attains its local strict maximum at $\hat{z}$.

*Proof of Theorem 2.2.1.* We only prove the subsolution case since the case of supersolution follows exactly in the same way.
(i) Since $\overline{u}$ is upper semicontinuous and $\overline{u}(z) < \infty$ for $z \in \mathcal{O}$, our goal is to prove

$$\varphi_t(\hat{z}) + F(\hat{z},\ \overline{u}(\hat{z}),\ \nabla\varphi(\hat{z}),\ \nabla^2\varphi(\hat{z})) \leq 0 \tag{2.2.13}$$

by assuming that $\varphi \in C^2(\mathcal{O})$ and $\hat{z} \in \mathcal{O}$ fulfill

$$\max_{\mathcal{O}}(\overline{u} - \varphi) = (\overline{u} - \varphi)(\hat{z}).$$

By Proposition 2.2.2 we may assume that $\overline{u} - \varphi$ has a strict local maximum at $\hat{z}$, i.e.,

$$(\overline{u} - \varphi)(z) < (u - \varphi)(\hat{z}), \ z \in B\backslash\{\hat{z}\}$$

for some compact neighborhood of $\hat{z}$.

We set $U_\varepsilon = u_\varepsilon^* - \varphi$ and observe that

$$\overline{U} = \limsup_{\varepsilon \to 0}{}^* \ U_\varepsilon = \overline{u} - \varphi.$$

We now apply Lemma 2.2.5 with $X = \mathcal{O}$ and $S = \{\hat{z}\}$ to get a subsequence of $\{U_\varepsilon\}$ (still denoted $\{U_\varepsilon\}$) and a sequence $\{z_\varepsilon\}$ in $\mathcal{O}$ satisfies

$$\begin{aligned} &U_\varepsilon(z) \le U_\varepsilon(z_\varepsilon) \quad \text{for} \ \ z \ \ \text{close to } z_\varepsilon, \\ &z_\varepsilon \to \hat{z} \quad \text{and} \quad U_\varepsilon(z_\varepsilon) \to \overline{U}(\hat{z}). \end{aligned} \tag{2.2.14}$$

Since $u_\varepsilon$ is a subsolution of (2.2.2) and $\varphi$ is a test function of $u_\varepsilon$ at $z_\varepsilon$, we obtain, by Proposition 2.2.2, that

$$\varphi_t(z_\varepsilon) + F_\varepsilon(z_\varepsilon \ u_\varepsilon^*(z_\varepsilon), \ \nabla\varphi(z_\varepsilon), \ \nabla^2\varphi(z_\varepsilon)) \le 0. \tag{2.2.15}$$

Sending $\varepsilon$ to 0 yields (2.2.13), since (2.2.1) holds and $z_\varepsilon \to \hat{z}, \ u_\varepsilon^*(z_\varepsilon) \to \overline{u}(\hat{z})$.

(ii) By Proposition 2.2.2 our goal is to prove differential inequalities in Definition 2.1.5 by assuming that $\overline{u} - \varphi$ has a strict local maximum at $\hat{z}$ for $\varphi \in C^2_F(\mathcal{O})$. By (i) we may assume $\nabla\varphi(\hat{z}) = 0$. We may take $\psi$ in (2.2.7) instead of $\varphi$.

Arguing as in (i), we observe that there is a subsequence of $u_\varepsilon$ (still denoted $u_\varepsilon$) and $z_\varepsilon \in \mathcal{O}$ satisfying (2.2.14). Since $u_\varepsilon$ is an $\mathcal{F}_\varepsilon$-subsolution of (2.2.2) we have (2.2.15) if $\nabla\psi(z_\varepsilon) \ne 0$ (i.e. $z_\varepsilon \ne \hat{z}$) and $\psi_t(z_\varepsilon) \le 0$ if $\nabla\psi(z_\varepsilon) = 0$. We send $\varepsilon$ to 0 and use (2.2.4) with $z_\varepsilon \to \hat{z}$, $u_\varepsilon^*(z_\varepsilon) \to \overline{u}(\hat{z})$ to get $\varphi_t(\hat{z}) = \psi_t(\hat{z}) \le 0$. □

We conclude this section by proving the equivalence of $\mathcal{F}$-subsolutions and subsolutions when (2.1.15) is fulfilled.

**Proposition 2.2.8.** *Assume that (2.1.15) holds for $F$ and that $F$ is degenerate elliptic. A function $u : \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ is an $\mathcal{F}$-subsolution of (2.1.5) (in $\mathcal{O}$) if and only $u$ is a subsolution of (2.1.5).*

*Proof.* Since a subsolution is always an $\mathcal{F}$-subsolution under (2.1.15) it suffices to prove that an $\mathcal{F}$-subsolution is a subsolution. Assume that $(\varphi, \hat{z}) \in C^2(\mathcal{O}) \times \mathcal{O}$ satisfies

$$\max_{\mathcal{O}}(u - \varphi) = (u - \varphi)(\hat{z}).$$

We may assume that $u-\varphi$ takes its strict maximum at $\hat{z}=(\hat{x},\hat{t})$ over $\mathcal{O}$ and $\nabla\varphi(\hat{z})=0$. For $\varepsilon>0$ we set

$$\begin{aligned}\Phi_\varepsilon(x,y,t) &= u(x,t)-\psi_\varepsilon(x,y,t),\\ \psi_\varepsilon(x,y,t) &= |x-y|^4/4\varepsilon+\varphi(y,t)\end{aligned}$$

and observe that

$$\overline{\Phi}(x,y,t)=(\limsup_{\varepsilon\to 0}{}^*\ \Phi_\varepsilon)(x,y,t)=\begin{cases} u(x,t)-\varphi(x,t) & \text{if} \quad x=y,\\ -\infty & \text{if} \quad x\neq y\end{cases}$$

for $(x,t),(y,t)\in\mathcal{O}$. Since $\overline{\Phi}$ takes its strict maximum at $(\hat{x},\hat{x},\hat{t})$, by convergence of maximum points (Lemma 2.2.5 and Remark 2.2.6 (i)) a maximizer $(x_\varepsilon,\ y_\varepsilon,\ t_\varepsilon)$ of $\Phi_\varepsilon$ converges to $(\hat{x},\hat{x},\hat{t})$ as $\varepsilon\to 0$. Since $\Phi_\varepsilon(x_\varepsilon,\ y,\ t_\varepsilon)$ takes its maximum at $y=y_\varepsilon$, we see

$$\nabla_y\ \psi_\varepsilon(x_\varepsilon,\ y_\varepsilon,\ t_\varepsilon)=0,\ \nabla_y^2\ \psi(x_\varepsilon,\ y_\varepsilon,\ t_\varepsilon)\geq O,$$

where $\nabla_y$ denotes the partial gradient in $y$. Since $\nabla|x|^4=4|x|^2x$, $\nabla^2|x|^4=4|x|^2\ I_N+8x\otimes x$, this is equivalent to say

$$-|x_\varepsilon-y_\varepsilon|^2(x_\varepsilon-y_\varepsilon)/\varepsilon+\nabla\varphi(y_\varepsilon,\ t_\varepsilon)=0, \tag{2.2.16}$$

$$|x_\varepsilon-y_\varepsilon|^2\ I_N+2(x_\varepsilon-y_\varepsilon)\otimes(x_\varepsilon-y_\varepsilon)+\varepsilon\nabla^2\varphi(y_\varepsilon,\ t_\varepsilon)\geq O \tag{2.2.17}$$

where $I_N$ denotes the $N\times N$ unit matrix. We note that $\psi_\varepsilon(\cdot,y_\varepsilon,\cdot)$ belongs to $\mathcal{A}_0$ as an function of $(x,t)$. This is equivalent to say that $\psi_\varepsilon(\cdot,y_\varepsilon,\cdot)\in C_F^2(\mathcal{O})$ by Proposition 2.1.8.

**Case A.** $\nabla\varphi(y_\varepsilon,\ t_\varepsilon)\neq 0$ for a sequence $\varepsilon=\varepsilon_j\to 0$.
Since $(x_\varepsilon,\ t_\varepsilon)$ is a maximum point of a function

$$(x,t)\mapsto u(x,t)-|x_\varepsilon-y_\varepsilon|^4/4\varepsilon-\varphi(x-(x_\varepsilon-y_\varepsilon),t)$$

for $\varepsilon=\varepsilon_j$ and since $u$ is an $\mathcal{F}$-subsolution, we have

$$\varphi_t(y_\varepsilon,\ t_\varepsilon)+F(x_\varepsilon,\ t_\varepsilon,\nabla\varphi(y_\varepsilon,\ t_\varepsilon),\nabla^2\varphi(y_\varepsilon,\ t_\varepsilon))\leq 0$$

for $\varepsilon=\varepsilon_j$. Sending $\varepsilon_j$ to zero yields

$$\varphi_t(\hat{x},\hat{t})+F_*(\hat{x},\hat{t},\nabla\varphi(\hat{x},\hat{t}),\nabla^2\varphi(\hat{x},\hat{t}))\leq 0.$$

**Case B.** $\nabla\varphi(y_\varepsilon,t_\varepsilon)=0$ for sufficiently small $\varepsilon>0$.
This condition implies $x_\varepsilon=y_\varepsilon$ so that $\nabla_x\psi_\varepsilon(x_\varepsilon,\ y_\varepsilon,\ t_\varepsilon)=0$, $\nabla_x^2\ \psi_\varepsilon(x_\varepsilon,\ y_\varepsilon,\ t_\varepsilon)=O$. Since $\Phi_\varepsilon(x,\ y_\varepsilon,\ t)$ takes its maximum at $(x_\varepsilon,\ t_\varepsilon)$ as a function of $(x,t)$, definition of an $\mathcal{F}$-subsolution (Proposition 2.1.7) yields

$$\partial_t\ \psi_\varepsilon(x_\varepsilon,\ y_\varepsilon,\ t_\varepsilon)\leq 0.$$

Sending $\varepsilon$ to zero yields

$$\varphi_t(\hat{x},\hat{t})\leq 0.$$

Since $x_\varepsilon = y_\varepsilon$, (2.2.16) and (2.2.17) yields

$$\nabla\varphi(y_\varepsilon,\ t_\varepsilon) = 0,\ \nabla^2\varphi(y_\varepsilon,\ t_\varepsilon) \geq O.$$

Sending $\varepsilon$ to zero again we see

$$\nabla\varphi(\hat{x},\hat{t}) = 0,\ \nabla^2\varphi(\hat{x},\hat{t}) \geq O.$$

Since $F$ is degenerate elliptic

$$\begin{aligned}\varphi_t(\hat{x},\hat{t}) + F_*(\hat{x},\hat{t},\nabla\varphi(\hat{x},\hat{t}),\nabla^2\varphi(\hat{x},\hat{t})) &\leq 0 + F_*(\hat{x},\hat{t},0,\nabla^2\varphi(\hat{x},\hat{t})) \\ &\leq 0 + F_*(\hat{x},\hat{t},0,O) = 0 \quad \text{by (2.1.15).} \quad \square\end{aligned}$$

## 2.3 Boundary value problems

So far we discuss viscosity solutions in an open set. We seek a natural definition of solutions for the boundary value problem. It is desirable that solutions satisfy stability principle. Suppose that $F_\varepsilon \in C^0(W)$ converges to $F$ uniformly on every compact set in $W$, where $W = \overline{\Omega} \times [0,T] \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$. Suppose that compact set with $J = \partial\Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$. Suppose that $u_\varepsilon$ solves

$$\begin{aligned} u_t + \ & F_\varepsilon(z,u,\nabla u,\nabla^2 u) \leq 0 \quad \text{in} \quad \Omega \times (0,T), \\ & B_\varepsilon(z,u,\nabla u,\nabla^2 u) \leq 0 \quad \text{on} \quad \partial\Omega \times (0,T)\end{aligned}$$

in a classical sense. What problem does $u = \limsup^*_{\varepsilon\to 0}\ u_\varepsilon$ solve (when $u(z) < \infty$ for all $z \in \overline{\Omega} \times (0,T)$)? We consider a function $\varphi \in C^2(\overline{\Omega} \times (0,T))$ such that $u - \varphi$ attains a strict maximum over $\overline{\Omega} \times (0,T)$ at $(\hat{t},\hat{x})$ with $\hat{x} \in \partial\Omega$, $0 < \hat{t} < T$. By Remark 2.2.6 (i) there is a sequence $z_\varepsilon = (t_\varepsilon, x_\varepsilon) \to (\hat{t},\hat{x})$ that satisfies

$$\max_{\overline{\Omega}\times(0,T)} (u_\varepsilon - \varphi) = (u_\varepsilon - \varphi)(t_\varepsilon, x_\varepsilon).$$

Since $u_\varepsilon$ solves (2.2.14), if $x_\varepsilon \in \partial\Omega$ then we expect

$$B_\varepsilon(z_\varepsilon,\ u_\varepsilon(z_\varepsilon),\ \nabla\varphi(z_\varepsilon),\ \nabla^2\varphi(z_\varepsilon)) \leq 0$$

under suitable monotonicity condition of $B_\varepsilon$ (Proposition 2.3.3). If $x_\varepsilon \in \Omega$ then

$$\varphi_t(z_\varepsilon) + F_\varepsilon(z_\varepsilon,\ u_\varepsilon(z_\varepsilon),\ \nabla\varphi(z_\varepsilon),\ \nabla^2\varphi(z_\varepsilon)) \leq 0$$

(under ellipticity condition of $F_\varepsilon$). Unfortunately, for sufficiently small $\varepsilon$, $x_\varepsilon$ may be interior point of $\Omega$ (Example 2.3.6), so as $\varepsilon \to 0$ the best we expect is either

$$\varphi_t(\hat{z}) + F(\hat{z},\ u(\hat{z}),\ \nabla\varphi(\hat{z}),\ \nabla^2\varphi(\hat{z})) \leq 0$$

*or*

$$B(\hat{z},\ u(\hat{z}),\ \nabla\varphi(z),\ \nabla^2\varphi(z)) \le 0$$

In other words

$$E(\hat{z},\ u(\hat{z}),\ \varphi_t(\hat{z}),\ \nabla\varphi(\hat{z}),\ \nabla^2\varphi(\hat{z})) \le 0$$

for $E(z,r,\tau,p,X) = (\tau + F(z,r,p,X)) \wedge B(z,r,p,X)$ where $a \wedge b = \min\{a,b\}$. This observation shows that the next general definition for viscosity solution is useful.

**Definition 2.3.1.** (i) Let $\mathcal{O}$ be a locally compact subset of $\mathbf{R}^d$ (so that every neighborhood of a point $\hat{z}$ of $\mathcal{O}$ includes some compact neighborhood of $\hat{z}$ in $\mathcal{O}$). Let $E$ be a lower (resp. upper) semicontinuous function on $\overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$ with values in $\mathbf{R} \cup \{-\infty\}$. (resp. $\mathbf{R} \cup \{+\infty\}$) A function $u : \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ (resp. $\mathbf{R} \cup \{+\infty\}$) is a (viscosity) *subsolution* (resp. supersolution) of

$$E(z,u,Du,D^2u) = 0 \tag{2.3.1}$$

in $\mathcal{O}$ if

(a) $u^*(z) < \infty$ (resp. $u_*(z) > -\infty$) for all $z \in \mathcal{O}$

(b) If $(\varphi,\hat{z}) \in C^2(\mathcal{O}) \times \mathcal{O}$ satisfies

$$\begin{aligned} &\max\nolimits_{\mathcal{O}}(u^* - \psi) = (u^* - \varphi)(\hat{z}) \\ &(\text{resp. } \min_{\mathcal{O}}(u^* - \varphi) = (u_* - \varphi)(\hat{z})), \end{aligned} \tag{2.3.2}$$

then

$$\begin{aligned} &E(\hat{z},u^*(\hat{z}),D\varphi(\hat{z}),D^2\varphi(\hat{z})) \le 0 \\ &(\text{resp. } E(\hat{z},u_*(\hat{z}),D\varphi(\hat{z}),D^2\varphi(\hat{z})) \ge 0), \end{aligned} \tag{2.3.3}$$

where $D\varphi = (\partial\varphi/\partial x_i)_{i=1}^d$, $D^2\varphi = (\partial^2\varphi/\partial x_i\partial x_j)_{1\le i,j\le d}$. Here $C^k(\mathcal{O})$ denotes the set of $C^k$ functions in some neighborhood of $\mathcal{O}$ in $\mathbf{R}^d$.

If $E$ is defined in a dense subset of $\overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$ (with $E_* < +\infty, E^* > -\infty$ on $\overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$) then we say that $u$ is a subsolution (resp. supersolution) of (2.3.1) in $\mathcal{O}$ if $u$ is a subsolution (resp. supersolution) of $E_*(z,u,Du,D^2u) = 0$ (resp. $E^*(z,u,Du,D^2u) = 0$) in $\mathcal{O}$.

(ii) Let $\Omega$ be a domain in $\mathbf{R}^N$ and $T > 0$. Let $F$ be a lower semicontinuous in $W = \overline{\Omega} \times [0,T] \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ with values in $\mathbf{R} \cup \{-\infty\}$. Let $B$ be lower semicontinuous in $J = \partial\Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ with values in $\mathbf{R} \cup \{-\infty\}$. Let $\mathcal{O}$ be an open set in $\overline{\Omega} \times (0,T)$. (For example we may take $\mathcal{O} = \overline{\Omega} \times (0,T)$). A function $u : \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ (resp. $\mathbf{R} \cup \{+\infty\}$) is a *subsolution* (resp. supersolution)

$$\begin{aligned} u_t + \ & F(z,u,\nabla u,\nabla^2 u) = 0 \quad \text{in} \quad \mathcal{O} \cap \Omega \times (0,T) \\ & B(z,u,\nabla u,\nabla^2 u) = 0 \quad \text{on} \quad \mathcal{O} \cap \partial\Omega \times (0,T) \end{aligned} \tag{2.3.4}$$

if $u$ is a subsolution (resp. supersolution) of

$$E(z,u,u_t,\nabla u,\nabla^2 u) = 0 \quad \text{in} \quad \mathcal{O}$$

in the sense of (i) with $E(z,r,\tau,p,X) = (\tau + F(z,r,p,X)) \wedge B(z,r,p,X)$. Note that $E$ is regarded as a lower semicontinuous function in $\overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$ with $d = N+1$.

**Remark 2.3.2.** By definition $u$ is subsolution of (2.3.4) if and only if
(a) $u^*(z) < \infty$ for $z \in \mathcal{O}$;
(b) If $(\varphi, \hat{z}) \in C^2(\mathcal{O}) \times \mathcal{O}$ satisfies (2.1.6), then (2.1.7) holds for $\hat{z} \in \Omega \times (0,T)$, and for $\hat{z} \in \partial\Omega \times (0,T)$ *either*

$$B(\hat{z}, u^*(\hat{z}),\ \nabla\varphi(\hat{z}),\ \nabla^2\varphi(\hat{z})) \leq 0$$

or (2.1.7) is valid.

The similar equivalent definition is available for supersolutions.

We warn the reader that even for degenerate elliptic $F$ a function $u \in C^2(\mathcal{O})$ satisfying

$$\begin{aligned} u_t + \quad & F(z,u,\nabla u,\nabla^2 u) \leq 0 \quad \text{in} \quad \mathcal{O} \cap (\Omega \times (0,T)) \\ & B(z,u,\nabla u,\nabla^2 u) \leq 0 \quad \text{on} \quad \mathcal{O} \cap (\partial\Omega \times (0,T)) \end{aligned} \tag{2.3.5}$$

pointwise may not be a viscosity subsolution in $\mathcal{O}$ unless we impose an extra condition on $B$. The main reason is that the condition.

$$\max_{\mathcal{O}}(u-\varphi) = (u-\varphi)(\hat{x},\hat{t}) \tag{2.3.6}$$

does not deduce $\nabla u(\hat{x},\hat{t}) = \nabla\varphi(\hat{x},\hat{t})$ if $\hat{x} \in \partial\Omega$. In fact if $\partial\Omega$ is $C^2$ near $\hat{x}$ the condition (2.3.6) implies either (i) or (ii) holds:
(i) $\nabla u(\hat{x},\hat{t}) = \nabla\varphi(\hat{x},\hat{t})$ and $\nabla^2 u(\hat{x},\hat{t}) \leq \nabla^2\varphi(\hat{x},\hat{t})$
(ii) $\nabla u(\hat{x},\hat{t}) = \nabla\varphi(\hat{x},\hat{t}) + \lambda\nu u(\hat{x})$ and

$$R_\nu \nabla^2 u(\hat{x},\hat{t}) R_\nu \leq R_\nu \nabla^2\varphi(\hat{x},\hat{t}) R_\nu - \lambda A_\nu \quad \text{for some} \quad \lambda > 0.$$

Here $\nu(\hat{x})$ is the outward unit normal and $R_\nu = I - \nu(\hat{x}) \otimes \nu(\hat{x})$ ; $A_\nu$ is the $N \times N$ symmetric matrix whose restriction on the tangent space $T_{\hat{x}}\partial\Omega$ is the Weingarten map of $\partial\Omega$ in the direction of $\nu(\hat{x})$ and $A_\nu \nu = 0$. By setting $v(x) = u(x,\hat{t}) - \varphi(x,\hat{t})$ this follows from the boundary version of the classical maximum principle :

**Maximum principle at the boundary.** *Let $\Omega$ be a domain in $\mathbf{R}^N$. Assume that $\partial\Omega$ is $C^2$ near $\hat{x} \in \partial\Omega$. Let $v$ be in $C^2(B_r(\hat{x}) \cap \overline{\Omega})$ with some $r > 0$. Assume that*

$$v(x) \leq v(\hat{x}) \quad \textit{for} \quad x \in B_r(\hat{x}) \cap \Omega.$$

*Then either*
*(i) $\nabla v(\hat{x}) = 0$ and $\nabla^2 v(\hat{x}) \leq 0$, or*
*(ii) $\nabla v(\hat{x}) = \lambda\nu(x)$ and $R_\nu\ \nabla^2 v(\hat{x}) R_\nu + \lambda A_\nu \leq 0$ for $\lambda > 0$ some $\lambda > 0$.*

*Proof.* We may assume $v(\hat{x}) = 0$. By translation and rotation of coordinates we may assume that $\hat{x} = 0$, $\nu(x) = (1,0,\cdots,0)$, $R_\nu =$ diag $(0,1\cdots 1), A\nu =$ diag $(0,\lambda_2,\cdots\lambda_N)$. Since $v$ has a local maximum at 0 over $\overline{\Omega} \cap B_r(0)$,

$$\frac{\partial v}{\partial x_1}(0) = \lambda \geq 0,\ \frac{\partial v}{\partial x_2}(0) = \cdots \frac{\partial v}{\partial x_N}(0) = 0.$$

By expression of $A_\nu$ for graphs, $\Omega$ is expressed as $x_1 < \psi(x_2, \cdots, x_N)$ near 0 with some $\psi \in C^2$ satisfying $\nabla \prime 2\psi(0) = A_\lambda$ with $\nabla' = (\partial_{x_2}, \cdots, \partial_{x_N})$. Since

$$v(x_1, \cdots, x_N) = \lambda x_1 + \frac{1}{2}\Sigma_{j=1}^N \frac{\partial^2 v}{\partial x_i \partial x_j}(0) x_i x_j + o(|x|^2) \quad \text{as} \quad |x| \to 0$$

and $v$ takes maximum 0 over $\overline{\Omega} \cap B_r(0)$ at 0, we see that for $\nabla^2 v(0) \leq O$ if $\lambda = 0$ and that

$$\lambda\psi(x_2, \cdots, x_N) + \frac{1}{2}\langle \nabla^2 v(0)x, x\rangle + o(|x|^2) \leq 0$$

for $\lambda > 0$. Since $\nabla'\psi(0) = 0$ and $\psi(0) = 0$, expanding $\psi$ around zero now yields

$$\lambda \nabla'^2 \psi(0) + \nabla'^2 v(0) \leq O.$$

This yields (ii). □

We give a sufficient condition for $B$ so that classical solution of (2.3.5) is a viscosity solution be restricting $B$ for the first order operators.

**Proposition 2.3.3.** *Let $u \in C^2(\mathcal{O})$ satisfies (2.3.5). Assume that $F$ is degenerate elliptic and that $B$ depends only on $(z, r, p) \in \mathcal{O} \times \mathbf{R} \times \mathbf{R}^N$. Assume that*

$$\lambda \mapsto B(z, r, p - \lambda\nu(x)) \tag{2.3.7}$$

*is nonincreasing in $\lambda \geq 0$. Then $u$ is a (viscosity) subsolution of (2.3.4).*

Indeed, if $\varphi$ satisfies (2.3.6) for $\hat{x} \in \partial\Omega$ so that (i) or (ii) holds, then by the monotonicity assumption on $B$ we have

$$\begin{aligned} B(\hat{z}, u(\hat{z}), \nabla\varphi(\hat{z})) &\leq B(\hat{z}, u(\hat{z}), \nabla\varphi(\hat{z}) + \lambda\nu(\hat{x})) \\ &= B(\hat{z}, u(\hat{z}), \nabla u(\hat{z})) \leq 0. \end{aligned}$$

**Remark 2.3.4.** (i) Under the same monotonicity assumption on $B$ a classical supersolution of (2.3.4) is also a viscosity supersolution.
(ii) As in Proposition 2.2.2 we may replace the maximum by a local maximum, a strict maximum or a local strict maximum in Definition 2.3.1 and the same remark applies to the minimum. The proof is identical with that of Proposition 2.2.2.
(iii) As in Proposition 2.2.3 we may replace class of test functions by $C^k(\mathcal{O})$ or $C^\infty(\mathcal{O})$ for $k \geq 2$ in Definition 2.3.1 (i) and even by $A^k(\mathcal{O})(k \geq 2)$ or $C^{2,1}(\mathcal{O})$ in Definition 2.3.1 (ii). The proof is essentially the same and it is based on Lemma 2.2.5 or Remark 2.2.6 (i).
(iv) As in Theorem 2.2.1 we have a stability result for the boundary value problem. Since the proof is identical with that of Theorem 2.2.1 (i), we only state the results (for subsolutions) which extends Theorem 2.2.1 (i).
(v) By definition localization property in §2.1.1 is still valid for (2.3.1).

**Theorem 2.3.5.** *Let $\mathcal{O}$ be a locally compact subset of $\mathbf{R}^d$.*
*(i) Assume that $E_\varepsilon$ and $E$ are lower semicontinuous on $\overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$ with values in $\mathbf{R} \cup \{-\infty\}$ for $\varepsilon > 0$. Assume that*

$$E \leq \liminf_{\varepsilon \to 0}{}_* E_\varepsilon \quad \textit{on} \quad \overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d.$$

*Assume that $u_\varepsilon$ is a subsolution of*

$$E_\varepsilon(z, u, Du, D^2u) \leq 0 \quad \textit{in} \quad \mathcal{O}.$$

*Then $\overline{u} = \limsup^*_{\varepsilon \to 0} u_\varepsilon$ solves (2.3.1) in $\mathcal{O}$ provided that $\overline{u}(z) < \infty$ for every $z \in \mathcal{O}$.*
*(ii) Under the notation of Definition 2.3.1 (ii) let $F_\varepsilon$ and $B_\varepsilon$ lower semicontinuous on $W$ and $J$ with values in $\mathbf{R} \cup \{-\infty\}$, respectively for $\varepsilon > 0$. Assume that*

$$F \leq \liminf_{\varepsilon \to 0}{}_* F_\varepsilon \quad \textit{in } W \quad \textit{and} \quad B \leq \liminf_{\varepsilon \to 0}{}_* B_\varepsilon \quad \textit{in } J.$$

*Let $u_\varepsilon$ be a subsolution of*

$$\begin{aligned} u_t + \quad & F_\varepsilon(z, u, \nabla u, \nabla^2 u) = 0 \quad \textit{in} \quad \mathcal{O} \cap (\Omega \times (0,T)) \\ & B_\varepsilon(z, u, \nabla u, \nabla^2 u) = 0 \quad \textit{on} \quad \mathcal{O} \cap (\partial\Omega \times (0,T)) \end{aligned}$$

*Then $\overline{u} = \limsup\limits_{\varepsilon \to 0}{}^* u_\varepsilon$ is a subsolution of (2.3.4) if $\overline{u}(z) < \infty$ for every $z \in \mathcal{O} \cap (\overline{\Omega} \times (0,T))$.*
*(The part (ii) is a trivial corollary of part (i)).*

**Example 2.3.6.** We give a simple example that a maximum point $z_\varepsilon$ of $u^*_\varepsilon - \varphi$ is not on $\partial\Omega \times (0,T)$ for all sufficiently small $\varepsilon$ even if $z_\varepsilon$ converges to $\hat{z} \in \partial\Omega \times (0,T)$ which is a maximum point of $\overline{u} - \varphi$ where we use the notation in Theorem 2.3.5 (ii). We set

$$u_\varepsilon(x,t) = \int_0^\infty \{g(t\varepsilon, x-y) + g(t\varepsilon, x+y)\} u_0(y) dy$$

with the Gauss kernel $g(t,x) = (4\pi t)^{-1/2} \exp(-|x|^2/4t)$ for $u_0 \in C^\infty[0,\infty)$ with $u_0'(0) > 0$. Clearly $u_\varepsilon \in C^\infty(\mathcal{O})$ is a classical solution of

$$\begin{aligned} & u_t - \varepsilon u_{xx} = 0 && \text{in} \quad \Omega \times (0,\infty), \\ & u_x = 0 && \text{on} \quad \partial\Omega \times (0,\infty) \end{aligned}$$

with $\Omega = (0,\infty)$ and $\mathcal{O} = \overline{\Omega} \times (0,\infty)$. (By Proposition 2.3.3 $u_\varepsilon$ is also a viscosity solution.) By the representation of $u_\varepsilon$ the limit $\overline{u} = \limsup^*_{\varepsilon \to 0} u_\varepsilon$ equals $u_0$. By the stability results (Theorem 2.3.5) $u_0$ is a viscosity subsolution of

$$\begin{aligned} & u_t = 0 && \text{in} \quad \Omega \times (0,\infty), \\ & u_x = 0 && \text{on} \quad \partial\Omega \times (0,\infty). \end{aligned}$$

However, evidently $u_0$ fails to fulfill $u_x(0) \leq 0$ in the strong sense by $u_{0x}(0) = u_0'(0) > 0$.

We take $\varphi(x,t) = u_0(x) + x^4 + (t - \hat{t})^2$ for $\hat{t} > 0$ so that $u_0 - \varphi$ takes a strict maximum at $(0, \hat{t}) = \hat{z}$ over $\mathcal{O}$. Let $z_\varepsilon$ be a maximum point of $u_\varepsilon - \varphi$ over $\mathcal{O}$. It conveges to $\hat{z}$ as

$\varepsilon \to 0$ by Lemma 2.2.5 and its remark. Suppose that there were subsequence $z_{\varepsilon_i}$ that belongs to $\partial\Omega \times (0,\infty)$. Since $u_{\varepsilon x}(0,t) = 0$, we would obtain $\varphi_x(z_{\varepsilon_i}) \le 0$. Sending $\varepsilon_i \to 0$ we would obtain $\varphi_x(0,\hat{t}) = u_0(0) \le 0$ which is a contradiction.

**Definition 2.3.7.** It is also possible to extend the notion of $\mathcal{F}$-solution for the boundary value problem. Let $F$ be continuous in $W_0 = \overline{\Omega} \times [0,T) \times \mathbf{R} \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$ with values in $\mathbf{R}$. Assume that $F$ fulfills (2.1.11) and $B \in C(J)$. We say $u : \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ is *$\mathcal{F}$-subsolution* of (2.3.4) if
(a) $u^*(z) < \infty$ for $z \in \mathcal{O}$
(b) if $(\varphi, \hat{z}) \in C^2_F(\mathcal{O}) \times \mathcal{O}$ satisfies (2.1.6), then

$$\begin{array}{ll} \varphi_t(\hat{z}) + F(\hat{z},\ u^*(\hat{z}),\ \nabla\varphi(\hat{z}),\ \nabla^2\varphi(\hat{z})) \le 0 & \text{if } \nabla\varphi(\hat{z}) \ne 0, \\ \varphi_t(\hat{z}) \le 0 & \text{otherwise} \end{array} \tag{2.3.8}$$

holds if $\hat{z} \in \Omega \times (0,T)$ and either (2.3.8) or

$$B(\hat{z}, u^*(\hat{z}),\ \nabla\varphi(\hat{z}),\ \nabla^2\varphi(\hat{z})) \le 0$$

if $\hat{z} \in \partial\Omega \times (0,T)$. The definition of $\mathcal{F}$-supersolution is similar so is omitted. Here $C^2_F(\mathcal{O})$ is the set of $\varphi \in C^2(\mathcal{O})$ compatible with $F$, i.e., $\varphi$ satisfies (2.1.12). The stability result Theorem 2.2.1 (ii) extends to the boundary value problem with a trivial modification.

**Example 2.3.8.** We conclude this section by giving examples of boundary condition that satisfies (2.3.7) other than the Dirichlet condition $B \equiv 0$. The operator $B = \langle p, \tau\rangle$ evidently fulfills (2.3.7). In this case the boundary condition $B = 0$ is called the homogeneous Neumann condition. The operator $B = \langle p, \nu\rangle - k(z)|p|$ fulfills (2.3.7) provided that $|k(z)| < 1$. The condition $B = 0$ is an oblique type boundary condition.

## 2.4 Perron's method

We give typical results on Perron's method for constructing solutions for a general equation (2.3.1) rather that for (2.1.5).

**Lemma 2.4.1** (Closedness under supremum). *Let $\mathcal{O}$ be a locally compact subset of $\mathbf{R}^d$. Let $E$ be a lower semicontinuous function on $\overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$ with values in $\mathbf{R} \cup \{-\infty\}$. Let $S$ be a set of subsolutions of*

$$E(z, u, Du, D^2u) = 0 \tag{2.4.1}$$

*in $\mathcal{O}$. Let $u$ be a function in $\mathcal{O}$ defined by*

$$u(z) = \sup\{v(z); v \in S\} \quad \textit{for} \quad z \in \mathcal{O}. \tag{2.4.2}$$

*If $u^*(z) < \infty$ for all $z \in \mathcal{O}$, then $u$ is a subsolution of (2.4.1) in $\mathcal{O}$.*

**Lemma 2.4.2** (Maximal subsolution). *Let $\mathcal{O}$ be as in Lemma 2.4.1. Let $E$ be defined in a dense set of $\overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$ with $E_* < \infty, E^* > -\infty$ everwhere. Assume that the* *equation (2.4.1) is degenerate elliptic (for subsolution) in the sense that all $u \in C^2(\mathcal{O}')$ satisfying*

$$E_*(z, u(z), Du(z), D^2u(z)) \leq 0 \quad \text{for} \quad z \in \mathcal{O}'$$

*is a (viscosity) subsolution of (2.4.1) in $\mathcal{O}'$ where $\mathcal{O}'$ is an open subset of $\mathcal{O}$. Let $h$ be a supersolution of (2.4.1). Let $S$ be the collection of all subsolutions $v$ of (2.4.1) in $\mathcal{O}$ that satisfies $v \leq h$ in $\mathcal{O}$. If $\tilde{v} \in S$ is not a supersolution of (2.4.1) in $\mathcal{O}$ with $\tilde{v}_* > -\infty$ in $\mathcal{O}$, then there are a function $w \in S$ and a point $z_0 \in \mathcal{O}$ that satisfies $\tilde{v}(z_0) < w(z_0)$.*

**Theorem 2.4.3** (Existence). *Let $\mathcal{O}$ be a locally compact subset of $\mathbf{R}^d$. Let $E$ be a densely defined function on $Z = \overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$ with $E_* < \infty$, $E^* > -\infty$ on $Z$. Assume that the equation (2.4.1) is degenerate elliptic (for subsolutions). Let $h_-$ and $h_+$ be a sub- and supersolution of (2.4.1), respectively with $h_+^* < \infty$, $h_{-*} > -\infty$ in $\mathcal{O}$. Suppose that $h_- \leq h_+$ in $\mathcal{O}$. Then there exists a solution $u$ of (2.4.1) that satisfies $h_- \leq u \leq h_+$ in $\mathcal{O}$.*

If we admit Lemmas 2.4.1 and 2.4.2, it is easy to prove Theorem 2.4.3. Indeed we take

$$S = \{v; v \text{ is a subsolution of (2.4.1) in } \mathcal{O} \text{ with } v \leq h_+ \text{ in } \mathcal{O}\}.$$

Since $h_- \leq h_+$, the set $S$ is not empty. Since $h_{-*} > -\infty$, the function $u$ defined by (2.4.2) fulfills $u_* > -\infty$ in $\mathcal{O}$. Since $h_+^* < \infty, u^* < \infty$ in $\mathcal{O}$. By Lemma 2.4.1 $u$ is a subsolution of (2.4.1). If $u$ were not a supersolution, then by Lemma 2.4.2 there would exist a function $w \in S$ and a point $z_0 \in \mathcal{O}$ with $u(z_0) < w(z_0)$. This contradicts the definition (2.4.2) of $u$. Thus $u$ is a solution of (2.4.1) in $\mathcal{O}$. The inequality $h_- \leq u \leq h_+$ follows by definition of $u$. Note that the comparison principle is unnecessary to construct a solution.

**Remark 2.4.4.** By Proposition 2.3.3 and localization property (§2.1.1) results in Lemmas 2.4.1, 2.4.2 and Theorems 2.4.3 apply the boundary value problem on $\mathcal{O} = \overline{\Omega} \times (0, T)$:

$$\begin{aligned} u_t + \quad & F(z, u, \nabla u, \nabla^2 u) = 0 && \text{in} \quad \Omega \times (0, T), \\ & B(z, u, \nabla u) = 0 && \text{on} \quad \partial\Omega \times (0, T) \end{aligned}$$

provided that $F$ is degenerate elliptic and $B$ satisfies the monotonicity assumption (2.3.7), where $\Omega$ is a domain in $\mathbf{R}^N$. The same remark applies (2.1.5) for degenerate elliptic $F$ by Proposition 2.1.2. In both examples $F$ need not be continuous as in Definition 2.1.4 (iii).

## 2.4.1 Closedness under supremum

We shall prove Lemma 2.4.1. Let $(\varphi, \hat{z}) \in C^2(\mathcal{O}) \times \mathcal{O}$ satisfy (2.3.2), i.e.,

$$\max_{\mathcal{O}}(u^* - \varphi) = (u^* - \varphi)(\hat{z}).$$

We must prove

$$E_*(\hat{z}, u^*(\hat{z}), D\varphi(\hat{z}), D^2\varphi(\hat{z})) \leq 0. \tag{2.4.3}$$

We may assume that $(u^* - \varphi)(\hat{z}) = 0$ by replacing $\varphi$ by $\varphi + (u^* - \varphi)(\hat{z})$. We set $\psi(z) = \varphi(z) + |z - \hat{z}|^4$ and observe that

$$(u^* - \psi)(z) \le -|z - \hat{z}|^4 \quad \text{for} \quad z \in \mathcal{O} \tag{2.4.4}$$

since

$$(u^* - \psi)(z) + |z - \hat{z}|^4 = (u^* - \varphi)(z) \le (u^* - \varphi)(\hat{z}) = 0.$$

By definition of upper semicontinuous envelope there is a sequence $\{z_k\}$ in $\mathcal{O}$ converging to $\hat{z}$ as $k \to \infty$ such that

$$a_k := (u^* - \psi)(z_k) \to (u^* - \psi)(\hat{z}) = 0.$$

By definition (2.4.2) of $u$ there exists a sequence $\{u_k\}$ in $S$ with $v_k(z_k) > u(z_k) - 1/k$. This implies

$$(v_k^* - \psi)(z_k) \ge (v_k - \psi)(z_k) > a_k - 1/k. \tag{2.4.5}$$

Since $v_k \le u$ in $\mathcal{O}$, (2.4.4) implies

$$(v_k^* - \psi)(z) \le -|z - \hat{z}|^4 \quad \text{for} \quad z \in \mathcal{O}. \tag{2.4.6}$$

Since $\mathcal{O}$ is locally compact there is a compact neighborhood of $\hat{z}$ denoted $B$. Since $v_k^* - \psi$ is upper semicontinuous and has an upper bound, $v_k^* - \psi$ attains its maximum over $B$ at some point $y_k \in B$. From (2.4.5) and (2.4.6) it now follows that

$$a_k - 1/k < (v_k^* - \psi)(z_k) \le (v_k^* - \psi)(y_k) \le -|y_k - \hat{z}|^4$$

for sufficiently large $k$ (so that $z_k \in B$). Since $a_k \to 0$ this implies that $y_k \to \hat{z}$ as $k \to \infty$ and that

$$\lim_{k\to\infty} (v_k^* - \psi)(y_k) = 0.$$

Hence

$$\lim_{k\to\infty} v_k^*(y_k) = \lim_{k\to\infty} \psi(y_k) = \psi(\hat{z}) = u^*(\hat{z}).$$

Since $v_k$ is a subsolution we see

$$E_*(y_k, v_k^*(y_k), D\psi(y_k), D^2\psi(y_k)) \le 0.$$

Since $E_*$ is lower semicontinuous and $y_k \to \hat{z}$, $v_k^*(y_k) \to u^*(\hat{z})$, sending $k \to \infty$ yields

$$E_*(\hat{z}, u^*(\hat{z}), D\psi(\hat{z}), D^2\psi(\hat{z})) \le 0.$$

We now obtain (2.4.3) since $D\psi(\hat{z}) = D\varphi(\hat{z})$, $D^2\psi(\hat{z}) = D^2\varphi(\hat{z})$.

We have proved that a class of subsolutions is essentially closed under the operation of supremum. The symmetric argument yields the closedness of supersolutions under the operation of infimum.

**Lemma 2.4.5** (Closedness under infimum). *Let $\mathcal{O}$ be a locally compact subset of $\mathbf{R}^d$. Let $E$ be an upper semicontinuous function on $\overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$ with values in $\mathbf{R} \cup \{+\infty\}$. Let $S$ be a set of supersolution of (2.4.1) in $\mathcal{O}$. Let $u$ be a function on $\mathcal{O}$ defined by*

$$u(z) = \inf\{v(z); v \in S\} \quad \text{for} \quad z \in \mathcal{O}.$$

*If $u_*(z) > -\infty$ for all $z \in \mathcal{O}$, then $u$ is a supersolutions of (2.4.1) in $\mathcal{O}$.*

### 2.4.2 Maximal subsolution

We shall prove Lemma 2.4.2. Since $\tilde{v}$ is not a supersolution of (2.4.1) there exist $\varphi \in C^2(\mathcal{O})$ and $\hat{z} \in \mathcal{O}$ that satisfies

$$\begin{array}{l} \min_{\mathcal{O}}(\tilde{v}_* - \varphi) = (\tilde{v}_* - \varphi)(\hat{z}) = 0, \\ E^*(\hat{z}, \tilde{v}_*(\hat{z}), D\varphi(\hat{z}), D^2\varphi(\hat{z})) = E^*(\hat{z}, \varphi(\hat{z}), D\varphi(\hat{z}), D^2\varphi(\hat{z})) < 0. \end{array} \tag{2.4.7}$$

Modifying $\varphi$ by $\varphi + |z - \hat{z}|^4$ as in the proof of Lemma 2.4.1 we may assume

$$(\tilde{v}_* - \varphi)(z) \geq |z - \hat{z}|^4 \quad \text{for} \quad z \in \mathcal{O}. \tag{2.4.8}$$

Evidently, $\tilde{v}_* \leq h_*$ in $\mathcal{O}$ so we see $\tilde{v}_*(\hat{z}) = \varphi(\hat{z}) < h_*(\hat{z})$. Indeed, if not, $\tilde{v}_*(\hat{z}) = h_*(\hat{z}) = \varphi(\hat{z})$ so that $\varphi$ would be a lower test function of $h_*$. Since $h$ is a supersolution, the inequality (2.4.7) is contradictory.

Since $E^*$ is upper semicontinuous and $\varphi \in C^2(\mathcal{O})$, for sufficiently small $\delta > 0$ we have

$$E^*(z, \varphi(z) + \delta^4/2,\ D\varphi(z),\ D^2\varphi(z)) \leq 0 \quad \text{for} \quad z \in B_{2\delta} \tag{2.4.9}$$

with $B_{2\delta} = B \cap B_{2\delta}(\hat{z})$ , where $B$ is a compact neighborhood of $\hat{z}$ and $B_r(z)$ is the closed ball of radius $r$ centered at $z$. (Such $B$ exists since $\mathcal{O}$ is locally compact.) Since $\tilde{v}_*(\hat{z}) < h_*(\hat{z})$, we may assume

$$\varphi(z) + \delta^4/2 \leq h_*(z) \quad \text{for} \quad z \in B_{2\delta} \tag{2.4.10}$$

by taking $\delta$ smaller if necessary. Since (2.4.1) is degenerate elliptic, (2.4.9) indicates that $\varphi + \delta^4/2$ is a subsolution in the interior int $B_{2\delta}$ of $B_{2\delta}$. From (2.4.8) it follows that

$$\tilde{v}(z) \geq \tilde{v}_*(z) - \delta^4/2 \geq \varphi(z) + \delta^4/2 \quad \text{for} \quad z \in B_{2\delta} \backslash B_\delta. \tag{2.4.11}$$

We now define $w$ by

$$w(z) = \begin{cases} \max\{\varphi(z) + \delta^4/2, \tilde{v}(z)\} & z \in B_\delta, \\ \tilde{v}(z) & z \in \mathcal{O} \backslash B_\delta. \end{cases}$$

To see $w$ is a subsolution one should be careful around a neighborhood of $\partial B_\delta$. It follows from (2.4.11) that

$$w(z) = \max\{\varphi(z) + \delta^4/2, \tilde{v}(z)\} \quad \text{for} \quad z \in B_{2\delta}$$

not only for $z \in B_\delta$. Since $\tilde{v}$ is a subsolution in int $B_{2\delta}$ by localization property (§2.1.1) and since $\varphi + \delta^4/2$ is a subsolution in int $B_{2\delta}$, Lemma 2.4.1 implies that $w$ is a subsolution in $B_{2\delta}$. By localization property $w$ is now a subsolution of (2.4,1) in $\mathcal{O}$. From (2.4.10) it follows that $w \in S$.

Since

$$0 = (\tilde{v}_* - \varphi)(\hat{z}) = \lim_{r \downarrow 0} \inf\{(\tilde{v} - \varphi)(z); z \in \mathcal{O} \quad \text{and} \quad |z - \hat{z}| \leq r\}$$

there is a point $z_0 \in B_\delta$ that satisfies

$$\tilde{v}(z_0) - \varphi(z_0) < \delta^4/2$$

which yields $v(z_0) < w(z_0)$. We have thus constructed $w \in S$ that satisfies $\tilde{v}(z_0) < w(z_0)$. The proof is now complete.

**Remark 2.4.6.** By the argument symmetric to that of Lemma 2.4.2 for a class of supersolutions greater than or equal to a given subsolution we observe that the infimum of the class is a solution if local boundedness condition is satisfied. We do not write a precise version of Lemma 2.4.2 for a class of supersolution. This property of minimal supersolution yields also a solution in Theorem 2.4.3. Indeed if the equation is degenerate elliptic (for supersolutions), i.e., $u \in C^2(\mathcal{O}')$ solving

$$E^*(z, u(z), Du(z), D^2u(z)) \geq 0 \quad \text{for} \quad z \in \mathcal{O}'$$

is a (viscosity) supersolution of (2.4.1) in $\mathcal{O}'$ for each open set $\mathcal{O}'$ in $\mathcal{O}$, then

$$u_-(z) = \inf\{v(z); v \text{ is a supersolution of (2.4.1) in } \mathcal{O} \text{ with } h_- \leq v \text{ in } \mathcal{O}\}$$

is a solution of (2.4.1) with $h_- \leq u_- \leq h_+$ in $\mathcal{O}$. As we have seen in the begining of §2,

$$u_+(z) = \sup\{v(z); v \text{ is a subsolution of (2.4.1) in } \mathcal{O} \text{ with } v \leq h_+ \text{ in } \mathcal{O}\}$$

is a solution of (2.4.1) with $h_- \leq u_+ \leq h_+$ in $\mathcal{O}$. By definition $u_-$ and $u_+$ are a minimal and a maximal solutions satisfying $h_- \leq u_\pm \leq h_+$, respectively. If $h_{-*} = h_+^*$ on some portion $\Sigma$ of the boundary of $\mathcal{O}$, we see $u_-^+ \leq u_+^* \leq h_+^* = h_{-*} \leq u_{-*} \leq u_{+*}$ on $\Sigma$. If the comparison principle holds, then

$$u_+^* \leq u_-^*, \; u_+^* \leq u_{+*}, \; u_-^* \leq u_{-*} \quad \text{in} \quad \mathcal{O} \cup \Sigma$$

so that $u_\pm \in C(\mathcal{O} \cup \Sigma)$ and $u_+ = u_-$. In other words the minimal and maximal solution agree and they are continuous.

### 2.4.3 Adaptation for very singular equations

Perron's method applies $\mathcal{F}$-solutions for very singular equations in Definition 2.1.5. Although it applied the boundary problems in Definition 2.3.7 if the problem is degenerate elliptic, we restrict ourselves the case without the boundary condition. We give rigorous statements and indication of the proof.

**Lemma 2.4.7** (Closedness under supremum). *Let $F$ and $\mathcal{F} \neq \emptyset$ be as in Definition 2.1.5. Assume that $\mathcal{F}$ is closed under positive multiplication. Let $S$ be a set of $\mathcal{F}$-subsolution of (2.1.5):*

$$u_t + F(z, u, \nabla u, \nabla^2 u) = 0$$

*in $\mathcal{O} = \Omega \times (0, T)$. Let $u$ be a function on $\mathcal{O}$ defined by (2.4.2). If $u^*(z) < \infty$ for all $z \in \mathcal{O}$, then $u$ is a subsolution of (2.4.1) in $\mathcal{O}$.*

The proof essentially parallels that of Lemma 2.4.1 so we only point out the place to be altered. For $(\varphi, \hat{z}) \in C_F^2(\mathcal{O}) \times (\mathcal{O})$ satisfying (2.3.2) we must prove

$$\begin{cases} \varphi_t(\hat{z}) + F(\hat{z}, u^*(\hat{z}), \nabla\varphi(\hat{z}), \nabla^2\varphi(\hat{z})) & \le 0 \quad \text{if } \nabla\varphi(\hat{z}) \ne 0, \\ \varphi_t(\hat{z}) & \le 0 \quad \text{otherwise.} \end{cases}$$

The proof of the first case is the same as that of Lemma 2.4.1 so we may assume that $\varphi$ is as in Remark 2.2.7 (B) with $\varphi(\hat{z}) = 0$ so that $\nabla\varphi(\hat{z}) = 0$. We set

$$\psi(z) = \varphi(z) + f(|x - \hat{x}|) + (t - \hat{t})^2.$$

Arguing as in §2.4.1 we end up with the existence of sequence $(x_k, t_k) \to (\hat{x}, \hat{t})(k \to \infty)$ with

$$u_k^*(x_k, t_k) \to \psi(\hat{z})$$

such that $v_k^* - \psi$ attains its local maximum at $y_k = (x_k, t_k)$. Since $v_k$ is an $\mathcal{F}$-subsolution, we have

$$\begin{cases} \psi_t(y_k) + F(y_k, v_k^*(y_k), \nabla\psi(y_k), \nabla^2\psi(y_k)) \le 0 & \text{if } \nabla\psi(y_k) \ne 0, \\ \psi_t(y_k) \le 0 & \text{otherwise} \end{cases}$$

By (2.2.7) $\nabla\psi(y_k) = 2\nabla(f(|x - x_k|),$

$$\nabla^2\psi(y_k) = 2\nabla^2(f(|x - x_k|))$$

for sufficiently large $k$. Since $f \in \mathcal{F}, y_k \to \hat{z}, v_k^*(y_k) \to \psi(\hat{z})$ we observe that

$$\lim_{k\to\infty} F(y_k, v_k^*(y_k), \nabla\psi(y_k), \nabla^2\psi(y_k)) = 0. \tag{2.4.12}$$

Thus sending $k \to \infty$ in (2.4.12) we end up with $\psi_t(\hat{z}) \le 0$.

**Lemma 2.4.8** (Maximal subsolution). *Assume the same hypotheses of Lemma 2.4.7 concerning $F$ and $\mathcal{F}$. Assume that $F$ is degenerate elliptic. Let $h$ be an $\mathcal{F}$-supersolution of (2.1.5) in $\mathcal{O} = \Omega \times (0, T)$. Let $S$ be the collection of all $\mathcal{F}$-subsolutions $v$ of (2.4.1) in $\mathcal{O}$ that satisfies $v \le h$ in $\mathcal{O}$. If $\tilde{v} \in S$ is not an $\mathcal{F}$-supersolution of (2.1.5) in $\mathcal{O}$ with $\tilde{v}_* > -\infty$ in $\mathcal{O}$, then there are a function $w \in S$ and a point $z_0 \in \mathcal{O}$ that satisfies $\tilde{v}(z_0) < w(z_0)$.*

The proof essentially parallels that of Lemma 2.4.2 so we give the idea of the way of modification. Since $\tilde{v}$ is not an $\mathcal{F}$-supersolution, there exist $\varphi \in C_F^2(\mathcal{O})$ and a point $\hat{z}$ that satisfies

$$\min_{\mathcal{O}}(\tilde{v}_* - \varphi) = (\tilde{v}_* - \varphi)(\hat{z}) = 0$$

but does not fulfill the desired inequality in Definition 2.1.5. We may assume $\nabla\varphi(\hat{z}) = 0$ so that $\varphi_t(\hat{z}) < 0$, otherwise, the desired $w$ and $z_0$ is constructed by the proof of Lemma 2.4.2; note, however one should take $\delta$ smaller so that $\nabla\varphi(z) \ne 0$ on $B_{2\delta}$. By Remark 2.2.7 we may assume $\varphi$ is a separable function:

$$\varphi(x, t) = f(|x - \hat{x}|) + g(t), \quad f \in \mathcal{F}$$

and obtain, instead of (2.4.8), that

$$(\tilde{v}_* - \varphi)(z) \geq f(|x - \hat{x}|) + (t - \hat{t})^2 \quad \text{near} \quad (\hat{x}, \hat{t})$$

by adding $f(|x - \hat{x}|) + (t - \hat{t})^2$ to $\varphi$.

Since $\nabla\varphi(\hat{z}) = 0$ and $\varphi_t(\hat{z}) < 0$, for sufficiently small $\delta > 0$ we have

$$\begin{aligned} \varphi_t(z) + \quad & F(z, \varphi(z) + f(\delta)/2, \nabla\varphi(z), \nabla^2\varphi(z)) \leq 0, \quad z \neq \hat{z}, \\ & \varphi(z) + f(\delta)/2 \leq h_*(z) \quad \text{for} \quad z \in B_{2\delta} \end{aligned}$$

by definition of $\mathcal{F}$. The first inequality implies that $\varphi + f(\delta)/2$ is an $\mathcal{F}$-subsolution of (2.1.5) in $B_{2\delta}$. We set

$$w(z) = \begin{cases} \max\{\varphi(z) + f(\delta)/2, \tilde{v}(z)\}, & z \in B_\delta, \\ \tilde{v}(z), & z \in \mathcal{O}\backslash B_\delta \end{cases}$$

and conclude $w$ is an $\mathcal{F}$-subsolution with $w = h$ and $v(z_0) < w(z_0)$ with some point $z_0 \in B_\delta$ as in the proof of Lemma 2.4.2, Lemmas 2.4.7 and 2.4.8 yield:

**Theorem 2.4.9** (Existence). *Assume that $F$ is continuous in $W_0 = \overline{\Omega} \times [0, T] \times \mathbf{R} \times (\mathbf{R}^N\backslash\{0\}) \times \mathbf{S}^N$ with values in $\mathbf{R}$. Assume that $\mathcal{F} \neq \emptyset$ and $\mathcal{F}$ is invariant under positive multiplication. Assume that $F$ is degenerate elliptic. Let $h_-$ and $h_+$ be an $\mathcal{F}$-sub- and supersolution of (2.1.5) in $\mathcal{O} = \Omega \times (0, T)$, respectively with $h_+^* < \infty, h_{-*} > -\infty$ in $\mathcal{O}$. If $h_- \leq h_+$ in $\mathcal{O}$, then there exists an $\mathcal{F}$-solution $u$ of (2.4.1) that satisfies $h_- \leq u \leq h_+$ in $\mathcal{O}$.*

## 2.5 Notes and comments

The notion of viscosity solution was first introduced by M. G. Crandall and P.-L. Lions (1981), (1983) in a different way for first order Hamilton-Jacobi equations with non-convex Hamiltonian. For a survey of early stage of the theory the reader is referred to a book of P.-L. Lions (1982). A few years later the theory extends to second order degenerate elliptic and parabolic equations. The User's Guide by M. G. Crandall, H. Ishii and P.-L. lions (1992) gives a nice review for the development of the theory. There is a nice but more concise review by H. Ishii (1994). A book by G. Barles (1994) covers the recent development of the theory mainly for the Hamilton-Jacobi equations.

A recent book of M. Bardi and I. Capuzzo-Dolcetta (1997) discusses applications of the theory of viscosity solutions to control theory, differential games. There is a nice lecture note by M. Bardi et al (1997) where various applications of viscosity solutions including a level set method are presented. We do not intend to exhaust references and rather to suggest the readers to consult these books.

Subjects in §2.1.1 and §2.1.2 are standard. The definition for general discontinuous functions is essentially due to H. Ishii (1985), (1987). There he constructed a solution by adjusting Perron's method for elliptic equations to viscosity solutions of the Hamilton-Jacobi equations. The advantage of the use of semicontinuous envelope is to get continuity of solution by comparison principle for semicontinuous functions as explained in §2.1.2 (Perron's method). The stronger version of the stability principle is due to G.Barles and B. Perthame (1987), (1989) and also H. Ishii (1989b). We postpone to discuss comparison theorems in the next Chapter.

The degenerate equations (2.1.5) with singularity at $\nabla u = 0$ is first studied by Y.-G. Chen, Y. Giga and S. Goto (1989), (1991a) to study level set equations of surface evolution equations and independently by L.C. Evans and J. Spruck (1991) for the level set equation of the mean curvature flow equations. The continuity condition (2.1.10) for comparison principle is explicitly stated in Y.-G. Chen, Y. Giga and S. Goto (1989), (1991a) and this property is also used in L.C. Evans and J. Spruck (1991). The conterexample for comparison principle for very singular equation is due to Y.-G. Chen, Y. Giga and S. Goto (1991b).

A suitable notion of viscosity solutions for very singular equations was first introduced by S. Goto (1994). The definition of $\mathcal{F}$-subsolution in §2.1.3 is apparently different from his and it is essentially due to H. Ishii and P. Souganidis (1995), where they only treated level set equations. It is not difficult to generalize this notion to other equations including $p$-Laplace diffusion equations as in M. Ohnuma and K. Sato (1997). In §2.1.3 we compare an $\mathcal{F}$-subsolution with a usual subsolution. Although it is elementary, Proposition 2.1.7 and 2.1.8 are not found in the literature. The first part of Lemma 2.1.9 is standard. The second part of Lemma 2.1.9 is essentially found in M. G. Crandall, L.C. Evans and P.-L. Lions (1984) where they showed equivalence of several definitions of viscosity solutions.

A convergence of maximum points and its various applications including strong stability principle are explained well in G. Barles (1994) for first order equations. Results in §2.2 is a straightforward and extension to singular and very singular equations. Since equation is an evolution type we note that separable type functions plays a role of a class of test functions as explained in Proposition 2.2.3. The stability results for very singular equation is essentially found in H. Ishii and P. Souganidis (1995) for level set equations and in M. Ohnuma and K. Sato (1997) for general equations. An alternate definition of viscosity subsolution in Remark 2.2.4 is due to L. Caffarelli (1989) and L. Wang (1990). Equivalence of $\mathcal{F}$-subsolution and subsolution is essentially due to G. Barles and C. Georgelin (1995) where they proved Proposition 2.2.8 from Proposition 2.1.7 for the level set equation of the mean curvature flow equation.

The definition of solution for the boundary value problem goes back to P.-L. Lions (1985). Materials in §2.3 is essentially taken from User's Guide with modification and adjustment for evolution peoblems. We do not consider Dirichlet problems in this weak sense in this book. There are also interesting other boundary conditions like state constraint problem as studied in H. M. Soner (1986). We do not touch these problems in this book. Those who are interested in these topics are encouraged to consult User's Guide and a book by W. Fleming and H. M. Soner (1993).

Perron's method presented in §2.4 is essentially due to H. Ishii (1987), where first order equations are treated. Its extension to various other equation is usually not difficult so in many cases it is stated without proof. For the reader's convenience we give a full proof for general second order equations and also for $\mathcal{F}$-solutions. For $\mathcal{F}$-solutions Perron's method is stated in H. Ishii and P. Souganidis (1995) without proof.

We won't mention any regularity theory for viscosity solutions. The reader is referred to nice books by L. Caffarelli and X. Cabré (1995) and by Q. Han and F.-H. Lin (1997) for such topics.

# Chapter 3

# Comparison principle

Comparison principle is a key step in the theory of viscosity solutions. We present various versions of comparison principle for singular degenerate equations which apply level set equations as well as other equations.

## 3.1 Typical statements

We consider an equation

$$u_t + F(z, u, \nabla u, \nabla^2 u) = 0 \tag{3.1.1}$$

in $Q = \Omega \times (0, T)$ for an open set $\Omega$ in $\mathbf{R}^N$ and $T > 0$. We list a typical set of assumptions which is fulfilled for degenerate parabolic level set equations in Chapter 1.
(F1) (Continuity) $F : W_0 = \overline{\Omega} \times [0, T] \times \mathbf{R} \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ is continuous.
(F2) (Degenerate ellipticity)

$$F(z, r, p, X) \leq F(z, r, p, Y) \quad \text{for } X \geq Y,\ X, Y \in \mathbf{S}^N$$

and $z \in \overline{\Omega} \times [0, T]$, $r \in \mathbf{R}$, $p \in \mathbf{R}^N \backslash \{0\}$.
For the class of level set equations satisfying (1.6.24), which includes the level set mean curvature flow equations (1.6.5), the singularity of $F$ at $p = 0$ is rather mild. To treat such equations it is reasonable assume
(F3) $-\infty < F_*(z, r, 0, O) = F^*(z, r, 0, O) < \infty$.
In general (F3) is not fulfilled for example for the level set Gaussian curvature flow equation (1.6.12) (see §1.6.5), a weaker assumption is necessary to treat such a problem.
(F3') $\mathcal{F}(F) \neq \emptyset$.
Here $\mathcal{F}$ is defined in §2.1.3 and may depend on $\Omega$ and $T$ if $F$ depends on $z$. In this case we consider $\mathcal{F}_\Omega$-viscosity solutions instead of usual viscosity solutions. We often drop the prefix $\mathcal{F}$ unless confusion occurs. For level set equations $F$ is independent of $r$ but in general we need to assume monotonicity in $r$.
(F4) (Monotonicity). For some constant $c_0$

$$r \mapsto F(z, r, p, X) + c_0 r$$

is a nondecreasing function.

### 3.1.1 Bounded domains

When $\Omega$ is bounded, comparison principle takes its naive form as in Chapter 2.
(BCP) Let $u$ and $v$ be sub- and supersolution of (3.1.1) in $Q = \Omega \times (0,T)$, where $\Omega$ is bounded. If $-\infty < u^* \leq v_* < \infty$ on $\partial_p Q$, then $u^* \leq v_*$ in $Q$.
If $F(x,t,r,p,X)$ is independent of $x \in \Omega$ it is easy to state the conditions so that the comparison principle (BCP) holds.

**Theorem 3.1.1.** *Assume that $\Omega$ is bounded and that $F = F(x,t,r,p,X)$ is independent of $x$. Assume that (F1)–(F3) and (F4) hold. Then the comparison principle (BCP) is valid. If we assume (F3') instead of (F3), then (BCP) is still valid (by replacing solutions by $\mathcal{F}_\Omega$-solutions) provided that $\mathcal{F}_\Omega$ is invariant under positive multiplication.*

**Corollary 3.1.2.** *Assume that $\Omega$ is bounded and that $F = F(x,t,r,p,X)$ is independent of $x$ and $r$. Assume that $F : [0,T] \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ is continuous and degenerate elliptic. Then (BCP) holds if $F$ is geometric.*

It is easy to see that Corollary 3.1.2 follows from Theorem 3.1.1 once we note:

**Lemma 3.1.3.** *Assume that $F : \overline{\Omega} \times [0,T] \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ is continuous and geometric, then $\mathcal{F}_\Omega = \mathcal{F}_\Omega(F)$ is nonempty provided that there exist $r_0 > 0$ and a continuous function $c \in C(0, r_0]$ such that*

$$|F(x,t,p,\pm I)| \leq c(|p|) \quad \textit{on} \quad \overline{\Omega} \times [0,T] \times (B_{r_0}(0) \setminus \{0\}).$$

*In particular, $\mathcal{F}_\Omega \neq \emptyset$ if $\Omega$ is bounded or $F$ is independent of $x$. Moreover, $f \in \mathcal{F}_\Omega$ implies $af \in \mathcal{F}_\Omega$ for $a > 0$.*

*Proof of Lemma 3.1.3.* We shall construct $f \in C^2[0,\infty)$ satisfying (2.1.11) with $f(0) = f'(0) = f''(0) = 0$, $f''(r) > 0$ for $r > 0$. We note that

$$\nabla_p(f(\rho)) = f'(\rho)p/\rho, \ \nabla_p^2(f(\rho)) = f''(\rho)\frac{p \otimes p}{\rho^2} + f'(\rho)\frac{1}{\rho}\left(I - \frac{p \otimes p}{\rho^2}\right)$$

with $\rho = |p| > 0$. Since $F$ is geometric, we see

$$\begin{aligned} & F(x,t,\nabla(f(\rho)), \ \nabla^2(f(\rho))) = F(x,t,\pm f'(\rho)\frac{p}{\rho}, \ \pm f'(\rho)\frac{I}{\rho}) \\ = \ & \frac{f'(\rho)}{\rho}F(x,t,p,\pm I), \ \rho = |p|. \end{aligned} \tag{3.1.2}$$

By assumption we have

$$|F(x,t,p,\pm I)| \leq c(|p|) \quad \text{on} \ \overline{\Omega} \times [0,T] \times (B_{r_0}(0) \setminus \{0\}).$$

We may assume that $c \in C^1(0, r_0]$ with $c > 0$ and that

$$(1/c)' > 0 \quad \text{in } (0,1], \ \lim_{r \downarrow 0} c(r) = \infty, \quad \text{and } \lim_{r \downarrow 0}(1/c)'(r) = 0.$$

Then $f : [0, r_0] \to \mathbf{R}$ defined by

$$f(r) = \begin{cases} \int_0^r \frac{s^2}{c(s)} ds & 0 < r \le r_0 \\ 0 & r = 0 \end{cases}$$

satisfies $f(0) = f'(0) = f''(0) = 0$ with $f''(r) > 0$ for $r > 0$. By definition of $c$ we now observe that

$$\begin{aligned} |F(x, t, \nabla_p(f(\rho)),\ \pm\nabla_p^2(f(\rho)))| &= \frac{f'(\rho)}{\rho}|F(x, t, p, \pm I)| \\ &\le \frac{f'(\rho)}{\rho}c(\rho) = \rho \to 0 \quad \text{as } \rho \to 0. \end{aligned} \tag{3.1.3}$$

We extend $f$ to $(0, \infty)$ in an appropriate way to get $f \in \mathcal{F}_\Omega(F)$.

If $F$ is independent of $x$ or $\Omega$ is bounded,

$$\begin{aligned} C(\rho) = \max \quad &\{\sup\{F(x, t, p, I); x \in \overline{\Omega}, t \in [0, T], |p| = \rho\}, \\ &\sup\{F(x, t, p, -I); x \in \overline{\Omega}, t \in [0, T], |p| = \rho\}\} \end{aligned}$$

is continuous in $(0, \infty)$ so the above argument implies that $\mathcal{F}_\Omega(F)$ is nonempty.

The property that $f \in \mathcal{F}_\Omega$ implies $af \in \mathcal{F}_\Omega$ is clear since

$$F(x, t, \lambda p, \lambda X) = \lambda F(x, t, p, X) \quad \text{for } \lambda > 0, p \in \mathbf{R}^N\backslash\{0\}, X \in \mathbf{S}^N, x \in \Omega, t \in [0, T].\Box$$

The proof of Theorem 3.1.1 is postponed in §3.4.

### 3.1.2 General domains

For a general domain $\Omega$ we replace the comparison principle for a bounded domain (BCP) by (CP) stated below.

(CP) Let $u$ and $v$ be sub- and supersolution of (3.1.1) in $Q = \Omega \times (0, T)$. Assume that $u$ and $-v$ are bounded from above on $Q$. Assume that

$$\begin{aligned} \lim_{\delta\to 0} \sup\{u^*(x, t) - v_*(y, s); \quad &|x - y| \le \delta,\ |t - s| \le \delta,\ \mathrm{dist}((x, t), \partial_p Q) \le \delta, \\ &\mathrm{dist}((y, s), \partial_p Q) \le \delta, \\ &(x, t), (y, s) \in \overline{\Omega} \times [0, T']\} \le 0 \end{aligned} \tag{3.1.4}$$

for each $T' \in (0, T)$ and that $u^* > -\infty$, $v^* < \infty$ on $\partial_p Q$. Then

$$\lim_{\delta\to 0} \sup\{u^*(x, t) - v_*(y, s);\ |x - y| \le \delta,\ |t - s| \le \delta, (x, t), (y, s) \in \overline{\Omega} \times [0, T']\} \le 0 \tag{3.1.5}$$

for each $T' \in (0, T)$.

If $\Omega$ is bounded, it is easy to check that $u^* \le v_*$ on $\partial_p Q$ is equivalent to (3.1.4) and $u^* \le v_*$ in $\overline{Q}$ is equivalent to (3.1.5). Thus (CP) and (BCP) is the same when $\Omega$ is bounded. Intuitively, (3.1.5) asserts the uniformity of $u^* \le v_*$.

**Theorem 3.1.4.** *Assume that $F(x, t, r, p, X)$ is independent of $x$. Assume that (F1)–(F3) and (F4) holds. Then the comparison principle (CP) is valid. If we assume (F3') instead of (F3), then (CP) is still valid (by replacing solutions by $\mathcal{F}_\Omega$-solutions) provided that $\mathcal{F}_\Omega$ is invariant under positive multiplications.*

**Corollary 3.1.5.** *Assume that $F = F(x, t, r, p, X)$ is independent of $x$ and $r$. Assume that $F : [0, T] \times (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ is continuous and degenerate elliptic. Then (CP) holds if $F$ is geometric.*

Again Corollary 3.1.5 follows from Theorem 3.1.4 together with Lemma 3.1.3. We shall give a proof of Theorem 3.1.4 in §3.4.2.

### 3.1.3 Applicability

In §1.6 we have studied various properties of level set equations of a surface evolution equation (1.6.1). Corollary 3.1.2 and Corollary 3.1.5 apply to the level set equation of (1.6.1) provided that

(i) $f$ in (1.6.1) is continuous in its variables; more precisely $f$ is continuous on $\Omega \times [0, T] \times \mathbf{E}$.
(ii) $f$ is independent of $x$.
(iii) (1.6.1) is degenerate parabolic.

Indeed, (F1) for $F_f$ (defined by (1.6.16)) follows from (i) as in Proposition 1.6.15. The ellipticity of $F_f$ follows from the definition of (iii); see (1.6.21). Geometricity follows from §1.6.2. The condition (F3) holds if the growth of $f$ with respect to $\nabla\mathbf{n}$ is at most linear (Proposition 1.6.18).

We list examples of (1.6.1) satisfying (i), (ii), (iii) for reader's convenience.

The quation (1.5.2) fulfills (i)–(iii) provided that $\beta > 0$ is continuous and $\gamma$ in (1.3.7) is convex and $C^2$ except the origin and that $a \geq 0$ and $c$ is independent of $x$ and is continuous on $[0, T]$. In particular, the mean curvature flow equation (1.5.4) and a Hamilton-Jacobi equation (1.5.6) evidently fulfill (i)–(iii) provided that $c$ in (1.5.6) is independent of $x$ and is continuous on $[0, T]$. The level set equations of these equations also fulfill (F3). Under the same assumption on $\gamma, a, c$ as above the equation (1.5.13) fulfills (i)–(iii) provided that $h$ is continuous and nondecreasing. Its level set equation fulfills (F3) if and only if growth of $h$ is at most linear.

The equation (1.5.8) fulfills (i)–(iii) provided that $g$ is continuous and that (1.5.8) is degenerate parabolic. The Gaussian curvature flow equation (1.5.9) and other related equation (1.5.10), (1.5.11), (1.5.12) fulfills (i)–(iii) provided that $e_m$ is interpreted by $\hat{e}_m$ (Theorem 1.6.10). In other words the modified form (1.6.22), (1.6.23) fulfills (i)–(iii). Here (F3) is in general not expected to hold for the level set equation.

Another typical example which is not geometric but Theorem 3.1.1 and Theorem 3.1.4 still apply is the $p$-Laplace equation of parabolic type:

$$u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

for $1 < p < 2$ as proved by M. Ohnuma and K. Sato (1997).

## 3.2 Alternate definition of viscosity solutions

### 3.2.1 Definition involving semijets

We recall the notion of the second-order semijets of a function which plays a role of derivatives up to the second-order in usual calculus. Semijets are infinitesimal quatities. We shall give an equivalent definition of viscosity solutions by using semijets. Such an infinitesimal interpretation of viscosity solutions in usuful to prove comparison principles.

**Definition 3.2.1** (Semijets). Let $\mathcal{O}$ be a locally compact subset of $\mathbf{R}^d$. Let $u : \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ be upper semicontinuous. Let $\hat{z}$ be a point in $\mathcal{O}$. An element $(q, Z) \in \mathbf{R}^d \times \mathbf{S}^d$ is called (the second-order) *superjet* of $u$ at $\hat{z}$ in $\mathcal{O}$ if $u(\hat{z})$ is finite and

$$u(z) - u(\hat{z}) \leq \langle q, z - \hat{z} \rangle + \frac{1}{2}\langle Z(z - \hat{z}), z - \hat{z} \rangle + o(|z - \hat{z}|^2) \tag{3.2.1}$$

for $z \in \mathcal{O}$ as $z \to \hat{z}$; here $o(h)$ denotes a function such that $o(h)/h \to 0$ as $h \to 0$. The set of all superjets of $u$ at $\hat{z}$ in $\mathcal{O}$ is denoted by $J^{2,+}_{\mathcal{O}} u(\hat{z}) (\subset \mathbf{R}^d \times \mathbf{S}^d)$. For a lower semicontinuous function $u : \mathcal{O} \to \mathbf{R} \cup \{+\infty\}$ an element $(q, Z) \in \mathbf{R}^d \times \mathbf{S}$ is called *a subjet* of $u$ at $\hat{z}$ in $\mathcal{O}$ if $(-q, -Z) \in J^{2,+}_{\mathcal{O}}(-u)(\hat{z})$. The set of all subjets of $u$ at $\hat{z}$ in $\mathcal{O}$ is denoted by $J^{2,-}_{\mathcal{O}} u(\hat{z})$, so that $J^{2,-}_{\mathcal{O}} u(\hat{z}) = -J^{2,+}_{\mathcal{O}}(-u)(\hat{z})$. By a semijet we mean either a superset or a subjet.

The set $J^{2,\pm}_{\mathcal{O}} u(\hat{z})$ does not vary even if $\mathcal{O}$ is replaced by a neighborhood $\mathcal{O}'$ of $\hat{z}$ in $\mathcal{O}$. In particular, $J^{2,\pm}_{\mathcal{O}} u(\hat{z})$ is independent of $\mathcal{O}$ if $\hat{z}$ is an interior point of $\mathcal{O}$, i.e., $\hat{z} \in \operatorname{int} \mathcal{O}$. In this case we often suppress the subscript $\mathcal{O}$ of $J^{2,\pm}_{\mathcal{O}} u(\hat{z})$ and simply write it by $J^{2,\pm} u(\hat{z})$. In general $J^{2,\pm}_{\mathcal{O}} u(\hat{z})$ may depends on $\mathcal{O}$. Indeed, if we consider $u \equiv 0$ in $\mathbf{R}$ we see

$$\begin{aligned} &J^{2,\pm}_{\mathbf{R}} u(0) = \{(0, Z); Z \geq 0\} \\ &J^{2,\pm}_{[0,\infty)} u(0) = \{(q, Z), Z \in \mathbf{R}\ q > 0\} \cup \{(0, Z); Z \geq 0\}. \end{aligned}$$

By definition a function $u$ has Taylor's expansion up to second order at $z \in \operatorname{int} \mathcal{O}$ if and only if

$$(J^{2,+} u)(\hat{z}) \cap (J^{2,-} u)(\hat{z}) \neq \emptyset.$$

(Actually, the intersection must be singleton if it is nonempty.) In particular, if $u$ is $C^2$ at $\hat{z} \in \operatorname{int} \mathcal{O}$, then

$$(J^{2,+} u)(\hat{z}) \cap (J^{2,-} u)(\hat{z}) = \{(Du(\hat{z}),\ D^2 u(\hat{z}))\},$$

which is the set of the second-order jets of $u$ at $\hat{z}$. Of course, the set $(J^{2,\pm}_{\mathcal{O}} u)(\hat{z})$ could be empty even if $u(\hat{z})$ is finite but this defines a mapping $J^{2,\pm}_{\mathcal{O}} u(\hat{z})$ from $\mathcal{O}$ to the set of all subsets of $\mathbf{R}^d \times \mathbf{S}^d$. Although the set $J^{2,\pm} u(\hat{z})$ could be empty for some $\hat{z}$, for generic $\hat{z}$ of $J^{2,\pm}_{\mathcal{O}} u(\hat{z})$ is nonempty.

**Lemma 3.2.2.** *The set of $z$ at which $J^{2,\pm}_{\mathcal{O}} u(\hat{z})$ is nonempty is dense in $\mathcal{O}' = \mathcal{O} \backslash \{z \in \mathcal{O}; u(z) = -\infty\}$. The assertion is still valid if $J^{2,\pm}_{\mathcal{O}} u$ is replaced by $J^{2,-}_{\mathcal{O}} u$.*

*Proof.* For a point $\hat{z} \in \mathcal{O}'$ we take a closed ball $B_r(\hat{z})$ of radius $r$ centered at $\hat{z}$ in $\mathbf{R}^d$ such that $\mathcal{O} \cup B_r(\hat{z})$ is compact. We take an upper semicontinuous function $\varphi$ on $\mathcal{O}$ which is $C^2$ in int $B_r(\hat{z})$ and observe that

$$\begin{aligned} & J_{\mathcal{O}}^{2,\pm}(u+\varphi)(z) = J_{\mathcal{O}}^{2,\pm}u(\hat{z}) + \{(D\varphi(z), D^2\varphi(z)\} \\ := \quad & \{(q + D\varphi(z), Z + D^2\varphi(z)), (q, Z) \in J_{\mathcal{O}}^{2,\pm}u(\hat{z})\}, \quad \text{for all } z \in \mathcal{O} \cap \mathrm{int} B_r(\hat{z}). \end{aligned}$$

We arrange $\varphi \equiv -\infty$ outside $B_r(\hat{z})$ and $-\varphi$ large near $\partial B_r(\hat{z})$ so that $u + \varphi$ attains its maximum over $\mathcal{O}$ at some point $z_r \in \mathcal{O} \cap$ int $B_r(\hat{z})$. By definition

$$\{(0, Z); Z \geq 0\} \subset J_{\mathcal{O}}^{2,\pm}(u+\varphi)(z_r)$$

so $J_{\mathcal{O}}^{-2,+}u(z_r)$ is not empty. We may take $r$ small so this implies that $\{z \in \mathcal{O}; J_{\mathcal{O}}^{2,+}(z) \neq \emptyset\}$ is dense in $\mathcal{O}'$. The same proof works for $J_{\mathcal{O}}^{2,-}u$ if we replace $+$ by $-$. □

For evolution equations (3.1.1) it is convenient to consider special component of semi-jets.

**Definition 3.2.3** (Parabolic semijets). Let $u$ be an upper semicontinuous function from a locally compact subset $\mathcal{O}$ in $\mathbf{R}^N \times \mathbf{R}$ to $\mathbf{R} \cup \{-\infty\}$. Let $\hat{z} = (\hat{x}, \hat{t}) \in \mathbf{R}^N \times (0, \infty)$ be a point in $\mathcal{O}$. An element $(t, p, X) \in \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ in called a *parabolic superjet* of $u$ at $\hat{z}$ in $\mathcal{O}$ if

$$\begin{aligned} u(z) - u(\hat{z}) \leq \tau(t - \hat{t}) \quad & + \langle q, x - \hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x - \hat{x}\rangle \\ & + o(|x-\hat{x}|^2 + |t - \hat{t}|), \ z = (x,t) \in \mathcal{O} \end{aligned} \tag{3.2.2}$$

as $z \to \hat{z}$, where $\langle , \rangle$ denotes the inner product in $\mathbf{R}^N$. The totality of parabolic superjets of $u$ at $\hat{z}$ in $\mathcal{O}$ is denoted $\mathcal{P}_{\mathcal{O}}^{2,+}u(\hat{z})$. For a lower semicontinuous function $u : \mathcal{O} \to \mathbf{R} \cup \{+\infty\}$, the $-\mathcal{P}_{\mathcal{O}}^{2,+}(-u)(\hat{z})$ is denoted $\mathcal{P}_{\mathcal{O}}^{2,-}u(\hat{z})$ and its element is called a *parabolic subjet* of $u$ at $\hat{z}$ in $\mathcal{O}$.

**Proposition 3.2.4.** *For $(p, \tau) \in \mathbf{R}^N \times \mathbf{R}, a \in \mathbf{R}$, $\ell \in \mathbf{R}^N, X \in \mathbf{S}^N$ if*

$$\left( \begin{pmatrix} p \\ \tau \end{pmatrix}, \begin{pmatrix} X & \ell \\ {}^t\ell & a \end{pmatrix} \right) \in J_{\mathcal{O}}^{2,+}u(\hat{x}, \hat{t}),$$

*then $(\tau, p, X) \in \mathcal{P}_{\mathcal{O}}^{2,+}u(\hat{x}, \hat{t})$, where $\mathcal{O}$ is a locally compact subset of $\mathbf{R}^N \times \mathbf{R}$ and ${}^t\ell$ denotes the transpose of a matrix $\ell$. Here $p$ and ${}^t\ell$ are column vectors.*

*Proof.* By definition

$$\begin{aligned} u(x,t) - u(\hat{x}, \hat{t}) \leq \tau(t - \hat{t}) \quad & + \langle p, x - \hat{x}\rangle + \frac{1}{2}a(t-\hat{t})^2 \\ & + (t - \hat{t})\langle \ell, x - \hat{x}\rangle + \frac{1}{2}\langle X(x-\hat{x}), x - \hat{x}\rangle + o(|t-\hat{t}|^2 + |x - \hat{x}|^2) \end{aligned}$$

as $(x,t) \to (\hat{x},\hat{t}), (x,t) \in \mathcal{O}$. We estimate the mixed term $(t-\hat{t})\langle \ell, x-\hat{x}\rangle$ by Young's inequality to get

$$\begin{aligned}(t-\hat{t})\langle \ell, x-\hat{x}\rangle &\le |\ell|\ |t-\hat{t}|\ |x-\hat{x}| \le |\ell|(\frac{2}{3}|t-\hat{t}|^{3/2} + \frac{1}{3}|x-\hat{x}|^3) \\ &= o(|t-\hat{t}| + |x-\hat{x}|^2).\end{aligned}$$

We thus observe that $(\tau, p, X) \in \mathcal{P}^{2,+}_{\mathcal{O}} u(\hat{x},\hat{t})$. □

We next recall the definition of the 'closures' of the set-valued mappings which is important to study comparison principle. We set

$$\begin{aligned}&\overline{J}^{2,+}_{\mathcal{O}} u(\hat{z}) = \{(q,Z) \in \mathbf{R}^d \times \mathbf{S};\quad \text{there is a sequence} \\ &(z_j, q_j, Z_j) \in \mathcal{O} \times \mathbf{R}^d \times \mathbf{S}^d \quad (j = 1,2,\cdots) \quad \text{such that} \\ &(q_j, Z_j) \in J^{2,+}_{\mathcal{O}} u(z_j) \text{ and } (z_j, u(z_j), q_j, Z_j) \to (\hat{z}, u(\hat{z}), q, Z) \text{ as } j \to \infty\}, \\ &\overline{J}^{2,-}_{\mathcal{O}} u(\hat{z}) := -(\overline{J}^{2,+}_{\mathcal{O}}(-u))(\hat{z}).\end{aligned}$$

These definitions are a little bit different from the standard closure of the set-valued mappings defined by the closure of the graph of the mappings since there is the extra condition of convergence $u(z_j) \to u(\hat{z})$ $(j \to \infty)$.

**Remark 3.2.5.** Clearly, the statement of Proposition 3.2.4 is still valid even if we replace $J^{2,+}_{\mathcal{O}}$ by $\overline{J}^{2,+}_{\mathcal{O}}$ and $\mathcal{P}^{2,+}_{\mathcal{O}}$ by $\overline{\mathcal{P}}^{2,-}_{\mathcal{O}}$ which is defined in Remark 3.2.9.

**Proposition 3.2.6** (Infinitesimal version of definitions of viscosity solutions). *(i) Let $\mathcal{O}$ be a locally compact subset of $\mathbf{R}^d$. Let $E$ be defined in a dense subset of $\overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$ with the property $E_* < +\infty$ on $\overline{\mathcal{O}} \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{S}^d$. A function $u : \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ (satisfying $u^*(z) < \infty$ for all $z \in \mathcal{O}$) is a subsolution of (2.3.1) in $\mathcal{O}$ if and only if*

$$E_*(\hat{z}, u^*(\hat{z}), q, Z) \le 0 \tag{3.2.3}$$

*for all $\hat{z} \in \mathcal{O}$ and $(q,Z) \in J^{2,+}_{\mathcal{O}} u^*(\hat{z})$.*
*(ii) Assume that $F$ satisfies (F1) and (F3'). A function $u : \mathcal{O} \to \mathbf{R} \cup \{-\infty\}$ (satisfying $u^*(z) < \infty$ for all $z \in \mathcal{O}$ ) is an $\mathcal{F}$-subsolution of (3.1.1) in $\mathcal{O}$ if and only if the following two conditions are fulfilled*
*(a)*

$$\tau + F(\hat{x}, \hat{t}, u^*(z), p, X) \le 0 \tag{3.2.4}$$

*for all $(\tau, p, X) \in \mathcal{P}^{2,+}_{\mathcal{O}} u(\hat{x},\hat{t})$ unless $p = 0$.*
*(b) $\varphi_t(\hat{z}) \le 0$ for all $(\varphi, \hat{z}) \in C^2_F(Q') \times Q'$ satisfying (2.1.6), for some neighborhood $Q'$ of $\hat{z}$ provided that $\mathcal{F}_\Omega$ is invariant under positive multiplication. Here $\mathcal{O}$ is an open set in $\Omega \times (0,T)$, where $\Omega$ is an open set in $\mathbf{R}^N$.*

These statements are easily verified by the following characterization of semijets; for (ii) we also invoke a localization property (Remark 2.1.10) and Proposition 2.2.3 (ii).

**Lemma 3.2.7.** *(i) For $\hat{z} \in \mathcal{O}$*

$$J^{2,+}_{\mathcal{O}} u^*(\hat{z}) = \{(D\varphi(\hat{z}), D^2\varphi(\hat{z})); \varphi \in C^2(\mathcal{O}) \ \text{ that satisfies} \max_{\mathcal{O}}(u^* - \varphi) = (u^* - \varphi)(\hat{z})\}.$$

*(ii) For $(\hat{x}, \hat{t}) \in Q$*

$$\begin{aligned}&\mathcal{P}^{2,+}_{\mathcal{O}} u^*(\hat{x}, \hat{t}) = \{(\varphi_t(\hat{x}, \hat{t}), \nabla\varphi(\hat{x}, \hat{t}), \nabla^2\varphi(\hat{x}, \hat{t})); \varphi \in C^{2,1}(Q) \ \text{ that satisfies}\\ &\max_{Q}(u^* - \varphi) = (u^* - \varphi)(\hat{x}, \hat{t})\}.\end{aligned}$$

*(iii) In (i) and (ii) $\mathcal{O}$ and $Q$ may be replaced by an (open) neighborhood of $\mathcal{O}'$ of $\hat{z}$ in $\mathcal{O}$ and $Q'$ of $(\hat{x}, \hat{t})$ in $Q$, respectively. In particular for $(\tau, p, X) \in \mathcal{P}^{2,+}_{\mathcal{O}} u^*(x, t)$, $p \neq 0$ there is always a neighborhood $Q'$ of $(\hat{x}, \hat{t})$ in $Q$ and $\varphi \in C^2(\mathcal{O}')$ that satisfies*

$$(\tau, p, X) = (\varphi_t(\hat{x}, \hat{t}), \nabla\varphi(\hat{x}, \hat{t}), \nabla^2\varphi(\hat{x}, \hat{t}))$$

*and $\nabla\varphi(\hat{z}) \neq 0$ for all $z \in Q'$.*

*Proof.* (i) Let $J$ denote the right hand side of the equality. We may assume $\hat{z} = 0$ by translation. If $u^* - \varphi$ takes it maximum at $\hat{z} = 0$, then

$$\begin{aligned} u^*(z) - u^*(0) \quad &\leq \varphi(z) - \varphi(0) = \langle D\varphi(0), z\rangle + \frac{1}{2}\langle D^2\varphi(0)z, z\rangle \\ &+ o(|z|^2) \quad \text{as } z \to 0 \end{aligned}$$

by Taylor's expansion. Thus $J^{2,+}_{\mathcal{O}} u^*(0) \supset J$.

The other side inclusion is less trivial. Assume now that $(q, Z) \in J^{2,+}_{\mathcal{O}} u^*(0)$, i.e.,

$$u^*(z) - u^*(0) \leq \langle q, z\rangle + \frac{1}{2}\langle Zz, z\rangle + \varepsilon(z), \ z \in \mathcal{O}$$

where $\varepsilon(z)/|z|^2 \to 0$ as $z \to 0$. The problem is that $\varepsilon(z)$ itself is not $C^2$ at all. We set

$$\omega_0(\sigma) = \sup\{|\varepsilon(z)/|z|^2; \ |z| \leq \sigma, z \in \mathcal{O}\},$$

so that $\omega_0$ is nondecreasing function from $[0, \infty)$ to $[0, \infty)$ with the property that $\omega_0$ is continuous at $\sigma = 0$ and $\omega_0(0) = 0$. By Lemma 2.1.9 there is $\theta \in C^2[0, \infty)$ that satisfies

$$\omega_0(\sigma)|\sigma|^2 \leq \theta(\sigma) \quad \text{for } \sigma \geq 0, \theta(0) = \theta'(0) = \theta''(0) = 0, \theta'''(\sigma) \geq 0 \quad \text{for } \sigma \geq 0.$$

Since $|\varepsilon(z)| \leq \omega_0(|z|)|z|^2$, we set a $C^2$ function by

$$\varphi(z) = \langle q, z\rangle + \frac{1}{2}\langle Zz, z\rangle + \theta(|z|)$$

to observe that $u^* - \varphi$ takes its maximum $u^*(0) - \varphi(0)$ over $\mathcal{O}$ at $\hat{z} = 0$ and that $q = D\varphi(0)$, $Z = D^2\varphi(0)$. Thus $J^{2,+}_{\mathcal{O}} u^*(0) \subset J$.

(ii) Let $\mathcal{P}$ denote the right hand side of the equality. By Proposition 3.2.4 and $J \subset J^{2,+}_{\mathcal{O}} u^*(\hat{z})$ the inclusion $\mathcal{P} \subset \mathcal{P}^{2,+}_{\mathcal{O}} u^*(\hat{x},\hat{t})$ is trivial.

To see the other inclusion we may assume $(\hat{x},\hat{t}) = (0,0)$. Assume that $(\tau,p,X) \in \mathcal{P}^{2,+}_{\mathcal{O}} u^*(0)$, i.e.,

$$u^*(x,t) - u^*(0) \le \langle \tau, t\rangle + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle + \varepsilon(x,t), (x,t) \in Q$$

where $\varepsilon(x,t)/(|x|^2 + |t|) \to 0$ as $|x| \to 0$, $|t| \to 0$. As for (i) we set

$$\omega_0(\sigma) = \sup\{|\varepsilon(x,t)/(|x|^2 + |t|)|;\ |x|^2 + |t| \le \sigma,\ (x,t) \in Q\}$$

and observe by Lemma 2.1.9 there is $\theta \in C^2[0,\infty)$ satisfying the same property as in (i). Since

$$|\varepsilon(x,t)| \le \omega_0((|x|^2 + |t|)^{1/2})(|x|^2 + |t|),$$

we set

$$\varphi(x,t) = \langle \tau, t\rangle + \langle p, x\rangle + \frac{1}{2}\langle Xx, x\rangle + \theta((|x|^2 + |t|)^{1/2})$$

(which is actually $C^{2,1}$ everywhere) to observe that $u^* - \varphi$ takes its maximum over $Q$ at $(\hat{x},\hat{t}) = 0$ and that $\tau = \varphi_t(0)$, $p = \nabla\varphi(0)$, $X = \nabla^2\varphi(0)$. Thus $\mathcal{P}^{2,+}_{Q} u^*(0) \subset \mathcal{P}$.
(iii) This follows from the proof of part (i) and (ii).

**Remark 3.2.8.** Corresponding to Proposition 3.2.6 we have infinitesimal version of definition of viscosity supersolutions in a symmetric way, since we have a characterization of $J^{2,-}$ and $\mathcal{P}^{2,-}$ corresponding to Lemma 3.2.7 by replacing max by min. Proposition 3.2 (ii) applies boundary value problems by taking $\mathcal{O}$ by $\overline{\Omega} \times (0,T)$. It is easy to extend Proposition 3.2.6 (ii) to boundary value problems for $\mathcal{F}$-solutions.

**Remark 3.2.9** (Closures of semijets). If $u$ is a subsolution of (2.3.1), so that (3.2.3) holds, then (3.2.3) still holds for all

$$(q,Z) \in \overline{J}^{2,+}_{\mathcal{O}} u^*(\hat{z}),\ \hat{z} \in \mathcal{O}$$

since $E_*$ is lower semicontinuous. The same remark applies (ii) of Proposition 3.2.6. If (a) holds, then (3.2.4) still holds for $(\tau,p,X) \in \overline{\mathcal{P}}^{2,+}_{\mathcal{O}} u^*(\hat{x},\hat{t})(\hat{x},\hat{t}) \in Q$ since $F(x,t,r,p,X)$ is continuous outside the set where $p = 0$. The set $\overline{\mathcal{P}}^{2,+}$ is defined in analogous way to $\overline{J}^{2,+}$:

$$\begin{aligned}
&\overline{\mathcal{P}}^{2,+}_{\mathcal{O}} u(\hat{z}) = \{(\tau,p,X) \in \mathbf{R} \times \mathbf{R}^N \times S^N;\ \text{there is a}\\
&\text{sequence } (z_j,\tau_j,p_j,X_j) \in Q \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N \quad (j = 1,2,\cdots)\\
&\text{such that } (\tau_j,p_j,X_j) \in \mathcal{P}^{2,+}_{\mathcal{O}} u(z_j) \quad \text{and}\\
&(z_j, u(z_j), \tau_j, p_j, X_j) \to (\hat{z}, u(\hat{z}), \tau, p, X)\\
&\text{as } j \to \infty\}.\\
&\overline{\mathcal{P}}^{2,-}_{\mathcal{O}} u(\hat{z}) = -\overline{\mathcal{P}}^{2,+}_{\mathcal{O}}(-u)(\hat{z}).
\end{aligned}$$

### 3.2.2 Solutions on semiclosed time intervals

When we study an evolution equation (3.1.1), it is sometimes convenient to consider solutions in $Q_* = \Omega \times (0, T]$ instead of $Q = \Omega \times (0, T)$, where $\Omega$ is an open set in $\mathbf{R}^N$.

**Theorem 3.2.10** (Extension). *(i) Assume that $F$ is lower semicontinuous in $W = \overline{\Omega} \times [0, T] \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$ with values in $\mathbf{R} \cup \{-\infty\}$. Let $u$ be a subsolution of (3.1.1) in $Q$. Then its upper semicontinuous envelope $u^*$ defined in $Q_*$ is a subsolution of (3.1.1) in $Q_*$ provided that $u^*(x, T) < \infty$ for all $x \in \Omega$.*
*(ii) Assume that $F$ fulfills (F1) and (F3') and that $a\mathcal{F}_\Omega \subset \mathcal{F}_\Omega$ for all $a > 0$. Let $u$ be an $\mathcal{F}_\Omega$-subsolution of (3.1.1) in $Q$. Then $u^*$ is an $\mathcal{F}_\Omega$-subsolution of (3.1.1) in $Q_*$ provided that $u^*(x, T) < \infty$ for all $x \in \Omega$.*

**Definition 3.2.11** (Left accessibility). Let $(y_0, t_0)$ be a point in $\mathbf{R}^m \times \mathbf{R}$. A function $w$ defined in $B_r(y_0) \times (t_0 - \delta, t_0]$ is called *left accessible* at $(y_0, t_0)$ if there are sequences $y_\ell \to y_0$, $t_\ell \to t_0 (\ell \to \infty)$ such that $t_\ell < t_0$ and $\lim_{\ell\to\infty} w(y_\ell, t_\ell) = w(y_0, t_0)$.

**Remark 3.2.12.** We note that by definition our extended function $u^*$ in $Q_*$ is always left accessible at any point $(x, T), x \in \Omega$. As we state later it turns out that $u^*$ is left acessible at any $(x_0, t_0), t_0 < T$ provided that $u$ is a subsolution (under extra assumptions on $F$). However, it is clear that $u^*$ for a general function $u$ in $Q$ may not be left accessible at a point $(x_0, t_0)$ for $t_0 < T$.

As usual, the symmetric statement corresponding to Theorem 3.2.10 holds for supersolutions. (The same remark applies following lemmas and its corollary in this section.)

*Proof.* (i) For $\varphi \in C^2(Q_*)$ let $(\hat{x}, \hat{t})$ be maximizer of $u^* - \varphi$ over $Q_*$. We may assume that $\hat{t} = T$ and $u^* - \varphi$ attains its strict maximum at $(\hat{x}, T)$. Since $u^*$ is left accessible at any point $(x, T), x \in \Omega$, it is easy to see that

$$\limsup_{\alpha\to\infty}{}^*(u^* - \varphi_\alpha) = u^* - \varphi \quad \text{on } Q_*$$

for $\varphi_\alpha(x, t) = \varphi(x, t) + \alpha/(T - t)$, $\alpha > 0$. Let $(x_\alpha, t_\alpha)$ be an maximizer of $u^* - \varphi_\alpha$ over $Q_*$. Since $\varphi_\alpha = +\infty$ at $t = T$, we see $t_\alpha < T$. By the convergence of maximum points (§2.2.2) $(x_\alpha, t_\alpha) \to (\hat{x}, T)$ and $u^*(x_\alpha, t_\alpha) - \alpha/(T - t_\alpha) \to u^*(\hat{x}, T)$ as $\alpha \to \infty$ (without taking subsequences since the convergence of $u^* - \varphi_\alpha$ is monotone). Since $\overline{\lim}_{\alpha\to\infty} u^*(x_\alpha, t_\alpha) \leq u^*(\hat{x}, T)$, we now have $u^*(x_\alpha, t_\alpha) \to u(\hat{x}, T)$ as $\alpha \to \infty$. Since $u$ is a subsolution in $Q$,

$$\frac{\partial \varphi_\alpha}{\partial t}(x_\alpha, t_\alpha) + F(x_\alpha, t_\alpha, u^*(x_\alpha, t_\alpha), \nabla\varphi(x_\alpha, t_\alpha), \nabla^2\varphi(x_\alpha, t_\alpha)) \leq 0 \tag{3.2.5}$$

Since $\partial\varphi_\alpha/\partial t > \partial\varphi/\partial t$, sending $\alpha \to \infty$ yields

$$\frac{\partial \varphi}{\partial t}(\hat{x}, T) + F(\hat{x}, T, u^*(\hat{x}, T), \nabla\varphi(\hat{x}, T), \nabla^2\varphi(\hat{x}, T)) \leq 0 \tag{3.2.6}$$

by lower semicontinuity of $F$.

(ii) The basic idea of the proof is the same as for (i). We may assume that $\Omega$ is bounded and $\mathcal{F}_\Omega \neq \emptyset$ and $a\mathcal{F}_\Omega \subset \mathcal{F}_\Omega$ for $a > 0$. We take $\varphi \in C_F^2(Q_*)$ and argue in the same way to get (3.2.5) if $\nabla\varphi(x_\alpha, t_\alpha) \neq 0$ and

$$\frac{\partial \varphi_\alpha}{\partial t}(x_\alpha, t_\alpha) < 0 \quad \text{if } \nabla\varphi(x_\alpha, t_\alpha) = 0 \tag{3.2.7}$$

since $\varphi_\alpha \in C_F^2(Q_*)$. If $\nabla\varphi(\hat{x}, T) \neq 0$, we get (3.2.6) from (3.2.5) so we may assume that $\nabla\varphi(\hat{x}, T) = 0$. Our goal is now to prove

$$\varphi_t(\hat{x}, T) \leq 0. \tag{3.2.8}$$

For $\varphi$ we take $\psi$ as in (2.2.7): so that $\psi(\cdot, t) \in \mathcal{F}_\Omega$:

$$\psi(x, t) = \varphi(\hat{x}, T) + \varphi_t(\hat{x}, T)(t - T) + 2f(|x - \hat{x}|) + \theta_1(|t - T|)$$

for some $f \in \mathcal{F}_\Omega$ and some $\theta \in C^2[0, \infty)$, $\theta(0) = \theta'(0) = 0$ and $\theta(r) > 0$ for all $r > 0$ so that

$$\begin{aligned} &\varphi(x, t) < \psi(\hat{x}, T) \quad \text{for} \quad x \neq \hat{x} \quad \text{or} \quad t < T \text{ and} \\ &\varphi(\hat{x}, T) = \psi(\hat{x}, T) \end{aligned}$$

for all $(x, t)$ in a neighborhood $\mathcal{O}' = \operatorname{int} B_\rho(\hat{x}) \times (T - \delta, T]$ of $(\hat{x}, T)$ with some $\rho > 0$ and $\delta > 0$. Since $(\hat{x}, T)$ is a strict maximizer of $u^* - \psi$ over $\mathcal{O}'$, as for $\varphi$, a maximizer $(y_\alpha, s_\alpha) \in \mathcal{O}'$ of $u^* - \psi_\alpha$ over $\mathcal{O}'$ converges to $(\hat{x}, T)$ as $\alpha \to \infty$, where $\psi_\alpha = \psi + \alpha/(T - t)$. Thus instead of (3.2.5) and (3.2.7) we have

$$\frac{\partial \psi_\alpha}{\partial t}(y_\alpha, s_\alpha) + F(y_\alpha, s_\alpha, u^*(y_\alpha, s_\alpha), \nabla\psi(y_\alpha, s_\alpha), \nabla^2\psi(y_\alpha, s_\alpha)) \leq 0 \tag{3.2.9}$$

if $\nabla\psi(y_\alpha, s_\alpha) \neq 0$

$$\psi_t(y_\alpha, s_\alpha) \leq \frac{\partial \psi_\alpha}{\partial t}(y_\alpha, s_\alpha) < 0 \quad \text{if } \nabla\psi(y_\alpha, s_\alpha) = 0. \tag{3.2.10}$$

We may assume that $\nabla\psi(y_\alpha, s_\alpha) \neq 0$ for sufficiently large $\alpha$ since otherwise (3.2.10) together with $\psi_t(\hat{x}, T) = \varphi_t(\hat{x}, T)$ implies (3.2.8). By definition of $\psi$ the inequality (3.2.9) is of form

$$\varphi_t(\hat{x}, T) + \theta_1'(T - t_\alpha) + F(y_\alpha, s_\alpha\ u^*(y_\alpha, s_\alpha), \nabla h(y_\alpha), \nabla^2 h(y_\alpha)) \leq 0 \tag{3.2.11}$$

where $h(y) = 2f(|y - \hat{x}|)$. Since $y_\alpha \to \hat{x}$ and $2f \in \mathcal{F}_\Omega$,

$$\lim_{\alpha\to\infty} F(y_\alpha, s_\alpha, u^*(y_\alpha, s_\alpha),\ \nabla h(y_\alpha),\ \nabla^2 h(y_\alpha))| = 0.$$

Thus letting $\alpha \to \infty$ in (3.2.11) yields (3.2.8). $\square$

**Lemma 3.2.13** (Localization). *(i) Assume the same hypothesis of Theorem 3.2.10 (i) concerning $F$. If $u$ is a subsolution of (3.1.1) in $Q = \Omega \times (0, T)$ (resp. $Q_* = \Omega \times (0, T]$)*

*then for any $T' < T$ (resp. $T' \le T$) $u$ is a subsolution of (3.1.1) in $Q' = \Omega \times (0, T']$ (resp. $\Omega \times (0, T')$).*
*(ii) Assume the same hypotheses of Theorem 3.2.10 (ii) concerning $F$. If $u$ is an $\mathcal{F}_\Omega$-subsolution of (3.1.1) in $Q$ (resp. $Q_*$) then for any $T' < T$ $u$ is an $\mathcal{F}_\Omega$-subsolution of (3.1.1) in $\mathcal{O}'$ (resp. $\Omega \times (0, T')$).*

*Proof.* Suppose that $u$ is a subsolution of (3.1.1) in $Q$. We may assyme that $\Omega$ is bounded. Assume that $u^* - \varphi$ attains its strict maximum at $(x_0, t_0)$ over $Q'$ for $\varphi \in C^2(Q')$. Extend $\varphi$ to a $C^2$ function on $Q$ (or $Q_*$) (still denoted $\varphi$) and set $\varphi_\delta = \varphi + g(t)/\delta$ for $\delta > 0$, where $g = 0$ for $t < t_0$ and $g = (t - t_0)^3$ for $t \ge t_0$ so that $g \in C^2(\mathbf{R})$. Let $(x_\delta, t_\delta)$ be a maximizer of $u^* - \varphi_\delta$. Then $t_\delta \le t_0$ since $g(t) = 0$ for $t \le t_0$ and $g(t) \ge 0$ for $t > t_0$. Since

$$\limsup_{\delta \to 0}{}^* (u^* - \varphi_\delta) = \begin{cases} u^* - \varphi & t \le t_0 \\ -\infty & t > t_0, \end{cases}$$

The convergence of maximum points (§2.2.2) implies that $(x_\delta, t_\delta) \to (x_0, t_0)$ and $u^*(x_\delta, t_\delta) - g(t_\delta)/\delta \to u^*(x_0, t_0)$ as $\delta \to 0$. Since $u^*$ is upper semicontinuous and $g \ge 0$, this convergence yields $g(t_\delta)/\delta \to 0$ since $g \ge 0$, so that $u^*(x_\delta, t_\delta) \to u^*(x_0, t_0)$. Since $u$ is a subsolution in $Q$ (or $Q_*$), we get (3.2.5) with $\alpha$ replaced by $\delta$. Since $\partial\varphi/\partial t \le \partial\varphi_\delta/\partial t$, sending $\delta \to 0$ yields

$$\varphi_t(x_0, t_0) + F(x_0, t_0, u^*(x_0, t_0), \nabla\varphi(x_0, t_0), \nabla^2\varphi(x_0, t_0)) \le 0.$$

Thus $u$ is a subsolution of (3.1.1) in $Q'$. The proof for the statement for locatization by an open set $\Omega \times (0, T')$ is easier so is omitted.
(ii) The idea of the proof is a combination of that of (i) and Theorem 3.2.10 (ii). It is safely left to the reader as an exercise. □

At this moment for a subsolution $u$ we wonder whether or not $u^*$ at $t = T' < T$ agrees with the upper semicontinuous envelope of the restriction of $u^*$ on $Q'$. In other words we wonder whether $u^*$ is left accessible at $(x, T')$ for all $x \in \Omega$, $T' < T$. As already seen in the counterexample in §2.1.2, there may be a subsolution which is not left accessible for singular degenerate parabolic equation. We have to restrict $F$ or modify the notion of solutions as in §2.1.3 to conclude the left accessibility of solutions. The next lemma is essentially found in a paper by Y.-G. Chen, Y. Giga and S. Goto (1991b). We do not give the proof here since we won't use this result. (The statement for $\mathcal{F}$-solutions are not included in the above article but the proof is easily extended to this case by replacing $|z_i - y_{0i}|^4$ and $|z_i - z_{i0}|^4$ by $f(|z_i - y_{0i}|)$ and $f(|z_i - z_{0i}|)$ for $f \in \mathcal{F}$, respectively without assuming the second inequality of (3.2.12) of course.)

**Lemma 3.2.14** (Accessibility). *Let $k$ be a positive integer. Let $T > 0$ and $y_{0i} \in \mathbf{R}^{Ni}$ and let $\Omega_i$ be an open set in $\mathbf{R}^{Ni}$ such that $y_{0i} \in \Omega_i$ for $1 \le i \le k$.*
*(i) Assume that $F = F_i : W_i \to \mathbf{R} \cup \{-\infty\}$ is lower semicontinuous and satisfies*

$$\begin{aligned} &F(x, t, r, p, X) > -\infty \quad \text{for } p \not\equiv 0, r \in \mathbf{R}, X \in S^N \\ &F(x, t, r, o, O) > -\infty \quad \text{for } r \in \mathbf{R} \end{aligned} \tag{3.2.12}$$

*with $N = N_i$ and $t = T$ for all $x$ near $y_{0i}(1 \le i \le k)$, where $W_i = \overline{\Omega}_i \times [0,T] \times \mathbf{R} \times \mathbf{R}^{Ni} \times \mathbf{S}^{Ni}$. Let $u_i$ be a subsolution of (3.1.1) with $F = F_i$ in $Q_i = \Omega_i \times (0,T]$. Then the function*

$$w(z,t) = \sum_{i=1}^{k} u_i^*(z_i, t)$$

*is left accessible at $(y_0, T)$, where $z = (z_1, \cdots z_k)$, $z_i \in \Omega_i$ and $y_0 = (y_{01}, \cdots, y_{0k})$.*
*(ii) Assume the same hypothesis of Theorem 3.2.10 (ii) concerning $F = F_i$ with $\Omega = \Omega_i$, $N = N_i$. Then $w$ is left accessible at $(y_0, T)$ provided that $u_i$ is an $\mathcal{F}_{\Omega_i}$-subsolution of (3.3.1) with $F = F_i$.*

**Corollary 3.2.15.** *Let $\Omega$ be an open set in $\mathbf{R}^N$.*
*(i) Assume that $F$ satisfies (3.2.12) for all $(x,t) \in Q = \Omega \times (0,T)$. If $u$ is a subsolution of (3.1.1) in $Q$, then $u^*$ is left accessible at each $(x,t) \in Q$.*
*(ii) Assume the same hypothesis of Theorem 3.2.10 (ii) concerning $F$. If $u$ is an $\mathcal{F}_\Omega$-subsolution of (3.1.1) in $Q$, then $u^*$ is left accessible at each $(x,t) \in Q$.*

This follows from localization and accessibility lemma (Lemmas 3.2.13 and 3.2.14). In Lemma 3.2.14 the conclusion for general $k$ cannot be reduced to the case of single function since the sum $u(x,t) + v(y,t)$ of two left accessible (upper semicontinuous) functions $u$ and $v$ may not be left accessible in general. Lemma 3.2.14 implies not only accessibility of subsolution $u$ itself but also accessibility of sum $u_1(x,t) + u_2(y,t)$ of two subsolutions or difference $u(x,t) - v(y,t)$ where $v$ is a supersolution (with assumptions of $F$ symmetric to (3.2.12)).

## 3.3 General idea for the proof of comparison principles

We consider a simple equation of non-evolution type to motivate the idea to establish comparison principles for viscosity sub- and supersolutions.

### 3.3.1 A typical problem

Let $\mathcal{O}$ be a bounded domain in $\mathbf{R}^d$. We consider

$$u + F(Du, D^2u) = 0 \quad \text{in } \mathcal{O}. \tag{3.3.1}$$

This equation has a comparison principle for degenerate elliptic $F$ for example of the following form.

**Theorem 3.3.1.** *Assume that $F : \mathbf{R}^d \times \mathbf{S}^d \to \mathbf{R}$ is degenerate elliptic and continuous. Let $u$ and $v$, respectively, be sub- and supersolutions of (3.3.1). If $-\infty < u^* \le v_* < \infty$ on $\partial\mathcal{O}$, then $u^* \le v_*$ in $\mathcal{O}$.*

**Classical idea.** If both $u$ and $v$ are $C^2$ in $\mathcal{O}$ and continuous in $\overline{\mathcal{O}}$, the proof is simple. We consider the difference $g(z) = u(z) - v(z)$. Assume that $g$ were positive somewhere in $\mathcal{O}$. Since $g$ is continuous and $\overline{\mathcal{O}}$ is bounded, $g$ attains its positive maximum over $\overline{\mathcal{O}}$ at some point $\hat{z} \in \overline{\mathcal{O}}$. Since on the boundary $\partial\mathcal{O}$ we have $g \leq 0$, so $\hat{z}$ must be an interior point of $\mathcal{O}$. A classical maximum principle for $C^2$ function implies

$$Dg(\hat{z}) = 0, \ D^2g(\hat{z}) \leq 0,$$

so that

$$Du(\hat{z}) = Dv(\hat{z}), D^2u(\hat{z}) \leq D^2v(\hat{z}). \tag{3.3.2}$$

Since $u$ and $v$ are classical sub- and supersolution of (3.3.1), respectively (cf. Proposition 2.2.1), we have

$$u(\hat{z}) + F(Du(\hat{z}), D^2u(\hat{z})) \leq 0 \leq v(\hat{z}) + F(Dv(\hat{z}), D^2v(\hat{z})).$$

By (3.3.2) and the degenerate ellipticity of $F$ we conclude $g(\hat{z}) = u(\hat{z}) - v(\hat{z}) \leq 0$ which contradicts the positivity of the maximum of $g$.

This argument can be easily generalized if one of $u^*$ and $v_*$ belongs to $C^2(\mathcal{O})$. For example, assume $v_* \in C^2(\mathcal{O})$. Since $g = u^* - v_*$ is upper semicontinuous and $\overline{\mathcal{O}}$ is compact as before $g$ attains its positive maximum at an interior point $\hat{z} \in \mathcal{O}$. By definition of $J^{2,+}$ instead of (3.3.2) we have

$$(Dv(\hat{z}), D^2v(\hat{z})) \in J^{2,+}_{\mathcal{O}} u^*(\hat{z}).$$

Since $u$ is a subsolution and $v$ is a classical supersolution,

$$u^*(\hat{z}) + F(Dv(\hat{z}), D^2v(\hat{z})) \leq 0 \leq v_*(\hat{z}) + F(Dv(\hat{z}), D^2v(\hat{z})),$$

which again yields a contradiction $g(\hat{z}) \leq 0$.

**Doubling variables.** If both $u$ and $v$ are not $C^2$, at least one of $J^{2,+}u^*(\hat{z})$ and $J^{2,-}v_*(\hat{z})$ may be empty so the classical idea does not work. If we are allowed to consider $J^{2,+}u^*$ and $J^{2,-}v_*$ at different points $\hat{z}, \hat{\zeta}$, there are more chance to have semijets. For this purpose we double the variables and consider

$$w(z, \zeta) = u(z) - v(\zeta) \tag{3.3.3}$$

instead of $g$; here and hereafter we suppress $*$ in both $u$ and $v$. However, since we are only interested in the behaviour of $w$ where $z$ is close to $\zeta$, we need to penalize around the diagonal set

$$\{(z, \zeta) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}; \ z = \zeta\}.$$

**Penalizing process.** We consider

$$\Phi_\alpha(z, \zeta) = w(z, \zeta) - \alpha\psi(z - \zeta) \tag{3.3.4}$$

for $\varphi \in C^2(\mathbf{R}^d)$ satisfying at least $\underline{\lim}_{|z|\to\infty}\psi(z) > 0$, $\psi(0) = 0$, and $\psi(z) > 0$ for $z \neq 0$ and $\alpha$ is a positive parameter so that the term $\alpha\varphi(z-\zeta)$ tends to infinity as $\alpha \to \infty$ unless $z = \zeta$. There are freedom to choose $\psi$ depending on the situation. For (3.3.1) one may take

$$\psi(z) = |z|^2/2.$$

We maximize the function $\Phi_\alpha$ over $\overline{\mathcal{O}} \times \overline{\mathcal{O}}$. As $\alpha \to \infty$ the maximizer and maximum approximate those of $g$. In fact, if $(z_\alpha, \zeta_\alpha) \in \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ is a maximizer of $\Phi_\alpha$ over $\overline{\mathcal{O}} \times \overline{\mathcal{O}}$, i.e.,

$$M_\alpha := \max_{\overline{\mathcal{O}}\times\overline{\mathcal{O}}} \Phi_\alpha = \Phi_\alpha(z_\alpha, \zeta_\alpha),$$

then
(i) $\lim_{\alpha\to\infty} \alpha\psi(z_\alpha - \zeta_\alpha) = 0$ and
(ii) $\lim_{\alpha\to\infty} M_\alpha = \max_{\overline{\mathcal{O}}}(u-v) = (u-v)(\hat{z})$ whenever $\hat{z}$ is an accumulation point of $\{x_\alpha\}$ as $\alpha \to \infty$.
(The proof is elementary. Since $w$ is bounded, by definition of maximizers

$$\overline{\lim}_{\alpha\to\infty}\alpha\psi(z_\alpha - \zeta_\alpha) < \infty$$

which in particular implies $z_\alpha - \zeta_\alpha \to 0$ as $\alpha \to \infty$. Let $\hat{z}$ be an accumulation point of $\{z_\alpha\}$ as $\alpha \to \infty$ so that $z_{\alpha_j} \to \hat{z}$, $\zeta_{\alpha_j} \to \hat{z}$ for some subsequence $\{\alpha_j\}$. Since

$$g(z) = g(z) - \alpha\psi(0) \leq w(z_\alpha, \zeta_\alpha) - \alpha\psi(z_\alpha - \zeta_\alpha) \quad \text{for all } z \in \overline{\mathcal{O}}$$

and $w$ is upper semicontinuous, letting $\alpha \to \infty$ yields

$$g(z) \leq g(\hat{z}) - \eta$$

where $\eta = \overline{\lim}_{j\to\infty}\alpha_j\psi(z_{\alpha_j} - \zeta_{\alpha_j})$. We may take $z = \hat{z}$ to conclude $\eta = 0$. This yields $\overline{\lim}_{\alpha\to\infty}\alpha\psi(z_\alpha - \zeta_\alpha) = 0$ and also $\lim_{\alpha\to\infty} M_\alpha = g(\hat{z}) \geq g(z)$ for all $z \in \overline{\mathcal{O}}$.)

**Relation of derivatives.** Using $\Phi_\alpha$ with $\psi(z) = |z|^2/2$ we sketch the proof of Theorem 3.3.1. Assume that $g$ were positive somewhere in $\mathcal{O}$. By approximation of maximum of $g$ by that of $\Phi_\alpha$, there is $\delta > 0$ such that $M_\alpha > \delta$ for sufficiently large $\alpha$. Since $g(z) \leq 0$ for $z \in \partial\mathcal{O}$ and the maximizer $(z_\alpha, \zeta_\alpha)$ of $\Phi_\alpha$ converges to $(\hat{z}, \hat{z})$ (by taking a subsequence), for sufficiently large $\alpha$ the maximum of $\Phi_\alpha$ does not attain on $\partial(\mathcal{O}\times\mathcal{O})$. We may assume that $(z_\alpha, \zeta_\alpha)$ is an (interior) point of $\mathcal{O}\times\mathcal{O}$ and $M_\alpha > 0$ and we fix $\alpha > 0$.

To see relations of derivatives we assume $u$ and $v$ are $C^2$ around ($\hat{z} = z_\alpha$ and $\hat{\zeta} = \zeta_\alpha$ respectively. Since $\Phi = \Phi_\alpha$ attains its maximum at $(\hat{z}, \hat{\zeta})$ a classical maximum principle for functions of $2d$ variables implies

$$Du(\hat{z}) = D_z\varphi(\hat{z}, \hat{\zeta}), Dv(\hat{\zeta}) = -D_\zeta\varphi(\hat{z}, \hat{\zeta}) \tag{3.3.5}$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq (D^2\varphi)(\hat{z}, \hat{\zeta}) \tag{3.3.6}$$

with $\varphi(z,\zeta) = \alpha\psi(z-\zeta)$, where $X = D^2u(\hat{z})$, $Y = D^2v(\hat{\zeta})$. For our choice of $\psi(z) = |z|^2/2$, this yields

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.3.7}$$

This implies $X \le Y$ since

$$\langle (X-Y)\rho, \rho\rangle = ({}^t\rho, {}^t\rho)\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\begin{pmatrix}\rho \\ \rho\end{pmatrix} \le \alpha({}^t\rho, {}^t\rho)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\begin{pmatrix}\rho \\ \rho\end{pmatrix} = 0$$

for $\rho \in \mathbf{R}^d$. The property $X \le Y$ also follows from (3.3.6) for general $\varphi$ provided that $\varphi$ is a function of $z - \zeta$. If $\varphi$ is a function of $z-\zeta$, (3.3.5) implies $Du(\hat{z}) = Dv(\hat{\zeta})$. Since $u$ and $v$ are classical sub- and supersolution near $\hat{z}, \hat{\zeta}$ respectively, we see

$$u(\hat{z}) + F(Du(\hat{z}), X) \le 0 \le v(\hat{\zeta}) + F(Dv(\hat{\zeta}), Y).$$

Since $X \le Y$ and $Du(\hat{z}) = Dv(\hat{\zeta})$, one get $u(\hat{z}) \le v(\hat{\zeta})$ as before by degenerate ellipticity of $F$ which yields a contradiction to $\Phi(\hat{z}, \hat{\zeta}) > 0$.

It is not difficult to extend the property $Du(\hat{z}) = Dv(\hat{\zeta})$ for semijets of functions. In fact, we conclude that there are elements of $J^{2,+}u(\hat{z})$ and $J^{2,-}v(\hat{\zeta})$ whose first derivative part is the same. So this observation yields a rigorous proof of Theorem 3.3.1 for the first order equations. The extension of (3.3.6) for semicontinuous functions is not trivial but surprisingly it is possible with some modification of results.

### 3.3.2 Maximum principle for semicontinuous functions

We give a maximum principle of type (3.3.5) and (3.3.6), which by now is standard, for semicontinuous functions.

**Theorem 3.3.2.** *Let $\mathcal{Z}_i$ be a locally compact subset of $\mathbf{R}^{Ni}$ for $i = 1, \cdots, k$. Let $u_i$ be an upper semicontinuous functions on $\mathcal{Z}_i$ with values in $\mathbf{R} \cup \{-\infty\}$. Set*

$$w(z) = u_1(z_1) + \cdots + u_k(z_k) \quad \text{for } z = (z_1, \cdots z_k) \in \mathcal{Z},$$

*where*

$$\mathcal{Z} = \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_k. \tag{3.3.8}$$

*For $\varphi \in C^2(\mathcal{Z})$ suppose that $\hat{z} = (\hat{z}_1, \cdots, \hat{z}_k) \in \mathcal{Z}$ be a point at which a maximum of $w - \varphi$ over $\mathcal{Z}$ is attained, i.e.,*

$$\max_{\mathcal{Z}}(w-\varphi) = (w-\varphi)(\hat{z}).$$

*Then for each $\lambda > 0$ there is $X_i \in \mathbf{S}^{Ni}$ such that*

$$(D_{z_i}\varphi(\hat{z}), X_i) \in \overline{J}^{2,+}_{\mathcal{Z}_i} u_i(\hat{z}_i) \quad \text{for } i = 1, \cdots, k \tag{3.3.9}$$

*and the block diagonal matrix with entries $X_i$ satisfies*

$$-(\frac{1}{\lambda} + |A|)I \le \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \le A + \lambda A^2, \tag{3.3.10}$$

*where* $A = D^2\varphi(\hat{z}) \in \mathbf{S}^N$, $N = N_1 + \cdots + N_k$.

The proof depends on a deep theory of real analysis. There is a nice presentation of the proof in the review paper by M. G. Crandall, H. Ishii and P.-L. Lions (1992), where $u_i$ is not allowed to take $-\infty$ but this small extension causes no technical problems. We do not give the proof. The maximum principle have a lot of modified version so that it applies for example for parabolic problems. The next version, whose proof paralles that of Theorem 3.3.2, is useful for our parabolic problem as pointed out by Mi-Ho Giga.

**Theorem 3.3.3.** *Assume the same hypothesis concerning* $u_i, \varphi$ *and* $\hat{z}$ *with orthogonal decomposition*

$$\mathbf{R}^N = \oplus_{i=1}^k V_i,\ V_i = \mathbf{R}^{Ni},\ \mathcal{Z}_i \subset V_i \quad \text{for } i = 1, \cdots k$$

*which corresponds to the decomposition of* $\mathcal{Z}$ *in (3.3.8). Let* $\{P_j\}_{j=1}^{\ell}$ *be a family of orthogonal projection on* $\mathbf{R}^N$ *that satisfies* $\sum_{j=1}^{\ell} P_j = I$*. Assume that* $V_i$ *is invariant under the operator of* $P_j$ *i.e.* $P_j V_i \subset V_i$ *for all* $i = 1, \cdots k, j = 1, \cdots \ell$*. Assume that* $W_j = P_j \mathbf{R}^N$ *is invariant under the operation of* $A$*, i.e.,* $AW_j \subset W_j$*. Then for each* $\lambda_j > 0$ $(1 \le j \le \ell)$ *there is* $X_i \in \mathbf{S}^{Ni}$ *that satisfies (3.3.9) and*

$$-\sum_{j=1}^{\ell} \left( \frac{1}{\lambda_j} + |AP_j| \right) P_j \le \begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_k \end{pmatrix} \le A + \sum_{j=1}^{\ell} \lambda_j A^2 P_j. \tag{3.3.11}$$

The condition (3.3.11) is more useful than (3.3.10) when we would like to handle spatial and time derivatives separately for evolutional problems. In (3.3.11) the $W_j$-component of the block diagonal matrix of $X_i$ is estimated not only by $|A|$ but also by $|AP_j|$.

**Completion of the proof of Theorem 3.3.1.** If the assertion were false, then as observed in §3.3.1 there is a maximizer $(\hat{z}, \hat{\zeta})$ in $\mathcal{O} \times \mathcal{O}$ of the penalized function $\Phi = \Phi_\alpha$ for fix $\alpha$ such that the maximum $M = M_\alpha$ of $\Phi$ is strictly positive. We apply Theorem 3.3.2 with $k = 2$, $\mathcal{Z}_1 = \mathcal{Z}_2 = \mathcal{O}$, $u_1 = u$, $u_2 = -v$, $\varphi(z, \zeta) = \alpha|z - \zeta|^2/2$. Since

$$D_z\varphi(\hat{z}, \hat{\zeta}) = -D_\zeta\varphi(\hat{z}, \hat{\zeta}) = \alpha(\hat{z} - \hat{\zeta}), A = \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$
$$A^2 = 2\alpha A,\ |A| = 2\alpha$$

and $\overline{J}_{\mathcal{O}}^{2,-} v = -\overline{J}_{\mathcal{O}}^{2,+}(-v)$, we conclude from (3.3.9) and (3.3.10) that for every $\lambda > 0$ there exists $X$ and $Y \in \mathbf{S}^N$ such that

$$(\alpha(\hat{z} - \hat{\zeta}), X) \in \overline{J}_{\mathcal{O}}^{2,+} u(\hat{z}),\ (\alpha(\hat{z} - \hat{\zeta}), Y) \in \overline{J}_{\mathcal{O}}^{2,-} v(\hat{\zeta}) \tag{3.3.12}$$

and

$$-(\frac{1}{\lambda} + 2\alpha) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \alpha(1 + 2\lambda\alpha) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.3.13}$$

In comparison with (3.3.7) the estimate from above for the matrix containg $X$ and $-Y$ is a little bit different but still our estimate concludes $X \leq Y$. Since $u$ and $v$ are sub- and supersolutions of (3.1.1) respectively, thanks to Remark 3.2.9 the relation (3.3.12) yields

$$u(\hat{z}) + F(\alpha(\hat{z} - \hat{\zeta}), X) \leq 0 \leq v(\hat{\zeta}) + F(\alpha(\hat{z} - \hat{\zeta}), Y).$$

Since $X \leq Y$ by (3.3.13), the degenerate ellipticity of $F$ implies that $u(\hat{z}) \leq v(\hat{\zeta})$ which contradicts $M = M_\alpha = \Phi_\alpha(\hat{z}, \hat{\zeta}) > 0$.

We do not use the lower bound in (3.3.13) in the proof. Its presence gives a control for bound for $|X|$ and $|Y|$ which is important in many more complicated problems. We presented the equation (3.3.1) to motivate the method so Theorem 3.3.1 is by no means optimal for our method. The method presented above already applies more general equation of form

$$E(z, u, Du, D^2u) = 0$$

as pointed out in the review paper by M. G. Crandall, H. Ishii and P.-L. Lions (1991). In the next section we shall prove comparison principles for spatially independent equation of evolution type (3.1.1) but having singularities at zeros of the gradient of solutions. For singular degenerate equation we should be careful in the way of penalizing.

## 3.4 Proof of comparison principles for parabolic equations

We now study comparison principles for parabolic equations. We use infinitesimal version (Proposition 3.2.6) definitions of viscosity solutions as in the proof of Theorem 3.3.1. However, since the equation may by singular where the gradient of solutions vanishes, an extra work is necessary.

### 3.4.1 Proof for bounded domains

We approximate maximum by penalization. We give its abstract form which also applies to maximizers $(z_\alpha, \zeta_\alpha)$ of $\Phi_\alpha$ in §3.3.1.

**Lemma 3.4.1.** *Let $Z_0$ be a closed subset of a metric space $Z$. Let $w$ be an upper semicontinuous function from $Z$ to $\mathbf{R} \cup \{-\infty\}$. Assume that*

$$\overline{\lim}_{\delta \to 0} \sup\{w(z); d(z, Z_0) \leq \delta, z \in Z\} \leq \sup_{Z_0} w. \tag{3.4.1}$$

*Let $\varphi_\sigma$ be a nonnegative continuous function on $Z$ parametrized by $\sigma = (\sigma_1, \cdots, \sigma_k) \in \mathbf{R}^k, \sigma_i \geq 0$ $(1 \leq i \leq k)$ such that $\varphi_\sigma = 0$ on $Z_0$ and that for any $\delta > 0$*

$$\lim_{|\sigma| \to \infty} \inf\{\varphi_\sigma(z);\ d(z, Z_0) \geq \delta\} = \infty. \tag{3.4.2}$$

*Let*

$$M_\sigma = \sup_Z (w(z) - \varphi_\sigma(z)) \tag{3.4.3}$$

*and assume that* $M_\sigma < \infty$ *for large* $|\sigma|$. *Let* $z_\sigma$ *be such that*

$$\lim_{|\sigma|\to\infty} (M_\sigma - (w(z_\sigma) - \varphi_\sigma(z_\sigma))) = 0 \tag{3.4.4}$$

*Then*
*(i)* $\lim_{|\sigma|\to\infty} \varphi_\sigma(z_\sigma) = 0$ *so that* $\lim_{|\sigma|\to\infty} d(z_\sigma, Z_0) = 0$ *and*
*(ii)* $\lim_{|\sigma|\to\infty} M_\sigma = \sup_{Z_0} w = w(\hat{z})$ *and* $\hat{z} \in Z_0$, *whenever* $\hat{z}$ *is an accumulation point of* $\{z_\sigma\}$ *as* $|\sigma| \to \infty$.

*Proof.* Since $M_\sigma < \infty$ for large $|\sigma|$, by definition (3.4.4) of $z_\sigma$

$$\eta = \overline{\lim}_{|\sigma|\to\infty} \varphi_\sigma(z_\sigma) < \infty$$

which in particular implies $d(z_\sigma, Z_0) \to 0$ (as $|\sigma| \to \infty$) by (3.4.2). By (3.4.1) and (3.4.4) we now conclude that

$$\begin{aligned} w(z) = \lim_{j\to\infty}(w(z) - \varphi_{\sigma_j}(z)) &\le \overline{\lim}_{j\to\infty}(w(z_{\sigma_j}) - \varphi(z_{\sigma_j})) \\ &\le \sup_{Z_0} w - \eta, \ z \in Z_0 \end{aligned}$$

if $\{\sigma_j\}$ is chosen such that $\eta = \lim_{j\to\infty} \varphi_{\sigma_j}(z_{\sigma_j})$. We take sup of the left hand side over $Z_0$ which forces $\eta = 0$. This yields (i) and $\overline{\lim}_{\sigma\to\infty} M_\sigma \le \sup_{Z_0} w$. Thus (ii) follows since $M_\sigma \ge \sup_{Z_0} w$ by definition and $w$ is upper semicontinuous at $z = \hat{z}$. □

*Proof of Theorem 3.1.1.* By the equivalence of an $\mathcal{F}$-subsolution and usual subsolution (Proposition 2.2.8) under (F1)–(F3) it suffices to prove the case that $u$ and $v$ are, respectively, $\mathcal{F}$-sub- and supersolutions of (3.1.1) in $Q$ under the assumptions (F1), (F2) (F3') and (F4). We suppress the word $\mathcal{F}$ in this proof.

We may assume that $u$ and $-v$ are upper semicontinuous from $\overline{\Omega} \times [0, T)$ to $\mathbf{R} \cup \{-\infty\}$ with $u \le v$ on $\partial_p Q$. Since $u$ and $-v < \infty$ in $\overline{\Omega} \times [0, T)$ the extended function $(u_{T'})^*$ and $(-v_{T'})^* < \infty$ at $t = T'$ and $(u_{T'})^* \le (v_{T'})_*$ on $\partial_p Q$, where $u_{T'}$ and $v_{T'}$ are, respectively, the restriction of $u$ and $v$ on $\overline{\Omega} \times [0, T')$ for $T' < T$. (If we use Corollary 3.2.15, it is easy to see that $(u_{T'})^* = u$, $(-v_{T'})^* = v$ up to $t = T'$ but we do not need this fact.) By rewriting $T'$ by $T$ we may assume that the extended functions $u^*, -v_*$ to $t = T$ does not take value $\infty$ on $\overline{\Omega} \times \{t = T\}$ and that $u^* \le v_*$ on $\partial_p \overline{Q}$, where $\partial_p \overline{Q} = \overline{\Omega} \times \{0\} \cup \partial\Omega \times [0, T]$. By Theorem 3.2.10 the extended function $u^*$ and $v_*$ are, respectively, sub- and supersolutions of (3.1.1) in $Q_* := \Omega \times (0, T]$. We shall denote $u^*$ and $v_*$ simply by $u$ and $v$.

We may also assume that $u$ and $v$ are bounded in $\overline{Q}$. Indeed, since $u$ and $-v$ are upper semicontinuous in $\overline{Q}$, they are bounded from above, i.e. $L = \sup_{\overline{Q}} u < \infty$, $S = \inf_{\overline{Q}} v > -\infty$. Since $-at - 1 + S$ is a subsolution of (3.1.1) in $Q$ for large $a > 0$, the function

$$u_S = u \vee (-at - 1 + S)$$

for such an $a$ is a subsolution of (3.1.1) by Lemma 2.4.7. Similarly, $v_L = v \wedge (at + 1 + L)$ us a supersolution for large $a > 0$. We may consider $u_S$ and $v_L$ instead of $u$ and $v$. So we may assume $u$ and $-v$ are bounded also from below.

In the monotonicity condition (F4) we may assume that $c_0 = 0$ by replacing $u$ by $\tilde{u}e^{c_0 t}$ since the equation for $\tilde{u}$ is

$$\tilde{u}_t + c_0\tilde{u} + e^{-c_0 t}F(x, t, e^{c_0 t}\tilde{u}, e^{c_0 t}\nabla\tilde{u}, e^{c_0 t}\nabla^2\tilde{u}) = 0;$$

the conditions (F1), (F2), (F3') are invariant under this transform.

As in the proof of Theorem 3.3.1 we double variables for $u - v$ and put penalizing term with small modifications. We take $f \in \mathcal{F}$ and set

$$\begin{aligned} w_\gamma(x, t, y, s) &= u(x, t) - v(y, s) - \gamma(t + s) \\ \varphi_{\alpha\beta}(x, t, y, s) &= \alpha f(|x - y|) + \beta|t - s|^2 \end{aligned} \tag{3.4.5}$$

for $\alpha, \beta, \gamma > 0, (x, t) \in \overline{Q}$, $(y, s) \in \overline{Q}$. Assume that the conclusion were false so that there would exist $(x_0, t_0) \in Q$ such that $u(x_0, t_0) - v(x_0, t_0) > 0$. Then there would exist $\gamma_0 > 0$ and $\delta_0 > 0$ that satisfies

$$\sup_Z w_\gamma \geq \delta_0 \quad \text{for all } 0 < \gamma < \gamma_0, \tag{3.4.6}$$

where $Z = \overline{Q} \times \overline{Q}$. Since $u - v \leq 0$ on $\partial_p\overline{Q}$, we may assume that

$$\sup\{w(x, t, y, s); (x, t, y, s) \in Z_0, (x, t) \in \partial_p\overline{Q}\} \leq \delta_0/2 \tag{3.4.7}$$

for $0 < \gamma < \gamma_0$ by taking $\gamma_0$ smaller, where

$$Z_0 = \{(x, t, y, s) \in \overline{Q} \times \overline{Q}; x = y, t = s\}. \tag{3.4.8}$$

We shall fix $\gamma < \gamma_0$ and suppress the subscript $\gamma$. Since $w$ is upper semicontinuous and $Z$ is compact there is a maximizer $z_\sigma \in Z$ of $w - \varphi_\sigma$, where $\sigma = (\alpha, \beta)$, $z_\sigma = (x_\sigma, t_\sigma, y_\sigma, s_\sigma) \in \overline{Q} \times \overline{Q}$. Since $Z_0$ is compact, the upper semicontinuity of $w$ implies (3.4.1). Since (3.4.2) is trivially fulfilled for $\varphi_\sigma$, we apply Lemma 3.4.1 with (3.4.3) and get $|x_\sigma - y_\sigma| \to 0$, $|t_\sigma - s_\sigma| \to 0$ as $|\sigma| \to \infty$. Since $Z$ is compact, the accumulation point of $\{z_\sigma\}$ always exists so that $x_\sigma \to \hat{x}$, $y_\sigma \to \hat{x}$, $t_\sigma \to \hat{t}$, $x_\sigma \to \hat{s}$ for some $(\hat{x}, \hat{t}) \in \overline{Q}$ by taking subsequences still denoted $x_\sigma$ and $t_\sigma$. By (ii) of Lemma 3.4.1 and (3.4.6) we see

$$\delta_0 \leq \lim_{|\sigma|\to\infty} M_\sigma = \sup_{Z_0} w = w(\hat{x}, \hat{t}, \hat{x}, \hat{t}).$$

Since (3.4.7) holds, this implies $(\hat{x}, \hat{t}) \notin \partial_p\overline{Q}$. In other words for sufficiently large $\sigma = (\alpha, \beta)$ say $\alpha > \alpha_0$, $\beta > \beta_0$ we observe that $(x_\sigma, t_\sigma, y_\sigma, s_\sigma) \in Q_* \times Q_*$, where $Q_* = \Omega \times (0, T]$.

We shall study behaviour of $u(x, t) - v(y, s)$ near $(x_\sigma, t_\sigma, y_\sigma, s_\sigma) \in Q_* \times Q_*$ for $\sigma = (\alpha, \beta)$, $\alpha > \alpha_0, \beta > \beta_0$.

**Case 1.** $x_{\sigma_j} = y_{\sigma_j}$ for some $\sigma_j \to \infty$. We fix $\sigma = \sigma_j$. Since

$$(w - \varphi_\sigma)(x, t, y_\sigma, s_\sigma) \leq (w - \varphi_\sigma)(x_\sigma, t_\sigma, y_\sigma, s_\sigma) = M_\sigma$$

for $(x, t) \in Q_*$, we see

$$\max_{Q_*}(u - \varphi^+) = (u - \varphi^+)(x_\sigma, t_\sigma)$$

if
$$\varphi^+(x,t) = \alpha f(|x-y_\sigma|) + \beta(t-s_\sigma)^2 + \gamma t.$$
Similarly, $(w-\varphi_\sigma)(x_\sigma, t_\sigma, y, s) \le M_\sigma$ implies
$$\min_{Q_*}(v-\varphi^-) = (v-\varphi^-)(y_\sigma, s_\sigma),$$
where
$$\varphi^-(y,s) = -\alpha f(|x_\sigma - y|) - \beta(t_\sigma - s)^2 - \gamma s.$$
Our assumption $x_\sigma = y_\sigma$ is equivalent to say that $\nabla\varphi^+(x_\sigma, t_\sigma) = 0$ and $\nabla\varphi^-(y_\sigma, s_\sigma) = 0$. Since $a\mathcal{F} \subset \mathcal{F}$ for $a > 0$, $\varphi^\pm \in C^2_F(Q_*)$. By the definition of $\mathcal{F}$-solutions in Chapter 2, we observe that
$$\varphi^+_t(x_\sigma, t_\sigma) = 2\beta(t_\sigma - s_\sigma) + \gamma \le 0 \le \varphi^-_t(y_\sigma, s_\sigma) = 2\beta(t_\sigma - s_\sigma) - \gamma.$$
This contradicts $\gamma > 0$ so Case 1 does not occur for all $\sigma = (\alpha, \beta)$, $\alpha > \alpha_0$, $\beta > \beta_0$.

**Case 2.** $x_\sigma \ne y_\sigma$ for sufficiently large $\sigma$. We set $\xi = (x,t), \eta = (s,y)$ and $\xi_\sigma = (x_\sigma, t_\sigma)$, $\eta_\sigma = (y_\sigma, s_\sigma)$. Since $w - \varphi_\sigma$ takes its maximum over $Q_* \times Q_*$ at $(\xi_\sigma, \eta_\sigma)$, we see
$$\left(\begin{pmatrix}\hat\varphi_\xi \\ \hat\varphi_\eta\end{pmatrix},\ A\right) \in J^{2,+}w(\xi_\sigma, \eta_\sigma)$$
$$A = \begin{pmatrix}\hat\varphi_{\xi\xi} & \hat\varphi_{\xi\eta} \\ \hat\varphi_{\eta\xi} & \hat\varphi_{\eta\eta}\end{pmatrix}$$
where $\hat\varphi_\xi = (D_\zeta\varphi_\sigma)(\xi_\sigma, \eta_\sigma)$, $\hat\varphi_{\xi\xi} = (D^2_\xi\varphi_\sigma)(\xi_\sigma, \eta_\sigma)$ and so on. We apply Theorem 3.3.3 with $k = 2$, $N_1 = N_2 = N+1$, $\mathcal{Z}_1 = \mathcal{Z}_2 = Q_*$, $\ell = 2$, where projection $P_1$ and $P_2$ are defined by
$$P_1 : \mathbf{R}^{N+1} \times \mathbf{R}^{N+1} \to \mathbf{R}^N \times \mathbf{R}^N, P_1(x,t,y,s) = (x,y)$$
and $P_2 = I - P_1$. (It is easy to check that $A$ satisfies $AW_j \subset W_j$ with $W_j = P_j\mathbf{R}^{2(N+1)}$.) Then we find that for each $\lambda = (\lambda_1, \lambda_2)$, $\lambda_j > 0 (j = 1,2)$ there exists $X_1, -Y_1 \in \mathbf{S}^{N+1}$ such that
$$(\hat\varphi_\xi, X_1) \in \overline{J^{2,+}}(u-\gamma t)(x_\sigma, t_\sigma) \tag{3.4.9}$$
$$(-\hat\varphi_\eta, Y_1) \in \overline{J^{2,-}}(v+\gamma s)(y_\sigma, s_\sigma)$$
$$-\sum_{j=1}^2\left(\frac{1}{\lambda_j} + |AP_j|\right)P_j \le \begin{pmatrix}X_1 & 0 \\ 0 & -Y_1\end{pmatrix} \le \left(A + \sum_{i=1}^2 \lambda_j A^2\right)P_j. \tag{3.4.10}$$
If we represent
$$X_1 = \begin{pmatrix}X & \ell_1 \\ {}^t\ell_1 & b_1\end{pmatrix} \quad Y_1 = \begin{pmatrix}Y & \ell_2 \\ {}^t\ell_2 & b_2\end{pmatrix}$$
by using $b_i \in \mathbf{R}$, $\ell_i \in \mathbf{R}^N (i = 1,2)$, $X, Y \in \mathbf{S}^N$, $(X,Y)$ can be regarded as a linear operator in $W_1 = P_1\mathbf{R}^{2(N+1)}$. Thus (3.4.10) yields
$$-\left(\frac{1}{\lambda_1} + |A_0|\right)I \le \begin{pmatrix}X & O \\ O & -Y\end{pmatrix} \le A_0 + \lambda_1 A_0^2 \tag{3.4.11}$$

where

$$A_0 = \begin{pmatrix} \hat{\varphi}_{xx} & \hat{\varphi}_{xy} \\ \hat{\varphi}_{yx} & \hat{\varphi}_{yy} \end{pmatrix}.$$

By Remark 3.2.9 the relation (3.4.9) yields

$$(\gamma + \hat{\varphi}_t, \hat{\varphi}_x, X) \in \overline{\mathcal{P}}^{2,+} u(x_\sigma, t_\sigma),$$
$$(-\gamma - \hat{\varphi}_s, -\hat{\varphi}_y, Y) \in \overline{\mathcal{P}}^{2,-} v(y_\sigma, s_\sigma).$$

Since $u$ and $v$ are sub and supersolutions, respectively and $\hat{\varphi}_x = -\hat{\varphi}_y \neq 0$ by the assumption $x_\sigma \neq y_\sigma$, we have

$$2\beta(t_\sigma - s_\sigma) + \gamma + F(t_\sigma, \hat{u}, \hat{\varphi}_x, X) \leq 0 \leq 2\beta(t_\sigma - s_\sigma) - \gamma + F(s_\sigma, \hat{v}, -\hat{\varphi}_y, Y), \quad (3.4.12)$$

where $\hat{u} = u(x_\sigma, t_\sigma), \hat{v} = v(y_\sigma, s_\sigma)$. Since $w(x_\sigma, t_\sigma, y_\sigma, s_\sigma) > 0$ by (3.4.6), we see $\hat{u} \geq \hat{v}$. Since $\varphi = \varphi_\sigma$ is a function of $x - y$ and $t - s$, (3.4.11) implies $X \leq Y$ as in §3.3. By mononicity (F4) with $c_0 = 0$ and (F2) the inequality (3.4.12) yields

$$2\gamma + F(t_\sigma, \hat{u}, \hat{\varphi}_x, X) - F(s_\sigma, \hat{u}, \hat{\varphi}_x, X) \leq 0, \quad (3.4.13)$$

since $\hat{\varphi}_x = -\hat{\varphi}_y$. If $F$ is independent of time, we already get a contradiction: $2\gamma \leq 0$.

If $F$ depends on $t$, we send $\beta$ of $\sigma = (\alpha, \beta)$ to infinity. By penalty argument as in Lemma 3.4.1 this time we observe that $t_\beta - s_\beta \to 0$ as $\beta \to \infty$. Since we fix $\alpha$, we write a subscript by $\beta$ instead of $\sigma$. We may assume that $x_\beta - y_\beta$ is bounded by the choice of $\alpha > \alpha_0, \beta > \beta_0$. There are two cases to discuss.

**Case A.** Assume that $x_\beta - y_\beta$ is bounded away from zero as $\beta \to \infty$. Then by (3.4.11) $X = X_\beta, Y = Y_\beta$ is bounded in the space $\mathbf{S}^N$; see the proof of Theorem 3.1.4, Case 2 for an explicit bound. Since $|x_\beta - y_\beta|$ is bounded from above, $\hat{\varphi}_x$ is bounded as $\beta \to \infty$. Since $u$ is bounded in $\overline{Q}$, $\hat{u}$ is bounded as $\beta \to \infty$. Thus by continuity of $F(t, r, p, X)$ outside $p = 0$, sending $\beta \to \infty$ in (3.4.13) implies $2\gamma \leq 0$ since $t_\beta - s_\beta \to 0$ as $\beta \to \infty$. This contradicts $\gamma > 0$.

**Case B.** Assume that $x_\beta - y_\beta \to 0$ as $\beta \to \infty$. Then we recall $\varphi^\pm$ in Case 1. Since $u$ and $v$ are sub- and supersolutions, respectively, we have

$$\begin{aligned} & \gamma + 2\beta(t_\beta - s_\beta) + F(t_\beta, u, \nabla\varphi^+, \nabla^2\varphi^+) \leq 0 \quad \text{at } (x_\beta, t_\beta) \\ \leq \; & -\gamma + 2\beta(t_\beta - s_\beta) + F(s_\beta, v, \nabla\varphi^-, \nabla^2\varphi^-) \quad \text{at } (y_\beta, s_\beta). \end{aligned}$$

This implies

$$\begin{aligned} & 2\gamma + F(t_\beta, u(x_\beta, t_\beta), \nabla\varphi^+(x_\beta), \nabla^2\varphi^+(x_\beta)) \\ & \quad - F(s_\beta, v(y_\beta, s_\beta), \nabla\varphi^-(y_\beta), \nabla^2\varphi^-(y_\beta)) \leq 0; \end{aligned}$$

note that $\nabla\varphi^+$, $\nabla^2\varphi^+$, $\nabla\varphi^-$, $\nabla^2\varphi^-$ is independent of the time variable. By definition of $\mathcal{F}$ sending $\beta \to \infty$ we see the limit involving $F$ equals zero. Again we get a contradiction to positivity of $\gamma$.

**Remark 3.4.2.** (i) From the proof to get comparison result stated in Theorem 3.1.1 we can weaken the definition of class $\mathcal{F}$ by replacing (2.1.11) by

$$\lim_{p\to 0}\sup_{z\in\overline{\mathcal{O}}}\sup_{|r|\le M}(F(z,r,\nabla f(|p|),\pm\nabla^2 f(|p|)|=0$$

for all $M>0$.
(ii) Instead of setting

$$w(x,t,y,s)=u(x,t)-v(y,s)-\gamma(t+s)$$

we may get

$$w(x,t,y,s)=u(x,t)-v(y,s)-\frac{\gamma}{T-t}-\frac{\gamma}{T-s}.$$

Indeed, M. Ohnuma and K. Sato (1997) established results of Theorem 3.1.1 when $F$ is independent of $t,r$ by this choice of $w$; they also assumed that $u^*$ and $-v^*$ are bounded from above at $t=T$. However, they did not use the fact that $u^*$ and $v_*$ are sub- and supersolutions on $Q_*$ in the proof.

### 3.4.2 Proof for unbounded domains

We shall prove Theorem 3.1.4 by adjusting the proof of Theorem 3.1.1.

*Proof of Theorem 3.1.4.* As in the proof of Theorem 3.1.1 it suffices to prove the case that $u$ and $v$ are, respectively, $\mathcal{F}$-sub- and supersolutions of (3.1.1) in $Q$ under the assumptions (F1), (F2), (F3') and (F4) with $c_0=0$. We suppress the word $\mathcal{F}$ in the proof. We may also assume that $u^*$ and $v_*$ are, respectively, sub- and supersolutions of (3.1.1) on $\Omega\times(0,T]$ and that (3.1.4) holds for $T'=T$ with the property that $u^*$ and $v_*$ are left accessible at $t=T$. We shall denote $u^*$ and $v_*$ simply by $u$ and $v$.

We set $w_\gamma$ and $\varphi_{\alpha\beta}$ as in (3.4.5). Assume that the conclusion were false so that

$$M\ge\theta_0:=\lim_{r\downarrow 0}\sup\{u(z)-u(\zeta);(z,\zeta)\in Z=\overline{Q}\times\overline{Q},\ |z-\zeta|<r\}>0,$$

where $M=\sup\{u(z)-u(\zeta);(z,\zeta)\in Z\}$; since $u$ and $-v$ are bounded from above, we see that $M<\infty$. Then for sufficiently small $\gamma>0$ we observe that

$$\mu_0:=\lim_{r\downarrow 0}\sup\{w_\gamma(z,\zeta);(z,\zeta)=(x,t,y,s)\in Z,\ |z-\zeta|<r\}>0.$$

We shall fix $\gamma$ such that $\mu_0>0$ and suppress that subscript $\gamma$. We define

$$\Phi_\sigma(x,t,y,s)=w(x,t,y,s)-\varphi_{\alpha\beta}(x,t,y,s)\ \text{ with } \sigma=(\alpha,\beta),\ \theta=\sup_Z\Phi_\sigma$$
$$\mu_1(r):=\sup\{\Phi_\sigma(x,t,y,s);(x,t,y,s)\in Z,|x-y|\le r\}\le\theta$$

and observe that there is $r_*>0$ independent of $\sigma$ such that

$$\mu_1(r)\ge 3\mu_0/4>0.$$

We shall argue as in Lemma 3.4.1. (Unfortunately, (3.4.1) may not hold when $Z_0$ is unbounded, so Lemma 3.4.1 does not directly apply to our setting.) Since $u$ and $-v$ are bounded from above on $Q$, $\Phi_\sigma(x,t,y,s) > 0$ implies

$$f(|x-y|) \leq \frac{M}{\alpha} \text{ and } (t-s)^2 \leq \frac{M}{\beta}. \tag{3.4.14}$$

By the hypothesis (3.1.4) on the values of $u$ and $v$ on the boundary there is $r_* > 0$ such that

$$\sup\{\Phi_\sigma(z,\zeta); (z,\zeta) \in \partial_p\overline{Q} \times \overline{Q} \cup \overline{Q} \times \partial_p\overline{Q},\ |z-\zeta| < r_*\} \leq \frac{\mu_0}{2}.$$

Take $\alpha_0, \beta_0$ so large that (3.4.14) with $\beta > \beta_0$ always implies $|z-\zeta| < r_*$ with $z = (x,t)$, $\zeta = (y,s)$. By the choice of $r_*$ we see that for all $\alpha > \alpha_0$ and $\beta > \beta_0$ if $z, \zeta \in \overline{Q}$ and $\Phi_\sigma(z,\zeta) > \mu_0/2$, then

$$z, \zeta \in Q_* = \Omega \times (0,T]. \tag{3.4.15}$$

Thus when we approximate the value $\mu_1$, we may assume that $z, \zeta$ are away from the parabolic boundary of $\overline{Q}$.

We shall distinguish two cases whether or not the supremum of $\Phi_\sigma$ is approximated at $x$ close to $y$ as $\beta \to \infty$. In the case $x$ close to $y$ the spatial gradient of the test function $\varphi_{\alpha\beta}$ approaches to zero so it roughly corresponds to the Case 1 in the proof of Theorem 3.1.1; see Remark 3.4.3. We always take $\sigma$ such that $\alpha, \beta$ satisfies $\alpha > \alpha_0$, $\beta > \beta_0$. We shall fix $\alpha$ but we shall later send $\beta$ to infinity. Note that $\theta$ and $\mu_1$ depends on $\beta$ so we sometimes write $\theta(\beta), \mu_1(r,\beta)$ to emphasize its dependence.

**Case 1.** For each $r \in (0, r_*)$ there is $\beta_r$ such that $\theta(\beta_r) = \mu_1(r, \beta_r)$ and $\beta_r \to \infty$ as $r \to 0$.

We first fix $r \in (0, r_*)$. By the definition of $\mu_1$ there is a sequence $\{(x_m, t_m, y_m, s_m)\}$ such that

$$\Phi_\sigma(x_m, t_m, y_m, s_m) \geq \theta(\beta) - \frac{1}{m} \text{ and } |x_m - y_m| \leq \frac{1}{m}.$$

with $\beta = \beta_r$. By taking a subsequence if necessary we may assume that $(t_m, s_m) \to (\hat{t}, \hat{s})$ and $x_m - y_m \to \hat{w}$ for some $\hat{t}, \hat{s} \in [0,T]$ and $\hat{w} \in \mathbf{R}^N$ satisfying $|\hat{\omega}| \leq r$. We now consider the function

$$\begin{aligned}
&\psi^+(x,t) = u(x,t) - \varphi^+(x,t),\\
&\varphi^+(x,t) = \alpha f(|x-y_m|) + \beta(t-s_m)^2 + f(|x-y_m-\hat{\omega}|) + (t-\hat{t})^2 + \gamma t.
\end{aligned}$$

Let $(\xi_m, \tau_m)$ be a maximum of $\psi^+$ over $\overline{Q}$. Such a point does exist since $\lim_{r\to\infty} f(r) = \infty$. Since

$$\psi^+(x_m, t_m) \leq \psi^+(\xi_m, \tau_m),$$

subtracting $v(y_m, s_m) + \gamma s_m$ from both hand sides yields

$$\begin{aligned}
&\Phi_\sigma(x_m, t_m, y_m, s_m) - f(|x_m - y_m - \hat{\omega}|) - (t_m - \hat{t})^2\\
\leq\ &\Phi_\sigma(\xi_m, \tau_m, y_m, s_m) - f(|\xi_m - y_m - \hat{\omega}|) - (\tau_m - \hat{t})^2.
\end{aligned}$$

Since $\Phi_\sigma \le \theta$, this implies

$$f(|\xi_m - y_m - \hat{\omega}|) + (\tau_m - \hat{t})^2 \le \theta(\beta) - \Phi_\sigma(x_m, t_m, y_m, s_m) + f(|x_m - y_m - \hat{\omega}|) + (t_m - \hat{t})^2 \quad (3.4.16)$$

and

$$\Phi_\sigma(x_m, t_m, y_m, s_m) \le \Phi_\sigma(\xi_m, \tau_m, y_m, s_m) + f(|x_m - y_m - \hat{\omega}|) + (t_m - \hat{t})^2. \quad (3.4.17)$$

The inequality (3.4.16) implies that $\tau_m \to \hat{t}$, $\xi_m - y_m \to \hat{\omega}$ since $\Phi_\sigma(x_m, t_m, y_m, s_m) \to \theta(\beta)$, $t_m \to \hat{t}$, $x_m - y_m \to \hat{\omega}$. Since $u$ is bounded, we may assume that $u(\xi_m, \tau_m) \to \hat{u}$ for some $\hat{u} \in \mathbf{R}$. The inequality (3.4.17) implies that

$$\Phi_\sigma(\xi_m, \tau_m, y_m, s_m) > \mu_0/2 \quad (3.4.18)$$

for sufficiently large $m$; we may assume that (3.4.18) holds for all $m$.

By (3.4.18) and (3.4.15) we observe that $(\xi_m, \tau_m) \in Q_*$ for all $m$. Since $u$ is a subsolution in $Q_*$ and since $u - \varphi^+$ is maximized at $(\xi_m, \tau_m) \in Q_*$, we have

$$\varphi_t^+ + F(\tau_m, u, \nabla\varphi^+, \nabla^2\varphi^+) \le 0 \quad \text{at} \quad (\xi_m, \tau_m)$$

if $p_m := \xi_m - y_m \ne 0$ and

$$\varphi_t^+(\xi_m, \tau_m) = 2\beta(\tau_m - s_m) + 2(\tau_m - \hat{t}) + \gamma \le 0$$

if $p_m = 0$. Since $\xi_m - y_m \to \hat{\omega}$, $\tau_m \to \hat{t}$, $s_m \to \hat{s}$, we send $m$ to infinity to get

$$2\beta(\hat{t} - \hat{s}) + \gamma + F(\hat{t}, \hat{u}, \alpha\nabla_p f(|p|), \alpha\nabla_p^2 f(|p|) \le 0 \quad \text{at} \quad p = \hat{\omega} \quad (3.4.19)$$

if $\hat{\omega} \ne 0$ and

$$2\beta(\hat{t} - \hat{s}) + \gamma \le 0 \quad (3.4.20)$$

if $\hat{\omega} = 0$ by the definition of $f \in \mathcal{F}$.

In the same way, we consider the function

$$\begin{aligned} &\psi^-(y, s) = -v(y, s) + \varphi^-(y, s), \\ &\varphi^-(y, s) = -\alpha f(|x_m - y|) - f(|x_m - y - \hat{\omega}|) - \beta(t_m - s)^2 - (s - \hat{s})^2 - \gamma s. \end{aligned}$$

and study the maximum point of $\psi^-$ over $Q_*$. By a similar argument as for $\psi^+$ we use the fact that $v$ is a supersolution to get

$$2\beta(\hat{t} - \hat{s}) - \gamma + F(\hat{s}, \hat{v}, \alpha\nabla_p f(|p|), -\alpha\nabla_p^2 f(|p|)) \ge 0 \quad \text{at} \quad p = \hat{\omega} \quad (3.4.21)$$

if $\hat{\omega} \ne 0$ and

$$2\beta(\hat{t} - \hat{s}) - \gamma \ge 0 \quad (3.4.22)$$

if $\hat{\omega} = 0$. In the latter case (3.4.20) and (3.4.22) yields a contradiction $2\gamma \le 0$ so we may assume that $\hat{\omega} \ne 0$. Subtracting (3.4.21) from (3.4.19) yields

$$2\gamma + F(\hat{t}, \hat{u}, \alpha\nabla_p f(|p|), \alpha\nabla_p^2(|p|)) - F(\hat{s}, \hat{v}, \alpha\nabla_p f(|p|), -\alpha\nabla_p^2 f(|p|)) \le 0$$

at $p = \hat{\omega}$. Note that $\hat{\omega}$ depends on $r$. Since $|\hat{\omega}| \le r$, we now send $r \to 0$ to get $2\gamma \le 0$ which is a contradiction.

**Case 2.** There is $r_0 \in (0, r_*)$ such that for sufficiently large $\beta$, say $\beta > \beta_1 \ge \beta_0$ for some $\beta_1$ the inequality $\theta(\beta) > \mu_1(r_0, \beta)$ holds. We define $\Psi_{\sigma\delta}$ by

$$\Psi_{\sigma\delta}(x,t,y,s) = \Phi_\sigma(x,t,y,s) - \delta|x|^2 - \delta|y|^2$$

for $\delta > 0$. Clearly, $\Psi_{\sigma\delta}$ attains a maximum at some point $(x_{\sigma\delta}, t_{\sigma\delta}, y_{\sigma\delta}, s_{\sigma\delta}) \in Z$ depending on $\delta$ and $\sigma$. By definition $\sup_Z \Psi_{\sigma\delta} \uparrow \theta$ as $\delta \to 0$. Thus, for sufficiently small $\delta$, say $\delta < \delta_0(\sigma)$ we obtain

$$\sup_Z \Psi_{\sigma\delta} > \mu_2 := \mu_1(r_0, \beta).$$

Since $\Psi_{\sigma\delta}(x_{\sigma\delta}, t_{\sigma\delta}, y_{\sigma\delta}, s_{\sigma\delta}) \ge \sup_Z \Psi_{\sigma\delta}$ and $\mu_2 \ge 3\mu_0/4$, we observe from (3.4.15) that

$$(x_{\sigma\delta}, t_{\sigma\delta}), (y_{\sigma\delta}, s_{\sigma\delta}) \in Q_*.$$

By definition of $\mu_1$ and (3.4.14) we obtain

$$f^{-1}(M/\alpha) \ge |x_{\sigma\delta} - y_{\sigma\delta}| > r_0. \tag{3.4.23}$$

Moreover, since $0 \le \Psi_{\sigma\delta}(x_{\sigma\delta}, t_{\sigma\delta}, y_{\sigma\delta}, s_{\sigma\delta}) \le M - \delta|x_{\sigma\delta}|^2 - \delta|y_{\sigma\delta}|^2$, we have

$$\delta(|x_{\sigma\delta}| + |y_{\sigma\delta}|) \to 0 \quad \text{as} \quad \delta \to 0. \tag{3.4.24}$$

Since $\Psi_{\sigma\delta} = (w - \delta|x|^2 - \delta|y|^2) - \varphi_\sigma$ attains its maximum at $(\hat{x}, \hat{t}, \hat{y}, \hat{s}) = (x_{\sigma\delta}, t_{\sigma\delta}, y_{\sigma\delta}, s_{\sigma\delta})$ over $Q_* \times Q_*$ we argue as in the Case 2 of the proof of Theorem 3.1.1 to get

$$\begin{aligned}
&(\gamma + \hat{\varphi}_t, \hat{\varphi}_x + 2\delta x_{\sigma\delta}, X_{\sigma\delta} + 2\delta I) \in \overline{\mathcal{P}}^{2,+} u(x_{\sigma\delta}, t_{\sigma\delta}) \\
&(-\gamma - \hat{\varphi}_s, -\hat{\varphi}_y - 2\delta y_{\sigma\delta}, Y_{\sigma\delta} - 2\delta I) \in \overline{\mathcal{P}}^{2,-} v(y_{\sigma\delta}, s_{\sigma\delta})
\end{aligned}$$

for some $X_{\sigma\delta}, Y_{\sigma\delta} \in \mathbf{S}^N$ satisfying (3.4.11). As in the proof of Lemma 3.1.3 a direct calculation yields

$$\begin{aligned}
A_0 &= \alpha \frac{f'(\rho)}{\rho} J + \alpha (f''(\rho) - \frac{f'(\rho)}{\rho}) Q \\
J &= \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, Q = \frac{1}{\rho^2} \begin{pmatrix} p \otimes p & -p \otimes p \\ -p \otimes p & p \otimes p \end{pmatrix}
\end{aligned}$$

with $p = x_{\sigma\delta} - y_{\sigma\delta}$ and $\rho = |p|$. Since

$$({}^t\xi, {}^t\eta) J \begin{pmatrix} \xi \\ \eta \end{pmatrix} = |\xi - \eta|^2, ({}^t\xi, {}^t\eta) Q \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\rho^2} |\langle \xi - \eta, p \rangle|^2, \quad \xi, \eta \in \mathbf{R}^N$$

so that $O \le Q \le J \le 2I$, we observe that $A_0 \ge \alpha f''(\rho) Q \ge O$. Since $A_0 \ge O$, we note that

$$|A_0| = \sup\{({}^t\xi, {}^t\eta) A_0 \begin{pmatrix} \xi \\ \eta \end{pmatrix}; |\xi|^2 + |\eta|^2 = 1, \quad \xi, \eta \in \mathbf{R}^N\}.$$

Since

$$A_0 \le \alpha \frac{f'(\rho)}{\rho} J + \alpha f''(\rho) Q \le \alpha K_1(\rho) J \le 2\alpha K_1(\rho) I$$

with $K_1(\rho) = f'(\rho)/\rho + f''(\rho)$, we have an estimate $|A_0| \le 2\alpha K_1(\rho)$. Since $J^2 = 2J$, $Q^2 = 2Q$, $JQ = QJ = 2Q$, $A_0^2$ is of the form

$$\begin{aligned} A_0^2 &= 2\alpha^2\{(\frac{f'(\rho)}{\rho})^2 J + (2\frac{f'(\rho)}{\rho} + f'(\rho) - \frac{f''(\rho)}{\rho})(f''(\rho) - \frac{f'(\rho)}{\rho})Q\} \\ &\le 2\alpha^2\{(\frac{f'(\rho)}{\rho})^2 J + (f''(\rho))^2 Q\} \le 2\alpha^2 K_2(\rho) J \end{aligned}$$

with $K_2(\rho) = (f'(\rho)/\rho)^2 + (f''(\rho))^2$. Thus (3.4.11) implies

$$-(\frac{1}{\lambda_1} + 2\alpha K_1(\rho))I \le \begin{pmatrix} X_{\sigma\delta} & O \\ O & -Y_{\sigma\delta} \end{pmatrix} \le (\alpha K_1(\rho) + 2\alpha^2 \lambda_1 K_2(\rho))J.$$

This yields $X_{\sigma\delta} \le Y_{\sigma\delta}$ as in the Case 2 of the proof of Theorem 3.1.1 and a bound

$$|X_{\sigma\delta}| \le 2\alpha K_1(\rho) + 4\alpha^2 K_2(\rho)\lambda_1 + \frac{1}{\lambda_1}.$$

with $\rho = |x_{\sigma\delta} - y_{\sigma\delta}|$. The right hand side is bounded as $\delta \to 0$ and $\beta \to \infty$ by (3.4.23). We thus have a bound for $X_{\sigma\delta}$ independent of $\delta$ and $\beta$. By (3.4.23) and (3.4.24) we observe that

$$\hat{\varphi}_x + 2\delta x_{\sigma\delta} \ne 0, \ -\hat{\varphi}_y - 2\delta y_{\sigma\delta} = \hat{\varphi}_x + 2\delta y_{\sigma\delta} \ne 0$$

for sufficiently small $\delta > 0$. Since $u$ and $v$ are sub- and supersolutions, we have

$$2\beta(t_{\sigma\delta} - s_{\sigma\delta}) + \gamma + F(t_{\sigma\delta}, \hat{u}, \hat{\varphi}_x + 2\delta x_{\sigma\delta}, X_{\sigma\delta} + 2\delta I) \le 0 \tag{3.4.25}$$
$$2\beta(t_{\sigma\delta} - s_{\sigma\delta}) - \gamma + F(s_{\sigma\delta}, \hat{v}, -\hat{\varphi}_y - 2\delta y_{\sigma\delta}, Y_{\sigma\delta} - 2\delta I) \ge 0. \tag{3.4.26}$$

By (F2) and (F4) with $c_0 = 0$, (3.4.26) with $X_{\sigma\delta} \le Y_{\sigma\delta}$ implies

$$2\beta(t_{\sigma\delta} - s_{\sigma\delta}) - \gamma + F(s_{\sigma\delta}, \hat{u}, \hat{\varphi}_y - 2\delta y_{\sigma\delta}, Y_{\sigma\delta} - 2\delta I) \ge 0. \tag{3.4.27}$$

Sending $\delta \to 0$, we obtain from (3.4.25) and (3.4.27) with a bound for $|X_{\sigma\delta}|$ and (3.4.23) that

$$2\beta(\overline{t}_\sigma - \overline{s}_\sigma) + \gamma + F(\overline{t}_\sigma, \overline{u}_\sigma, \alpha f'(|\overline{p}_\sigma|), \overline{X}_\sigma) \ge 0 \tag{3.4.28}$$
$$2\beta(\overline{t}_\sigma - \overline{s}_\sigma) - \gamma + F(\overline{s}_\sigma, \overline{u}_\sigma, \alpha f'(|\overline{p}_\sigma|), \overline{X}_\sigma) \le 0 \tag{3.4.29}$$

for some $\overline{t}_\sigma, \overline{s}_\sigma \in [0, T]$, $\overline{u}_\sigma \in \mathbf{R}$, $\overline{X}_\sigma \in \mathbf{S}^N$, $\overline{p}_\sigma \in \mathbf{R}^N$ by taking a subsequence if necessary; here we invoke the assumption that $u$ is bounded so that $\overline{u}_\sigma$ exists. Sending $\beta \to \infty$ so that $\overline{t}_\sigma - \overline{s}_\sigma \to 0$ by (3.4.14), we now obtain from (3.4.28) and (3.4.29) that $2\gamma \le 0$ since $\overline{X}_\sigma, |\overline{p}_\sigma|, |\overline{p}_\sigma|^{-1}$ are still bounded as $\beta \to \infty$. This contradicts $\gamma > 0$ so we have proved that $\theta_0 \le 0$. □

**Remark 3.4.3.** The classification into Case 1 and Case 2 for bounded domains (Theorem 3.1.1) is slightly different from that for unbounded domains (Theorem 3.1.4). If we classify into the following two cases for bounded domains with fixed $\alpha$, this exactly corresponds the classification for unbounded domains.

**Case 1.** For each $r > 0$ there is $\beta_r \to \infty$ (as $r \to 0$) and a maximum $z_{\alpha\beta_r}$ of $w - \varphi_{\alpha\beta_r}$ in $Q_* \times Q_*$ such that $|x_{\alpha\beta_r} - y_{\alpha\beta_r}| \leq r$, where $z_{\alpha\beta} = (x_{\alpha\beta}, t_{\alpha\beta}, y_{\alpha\beta}, s_{\alpha\beta})$.

**Case 2.** There is $r_0 > 0$ such that for sufficiently large $\beta$ any maximizer $z_{\alpha\beta}$ of $w - \varphi_{\alpha\beta}$ satisfies $|x_{\alpha\beta} - y_{\alpha\beta}| > r_0$.

## 3.5 Lipschitz preserving and convexity preserving property

We study some general properties of solutions of the Cauchy problem for (3.1.1). As we shall see in §4.3 for a given $u_0 \in \mathrm{BUC}(\mathbf{R}^N)$ there exists a unique viscosity solution $u \in \mathrm{BUC}(\mathbf{R}^N \times [0, T))$ of (3.1.1) satisfying $u|_{t=0} = u_0$ for example under the assumption of Corollary 3.1.5. Here the space of all bounded, uniformly continuous function on $D \subset \mathbf{R}^d$ is denoted by $\mathrm{BUC}(D)$. We first observe that global Lipschitz continuity is preserved for spatial homogeneous equations.

**Theorem 3.5.1.** *Assume that $F(r, p, X)$ is independent of $x$ and $r$. Assume that (F1)–(F2) and (F3) (or (F3') with the assumption that $\mathcal{F}_{\mathbf{R}^N}$ is invariant under positive multiplications.) If $u \in BUC(\mathbf{R}^N \times [0, T))$ is an ($\mathcal{F}$-) solution of (3.1.1) in $\mathbf{R}^N \times (0, T)$ with some constant $L > 0$ satisfying*

$$|u(x, 0) - u(y, 0)| \leq L|x - y| \tag{3.5.1}$$

*for all $x, y \in \mathbf{R}^n$, then*

$$|u(x, t) - u(y, t)| \leq L|x - y| \tag{3.5.2}$$

*for all $x, y \in \mathbf{R}^n$, $t \in [0, T)$.*

*Proof.* Since $F$ is independent of $x$ and $u$, we see

$$v(x, t) = u(x + h, t) + L|h|, \quad h \in \mathbf{R}^n$$

is also a viscosity solution in $\mathrm{BUC}(\mathbf{R}^n \times [0, T))$ of (3.1.1) in $\mathbf{R}^n \times (0, T)$. By the assumption of the initial data (3.5.1) and uniform continuity we see that $u$ and $v$ satisfy (3.1.4). By (CP) we have

$$u(x, t) \leq v(x, t)$$

or

$$u(x, t) - u(x + h, t) \leq L|h|$$

for all $x \in \mathbf{R}^N$, $t \in [0,T)$. A symmetric argument comparing $u(x+h,t) - L|h|$ and $u(x,t)$ yields

$$u(x,t) - u(x+h,t) \geq -L|h|.$$

We thus prove (3.5.2). □

We next study whether or not concavity of initial data $u_0$ is preserved as time develops.

**Theorem 3.5.2.** *Assume that $F = F(x,t,r,p,X)$ is independent of $x$ and $r$ and satisfies (F1), (F2). Assume that $F$ is geometric. Assume that*

$$X \mapsto F(t,p,X) \ \text{ is convex on } \ \mathbf{S}^N \tag{3.5.3}$$

*for all $p \in \mathbf{R}^N \setminus \{0\}$, $t \in [0,T]$. Let $u \in C(\mathbf{R}^N \times [0,T))$ be an $\mathcal{F}$-solution of (3.1.1) in $\mathbf{R}^N \times (0,T)$ satisfying (3.5.2) with some $L > 0$. If $u(x,0)$ is concave, then*

$$u(t,x) + u(y,t) - 2u(z,t) \leq L|x+y-2z| \tag{3.5.4}$$

*holds for all $x,y,z \in \mathbf{R}^N$, $0 \leq t \leq T$. In particular, $x \mapsto u(x,t)$ is concave for all $t \in [0,T)$.*

We shall prove Theorem 3.5.2 in several steps. Our first observation is the equivalence of concavity and (3.5.4) type inequality when a function is globally Lipschitz.

**Lemma 3.5.3.** *Assume that $v_0$ is globally Lipschitz in $\mathbf{R}^N$ with constant $L$. Then $v_0$ is concave if and only if the inequality*

$$v_0(x) + v_0(y) - 2v_0(z) \leq L|x+y-2z| \tag{3.5.5}$$

*holds for all $x,y,z \in \mathbf{R}^N$.*

*Proof.* If $v_0$ is concave, then

$$\begin{aligned}
& v_0(x) + v_0(y) - 2v_0(z) \\
= \ & v_0(x) + v_0(y) - 2v_0(z)(\frac{x+y}{2}) + 2\{v_0(\frac{x+y}{2}) - v_0(z)\} \\
\leq \ & 2\{v_0(\frac{x+y}{2}) - v_0(z)\}.
\end{aligned}$$

Since $v_0$ is globally Lipschitz, the right hand side is dominated by $2L|\frac{x+y}{2} - z|$. We thus obtain (3.5.5).

The inequality (3.5.5) yieldss

$$\frac{v_0(x) + v_0(y)}{2} \leq v_0(\frac{x+y}{2})$$

by taking $z = (x+y)/2$. This says that $v_0$ is mid-concave. Successive use of this inequality yields

$$\lambda v_0(x) + (1-\lambda)v_0(y) \leq v_0(z), \ z = \lambda x + (1-\lambda)y$$

for all $\lambda \in (0,1)$ of the form $\lambda = k2^{-h}$ with positive integers $h$ and $k$. Since $v_0$ is continuous, this implies concavity of $v_0$. $\Box$

*Proof of Theorem 3.5.2.* We suppress the word $\mathcal{F}$ in the proof. For $M > 0$ and $R > 0$ let $\theta^M$ and $\theta_R$ be functions on $\mathbf{R}$ of the form

$$\theta^M(\eta) = \min(M, \eta), \theta_R(\eta) = \max(-R, \eta).$$

Since $F$ is geometric, by the invariance under change of dependent variables proved later in §4.2.1 we see that

$$u^M(x,t) = (\theta^M \circ u)(x,t) = \min(M, u(x,t))$$
$$u_R(x,t) = (\theta_R \circ u)(x,t) = \max(-R, u(x,t))$$

are solutions of (3.1.1) in $Q = \mathbf{R}^N \times (0,T)$ satisfying (3.5.2). Since $u^M$ is still concave in $x$, we may assume that $u$ is bounded from *above* in $\mathbf{R}^N \times (0,T)$. As in the proof of Theorem 3.1.1 we may assume that $u$ is a solution of (3.1.1) in $Q_* = \mathbf{R}^N \times (0,T]$ by taking $T$ smaller. By replacing $u$ by $u/L$ we may assume that $L = 1$.

Our goal is to prove

$$u(x,t) + u(y,t) - 2u(z,t) \le |x + y - 2z| \tag{3.5.6}$$

for all $x, y, z \in \mathbf{R}^N$, $t \in [0,T]$. We take $f \in \mathcal{F}$. For $\kappa > 0$ there is a unique $A(\kappa) > 0$ such that

$$\eta \le A(\kappa) f(\eta) + \kappa =: N_\kappa(\eta) \quad \text{for all} \quad \eta > 0$$

and the equality attains only at one point $\eta = \eta(\kappa) > 0$. We thus observe that

$$\eta = \inf_{\kappa > 0} N_\kappa(\eta) \tag{3.5.7}$$

for $\eta > 0$. We set

$$w_\gamma^{\kappa R}(x, s_1, y, s_2, z, s_3) = u(x, s_1) + u(y, s_2) - 2u_R(z, s_3) - N_\kappa(|x + y - 2z|) - \gamma s_1$$

for $R > 0$, $\gamma \ge 0$, $\kappa > 0$; we use $w_\gamma^{\kappa\infty}$ when $u$ replaces $u_R$.

Assume that the conclusion (3.5.6) were false. By (3.5.7) there would exist $\kappa = \kappa_0$ such that

$$\sup\{w_0^{\kappa_0\infty}(x,t,y,t,z,t); x,y,z \in \mathbf{R}^N, t \in [0,T]\} > 0.$$

We may replace $u(z,t)$ by $u_R(z,t)$ for large $R$ and observe that

$$\begin{aligned}\theta_0 := \lim_{r \downarrow 0} \sup\{&w_0^{\kappa_0\infty}(x, s_1, y, s_2, z, s_3); x, y, z \in \mathbf{R}^N, s_1, s_2, s_3 \in [0,T],\\ &|s_1 - s_2| \le r, |s_2 - s_3| < r\} > 0.\end{aligned}$$

Since $u$ is bounded from above on $\overline{Q} = \mathbf{R}^N \times [0,T]$, i.e. $K := \sup_{\overline{Q}} u < \infty$, $\theta_0 \le 2K + 2R < \infty$. Then for sufficiently small $\gamma > 0$ we observe that

$$\mu_0 := \lim_{r \downarrow 0} \sup\{w_\gamma^{\kappa_0\infty}(x, s_1, y, s_2, z, s_3); (x, s_1), (y, s_2), (z, s_3) \in \overline{Q}, |s_1 - s_2| < r, |s_2 - s_3| < r\} > 0.$$

We shall fix $R$ and $\gamma$ such that $\mu_0 > 0$ and suppress the subscript $\gamma$ and the superscript $R$ as well as $\kappa_0$. We define

$$\begin{aligned}&\Phi_\beta(x, s_1, y, s_2, z, s_3) := w(x, s_1, y, s_2, z, s_3) - \varphi_\beta(s_1, s_2, s_3)\\&\varphi_\beta(s_1, s_2, s_3) := \beta(|s_1 - s_3|^2 + |s_2 - s_3|^2), \quad \theta(\beta) = \sup_W \Phi_\beta,\end{aligned}$$

where $W = \overline{Q} \times \overline{Q} \times \overline{Q}$.

We shall prove that for sufficiently large $\beta$, say $\beta > \beta_0$, if $(x, s_1)$, $(y, s_2)$, $(z, s_3) \in \overline{Q}$ satisfies

$$\Phi_\beta(x, s_1, y, s_2, z, s_3) > \mu_0/2, \tag{3.5.8}$$

then $s_1 > 0$, $s_2 > 0$ and $s_3 > 0$. By (3.5.5) and (3.5.7) we observe that

$$u_0(x) + u_0(y) - 2u_{R0}(z) - N_\kappa(|x + y - 2z|) \le 0, \quad x, y, z \in \mathbf{R}^N \tag{3.5.9}$$

for all $\kappa > 0$ where $u_0(x) = u(x, 0)$, $u_{R0}(x) = u_R(x, 0) = \max(-R, u_0(x))$. The left hand side of (3.5.9) is always negative if either $u_0(x) \le R_1$ or $u_0(y) \le R_1$ for $R_1 = K + 2R$. Thus (3.5.9) is equivalent to

$$u_{R_10}(x) + u_{R_10}(y) - 2u_{R0}(z) - N_\kappa(|x + y - 2z|) \le 0, \quad x, y, z \in \mathbf{R}^N \tag{3.5.10}$$

for all $\kappa > 0$. Since $u_{R_*0}$ is in BUC($\mathbf{R}^N$), $u_{R_*}$ is in BUC $(\mathbf{R}^N \times [0, T))$ for $R_* > 0$ by the uniqueness of solutions. By this uniform continuity and (3.5.10) there exists a small $r_0 > 0$ such that

$$\begin{aligned}&\sup\{u_{R_1}(x, s_1) + u_{R_1}(y, s_2) - 2u_R(z, s_3) - N_{\kappa_0}(|x + y - 2z|) - \gamma s_1 - \varphi_\beta(s_1, s_2, s_3);\\&s_1 s_2 s_3 = 0,\ |s_1 - s_3| \le r_0,\ |s_2 - s_3| \le r_0,\ (x, s_1), (y, s_2), (z, s_3) \in \overline{Q}\} \le \mu_0/2.\end{aligned}$$

By the choice of $R_1$ this implies

$$\begin{aligned}&\sup\{\Phi_\beta(x, s_1, y, s_2, z, s_3); s_1 s_2 s_3 = 0,\ |s_1 - s_3| \le r_0,\ |s_2 - s_3| \le r_0,\\&(x, s_1), (y, s_2),\ (z, s_3) \in \overline{Q}\} \le \mu_0/2.\end{aligned} \tag{3.5.11}$$

Since $\Phi_\beta \le 2K + 2R$, $\Phi_\beta(x, s_1, y, s_2, z, s_3) > 0$ implies

$$(s_1 - s_3)^2 + (s_2 - s_3)^2 \le 2(K + R)/\beta. \tag{3.5.12}$$

Take $\beta_0$ so large that (3.5.2) with $\beta > \beta_0$ implies that $|s_1 - s_3| \le r_0$ and $|s_2 - s_3| \le r_0$. Then from (3.5.11) it follows that (3.5.8) implies $s_1 > 0$, $s_2 > 0$, $s_3 > 0$ if $\beta > \beta_0$.

We shall distinguish two cases whether or not the supremum of $\Phi_\beta$ is approximated at $x, y, z$ with the property that $x + y - 2z$ close to zero. We shall always assume that $\beta > \beta_0$. We set

$$\mu_1(r, \beta) := \sup\{\Phi_\beta(x, s_1, y, s_2, z, s_3); |x + y - 2z| \le r, (x, s_1), (y, s_2), (z, s_3) \in \overline{Q}\} \le \theta(\beta).$$

By the definition of $\mu_0$ there exists $r_* > 0$ (independent of $\beta$) such that $\mu_1(r, \beta) \ge 3\mu_0/4$ for $r < r_*$.

**Case 1.** For each $r \in (0, r_*)$ there is $\beta_r$ such that $\theta(\beta_r) = \mu_1(r, \beta_r)$ and $\beta_r \to \infty$ as $r \to 0$.

We argue in the similar way in the proof (Case 1) of Theorem 3.1.4. We first fix $r \in (0, r_*)$. By the definition of $\mu_1$ there is a sequence $\{a_m\}$ with $a_m = (x_m, s_m^1, y_m, s_m^2, z_m, s_m^3)$ such that

$$\Phi_\beta(a_m) \geq \theta(\beta) - 1/m \quad \text{and} \quad |x_m + y_m - 2z_m| \leq r$$

with $\beta = \beta_r$. We may assume that $(s_m^1, s_m^2, s_m^3) \to (\hat{s_1}, \hat{s_2}, \hat{s_3})$, $x_m + y_m - 2z_m \to \hat{\omega}$ for some $\hat{s_1}, \hat{s_2}, \hat{s_3} \in [0, T]$ and $\hat{\omega} \in \mathbf{R}^N$ with $|\hat{\omega}| \leq r$ by taking a subsequence. We now consider the function

$$\begin{aligned}
&\psi^+(x,t) = u(x,t) - \varphi^+(x,t)\\
&\varphi^+(x,t) = Af(|x + y_m - 2z_m|) + \beta(t - s_m^3)^2\\
&+ f(|x + y_m - 2z_m - \hat{\omega}|) + (t - \hat{s_1})^2 + \gamma t, A = A(\kappa_0).
\end{aligned}$$

Let $(\xi_m, \tau_m)$ be a maximum of $\psi^+$ over $\overline{Q}$. Such a point exists since $\lim_{r\to\infty} f(r) = \infty$.

$$\psi^+(x_m, s_m^1) \leq \psi^+(\xi_m, \tau_m),$$

ading $u(y_m, s_m^2) - 2u(z_m, s_m^3) - (s_m^2 - s_m^3)^2$ to both head sides yields

$$\begin{aligned}
&\Phi_\beta(a_m) - f(|x_m + y_m - 2z_m - \hat{\omega}|) - (s_m^1 - \hat{s_1})^2\\
&\quad \leq \Phi_\beta(b_m) - f(|\xi_m + y_m - 2z_m - \hat{\omega}|) - (\tau_m - \hat{s_1})^2.
\end{aligned}$$

with $b_m = (\xi_m, \tau_m, y_m, s_m^2, z_m, s_m^3)$. Since $\Phi_\sigma \leq \theta$, this implies

$$f(|\xi_m + y_m - 2z_m - \hat{\omega}|) + (\tau_m - \hat{s_1})^2 \leq \theta(\beta) - \Phi_\beta(a_m) + f(|x_m + y_m - 2z_m - \hat{\omega}|) + (s_m^1 - \hat{s_1})^2 \tag{3.5.13}$$

and

$$\Phi_\beta(a_m) \leq \Phi_\beta(b_m) + f(|x_m + y_m - 2z_m - \hat{\omega}|) + (s_m^1 - \hat{s_1})^2. \tag{3.5.14}$$

The inequality (3.5.13) implies that $\xi_m + y_m - 2z_m \to \hat{\omega}$, $\tau_m \to \hat{s_1}$ as $m \to \infty$ since $x_m + y_m - 2z_m \to \hat{\omega}$, $s_m^1 \to \hat{s_1}$. The inequality (3.5.14) implies that

$$\Phi_\beta(b_m) > \mu_0/2 \tag{3.5.15}$$

for sufficiently large $m$ since $\Phi_\beta(a_m) \to \theta(\beta)$. We may assume that (3.5.15) holds for all $m$.

Since (3.5.15) implies (3.5.8), we observe that $\tau_m > 0$ for all $m$ so that $(\xi_m, \tau_m) \in Q_*$. Since $u$ is a subsolution in $Q_*$ and since $u - \varphi^+$ is maximized at $(\xi_m, \tau_m) \in Q_*$, we have as in the proof of Theorem 3.1.4

$$\varphi_t^+ + F(\tau_m, \nabla\varphi^+, \nabla^2\varphi^+) \leq 0 \quad \text{at} \quad (\xi_m, \tau_m)$$

if $p_m := \xi_m + y_m - 2z_m \neq 0$ and

$$\varphi_t^+(\xi_m, \tau_m) = 2\beta(\tau_m - s_m^3) + 2(\tau_m - \hat{s_1}) + \gamma t \leq 0$$

if $p_m = 0$. Since $\xi_m + y_m - 2z_m \to \hat{\omega}$, $\tau_m \to \hat{s_1}$, $s_m^i \to \hat{s_i}$, we send $m$ to infinity to get

$$2\beta(\hat{s_1} - \hat{s_3}) + \gamma + F(\hat{s_1}, A\nabla_p f(|p|), A\nabla_p^2 f(|p|) \leq 0 \quad \text{at} \quad p = \hat{\omega} \tag{3.5.16}$$

if $\hat{\omega} \neq 0$ and

$$2\beta(\hat{s_1} - \hat{s_3}) + \gamma \leq 0$$

if $\hat{\omega} = 0$ by the definition of $f \in \mathcal{F}$.

In the same way we study the maximum of

$$\begin{aligned}
&\psi_+(y, s) = u(y, s) - \varphi_+(y, s) \\
&\varphi_+(y, s) = Af(|x_m + y - 2z_m|) + \beta(s - s_m^3)^2 + f(|x_m + y - 2z_m - \hat{\omega}|) + (s - \hat{s_2})^2
\end{aligned}$$

and obtain

$$2\beta(\hat{s_2} - \hat{s_3}) + F(\hat{s_2}, A\nabla_p f(|p|), A\nabla_p^2 f(|p|)) \leq 0 \quad \text{at} \quad p = \hat{\omega} \tag{3.5.17}$$

if $\hat{\omega} \neq 0$ and $2\beta(\hat{s_2} - \hat{s_3}) \leq 0$ if $\hat{\omega} = 0$.

In the same way we study the maximum of

$$\begin{aligned}
&\psi^-(z, s_3) = -u(z, s_3) + \varphi^-(z, s_3) \\
&2\varphi^-(z, s_3) = -Af(|x_m + y_m - 2z|) - \beta(s_m^1 - s_3)^2 - \beta(s_m^2 - s_3)^2 \\
&-f(|x_m + y_m - z + \hat{\omega}|) - (s_3 - \hat{s_3})^2
\end{aligned}$$

and obtain

$$\beta(\hat{s_1} - \hat{s_3}) + \beta(\hat{s_2} - \hat{s_3}) + F(\hat{s_3}, \frac{A}{2}\nabla_p f(|p|), -\frac{A}{2}\nabla_p^2 f(|p|)) \geq 0 \quad \text{at} \quad p = \hat{\omega} \tag{3.5.18}$$

if $\hat{\omega} \neq 0$ and $\beta(\hat{s_1} - \hat{s_3}) + \beta(\hat{s_2} - \hat{s_3}) \leq 0$ if $\hat{\omega} = 0$. If $\hat{\omega} = 0$, we easily get contradiction: $\gamma \leq 0$ so we may assume that $\hat{\omega} \neq 0$. We consider (3.5.16) + (3.5.17) $-2\times$ (3.5.18) and send $r \to 0$. Since $|\hat{\omega}| \leq r$, this yields a contradiction: $\gamma \leq 0$.

**Case 2.** There is $r_0 \in (0, r_*)$ such that for sufficiently large $\beta$, say $\beta > \beta_1$ for some $\beta_1$ the inequality $\theta(\beta) > \mu_1(r_0, \beta)$ holds.

We set $\mu_2 = \mu(r_0, \beta)$ for $\beta > \beta_1$. For $\delta > 0$ we introduce

$$\Psi_{\beta r}(a) = \Phi_\beta(a) - \delta(|x|^2 + |y|^2 + |z|^2).$$

with $a = (x, s_1, y, s_2, z, s_3)$. By definition

$$\sup_W \Psi_{\beta\delta} \uparrow \theta \quad \text{as} \quad \delta \downarrow 0.$$

Thus there is $\delta_0 = \delta_0(\beta) > 0$ that satisfies

$$\sup_W \Psi_{\beta\delta} > \max(\mu_2, \mu_0/2) \quad \text{for} \quad 0 < \delta < \delta_0$$

since $\mu_0 \leq \theta$. Clearly, $\Psi_{\beta\delta}$ attains a maximum at some points $\hat{\xi} = (\hat{x}, \hat{s_1}, \hat{y}, \hat{s_2}, \hat{z}, \hat{s_3}) \in W$ depending on $\beta$ and $\delta$. We suppress the dependence with respect to $\beta$ and $\delta$ for simplicity of notation but we shall later send $\delta$ to zero first and $\beta$ to infinity. Since (3.5.8) is fulfilled at $\hat{\xi}$, we see that $\hat{s_i}$ is positive for $i = 1, 2, 3$. By the definition of $\mu_2$ we observe that

$$f^{-1}(2(K+R)/A(\kappa_0)) \geq |\hat{x} + \hat{y} - 2\hat{z}| > r_0; \tag{3.5.19}$$

the left inequality follows from the fact that $\Psi_{\beta\delta}$ is bounded from above by $2(K+R)$ and $\Psi_{\beta\delta}(\hat{\xi}) > 0$. This fact also yields the bound of $\delta(|\hat{x}|^2 + |\hat{y}|^2 + |\hat{z}|^2)$ so in particular we have

$$\delta(|\hat{x}| + |\hat{y}| + |\hat{z}|) \to 0 \quad \text{as} \quad \delta \to 0. \tag{3.5.20}$$

Since $s_i > 0$ for $\delta < \delta_0(\beta)$, $i = 1, 2, 3$, the function

$$\begin{aligned} \Psi_{\beta\delta} &= u(x, s_1) + u(y, s_2) - 2u(z, s_3) - \delta(|x|^2 + |y|^2 + |z|^2) - \psi, \\ \psi &= N_{\kappa_0}(|x + y - 2z|) + \varphi_\beta(s_1, s_2, s_3) \end{aligned}$$

attains its maximum at $\hat{\xi} = (\hat{x}, \hat{s_1}, \hat{y}, \hat{s_2}, \hat{z}, \hat{s_3})$ over $Q_* \times Q_* \times Q_*$ for $\delta < \delta_0(\beta)$. We now apply Theorem 3.3.3 with $k = 3$, $N_1 = N_2 = N_3 = N + 1$, $\mathcal{Z}_1 = \mathcal{Z}_2 = \mathcal{Z}_3 = Q_*$; $\ell = 2$ with projection $P_1$ and $P_2$ defined by $P_1(x, s_1, y, s_2, z, s_3) = (x, y, z)$ and $P_2 = I - P_1$. As in the Case 2 of the proof of Theorem 3.1.1 for each $\lambda_1 > 0$ there exists $X, Y, Z \in \mathbf{S}^N$ such that

$$-(\frac{1}{\lambda_1} + |B_0|) \leq I \leq \begin{pmatrix} X & O & O \\ O & Y & O \\ O & O & -Z \end{pmatrix} \leq B_0 + \lambda_2 B_0^2, \tag{3.5.21}$$

$$(\gamma + \hat{\psi}_{s_1}, \hat{\psi}_x + 2\delta\hat{x}, X + 2\delta I) \in \overline{\mathcal{P}}^{2,+} u(\hat{x}, \hat{s_1}), \tag{3.5.22}$$

$$(\hat{\psi}_{s_2}, \hat{\psi}_y + 2\delta\hat{y}, Y + 2\delta I) \in \overline{\mathcal{P}}^{2,+} u(\hat{y}, \hat{s_2}), \tag{3.5.23}$$

$$(-\frac{\hat{\psi}_{s_3}}{2}, -\frac{1}{2}(\hat{\psi}_z + 2\delta\hat{z}), \frac{1}{2}(Z - 2\delta I)) \in \overline{\mathcal{P}}^{2,-} u(\hat{x}, \hat{s_3}), \tag{3.5.24}$$

with

$$B_0 = \begin{pmatrix} \hat{\psi}_{xx} & \hat{\psi}_{xy} & \hat{\psi}_{xz} \\ \hat{\psi}_{yx} & \hat{\psi}_{yy} & \hat{\psi}_{yz} \\ \hat{\psi}_{zx} & \hat{\psi}_{zy} & \hat{\psi}_{zz} \end{pmatrix}, \tag{3.5.25}$$

where $\hat{\psi}_x$ denotes $\nabla_x \psi$ evaluated at $\hat{\xi}$ and so on. We shall fix $\lambda_1$. Note that $X, Y, Z$ may depend on $\delta$ and $\beta$.

We shall derive a bound for $X, Y, Z$. As in the Case 2 of the proof of Theorem 3.1.4 a direct calculation of (3.5.25) yields

$$\begin{aligned} B_0 &= A\{\frac{f'(\rho)}{S} S + (f''(\rho) - \frac{f'(\rho)}{\rho}) R\}, \quad A = A(\kappa_0) \\ S &= \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}, \quad R = \frac{1}{\rho^2} \begin{pmatrix} p \otimes p & p \otimes p & -2p \otimes p \\ p \otimes p & p \otimes p & -2p \otimes p \\ -2p \otimes p & -2p \otimes p & 4p \otimes p \end{pmatrix} \end{aligned}$$

with $p = \hat{x} + \hat{y} - 2z$, $\rho = |p|$ since

$$({}^t\xi, {}^t\eta, {}^t\zeta)S \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = |\xi + \eta - 2\zeta|^2, \ ({}^t\xi, {}^t\eta, {}^t\zeta)R \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = |\langle \xi + \eta - 2\zeta, p\rangle|^2, \ \xi, \eta, \zeta \in \mathbf{R}^N$$

so that $O \leq R \leq S (\leq 6I)$, we observe that

$$AK_1(\rho)S \geq B_0 \geq Af''(\rho)R \geq O \tag{3.5.26}$$

with $K_1(\rho) = f'(\rho)/\rho + f''(\rho)$. We thus have an estimate

$$|B_0| \leq AK_1(\rho)|S| = 6A(\kappa_0)K_1(\rho) \tag{3.5.27}$$

since $|S| = 6$. Since $S^2 = 6S$, $R^2 = 6R$, $RS = SR = 6R$, as in the case 2 of the proof of Theorem 3.1.4

$$\begin{aligned} B_0^2 &= 6A^2\{(\frac{f'(\rho)}{\rho})^2 S + ((f''(\rho))^2 - (\frac{f'(\rho)}{\rho})^2)R\} \\ &\leq 6A^2 K_2(\rho) S \end{aligned} \tag{3.5.28}$$

with $K_2(\rho) = (f'(\rho)/\rho)^2 + (f''(\rho))^2$. Applying (3.5.26)–(3.5.28) to (3.5.21) we obtain

$$-(\frac{1}{\lambda_1} + 6AK_1(\rho))I \leq \begin{pmatrix} X & O & O \\ O & Y & O \\ O & O & -Z \end{pmatrix} \leq A(K_1(\rho) + 6\lambda_1 AK_2(\rho))S.$$

This inequality implies that $X + Y \leq Z$ since

$$({}^t\xi, {}^t\xi, {}^t\xi)S \begin{pmatrix} \xi \\ \xi \\ \xi \end{pmatrix} = |\xi + \xi - 2\xi|^2 = 0.$$

Moreover it provides a bound for $X$ and $Y$

$$|X|, |Y|, |Z| \leq 6AK_1(\rho) + 12A^2K_2(\rho)\lambda_1 + \frac{1}{\lambda_1} \tag{3.5.29}$$

with $\rho = |\hat{x} + \hat{y} - 2\hat{z}|$. By (3.5.19) the right hand side of (3.5.29) is bounded as $\delta \to 0$ and $\beta \to 0$.

By (3.5.19) and (3.5.20) we observe that

$$\hat{\psi}_x + 2\delta\hat{x} \neq 0, \quad \hat{\psi}_y + 2\delta\hat{y} \neq 0, \quad -\hat{\psi}_z - 2\delta\hat{z} \neq 0$$

for sufficiently small $\delta > 0$. Since $u$ is a solution, the relation (3.5.22)–(3.5.23) yields

$$2\beta(\hat{s}_1 - \hat{s}_3) + \gamma + F(\hat{s}_1, \hat{\psi}_x + 2\delta\hat{x}, X + 2\delta I) \leq 0, \tag{3.5.30}$$

$$2\beta(\hat{s}_2 - \hat{s}_3) \quad + F(\hat{s}_2, \hat{\psi}_y + 2\delta\hat{y}, Y + 2\delta I) \leq 0. \tag{3.5.31}$$

Since $X+Y\le Z$, the relation (3.5.24) with degenerate ellipticity yields

$$\beta(\hat{s_1}-\hat{s_3})+\beta(\hat{s_2}-\hat{s_3})+F(\hat{s_3},-\frac{1}{2}(\hat{\psi}_z+2\delta\hat{z}),\frac{1}{2}(X+Y-2\delta I))\ge 0. \qquad (3.5.32)$$

Since $\hat{\psi}_x=\hat{\psi}_y=-\hat{\psi}_z/2=Af'(\rho)p/\rho$, sending $\delta\to 0$ in the inequality (3.5.30) + (3.5.31) $-2\times$ (3.5.32) yields

$$\gamma+F(\overline{s}_1,\mu,\overline{X})+F(\overline{s}_2,\overline{\mu},\overline{Y})-2F(\overline{s}_3,\mu,(\overline{X}+\overline{Y})/2))\le 0$$

with $\mu=Af'(|\overline{p}|)\overline{p}/|\overline{p}|$ with some $\overline{X},\overline{Y}\in\mathbf{S}^N$, $\overline{p}\in\mathbf{R}^N$, $\overline{s}_1,\overline{s}_2,\overline{s}_3\in[0,T]$. Since $\overline{X},\overline{Y}$ are bounded as $\beta\to\infty$ by (3.5.29) and $|\overline{p}|^{-1}$, $|\overline{p}|$ is bounded as $\sigma\to\infty$ by (3.5.19), sending $\beta\to\infty$ implies

$$\gamma+F(\tilde{s},\tilde{\mu},\tilde{X})+F(\tilde{s},\tilde{\mu},\tilde{Y})-2F(\tilde{s},\tilde{\mu},(\tilde{X}+\tilde{Y})/2)\le 0$$

by (3.5.12) for some $\tilde{s}\in[0,T]$, $\tilde{X},\tilde{Y}\in\mathbf{S}^N$, $\tilde{\mu}\in\mathbf{R}^N$. Since $X\mapsto F(t,r,X)$ is convex in $X$, this yields $\gamma\le 0$ which is a contradiction. We have thus proved $\theta_0\le 0$ to get (3.5.6). □

**Corollary 3.5.4.** *Assume that $F=F(x,t,r,p,X)$ is independent of $x$ and $r$ and satisfies (F1), (F2). Assume that $F$ is geometric. Assume that*

$$X\mapsto F(t,p,X)\quad \textit{is concave on } \mathbf{S}^N$$

*for all $p\in\mathbf{R}^N\setminus\{0\}$, $t\in[0,T]$. Let $u\in C(\mathbf{R}^N\times[0,T])$ be an $\mathcal{F}$-solution of (3.1.1) in $\mathbf{R}^N\times(0,T)$ satisfying (3.5.2) with some $L>0$. If $u(x,0)$ is convex, then $x\mapsto u(x,t)$ is convex for all $t\in[0,T]$.*

This immediately follows from Theorem 3.5.2 by replacing $u$ by $-u$ since the concavity of $X\mapsto F(t,p,X)$ is equivalent to that of $X\mapsto F(t,p,-X)$ .

**Remark 3.5.5**(Applicability). The concavity (resp. convexity of $X\mapsto F_f(p,X)$ is fulfilled if and only if $f$ in (1.6.1) is convex (resp. concave) in $\nabla\mathbf{n}$ since $F_f$ is defined by (1.6.4). Our Theorem 3.5.2 and Corollary 3.5.4 apply to the level set equations of (1.5.2), (1.5.4), (1.5.6) provided that conditions in §3.1.3 is fulfilled.

For the affine curvature flow equation (1.5.14) or its generalization $V=(\kappa_+)^\alpha$, $\alpha>0$ one should take the orientation $\mathbf{n}$ inward for convex curves so that the equation is not trivial one $V\equiv 0$. For a convex initial curves $\Gamma_0$ let $u_0$ be a function representing $\Gamma_0$ i.e. $\Gamma_0=\partial\{u_0>0\}=\partial\{u_0<0\}$ with $\mathbf{n}=-\nabla u/|\nabla u|$. One can take $u_0$ so that it is convex; however it is impossible to take concave $u_0$. The convexity of the solution $u$ of the level set equation of $V=(\kappa_+)^\alpha$ is preserved if $\alpha\ge 1$ by Corollary 3.5.4.

Since

$$[\det(\frac{X+Y}{2})]^{1/m}\ge\frac{1}{2}((\det X)^{1/m}+(\det Y)^{1/m})$$

for $X\ge O$, $Y\ge O$, $X,Y\in\mathbf{S}^m$ (see e.g. D. S. Mintrinović (1970)), for the equation $V=(\kappa_1^+\cdots\kappa_{N-1}^+)^{1/(N-1)}$ in $\mathbf{R}^N$ the right hand side is concave in $\nabla\mathbf{n}$ so that $F_f(p,X)$ is convex in $X$. Unfortunately, we again have to consider convex initial data $u_0$ for convex initial hypersurface $\Gamma_0$ so neither Theorem 3.5.2 nor Corollary 3.5.4 apply to conclude that solution $u$ is convex in $x$. So it is unlikely that Theorem 3.5.2 and Corollary 3.5.4 apply for surface evolution equation by principal curvatures like (1.6.22), (1.6.23).

## 3.6 Spatially inhomogeneous equations

We present a few versions of comparison principles for (3.1.1) when $F$ also depends on the spatial variable $x$. We restrict ourselves to the case when $\Omega$ is bounded since the results for unbounded $\Omega$ is very complicated to state. We first

### 3.6.1 Inhomogeneity in first order perturbation

The next result applies when $F$ is of the form

$$F(x,t,r,p,X) = F_0(t,r,p,X) + F_1(x,t,p) \tag{3.6.1}$$

with $F_0$ satisfying the assumptions on $F$ of Theorem 3.1.1 (including (F3)) and $F_1 \in C(\overline{\Omega} \times [0,T] \times \mathbf{R}^N)$ that satisfies

$$|F_1(x,t,p) - F_1(y,t,q)| \leq C(1+|p|)|x-y| \tag{3.6.2}$$

with some constant $C > 0$ independent of $x, y \in \overline{\Omega}$, $p \in \mathbf{R}^N$. In particular, it applies the level set equation of (1.5.2) even when the constancy property of $C$ in §3.1.2 is relaxed as a uniform Lipschitz continuity in $x$:

$$|c(x,t) - c(y,t)| \leq C|x-y|.$$

**Theorem 3.6.1.** *Assume that $\Omega$ is bounded. Assume that (F1)–(F3) and (F4) hold. Assume that for $R > 0$ there is a modulus $\omega_R$ such that*

$$|F(x,t,r,p,X) - F(y,t,r,p,Y)| \leq \omega_R(|x-y|(|p|+1)) \tag{3.6.3}$$

*holds for $x, y \in \overline{\omega}$, $t \in [0,T]$, $|r| \leq R$, $p \in \mathbf{R}^N \setminus \{0\}$, $X, Y \in \mathbf{S}^N$. Then the comparison principle (BCP) is valid. If we assume (F3') instead of (F3), then (BCP) is still valid (by replacing solutions by $\mathcal{F}_\Omega$-solutions) provided that*
*(i) $\mathcal{F}_\Omega$ is invariant under positive multiplication;*
*(ii) there exists $f \in \mathcal{F}_\Omega$ such that $f'(\rho)\rho/f(\rho)$ is bounded on $[0,1]$.*

**Corollary 3.6.2.** *Assume that $\Omega$ is bounded and that $F$ satisfies (F1), (F2) and (3.6.3). Assume that $F$ satisfies*

$$|F(x,t,p,\pm I)| \leq c_0|p|^{-\eta} \quad on \quad \overline{\Omega} \times [0,T] \times (B_{r_0}(0) \setminus \{0\}) \tag{3.6.4}$$

*for some $r_0 > 0$, $c_0 > 0$, $\eta > 0$. Then (BCP) holds if $F$ is geometric.*

From the proof of Lemma 3.1.3 it is clear that the assumption on the existence of $c$ guarantees the existence of $f \in \mathcal{F}_\Omega$ satisfying (ii) of Theorem 3.6.1. The assumption (i) of Theorem 3.6.1 is automatically fulfilled since $F$ is geometric as proved in Lemma 3.1.3. Thus, Corollary 3.6.2 follows from Theorem 3.6.1.

*Proof of Theorem 3.6.1.* We augue in the same way as in the proof of Theorem 3.1.1 right before formula (3.4.12), where $x$-dependence should be taken into account. Instead of (3.4.13) we obtain

$$2\gamma + F(x_\sigma, t_\sigma, \hat{u}, \hat{\varphi}_x, X) - F(y_\sigma, s_\sigma, \hat{u}, \hat{\varphi}_x, X) \le 0. \tag{3.6.5}$$

We send $\beta \to \infty$ to get $t_{\alpha\beta} - s_{\alpha\beta} \to 0$. If $x_{\alpha\beta} - y_{\alpha\beta} \to 0$ as $\beta \to \infty$, we get a contradiction as in the Case B of the proof of Theorem 3.1.1. Thus we may assume that $x_{\alpha\beta} \to x_\alpha$, $y_{\alpha\beta} \to y_\alpha$ with $x_\alpha \ne y_\alpha$. We may also assume that

$$\begin{aligned}&t_{\alpha\beta} \to \hat{t}, s_{\alpha\beta} \to \hat{t}, \hat{u} \to u_\alpha \in \mathbf{R}, \hat{\varphi}_x \to q_\alpha := \alpha f'(|p_\alpha|)p_\alpha/|p_\alpha| \quad \text{with} \;\; p_\alpha = x_\alpha - y_\alpha;\\ &X \to X_\alpha \in \mathbf{S}^N \quad \text{(by 3.4.11)}\end{aligned}$$

as $\beta \to \infty$ by taking a subsequence. Thus

$$2\gamma + F(x_\alpha, \hat{t}, u_\alpha, q_\alpha, X_\alpha) - F(y_\alpha, \hat{t}, u_\alpha, q_\alpha, X_\alpha) \le 0. \tag{3.6.6}$$

By (3.6.3) with $R = \sup_{\overline{Q}} |u| + 1$ this inequality yields

$$2\gamma - \omega_R(|x_\alpha - y_\alpha|(|q_\alpha| + 1)) \le 0. \tag{3.6.7}$$

Since $M_\sigma \to \sup_{Z_0} w$ (as $\sigma \to \infty$ by Lemma 3.4.1), we observe that $\alpha f(|x_\alpha - y_\alpha)) \to 0$ as $\alpha \to \infty$. By the assumption (ii)

$$|x_\alpha - y_\alpha||q_\alpha| = \alpha f'(|p_\alpha|)|p_\alpha| \le C\alpha \; f(|p_\alpha|)$$

with some $C$ independent of $\alpha$. Thus $|x_\alpha - y_\alpha||q_\alpha| \to 0$ as $\alpha \to \infty$. Since $|p_\alpha| \to 0$ as $\alpha \to 0$, (3.6.7) now yields $2\gamma \le 0$ which is contradiction. □

**Remark 3.6.3.** (i) The assumption (3.6.4) is fulfilled for $F = F_f$ of (1.6.19) provided that $g$ is $\hat{e}_m$ or $\hat{e}_m/e_\ell$ ($m > \ell$). If $F_0$ in (3.6.1) is equal to such an $F_f$, $F$ of (3.6.1) satisfies the assumption of Corollary 3.6.2 provided that $F_1$ is geometric and satisfies (3.6.2). For example, Corollary 3.6.2 guarantees that (BCP) holds for the level set equation of

$$V = K + c(x,t)$$

if $c$ satisfies the uniform (in $t$) Lipschitz continuity in $x$ provided that the Gaussiaan curvature $K$ is interpreted as its modified form $\hat{e}_{N-1}(\kappa_1, \cdots, \kappa_{N-1})$.

(ii) It is of course possible to prove (CP) when $\Omega$ is unbounded. The proof is similar to that of Theorem 3.1.4. Since we shall send $\alpha \to \infty$, we modify the classification Case 1, Case 2 as follows.

**Case 1.** There is a sequence $\alpha_j \to \infty$ such that for each $r \in (0, r_*)$ there is $\beta_r \to \infty$ such that $\theta(\beta_r) = \mu_1(r, \beta_r)$ as $r \to 0$.

**Case 2.** For sufficiently large $\alpha$, say $\alpha > \alpha_0$, there is $r_\alpha \in (0, r_*)$ such that $\theta(\beta) > \mu_1(r_\alpha, \beta)$ for all $\beta > \beta_\alpha$ with some $\beta_\alpha$.

### 3.6.2 Inhomogeneity in higher order terms

Unfortunately, results in §3.6.1 does not apply to (3.1.1) when a spatial imhomogeneity appears in the order term. For example, it does not apply to $F(x,t,p) = -a(x)$ trace $X$ unles $a \geq 0$ is a constant. The next result applies to such an $F$.

**Theorem 3.6.4.** *Assume that $\Omega$ is bounded. Assume that (F1)–(F3) and (F4) hold. Assume that for $R > 0$ there is a modulus $\omega_R$ such that*

$$F(x,t,r,p,X) - F(y,t,r,p,Y) \geq -\omega_R(|x-y|(|p|+1) + \mu|x-y|^2) \tag{3.6.8}$$

*for $x, y \in \overline{\Omega}$, $t \in [0,T]$, $p \in \mathbf{R}^N \setminus \{0\}$, $X, Y \in \mathbf{S}^N$, $|r| \leq R$, $\mu > 0$ whenever*

$$-3\mu I \leq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq 3\mu J, \ J = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.6.9}$$

*Then (BCP) holds.*

**Remark 3.6.5.** The assumption (F2) follows from (3.6.8)–(3.6.9) so it is redundent. Indeed,

$$\begin{aligned} {}^t\xi X\xi - {}^t\eta Y\eta &= {}^t(\xi-\eta)X(\xi-\eta) + 2{}^t\eta X(\xi-\eta) + {}^t\eta(X-Y)\eta \\ &\leq |X||\xi-\eta|^2 + 2|X||\xi-\eta||\eta| + {}^t\eta(X-Y)\eta \\ &\leq |X|(1+\varepsilon^{-1})|\xi-\eta|^2 \end{aligned}$$

For $\xi, \eta \in \mathbf{R}^n$, $\varepsilon > 0$. If $X \leq Y$, this implies

$$\begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq (1+\varepsilon^{-1})|X|J + \varepsilon|X|I. \tag{3.6.10}$$

(By the way estimate (3.6.10) where $\varepsilon|X|$ replaced by $|X-Y| + \varepsilon|X|$ in front of $I$ holds for arbitrary $X, Y \in \mathbf{S}^N$.) This evidently implies

$$\begin{pmatrix} X_\varepsilon & O \\ O & -Y_\varepsilon \end{pmatrix} \leq (1+\varepsilon^{-1})|X|J$$

with $X_\varepsilon = X - \varepsilon|X|I$, $Y_\varepsilon = Y + \varepsilon|X|I$. Thus $X_\varepsilon$, $Y_\varepsilon$ satisfies (3.6.9) by a suitable choice of $\mu = \mu_\varepsilon$. By (3.6.8) we have

$$F(x,t,r,p,X_\varepsilon) - F(x,t,r,p,Y_\varepsilon) \geq 0.$$

Sending $\varepsilon$ to zero yields (F2).

We give a typical example of $F$ to which Theorem 3.6.4 applies.

**Corollary 3.6.6.** *Assume that $\Omega$ is bounded. Assume that $F$ is of the form (3.6.1) with $F_1 \in C(\overline{\Omega} \times [0,T] \times \mathbf{R}^N)$ satisfying (3.6.2). Assume that $F_0$ is of the form*

$$F_0(x,t,r,p,X) = -\mathit{trace}({}^tA(x,t,p)XA(x,t,p)) \tag{3.6.11}$$

*and $A(x,t,p)$ is an $\mathbf{S}^N$-valued bounded continuous functions in $\overline{\Omega} \times [0,T] \times (\mathbf{R}^N \setminus \{0\})$. Assume that $A \geq O$ and satisfies*

$$|A(x,t,p) - A(y,t,p)| \leq C|x-y| \tag{3.6.12}$$

*with $C > 0$ independent of $p \in \mathbf{R}^N \setminus \{0\}$, $x, y \in \overline{\Omega}$, $C \in [0,T]$. Then (BCP) holds for (3.1.1).*

*Proof of Corollary 3.6.6.* Since (F3) and (F4) are clearly fulfilled, it suffices to prove that (3.6.9) implies (3.6.8). Such property is closed under addition of $F$. Thus, we may assume that $F = F_0$, since the assertion is clear for $F_1$.

From (3.6.9) we see $X \leq Y$, i.e., ${}^t\xi X\xi - {}^t\eta Y\eta \leq \mu|\xi - \eta|^2$. This implies

$$F_0(x,t,r,p,X) - F_0(y,t,r,p,Y) \geq -\mu||A(x,t,p) - A(y,t,p)||_2^2$$

where $||\cdot||_2$ denotes the Hilbert-Schmidt norm. By (3.6.12) the estimate (3.6.8) is fulfilled by choosing $\omega_R(x) = Cs$.

*Proof of Theorem 3.6.4.* We take $f(\eta) = \eta^4 \in \mathcal{F}$ argue as in the proof of Theorem 3.6.1 right before the formula (3.6.5); we invoke (F3) to handle Case 1 and Case A. Instead of (3.6.5) we obtain

$$2\gamma + F(x_\sigma, t_\sigma, \hat{u}, \hat{\varphi}_x, X) - F(y_\sigma, s_\sigma, \hat{u}, \hat{\varphi}_x, -Y) \leq 0$$

without using $X \leq Y$. We send $\beta \to \infty$ and obtain

$$2\gamma + F(x_\alpha, \hat{t}, u_\alpha, q_\alpha, X_\alpha) - F(y_\alpha, \hat{t}, u_\alpha, q_\alpha, -Y_\alpha) \leq 0 \tag{3.6.13}$$

instead of (3.6.6), where $q_\alpha = f'(|p_\alpha|)p_\alpha/|p_\alpha|(= 4|p_\alpha|^2 p_\alpha)$, $p_\alpha = x_\alpha - y_\alpha$.

In the estimate (3.4.11) we may take $\lambda_1$, $\lambda_1 = \alpha K_1(\rho)$, $K_1(\rho) = f'(\rho)\rho + f''(\rho)$ with $p = x_\sigma - y_\sigma$ to get a bound

$$-3\alpha K_1(\rho_\alpha)I \leq \begin{pmatrix} X_\alpha & O \\ O & -Y_\alpha \end{pmatrix} \leq 3\alpha K_1(\rho_\alpha)J$$

with $\rho_\alpha = |x_\alpha - y_\alpha|$; see the estimate of

$$\begin{pmatrix} x_{\sigma\delta} & 0 \\ -0 & -Y_{\sigma\delta} \end{pmatrix}$$

in the proof of Theorem 3.1.4, Case 2. Thus $X_\alpha, Y_\alpha \in \mathbf{S}^N$ fulfills (3.6.9) with $\mu = \alpha K_1(\rho_\alpha)$. Applying (3.6.8) to (3.6.13) we obtain

$$2\gamma - \omega_R(|x_\alpha - y_\alpha|)(|q_\alpha| + 1) + \alpha K_1(\rho_\alpha)|x_\alpha - y_\alpha|^2) \leq 0 \tag{3.6.14}$$

by taking $R = \sup_{\overline{Q}}|u| + 1$. We have observed that $\alpha f(|x_\alpha - y_\alpha|) = \alpha|\rho_\alpha|^4 \to 0$ as $\alpha \to \infty$ in the proof of Theorem 3.6.1. Thus, the quantity in $\omega_R$ tends to zero as $\alpha \to \infty$. Sending $\alpha \to \infty$ in (3.6.14) now yields a contradiction $2\gamma \leq 0$. Thus (BCP) has been proved. $\square$

**Example.** By Corollary 3.6.1 wee observe that (BCP) holds for the level set equation of (1.5.2) provided that $\beta > 0$ is continuous and $\gamma$ in (1.3.7) is convex and $C^2$ except the origin and that $a \geq 0$ and $c \in C(\overline{\Omega} \times [0,T])$ satisfies the Lipschitz continuity in $x$ uniformly in $t$, i.e.

$$|c(x,t) - c(y,t)| \leq C|x-y|.$$

We seek a sufficient condition for (BCP) when $\beta$ and $\gamma$ are also depends on $\gamma$. We first recall the level set equation of

$$\beta(x, \mathbf{n})V = -a \ \mathrm{div}_{\Gamma_t}\xi(x,\mathbf{n}) + c(x,t). \tag{3.6.15}$$

where $\xi = \nabla_p\gamma$, $p \mapsto \gamma(x,p)$ is convex, positively homogeneous of degree one. If we write its level set equation in the form of (3.1.1),

$$F(x,t,p,X) = -\frac{a}{\beta(x, -\frac{p}{|p|})}\{\mathrm{trace}\nabla_p^2\gamma(x,-p)X + \mathrm{trace}\nabla_x\nabla_p\gamma(x,-p)\}|p| + c|p|, \tag{3.6.16}$$

$$F_0(x,p,X) = -\frac{a}{\beta(x, -\frac{p}{|p|})}(\mathrm{trace}\nabla_p^2\gamma(x,-p)X))|p|, \tag{3.6.17}$$

$$F_1(x,t,p,X) = -\frac{a}{\beta(x, -\frac{p}{|p|})}(\mathrm{trace}\nabla_x\nabla_p\gamma(x,-p))|p| + c|p|. \tag{3.6.18}$$

The condition (3.6.2) for $F_1$ in (3.6.18) is fulfilled if (and only if)

$$M(x,t,p) = -a\frac{\mathrm{trace}\nabla_x\nabla_p\gamma(x,-p)}{\beta(x,-p/|p|)} + c(x,t) \tag{3.6.19}$$

is Lipschitz continuous in $x$ uniformly in $t, p$, i.e.

$$|M(x,t,p) = -M(y,t,p)| \leq C|x-y| \tag{3.6.20}$$

with $C$ independent of $x, y \in \overline{\Omega}$, $t \in [0,T]$, $p \in \mathbf{R}^N \setminus \{0\}$. To apply Corollary 3.6.6 we shall find a condition that

$$A(x,p) = (\nabla_p^2\gamma(x, -p|p|))^{1/2}\frac{a^{1/2}}{\beta^{1/2}} \tag{3.6.21}$$

satisfies (3.6.12) so that $F_0$ in (3.6.17) is written of the form (3.6.11) with (3.6.12). Since $A$ is positively homogeneous of degree zero in $p$, it suffices to check the condition that (3.6.12) for $p \in S^{N-1}$. The next result shows a sufficient condition for (3.6.12).

**Lemma 3.6.7.** *Let $\Omega$ be an open set in $\mathbf{R}^N$. Assume that $\gamma = \gamma(x,p)$, $x \in \overline{\Omega}$, $p \in \mathbf{R}^N$ is convex and positively homogeneous of degree one on $\mathbf{R}^N$ as a function of $p$. Assume that $\gamma \in C(\overline{\Omega} \times \mathbf{R}^N)$ is $C^2$ as a function of $p$ except the origin and $\Gamma = \nabla_p^2\gamma \in C(\overline{\Omega} \times (\mathbf{R}^N \setminus \{0\})$. Then the square root $\Gamma^{1/2}$ of $\Gamma$ is Lipschitz continuous on $\overline{\Omega} \times S^{N-1}$ (respectively, Lipschitz continuous in $x \in \Omega$ uniformly in $p \in S^{N-1}$) if one of following condition holds*
*(i) $\Gamma$ is $C^1$ and its derivative $\nabla_p\Gamma$, $\nabla_x\Gamma$ (resp. $\nabla_x\Gamma$) is Lipschitz continuous on $\overline{\Omega} \times S^{N-1}$ (resp. Lipschitz continuous in $x \in \overline{\Omega}$ uniformly in $p \in S^{N-1}$).*

*(ii) All eigenvalues of $\Gamma(x,p)+p\otimes p$ is uniformly away from zero in $\overline{\Omega}\times S^{N-1}$ and $\Gamma$ has the corresponding Lipschitz regularity.*

*Proof.* If (i) holds, $\Gamma$ has the Lipschitz property in $\overline{\Omega}\times U$ where $U$ is some tubular neighborhood of $S^{N-1}$. Since $\Gamma \geq 0$, this implies that $\Gamma^{1/2}$ has the desired Lipschitz continuity; see M. G. Crandall, H. Ishii, P. L. Lions (1992) [Example 3.6].

By homogeneity all eigenvalues of $\gamma$ the matrix $\Gamma(x,p)$ has a zero eigenvalue with eigenvector $p\in S^{N-1}$. If (ii) holds, $(\Gamma+p\otimes p)^{1/2}$ is positive so it has the desired Lipschitz property. Since

$$(\Gamma(x,p))^{1/2} = Q_p((\Gamma(x,p)+p\otimes p)^{1/2}), \quad p\in S^{N-1}$$

with $Q_p(X)=R_p\times R_p$, $R_p = I-p\otimes p/|p|^2$, this yields the Lipschitz property of $(\Gamma(x,p))^{1/2}$. $\square$

Summarizing these results, we conclude that $F$ in (3.6.16) satisfies the assumptions of Corollary 3.6.6 if (i) or (ii) of Lemma 3.6.7 is fulfilled for $\Gamma$ and (3.6.19)–(3.6.20) as well as the Lipschitz continuity of $\beta\in C(\overline{\Omega}\times S^{N-1})$ in $x$ uniformly in $p$ and $\beta\geq\beta_0>0$ on $\overline{\Omega}\times S^{N-1}$ with some constant $\beta_0$.

**Remark 3.6.8.** It seems to be nontrivial to allow spatially inhomogeneity in the second order term for (BCP) if the singularity at $\nabla u=0$ in (3.1.1) is very strong so that (F3) is violated. We consider the level set equation of $V=((a(x))^2H)^3$, where $a\in C^\infty(\overline{\Omega})$ such that $\inf_{\overline{\Omega}} a>0$. It is of the form

$$u_t + F(x,\nabla u,\nabla^2 u)=0$$

with

$$F(x,p,X) = -[G(x,p,X)]^3/|p|^2, G(x,p,X)=\mathrm{trace}((aI)X(aI)).$$

We argue as in the proof of Theorem 3.6.4. for $f(\eta)=\eta^6\in\mathcal{F}$ to obtain (3.6.13) with the estimate

$$-3\mu I \leq \begin{pmatrix} X_\alpha & O \\ O & -Y_\alpha \end{pmatrix} \leq 3\mu J,\ \mu=\alpha K_1(\rho_\alpha).$$

This relation yields

$$G(y_\alpha,q_\alpha,Y_\alpha)-G(x_\alpha,q_\alpha,X_\alpha)\geq -N\mu|a(x_\alpha)-a(y_\alpha)|^2.$$

It also yields

$$|G(x_\alpha,q_\alpha,X_\alpha)|,|G(y_\alpha,q_\alpha,Y_\alpha)|\leq C_1K_1(\rho_\alpha).$$

Since

$$\begin{aligned} F(r,p,X)-F(y,p,Y) &= \frac{1}{|p|^2}\{G(x,p,X))^2+G(x,p,X)G(y,p,Y)+(G(y,p,Y))^2\} \\ &\times\{G(y,p,Y)-G(x,p,X)\}, \end{aligned}$$

plugging $x_\alpha, y_\alpha, X_\alpha, Y_\alpha, q_\alpha$ yields

$$F(x_\alpha, q_\alpha, X_\alpha) - F(y_\alpha, q_\alpha, Y_\alpha) \geq -\alpha C_2 K_1(\rho_\alpha) \frac{K_1(\rho_\alpha)^2}{(f'(\rho_\alpha))^2} \rho_\alpha^2 \tag{3.6.22}$$

if $a$ in Lipschitz continuous. Here $C_1$ and $C_2$ are constants independent of $\alpha$. For the choice of $f$ we see $f''(\eta)\eta^2/30 = f'(\eta)\eta/5 = f(\eta)$ so $\alpha K_1(\rho_\alpha)\rho_\alpha \to 0$ as $\alpha \to \infty$ since $\alpha f(\rho_\alpha) \to 0$. However, there is no control on $K_1(\rho_\alpha)/q_\alpha$, $1/\rho_\alpha$ as $\alpha \to \infty$ so it is not clear that the right hand side of (3.6.22) tends to zero as $\alpha \to \infty$.

## 3.7 Boundary value problems

We consider the boundary value problem

$$u_t + F(x, t, u, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = \Omega \times (0, T), \tag{3.7.1}$$

$$B(x, \nabla u) = 0 \quad \text{on} \quad S = \partial\Omega \times (0, T). \tag{3.7.2}$$

When $\Omega$ is convex with $C^2$ boundary $\partial\Omega$ and (3.7.2) is the homogeneous Neumann boundary condition i.e., $B(x, p) = \langle \nu(x), p \rangle$, where $\nu$ is the outward unit normal of $\partial\Omega$, it is easy to state a condition to guarantee the comparison principle:

(BCPB) Let $u$ and $v$ be sub and supersolutions of (3.7.1)–(3.7.2) in $\overline{\Omega} \times (0, T)$, where $\Omega$ is bounded. If $-\infty < u^* \leq v_* < \infty$ at $t = 0$, then $u^* \leq v_*$ in $Q$.

**Theorem 3.7.1.** *Assume that $\Omega$ is a convex domain with $C^2$ boundary in $\mathbf{R}^N$. Assume that $F$ is independent of $x$. Assume that (F1)–(F3) and (F4) holds. Then the comparison principle (BCPB) holds. If we assume (F3') instead of (F3), then (BCPB) is still valid (by replacing solutions by $\mathcal{F}_\Omega$-solutions) provided that $\mathcal{F}_\Omega$ is invariant under positive multiplication.*

By Lemma 3.1.3 this result yields:

**Corollary 3.7.2.** *Assume that $\Omega$ is a convex domain with $C^2$ boundary in $\mathbf{R}^N$. Assume that $F$ is independent of $x$ and $r$. Assume that $F : [0, T] \times (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ is continuous and degenerate elliptic. Then (BCPB) holds if $F$ is geometric.*

*Proof of Theorem 3.7.1.* To simplify the presentation we give a proof when $F$ is also independent of $r$.

We first construct a 'barrier' of the boundary. Let $d(x, \partial\Omega)$ be the distance of $x$ and the boundary $\partial\Omega$. As well-known $d(x, \partial\Omega)$ is $C^2$ near $\partial\Omega$ for $x \in \overline{\Omega}$; see e.g. D. Gilbarg and N. Trudinger (1983) [Lemma 14.16]. We set $b(x) = -d(x, \partial\Omega)$ near $\partial\Omega$ and extend to $\Omega$ in a suitable way so that $b \in C^2(\overline{\Omega})$ and $b < 0$ in $\Omega$. Clearly $b$ fulfills $\nu(x) = \nabla b(x)$. We set $g(x) = b(x) - \inf_{\overline{\Omega}} b$ so that $g \geq 0$ in $\overline{\Omega}$.

We argue in the same way in the proof of Theorem 3.1.1. Let $z_\sigma = (x_\sigma, t_\sigma, y_\sigma, s_\sigma) \in \overline{Q} \times \overline{Q}$ be a maximizer of $w - \varphi_\sigma$. We invoke the assumption that $u \leq v$ at $t = 0$ and observe that $t_\sigma > 0$, $s_\sigma > 0$ for large $\sigma$, say $\alpha > \alpha_0$, $\beta > \beta_0$. The main difference from the proof of Theorem 3.1.1 is that $x_\sigma$ and $y_\sigma$ may be on $\partial\Omega$.

We devide the situation into two cases following Remark 3.4.3. We fix $\alpha > \alpha_0$ and suppress the subscript $\alpha$.

**Case 1.** For each $r > 0$ there is $\beta_r \to \infty$ (as $r \to 0$) and a maximizer $z_{\beta_r}$ of $w - \varphi_{\beta_r}$ in $Q_{**} \times W_{**}$ such that $|x_{\beta_r} - y_{\beta_r}| \le r$, where $z_\beta = (x_\beta, t_\beta, y_\beta, s_\beta)$ and $Q_{**} = \overline{\Omega} \times (0, T]$. Since

We first fix $r$ and denote $\beta_r$ simply by $\beta$.

$$(w - \varphi_\beta)(x, t, y_\beta, s_\beta) \le (w - \varphi_\beta)(x_\beta, t_\beta, y_\beta, s_\beta)$$

for $(x, t) \in Q_{**}$, we see

$$\max_{Q_{**}}(u - \varphi^+) = (u - \varphi^+)(x_\beta, t_\beta)$$

if

$$\varphi^+(x, t) = \alpha f(|x - y_\beta|) + \beta(t - s_\beta)^2 + \gamma t.$$

Our goal is to prove

$$\varphi_t^+(x_\beta, t_\beta) + F(t_\beta, \alpha\nabla_p f(|p|), \alpha\nabla_p^2 f(|p|)) \le 0 \quad \text{at} \quad p = p_\beta \tag{3.7.3}$$

if $p_\beta := x_\beta - y_\beta \neq 0$ and

$$\varphi_t^+(x_\beta, t_\beta) = 2\beta(t_\beta - s_\beta) + \gamma \le 0 \tag{3.7.4}$$

if $p_\beta = 0$. Since $u$ is a subsolution in $Q$, this is clear if $x_\beta \in \Omega$. So we may assume that $x_\beta \in \partial\Omega$. We modify $\varphi^+$ to

$$\tilde{\varphi^+} = \varphi^+ + f(|x - x_\beta|) + |t - t_\beta|^2$$

so that $u - \tilde{\varphi^+}$ takes a strict maximum at $x = x_\beta$. We set $\varphi_\delta^+ = \tilde{\varphi^+} + \delta b$. By the convergence of maximum point (Lemma 2.2.5), there is a maximizer $(x^\delta, t^\delta)$ of $u - \varphi_\delta^+$ in $\overline{Q}$ converging to $(x_\beta, t_\beta)$ as $\delta \downarrow 0$.

If $y_\beta \neq x_\beta$, then $\nabla\varphi_\delta^+(x^\delta, t^\delta) \neq 0$ for small $\delta > 0$. If $x^\delta \in \partial\Omega$, then

$$\langle \nu(x^\delta), \nabla\tilde{\varphi^+}(x^\delta)\rangle \ge 0$$

since $\Omega$ is convex; we suppress $t^\delta$ since $\nabla\tilde{\varphi^+}$ does not depend on $t$. Since $\langle \nu(x), \nabla b(x)\rangle = 1$, this implies

$$\langle \nu(x^\delta), \nabla\varphi_\delta^+(x^\delta)\rangle \ge \delta > 0 \tag{3.7.5}$$

We recall Definition 2.3.7 with $\varphi = \varphi_\delta^+$ near $(x_\beta, t_\beta)$. The estimate (3.7.5) guarantees that

$$\varphi_{\delta t}^+ + F(t^\delta, \nabla\varphi_\delta^+, \nabla^2\varphi_\delta^+) \le 0 \quad \text{at} \quad (x^\delta, t^\delta) \tag{3.7.6}$$

even if $x^\delta \in \partial\Omega$. Sending $\delta \to 0$ we obtain (3.7.3).

If $y_\beta = x_\beta$, then there might be a chance that $\nabla\varphi_\delta^+(x^\delta, t^\delta) = 0$ i.e.

$$(\alpha+1)\nabla_x f(|x-y_\beta|) + \delta\nabla b(x) = 0 \quad \text{at} \quad x = x^\delta.$$

Since $b = -d(x, \partial\Omega)$ near $\partial\Omega$, such $x^\delta$ is of the form $x^\delta = x_\beta - C(\delta)\nu(x_\beta)$ with some $C(\delta) > 0$ for small $\delta$. In other words $x^\delta$ lies on the line through $x_\beta$ with direction $-\nu(x_\beta)$. Let $\theta$ be a strictly concave, increasing $C^2$ function on $[0,\infty)$ satisfying $\theta(0) = 0$, $\theta(C(\delta)) = C(\delta)$. For a suitable choice of such $\theta$ (depending on $\delta$)

$$u - \tilde{\varphi}^+ + \delta\theta(-b)$$

takes its maximum at some $(\xi^\delta, \tau^\delta) \to (x_\beta, t_\beta)$ as $\delta \to 0$ and that $\nabla\tilde{\varphi}^+(x^\delta) \neq \delta\nabla\theta(-b)(x^\delta)$ since $f$ is strictly convex. In a similar argument to obtain (3.7.6) we obtain (3.7.6) at $(\xi^\delta, \tau^\delta)$ even if $\xi^\delta \in \partial\Omega$. Sending $\delta \to 0$ yields (3.7.4).

We observe that

$$\min_{Q_{**}}(v - \varphi^-) = (v - \varphi^-)(y_\beta, s_\beta)$$

with $\varphi^-(y,s) = -\alpha f(|x_\beta - y|) - \beta(t_\beta - s)^2 - \gamma s$. We argue in a similar way to obtain

$$\varphi_t^-(y_\beta, s_\beta) + F(s_\beta, \alpha\nabla_p f(|p|), -\alpha\nabla_p^2 f(|p|)) \geq 0 \quad \text{at} \quad p = p_\beta \tag{3.7.7}$$

if $p_\beta \neq 0$ and

$$\varphi_t^-(y_\beta, s_\beta) = 2\beta(t_\beta - s_\beta) - \gamma \geq 0 \tag{3.7.8}$$

if $p_\beta = 0$. If $p_\beta = 0$, (3.7.4) and (3.7.8) yields a contradiction $2\gamma \leq 0$ so we may assume that $p_\beta \neq 0$. Subtracting (3.7.7) from (3.7.3) and sending $r \to 0$ so that $p_{\beta_r} \to 0$ (since $|p_{\beta_r}| < r$), we obtain a contradiction $2\gamma \leq 0$.

**Case 2.** Then is $r_0 > 0$ such that for sufficiently large $\beta$ any maximizer $z_\beta$ of $w - \varphi_\beta$ in $Q_{**} \times Q_{**}$ satisfies $|z_\beta| > r_0$.

We define $\Psi_{\beta\delta}$ by

$$\Psi_{\beta\delta}(x,t,y,s) = \Phi_\beta(x,t,y,s) - \delta(g(x) + g(y)).$$

for $\delta > 0$. Let $\hat{z} = (\hat{x}, \hat{t}, \hat{y}, \hat{s})$ be a maximizer of $\Psi_{\beta\delta}$ on $\overline{Q}\times\overline{Q}$. Clearly, $\sup_Z \Psi_{\beta\delta} \uparrow \sup_Z \Phi_\beta$ as $\delta \downarrow 0$, so as in §3.4 for sufficiently small $\delta$, say $\delta < \delta_0(\beta)$, we see that $\hat{z} \in Q_{**} \times Q_{**}$ and $|\hat{x} - \hat{y}| > r_0$. (Note that $\hat{z}$ depends on $\beta$ and $\delta$.)

Since $\Psi_{\beta\delta} = w - \delta(g(x) + g(y)) - \varphi_\beta$ attains its maximum at $\hat{z}$ over $Q_{**} \times Q_{**}$, we apply Theorem 3.3.3 with $k = 2, N_1 = N_2 = N_1 + 1$, $\mathcal{Z}_1 = \mathcal{Z}_2 = Q_{**}$, $\ell = 2$ to $w - \varphi = \Psi_{\beta\delta}$. Then we conclude as in the proof of Theorem 3.1.1 and 3.1.4 (Case 2), that for each $\lambda_1 > 0$ there is $X, Y \in \mathbf{S}^N$ such that (3.4.11) holds and

$$(\gamma + \hat{\varphi}_t, \hat{\varphi}_x + \delta\nabla g(\hat{x}), X + \delta\nabla^2 g(\hat{x})) \in \overline{\mathcal{P}}^{2,+}_{Q_{**}} u(\hat{x}, \hat{t}), \tag{3.7.9}$$

$$(-\gamma + \hat{\varphi}_t, \hat{\varphi}_y - \delta\nabla g(\hat{y}), Y - \delta\nabla^2 g(\hat{y})) \in \overline{\mathcal{P}}^{2,-}_{Q_{**}} v(\hat{y}, \hat{s}). \tag{3.7.10}$$

Since $|\hat{x} - \hat{y}| > r_0$ for small $\delta$, $\hat{\varphi}_x + \delta\nabla g(\hat{x}) \neq 0$, $-\hat{\varphi}_y - \delta\nabla g(\hat{y}) \neq 0$.

If $\hat{x}$ is on the boundary, because of the presence of $\hat{g}$ and the convexity of the domain

$$\langle \hat{\varphi}_x + \delta \nabla g(\hat{x}), \nu(x) \rangle \geq \delta > 0.$$

Thus the alternate requirement $B \leq 0$ in Definition 2.3.7 is not fulfilled. So (3.7.9) implies

$$\gamma + \hat{\varphi}_t + F(\hat{t}, \hat{\varphi}_x + \delta \nabla g(\hat{x}), X + \delta \nabla^2 g(\hat{x})) \leq 0 \tag{3.7.11}$$

either for $\hat{x} \in \Omega$ or $\hat{x} \in \partial\Omega$ since $u$ is a subsolution in $Q_{**}$. Similarly for $\hat{y} \in \partial\Omega$

$$\langle -\hat{\varphi}_y - \delta \nabla g(\hat{y}), \nu(y) \rangle \leq -\delta < 0$$

so (3.7.10) yields

$$-\gamma + \hat{\varphi}_t + F(\hat{s}, -\hat{\varphi}_y - \nabla g(\hat{y}), Y - \delta \nabla^2 g(\hat{y})) \geq 0 \tag{3.7.12}$$

since $v$ is a supersolution. Subtracting (3.7.12) from (3.7.11), we send $\delta \to 0$ and then send $\beta \to \infty$ to get a contradiction $2\gamma \leq 0$ as in the proof of Theorem 3.1.4 Case 2. We omit the detail. □

**Remark 3.7.3** There are several directions of extension of Theorem 3.7.1 by removing the convexity or handling more general boundary condition. However, all such extensions so far exclude very singular equations and forced to assume (F3). We give a typical result.

**Theorem 3.7.4** *Assume that $\Omega$ is a bounded domain in $\mathbf{R}^N$ with smooth boundary. We consider the level set mean curvature flow equation with the prescribed contract angle boundary condition (in §1.6.1), i.e.,*

$$\begin{cases} u_t - \Delta u + \sum \frac{u_{x_j} u_{x_j}}{|\nabla u|^2} u_{x_i x_j} = 0 & \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial \nu} + z|\nabla u| = 0 & \text{on } \partial\Omega \times (0,T) \end{cases}$$

*with a smooth function $z = z(x)$ satisfying $\sup_{\partial\Omega} |z| < 1$. Then (BCPB) holds.*

The level set mean curvature flow equation can be extended to (1.6.10) provided that $\beta \in C^1(S^{N-1})$ and $\gamma \in C^4(S^{N-1})$ so that the assumption (i) of Lemma 3.6.7 is fulfilled. We do not give the proof of Theorem 3.7.4 because it is very technical. The reader is referred to papers of H. Ishii and M.-H. Sato (2001) and G. Barles (1999) for more general results as well as the proof of Theorem 3.7.4.

## 3.8 Notes and comments

A version of comparison principles for viscosity solutions was first proved by M. G. Crandall and P.-L. Lions (1983) and then by M. G. Crandall, L. C. Evans and P.-L. Lions (1984) for first-order equations. Some comparison principles were proved for a special second order equations called Hamilton-Jacobi-Bellman equation by P.-L. Lions (1983), (1984) by ad hoc stochastic control method. However, a general theory for the second order equations (even without singularity) remained open for quite a while until R. Jensen (1998) developed several key ideas to overcome difficulties. Then H. Ishii (1989a) extended the theory to include more examples by introducing matrix inequalities of the general form (3.3.7); see also H. Ishii and P.-L. Lions (1990). A maximum principle (Theorem 3.3.2) for semicontinuous functions is due to M. Crandall and H. Ishii (1990). This work also introduced the notion of semijets $J^{2,\pm}$ and $\overline{J}^{2,\pm}$. For more detailed information of the development of the theory for comparison principles the reader is referred to the User's Guide by M. G. Crandall, H. Ishii and P.-L. Lions (1992). See also B. Kawohl and N. Kutev (2000) for further examination of structures of equations.

A level set equation has a sengularity at $\nabla u = 0$ if it is second order. For such an equation comparison principle was first proved by Y.-G. Chen, Y. Giga and S. Goto (1991a) and independently by L. C. Evans and J. Spruck (1991) (for the level set mean curvature flow equation). Theorem 3.1.1 under (F3) is due to Y.-G. Chen, Y. Giga and S. Goto (1991a). The results were extended to unbounded domains and spatially inhomogeneous problems by Y. Giga, S. Goto, H. Ishii and M.-H. Sato (1991) which adjusts elliptic version by M.-H. Sato (1990) to parabolic one. This work includes Theorem 3.1.4 under (F3). The extension to very singular equations (without (F3)) was established by S. Goto (1994) and by H. Ishii and P. E. Souganidis (1995) for geometric equations. The methods are different and the former work seems to be limited to bounded domains while the latter work applies to general domains. Also the latter work introduced the notion of $\mathcal{F}$-solution and it is not limited to geometric equations. In fact M. Ohnuma and K. Sato (1997) extended their theory to non geometric equations. When $F$ is independent of $t$, Theorem 3.1.4 without (F3) is due to M. Ohnuma and K. Sato (1997) and Corollary 3.1.4 without (F3) is due to H. Ishii and P. E. Souganidis (1995).

Contents of §3.2.1 is essentially taken from User's Guide with spetial attention to parabolic semijets. Extension of solution defined in $\Omega \times (0,T)$ to $\Omega \times (0,T]$ is often important. It is stated in several papers including Y.-G. Chen, Y. Giga and S. Goto (1991b) and H. Ishii and P. E. Souganidis (1995) (for $\mathcal{F}$-solutions). Remaining results in §3.2.2 are taken from the work of Y.-G. Chen, Y. Giga and S. Goto (1991b).

The contents of §3.3.1 is essentially taken from User's Guide except Theorem 3.3.3 that is very useful to handle parabolic problems. The proof of Theorem 3.1.1 is more transparent than the original proofs under (F3) of Y.-G. Chen, Y. Giga and S. Goto (1991a) and L. C. Evans and J. Spruck (1991) since during their researches Theorem 3.3.2 was not available. The proof of Theorem 3.1.4 without (F3) is slightly different from H. Ishii and P. E. Souganidis (1995) or M. Ohnuma and K. Sato (1997) since $F$ depends on $t$. Even under (F3) it is different from that of Y. Giga, S. Goto, H. Ishii and M.-H. Sato (1991) since we rather use Theorem 3.3.3 instead of usual parabolic version of maximum principle (due to M. Crandall and H. Ishii (1990)).

The Lipschitz preserving property is clear if $F$ is spatially homogeneous, i.e. $F$ is independent of $x$. The convexity preserving property is more difficult to obtain. When $F$ satisfies (F3) and is independent of $t$, this property was first proved by Y. Giga, S. Goto, H. Ishii and M.-H. Sato (1991) by adjusting idea of H. Ishii and P.-L. Lions (1990) for singular equations. A statement similar to Theorem 3.5.2 is stated by H. Ishii and P. E. Souganidis (1995) without proof when $F$ is independent of time. We here give a complete proof.

When $F$ depends explicitly on the spatial variable $x$ it is hard to state the results in a simple way. When $x$-dependence appears in fitst order terms, it is relatively easy to state (Theorem 3.6.1). Although this is not explicitly stated in the literature without assuming (F3), the proof seems to be standard. When $x$-dependence appears in top order term, Theorem 3.6.4 is considered as a variant of comparison results of Y. Giga, S. Goto, H. Ishii and M.-H. Sato (1991) and of G. Barles, H. M. Soner and P. E. Souganidis (1993). Even for nonsingular equation the power 2 of $\mu|x-y|^2$ in (3.6.8) is optimal in the sense that we cannot replace $\mu|x-y|^2$ by $\mu|x-y|^k$ with $k < 2 (k > 0)$. This is already pointed out by H. Ishii (1989a) [Theorem 3.3]. As mentioned in Remark 3.6.8 it is nontrivial to extend spatially in homogeneity in the second order term when (F3) is violated.

**Boundary value problems.** The Neumann type boundary problem, i.e. $B(x,p) = \langle \nu(x), p\rangle$ was proposed for viscosity solutions first by P.-L. Lions (1982) and established a comparison principle for first order equations. It was extended to second order equations with more complicated boundary condition by G. Barles (1993) and by H. Ishii (1991) including oblique type boundary conditions, when the equation has no singularities at $\nabla u = 0$. The first work for singular equation was done by M.-H. Sato (1994), where he proved Theorem 3.7.1 under (F3). Extention to $\mathcal{F}$-solutions seems to be nontrivial from his proof so we provide a detailed proof. Under (F3) the convexity assumption on domain was successfully removed by Y. Giga and M.-H. Sato (1993) at the expense of restricting a class of $F$. Later, this result was generalized to oblique type problem where $B(x,p) = \langle \nu(x), p\rangle + z|p|$ with a constant $z \in (-1,1)$ by M.-H. Sato (1996) when the domain is a half space. For a general domain and a general boundary condition $B(x, \nabla u) = 0$ the comparison principle has been proved by G. Barles (1999) and H. Ishii and M.-H. Sato (2001) by a different method and for a different generality. Such a comparison principle is useful not only to the level set method itself but also to stability analysis of stationary solution as presented in S.-I. Ei, M.-H. Sato and E. Yanagida (1996).

**Level set anisotropic mean curvature flow equation.** When we consider the level set equation (1.6.10) of general evolution equation of isothermal interface (1.5.2), our theory in this Chapter requires at least $C^2$ regularity of the interfacial energy $\gamma$ on $S^{N-1}$ as mentioned in §3.1.3. In applied problems if is sometimes too restrictive. There are several extensions to relax regularity assumptions on $\gamma$. For example, M. E. Gurtin, H. M. Soner and P. E. Souganidis (1995) and M. Ohnuma and M.-H. Sato (1993) independently relax the assumption in the way that $\nabla\gamma$ is Lipschitz but may not be $C^1$ in a finitely many points on $S^1$ when $N = 2$. See also Y. Giga (1994), for a comparison principle of graph-like solutions. There is also higher dimensional extension by H. Ishii (1996).

When $\gamma$ is not $C^1$, the equation has nonlocal nature. A typical example is the case when Frank diagram of $\gamma$ is a convex polyhedra. When $N = 2$, such energy is called crystalline and in this case the evolution law is not expected to be local. Nevertheless, when $N = 2$, a comparison principle has been established by introducing appropriate notion of viscosity solutions by M.-H. Giga and Y. Giga (2001); for announcement see M.-H. Giga and Y. Giga (1998b). The theory developed there establishs the level set method as well as several convergence results. In particular it provides the convergence of crystalline algorithm and asserts that crystalline flow is obtained as a limit of motion by smooth approximated interfacial energy as discussed in M.-H. Giga and Y. Giga (2000). See also K. Ishii and H.-M. Soner (1999) for convergence of crystalline algorithm for curve shortening equation. Such convergence results were first proved by P. M. Gigão and R. V. Kohn (1994) and T. Fukui and Y. Giga (1996) for graph-like functions. For curve shortening equation P. E. Girão (1995) proved the convergence for convex curves; see also a review of P. E. Girão and R. V. Kohn (1996). The crystalline motion was first proposed by S. B. Angenent and M. E. Gurtin (1989) and independently by J. Taylor (1991).

The theory of M.-H. Giga and Y. Giga (2001) is based on the corresponding theory for graph-like functions developed by M.-H. Giga and Y. Giga (1998a), (1999). An idea of converting the problem to that of graph-like functions may be useful to analyse geometric equations. For the background of these materials the reader is referred to a review article of Y. Giga (2000) and references cited there as well as recent development of the theory.

# Chapter 4

# Classical level set method

We introduce a notion of generalized solutions for surface evolution equations which tracks the evolution after singularities develop. For this purpose we solve the level set equations (Chapter 1) globally in time in the sense of viscosity solutions developed in Chapters 2 and 3. We also prove that each level set of solution is determined by the corresponding level set of initial data and is independent of choice of initial data for solutions of level set equations. This uniqueness of level set is fundamental to define a notion of generalized solution by a level set of solutions of level set equations. We also study various general properties of solutions. In particular we explain what is called fattening phenomena; a level set of solutions of the level set equations may have interior point even if the corresponding level set of initial data has no interior points.

## 4.1 Brief sketch of a level set method

We consider a surface evolution equation of the form

$$V = f(z, \mathbf{n}, \nabla \mathbf{n}) \quad \text{on} \quad \Gamma_t, \tag{4.1.1}$$

where $f : \mathbf{R}^N \times [0, T] \times \mathbf{E} \to \mathbf{R}$ is a given function as in Chapter 1. Here $\mathbf{E}$ is a bundle defined by

$$\mathbf{E} = \{(p, Q_p(X));\ p \in S^{N-1}, X \in \mathbf{S}^N\},$$

where $Q_p(X) = (I - p \otimes p)X(I - p \otimes p)$ for unit vector $p$. Examples of surface evolution equations including the Hamilton-Jacobi equations, the mean curvature flow equation and its anisotropic version, trancated Gaussian and symmetric curvature flow equation, the affine curvature flow equations enjoy at least following two properties
(f1) (Continuity) $f : \mathbf{R}^N \times [0, T] \times \mathbf{E} \to \mathbf{R}$ is continuous
(f2) (Degenerate ellipticity)

$$f(z, p, Q_p(X)) \leq f(z, p, Q_p(Y)) \quad \text{whenever } Q_p(Y) \geq Q_p(Y).$$

In this book we consider (4.1.1) satisfying (f1) and (f2) i.e., continuous and degenerate parabolic surface evolution equations. We associate the level set equaton of (4.1.1):

$$u_t + F(z, \nabla u, \nabla^2 u) = 0, \tag{4.1.2}$$

where $F$ is given by

$$F(z,p,X) = -|p|f(z,-\hat{p},-Q_{\hat{p}}(X)/|p|),\ \hat{p} = p/|p|. \tag{4.1.3}$$

We would like to solve (4.1.1) with given initial hypersurface globally in time with aid of the level set equation. Our level set method is summarized as follows. To simplify the argument we assume that $\Gamma_t$ is a compact hypersurface (without boundary) so that $\Gamma_0 = \partial D_0$ for some bounded open set in $\mathbf{R}^N$.

1. **Initial auxiliary function**. We take an auxiliary function $u_0$ which is at least continuous in $\mathbf{R}^N$ such that

$$\Gamma_0 = \{x \in \mathbf{R}^N, u_0(x) = 0\},\ D_0 = \{x \in \mathbf{R}^N, u_0(x) > 0\}. \tag{4.1.4}$$

The assumption that $u_0$ is positive in $D_0$ gives the orientation of $\Gamma_0$. Formally, outward unit normal from $D_0$ equals $\mathbf{n} = -\nabla u_0/|\nabla u_0|$ by this choice of $u_0$. For convenience we often arrange that $u_0$ equals a negative constant $-\alpha$ outside some big ball.

2. **Global unique solvability of level set equations.** We solve (4.1.2) with initial data $u(x,0) = u_0(x)$ globally in time in the sense of viscosity solutions.

3. **Level set solution**. For the global solution $u$ of (4.1.2) with initial data $u_0$ we set

$$\begin{aligned}\Gamma &= \{(x,t) \in \mathbf{R}^N \times [0,T);\ u(x,t) = 0\},\\ D &= \{(x,t) \in \mathbf{R}^N \times [0,T);\ u(x,t) > 0\}\end{aligned}$$

and expect that the cross-section

$$\begin{aligned}\Gamma(t) &= \{x \in \mathbf{R}^N;\ (x,t) \in \Gamma\},\\ D(t) &= \{x \in \mathbf{R}^N;\ (x,t) \in D\}\end{aligned}$$

is a kind of generalized solution. (The orientation $\mathbf{n}$ of $\Gamma(t)$ is formally taken so that it is outward from $D(t)$ and $\mathbf{n} = -\nabla u/|\nabla u|$.) The sets $\Gamma$ and $D$ are a kind of weak or generalized solutions with initial data $\Gamma_0$ and $D_0$. Since (4.1.2) is invariant in addition of a constant, the value zero in the definition of $D$ and $\Gamma$ may be replaced by other value $c$, of course.

Step 1 is easy to be implemented by taking

$$u_0(x) = \max(\mathrm{sd}(x,\partial D_0), -1),$$

where sd denotes the signed distance of $\partial D_0$ defined by

$$\mathrm{sd}(x,\partial D_0) = \begin{cases} \mathrm{dist}(x,\partial D_0), & x \in D_0\\ -\mathrm{dist}(x,\partial D_0), & x \notin D_0. \end{cases}$$

The global solvability of Step 2 is one of main topic of this Chapter. Actually, Step 2 is implemented with aid of theory of viscosity solutions. Since our definition of $\Gamma$ and $D$ in

Step 3 is extrinsic, to see that our definition is well-defined, we should check that $\Gamma$ and $D$ are determined by $\Gamma_0$ and $D_0$ respectively and independent of the choice of $u_0$ satisfying (4.1.4). We call this part of problem as "uniqueness of generalized evolutions" by naming that $\Gamma$ and $D$ are generalized evolutions. In the next section we discuss the problem of uniqueness of generalized evolutions.

We conclude this section by giving rigorous definitions of generalized evolutions. There are at least two ways to define. The first one only applies evolution of bounded sets (or its complement) so it is rectrictive. However, the proof for the uniqueness depends on comparison principle (BCP) in a bounded domain other than the invariance (Theorem 4.2.1) and it is in structive. The second one is very general but we need the comparison principle in $\mathbf{R}^N$ and the global solvability for (4.1.2) to prove the uniqueness of the generalized evolutions. Moreover the proof is not intuitive. We only give the proof for the first one in §4.2 and postpone the proof for the second one in §4.3.

**Definition 4.1.1.** Let $D_0$ be a bounded open set in $\mathbf{R}^N$. An open set $D$ in $Z = \mathbf{R}^N \times [0,T)$ is called a *(generalized ) open evolution* of (4.1.1) with initial data $D_0$ if there exist a ($\mathcal{F}_{\mathbf{R}^N}$-)solution $u \in K_\alpha(Z)$ of (4.1.2) in $Z$ that satisfies

$$D = \{(x,t) \in Z; u(x,t) > 0\},\ D_0 = \{x \in \mathbf{R}^N; u(x,0) > 0\},$$

for some $\alpha \leq 0$, where $K_\alpha(Z)$ denotes the space of all real-valued continuous function $u$ on $Z$ such that $u \equiv \alpha$ outside $B_R(0) \times [0,T)$ for some $R > 0$.

Let $F_0$ be a bounded closed set in $\mathbf{R}^N$. A closed set $E$ in $Z$ is called a *(generalized) closed evolution* of (4.1.1) with initial data $E_0$ if there exists a ($\mathcal{F}_{\mathbf{R}^N}$-)solution $u \in K_\alpha(Z)$ of (4.1.2) in $Z$ that satisfies

$$E = \{(x,t) \in Z; u(x,t) \geq 0\},\ E_0 = \{x \in \mathbf{R}^N, u(x,0) \geq 0\}$$

for some $\alpha < 0$. If $E_0 = \overline{D}_0$, the set $E \backslash D = \Gamma$ is called a *(generalized) interface evolution* of (4.1.1) with initial data $\Gamma_0 = E_0 \backslash D_0$.

**Definition 4.1.2.** We replace $K_\alpha(Z)$ by $\mathrm{BUC}(Z)$ to define generalized evolutions for arbitrary open and closed sets in $Z$. Here $\mathrm{BUC}(Z)$ denotes the space of all bounded uniformly continuous functions. By definitions if $D$ is an open evolution of (4.1.2) with initial data $D_0$ in Definition 4.1.1, it should be so in the sense of this definition since $K_\alpha(Z) \subset \mathrm{BUC}(Z)$.

**Remark 4.1.3.** We are tempting to use the word 'level set solution' to describe 'generalized evolution'. The reason we did not use this word is that we use 'level set solution' in Chapter 5 in a different sense although it turns out both notions are equivalent (Proposition 5.2.8 and Remark 5.2.9). We often suppress the word 'generalized'. The word 'evolution' to describe $D$ and $E$ was used by S. Altschuler, S. Angenent and Y. Giga (1995).

## 4.2 Uniqueness of bounded evolutions

We shall study whether level set solutions $D$ and $E$ are determined by $D_0$ and $E_0$ respectively and are independent of choice of an auxiliary function $u$.

### 4.2.1 Invariance under change of dependent variables.

We first study special invariance of geometric equations. As observed in §1.6.4 if $u$ solves a geometric equation so does $\theta(u)$ for any $\theta$ with $\theta' \geq 0$ at least formally. We state this property in a rigorous way.

**Theorem 4.2.1.** (Invariance) *Assume that $F : W_0 = \overline{\Omega} \times [0,T] \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ is continuous and geometric, where $\Omega$ is an open set in $\mathbf{R}^N$. If $u$ is an $\mathcal{F}_\Omega$-subsolution (resp. $\mathcal{F}_\Omega$-supersolution) of*

$$u_t + F(x,t,\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = \Omega \times (0,T). \tag{4.2.1}$$

*Then the composite function $\theta \circ u = \theta(u)$ is also an $\mathcal{F}_\Omega$-subsolution (resp. $\mathcal{F}_\Omega$- supersolution) of (4.2.1) provided that $\theta : \mathbf{R} \to \mathbf{R}$ is continuous and nondecreasing. One may weaken the assumption on continuity of $\theta$ by uppersemicontinuity (resp. lowersemicontinuity) with values $\mathbf{R} \cup \{-\infty\}$ (resp. $\mathbf{R} \cup \{+\infty\}$).*

To prove this result we need invariance of class $C_F^2$ under change of dependent variables.

**Lemma 4.2.2.** *Assume the same hypothesis of Theorem 4.2.1 concerning $F$ and $\Omega$.*

*(i) If $g \in \mathcal{F}_\Omega$ and $\theta : \mathbf{R} \to \mathbf{R}$ is $C^2$ with $\theta' > 0$, $\theta'' \geq 0$ everywhere, then $\theta \circ g \in \mathcal{F}_\Omega$.*

*(ii) If $\varphi \in C_F^2(Q)$, then $\theta \circ \varphi \in C_F^2(Q)$ for any $\theta \in C^2(\mathbf{R})$ with $\theta' > 0$.*

*Proof.* (i) Since $F$ is geometric, for $f = \theta \circ g$ and $g$ we see by (3.1.2) (in the proof of Lemma 3.1.3)

$$\begin{aligned} F(z, \nabla_p f(\rho), \pm \nabla_p^2 f(\rho)) &= \frac{f'(\rho)}{\rho} F(z,p,\pm I), \\ F(z, \nabla_p g(\rho), \pm \nabla_p^2 g(\rho)) &= \frac{g'(\rho)}{\rho} F(z,p,\pm I) \end{aligned}$$

for $\rho = |p| > 0$. Since $f'(\rho) = \theta'(g(\rho))g'(\rho)$, the preceeding two formula imply that

$$\lim_{p \to 0} \sup_{z \in Q} |F(z, \nabla_p f(\rho), \pm \nabla_p^2 f(\rho))| = 0$$

if the same formula holds for $g$ instead of $f$. Other conditions $f(0) = f'(0) = f''(0)$, $f''(r) > 0$ for $r > 0$ follows from $\theta' > 0, \theta'' \geq 0$ and the corresponding properties for $g$.

(ii) This can be proved independent of (i). We set $\psi = \theta \circ \varphi$. We note that $\nabla \varphi(z) = 0$ is equivalent to $\nabla \psi(z) = 0$ since $\theta' > 0$. Thus it suffices to check the behaviour of $\psi$ near point $\hat{z}$ where $\nabla\varphi(\hat{z}) = 0$. We may assume that $\theta(0) = 0, \varphi(\hat{z}) = 0, \varphi_t(\hat{z}) = 0$. Our goal is to prove

$$|\psi(z)| \le f(|x - \hat{x}|) + \omega_1(|t - \hat{t}|)$$

as $z = (x,t) \to \hat{z} = (\hat{x}, \hat{t})$ in $Q$ for some $f \in \mathcal{F}_\Omega$ and $\omega_1$ with $\omega_1(\sigma)/\sigma \to 0$ as $\sigma \to 0$, $\omega_1(0) = 0$. Since $\varphi \in C^2_F(Q)$, there is $g \in \mathcal{F}_\Omega$ and $\overline{\omega}_1$ with $\overline{\omega}_1(\sigma)/\sigma \to 0$ as $\sigma \to 0$ with $\overline{\omega}_1(0) = 0$ such that

$$|\varphi(z)| \le g(|x - \hat{x}|) + \overline{\omega}_1(|t - \hat{t}|)$$

as $(x,t) \to (\hat{x}, \hat{t})$ in $Q$. This implies

$$|\psi(z)| = |\theta \circ \varphi(z)| \le \theta(g(|x - \hat{x}|) + \overline{\omega}_1(|t - \hat{t}|)).$$

Since $\theta(0) = 0$ and $\theta'(0) = \gamma > 0$, the right hand side is dominated by

$$2\gamma g(|x - \hat{x}|) + 2\gamma\overline{\omega}_1(|t - \hat{t}|)$$

for $(x,t)$ close to $(\hat{x}, \hat{t})$. Since $ag \in \mathcal{F}$, this yields the desired estimate for $|\psi(z)|$ with $f = 2\gamma g$, $\omega_1 = 2\gamma\overline{\omega}_1$. $\square$

*Proof of Theorem 4.2.1.* We may assume that $\mathcal{F}_\Omega$ is not empty. The property $(\theta \circ u)^*(z) < \infty$ for $z \in Q$ is trivial since $u^*(z) < \infty$ for $z \in Q$.

1. We first discuss the case when $\theta \in C^2(\mathbf{R})$ and $\theta' > 0$ everwhere. Assume that $\varphi \in C^2_F(Q)$ and $z = (x,t) \in Q$ satisfies

$$\max_Q(\theta \circ u - \varphi) = \theta(u(z)) - \varphi(z) = 0.$$

Since $\theta \in C^2$ and $\theta' > 0$ the inverse function $h = \theta^{-1}$ is also $C^2$ and $h' > 0$. By definition

$$\max_Q(u - \psi) = u(z) - \psi(z) = 0, \tag{4.2.2}$$

where $\psi = h \circ \varphi$. By Lemma 4.2.2(ii) $\psi = h \circ \varphi \in C^2_F(Q)$; we do not use Lemma 4.2.2 (i). If $\nabla\varphi(z) = p \neq 0$, geometricity of $F$ implies that

$$F(z, \nabla\psi(z), \nabla^2\psi(z)) = F(z, \mu p, \mu X + \sigma p \otimes p) = \mu F(z, p, X),$$

where $X = \nabla^2\varphi(z)$, $\mu = h'(\varphi(z))$, $\sigma = h''(\varphi(z))$. Since $u$ is a subsolution of (4.2.1), it follows from (4.2.2) that

$$\begin{aligned} 0 &\ge \frac{\partial\psi}{\partial t}(z) + F(z, \nabla\psi(z), \nabla^2\psi(z)) \\ &= h'(\varphi(z))\left(\frac{\partial\varphi}{\partial t}(z) + F(z, \nabla\varphi(z), \nabla^2\varphi(z))\right), \end{aligned}$$

which implies the condition for subsolution when $\nabla\varphi(z) = p \neq 0$. If $\nabla\varphi(z) = p = 0$ then $\nabla(h \circ \varphi)(z) = 0$. Since $u$ is an $\mathcal{F}$-subsolution, we see

$$0 \ge \frac{\partial\psi}{\partial t}(z) = h'(\varphi(z))\frac{\partial\varphi}{\partial t}(z)$$

so $\theta \circ u$ is now an $\mathcal{F}$-subsolution.

2. For general $\theta$ we take an approximate sequence $\theta_m \in C^2(\mathbf{R})$ with $\theta'_m > 0$ such that $\theta = \limsup^* \theta_m$; then existence of such a sequence is guaranteed in Lemma 4.2.3 below. Since $u_m = \theta_m \circ u$ is an $\mathcal{F}$-subsolution and $(\theta \circ u)^* = \limsup^* u_m$, the stability result (Theorem 2.2.1) in Chapter 2 implies that $u$ is an $\mathcal{F}$-subsolution.

The proof of supersolution is symmetric so is omitted. □

**Lemma 4.2.3.** *Let $\theta$ be an upper semicontinuous nondecreasing function defined on $\mathbf{R}$ with values in $\mathbf{R} \cup \{-\infty\}$. Then there is a smooth $\theta_m$ with $\theta'_m > 0$ such that $\theta = \limsup^* \theta_m$. If $\theta : \mathbf{R} \to \mathbf{R}$ is moreover continuous, the sequence $\theta_m$ can be arranged so that $\theta_m$ converges to $\theta$ uniformly in $\mathbf{R}$.*

*Proof.* 1. We approximate $\theta$ by nondecreasing piecewise linear functions. We may assume that $\theta \not\equiv -\infty$ since otherwise $\theta_m(t) = -m + t/m$ gives a desired sequence. Let $I$ be the infinite interval such that

$$I = \{t \in \mathbf{R},\ \theta(t) > -\infty\}.$$

For an integer $j$ and positive integer $m$ we set

$$a_j^{(m)} = \inf\{t \in I;\ \theta(t) \geq j/k\}$$

with convention that $a_j^{(m)} = +\infty$ if there is no element $t \in I$ that satisfies $\theta(t) \geq j/k$. Since $\theta$ is nondecreasing, $a_j^{(m)} \leq a_{j+1}^{(m)}$ for any integer $j$ and positive integer $m$ and $\{a_j^{(m)}\}_{j=-\infty}^{\infty}$ has no accumulation points in the interior of $I$. We divide into two cases depending on whether or not $\theta(t) \to -\infty$ as $t \downarrow \gamma = \inf I$.
(i) (the case $\lim_{t\downarrow\gamma} \theta(t) = -\infty$) We take

$$\vartheta_m(t) = (j_+ + 1)/m \quad \text{at} \quad t = a_j^{(m)} \in I$$

where $j_+$ is a number such that

$$j_+ = \max\{\ell;\ a_\ell^{(m)} = a_j^{(m)}\}.$$

We assign the value of $\vartheta_m$ for $t \neq a_j^{(m)}$ such that $\vartheta_m$ is continuous, piecewise linear (linear outside $\{a_j^{(m)}\}_{j=-\infty}^{\infty}$) in $I = (\gamma, \infty)$. To extend $\vartheta_m$ outside $I$ we set

$$\theta_m(t) = \begin{cases} \min(-m, \vartheta_m(t)) & t \in I, \\ -m & t \notin I. \end{cases}$$

Since $\lim_{t\downarrow\gamma} \theta(t) = -\infty, \theta_m(t)$ is continuous, piecewise linear and nondecreasing. By definition $\vartheta_m \geq \theta$, and if $\theta$ is upper semicontinuous it is not difficult to see that

$$\theta = \limsup_{m\to\infty}{}^* \theta_m.$$

(ii) (the case $\lim_{t\downarrow\gamma}\theta(t)=\beta>-\infty$) We set $\vartheta_m$ as in (i) so that $\vartheta_m$ is well-defined on $I=[\gamma,\infty)$. We set

$$\theta_m(t)=\begin{cases}\vartheta_m(t) & \text{for} \quad t\geq\gamma\\ m(t-\gamma)+(j_1+1)/m & \text{for} \quad t\leq\gamma,\end{cases}$$

where $j_1=\max\{\ell; a_\ell^{(m)}=\gamma\}$. The function $\theta_m\geq\theta$ is continuous, piecewise linear and nondecreasing. The convergence $\theta=\limsup^*\theta_m$ is not difficult to prove. In both cases (i) and (ii) we remark that the converge $\theta_m\to\theta$ is uniform if $\theta:\mathbf{R}\to\mathbf{R}$ is continuous by this construction.

2. We approximate the nondecreasing piecewise linear function $\theta_m$ by nondecreasing $C^2$ function from above by mollifying $\theta_m$ near nondifferentiable points. We still denote $C^2$ approximation of $\theta_m$ by $\theta_m$.

3. We approximate the nondecreasing $C^2$ function $\theta_m$ by a $C^2$ function whose derivative is always positive. We take a positive bounded $C^2$ function $r$ on $\mathbf{R}$ whose derivative is positive everwhere; a typical example of such an $r$ is $r(t)=\text{Arctan}\ t+\pi/2$

$$\tilde{\theta_m}(t)=\theta_m(t)+r(t)/m$$

to get $\theta=\limsup^*\tilde{\theta}_m$ and $\tilde{\theta}_m\geq\theta$. This $\tilde{\theta}_m\in C^2(\mathbf{R})$ now satisfies $\tilde{\theta}'_m(t)>0$ for all $t\in\mathbf{R}$. We thus obtained the desired sequence. Again if $\theta:\mathbf{R}\to\mathbf{R}$ is continuous the convergence $\tilde{\theta}_m\to\theta$ is uniform which is inherited from Step 1. □

**Remark 4.2.4.** The invariance property stated in Theorem 4.2.1 is easily extended to more general geometric equations including boundary value problems. We shall state these results without proof since the proofs are essentially the same as that of Theorem 4.2.1.

**Theorem 4.2.5.** (Invariance for general equations) *Let $E(z,\cdot,\cdot)$ be a real-valued function defined in a dense set of $W$ of $\mathbf{R}^d\times\mathbf{S}^d$ for $z\in\mathcal{O}$, where $\mathcal{O}$ is a local compact subset $\mathbf{R}^d$. Assume that the equation*

$$E(z,Du,D^2u)=0 \quad \textit{in} \quad \mathcal{O} \tag{4.2.2}$$

*is geometric in the sense of Definition 1.6.13. If $u$ is a subsolution (resp. supersolution) of (4.2.2) in $\mathcal{O}$, then $\theta\circ u$ is also a subsolution (resp. supersolution) if $\theta$ is nondecreasing, uppersemicontiuous (resp. lowersemicontinuous) on $\mathbf{R}$ with values in $\mathbf{R}\cup\{-\infty\}$ (resp. $\mathbf{R}\cup\{+\infty\}$).*

This results applies to level set equations for surface equations with boundary conditions. For boundary value problems of very singular equations we also get invariance by interpreting solution by $\mathcal{F}$-solutions in Definition 2.3.7.

### 4.2.2 Orientation-free surface evolution equations

As we already observed in Chapter 1, there are a class of surface evolution equations invariant under the change of orientations of hypersurfaces. Such as surface equation is called orientation-free which is rigrously defined in Definition 4.2.6 below. Examples include the mean curvature flow equation. For such a class of equations we shall show that $\theta \circ u$ is a solution if $u$ is a solution without assuming that $\theta$ is nondecreasing in Theorem 4.2.1, (where $\theta$ is assumed to be continuous).

**Definition 4.2.6.** If $f$ in (4.1.1) fulfills

$$f(z, -p, -Q_p(X)) = -f(z, p, Q_p(X)) \tag{4.2.3}$$

for all $z \in \mathbf{R}^N \times [0, T], (p, Q_p(X)) \in \mathbf{E}$, the equation (4.1.1) is called *orientation-free*. It is clear that this condition is equivalent to the property

$$F(z, -p, -X) = -F(z, p, X) \tag{4.2.4}$$

for all $z \in \mathbf{R}^N \times [0, T]$, $p \in \mathbf{R}^N \backslash \{0\}$, $X \in \mathbf{S}^N$ where $F$ is in (4.1.3). So if (4.2.4) holds for geometric $F$, then the equation (4.2.1) is also called *orientation-free*.

**Theorem 4.2.7.** *Assume the same hypothesis of Theorem 4.2.1 and $\Omega$. Assume that (4.2.1) is orientation-free. If $u$ is an $\mathcal{F}_\Omega$-solution of (4.2.1), then so is $\theta \circ u$ provided that $\theta : \mathbf{R} \to \mathbf{R}$ is continuous.*

*Proof.* 1. We suppress the word $\mathcal{F}_\Omega$ in the proof. We first note that $(\theta \circ u)^* < \infty$ and $(\theta \circ u)_* > -\infty$ since $-\infty < u_* \leq u^* < +\infty$. By (4.2.4) and definition of sub- and supersolutions we see that $-u$ is a subsolution (resp. supersolution) if $u$ is a supersolution (resp. subsolution).

2. By Theorem 4.2.1 and Step 1 $\theta \circ u$ is a solution if $\theta : \mathbf{R} \to \mathbf{R}$ is continuous and either nondecreasing or nonincreasing and $u$ is a solution.

3. For general continuous $\theta$ we may assume that $u$ is bounded on $Q = \Omega \times (0, T)$. Indeed, for a given $M > 0$ by Theorem 4.2.1 $u_M = \sigma_M \circ u$ for $\sigma_M(t) = (t \wedge M) \vee (-M)$ is a solution if $u$ is a solution. If $\theta(u_M)$ is a solution, so is $\theta(u)$ by stability principle (§2.2.1) since $\theta(u)$ is given as a limit of $\theta(u_M)$ as $M \to \infty$.

4. We approximate $\theta$ by a polynomial $\theta_m$ uniformly on $[-M-1, M+1]$ by Weierstrass' approximation theorem. Thus again by stability principle we may assume that $\theta$ is a polynomial on $[-M-1, M+1]$.

5. Since $\theta$ is polynomial there are only finite number of local maximizers and minimizers of $\theta$ in $(-M-1, M+1)$. We may assume that $\theta$ has either a maximizer or a minimizer since otherwise $\theta$ is either nondecreasing or nonincreasing. By the symmetry of the argument we may assume that $\theta$ has a maximizer in $(-M-1, M+1)$. For given $m$ we trancate $\theta$ near a maximizer $r$; we set

$$\theta_m(t) = \min(\theta(t), \theta(r) - 1/m)$$

for all $t \in (s_-, s_+)$, where $(s_-, s_+)$ is a maximal interval where $\theta$ takes no local maximum or minimum in $(s_-, s_+)$ other than at $t = r$. Similarly, we also trancate $\theta$ near a minimizer $r$ by the value $\theta(r) + 1/m$ and define new $\theta_m$. Our $\theta_m$ has the property that $\theta_m$ is constant on some closed interval where $\theta_m$ takes either local maximum and minimum at least for sufficiently large $m$. Again by stability principle we may assume that $\theta$ is constant on some closed interval where $\theta$ takes either local maximum and minimum and number of such intervals are finite in $[-M-1, M+1]$.

6. Assume that $|u| \leq M$ and $\theta$ is the trancated function obtained at the end of Step 5. Assume that

$$\max(\theta \circ u - \varphi) = (\theta \circ u - \varphi)(\hat{x}, \hat{t})$$

for some $(\hat{x}, \hat{t}) \in Q$ and $\varphi \in C_F^2(Q)$. By the choice of $\theta$ the function $\theta$ is either nondecreasing or nonincreasing in a neighborhood $[\hat{u} - 2\delta, \hat{u} + 2\delta]$ for some small $\delta > 0$, where $\hat{u} = u^*(\hat{x}, \hat{t})$. By invariance (Theorem 4.2.1)

$$u_\delta = (u \wedge (\hat{u} + \delta)) \vee (\hat{u} - \delta)$$

is also a solution. We modify $\theta$ outside $[\hat{u} - \delta, \hat{u} + \delta]$ to get a nonincreasing continuous function $\tilde{\theta}$ that satisfies $\theta = \tilde{\theta}$ on $[\hat{u} - \delta, \hat{u} + \delta]$, since $\theta \circ u_\delta = \tilde{\theta} \circ u_\delta$.

Step 2 implies that $\theta \circ u_\delta$ is a solution. Thus in particular we have the desired inequality for $\varphi$ at $(\hat{x}, \hat{t})$ for subsolution since

$$\max(\theta \circ u_\delta - \varphi) = (\theta \circ u_\delta - \varphi)(\hat{x}, \hat{t}).$$

We thus conclude that $\theta \circ u$ is a subsolution. The proof for supersoluton is symmetric so is not repeated. □

### 4.2.3 Uniqueness

In this subsection we prove the uniqueness of generalized evolutions defined in Definition 4.1.1. As we will see in §4.3, it is possible to prove the uniqueness of generalized evolutions defined in Definition 4.1.2 and practically and logically speaking this is enough since Definition 4.1.2 is more general than Definition 4.1.1. However, the uniqueness proof for generalized evolutions in the sense of Definition 4.1.2 needs several properties of equations including global solvability of (4.1.2) other than the comparison principle and the invariance property. The proof for evolutions in Definition 4.1.2 is intuitive and instructive and it only depends on the comparison principle in a bounded domain and the invariance property (Theorem 4.1.2). So we present the proof here.

**Theorem 4.2.8.** *Assume that the level set equation (4.1.2) of surface evolution equation (4.1.1) (satisfying (f1)) has the comparison principle (BCP) in every ball. (This implicitly assumes (f2).) Then there is at most one open (resp. closed) evolution of Definition 4.1.1 for a given initial bounded open (resp. closed) set in* $\mathbf{R}^N$.

For the proof we prepare an elementary fact for level sets.

**Lemma 4.2.9.** *Let $Y$ be a closed set in $\mathbf{R}^N$. For $u_0, v_0 \in C(Y)$ assume that the set $\{u_0 > 0\}(= \{x \in Y; u_0(x) > 0\})$ is included in $\{v_0 > 0\}$. Assume that the boundary of $\{u_0 > 0\}$ is compact. (If $\{v_0 > 0\}$ includes a neighborhood of infinity, we further assume that $\liminf_{|x|\to\infty} v_0(x) > 0$ and that $u_0$ is bounded from above.) Then there exists a nondecreasing function $\theta \in C(\mathbf{R})$ such that $\theta(s) = 0$ (for $s \le 0$) and $\theta(s) > 0$ (for $s > 0$) and*

$$u_0 \le \theta \circ v_0 \quad \text{in} \quad Y.$$

*Proof.* We define $\overline{\theta} : [0,\infty) \to [0,\infty)$ by

$$\overline{\theta}(\sigma) = \sup\{u_0(y) \vee 0;\ y \in V(\sigma)\}, V(\sigma) = \{y \in Y;\ 0 \le v_0(y) \le \sigma\}.$$

We extend $\overline{\theta}$ outside $[0,\infty)$ by zero and the extended $\overline{\theta}$ is still denoted $\overline{\theta}$. By definition $\overline{\theta}$ is nondecreasing and $u_0 \le \overline{\theta} \circ v_0$ in $Y$. Our assumption on $v_0$ implies that $V(\sigma)$ is compact for small $\sigma \ge 0$. Thus $\overline{\theta} \to 0$ as $\sigma \to 0$ since otherwise it would contradict the assumption that $u_0(y) > 0$ implies $v_0(y) > 0$ for all $y \in Y$. By Lemma 2.1.9 there is a nondecreasing function $\theta \in C(\mathbf{R})$ that satisfies $\theta \ge \overline{\theta}$ in $\mathbf{R}$ and $\theta(\sigma) = \overline{\theta}(\sigma)$ for $\sigma \le 0$. This $\theta$ fulfills all desired properties since $\theta \ge \overline{\theta}$ implies that $u_0 \le \overline{\theta} \circ v_0 \le \theta \circ v_0$ in $Y$. □

*Proof of Theorem 4.2.8.* Let $u \in K_\alpha(Z)$ and $v \in K_\beta(Z)$ be solutions of (4.1.2), where $\alpha, \beta \le 0$. Assume that

$$D_0 = \{u_0 > 0\} = \{v_0 > 0\}$$

where $u_0(x) = u(x,0)$, $v_0(x) = v(x,0)$ and that $D_0$ is bounded. By Lemma 4.2.9 there exists a nondecreasing continuous function $\theta : \mathbf{R} \to \mathbf{R}$ such that $\theta(s) = 0$ for $s \le 0$, $\theta(s) > 0$ for $s > 0$ and that $u_0 \le \theta \circ v_0$ in $\mathbf{R}^N$. By the invariance (Theorem 4.2.1) the function $w := \theta \circ v \in K_0(Z)$ is a solution of (4.1.2). By definition there is a ball $B_R(0)$ such that $w \equiv 0$ and $u \equiv \alpha$ outside $B_R(0) \times [0,T)$. Since $u_0 \le \theta \circ v_0$ implies that $u \le w$ initially, the comparison principle (BCP) for $\Omega = B_R(0)$ implies that $u \le w$ on $B_R(0) \times [0,T)$. It now follows that $\{u > 0\}(= \{(x,t) \in Z; u(x,t) > 0\})$ is included in $\{w > 0\}$. Since the two sets $\{w > 0\}$ and $\{v > 0\}$ agree with each other, $\{u > 0\}$ is included in $\{v > 0\}$. If we exchange the role of $u_0$ and $v_0$, the opposite inclusion holds. Thus we see $D = \{u > 0\}$ is determined by $D_0$ and is independent of the choice of $u_0$.

The proof for closed evolutions is symmetric if we consider the complement set $\{-u > 0\}$ of $\{u \ge 0\}$. □

**Remark 4.2.10.** (Orientation-free equations) If the equation (4.1.1) is orientation-free, then $|u|$ solves (4.1.2) if $u$ solves (4.1.2) by Theorem 4.2.7. Thus we may assume that $u \ge 0$ in Definitions 4.2.1 and 4.2.1 to define a generalized interface evolution $\Gamma$ or zero level set of $u$. If $\Gamma$ is bounded, then as in the same way to prove Theorem 4.2.8 we see that the set $\{u > 0\}$ is determined by $\{u_0 > 0\}$ and is independent of the choice of $u_0$. In other words $\Gamma$ is uniquely determined by $\Gamma_0 = \Gamma(0)$. Note that we do not need to assume that $\Gamma_0$ is contained in a boundary of some bounded set in this argument. (Even if $\Gamma$ is not bounded, $\Gamma$ is determined by $\Gamma_0$ once uniqueness for arbitary open evolutions are established.)

### 4.2.4 Unbounded evolutions

We now prove the uniqueness of evolutions in the sense of Definition 4.1.2 admitting the global solvability of (4.1.2).

**Theorem 4.2.10.** *Assume that the level set equation (4.1.2) of surface evolution equation (4.1.1) satisfying (f1) has the comparison principle (CP) in* $\mathbf{R}^N$*. Assume that for given data* $g \in BUC(\mathbf{R}^N)$ *there is a solution* $w \in BUC(Z)$ *of (4.1.2) with* $w(x,0) = g(x)$*. Then there is at most one open (resp. closed) evolution of Definition 4.1.2 for a given initial open (resp. closed) set in* $\mathbf{R}^N$*.*

To show this statement we need the monotone convergence result stated below. By $a_m \uparrow a$ (as $m \to \infty$) we mean the convergence is monotone i.e. $a_m \le a_{m+1}$ and $\lim_{m\to\infty} a_m = a$, where $a_m, a \in \mathbf{R}$.

**Lemma 4.2.11** (Monotone convergence). *Assume the same hypotheses of Theorem 4.2.10 concerning (4.1.2). Assume that* $u_{0m} \uparrow u_0$ *where* $u_{0m}$*,* $u_0 \in BUC(\mathbf{R}^N)$*. Let* $u_m$ *and* $u$ *be the* $\mathcal{F}_{\mathbf{R}^N}$*-solutions of (4.1.2) with initial data* $u_{0m}$ *and* $u_0$ *respectively. Then* $u_m \uparrow u$*.*

We postpone the proof of Lemma 4.2.11 is §4.6.

*Proof of Theorem 4.2.10.* Let $u \in \mathrm{BUC}(Z)$ and $v \in \mathrm{BUC}(Z)$ be solutions of (4.1.2). Assume that $\{u_0 > 0\} = \{v_0 > 0\}$, where $u_0(x) = u(x,0)$, $v_0(x) = v(x,0)$. Our goal is to prove $\{u > 0\} = \{v > 0\}$. By the invariance (Theorem 4.2.1) $u_+ = \theta \circ u, v_+ = \theta \circ v$ where $\theta(\sigma) = \max(\sigma, 0)$ are solutions of (4.1.2) so we may assume that $u \ge 0$ and $v \ge 0$ in $Z$.

For $m = 1, 2, \cdots$ we set

$$g_m = u_0 \wedge m v_0$$

and let $w_m \in \mathrm{BUC}(Z)$ be a solution with initial data $g_m$; here the assumption of the global solvability is invoked. Clearly, $g_m(x) \uparrow u_0(x)$ for all $x \in \mathbf{R}^N$. By the monotone convergence lemma (Lemma 4.2.11) we conclude that $w_m(z) \uparrow u(z)$ for all $z \in Z$ as $m \to \infty$. Since $g_m \le m v_0$ in $\mathbf{R}^N$ and $mv$ is the solution of (4.1.2) (by the invariance) with initial data $mv_0$, the comparison principle (CP) yields that $w_m \le mv$ in $Z$.

Since $w_m(z) \uparrow u(z)$ for $z \in Z$, for a given point $z \in \{u > 0\}$ we see $w_m(z) > 0$ for some $m$. Since $w_m \le mv$, this implies $v(z) > 0$. Thus we have proved that $\{u > 0\}$ is included in $\{v > 0\}$. If we exchange the role of $u$ and $v$ the opposite inclusion holds. Thus the set $\{u > 0\}$ is determined by $D_0 = \{u_0 > 0\}$ and is independent of the choice of $u_0$.

The proof the closed evolutions is symmetric as in §4.2.3. □

**Remark 4.2.12.** For orientation-free equation thanks to Theorem 4.2.9 it is possible to prove that each $\ell$-level set of solution of (4.1.2) is determined by its initial shape independent of $\{u > \ell\}$ and $\{u < \ell\}$.

## 4.3 Existence by Perron's method

We shall prove the existence of a global-in-time solution for the Cauchy problem of (4.1.2) or (4.2.1) by Perron's method developed in §2.4. For $\alpha \in \mathbf{R}$ let $K_\alpha(\mathbf{R}^N)$ be the space of all real-valued continuous function that equals $\alpha$ outside some ball $B_R(0)$.

**Theorem 4.3.1.** *Assume that $F : \mathbf{R}^N \times [0,T] \times (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ $(0 < T < \infty)$ is continuous and geometric. Assume that (4.2.1) with any ball (BCP). Assume that there is $c \in C(0, r_0]$ with some constant $r_0 > 0$ that satisfies*

$$|F(x,t,p,\pm I)| \leq c(|p|) \quad \text{on} \quad \mathbf{R}^N \times [0,T] \times (B_{r_0}(0) \setminus \{0\}). \tag{4.3.1}$$

*Then for $\alpha \in \mathbf{R}$ and each $u_0 \in K_\alpha(\mathbf{R}^N)$ there exists a unique $\mathcal{F}_{\mathbf{R}^N}$-solution $u \in K_\alpha(Z)$ of (4.2.1) in $\mathbf{R}^N \times (0,T)$ with $u(x,0) = u_0(x)$ for all $x \in \mathbf{R}^N$, where $Z = \mathbf{R}^N \times [0,T)$.*

If $F$ is degenerate elliptic and independent of the spatial variables $x$, then (BCP) holds (Corollary 3.1.2). Moreover, it is easy to check the existence of $c$ in (4.3.1); see the proof of Lemma 3.1.3. Since the existence of initial (Lipschitz continuous) auxiliary function $u_0$ is clear (§4.1), we see that level set solutions exists globally for (4.1.1) satisfying (f1), (f2) provided that $f$ is independent of $x$.

**Corollary 4.3.2.** *Assume that $f : [0,T] \times \mathbf{E} \to \mathbf{R}$ satisfies (f1) and (f2) with $T < \infty$. Then for each bounded open (resp. closed) set $D_0$ (resp. $E_0$) there exists a unique bounded open (closed) evolution $D$ (resp. $E$) of (4.1.1) with initial data $D_0$ (resp. $E_0$). (The sets $D$ and $E$ are considered as a subset of $Z = \mathbf{R}^N \times [0,T)$.)*

We shall prove Theorem 4.3.1 by constructing suitable sub- and supersolutions and applying Theorem 2.4.9. For this purpose we prepare special, radial sub- and supersolutions.

**Lemma 4.3.3** (Radial solutions). *Assume that same hypotheses of Theorem 4.3.1 concerning $F$ (except (BCP)) but with degenerate ellipticity. Then $\mathcal{F}_{\mathbf{R}^N} \neq \emptyset$. Moreover, for $h \in \mathcal{F}_{\mathbf{R}^N}$ with $\sup_{\mathbf{R}} h' < \infty$ and each $A > 0$ there is a constant $M = M(A,T,F,h)$ such that $V^+(x-\xi,t)$ and $v^+(x-\xi,t)$ (resp. $V^-(x-\xi,t)$ and $v^-(x-\xi,t)$) are $\mathcal{F}_{\mathbf{R}^N}$-supersolutions (subsolutions) of (4.2.1) in $\mathbf{R}^N \times (0,T)$ if $V^\pm$, $v^\pm$ is of form*

$$V^\pm(x,t) = \pm(Mt + Ah(|x|)),\ v^\pm(x,t) = \pm(Mt - Ah(|x|)). \tag{4.3.2}$$

*Proof.* By Lemma 3.1.3 the set $\mathcal{F}_{\mathbf{R}^N}$ is nonempty since $F$ is geometric and (4.3.1) holds. (By definition of $\mathcal{F}_{\mathbf{R}^N}$ there is $h \in \mathcal{F}_{\mathbf{R}^N}$ such that $\sup_{\mathbf{R}} h' < \infty$ if $\mathcal{F}_{\mathbf{R}^N} \neq \emptyset$.)

We shall prove that $c$ in (4.3.1) can be extended to a continuous function on $(0,\infty)$ so that (4.3.1) holds on $\mathbf{R}^N \times [0,T] \times (\mathbf{R}^N \setminus \{0\})$ and that $c(\rho)/\rho$ is constant on $[r_0,\infty)$. We may assume $r_0 = 1$ by replacing $p$ by $p/r_0$. Since $F$ is degenerate elliptic and geometric, we see

$$\begin{aligned} F(x,t,p,I) &= |p|F(x,t,p/|p|, I/|p|) \\ &\geq |p|F(x,t,p/|p|, I) \geq -|p|c(1) \quad \text{for} \quad |p| \geq 1. \end{aligned}$$

Similarly, we have

$$\begin{aligned} F(x,t,p,I) &= F(x,t,p,O) \le |p|F(x,t,p/|p|,O) \\ &\le |p|F(x,t,-(-p/|p|),-I) \le |p|c(1), \quad p \in \mathbf{R}^N. \end{aligned}$$

These two inequalities yields

$$|F(x,t,p,I)| \le |p|c(1), \quad |p| \ge 1$$

Symmetric argument yields $|F(x,t,p,-I)| \le |p|c(1)$, $|p| \ge 1$. Thus the value of $c(\rho)$ for $\rho \ge 1$ can be defined by $\rho c(1)$ so that (4.3.1) holds on $\mathbf{R}^N \times [0,T] \times (\mathbf{R}^N \setminus \{0\})$.

As in the proof of Lemma 3.1.3 by (3.1.3) we see

$$|F(x,t,\nabla_p(h(\rho)),\ \pm\nabla_p^2(h(\rho))| \le \frac{h'(\rho)}{\rho}c(\rho) = h'(\rho)c(1)$$

for $\rho = |p| \ge 1$. If $h \in \mathcal{F}_{\mathbf{R}^N}$ and $\sup h' < \infty$, the definition of $\mathcal{F}_{\mathbf{R}^N}$ and the above inequality yields

$$B := \sup_{0 \le t \le T} \sup_{x,p \in \mathbf{R}^N} |F(x,t,\nabla_p(h(\rho)),\ \pm\nabla_p^2(h(\rho)))| < \infty.$$

We take $M \ge AB$ to observe that

$$\begin{aligned} &V_t^+(x-\xi,t) + F(x,t,\nabla_x(V^+(x-\xi,t)),\nabla_x^2(V^+(x-\xi,t))) \\ &\ge M - AB \ge 0. \end{aligned}$$

Since $V_t^+ = M > 0$ and $V^+$ is $C^2$ in $\mathbf{R}^N \times (0,T)$, we see that $V^+$ is an $\mathcal{F}_{\mathbf{R}^N}$-supersolution of (4.2.1) in $\mathbf{R}^N \times (0,T)$ by Remark 2.1.10. The proof for $v^+$, $V^-$, $v^-$ is similar so is omitted. $\Box$

**Lemma 4.3.4.** *Assume the same hypotheses of Lemma 4.3.3 concerning $F$. Let $u_0$ be a uniformly continuous function in $\mathbf{R}^N$. There is a $\mathcal{F}_{\mathbf{R}^N}$-sub- and supersolution $u^-$ and $u^+$ of (4.2.1) in $\mathbf{R}^N \times (0,T)$ with initial data $u_0$ which satisfies $u^+(x,t) \ge u_0(x) \ge u^-(x,t)$ for all $x \in \mathbf{R}^N, t \in [0,T)$,*

$$\lim_{\substack{t \to 0 \\ \delta \to 0}} \sup_{|x-y| \le \delta} |u^\pm(y,t) - u_0(x)| = 0, \tag{4.3.3}$$

*and $u^\pm$ is locally bounded in $\mathbf{R}^N \times [0,T)$ if $u_0$ is bounded. Moreover, $u^\pm$ is locally bounded in $\mathbf{R}^N \times [0,T)$.*

*Proof.* Since $u_0$ is uniformly continuous, there is a modulus $\omega$ such that

$$u_0(x) - u_0(\xi) \le \omega(|x-\xi|),\ x,\xi \in \mathbf{R}^N.$$

Since $\mathcal{F}_{\mathbf{R}^N} \ne \emptyset$, there is $h \in \mathcal{F}_{\mathbf{R}^N}$ with $\sup_{\mathbf{R}} h' < \infty$. For each $\delta > 0$ there is $A_\delta > 0$ that satisfies $\omega(s) \le \delta + A_\delta h(s)$ for $s \ge 0$. Thus,

$$u_0(x) \le u_0(\xi) + \delta + A_\delta h(|x-\xi|).$$

By Lemma 4.3.3 for some $M$ depending on $A_\delta$ the function

$$\begin{aligned} w_\delta^\xi(x,t) &= u_0(\xi) + \delta + A_\delta h(|x-\xi|) + Mt \\ &= u_0(\xi) + \delta + V^+(x-\xi, t),\ \xi \in \mathbf{R}^N \end{aligned}$$

is an $\mathcal{F}_{\mathbf{R}^N}$-supersolution of (4.2.1) in $\mathbf{R}^N \times (0,T)$. Now we set

$$u^+(x,t) = \inf\{w_\delta^\xi(x,t); \delta \in (0,1),\ \xi \in \mathbf{R}^N\} \tag{4.3.4}$$

and observe that $u^+$ is an $\mathcal{F}_{\mathbf{R}^N}$-supersolution of (4.2.1) in $\mathbf{R}^N \times (0,T)$ by closedness under infimum (Lemma 2.4.5 and Lemma 2.4.7). Since $w_\delta^\xi(x,t) \geq u_0(x)$, $x \in \mathbf{R}^N$, $t > 0$, we have $u^+(x,t) \geq u_0(x)$ for $x \in \mathbf{R}^N$, $t > 0$. In particular, $u^+$ is locally bounded from below.

We shall prove (4.3.3) and locally boundedness of $u^+$ from above. Since $u^+(x,t) \geq w_\delta^x(x,t)$ for $\delta \in (0,1)$,

$$u^+(x,t) - u_0(x) \leq w_\delta^x(x,t) - u_0(x) \leq \delta + Mt.$$

(This in particular implies that $u^+$ is locally bounded from above.) Here $M$ depends only on $\delta$; it is independent of $x$. Thus

$$\limsup_{t\to 0} \sup_{x\in\mathbf{R}^N} (u^+(x,t) - u_0(x)) = 0.$$

Since $u_0$ is uniformly continuous and $u^+(x,t) \geq u_0(x)$, this yields (4.3.3) for $u^+$. If $u_0$ is bounded, it it easy to see that $u^+$ in (4.3.4) is bounded. Construction of $u^-$ is symmetric. We use $V^-$ in (4.3.2) instead of $V^+$. The proof of the property for $u^-$ is the same as for $u^+$. $\square$

*Proof of Theorem 4.3.1.* For $u_0 \in K_\alpha(\mathbf{R}^N)$ and $h \in \mathcal{F}_{\mathbf{R}^N}$ with $\sup_{\mathbf{R}} h' < \infty$ we take $\sigma$ large such that

$$u_0(x) < \sigma - h(|x|) \text{ for } x \text{ satisfying } u_0(x) \neq \alpha.$$

Let $v^+$ be as in (4.3.2) with $A = 1$. By Lemma 4.3.3 $v^+$ is an $\mathcal{F}_{\mathbf{R}^N}$-supersolution of (4.2.1) so that $v^+ + \sigma$ is also an $\mathcal{F}_{\mathbf{R}^N}$-supersolution. By the invariance (Theorem 4.2.1)

$$w^+(x,t) = \max\{v^+(x,t) + \sigma, \alpha\}$$

is an $\mathcal{F}_{\mathbf{R}^N}$-supersolution of (4.2.1) in $\mathbf{R}^N \times (0,T)$. Evidently, $w^+$ is continuous in $\mathbf{R}^N \times [0,T)$. By the choice of $\sigma$ we see

$$u_0(x) \leq w^+(x,0) \quad \text{for all} \ \ x \in \mathbf{R}^N.$$

Since $h(s) \to \infty$ as $s \to \infty$.

$$\operatorname{spt}(w^+ - \alpha) \subset \Omega' \times [0,T) \quad \text{with} \ \ \Omega' = \operatorname{int} B_R$$

for sufficiently large ball $B_R = B_R(0)$. Similarly, one can construct an $\mathcal{F}_{\mathbf{R}^N}$-supersolution $w^-$ of (4.2.1) in $\mathbf{R}^N \times (0,T)$ that satisfies

$$u_0(x) \geq w^-(x,0) \quad \text{for all} \ \ x \in \mathbf{R}^N$$

and spt $(w^- - \alpha) \subset \Omega' \times [0,T)$ by taking $R$ larger if necessary.

Let $u^\pm$ be functions in Lemma 4.3.4 for uniformly continuous function $u_0 \in K_\alpha(\mathbf{R}^N)$. We set

$$U^+(x,t) = \min(u^+(x,t),\ w^+(x,t)),$$
$$U^-(x,t) = \max(u^-(x,t),\ w^-(x,t)).$$

Since $u^+$, $w^+$ are $\mathcal{F}_{\mathbf{R}^N}$-supersolutions of (4.2.1), so is $U^+$ by closedness of infimum for supersolutions (Lemma 2.4.5 and Lemma 2.4.7). A symmetric argument yields that $U^-$ is an $\mathcal{F}_{\mathbf{R}^N}$- supersolution (4.2.1) in $\mathbf{R}^N \times (0,T)$. By the choice of $w^\pm$

$$U^\pm(x,t) = \alpha \quad \text{on} \quad (B_{2R} \setminus B_R) \times [0,T) \quad \text{and} \quad U^- \le U^+ \quad \text{on} \quad B_{2R} \times (0,T).$$

Since $U^\pm$ is locally bounded on $Z$, we apply Perron's method (Theorem 2.4.9) in $\mathcal{O} = \Omega \times (0,T)$ with $\Omega = \mathrm{int} B_{2R}$ to get a solution $u$ of (4.2.1) in $\mathcal{O}$ that satisfies $U^- \le u \le U^+$ in $\mathcal{O}$. Since $u = \alpha$ on $(B_{2R} \setminus B_R) \times [0,T)$, we extend $u$ in $(\mathbf{R}^N \setminus B_R) \times [0,T)$ by $\alpha$ and conclude that the extended function (still denoted $u$) is a solution of (4.2.1) in $\mathbf{R}^N \times (0,T)$.

It remains to prove that $u|_{t=0} = u_0$ and $u \in K_\alpha(Z)$. We shall use (BCP). Since $U^\pm$ is continuous at $t = 0$ by Lemma 4.3.4, and $U^\pm|_{t=0} = u_0$, we see that $u$ is continuous at $t = 0$. In other words $u^* \le u_*$ on $B_{2R}$ at $t = 0$. Since $u = \alpha$ in $(B_{2R} \setminus B_R) \times [0,T)$, the relation $u^* \le u_*$ holds on $\partial B_{2R} \times [0,T)$. We now apply (BCP) to get $u^* \le u_*$ in $B_{2R} \times [0,T)$. Thus $u^* = u_*$ which implies that $u$ is continuous in $B_{2R} \times [0,T)$. Since $u = \alpha$ in $(\mathbf{R}^N \setminus B_R) \times [0,T)$, we conclude $u \in K_\alpha(Z)$. Since $U^- \le u \le U^+$ and $U^+|_{t=0} = u_0$, the initial value of $u$ must be $u_0$. (The uniqueness of solutions follows from (BCP).) The proof is now complete. □

There is another version of existence results with initial data not necessarily in $K_\alpha(\mathbf{R}^N)$ but in a larger space $\mathrm{UC}_*(\mathbf{R}^N) = \{u_0 \in C(\mathbf{R}^N);\ (u_0)_M$ is uniformly continuous in $\mathbf{R}^N$ i.e., $(u_0)_M \in \mathrm{UC}(\mathbf{R}^N)$ for every $M > 0\}$, where $(u_0)_M = (u_0 \wedge M) \vee (-M)$. Such a function $u_0$ may not be a constant at space infinity. The space $\mathrm{UC}_*(Z)$ is defined by replacing $\mathbf{R}^N$ by $Z$.

**Theorem 4.3.5.** *Assume that $F : \mathbf{R}^N \times [0,T] \times (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ $(0 < T < \infty)$ is continuous and geometric. Assume that (4.2.1) satisfies (CP) with $\Omega = \mathbf{R}^N$. Assume that there is $c \in C(0,r_0)$ with some $r_0 > 0$ satisfying (4.3.1). Then for $u_0 \in UC_*(\mathbf{R}^N)$ there exists a unique $\mathcal{F}_{\mathbf{R}^N}$-solution $u \in UC_*(Z)$ of (4.2.1) in $\mathbf{R}^N \times (0,T)$ with $u|_{t=0} = u_0$. If $u_0 \in BUC(\mathbf{R}^N)$, then $u \in BUC(Z)$.*

*Proof.* For $u_0 \in \mathrm{UC}_*(\mathbf{R}^N)$ we take $u^\pm$ as in Lemma 4.3.4. We apply Theorem 2.4.9 for (4.2.1) with $\mathcal{O} = \mathbf{R}^N \times (0,T)$ to get an $\mathcal{F}_{\mathbf{R}^N}$-solution $u$ of (4.2.1) in $\mathcal{O}$ satisfying $u^- \le u \le u^+$ in $\mathcal{O}$. Since $u^\pm$ satisfies (4.3.3), we see $u|_{t=0} = u_0$. If $u_0$ is bounded, then $u^\pm$ is bounded in $(Z)$ by Lemma 4.3.4 to get $u$ is bounded in $Z$.

It remains to prove that $u \in \mathrm{UC}_*(Z)$. Here we use the comparison principle (CP). By invariance (Theorem 4.2.1) $u_M$ is an $\mathcal{F}_{\mathbf{R}^N}$-sulution of (4.2.1) so we may assume that $u$ is bounded. By (4.3.3) we see

$$\lim_{\substack{t\to 0\\ s\to 0\\ \delta\to 0}} \sup_{|x-y|\le\delta} (u^*(x,t) - u_*(y,s)) = 0.$$

We now apply (CP) to get

$$\lim_{\delta\to 0}\sup\{u^*(x,t)-u_*(y,s));\ |x-y|\le\delta,\ |t-s|\le\delta,\ 0\le t,\ s<T\}\le 0.$$

This yields $u\in \mathrm{BUC}(Z)$. The uniqueness of solutions follows from (CP). □

From Theorem 4.3.5 with Remark 4.3.7 it easily follows an existence result.

**Corollary 4.3.6.** *Assume that $f:[0,T]\times\mathbf{E}\to\mathbf{R}$ satisfies (f1) and (f2) with $T<\infty$. Then for each open (resp. closed) set $D_0$ (resp. $E_0$) there exists a unique open (closed) evolution $D$ (resp. $E$) of (4.1.1) with initial data $D_0$ (resp. $E_0$).*

**Remark 4.3.7**(Condition (4.3.1)). If we examine the proof, we realize that the condition (4.3.1) is actually unnecessary. Indeed, by Lemma 3.1.3 there always exists $c_R\in C(0,r_0)$ with some $r_0>0$ satisfying

$$|F(x,t,p,\pm I)|\le C(|p|)\quad\text{for}\quad x\in B_R, t\in[0,T], p\in B_{r_0}\setminus\{0\} \tag{4.3.5}$$

for every $R>0$. This is enough to carry out the proof although Lemma 4.3.3 and Lemma 4.3.4 should be modified. So Corollary 4.3.2 can be extended to spatially inhomogeneous equation like (3.6.15) provided it satisfies (BCP). There is no need to assume (4.3.1).

By the way the condition (4.3.1) is interpreted as control of speed of the growth of a unit ball moved by (4.1.1).

**Remark 4.3.8**(Exact solutions). Exact solutions in §1.7 are actually an interface evolution. For example, in §1.7.1 (level set approach) we pointed out that

$$u(x,t)=-(t+|x|^2/2(N-1))(+\text{const.})$$

is a solution of the level set mean curvature flow equation (1.6.5). As anticipated, it is easy to check that this function $u$ solves (1.6.5) in the viscosity sense. So we conclude that the shrinking sphere defined by (1.7.2)–(1.7.3) is actually an interface evolution. In Chapter 5 we shall show that smooth solution $\{\Gamma_t\}$ is an interface evolution so the notion of our generalized solution is consistent with classical solutions.

## 4.4 Existence by approximation

Perhaps it is more standard than Perron's method to construct solutions as a limit of solutions of approximate equations. For example, when we are asked to solve first order equation

$$u_t+F(\nabla u)=0,\quad u|_{t=0}=u_0 \tag{4.4.1}$$

globally in time, we often consider a regularized problem

$$u_t^\varepsilon+F(\nabla u^\varepsilon)=\varepsilon\Delta u^\varepsilon,\quad u^\varepsilon|_{t=0}=u_0 \tag{4.4.2}$$

for $\varepsilon > 0$ and construct the solution $u$ of the original problem (4.4.2) as a limit of solution $u^\varepsilon$ of (4.4.2) as $\varepsilon \to 0$. This method is called a vanishing viscosity method since the parameter $\varepsilon$ resembles the viscosity in equations of fluid dynamics. The viscosity solution is obtained as the limt of such a problem. The name of 'viscosity solution' stems from this type of a vanishing viscosity method. For more background the reader is referred to a book of P.-L. Lions (1982).

For spatially homogeneous level set equations as pointed out by Y.-G. Chen, Y. Giga and S. Goto (1989), we approximate the equation by a strictly (or uniformly) parabolic equation and observe that a solution of level set equation is obtained as local uniform limit of solutions of the approximate problem. In the above paper by Y.-G. Chen, Y. Giga and S. Goto, the way of approximation was not mentioned. For the level set mean curvature flow equation L. C. Evans and J. Spruck (1991) solved a strictly parabolic equation of the form

$$u_t - \sqrt{|\nabla u|^2 + \varepsilon^2}\ \mathrm{div}\left(\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}}\right) = 0 \tag{4.4.3}$$

or equivalently

$$u_t - \Delta u + \sum_{1 \le i,j \le n} \frac{u_{x_i} u_{x_j}}{\varepsilon^2 + |\nabla u|^2} = 0 \tag{4.4.4}$$

with a suitable initial data. Then they obtained a solution of the level set equation (1.6.7) with initial data $u_0 \in K_\alpha(\mathbf{R}^N)$ as a local uniform limit of solution of (4.4.4).

This method consists of two parts:

(i) **Solvability of approximate problem.** One has to solve (4.4.3).

(ii) **Limiting procedure.** One has to prove that desired solution of (1.6.7) is obtained as a limit of approximate solution $u^\varepsilon$ of (4.4.4).

For the first part (i) we need the theory of parabolic equations; see e.g. a book of O. A. Ladyzhenskaya, V. Solonnikov and N. Ural'ceva (1968) or A. Lunardi (1995). So do not touch this problem here. We give a precise statement for part (ii).

**Theorem 4.4.1.** *Assume that $u^\varepsilon \in C(\mathbf{R}^N \times (0,\infty))$ is a unique smooth solution of (4.4.3) with initial data $u_0 \in BUC(\mathbf{R}^N)$. Then there exists $u \in C(\mathbf{R}^N \times [0,\infty))$ such that $u \in BUC(\mathbf{R}^N \times [0,T))$ for every $T > 0$ and that $u$ is obtained as a local uniform limit of $u^\varepsilon$ in $\mathbf{R}^N \times [0,\infty)$. Moreover, $u$ is a viscosity solution of (1.6.7).*

The statement is actually a special version of our convergence result (Theorem 4.6.3). It is easy to check assumptions of Theorem 4.6.3 are fulfilled.

However, there is a more classical way to prove such a statement when $u_0$ is more regular. Indeed, if $u_0 \in C^2(\mathbf{R}^N) \cap K_\alpha(\mathbf{R}^N)$ by the maximum principle, one get a uniform bound for $|\nabla u^\varepsilon|$, $|u^\varepsilon|$, $|u^\varepsilon_t|$ in $\mathbf{R}^N \times (0,\infty)$. By Ascoli-Arzèla's compactness theorem, $u^{\varepsilon_j} \to u$ locally uniformly in $\mathbf{R}^N \times [0,\infty)$ with some function $u$ by taking a subsequence $\varepsilon_j \to 0$. By a stability principle with local uniform convergence (§2.1.2) we conclude that $u$ is a viscosity solution of (1.6.7) with initial data $u_0$. By the uniqueness of initial value problem for (1.6.7), the limit $u$ is independent of the choice of subsequence so we obtain a full convergence. The reader is referred to the work of L. C. Evans and J. Spruck (1991) for details of this type of argument.

## 4.5 Various properties of evolutions

We study various general properties of level set solutions.

**Assumptions on well-posedness (W).** Assume that $f : \mathbf{R}^N \times [0,T] \times \mathbf{E} \to \mathbf{R}$ satisfies (f1) and (f2). Assume that (4.1.2) with (4.1.3) satisfies (CP) with $\Omega = \mathbf{R}^N$ and (4.3.1) with some $c$. (The last assumption is fulfilled if for example, $f$ is independent of the space variables.)

By Theorem 4.2.10 and Theorem 4.3.1 if we assume (W), then for each open set $D_0$ in $\mathbf{R}^N$ there is a unique open evolution $D(\subset \mathbf{R}^N \times [0,T))$ if (4.1.1) with $D(0) = D_0$, where $D(t)$ denotes the cross-section of $D$ at time $t$, i.e.,

$$D(t) = \{x \in \mathbf{R}^N; (x,t) \in D\}.$$

By translation in time under (W) there is a unique open evolution $\tilde{D} \subset \mathbf{R}^N \times [s,T)$ of (4.1.1) with $\tilde{D}(s) = D_0$ where $s$ is a given positive number. Let $U(t,s)$ denote the mapping: $D_0 \mapsto \tilde{D}(t)$. Similary, let $M(t,s)$ denote the mapping which maps a closed set $E_0$ to $\tilde{E}(t)$ where $\tilde{E}$ is a closed level set solution with $\tilde{E}(s) = E_0$. By unique existence of evolution we have a semigroup property.

**Theorem 4.5.1** (Semigroup property). *Assume (W). Then*

$$U(t,\tau) \circ U(\tau,s) = U(t,s),\ M(t,\tau) \circ M(\tau,s) = M(t,s)$$

*for all $s, \tau, t$ satisfying $0 \le s \le \tau \le t < T$.*

The operators $M$ and $U$ have order preserving properties which follows from (CP).

**Theorem 4.5.2** (Order preserving property). *Assume (W). Let $D_0$ and $D_0'$ be two open sets in $\mathbf{R}^N$ and let $E_0$. $E_0'$ be two closed sets in $\mathbf{R}^N$.*

*(i) $D_0 \subset D_0'$ implies $U(t,s)D_0 \subset U(t,s)D_0'$;*

*(ii) $E_0 \subset E_0'$ implies $M(t,s)E_0 \subset M(t,s)E_0'$;*

*(iii) $D_0 \subset E_0$ implies $U(t,s)D_0 \subset M(t,s)E_0$;*

*(iv) if $E_0 \subset D_0$ and dist $(E_0, \partial D_0) > 0$, then $M(t,s)E_0 \subset U(t,s)D_0$
for all $t, s \in [0,T]$ satisfying $t \ge s$.*

*Proof.* (i) We take

$$u_0(x) = \max(\mathrm{sd}(x,\partial D_0), -1),$$
$$v_0(x) = \max(\mathrm{sd}(x,\partial D_0'), -1)$$

as initial auxiliary function for $D_0$ and $D'_0$ so that $u_0 \le v_0$. Since the solutions $u$ and $v$ starting from $u_0$ and $v_0$ at $t = s$ are bounded, uniformly continuous on $\mathbf{R}^N \times [s, T')$ for any $T' < T$, $u$ and $v$ satisfies the assumptions of (CP). Thus $u \le v$ on $\mathbf{R}^N \times [s, T)$ which implies $U(t,s)D_0 \subset U(t,s)D'_0$.

(ii), (iii) The proof is similar to (i).

(iv) We may assume $s = 0$. We take

$$u_0(x) = \begin{cases} \mathrm{dist}(x, \partial D_0)/\{\mathrm{dist}(x, \partial D_0) + \mathrm{dist}(x, E_0)\}, & x \in D_0 \\ -(\mathrm{dist}(x, \partial D_0), 1) & x \notin D_0 \end{cases}$$

so that $D_0 = \{u_0 > 0\}$ and $E_0 = \{u_0 \ge 1\}$. By the assumption $\mathrm{dist}(E_0, \partial D_0) > 0$, $u_0$ is a Lipschitz continuous function. Let $u$ be the solution of (4.1.2) with initial data $u$. Since $u - 1$ also solves (4.1.2), $M(t,0)E_0 = \{x \in \mathbf{R}^N; u(x,t) \ge 1\}$. Since $U(t,0)D'_0 = \{u(\cdot,t) \ge 0\}$, it is clear that $M(t,0)E_0 \subset U(t,0)D_0$.

We shall study convergence properties of level set solutions. Below we use following notation. Let $\{A_j\}_{j\ge1}$ be a sequence of sets and $B$ and a set. By $A_j \uparrow B$ we mean that $A_j \subset A_{j+1}$ and $\cup_{j\ge1} A_j = B$. Similarly, by $A_j \downarrow B$ we mean that $A_j \supset A_{j+1}$ and $\cap_{j\ge1} A_j = B$.

**Lemma 4.5.3** (Approximation). *Let $D$ be an open evolution. There exist two sequences of open evolution $\{D'_k\}_{k\ge1}$ and closed evolution $\{E'_k\}_{k\ge1}$ such that*

$$D'_k \uparrow D \quad \text{and} \quad D'_k \subset E'_k \subset D'_{k+1}. \tag{4.5.1}$$

*Proof.* Let $u$ be a solution of (4.1.2) such that $\{u > 0\} = D$. Then we define

$$D'_k = \left\{(x,t); u(x,t) > \frac{1}{2^k}\right\}, \ E'_k = \left\{(x,t); u(x,t) \ge \frac{1}{2^k}\right\}.$$

These sets clearly fulfill (4.5.1). Since $u - 2^{-k}$ is a solution of (4.1.2), $D'_k$ and $E'_k$ are open and closed evolutions. □

**Theorem 4.5.4** (Monotone convergence). *(i) Let $D$ and $\{D_j\}_{j\ge1}$ be open evolutions with initial data $D_0$ and $D_{j0}$. If $D_{j0} \uparrow D_0$, then $D_j \uparrow D$.*

*(ii) Let $E$ and $\{E_j\}_{j\ge1}$ be closed evolution with initial data $E_0$ and $E_{j0}$. If $E_{j0} \downarrow E_0$, then $E_j \downarrow E$.*

*Proof.* (i) The proof is easy if $D_0$ and is bounded. Let $D'_k$ and $E'_k$ be the approximating open and closed evolutions for $D$ which were constructed in Lemma 4.5.3. If $D_0$ is bounded, then $E'_k(0)$ is compact, so there is a $j_k \ge 1$ such that $E'_k(0) \subset D_{j_k 0}$. By comparison (Theorem 4.5.2 (iv)), we have $E'_k \subset D_{j_k}$. The sequence $E'_k$ was constructed so that $D'_k \subset E'_k \uparrow D$. Thus $D_{j_k} \uparrow D$, which proves (i) when $D_0$ is bounded. In general, we use Lemma 4.2.11 with

$$\begin{aligned} u_{0j}(x) &= (\mathrm{sd}(x, D_{j0}) \wedge 1) \vee (-1) \\ u_0(x) &= (\mathrm{sd}(x, D_0) \wedge 1) \vee (-1) \end{aligned}$$

and observe that the solution $u_j$ with initial data $u_{0j}$ satisfies $u_j \uparrow u$, where $u$ is the solution with initial data $u_0$. Since $D_j = \{u_j > 0\}$ and $D = \{u > 0\}$, $u_j \uparrow u$ implies $D_j \uparrow D$.

(ii) The proof is similar so is omitted. □

We shall study continuity of $D(t)$ and $E(t)$ for an open and closed evolution as a function of time $t$. To formulate continuity of open-set valued functions we define the $\varepsilon$-core $\mathcal{C}_\varepsilon(W)$ of an open set $W \subset \mathbf{R}^N$ to be

$$\mathcal{C}_\varepsilon(W) = \{x \in \mathbf{R}^N;\ \mathrm{dist}(x, W^c) \geq \varepsilon\},$$

where $W^c = \mathbf{R}^N \setminus W$. This concept is dual to that of an $\varepsilon$-neighborhood $\mathcal{N}_\varepsilon(Y)$ of a closed set $Y$ in $\mathbf{R}^N$ in the sense that $\mathcal{C}_\varepsilon(W) = (\mathcal{N}_\varepsilon(W^c))^c$.

**Theorem 4.5.5** (Continuity in time). *Let $D$ and $E$ be open and closed evolutions.*

*(ia) $D(t)$ is a lower semicontinuous function of $t \in [0,T)$, in the sense that for any $t_0 \geq 0$, and sequence $x_n \in (D(t_n))^c$ with $x_n \to x_0$, $t_n \to t_0$ the limit $x_0 \in (D(t_0))^c$. If $D(0)$ is bounded so that $\mathcal{C}_\varepsilon(D(t_0))$ is compact, this implies that for any $t_0 \geq 0$, $\varepsilon > 0$ there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies $D(t) \supset \mathcal{C}_\varepsilon(D(t_0))$.*

*(ib) $E(t)$ is a upper semicontinuous function of $t \in [0,T)$, in the sense that for any $t_0 \geq 0$ and sequence $x_n \in E(t_n)$ with $x_0 \to x_0$, $t_n \to t_0$ the limit $x_0 \in E(t_0)$. If $E(0)$ is bounded so that $\mathcal{N}_\varepsilon(E(t_0))$ is compact, this implies that for any $t_0 \geq 0$, $\varepsilon > 0$, there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies $E(t) \subset \mathcal{N}_\varepsilon(E(t_0))$.*

*(iia) $D(t)$ is left upper semicontinuous in $t$ in the sense that for any $t_0 \in (0,T)$, $x_0 \in (D(t_0))^c$ there is a sequence $x_n \to x_0$ and $t_n \downarrow t_0$ with $x_n \in (D(t_n))^c$. Moreover, for any $t_0 \in (0,T)$, $\varepsilon > 0$ there exists a $\delta > 0$ such that $t_0 - \delta < t < t_0$ implies $\mathcal{C}_\varepsilon(D(t)) \subset D(t_0)$.*

*(iib) $E(t)$ is left lower semicontinuous in $t$ in the sense that for any $t_0 \in (0,T)$, $x_0 \in E(t_0)$ there is a sequence $x_n \to x_0$ and $t_n \downarrow t_0$ with $x_n \in (E(t_0))$. Moreover, for any $t_0 \in (0,T)$, $\varepsilon > 0$ there exists a $\delta > 0$ such that $t_0 - \delta < t < t_0$ implies $\mathcal{N}_\varepsilon(E(t)) \supset E(t_0)$.*

For any two closed sets $C_1, C_2 \subset \mathbf{R}^N$ the Hausdorff distance between them is defined by

$$d_H(C_1, C_2) = \inf\{\varepsilon > 0;\ C_1 \subset \mathcal{N}_\varepsilon(C_2) \quad \text{and} \quad C_2 \subset \mathcal{N}_\varepsilon(C_1)\}.$$

For open subsets $U_1, U_2 \subset \mathbf{R}^N$ we define

$$d_H^*(U_1, U_2) = \inf\{\varepsilon > 0;\ \mathcal{C}_\varepsilon(U_1) \subset U_2 \quad \text{and} \quad \mathcal{C}_\varepsilon(U_2) \subset U_1\}.$$

By definition $d_H^*(U_1, U_2) = d_H(U_1^c, U_2^c)$ so $d_H^*$ is a metric of open sets in $\mathbf{R}^N$. Theorem 4.5.5 implies that $D(t)$ and $E(t)$ are left continuous functions with respect to the Hausdorff metric $d_H^*$ and $d_H$ provided that $D_0$ and $E_0$ are bounded.

*Proof.* (ia), (ib) follows from the fact that $D, E$ are open and closed sets in $\mathbf{R}^N \times [0,T)$. Since (iia) and (iib) are dual to each other, we only prove (iia). Let $V^-$ be a radial subsolution defined in (4.3.2) with some $M$ and $A$. For $\varepsilon > 0$ and $t_0 > 0$ we define $\delta = Ah(\varepsilon)/M$. For given $x_0 \in \mathcal{C}_\varepsilon(D(t))$ we set

$$W_\tau = \{(y,s);\ V^-(x - x_0, s - \tau) > 0\}.$$

For $t \in (t_0 - \delta, t_0)$ we take $\tau > t_0$ such that $\tau - t < \delta$. By the choice of $\delta$ we see $W_\tau(t) \subset D(t)$. Since $V^-$ is a subsolution of (4.1.2), by Remark 4.5.14 (ii) on Theorem 4.5.2, we have $W_\tau(t_0) \subset D(t_0)$ so that $x_0 \in D(t_0)$. $\square$

**Fattening and regularity.** If initial closed set $E_0 = \overline{D_0}$ and $D_0$ is open, then one would expect that $E = \overline{D}$, where $E$ (resp. $D$) is a closed evolution with initial data $D_0$ (resp. $D_0$). However, unfortunately this is not true in general even if the equation (4.1.1) is the curve shortening equation $V = k$ and the initial data is compact. Thus we call an open evolution $D$ is *regular* if $E = \overline{D}$. We say that an interface evolution $\Gamma = E \setminus D$ is fattens if $\Gamma(t)$ has an interior point for some $t > 0$ although $\Gamma(0) = E_0 \setminus D_0$ has no interior. If $\Gamma(t_0)$ has an interior at some $t_0$, then $\Gamma$ has an interior in $\mathbf{R}^N \times [0,T)$. Indeed, for $x_0 \in \operatorname{int}(\Gamma(t_0))$ we take $\tau$ slightly larger than $t_0$ so that $W_\tau(t_0) \subset \Gamma(t_0)$ and $D \cap W_\tau \cap \{t \geq t_0\} = \emptyset$. Then $W_\tau(t) \subset \Gamma(t)$ for $t(> t_0)$ close to $t_0$. Thus $\Gamma$ has an interior in $\mathbf{R}^N \times [0,T)$. If $D$ is regular, then the interface evolution $\Gamma$ with initial data $\Gamma(0) = \partial D_0$ does not fatten. The converse is not clear by our observations given so far.

Applying monotone convergence theorem to $D_0$ with $\Gamma_0 = \partial D_0$ and $E_0 = \overline{D_0}$, we see that evolutions $D$ and $E$ are obtained as a limit of evolutions approximating $D_0$ from the interior and $E_0$ from the exterior, respectively. If $\Gamma(t)$ has an interior point, these two "solutions" do not agree. In particular, "continuity of solutions with respect to initial data", which usually is expected for differential equations generally, is not valid in this case. The situation $\overline{D} \neq E$ can be also interpreted as a loss of uniqueness (§5.2.1).

For the curve-shortening equation $V = k$, if the initial data $\Gamma_0 = \partial D_0$ has the shape of figure "8" (embedded in $\mathbf{R}^2$), then $\Gamma$ fattens instanteneously. This is first observed by Evans and Spruck (1991). It is intuitively clear that a solution approximating from the interior does not agree with one approximated from the exterior. In §5.2.1 we give a rigorous proof of fattening when $\Gamma_0$ consists of two lines crossing at one point with right angle.

We give several criteria of nonfattening or regularity. Note that $\overline{D}(t)$ does not represent the closure of $D(t)$ in $\mathbf{R}^N$. It is cross-section of the closure of $D$ in $\mathbf{R}^N \times [0,T)$.

**Theorem 4.5.6** (Monotone motion). *Assume $f$ in (4.1.1) is independent of $t$. Assume (W). Assume that $D_0$ is a bounded open set. If $M(h,0)\overline{D_0} \subset D_0$ for sufficiently small $h$, then $D$ is regular, where $D$ is an open evolution with initial data $D_0$.*

*Proof.* Since the equation is autonomous, $M(t,s) = M(t-s,0)$ $t \geq s \geq 0$ so we write $M_h = M(h,0)$ so that $M(t,s) = M(t-s)$. We use similar convention for $U(t,s)$.

By order-preserving (Theorem 4.5.2) we see

$$M_h(\overline{D_0}) \subset D_0 \quad \text{implies} \quad M_t M_h(\overline{D_0}) \subset U_t(D_0).$$

The semigroup property implies $M_t M_h(\overline{D_0}) = E(t+h)$, when $E$ is a closed evolution with initial data $\overline{D_0}$. Thus $E(t+h) \subset D(t)$ for $t > 0$. Iteration of this argument then shows that $E(t) \subset D(s)$ for all $t > s \geq 0$.

We next note that $\overline{D}(t') \downarrow \overline{D}(t)$ are $t' \uparrow t$ since $\overline{D}(t')$ is decreasing in time and $D(t)$ is left continuous as sets by Theorem 4.5.5. Thus $\cap_{0<t'<t}\overline{D}(t') = \overline{D}(t)$.

Since $E(t) \subset D(s)$, we have

$$E(t) \subset \bigcap_{0<t'<t} D(t) \subset \bigcap_{0<t'<t} \overline{D}(t') = \overline{D}(t).$$

Hence $E(t) \subset \overline{D}(t)$ for all $t > 0$ and $E \subset \overline{D}$. Since the converse inclusion $\overline{D} \subset E$ is true by assumption, this completes the proof. □

**Theorem 4.5.7.** *Assume that $f$ is independent of $t$. Assume $(W)$. Let $D_0$ be smoothly bounded domain such that $f(x, \mathbf{n}, \nabla\mathbf{n}) < 0$ on $\partial D_0$. Denote the open and closed evolutions with initial data $D_0$ and $\overline{D_0}$ by $D$ and $E$, respectively. Then $E(t) \subset D(s)$ for all $t > s \geq 0$ and $\overline{D} = E$.*

*Proof.* By Theorem 4.5.6 it suffices to prove that $M_h\overline{D_0} \subset D_0$ for small $h > 0$. We set $\sigma = \inf_{\partial D_0}(-f(x, \mathbf{n}, \nabla\mathbf{n})) > 0$ and observe that

$$\psi(x,t) = \mathrm{sd}(x, \partial D_0) - \sigma t$$

is a supersolution of (4.1.2). By comparison (CP),

$$M_h\overline{D_0} \subset \{x \in \mathbf{R}^N; \psi(x,h) \geq 0\},$$

which yields $M_h\overline{D_0} \subset D_0$ for $h > 0$.

**Corollary 4.5.8.** *Assume that $f$ does not depend on $t$. Assume that $f$ is independent of $\nabla\mathbf{n}$ (so that (4.1.1) is of the first order.) Assume (W). If $f$ does not change sign, then $D$ is regular for any bounded open initial data $D_0$.*

In general one get several criterion based on invariance of equations. It can be written in fomally as follows. If the equation is invariant under a semigroup of actions $\{S_h\}$. If $S_h\overline{D_0} \subset D_0$ for small $h > 0$, then $D$ is regular. The condition $S_h\overline{D_0} \subset D_0$ is fulfilled if the generator of $S_h$ is negative on $\partial D_0$.

We give another example of such situation. For a set $G$ in $\mathbf{R}^N \times [0, \infty)$, let $\mathcal{D}^m_\lambda(G)$ denote

$$\mathcal{D}^m_\lambda(G) = \{(\lambda_x, \lambda^{m+1}t);\ (x,t) \in G\},$$

where $m \in \mathbf{R}$, $\lambda > 0$. Similarly, for a set $G_0$ in $\mathbf{R}^N$ let $\mathcal{D}^m_{\lambda,0}(G_0)$ denote

$$\mathcal{D}_{\lambda,0}(G_0) = \{\lambda x, x \in G_0\}.$$

The condition $\mathcal{D}_{1-h,0}(\overline{G_0}) \subset G_0$ for small $h > 0$ is a kind of starsharpness.

**Theorem 4.5.9** (Dilation invariant equations). *Assume that $f$ in (4.1.1) is independent of $x$ and $t$. Assume that $f(p, Q_p(X))$ is positively homogeneous of degree $m \in \mathbf{R}$ in $Q_p(X)$, i.e.,*

$$f(p, \mu Q_p(X)) = \mu^m f(p, Q_p(X)), \quad \mu > 0$$

*for all $(p, Q_p(X)) \in \mathbf{E}$. Assume (W). Let $D_0$ be an bounded openset in $\mathbf{R}^N$. If $\mathcal{D}_{1-h,0}(\overline{D_0} \subset D_0$ for sufficiently small $h > 0$, then $D$ is regular, where $D$ is an open evolution with initial data $D_0$.*

*Proof.* Let $F$ be the operator defined by (4.1.3). By homogeneity of $f$ we have

$$F(\mu p, \mu^2 X) = \mu^{m+1} F(p, X), \ \mu > 0.$$

Let $E$ be the closed evolution with initial data $\overline{D_0}$. Then by homogeneity of $F$ the set $\mathcal{D}^m_{1-h}(E)$ is a closed evolution with initial data $\mathcal{D}_{1-h,0}(\overline{D_0})$. Sending $h$ to zero yields $E \subset \overline{D}$. Since $\overline{D} \subset E$ is trivial, $D$ is regular. □

**Corollary 4.5.10.** *Assume the same hypotheses of Theorem 4.5.9 concerning $f$. Let $D_0$ be a smoothly bounded domain such that $\langle x, \mathbf{n}\rangle > 0$ on $\partial D_0$. Let $D$ be the open evolution with initial data $D_0$. Then $D$ is regular.*

*Proof.* Since $\langle -x, \mathbf{n}\rangle$ is the generator of group $\{\mathcal{D}_{e^{-q},0}\}$ at $q = 0$, it is clear that $\mathcal{D}_{1-h,0}(\overline{D_0}) \subset D_0$ for small $h > 0$. Thus $D$ is regular by Theorem 4.5.9. □

We give another criterion when $f$ is rotationally symmetric in the sense that

$$f({}^tRx, t, Rp, {}^tRQ_p(X)R) = f(x, t, p, Q_p(X)).$$

for any rotation matrix $R$. As well known a one parameter group $\{R(\lambda)\}$ of rotation is generated by a skew-symmetric matrix so it is of form $\{e^{\lambda A}\}_{\lambda \in \mathbf{R}}$ with skew-symmetric matrix $A$. Similarly to Theorem 4.5.9 we see that if $f$ is rotationally symmetric $e^{\lambda A}(\overline{D_0}) \subset D_0$ for small $\lambda > 0$ implies the regularity of $D$. The condition $e^{\lambda A}(\overline{D_0}) \subset D_0$ is fulfilled if $\langle Ax \cdot \mathbf{n}\rangle < 0$ on $\partial D_0$ if $D_0$ in smooth. Below we give a criterion of regularity when $f$ is independent of $t, x$, dilation invariant and rotationally symmetric.

**Corollary 4.5.11.** *Assume the same hypotheses of Theorem 4.5.9 concerning $f$. Assume moreover, $f$ is rotationally symmetric. Assume that $D_0$ is a smoothly bounded domain. Assume that there is a nonnegative constant $c_1, c_2$ and a skew-symmetric matrix $A$ such that*

$$c_1 f(\mathbf{n}, \nabla \mathbf{n}) + \langle Ax, \mathbf{n}\rangle - c_2\langle x, \mathbf{n}\rangle < 0 \ \text{ on } \ \partial D_0.$$

*Then an open evolution $D$ with initial data $D_0$ is regular.*

*Proof.* The condition on $\partial D_0$ guarantees that

$$(M_{c_1 h}\mathcal{D}_{1-c_2 h,0} d^{hA})(\overline{D_0}) \subset D_0$$

for small $h > 0$. By invariance of the equation $E_h = \mathcal{D}^m_{1-c_2h} e^{hA}(E)$ is a closed evolution if $E$ is the closed evolution with initial data $\overline{D_0}$. By comparison $E_h(t+h) \subset D(t)$, $t \geq 0$ for small $h > 0$. In other words

$$E_h(t) \subset D(t-h) \;\text{ for }\; t - h \geq 0$$

for small $h$. Since $\overline{D}$ is left continuous and $E_h(t) \to E(t)$ as $h \to 0$, we see $E \subset \overline{D}$. Thus $D$ is regular.

Finally, we give another type of regularity criterion for orientation free equations when an open evolutions is a disjoint sum of regular open evolutions.

**Lemma 4.5.12.** *Assume (W) and (4.1.2) is orientation free. Let $U$ be a bounded open set in $\mathbf{R}^N$ that may be written as the union of a finite number of disjoint open sets $U^1, \ldots, U^k$. Denote the open evolutions with initial data $U$ and $U^i$ by $D$ and $D^i$, respectively ($1 \leq i \leq k$). Let $E^i$ be the closed evolution with initial data $\overline{U}^i$ and assume that $E^i = \overline{D_i}$.*

*Assume that there is a sequence of open covers $\{U^i_\alpha, \ldots, U^k_\alpha\}_{\alpha \geq 1}$ of $U$ which satisfies:*

*(i) $U^i_\alpha \supset U^i_{\alpha+1}$ and $\overline{U}^i = \cap_{\alpha \geq 1} \overline{U}^i_\alpha$ for $i = 1, \ldots, k$;*

*(ii) the sets $U^1_\alpha, \ldots, U^k_\alpha$ are pairarise disjoint for $\alpha \geq 1$.*

*Finally, let $E$ be a closed evolution for which a double sequence $\{t_{\alpha,\ell}\}_{\alpha,\ell \geq 1}$ exists such that $t_{\alpha,\ell} \downarrow 0$ as $\ell \to \infty$ and*

$$E(t_{\alpha,\ell}) \subset U^1_\alpha \cup \cdots U^k_\alpha.$$

*Then $\overline{D} = E$.*

*Proof.* Let $D^i_\alpha$ be an open evolution with initial data $U^i_\alpha$ and let $D_\alpha$ be the open evolution with initial data $\cup_{i=1}^k U^i_\alpha$. Since the equation is orientation-free, we apply the separation lemma (stated after this proof as Lemma 4.5.13) to obtain that $D_\alpha = \cup_{i=1}^k U^i_\alpha$.

By order preserving property (Theorem 4.5.2 (iv)) we see

$$E(t) \subset D_\alpha(t - t_{\alpha,\ell}) \subset \cup_{i=1}^k D_\alpha(t - t_{\alpha\ell}) \subset \cup_{i=1}^k \overline{D}_\alpha(t - t_{\alpha,\ell})$$

for $t > t_{\alpha,\ell}$. By continuity letting $\ell \to \infty$ yields

$$E(t) \subset \cup_{i=1}^k \overline{D}^i_\alpha(t) \;\text{ for }\; t > 0.$$

On the other hand $\overline{D}^i_\alpha \subset E^i_\alpha$, where $E^i_\alpha$ is the closed evolution with initial data $\overline{U}^i_\alpha$. Since $\overline{U}^i_\alpha \downarrow U^i$, by monotone convergence (Theorem 4.5.4) we have $E^i_\alpha \downarrow E^i$, where $E^i$ is the closed evolution with initial data $\overline{U}^i$.

Since $E(t) \subset \cup_{i=1}^k \overline{D}^i_\alpha(t)$, letting $\alpha \to \infty$ to get $E(t) \subset \cup_{i=1}^k E^i(t)$ for all $t > 0$. Since we have assumed $E^i = \overline{D}^i$, we get

$$E(t) \subset \cup_{i=1}^k \overline{D}^i(t) \;\text{ for }\; t > 0.$$

By order preserving property $D^i \subset D$ for all $i = 1, \ldots, k$ so we now obtain $E(t) \subset \overline{D}(t)$ for all $t \geq 0$ i.e., $E \subset \overline{D}$. Thus $\overline{D} = E$ since $D \subset E$ is trivial. □

**Lemma 4.5.13** (Separation). *Assume (W) and (4.1.1) is orientation-free. Let $D^i$ be an open evolution ($i = 1, 2$).*

*(i) If $D_1(0)$ and $D_2(0)$ are disjoint so are $D_1$ and $D_2$.*

*(ii) Let $D$ be the open evolution with initial data $D_1(0) \cup D_2(0)$. If $D_1(0)$ and $D_2(0)$ are disjoint then $D = D_1 \cup D_2$.*

*Proof.* (i) Let $d(\geq 0)$ be the distance between $D_1(0)$ and $D_2(0)$ and let $d_j(x)$ denote the distance from $x \in \mathbf{R}^N$ to $\partial(D_j(0))(j = 1, 2)$. Define

$$u_0(x) = \begin{cases} d_1(x) + d/2 & \text{for} \quad x \in D_1(0) \\ -d_2(x) - d/2 & \text{for} \quad x \in D_2(0) \\ (d_2(x) \wedge \frac{d}{2}) - (d_1(x) \wedge \frac{d}{2}) & \text{otherwise.} \end{cases}$$

We claim that $u_0$ is Lipschitz with constant 1. In the open sets $D_j(0)(j = 1, 2)$ this is clear since the distance to a set is always Lipschitz with constant 1. Outside of the closure of the $D_i(0)$ we always have $d_1(x) + d_2(x) \geq d$, by the triangle inequality. So the open sets

$$V_i = \{x \in \mathbf{R}^N \setminus (\overline{D_1(0)} \cup \overline{D_2(0)}); d_i(x) < d/2\}$$

are disjoint. On these sets $u_0(x) = \pm(d_i(x) - d/2)$ so that $u_0$ is also Lipschitz with constant 1 on these sets. Finally $u_0(x)$ vanishes outside $D_1(0) \cup D_2(0) \cup V_1 \cup V_2$. Collecting these separate Lipschitz estimates we find that $u_0$ is Lipschitz with constant 1 in $\mathbf{R}^N$.

Since $u_0$ is Lipschitz, there is a unique solution $u$ of (4.1.2) which is uniformly continuous in $\mathbf{R}^N \times [0, T']$ for every $T' < T$. Since $\pm u - c(c \in \mathbf{R})$ is also solution of (4.1.2) by invariance and orientation free property,

$$\begin{aligned} D_1 &= \{(x, t) \in [0, T') \times \mathbf{R}^N;\ u(x, t) > d/2\}, \\ D_2 &= \{(x, \tau) \in [0, T') \times \mathbf{R}^N;\ u(x, t) > -d/2\} \end{aligned}$$

are open evolutions with initial data $D_1(0)$ and $D_2(0)$. Clearly $D_1$ and $D_2$ are disjoint. Since $u$ is uniformly continuous, the distance of $D_1(t), D_2(t)$ is uniformly positive for all $t \in [0, T']$ if $d > 0$. This property will be used in the proof of (ii).

(ii) There are sequences of open sets $D_{i0}^j \uparrow D_i(0)(i = 1, 2)$ such that the distance from $D_{i0}^j$ to $\partial(D_i(0))$ is positive. Let $u_i^j$ be the solution of (4.1.2) such that $u_i^j|_{t=0} = u_{0i}^j$ is uniformly continuous and that $u_{0i}^j(x) > 0$ for $x \in D_{i0}^j$ and $u_{0i}^j(x) = 0$ for $x \notin D_{i0}^j$. The open evolution $D_i^j$ with initial data $D_{i0}^j$ is given

$$D_i^j = \{(x, t); u_i^j(x, t) > 0\}$$

and $u_i^j \geq 0$ everywhere. Since the distance of $D_{10}^j$ and $D_{20}^j$ is positive, the distance of $D_1^j, D_2^j$ are positive in $\mathbf{R}^N \times [0, T']$. Thus $u_1^j \vee u_2^j$ is also solution of (4.1.2) so the open evolution $D^j$ with initial data $D_{10}^j \cup D_{20}^j$ equals $D_1^j \cup D_2^j$. By monotone convergence we see $D^j \uparrow D$ and $D_i^j \uparrow D_i$. Thus $D = D_1 \cup D_2$. □

**Remark 4.5.14.** (i) If we consider only bounded evolutions $D$ and $E$ in the sense that $D(0)$ and $E(0)$ are bounded, we may replace (CP) by (BCP) in the assumption (W). By Remark 4.3.7 the requirement of (4.3.1) is unnecessary since (4.3.5) is always fulfilled.

(ii) The order preserving property of Theorem 4.5.2 can be strengthened by introducing super level sets of sub and supersolutions instead of solutions. We just give below a typical result corresponding to Theorem 4.5.2 (i) when $D$ and $D'$ are bounded. Assume that a bounded open set $D$ and a bounded closed set in $Z = \mathbf{R}^N \times [0,T)$ are of the form

$$\begin{aligned} D &= \{(x,t) \in Z;\ u(x,t) > 0\} \\ D' &= \{(x,t) \in Z;\ v(x,t) \geq 0\} \end{aligned} \tag{4.5.2}$$

for some subsolution $u$ and supersolution $v$ of (4.1.2) and (4.1.3) such that $u$ and $v$ are *continuous* in $Z$ and $u|_{t=0}$, $v|_{t=0} \in K_\alpha(\mathbf{R}^N)$ for some $\alpha < 0$. If (BCP) holds, then $D \subset D'$ holds provided that $D(0) \subset D'(0)$. This is easy to prove by (BCP) since we may assume that $u \leq v$ at $t = 0$ by the invariance Theorem 4.2.1 and Lemma 4.2.9.

We are tempting to introduce a notion of subsolution for sets $D$ if it is of the form (4.5.2). However, this notion is not the same as the notion of set-theoretic subsolution defined in Definition 5.1.1. It agrees with the level set subsolution in Definition 5.2.2 as proved in Proposition 5.2.3. The difference of these two notions stems from fattening phenomena, as explained in Chapter 5.

## 4.6 Convergence properties for level set equations

We shall study whether solutions of approximate equation

$$V = f_\varepsilon(z, \mathbf{n}, \nabla \mathbf{n}) \tag{4.6.1}$$

converges to the solution of

$$V = f(z, \mathbf{n}, \nabla \mathbf{n}) \tag{4.6.2}$$

when $f_\varepsilon$ tends to $f$ as $\varepsilon \to 0$ in certain sense. We first discuss convergence of solutions of level set equations.

**Theorem 4.6.1** (Convergence). *Assume that $F_\varepsilon$, $F : \mathbf{R}^N \times [0,T] \times (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N \to R$ ($0 < T < \infty$, $0 < \varepsilon < 1$) are continuous and geometric. Assume that there is $c \in C(0, r_0)$ that satisfies (4.3.1). Assume that there is a constant $C$ such that*

$$|F_\varepsilon(x,t,p,\pm I)| \leq C|F(x,t,p,\pm I)| \quad \text{for} \quad (x,t,p) \in \mathbf{R}^N \times [0,T] \times (B_{r_1}(0) \setminus \{0\}) \tag{4.6.3}$$

*for some $r_1 \in (0, r_0)$. Assume that $F_\varepsilon \to F$ locally uniformly in $\mathbf{R}^N \times [0,T] \times (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N$. Assume that $u_{0\varepsilon} \in BUC(\mathbf{R}^N)$ converges to $u_0 \in BUC(\mathbf{R}^N)$ uniformly in $\mathbf{R}^N$ as $\varepsilon \to 0$. Let $u_\varepsilon \in BUC(\mathbf{R}^N \times [0,T))$ be the $\mathcal{F}_{\mathbf{R}^N}$-solution of*

$$u_t + F_\varepsilon(z, \nabla u, \nabla^2 u) = 0 \tag{4.6.4}$$

*with initial data $u_{0\varepsilon}$. Then $\lim_{\varepsilon\to 0} u_\varepsilon = u \in BUC(\mathbf{R}^N \times [0,T))$ exists and $u$ is the $\mathcal{F}_{\mathbf{R}^N}(F)$-solution of the limit equation*

$$u_t + F(z, \nabla u, \nabla^2 u) = 0 \tag{4.6.5}$$

*with initial data $u_0$ provided that (4.6.4), (4.6.5) satisfies (CP) with $\Omega = \mathbf{R}^N$. Moreover, the convergence is locally uniform in $\mathbf{R}^N \times [0,T)$.*

*Proof.* By (4.6.3) and (3.1.2) we see that $\mathcal{F}_{\mathbf{R}^N}(F_\varepsilon) \supset \mathcal{F}_{\mathbf{R}^N}(F)$ and (2.2.4) is fulfilled. (The existence of $c$ guarantees that $\mathcal{F}_{\mathbf{R}^N}(F) = \emptyset$ by Lemma 3.1.3.) We now apply the stability results (Theorem 2.2.1) and observe that

$$\overline{u} = \limsup{}^* u_\varepsilon \quad \text{and} \quad \underline{u} = \liminf{}_* u_\varepsilon$$

are $\mathcal{F}_{\mathbf{R}^N}(F)$-sub- and supersolutions of (4.2.1) provided that $\overline{u} < \infty$ and $\underline{u} > \infty$.

We shall prove that

$$\overline{u} < \infty \quad \text{and} \quad \varlimsup_{t\downarrow 0} \sup_{\xi} (\overline{u}(\xi,t) - u_0(\xi)) \le 0.$$

Since $u_0^\varepsilon$ converges to $u_0$ uniformly in $\mathbf{R}^N$ and $u_0^\varepsilon$, $u_0 \in \mathrm{BUC}(\mathbf{R}^N)$, there is a modulus of continuity independent of $\varepsilon \in (0,1)$ such that

$$u_0^\varepsilon(x) - u_0^\varepsilon(\xi) \le \omega(|x-\xi|), \ \varepsilon \in (0,1), \ x \in \mathbf{R}^N.$$

By the assumptions (4.3.1) and (4.6.3) there is $h \in \mathcal{F}_{\mathbf{R}^N}(F) \subset \mathcal{F}_{\mathbf{R}^N}(F_\varepsilon)$ such that $\sup h' < \infty$. Moreover,

$$B_\varepsilon := \sup_{0\le t<T} \sup_{x,p\in\mathbf{R}^N} |F(x,t,\nabla_p(h(\rho)), \pm\nabla_p^2 h(\rho))| < \infty, (\rho = |\rho|) \tag{4.6.6}$$

is bounded on $\varepsilon \in (0,1)$, i.e.

$$B = \sup_{0<\varepsilon<1} B_\varepsilon < \infty. \tag{4.6.7}$$

For each $\delta > 0$ there is $A_\delta > 0$ such that $\omega(s) \le \delta + A_\delta s^2$, $s \ge 0$. Thus

$$u_0^\varepsilon(x) \le u_0^\varepsilon(\xi) + \delta + A_\delta h(|x-\xi|).$$

Since $B$ is independent of $\varepsilon$, by Lemma 4.3.3 and its proof the function $V^+(x) = A_\delta h(|x|) + M_\delta t$ is an $\mathcal{F}_{\mathbf{R}^N}(F_\varepsilon)$-supersolution of (4.6.4) (independent of $\varepsilon$). So the function

$$w_\delta^\varepsilon(x,t) = u_0^\varepsilon(\xi) + \delta + V^+(x-\xi,t)$$

is also an $\mathcal{F}_{\mathbf{R}^N}(F_\varepsilon)$-supersolution of (4.6.4). Since $u^\varepsilon$ and $w^\varepsilon\delta$ is uniformly continuous on $R^N \times [0,T)$, by (CP) we have $u^\varepsilon \le w^\varepsilon\delta$. In particular, $\overline{u} < \infty$ on $\mathbf{R}^N \times [0,T)$. Since $u^\varepsilon \le w^\varepsilon\delta$, we see

$$\overline{u}(\xi,t) \le \lim_{\varepsilon\to 0} u_0^\varepsilon(\xi) + \delta + Mt.$$

Thus

$$\overline{\lim_{t\downarrow 0}} \sup_{\xi\in\mathbf{R}^N} (\overline{u}(\xi,t) - u_0(\xi)) \le \delta$$

since $\delta > 0$ is arbitrary, we get

$$\overline{\lim_{t\downarrow 0}} \sup_{\xi\in\mathbf{R}^N} (\overline{u}(\xi,t) - u_0(\xi)) \le 0.$$

A symmetric argument yields $\underline{u} > -\infty$ and

$$\overline{\lim_{t\downarrow 0}} \sup_{\xi\in\mathbf{R}^N} (u_0(\xi) - \underline{u}(\xi,t)) \le 0.$$

In partiuclar $\overline{u}|_{t=0} = \underline{u}|_{t=0} = u_0$. Since $u_0$ is uniformly continuous, these two inequalities yields

$$\sup_{0\le t,s\le\varepsilon} \sup_{|\xi-\eta|\le\varepsilon} (\overline{u}(\xi,t) - \underline{u}(\xi,s)) \to 0 \ \text{ as } \ \eta \to 0.$$

We are now in position to apply (CP) to get $\overline{u} \le \underline{u}$. The converse inequality is trivial so $\overline{u} = \underline{u}$. By definition of $\limsup^*$, $\limsup_*$ this implies that $u^\varepsilon$ converges to $u$ locally uniformly in $\mathbf{R}^N \times [0,T)$ as $\varepsilon \to 0$. Moreover $u|_{t=0} = u_0$ and $u$ solves (4.6.5) since $u = \underline{u} = \overline{u}$ is both $\mathcal{F}_{\mathbf{R}^N}(F)$-sub- and supersolution. □

**Remark 4.6.2.** (i) We may replace (4.6.3) by $\mathcal{F}_{\mathbf{R}^N}(F) \subset \mathcal{F}_{\mathbf{R}^N}(F_\varepsilon)$ in Theorem 4.6.1. If $f_\varepsilon$ and $f$ are independent of $x,t$ in (4.6.1), (4.6.2) and

$$|f_\varepsilon(p, Q_p(X))| \le A(1 + |Q_p(X)|), \quad \varepsilon \in (0,1)$$

with constant $A > 0$, then $\mathcal{F}(F_{f_\varepsilon}) = \mathcal{F}(F_f)$ $(= \{f \in C^2(\mathbf{R}^N);\ f(0) = f'(0) = f''(0) = 0\ f'' > 0\})$. So in this situation we have convergence of solutions of level set equations.

(ii) If we fully use (3.1.5) for $\overline{u}$ and $\underline{u}$, we can also prove that $u^\varepsilon \to u$ uniformly in $\mathbf{R}^N \times [0,T']$ for $T' < T$.

(iii) If $u_0^\varepsilon \in K_{\alpha_\varepsilon}(\mathbf{R}^N)$ and spt $(u_0^\varepsilon - \alpha_\varepsilon) \subset$ int $B_R$ for some $R$ independent of $\varepsilon \in (0,1)$, then there is $R^1 \ge R$ (independent of $\varepsilon$) that satisfies

$$\mathrm{spt}(u^\varepsilon - \alpha_\varepsilon) \subset \ \mathrm{int} B_{R'} \times [0,T).$$

(In particular, the convergence $u^\varepsilon \to u$ is uniform in $R^N \times [0,T']$, $T' < T$.) This uniform estimate of the support $u^\varepsilon - \alpha_\varepsilon$ can be proved by constructing a suitable super- and subsolution as in the proof of Theorem 4.3.1. Instead of $w^+$ we use here

$$w_\varepsilon^+(x,t) = \max\{v^+(x,t) + \sigma, \alpha_\varepsilon\},$$

where $\sigma$ is chosen such that $u_0^\varepsilon(x) \le \sigma - h(|x|)$ for $x$ satisfying $u_0^\varepsilon(x) \ne \alpha_\varepsilon$. We estimate $w_t^- \le u^t \le w_\varepsilon^+$ by constructing $w_\varepsilon^-$ similarly and observe that $u^\varepsilon - \alpha_\varepsilon = \alpha_\varepsilon$ outside $\Omega' \times [0,T)$, where $\Omega'$ is a ball.

(iv) Geometricity of $F_\varepsilon$ and $F$ is not necessary if there exists $V^\pm$ (independent of $\varepsilon$) in the proof of Theorem 4.6.1 and if assumptions in Theorem 2.2.1 are fulfilled. We give another version of convergence results whose proof parallels that of Theorem 4.6.1.

**Theorem 4.6.3** (Convergence without geometricity). *Assume that $F$, $F_\varepsilon : \mathbf{R}^N \times [0,T] \times (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N \to \mathbf{R}$ $(0 < T < \infty, 0 < \varepsilon < 1)$ are continuous. Assume that $\mathcal{F}_{\mathbf{R}^N}(F) \subset \mathcal{F}_{\mathbf{R}^N}(F_\varepsilon)$ and fulfills (2.2.4). Assume that $B$ in (4.6.6), (4.6.7) is finite for some $h \in \mathcal{F}_{\mathbf{R}^N}(F)$. Assume that $F_\varepsilon \to F$ locally uniformly in $\mathbf{R}^N \times [0,T] \times (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N$. Assume that $u_{0\varepsilon} \in BUC(\mathbf{R}^N)$ converges to $u_0 \in BUC(\mathbf{R}^N)$ uniformly in $\mathbf{R}^N$ as $\varepsilon \to 0$. Let $u_\varepsilon \in BUC(Z)$ be the $\mathcal{F}_{\mathbf{R}^N}(F_\varepsilon)$-solution of (4.6.4) with initial data $u_{0\varepsilon}$. Then $\lim_{\varepsilon\to 0} u^\varepsilon = u \in BUC(Z)$ exists and $u$ is the $\mathcal{F}_{\mathbf{R}^N}(F)$-solution of (4.6.5) with initial data $u_0$ provided that (4.6.4), (4.6.5) satisfies (CP) with $\Omega = \mathbf{R}^N$. Moreover the convergence is locally uniformly in $Z = \mathbf{R}^N \times [0,T)$.*

We take this opportunity to prove Lemma 4.2.11. We suppress the word $\mathcal{F}_{\mathbf{R}^N}$ in the proof.

*Proof of Lemma 4.2.11.* Since $u_{0m} \in \mathrm{BUC}(\mathbf{R}^N)$ and the solution $u_m$ with initial data $u_{0m}$ also belongs to $\mathrm{BUC}(Z)$, we see (CP) is applicable to conclude $u_m \le u_{m+1} (\le u)$ in $Z = \mathbf{R}^N \times [0,T)$. We set

$$v(z) = \sup_{m\ge 1} u_m(z) (= \lim_{m\to\infty} u_m(z)), \quad z \in Z.$$

Since $u_m$ is continuous, $v$ is lower semicontinuous in $Z$. Thus

$$\begin{aligned} v(z) \ &= \lim_{r\downarrow 0} \inf\{v(\zeta); \zeta \in Z,\ |\zeta - z| \le r\} \\ &\ge \lim_{r\downarrow 0} \inf\{u_k(\zeta); \zeta \in Z,\ |\zeta - z| \le r,\ k \ge \frac{1}{r}\} \\ &\ge \lim \inf\{u_m(\zeta); \zeta \in Z, |\zeta - z| \le r\}, \quad m = 1, 2, \dots . \end{aligned}$$

The last quatity equals $u_m(z)$ so sending $m \to \infty$ yields

$$v(z) \ge \lim_{r\downarrow 0} \inf\{u_k(\zeta); \zeta \in Z; |\zeta - z| \le r,\ k \ge \frac{1}{r}\} \ge v(z).$$

Thus we conclude

$$v(z) = (\liminf_{m\to\infty}{}_* u_m)(z).$$

By the stability (Theorem 2.2.1) $v$ is a supersolution of (4.2.1).

We shall prove that

$$\lim_{r\downarrow 0} \sup\{(u(x,t) - v(y,s);\ |x-y| \le r,\ t \vee s \le r\} \le 0.$$

Let $h \in \mathcal{F}_{\mathbf{R}^N}$ with $\sup h' < \infty$. As in the proof of Lemma 4.3.4 for each $\delta > 0$ there are positive constants $A_\delta$ and $M_\delta = M_{A_\delta}$ such that

$$w_\delta^\xi(x,t) = u_0(\xi) - \delta - A_\delta h(|x-\xi|) - Mt$$

is a subsolution of (4.2.1) in $\mathbf{R}^N \times (0,T)$ and

$$u_0(x) - \delta \ge w_\delta^\xi(x,0).$$

By Dini's theorem $u_{m0} \to u_0$ locally uniformly in $\mathbf{R}^N$. Since $\{u_{m0}\}$ is bounded for below and $w_\delta^\xi$ there is $\ell(\delta, \xi)$ such that

$$u_{m0} \geq w_\delta^\xi(\cdot, 0) \ \text{ in } \ \mathbf{R}^N \ \text{ for } \ m \geq \ell.$$

Since both $w_\delta^\xi$ and $u_m$ is uniformly continuous in $\mathbf{R}^N \times [0, T']$, $T' < T$, we see, by (CP), that $u_m \geq w_\delta^\xi$ for all $m \geq \ell$. This yields $v \geq w_\delta^\xi$ in $\mathbf{R}^N \times [0, T)$ for $\delta \in (0, 1)$, $\xi \in \mathbf{R}^N$. In particular, $v(y, s) \geq w^y(y, s)$. Thus

$$u(x, t) - v(y, s) \leq u(x, t) - w^y(y, s) \leq u(x, t) - u_0(y) + \delta + M_\delta s.$$

Since $u$ is uniformly continuous in $Z$, it is clear that

$$\lim_{r \downarrow 0} \sup\{u(x, t) - u_0(y); |x - y| \leq r, \ t \leq r\} \leq 0.$$

We thus obtain that

$$\lim_{r \downarrow 0} \sup\{u(x, t) - v(y, s); |x - y| \leq r, \ t \vee s \leq r\} \leq \delta.$$

Since $\delta > 0$ is arbitrary, we conclude

$$\lim_{r \downarrow 0} \sup\{u(x, t) - v(y, s); |x - y| \leq r, \ t \vee s \leq r\} \leq 0$$

and apply (CP) to get $u \leq v$ in $Z$. Since $v \leq u$, this implies $u \equiv v$. Thus we have proved that $u_m \uparrow u$. $\square$

The convergence result (Theorem 4.6.1) yields the convergence of evolutions of (4.6.1) to (4.6.2) provided that fattening does not occur.

**Theorem 4.6.4** (Convergence of evolutions). *Assume that (4.6.1) and (4.6.2) fulfills (W). Assume that $f_\varepsilon$ convergence to $f$ as $\varepsilon \to 0$ locally uniformly on $\mathbf{R}^N \times [0, T] \times \mathbf{E}$. Assume that $F_{f_\varepsilon}$ and $F_f$ satisfies $\mathcal{F}_{\mathbf{R}^N}(F_f) \subset \mathcal{F}_{\mathbf{R}^N}(F_{f_\varepsilon})$ for $\varepsilon \in (0, 1)$. Let $E_0$ and $D_0^\varepsilon$ be compact sets in $\mathbf{R}^N$. Let $E^\varepsilon$ and $E$ be closed evolution of (4.6.1) and (4.6.2) with initial data $E_0^\varepsilon$ and $E_0$, respectively. Assume that $d_H(E_0^\varepsilon, E_0) \to 0$ as $\varepsilon \to 0$. Assume that int $E$ is regular. Then*

*(i) $d'_H(E^\varepsilon, E) \to 0$ as $\varepsilon \to 0$, where $d'_H$ denotes the Hausdorff distance in $\mathbf{R}^N \times [0, T']$, $T' < T$.*

*(ii) Assume that $E$ is strongly regular in the sense that $E(t) = \overline{D(t)}$ for all $t \in [0, T']$ where $D$ is the open evolution of (4.6.1) with initial data int $E_0$. Then $d_H(E^\varepsilon(t), E(t)) \to 0$ as $\varepsilon \to 0$ and the convergence is uniform in $t \in [0, T']$.*

For the proof we need a lemma on level sets.

**Lemma 4.6.5.** *Let $Z$ and $Y$ be compact metric spaces. Assume that $f_\lambda^\varepsilon \in C(Z)$ converges to $f_\lambda \in C(Z)$ as $\varepsilon \to 0$ uniformly in $\lambda \in Y$, where $\varepsilon \in (0, 1)$. For a given*

*$\ell \in \mathbf{R}$ assume that $\{z \in Z; f_\lambda(z) \geq \ell\} := \{f_\lambda \geq \ell\} = \overline{\{f_\lambda > \ell\}}$. Assume that the sets $\{f_\lambda \geq \ell\}$ and $\{f_\lambda^\varepsilon \geq \ell\}$ are compact in $Z$ for all $\lambda \in Y$. Assume that for small $\varepsilon > 0$ the set $\{f_\lambda^\varepsilon \geq \ell\}$ is continuous in $\lambda \in Y$ with respect to the Hausdorff metric $d_H$ in $Z$. Then $d_H(\{f_\lambda^\varepsilon \geq \ell\}, \{f_\lambda \geq \ell\}) \to 0$ as $\varepsilon \to 0$ uniformly in $\lambda \in Y$.*

*Proof.* We first note that for each $\eta > 0$ there is $\varepsilon_0 \in (0,1)$ such that $\{f_\lambda^\varepsilon \geq \ell\} \subset \{f_\lambda \geq \ell\}_\eta$ for all $\varepsilon \in (0, \varepsilon_0)$, $\lambda \in Y$, where $A_\eta = \{z \in Z; \mathrm{dist}(z, A) \leq \eta\}$ for a subset $A$ of $Z$. If not, for some $\eta > 0$ there is a sequence $\varepsilon_j \to 0$, $\{\lambda_j\} \subset Y$, $\{z_j\} \subset Z$ that satisfies dist $(z_j, \{f_{\lambda_j} \geq \ell\}) > \eta$ and $z_j \in \{f_{\lambda_j}^{\varepsilon_j} \geq \ell\}$. We may assume that $\lambda_j \to \lambda_0$, $z_j \to z_0$ as $j \to \infty$ by taking a subsequence. Since $f_\lambda^\varepsilon$ converges to $f_\lambda$ uniformly in $Z$ and for $\lambda \in Y$, we have $f_{\lambda_j}^{\varepsilon_j}(z_j) \to f_{\lambda_0}(z_0)$. Since $f_{\lambda_j}^{\varepsilon_j}(z_j) \geq \ell$, this implies $f_{\lambda_0}(z_0) \geq \ell$. This is absurd since dist $(z_0, \{f_{\lambda_0} \geq \ell\}) = \lim_{j\to\infty}$ dist $(z_j, \{f_{\lambda_j} \geq \ell\}) \geq \eta$ by continuity of $\{f_\lambda \geq \ell\}$ in $\lambda \in Y$. Thus, $\{f_\lambda^\varepsilon \geq \ell\} \subset \{f_\lambda \geq \ell\}_\eta$ for small $\varepsilon > 0$ uniformly in $\lambda \in Y$.

It remains to prove that $\{f_\lambda \geq \ell\} \subset \{f_\lambda^\varepsilon \geq \ell\}_\eta$ for sufficiently small $\varepsilon$ (uniformly in $\lambda \in Y$). If not, there is $r > 0$ and $\lambda_j, z_j$ with $z_j \in \{f_\lambda \geq \ell\}$ such that $\{f_{\lambda_j}^{\varepsilon_j} \geq \ell\} \cap B_r(z_j) = \emptyset$ for some sequence $\varepsilon_j \to 0$. By continuity of $\{f_\lambda \geq \ell\}$ in $\lambda$ we may assume that $\lambda_j \to \lambda_0$ and $z_j \to z_0 \in \{f_{\lambda_0} \geq \ell\}$ so that $\{f_{\lambda_j}^{\varepsilon_j} \geq \ell\} \cap B_{r/2}(z_0) = \emptyset$. Since $f_\lambda^\varepsilon \to f_\lambda$ uniformly in $Z$ and for $\lambda \in Y$, this implies that $f_{\lambda_0} \leq \ell$ on $B_{r/2}(z_0)$. This contradicts $\overline{\{f_{\lambda_0} > \ell\}} = \{f_{\lambda_0} \geq \ell\}$. □

**Lemma 4.6.6** (strongly regular evolution). *Assume (W). Assume that $E_0 = E(0)$ is compact. If $E$ is strongly regular, then $t \mapsto E(t)$ is continuous as a set-valued function on $[0, T)$.*

*Proof.* Since $t \mapsto E(t)$ is left continuous and upper semicontinuous, it suffices to prove that $E(t)$ is right lower semicontinuous in $t$.

Assume that $E(t)$ is not right continuous at some point $t_0 \in [0, T)$. Then there is a point $x_0 \in E(t_0)$ and a ball $B_r(x_0)$ such that $E(t_j) \cap B_r(x_0) = \emptyset$ for some $t_j \downarrow t_0$. Since $\overline{D(t_0)} = E(t_0)$ there is a ball $B \subset D(t_0) \cap E(t_0)$. Comparing a special subsolution we conclude that an open evolution starting with int $B$ at $t = t_0$ contains a center of $B$ for $[t_0, t_0 + \delta)$ for some $\delta > 0$. Thus by comparison we see $D(t_j) \cap B_r(x_0) \neq \emptyset$ which contradicts $E(t_j) \cap B_r(x_0) = \emptyset$. Therefore $E(t)$ is right continuous on $[0, T)$. □

*Proof of Theorem 4.6.4.* For $E_0^\varepsilon$ we set

$$u_0^\varepsilon(x) = (\mathrm{sd}(x, \partial D_0^\varepsilon) \wedge 1) \vee (-1).$$

Since $d_H(E_0^\varepsilon, E_0) \to 0$ as $\varepsilon \to 0$, we see that $u_0^\varepsilon$ converges to

$$u_0(x) = (\mathrm{sd}(x, \partial E) \wedge 1) \vee (-1).$$

Let $u^\varepsilon$ be the solution of (4.6.4) with $u_0^\varepsilon|_{t=0} = u_0^\varepsilon$ and let $u$ be the solution of (4.6.5) with $u|_{t=0} = u_0$. Our assumption $f_\varepsilon \to f$ guarantees the convergence $F_{f_\varepsilon} \to F_f$ of Theorem 4.6.1. By the convergence result (Theorem 4.6.1 and Remark 4.6.2 (i)) $u^\varepsilon$ converges to $u$ uniformly in $\mathbf{R}^n \times [0, T']$ for every $T' < \infty$. Since $E^\varepsilon = \{u^\varepsilon \geq 0\}$, $E = \{u \geq 0\}$, we apply Lemma 4.6.5 with $Z = B_R \times [0, T']$ so that $\{u^\varepsilon \geq 0\} \subset Z$ and $Y = \emptyset$ and $f^\varepsilon = u^\varepsilon$ to get (i). To show (ii) it suffices to take $Z = B_R$, $Y = [0, T']$ and $f_\lambda^\varepsilon = u^\varepsilon(,\lambda)$, since the strong regularity implies the continuity of $E(t)$ by Lemma 4.6.6. □

**Remark 4.6.7.** (i) As an application of Theorem 4.6.4 (i) we have the convergence of the extinction time $T^\varepsilon$ of $E^\varepsilon$ so that of $E$ as $\varepsilon = 0$. Here we define

$$T_\varepsilon = \sup\{t; E_\varepsilon(t) \neq \emptyset\},\ T_0 = \sup\{t; E(t) \neq \emptyset\}.$$

(ii) It is possible to replace $E^\varepsilon$ by the interface ecolution $\Gamma^\varepsilon = E^\varepsilon \setminus D^\varepsilon$ in Theorem 4.6.4 with trivial modifications.

(iii) In Theorem 4.6.4 (ii) the strong regularity assumption on $[0, T']$ is actually stronger than $E = \overline{D}$. For example if $E(t)$ is a shrinking disk solving the mean curvature flow equation $V = H$ (see §1.7 and Remark 4.3.6) then at the extinction time $T$ the set $E(T)$ is a singleton but $D(T) = \emptyset$. This $E(T)$ is not right continuous at $t = T$, so continuity in $t$ does not follow from $E = \overline{D}$. (By definition it is clear that strong regularity implies $E = \overline{D}$.)

## 4.7 Notes and comments

We take this opportunity to review that a scope of equations to which our theory applies.

**Unique existence of generalized evolutions.** Corollary 4.3.2 and Corollary 4.3.6 covers all spatially homogeneous surface evolution equations satisfying (f1) and (f2). This class of equation is the same as in §3.1.3 so it includes examples mentioned there.

Corollary 4.3.2 can be extended to (4.1.1) when $f$ depends on $x$ provided that (BCP) holds. Several class of equations satisfying (BCP) has bee studied in §3.6.

For Corollary 4.3.6 we further needs the uniform control (4.3.1) as well as (CP) when the equation is spatially inhomogeneous.

Corollary 4.3.2 can be also extended to (4.1.1) in a domain (not necessarily bounded) with prescribed contact angle on the boundary provided that (BCPB) holds. Several examples are provided in §3.7.

Although we do not mention the one corresponding to (CP) for boundary value problems, Corollary 4.3.6 also can be extended to some boundary value problems. However, it is not explicit what kind of equation satisfies such a comparison principle in the literature.

Since the assumption (W) is essentially fulfilled (see Remark 4.5.14(i)) under situation mentioned above, results in §4.5 applies to these problems.

**Orientation-free equations.** We list examples of orientation-free equations. The equation (1.5.2) is orientation-free if $\beta(p) = \beta(-p)$, $\gamma(p) = \gamma(-p)$ for $p \in S^{N-1}$ and $c \equiv 0$. In particular the mean curvature flow equation (1.5.4) is orientation-free. Although the Gaussian curvarure flow equation (1.5.9) as well as (1.5.12) is orientation-free, our modified equations (1.6.22), (1.6.23) are not orientation-free. The equation (1.5.13) is orientation-free if $h(\sigma, p) = -h(-\sigma, p)$, $h(\sigma, p) = h(\sigma, -p)$, $\gamma(p) = \gamma(-p)$ for $p \in S^{N-1}$, $\sigma \in \mathbf{R}$. The equation (1.5.14) is not orientation-free although the equation $V = |k|^{\alpha-1}k$ for $\alpha > 0$ is orientation-free.

**Convergence.** We give a typical example to which Theorem 4.6.4 applies. We consider (1.5.2) parametrized by $\varepsilon$:

$$\beta_\varepsilon(\mathbf{n})V = -a_\varepsilon \mathrm{div}_{\Gamma_t}(\xi_\varepsilon(\mathbf{n})) - c_\varepsilon. \tag{4.7.1}$$

Here $c_\varepsilon$ is assumed to be independent of $x$ and $t$ for simplicity. Assume that $c_\varepsilon \to c$ in $\mathbf{R}$ and $a_\varepsilon \to a \in [0,\infty)$ as $\varepsilon \to 0$. If $\beta_\varepsilon \to \beta$, $\partial_p^\alpha \gamma_\varepsilon \to \partial_p^\alpha \gamma$ for $|\alpha| \le 2$ as $\varepsilon \to 0$ and the convergence is uniform on $S^{N-1}$, then $f_\varepsilon$ converges to $f$ locally uniformly in $\mathbf{E}$ if (4.7.1) is written of the form $V = f_\varepsilon(\mathbf{n}, \nabla\mathbf{n})$ and

$$\beta(\mathbf{n})V = -a\ \mathrm{div}_{\Gamma_t}\xi(\mathbf{n}) - c \tag{4.7.2}$$

is written of the form $V = f(\mathbf{n}, \nabla\mathbf{n})$. In this example as in Remark 4.6.2 it is easy to see that $\mathcal{F}(F_{f_\varepsilon}) = \mathcal{F}(F_f)$. So if both (4.7.1) and (4.7.2) fulfill (W), Theorem 4.6.4 is applicable.

We first point out that there are two review articles on mathematical analysis for the level set method – the article by Y. Giga (1995a) and the article by L. C. Evans published in the lecture note of M. Bardi et al (1997). These articles present main ideas only and did not give details. The present Chapter gives details and clarify the class of equations to which the method applies.

The invariance (Theorem 4.2.1) goes back to the work of Y.-G. Chen, Y. Giga and S. Goto (1991a) where they consider conventional viscosity solutions instead of $\mathcal{F}$-solutions. Theorem 4.2.7 for the level set mean curvature flow equation is due to L. C. Evans and J. Spruck (1991). The proofs given here are based on simplication by Y.-G. Chen, Y. Giga and S. Goto (1991c). Theorem 4.2.8 is essentially due to Y.-G. Chen, Y. Giga and S. Goto (1991a), where they assumed that $f(x, t, \mathbf{n}, \nabla\mathbf{n})$ is independent of $x$ and satisfies (f1), (f2) so that (BCP) holds. For the case of the level set mean curvature flow equation such uniqueness with Remark 4.2.12 is proved by L. C. Evans and J. Spruck (1991). Extention to unbounded evolutions (§4.2.4) is due to H. Ishii and P. E. Souganidis (1995) as well as $\mathcal{F}$-solutions.

The existence by Perron's method (§4.3) is essentially due to Y.-G. Chen, Y. Giga and S. Goto (1991a) except extensions to $\mathcal{F}$-solutions and unbounded evolutions. Theorem 4.3.5 is due to H. Ishii and P. E. Souganidis (1995). The existence by approximation (§4.4) is due to L. C. Evans and J. Spruck (1991).

Semigroup properties was first stated exlicitly by L. C. Evans and J. Spruck (1991). The results from Theorem 4.5.2 to Corollary 4.5.8 and in Lemma 4.5.12 are taken from the work of S. Altschuler, S. B. Angenent and Y. Giga (1995), where they discussed only for the mean curvature flow equations; extension to general equation is straightforward. For fattening phenomena references are given in §5.6. The results from Theorem 4.5.9 to Corollary 4.5.11 are due to G. Barles, H. M. Soner and P. E. Souganidis (1993).

The strategy to prove convergence of viscosity solutions only by bound for maximum norm without estimating derivatives goes back to G. Barles and B. Perthame (1987), (1988) and independently by H. Ishii (1989b). Weaker version of convergence is stated in the work of F. Camilli (1998), who also discussed the convergence of level sets. His proof is different from ours. It seems that Theorem 4.6.1 was not stated in the literature.

Derivation of Theorem 4.6.4 from Theorem 4.6.1 presented here is due to M.-H. Giga and Y. Giga (2001). When $N = 2$, for (4.7.1)–(4.7.2) they proved a stronger result without assuming convergence of derivative of $\gamma$; they only assume the uniform convergence of $\gamma_\varepsilon$ to $\gamma$ on $S^{N-1}$.

# Chapter 5

# Set-theoretic approach

For surface evolution equations we have introduced generalized solutions as a level set of auxiliary functions. In this chapter we introduce various notion of solutions for surface evolution equations without using auxiliary functions. It turns out that the notion of solutions for evolving sets does not even need level set equations. It only needs surface evolution equations. Since the notion is directly related to evolving sets rather through auxiliary functions, this approach is called intrinsic or set-theoretic. We moreover compare our new notion of solutions with generalized solutions defined in the preceeding chapters. The last part of this chapter is devoted to the study of barrier solutions. It turns out the notion is important to prove the comparison principle via local existence of classical solutions for surface evolution equations without using comparison results in Chapter 3.

## 5.1 Set-theoretic solutions

We consider a surface evolution equation

$$V = f(z, \mathbf{n}, \nabla \mathbf{n}) \quad \text{on} \quad \Gamma_t. \tag{5.1.1}$$

Here $f(z, \cdot, \cdot)$ for $z \in \mathbf{R}^N \times [0, T]$ is a given function defined in

$$\mathbf{E} = \{(p, Q_p(X));\ p \in S^{N-1},\ X \in \mathbf{S}^N\},$$

where $Q_p(X) = (I - p \otimes p)X(I - p \otimes p)$. We recall a part of assumptions on $f$ of the preceeding chapter which we still use in this chapter.

(f1) $f$ is continuous in each variables, i.e. $f : \mathbf{R}^N \times [0, T] \times \mathbf{E} \to \mathbf{R}$ is continuous.
(f2) $f$ is degenerate elliptic in the sense that

$$f(z, p, Q_p(X)) \le f(z, p, Q_p(Y)) \quad \text{whenever} \quad Q_p(X) \ge Q_p(Y).$$

If (f2) is fulfilled, (5.1.1) is called a degenerate parabolic equations. We associate the level set equation of (5.1.1):

$$\begin{aligned} &u_t + F(z, \nabla u, \nabla^2 u) = 0, \\ &F(z, p, X) = -|p| f(z, -\hat{p}, -Q_{\hat{p}}(X)/|p|),\ \hat{p} = p/|p|. \end{aligned} \tag{5.1.2}$$

As already observed, (f1) is equivalent to the condition that $F$ is continuous in $\mathbf{R}^N \times [0,T] \times (\mathbf{R}^N \backslash \{0\}) \times \mathbf{S}^N$; (f2) is equivalent to the degenerate ellipticity of $F$. Since (5.1.2) is a level set equation, the set $\mathcal{F}_\Omega$ in chapter 2 is nonempty for each bounded open set $\Omega$ of $\mathbf{R}^N$ if (f1) is fulfilled (cf. Lemma 3.1.3). Moreover $\mathcal{F}_\Omega$ is invariant under positive multiplication. Thus stability property as well as Perron's method is available for $\mathcal{F}_\Omega$-solutions just as usual viscosity solutions. We often suppress $\mathcal{F}$ of an $\mathcal{F}_\Omega$-solution since we only treat level set equations in this chapter.

### 5.1.1 Definition and its characterization

Let $\chi_D$ denote the characteristic function of a set $D$, i.e.

$$\chi_D(z) = \begin{cases} 1 & z \in D, \\ 0 & \text{otherwise.} \end{cases}$$

If $D$ is a set in a metric space, it is easy to see that

$$(\chi_D)^* = \chi_{\overline{D}},\ (\chi_D)_* = \chi_{\text{int } D}.$$

**Definition 5.1.1.** Let $G$ be a set in $\mathbf{R}^N \times J$, where $J$ is an open interval in $(0,T)$. We say that $G$ is a *set-theoretic subsolution* (resp. *supersolution*) of (5.1.1) if $\chi_G$ is a subsolution (resp. supersolution) of (5.1.2) in $\mathbf{R}^N \times J$. If $G$ is both a set-theoretic sub- and supersolution of (5.1.1) $G$ is called a *set-theoretic solution* of (5.1.1).

Note that we rather consider an enclosed set by evolving surface $\Gamma_t$ than $\Gamma_t$ itself. This is because we have to fix orientations. In our definition it turns out that the normal $\mathbf{n}$ is taken outward from $G(t)$, where $G(t)$ is the cross section of $G$ at time $t$:

$$G(t) = \{x \in \mathbf{R}^N;\ (x,t) \in G\}.$$

To define a set-theoretic subsolution we have used the level set equation (5.1.2). There is a nice characterization of set-theoretic subsolutions without using (5.1.2).

**Theorem 5.1.2.** *Let $J$ be an open interval in $(0,T)$. Under the continuity assumption (f1) a set $G$ in $\mathcal{O} = \mathbf{R}^N \times J$ is a set-theoretic subsolution of (5.1.1) if and only if the following two conditions are fulfilled.*
*(i) Assume that a smoothly evolving hypersurface $\{S_t\}$ around $(x_0,t_0) \in \overline{G}$ has only intersection with $\overline{G}(t)$ at $x_0 \in (\partial G)(t_0)$ around $(x_0,t_0)$. Let $\mathbf{n}_{S_t}$ denote the smooth unit normal vector field of $S_t$ such that $\mathbf{n}_{S_t}(x_0)$ directs outward from $\overline{G}(t)$ at $t = t_0$. Let $V_{S_t}$ denote the normal velocity of $S_t$ in the direction of $\mathbf{n}_{S_t}$. Then*

$$V_{S_t} \le f(x,t,\mathbf{n}_{S_t},\ \nabla \mathbf{n}_{S_t}) \quad \text{at} \quad x = x_0,\ t = t_0. \tag{5.1.3}$$

*(ii) (left accessibility)* *For each* $(x_0, t_0) \in \overline{G}$ *there is a sequence* $(x_j, t_j)$ *converging to* $(x_0, t_0)$ *as* $j \to \infty$ *with* $t_j < t_0$ *and* $(x_j, t_j) \in \overline{G}$. *Here* $\overline{G}$ *and* $\partial G$ *denote the closure and the boundary of* $G$ *in* $\mathcal{O} = \mathbf{R}^N \times J$.

**Remark 5.1.3.** In general $\overline{G}(t)$ does not agree with $\overline{G(t)}$, the closure of $G(t)$ in $\mathbf{R}^N$.

*Proof.* We first prove that (i) and (ii) if $G$ is a subsolution. By rotation we may assume that $S_t$ is represented as the graph of a smooth function $\psi$ near $(x_0, t_0)$ of the form

$$\begin{aligned} x_N &= \psi(x', t), \qquad x = (x', x_N), \\ x_{0N} &= \psi(x'_0, t_0), \quad \nabla'\psi(x'_0, t_0) = 0, \end{aligned}$$

where $\nabla'$ denotes the gradient in $x'$ variables. We may assume that $\mathbf{n}_{S_t}(x_0) = (0, \cdots 0, 1)$ at $t = t_0$. We then set $\varphi(x, t) = \psi(x', t) - x_N$ and observe that $\chi_{\overline{G}} - \varphi$ takes its strict local maximum 1 at $(x_0, t_0)$ since $S_t$ has intersection with $\overline{G}(t)$ only at $x_0$ with $t = t_0$. Since $\chi_{\overline{G}}$ is a subsolution and $\nabla\varphi(z_0) \neq 0$, we see

$$\varphi_t(z_0) + F(z_0, \nabla\varphi(z_0), \nabla^2\varphi(z_0)) \leq 0 \quad \text{with} \quad z_0 = (x_0, t_0)$$

by Proposition 2.2.2. Since $S_t$ is given by $\varphi = 0$ near $(x_0, t_0)$ the last inequality is a level set representation of (5.1.3). We thus obtain (i). It remains to prove (ii). If (ii) were false, there would exist a point $(x_0, t_0) \in \overline{G}$, $\delta > 0$ and a small ball $B_r(x_0)$ that does not interset $\overline{G}(t)$ for $t_0 - \delta < t < t_0$. Thus for any $M > 0$ the function $\chi_{\overline{G}} - \varphi$ with $\varphi(t) = M(t - t_0)$ would take its maximum 1 at $(x_0, t_0)$ over some neighborhood of $(x_0, t_0)$ contained in $B_r(x_0) \times (t_0 - \delta, \ t_0 + \delta)$. We set $\mathcal{O}' = \operatorname{int} B_r(x_0) \times (t_0 - \delta, \ t_0 + \delta)$. Since $\chi_{\overline{G}}$ is an $\mathcal{F}$-subsolution and $\varphi \in C^2_F(\mathcal{O}')$ with $\nabla\varphi \equiv 0, \nabla^2\varphi \equiv O$, by definition we have $\varphi_t(x_0, t_0) \leq 0$ which contradicts $M > 0$.

We next prove that $\chi_{\overline{G}}$ is an $\mathcal{F}$-subsolution if (i) and (ii) are fulfilled. Suppose that $(\varphi, z_0) \in C^2_F(\mathcal{O}') \times \mathcal{O}'$

$$\max_{\mathcal{O}'}(\chi_{\overline{G}} - \varphi) = (\chi_{\overline{G}} - \varphi)(z_0) = 0.$$

We may assume that $z_0 \in \partial G$ since otherwise $\nabla\varphi(z_0) = 0$ and $\varphi_t(z_0) = 0$. We may assume that $\varphi \in C^\infty_F(\mathcal{O}')$ by Proposition 2.2.3 and that $\chi_{\overline{G}} - \varphi$ attains its strict maximum at $z_0$ by Proposition 2.2.2 since $\mathcal{F}$ is invariant under positive multiplication by geometricity of $F$. If $\nabla\varphi(z_0) \neq 0$, then by the implicit function theorem

$$S_t = \{x \in \mathbf{R}^N; \ \varphi(x, t) = \varphi(z_0)\}$$

is a smoothly evolving hypersurface around $z_0 = (x_0, t_0)$. Since $z_0 \in \partial G$ is a strict maximum point of $\chi_{\overline{G}} - \varphi$, $\{S_t\}$ has only intersection with $\partial G$ at $z_0$ near $z_0$. Since $S_t$ satisfies (5.1.3) at $z_0$, as its level set representation we obtain

$$\varphi_t(z_0) + F(z_0, \ \nabla\varphi(z_0), \ \nabla^2\varphi(z_0)) \leq 0;$$

note that $\mathbf{n}_{S_t} = -\nabla\varphi(z_0)/|\nabla\varphi(z_0)|$. It remains to prove that $\varphi_t(z_0) \le 0$ for $z_0 \in \partial G$ when $\nabla\varphi(z_0) = 0$. Suppose that $\varphi_t(z_0) > 0$. Then $\varphi(x,t) < \varphi(z_0) = 1$ near $(x_0, t_0)$ if $t < t_0$. Since $\chi_{\overline{G}} - \varphi \le 0$, this would imply

$$\chi_{\overline{G}} = 0 \quad \text{on} \quad B_r(x_0) \times (t_0 - \delta, t_0)$$

for small $r > 0, \delta > 0$. This contradicts (ii).

**Remark 5.1.4.** (i) Of course there is an equivalent characterization for a set-theoretic supersolution corresponding to Theorem 5.1.2. One should replace (5.1.3) by

$$V_{S_t} \ge f(x,\ t,\ \mathbf{n}_{S_t},\ \nabla\mathbf{n}_{S_t}) \quad \text{at} \quad x = x_0,\ t = t_0,$$

where $S_t$ has only intersection with $\overline{\mathcal{O}\backslash G}(t)$ at $x_0 \in \underline{(\partial(\mathcal{O}\backslash G))(t_0)}$ and $\mathbf{n}_{S_t}(x_0)$ directs outward from $\mathrm{int}G(t)$ at $t = t_0$. In (ii) one should replace $\overline{G}$ by $\overline{\mathcal{O}\backslash G}$. The proof for supersolutions parallels that for subsolutions.

(ii) We have defined a notion of set-theoretic solutions only for set in $\mathbf{R}^N \times J$ but of course one may replace $\mathbf{R}^N$ by an open set $\Omega$ in $\mathbf{R}^N$. The statements of Theorem 5.1.2 are still valid if one replaces $\mathbf{R}^N$ by $\Omega$. It is also possible to define a notion of set-theoretic solutions for the boundary value problem. For example if we impose the right angle boundary condition

$$\Gamma_t \perp \partial\Omega$$

to (5.1.1) in $\Omega$, we say that $G$ in $\overline{\Omega} \times J$ is a *set-theoretic subsolution* if $\chi_{\overline{G}}$ is a subsolution of (5.1.2) in $\Omega \times J$ with $\partial u/\partial\nu = 0$. We have a characterication of a set-theoretic subsolution corresponding to Theorem 5.1.2. The statement is almost the same with $\mathbf{R}^N$ replaced by $\overline{\Omega}$ except in condition (ii) at $x_0 \in \partial\Omega \cap \partial G(t_0)$ we only require either (5.1.3) or

$$\langle -\mathbf{n}_{S_t}(x_0),\ \nu\rangle \le 0 \quad \text{with} \quad t = t_0.$$

(iii) From the proof and Proposition 2.2.3 it is easy to observe that smoothness of $S_t$ in Theorem 5.1.2 may be replaced by the condition that $S_t$ is a $C^{2,1}$ hypersurface as definition of viscosity solutions. We may also replace the assumption that $\{S_t\}$ has only intersection with $G(t)$ at $x_0 \in (\partial G)(t_0)$ by the condition that $\{S_t\}$ is in $(\mathcal{O}\backslash \text{ int } G)(t)$ near $t = t_0$ that satisfies

$$V_{S_t} \le f(x_0,\ t_0,\ \mathbf{n}_{S_t}(y_0),\ \nabla\mathbf{n}_{S_t}(y_0)) \quad \text{at} \quad t = t_0$$

for $y_0 \in S_{t_0}$ with

$$\mathrm{d}(y_0,\ \partial G(t_0)) \le \mathrm{d}(y,\ \partial G(t)) \quad \text{for all } y \in S_t,$$

where $t$ is close to $t_0$, provided that $y_0$ is not a geometric boundary point of $S_{t_0}$.

(iv) As we observed in Chapter 3, $u$ is a subsolution of (5.1.2) in $\mathbf{R}^N \times (t_1, t_2]$ for all $t_1 < t_2$ satisfying $(t_1, t_2] \subset J$. It is easy to observe from the proof of Theorem 5.1.2 that this remark relaxes the assumption on test surface $\{S_t\}$ in Theorem 5.1.2 (i). We require only for $t \le t_0$ that $\{S_t\}$ has only intersection with $\overline{G}(t)$ at $x_0 \in \partial G(t_0)$ around $(x_0, t_0)$; $\{S_t\}$ is allowed to intersect $\overline{G}(t)$ for $t > t_0$.

**Corollary 5.1.5.** *Assume (f1) and (f2). Let $G$ be a subset of $\mathcal{O} = \mathbf{R}^N \times J$. If $\partial G$ is a smoothly evolving hypersurface in $(0, T)$ and fulfills*

$$V \leq f(x,\ t,\ \mathbf{n},\ \nabla \mathbf{n})$$

*on $\partial G$, where $\mathbf{n}$ directs outward to $G$. Then $G$ is a set-theoretic subsolution of (5.1.1).*

This is easy to prove once we admit the characterization (Theorem 5.1.2). We leave the proof to the reader.

## 5.1.2 Characterization of solutions of level set equations

The notion of set-theoretic solutions is important to characterize solutions of level set equations. Here we give only its simplest form for a function in $\mathbf{R}^N \times (0, T)$ but as remarked in Remark 5.1.4 its extension to the boundary value problem is straightforward.

**Theorem 5.1.6.** *Assume the continuity (f1) of $f$ in (5.1.1). Let $J$ be an open interval in $(0, T)$. Let $u : \mathbf{R}^N \times J \to \mathbf{R} \cup \{-\infty\}$ (resp. $\mathbf{R} \cup \{+\infty\}$) satisfy $u^* < \infty$ (resp. $u_* > -\infty$) on $\mathbf{R}^N \times J$. Then $u$ is a subsolution (resp. supersolution) of (5.1.2) in $\mathbf{R}^N \times J$ if and only if each superlevel set*

$$G_c = \{(x, t) \in \mathbf{R}^N \times J;\ u^*(x, t) \geq c \quad (\text{resp.} \quad u_*(x, t) > c)\}$$

*is a set-theoretic subsolution (resp. supersolution ) of (5.1.1) in $\mathbf{R}^N \times J$ for all $c \in \mathbf{R}$.*

*Proof.* We have proved several fundamental properties of the level set equation (5.1.2) in Chapters 2 and 4. We list a part of them for further citation in this section.

(SP) **Stability principle**. Assume that $u_\varepsilon$ is a subsolution (resp. supersolution) of (5.1.2) in $\mathbf{R}^N \times J$ for $\varepsilon > 0$. Then $\overline{u} = \limsup_{\varepsilon \to 0}{}^* u_\varepsilon$ (resp. $\underline{u} = \liminf_{\varepsilon \to 0}{}_* u_\varepsilon$) is a subsolution (resp. supersolution) in $\mathbf{R}^N \times J$ provided that $\overline{u} < \infty$ (resp. $\underline{u} > -\infty$) on $\mathbf{R}^N \times J$.

(CL) **Closedness under supremum and infimum**. Assume that $S$ is a set of subsolutions (resp. supersolutions) of (5.1.2) in $\mathbf{R}^N \times J$. Then

$$\begin{aligned} u(x,t) \quad &= \sup\{v(x,t),\ v \in S\} \\ (\text{resp. } u(x,t) \quad &= \inf\{v(x,t),\ v \in S\}) \end{aligned}$$

is a subsolution (resp. supersolution) of (5.1.2) provided that $u^* < \infty$ (resp. $u_* > -\infty$) in $\mathbf{R}^N \times J$.

(I) **Invariance**. Assume that $\theta$ is continuous and nondecreasing function from $\mathbf{R}$ into $\mathbf{R}$. If $u$ is a subsolution (resp. supersolution) of (5.1.2) in $\mathbf{R}^N \times J$, so does $\theta(u)$.

The properties (SP) and (CL) have been proved in Chapter 2; these properties hold for nongeometric equations, too. The property (I) has been proved in Chapter 4 and it reflects the property that the level set equation is geometric. Assume that $u$ is a subsolution so that $u^*$ is a subsolution. We approximate the Heaviside function by

$$\theta_\varepsilon(\zeta) = \begin{cases} 1, & \zeta \ge 0, \\ (\zeta+\varepsilon)/\varepsilon, & -\varepsilon \le \zeta \le 0, \\ 0, & \zeta \le -\varepsilon \end{cases} \qquad \text{for} \quad \varepsilon > 0.$$

For $c \in \mathbf{R}$ we set

$$v_\varepsilon(x,t) = \theta_\varepsilon(u^*(x,t) - c)$$

and observe that

$$\limsup_{\varepsilon\to 0}{}^* v_\varepsilon = \chi_{G_c}.$$

By the invariance (I) $v_\varepsilon$ is a subsolution of (5.1.2) in $\mathbf{R}^N \times J$. By the stability (SP) $\chi_{G_c}$ is now a subsolution of (5.1.2) i.e., $G_c$ is a set-theoretic subsolution.

The converse is easy to prove. We first note that for a closed set $G$ a function

$$I_G(x,t) = \begin{cases} 0 & (x,t) \in G \\ -\infty & \text{otherwise} \end{cases}$$

is a subsolution of (5.1.2) if $\chi_G$ is a subsolution. (Indeed if we take

$$w_\varepsilon(x,t) = \tilde{\theta}_\varepsilon(\chi_G(x,t)) \quad \text{with} \quad \tilde{\theta}_\varepsilon(\zeta) = \begin{cases} 0, & \zeta \ge 1, \\ -(\zeta-1)/\varepsilon, & \zeta \le 1, \end{cases}$$

then

$$\limsup_{\varepsilon\to 0}{}^* w_\varepsilon = I_G.$$

By (I) and (SP), we see $I_G$ is a subsolution.) Since $u^*$ is upper semicontinuous so that $G_c$ is closed, $I_{G_c}$ is a subsolution of (5.1.2) for each $c \in \mathbf{R}$. Let $u^* - \varphi$ take its maximum at $(\hat{x},\hat{t}) \in \mathbf{R}^N \times J$, where $\varphi \in C^2_{F_f}(\mathbf{R}^N \times J)$. We may assume that $(u^* - \varphi)(\hat{x},\hat{t}) = 0$ and set $c = u^*(\hat{x},\hat{t})$. Since $I_{G_c} - \varphi$ takes its maximum at $(\hat{x},\hat{t}) \in \mathbf{R}^N$, we obtain a desired inequality of $\varphi$ at $(\hat{x},\hat{t})$, i.e.

$$\begin{aligned} &\varphi_t + F_f(\hat{x},\hat{t},\nabla\varphi,\nabla^2\varphi) \le 0 \quad \text{at} \quad (\hat{x},\hat{t}) \quad \text{if} \quad \nabla\varphi(\hat{x},\hat{t}) \ne 0, \\ &\varphi_t(\hat{x},\hat{t}) \le 0 \quad \text{if} \quad \nabla\varphi(\hat{x},\hat{t}) = 0. \end{aligned}$$

We give another proof. It is easy to observe that

$$u^*(x,t) = \sup_c (I_{G_c}(x,t) + c)$$

with interpretation that $-\infty + r = -\infty$ for $r \in \mathbf{R}$. Since $I_{G_c} + c$ is a subsolution of (5.1.2) by (I), the closedness (CL) under supermum implies that $u^*$ is a subsolution of (5.1.2) in $\mathbf{R}^N \times J$.

The proof for supersolutions parallels that for subsolutions so is omitted.

### 5.1.3 Characterization by distance functions

We shall characterize a set-theoretic subsolution (resp. supersolution) by signed distance function. For a given set $A$ in $\mathbf{R}^N$ we associate the signed distance function

$$\mathrm{sd}(x,A)=\begin{cases} \mathrm{d}(x,A^c), & x\in A,\\ -\mathrm{d}(x,A), & x\in A^c\end{cases}$$

where $A^c$ denotes the complement of $A$. We use the convention that $\mathrm{sd}(x,A)\equiv-\infty$ if $A$ is empty and $\mathrm{sd}(x,A)\equiv\infty$ if $A^c$ is empty.

**Theorem 5.1.7.** *Assume the continuity (f1) of $f$ in (5.1.1). Let $J$ be an open interval in $(0,T)$. Let $G$ be a set in $\mathcal{O}=\mathbf{R}^N\times J$. Then $G$ is a set-theoretic subsolution (resp. supersolution) of (5.1.1) if and only if $u\equiv\mathrm{sd}\wedge 0$ (resp. $u\equiv\mathrm{sd}\vee 0$) is a subsolution (resp. supersolution) of*

$$u_t+F(x-u\nabla u,\ t,\ \nabla u,\ \nabla^2 u)=0\quad in\quad \mathcal{O}, \tag{5.1.4}$$

*where* $\mathrm{sd(x,t)=sd(x,G(t))}$.

*Proof.* Assume that $G$ is a set-theoretic subsolution. We shall prove that $u^*$ is a subsolution of (5.1.4). We may assume that $G$ is closed so that $u^*=u$ since $u^*(x,t)=\mathrm{sd}(x,\overline{G}(t))\wedge 0$. Note that $u(x,t)=0$ is equivalent to $(x,t)\in G$. Suppose that $(\varphi,(\hat{t},\hat{x}))\in C^2_F(\mathcal{O}')\times\mathcal{O}'$ satisfies

$$\max_{\mathcal{O}'}(u-\varphi)=(u-\varphi)(\hat{t},\hat{x})=0,$$

with $\mathcal{O}'=\Omega\times(0,T)$, where $\Omega$ is a bounded open set.

**Case 1.** If $(\hat{x},\hat{t})\in G$, then $\varphi$ is an upper test function of $\chi_G$ at $(\hat{x},\hat{t})$. Since $\chi_G$ is a subsolution of (5.1.2) and $u(\hat{x},\hat{t})=0$, it follows that

$$\varphi_t+F(\hat{x},\hat{t},\ \nabla\varphi,\ \nabla^2\varphi)\le 0\quad\text{at}\quad(\hat{x},\hat{t})$$

or

$$\varphi_t(\hat{x},\hat{t})+F(\hat{x}-u^*(\hat{x},\hat{t})\nabla\varphi(\hat{x},\hat{t}),\ \hat{t},\nabla\varphi(\hat{x},\hat{t}),\ \nabla^2\varphi(\hat{x},\hat{t}))\le 0. \tag{5.1.5}$$

provided that $\nabla\varphi(\hat{x},\hat{t})\ne 0$. If $\nabla\varphi(\hat{x},\hat{t})=0$, then we get $\varphi_t(\hat{x},\hat{t})\le 0$ instead of (5.1.5).

**Case 2.** If $(\hat{x},\hat{t})\notin G$, then $u(\hat{x},\hat{t})=\mathrm{sd}(\hat{x},\hat{t})(=-\delta<0)$. We first observe that $\nabla\varphi(\hat{x},\hat{t})\ne 0$. Indeed, by definition of $(\hat{x},\hat{t})$ we see

$$\mathrm{sd}(x,G(t))-\varphi(x,t)\le\mathrm{sd}(\hat{x},G(\hat{t}))-\varphi(\hat{x},\hat{t}).$$

near $(\hat{x},\hat{t})$. Setting $t=\hat{t}$ and expending $\varphi(x,\hat{t})$ near $\hat{x}$ yields

$$\mathrm{d}(\hat{x},G(\hat{t}))-\mathrm{d}(x,G(\hat{t}))\le\langle\nabla\varphi(\hat{x},\hat{t}),\ x-\hat{x}\rangle+o(|x-\hat{x}|)$$

as $x \to \hat{x}$. Let $x_0$ be a point in $G(\hat{t})$ that satisfies

$$\delta = \mathrm{d}(\hat{x}, G(\hat{t})) = |x_0 - \hat{x}|.$$

We take $x = \hat{x} + \sigma(x_0 - \hat{x})$ for $\sigma$ close to zero and observe that $\mathrm{d}(x, G(\hat{t})) = (1-\sigma)\delta$. For this choice of $x$ we arrive at

$$\delta - (1-\sigma)\delta \leq \langle \nabla\varphi(\hat{x}, \hat{t}),\ \sigma(x_0 - \hat{x})\rangle + o(\sigma\delta)$$

as $\sigma \to 0$. This implies that

$$\nabla\varphi(\hat{x}, \hat{t}) = (x_0 - \hat{x})/|x_0 - \hat{x}| \neq 0. \tag{5.1.6}$$

We next observe that $\psi(x,t) < 0$ implies $u(x,t) < 0$ if we set

$$\psi(x,t) = \varphi(x + \hat{x} - x_0,\ t) + \delta.$$

Indeed, the triangle inequality implies

$$-u(x + \hat{x} - x_0,\ t) + u(x,t) \leq |\hat{x} - x_0| = \delta.$$

Since $u - \varphi$ takes its zero maximum at $(\hat{x}, \hat{t})$, $\psi(x,t) < 0$ implies

$$-u(x + \hat{x} - x_0,\ t) \geq -\varphi(x + \hat{x} - x_0,\ t) > \delta.$$

Combining these two inequalities yields $u(x,t) < 0$.

We are now in position to prove (5.1.5) for $(\hat{x}, \hat{t}) \notin G$. Since $\psi(x,t) < 0$ implies $u(x,t) < 0$ and $\psi(x_0, \hat{t}) = 0$, $\chi_G - \psi$ takes its maximum at $(x_0, \hat{t})$. Since $\chi_G$ is a subsolution of (5.1.2) and $\nabla\varphi(x_0, \hat{t}) = \nabla\varphi(\hat{x}, \hat{t}) \neq 0$, we have

$$\psi_t(x_0, \hat{t}) + F(x_0, \hat{t},\ \nabla\psi(x_0, \hat{t}),\ \nabla^2\psi(x_0, \hat{t})) \leq 0.$$

By (5.1.6) we see

$$x_0 = \hat{x} + \delta\nabla\psi(x_0, \hat{t}) = \hat{x} - u(\hat{x}, \hat{t})\nabla\varphi(\hat{x}, \hat{t}).$$

We thus obtain (5.1.5) for $\varphi$.

It remains to prove that $G$ is a set-theoretic subsolution of (5.1.1) if $u$ is a subsolution of (5.1.4). We may again assume that $G$ is closed. The proof is already contained in Theorem 5.1.6 of $f$ is independent of $x$ since

$$G = \{(x,t) \in \mathbf{R}^N \times (0,T);\ u(x,t) \geq 0\}.$$

If $f$ depends on $x$, $\theta(u)$ for nonincreasing function $\theta$ may not be a subsolution of (5.1.4). So Theorem 5.1.6 does not apply. However, as in the proof of the invariance lemma in Chapter 4, $w = \theta(u)$ is a subsolution of

$$w_t + F(x - \rho(w)\rho'(w)\nabla w,\ t,\ \nabla w,\ \nabla^2 w) = 0 \quad \text{in} \quad \mathcal{O}$$

for $\rho = \theta^{-1}$ provided that $\theta \in C^1(\mathbf{R})$ with $\theta' > 0$. We take $\tilde{\theta}_\varepsilon$ as in the proof of Theorem 5.1.6 and observe that $w_\varepsilon = \overline{\theta}_\varepsilon(u)$ with $\overline{\theta}_\varepsilon(\sigma) = \tilde{\theta}_\varepsilon(\sigma + 1)$ is a subsolution of

$$w_t + F(x - \rho_\varepsilon(w)\rho'_\varepsilon(w)\nabla w,\ t,\ \nabla w,\ \nabla^2 w) = 0 \quad \text{in} \quad \mathcal{O},$$

where $\rho_\varepsilon = \overline{\theta}_\varepsilon^{\,-1}$ with interpreted that $\rho_\varepsilon(\sigma) = 0$ for $\sigma \geq 0$ so that $\rho_\varepsilon \rho'_\varepsilon$ is interpreted to be continuous at $\sigma = 0$. Since $\overline{\theta}_\varepsilon$ is not $C^1$ at zero we should approximate $\overline{\theta}_\varepsilon$ by $C^1$ function to get this formula by the stability results. By using the stability result we conclude that $I_G = \limsup\limits_{\varepsilon \to 0}{}^* w_\varepsilon$ is a subsolution of (5.1.2) since $\rho_\varepsilon(\sigma) = \varepsilon\sigma \to 0$ as $\varepsilon \to 0$ for $\sigma < 0$ and $\rho_\varepsilon(w) = 0$ for $w > 0$.

The proof for supersolution is similar so is omitted. $\square$

**Corollary 5.1.8.** *Assume the continuity (f1) of $f$ in (5.1.1). The function $u = \mathrm{sd} \wedge 0$ is a subsolution of (5.1.4) in $\mathcal{O}$ if and only if condition (ii) for $G$ of Theorem 5.1.2 is fulfilled and $u^*$ satisfies the left accessibility property : for each $(x_0, t_0) \in \mathbf{R}^N \times (0, T)$ there is a sequence $(x_j, t_j)$ converging to $(x_0, t_0)$ as $j \to \infty$ with $t_j < t_0$ such that $u(x_j, t_j) \to u(x_0, t_0)$. (Similar assertion holds for* $\mathrm{sd} \vee 0$.*)*

*Proof.* Since the left accessibility of $\overline{G}$ is equivalent to the left accessibility of $u^*$, this follows from Theorems 5.1.2 and 5.1.7.

### 5.1.4 Comparison principle for sets

We shall review comparison principle obtained in Chapter 3 from the point of set-theoretic solutions. When $F$ in (5.1.2) is degenerate elliptic, we have proved under reasonable assumptions the comparison principle. We give a slightly different version of (CP) and (BCP) stated in Chapter 3.

(CP) Let $u$ and $v$ be sub- and supersolution of (5.1.2) in $\mathcal{O} = \mathbf{R}^N \times (0, T)$, respectively.
(i) Assume that $u$ and $-v$ are bounded from above on $\mathcal{O}$. Assume that

$$\lim_{\delta \downarrow 0} \sup\{u^*(x,t) - v_*(y,s);\ (x,t),(y,s) \in \mathbf{R}^N \times [0,T),\ |x - y| \leq \delta,\ t \leq \delta,\ s \leq \delta\} \leq 0, \tag{5.1.7}$$

for each $T' \in (0, T)$ and $u^* > -\infty$, $v^* < \infty$ on $\partial_p \mathcal{O}$. Then

$$\lim_{\delta \downarrow 0} \sup\{u^*(x,t) - v_*(y,s);\ (x,t),(y,s) \in \mathbf{R}^N \times [0,T'],\ |x - y| \leq \delta,\ |t - s| \leq \delta\} \leq 0, \tag{5.1.8}$$

for each $T' < T$.

(ii) If $u^* \leq v_*$ at $t = 0$, then $u^* \leq v_*$ on $\mathbf{R}^N \times [0, T)$, provided that $u(x,t)$ and $v(x,t)$ are constant outside $B_R(0) \times (0, T)$ for some large $R > 0$.

Of course, the second property follows from the first. The first property is nothing but (CP) in Chapter 3 when $\Omega = \mathbf{R}^N$. The property (CP (i), (ii)) holds for (5.1.2) for example when (5.1.1) is degenerate parabolic with continuous $f$ independent of the space variable $x$. The second property essetially follows from (BCP). See Chapter 3 for more details.

We give a comparison principle for bounded set-theoretic solutions.

(BCPS) Let $E$ and $D$ be set-theoretic sub- and supersolutions of (5.1.1) in $\mathcal{O}$, respectively. If $E^*(0) \subset D_*(0)$, then $E^* \subset D_*$ provided that $D$ (or $\mathcal{O}\backslash D$) and $E$ (or $\mathcal{O}\backslash E$) are bounded in $\mathcal{O}$. Here $E^*$ denotes the closure of $E$ as a set in $\mathbf{R}^N \times [0,T)$ and $D_*$ denotes the complement of $(\mathcal{O}\backslash D)^*$ in $\mathbf{R}^N \times [0,T)$.

This follows from (CP (ii)) by setting $u = \chi_E$, $v = \chi_D$. It turns out that (BCPS) is equivalent to (CP (ii)).

**Lemma 5.1.9.** *Assume the continuity (f1) of $f$ in (5.1.1). The property (BCPS) for (5.1.1) holds if and only if (CP (ii)) holds for (5.1.2), where $\mathcal{O} = \mathbf{R}^N \times (0,T)$.*

*Proof.* It suffices to prove that (BCPS) implies (CP (ii)). For $c, d \in \mathbf{R}$ we set

$$E_c = \{(x,t) \in \mathbf{R}^N \times [0,T);\ u^*(x,t) \geq c\},$$
$$D_d = \{(x,t) \in \mathbf{R}^N \times [0,T);\ v_*(x,t) > d\}.$$

Since $u$ (resp. $v$) is constant outside $B_R(0) \times (0,T)$, $E_c$ or $\mathcal{O}\backslash E_c$ (resp. $D_d$ or $\mathcal{O}\backslash D_d$) is bounded in $\mathcal{O}$. By Theorem 5.1.6 $E_c$ and $D_d$ are set-theoretic sub- and supersolutions in $\mathcal{O}$, respectively. By definition of $u^*$ and $v_*$ $E_c = (E_c \cap \mathcal{O})^*$ and $D_d = (D_d \cap \mathcal{O})_*$. Since $u^* \leq v_*$ at $t = 0$, $E_c(0) \subset D_{c-\delta}(0)$ for all $c \in \mathbf{R}$, $\delta > 0$. By (BCPS) we see

$$E_c \subset D_{c-\delta}$$

for all $c \in \mathbf{R}$, $\delta > 0$. This implies that $u^* \leq v_*$ in $\mathbf{R}^N \times [0,T)$, since otherwise there would exist a point $(x_0,t_0) \in \mathcal{O}$ with $u^*(x_0,t_0) = c > v_*(x_0,t_0) = c - \delta$ for some $c \in \mathbf{R}$, $\delta > 0$ so that $(x_0,t_0) \in E_c$ but $(x_0,t_0) \notin D_{c-\delta}$. $\square$

We now derive comparison principle for set-theoretic solutions corresponding to (CP (i)) which is not necessarily bounded. It is not difficult to prove that (5.1.7) with $u = \chi_E$ and $v = \chi_D$ is equivalent to

$$\inf\{\mathrm{dist}(E_*(t),\ (\mathcal{O}\backslash D)^*(s));\ 0 \leq t \leq \varepsilon_0,\ 0 \leq s \leq \varepsilon_0\} \geq \varepsilon_0 \tag{5.1.9}$$

for some small $\varepsilon_0 > 0$. Similarly, the condition (5.1.8) for $u = \chi_E$ and $v = \chi_D$ is equivalent to say that for each $0 \leq T' < T$

$$\inf\{\mathrm{dist}(E_*(t),\ (\mathcal{O}\backslash D)^*(s));\ |t-s| \leq \varepsilon_1,\ 0 \leq t,\ s \leq T'\} \geq \varepsilon_1 \tag{5.1.10}$$

for some $\varepsilon_1 > 0$ (which may depend on $T'$), where

$$\mathrm{dist}(A, B) = \inf\{\mathrm{dist}(x, y);\ x \in A,\ y \in B\}$$

and the closure is taken in $\mathbf{R}^N \times [0, T)$ not in $\mathcal{O}$.

We now propose comparison principle for general set-theoretic solutions.

(CPS) Let $E$ and $D$ be set-theoretic sub- and supersolution of (5.1.1) in $\mathcal{O}$, respectively. If (5.1.9) holds for some $\varepsilon_0 > 0$, then (5.1.10) holds for some $\varepsilon_1$.

**Lemma 5.1.10.** *Assume the continuity (f1) of $f$ in (5.1.1). The property (CPS) for (5.1.1) holds if and only if (CP (i)) for (5.1.2) holds.*

*Proof.* It suffices to prove that (CPS) implies (CP (i)). We set $E_c$ and $D_d$ as in the proof of Lemma 5.1.9. If (5.1.8) were false, then there would exist a sequence $c_j \in \mathbf{R}, (x_j, t_j),\ (y_j, s_j) \in \mathbf{R}^N \times [0, T']$ and a constant $\eta > 0$ that satisfies

$$\begin{aligned} &u^*(x_j, t_j) = c_j > v_*(y_j, s_j) + \eta, \\ &|x_j - y_j| \to 0,\ |t_j - s_j| \to 0 \quad \text{as} \quad j \to \infty. \end{aligned}$$

In particular $x_j \in E_{c_j}(t_j)$ and $y_j \notin D_{c_j - \eta/2}(s_j)$, so that

$$\mathrm{dist}(E_{c_j}(t_j),\ (Z \backslash D_{c_j - \eta/2})(s_j)) \to 0 \tag{5.1.11}$$

as $j \to \infty$, where $Z = \mathbf{R}^N \times [0, T']$. Since (5.1.7) holds, there is $\varepsilon_0 > 0$ independent of $c_j$ that satisfies

$$\inf\{\mathrm{dist}(E_{c_j}(t),\ (Z \backslash D_{c_j - \eta/2})(s)),\ 0 \le t,\ s \le \varepsilon_0\} \ge \varepsilon_0. \tag{5.1.12}$$

By definition of upper and lower semiconvergence there is a closet set $E$ in $Z$ and an open set $D$ in $Z$ that satisfies

$$\chi_E = \limsup_{j \to \infty}{}^* \chi_{E_{c_j}},\ \chi_D = \liminf_{j \to \infty}{}_* \chi_{D_{c_j} - \eta/2}.$$

By (5.1.11) we would obtain

$$\mathrm{dist}(E(\hat{t}),\ (Z \backslash D)(\hat{t})) = 0 \tag{5.1.13}$$

for some $\hat{t} \in [0, T']$ which is an accumulation point of $t_j$, we also observe that $s_j \to \hat{t}$ since $t_j - s_j \to 0$. By (5.1.12) we see

$$\inf\{\mathrm{dist}(E(t),\ (Z \backslash D)(s)),\ 0 \le t,\ s \le \varepsilon_0\} \ge \varepsilon_0 \tag{5.1.14}$$

which in particular implies $\hat{t} > 0$.

Since the stability principle (SP) holds, $E$ and $D$ are still set-theoretic sub- and supersolutions of (5.1.1), respectively. By (CPS) (5.1.14) implies

$$\inf\{\mathrm{dist}(E(t),\ (Z\backslash D)(s));\ |t-s| \le \varepsilon_1\} \ge \varepsilon_1$$

for some $\varepsilon_1 > 0$ which contradicts (5.1.13). □

**Remark 5.1.11.** **(i)** In the comparison principle (CPS) for set-theoretic solutions we have assumed (5.1.9). This assumption needs property of solutions not only at time zero but also near time zero. However, if we assume uniformly upper semicontinuity of $E_*(t)$ and $(\mathcal{O}\backslash D)^*(t)$ at $t = 0$ (5.1.9) follows from a simple condition

$$\mathrm{dist}(E^*(0),\ ((\mathcal{O}\backslash D)^*)(0)) > 0. \tag{5.1.15}$$

Here we say that $E^*(t)$ is *uniformly upper semicontinuous* at $t = 0$ if

$$\lim_{t\to 0}\ \sup\{\mathrm{dist}(x,\ E^*(0));\ x \in E^*(t)\} = 0.$$

Of course this property always holds if $E^*(0)$ is compact in $\mathbf{R}^N$.

(ii) So far we have taken initial data at $t = 0$ in (CPS) and (BCPS). Sometimes it is necessary to take initial data at $t = t_0 \in (0, T)$. We replace 0 in (CPS) and (BCPS) by $t_0$ and refer these conditions as (CPS $t_0$) and (BCPS $t_0$), respectively. Of course the statements of Lemma 5.1.9 and Lemma 5.1.10 are still valid for (BCPS $t_0$) and (CPS $t_0$) if we replace (CP) in an appropriate way.

### 5.1.5 Convergence of sets and functions

In previous subsections we compared various aspects of set-theoretic solutions and solutions of level set equations. Here we compare the upper semiconvergence of functions and its level set. For a family $\{E^\varepsilon\}_{\varepsilon>0}$ of sets in a metric space $X$ it is easy to see that

$$\limsup_{\varepsilon\to 0}\ E^\varepsilon = \{x \in X;\ (\limsup_{\varepsilon\to 0}{}^*\ \chi_{E_\varepsilon})(x) = 1\}.$$

**Lemma 5.1.12.** *Let $u^\varepsilon(\varepsilon > 0)$ be an upper semicontinuous function on $X$ with values in $\{-\infty\} \cup \mathbf{R}$. For $c \in \mathbf{R}$ let $E_c^\varepsilon$ denote*

$$E_c^\varepsilon = \{x \in X;\ u^\varepsilon(x) \ge c\}.$$

*Let $\overline{u}$ and $E_c$ be defined by*

$$\overline{u} = \limsup_{\varepsilon\to 0}{}^*\ u^\varepsilon,\ E_c = \limsup_{\substack{\varepsilon\to 0\\ \ell\to c}}\ E_\ell^\varepsilon.$$

*Then*

$$E_c = \{x \in X;\ \overline{u}(z) \geq c\}.$$

*Proof.* The condition $\chi_{\overline{u}\geq c}(z) = 1$ is equivalent to saying that there exists a sequence $\varepsilon_j \to 0$, $z_j \to z$ such that $\liminf_{j\to\infty} u^{\varepsilon_j}(z_j) \geq c$. We observe that the last inequality is equivalent to $\lim_{j\to\infty} \chi_{E^{\varepsilon_j}_{\ell_j}}(z_j) = 1$ with $\ell_j \to c$ by setting $\ell_j = \min(u^{\varepsilon_j}(z_j), c)$. Thus the result follows. □

This lemma together with Theorem 5.1.6 implies that the stability results in chapter 2 is equivalent to the stability of sets in $X = \mathbf{R}^N \times (0,T)$ or characteristic functions. We left its explicit form to the reader.

# 5.2 Level set solutions

For a given set $G_0$ in $\mathbf{R}^N$ we seek a set-theoretic solution $G \subset \mathbf{R}^N \times [0,T)$ of (5.1.1) in $\mathbf{R}^N \times (0,T)$ with initial data $G_0$. To be precise we say that the initial data of $G$ equals $G_0$ if $G^*(0) = \overline{G_0}$ and $(G_*)(0) = \operatorname{int} G_0$. Unfortunately, for a given initial data there may be several set-theoretic solutions, so solutions for initial value problem of (5.1.1) may not be unique. For this reason we shall introduce notion of level set sub- and supersolutions.

## 5.2.1 Nonuniqueness

We first give a simple example of nonuniqueness for the curve shortening equation $V = k$ in $\mathbf{R}^2$.

**Example 5.2.1.** Let $h = h(z,t)$ be the unique smooth solution of

$$\begin{cases} \partial_t h = \dfrac{h_{zz}}{1+h_z^2}, & t > 0,\ z > 0, \\ h_z(0,t) = 0, & t > 0, \\ \lim\limits_{\zeta\to\infty} h_z(\zeta,t) = 1, & t > 0, \\ h(z,0) = z. & \end{cases}$$

Such a solution exists globally in time. It can be constructed by an approximate argument; see for example the work of K. Ecker and G. Huisken (1989). Define

$$\begin{aligned} D_1 &= \{(x,y,t) \in \mathbf{R}^2 \times [0,\infty);\ |y| < h(|x|,t)\}, \\ D_2 &= \{(x,y,t) \in \mathbf{R}^2 \times [0,\infty);\ |x| > h(|y|,t)\}, \\ D_3 &= \{(x,y,t) \in \mathbf{R}^2 \times [0,\infty);\ |y| < |x|\}. \end{aligned}$$

Since $\partial D_j$ solves $V = k$ in the classical sense, by Corollary 5.1.5 $D_1$ and $D_2$ are set-theoretic solutions of $V = k$. By Theorem 5.1.2 and Remark 5.1.4 (i) the set $D_3$ is also a set-theoretic solution. Indeed, for example to see that $\overline{D_3}$ is a set theoretic subsolution we observe that all $\{S_t\}$ in Theorem 5.1.2 fulfills (5.1.3) with $V_{S_t} = 0$. Clearly,

$$D_j^*(0) = D_0, \quad D_{j*}(0) = D_0 \quad (j = 1, 2, 3)$$

with $D_0 = \{(x,y) \in \mathbf{R}^2;\ |x| > |y|\}$. Thus all $D_j$ is a set-theoretic solution of $V = k$ with initial data $D_0$. There is also a similar type example of nonuniqueness for bounded initial data. There are several criteria for uniqueness, but we do not give them here.

### 5.2.2 Definition of level set solutions

We introduce notion of a level set subsolution for an open set in $\mathcal{O} = \mathbf{R}^N \times (0, T)$.

**Definition 5.2.2.** Let $D$ be an open set in $Z = \mathbf{R}^N \times [0, T)$. We say that $D$ is a *level set subsolution* of (5.1.1) in $\mathcal{O} = \mathbf{R}^N \times (0, T)$ if there is a sequence $\{E_j\}_{j=1}^{\infty}$ of closed sets in $Z$ that satisfies

(i) $E_j \subset E_{j+1}$ for $j \geq 1$ and $\bigcup_{j=1}^{\infty} E_j = D$,

(ii) $E_j$ is a set-theoretic subsolution of (5.1.1) in $\mathcal{O}$ for $j \geq 1$,

(iii) $\inf\{\mathrm{dist}(E_j(t),\ (\partial D)(t));\ 0 \leq t < T\} > 0$ for each $j \geq 1$.

For an open set $D$ in $\mathcal{O}$ we say that $D$ is an *(open) level set subsolution* of (5.1.1) in $\mathcal{O}$ if there is an open set $D'$ in $Z$ which is a level set subsolution of (5.1.1) in $\mathcal{O}$ with $D = \mathcal{O} \cap D'$. If $D$ in $\mathcal{O}$ fulfills (i), (ii), (iii) with $\{E_j\}_{j=1}^{\infty} \subset \mathcal{O}$ and $0 \leq t$ in (iii) is replaced by $0 < t$, then such $D'$ always exists by setting $D' = \cup_{j=1}^{\infty} E_j^*$, where $E_j^*$ is the closure of $E_j$ in $Z$. Note that $D'$ is not uniquely determined by $D$.

Similarly for closed set $E$ in $Z$ we say that $E$ is a *(closed) level set supersolution* of (5.1.1) in $\mathcal{O}$ if there is a sequence $\{D_j\}_{j=1}^{\infty}$ of open sets in $Z$ that satisfies

(i) $D_{j+1} \subset D_j$ for $j \geq 1$ and $\bigcap_{j=1}^{\infty} D_j = E$,

(ii) $D_j$ is a set-theoretic supersolution of (5.1.1) for $j \geq 1$,

(iii) $\inf\{\mathrm{dist}((\partial D_j)(t)), E(t));\ 0 \leq t < T\} > 0$ for each $j \geq 1$. We define a closed level set supersolution for a closed set in $\mathcal{O}$ in a similar way as defining a level set subsolution.

Since

$$\chi_{\overline{D}} = \limsup_{j \to \infty}{}^* \chi_{E_j},$$

we see that $\overline{D}$ is a subsolution by the stability principle (SP). This justifies the wording 'subsolution' since $D$ is a set-theoretic subsolution. The same remark applies to level set supersolutions.

It is also possible to define a level set solution in $\mathbf{R}^N \times [t_0, t_1)$ for any $[t_0, t_1) \subset [0, T)$ by replacing $[0, T)$ by $[t_0, t_1)$. We next characterize the solutions as a level set of an auxiliary function.

**Proposition 5.2.3.** *Assume the continuity (f1) of $f$ in (5.1.1). Let $D$ (resp. $E$) be a(n) open (resp. closed) set in $\mathcal{O} = \mathbf{R}^N \times (0,T)$. Then $D$ (resp. $E$) is a level set subsolution (resp. supersolution) if and only if there is an upper semicontinuous subsolution (resp. a lowersemicontinuous supersolution) $u$ of (5.1.2) in $\mathcal{O}$ with properties*

*(i) $D = \{(x,t) \in \mathcal{O}; u(x,t) > 0\}, \overline{D} = \{(x,t) \in \mathcal{O}; u(x,t) \geq 0\}$, (resp. $E = \{(x,t) \in \mathcal{O}; u(x,t) \geq 0\}$, int $E = \{(x,t) \in \mathcal{O}; u(x,t) > 0\}$)*

*(ii) $u(x,t)$ is uniformly continuous in $x$ on its zero level set in $\overline{D}(t)$ (resp. $(\mathcal{O}\backslash \mathrm{int}E)(t)$) uniformly in $t \in (0,T)$, i.e., there is a modulus $m$ that satisfies*

$$u(x,t_0) \leq m(|x - x_0|)$$

*for $x, x_0 \in \overline{D}(t_0), t_0 \in (0,T)$ with $u(x_0,t_0) = 0$, $u(x,t_0) \geq 0$ (resp. $u(x,t_0) \leq 0$).*

*Proof.* We only give the proof for $D$ since the proof for $E$ is similar. Assume first that $D$ is a level set subsolution and that an approximate sequence $\{E_j\}_{j=1}^{\infty}$ is taken as in Definition 5.2.2. We set

$$v(x,t) = \sup\{(\frac{1}{j} + 1)\chi_{E_j} - 1;\ j = 1, 2, \cdots\}$$

and observe that $v$ is a subsolution of (5.1.2) by the closedness under supremum (CL) since $(j^{-1} + 1)\chi_{E_j} - 1$ is a subsolution by the invariance (I). Since $E_j \subset E_{j+1}$ for $j \geq 1$, $v$ is upper semicontinuous in $D = \bigcup_{j=1}^{\infty} E_j$ and moreover

$$v^*(x,t) = \begin{cases} v(x,t), & (x,t) \in D, \\ 0, & (x,t) \in \partial D, \\ -1, & (x,t) \in \mathcal{O}\backslash \overline{D}. \end{cases}$$

If we set $u = v^*$, the property (i) is fulfilled. The uniform continuity (ii) follows from Definition 5.2.2(iii). Indeed, let $m_0 : [0,\ \infty) \to [0, \infty)$ be a nondecreasing function of form

$$m_0(\sigma) = \sup\{u(x,t);\ d(x, \partial D(t)) \leq \sigma,\ (x,t) \in \overline{D}\}.$$

By (iii) of Definition 5.2.2 and $\cup_{j=1}^{\infty} E_j = D$ we see $m_0(\sigma) \to 0$ as $\sigma \to 0$. By Lemma 2.1.9 there is a modulus $m \geq m_0$. Thus $u(x,t_0) \leq m(|x - x_0|)$ for $x_0 \in \partial D(t_0),\ x \in \overline{D}(t_0),\ t_0 \in 0, T)$.

We now prove the converse. Let $u$ be an upper semicontinuous subsolution $u$ of (5.1.2) satisfying (i) and (ii). We set

$$E_j = \{(x,t) \in \mathcal{O};\ u(x,t) \geq 1/j\}$$

and observe that $E_j$ is a subsolution of (5.1.1) by Theorem 5.1.6. By definition $E_j \subset E_{j+1}$ and $\cup_{j=1}^{\infty} E_j^* = D_*$. Since

$$1/j \leq u(x,t_0) \leq m(|x - x_0|)$$

for all $x_0 \in \partial D(t_0),\ t_0 \in (0,T),\ x \in E_j(t_0)$, the property (ii) now follows.

**Remark 5.2.4.** If $f$ in (5.1.1) is independent of $x$ so that $F$ in (5.1.2) is independent of $x$, then by Theorem 5.1.7 sd $\wedge 0$ is a subsolution of (5.1.2) for $\mathrm{sd}(x,t) = \mathrm{sd}(x, D(t))$ where $D$ is a level set subsolution. We take

$$w(x,t) = \max(\mathrm{sd} \wedge 0,\ v)$$

with $v$ in the proof of Proposition 5.2.3 and observe from (CL) that $w$ is an upper semicontinuous subsolution and that $w(x,t)$ is uniformly continuous in $x$ on its zero level set (uniformly in $t \in (0,T)$) without restricting $x$ in $\overline{D}(t)$. This is because sd is always Lipschitz continuous.

### 5.2.3 Uniqueness of level set solution

We give comparison principle for level set sub- and supersolutions. For a set $D$ in $\mathbf{R}^N \times [0,T)$ we often denote its intersection with $\mathbf{R}^N \times (0,T)$ still by $D$.

**Theorem 5.2.5.** *Assume that (CPS) holds. Let $G$ be an open (resp. closed) set in $Z = \mathbf{R}^N \times [0,T)$. Let $G$ be a set-theoretic supersolution (resp. subsolution) of (5.1.1) in $\mathcal{O} = \mathbf{R}^N \times (0,T)$. Let $D$ be an open (resp. closed) set in $Z$. Assume that $(Z\backslash G)(t)$ and $\overline{D}(t)$ are (resp. $G(t)$ and $(Z\backslash \text{ int } D(t))$ uniformly upper semicontinuous. Assume that $D$ is a level set subsolution (resp. supersolution) of (5.1.1) in $\mathcal{O}$. If $D(0) \subset G(0)$ (resp. $D(0) \supset G(0)$), then $D \subset G$ (resp. $G \subset D$).*

*Proof.* Again we only give the proof when $G$ is a set-theoretic supersolution and $D$ is a level set subsolution since the proof for the other case is similar.

Let $\{E_j\}_{j=1}^{\infty} \subset Z = \mathbf{R}^N \times [0,T)$ be a sequence approximately $D$ in Definition 5.2.2. Since $\overline{D}(t)$ is uniformly upper semicontinuous at $t=0$, the property (iii) in Definition 5.2.2 yields

$$\inf\{\text{dist }(E_j(t),\ (Z\backslash D)(0)),\ 0 \le t \le \varepsilon_1\} > 0$$

for some small $\varepsilon_1 > 0$. Since $(Z\backslash G)(t)$ is uniformly upper semicontinuous at $t=0$, and $D(0) \subset G(0)$, the preceeding estimate yields

$$\inf\{\text{dist }(E_j(t),\ (Z\backslash G)(s));\ 0 \le t \le \varepsilon_0,\ 0 \le s \le \varepsilon_0\} \ge \varepsilon_0$$

for some small $\varepsilon_0 > 0$. Since $E_j$ is a set-theoretic subsolution and $G$ is a set-theoretic supersolution, the comparison principle (CPS) yields $E_j \subset G$. Since $D = \cup_{j=1}^{\infty} E_j$, this implies that $D \subset G$.

**Corollary 5.2.6.** *Assume that (CPS) holds. Let $D_0$ (resp. $E_0$) be a(n) open (resp. closed) set in $\mathbf{R}^N$. There is at most one level set subsolution $D$ (in $Z$) (resp. supersolution $E$) which is also a set-theoretic supersolution (resp. subsolution) of (5.1.1) in $\mathcal{O} = \mathbf{R}^N \times (0,T)$ with $D(0) = D_0$ (resp. $E(0) = E_0$) provided that $(Z\backslash D)(t)$ and $\overline{D}(t)$ (resp. $E(t)$ and $(Z\backslash \text{ int } E)(t)$) are uniformly upper semicontinuous at $t=0$.*

This follows from Theorem 5.2.5. Indeed, if $\tilde{D}$ and $D$ are both a level set subsolution and a set-theoretic supersolution (with $D(0) = \tilde{D}(0) = D_0$) that satisfies the uniform upper semicontinuity at $t = 0$, then applying Theorem 5.2.5 with $G = \tilde{D}$ yields $\tilde{D} \supset D$. A symmetric argument yields $D \supset \tilde{D}$.

**Definition 5.2.7.** Let $D_0$ (resp. $E_0$) be a(n) open (resp. closed) set in $\mathbf{R}^N$. A(n) open set $D$ (resp. closed set $E$) in $Z$ is called a *level set solution* of (5.1.1) with initial data $D_0$ (resp. $E_0$) if $D$ (resp. $E$) is simultaneously a level set subsolution (resp. supersolution) and a set-theoretic supersolution (resp. subsolution) with $D(0) = D_0$ (resp. $E(0) = E_0$) and $(Z \backslash D)(t)$ and $\overline{D}(t)$ (resp. $E(t)$ and $(Z \backslash \operatorname{int} E)(t)$) are uniformly upper semicontinuous at $t = 0$.

By Corollary 5.2.6 for $D_0$ (resp. $E_0$) a level set solution $D$ (resp. $E$) with $D(0) = D_0$ (resp. $E(0) = E_0$) is unique. Note that the uniformly upper semicontinuity is always fulfilled if one of $Z \backslash D$ or $\overline{D}$ (resp. $E$ and $Z \backslash \operatorname{int} E$) is bounded. We shall compare a generalized evolution of (5.1.1) with a level set solution.

**Proposition 5.2.8.** *Let $D$ (resp. $E$) in $Z = \mathbf{R}^N \times [0, T)$ be a generalized open (resp. closed) evolution of (5.1.1) with initial data $D(0) = D_0$ (resp. $E(0) = E_0$). Then $D$ (resp. $E$) is a level set solution of (5.1.1) with $D(0) = D_0$ (resp. $E(0) = E_0$).*

*Proof.* Again we only discuss open evolution $D$ since the proof for $E$ is similar. By definition there is a solution $u : Z \to \mathbf{R}$ (of (5.1.2)) that belongs to $BUC(\mathbf{R}^N \times [0, T'])$ for every $T' < T$ and it satisfies

$$D = \{(x, t) \in Z;\ u(x, t) > 0\}.$$

Since $u$ is a supersolution, $D$ is a set-theoretic supersolution by Theorem 5.1.6. If we take

$$E_j = \{(x, t) \in Z;\ u(x, t) \geq 1/j\},\ j = 1, 2, \cdots,$$

then $E_j$ fulfills properties (i), (ii), (iii) of Definition 5.2.2. Indeed, the property (iii) follows from $u \in BUC(\mathbf{R}^N \times [0, T'])$. The property (ii) follows from Theorem 5.1.6. The property (i) is clear by definition. Thus $D$ is a level set subsolution.

The uniformly upper semicontinuity at $t = 0$ follows from the fact that $u \in BUC(\mathbf{R}^N \times [0, T'])$ so $D$ is a level set solution.

**Remark 5.2.9.** By uniqueness (Corollary 5.2.6) there are no level set solutions other than generalized evolution. We may arrange the definition of level set solution more restrictive so that Proposition 5.2.8 holds. If we assume that $E_j$'s in Definition 5.2.2 is bounded, the notion of a level set subsolution comes to be more restrictive than we used above unless $D$ is bounded. However, the statement of comparison and uniqueness (Theorem 5.2.5 and Corollary 5.2.6) still hold for these restrictive level set solutions. As observed in Chapter 4, a generalized open evolution can be approximated by bounded subsolution $E_j$ form inside in many cases, for example the case when (5.1.2) satisfies the comparison principle (CP) as well as the continuity (f1) of $f$. Thus the statement of Proposition 5.2.8 is still valid for level set solutions in the restrictive sense.

**Corollary 5.2.10.** (Uniqueness) *Assume that (CPS) holds. Let $D$ be an (open) level set solution of (5.1.1) with initial data $D_0$. Assume that $\overline{D}$ is the (closed) level set solution of (5.1.1) with initial data $\overline{D_0}$. If $G \subset Z = \mathbf{R}^N \times [0,T)$ is a set-theoretic subsolution in $\mathcal{O} = \mathbf{R}^N \times (0,T)$ with $\overline{G}(0) = \overline{D_0}$ and a set-theoretic supersolution in $\mathcal{O}$ with int $G(0) = D_0$, then int $G = D$ and $\overline{G} = \overline{D}$ provided that $\overline{G}(t)$ and $(Z\backslash G)(t)$ are uniform upper semicontinuous at time zero. In particular, an open set-theoretic solution with initial data $D_0$ is unique.*

We characterize a level set solution (open in $Z$) by smallest supersolution with the same initial data.

**Theorem 5.2.11.** *Assume that (CPS) holds. For given an open (resp. closed) set $D_0$ (resp. $E_0$) in $\mathbf{R}^N$ let $S$ be the set of all open (resp. closed) set $G$ in $Z = \mathbf{R}^N \times [0,T)$ that satisfies*

*(i) $D_0 \subset G(0)$ (resp. $G(0) \subset E(0)$).*

*(ii) $G$ is a set-theoretic supersolution (resp. subsolution) of (5.1.1) in $\mathcal{O}$.*

*(iii) $(Z\backslash G)(t)$ (resp. $G(t)$) is uniformly upper semicontinuous at $t = 0$.*

*Let $D$ (resp. $E$) be the level set solution with initial data $D_0$ (resp. $E_0$). Then*

$$D = \bigcap\{G;\ G \in S\}$$

$$(\text{resp.}\qquad E = \bigcup\{G;\ G \in S\}.)$$

*If $\partial D_0$ (resp. $\partial E_0$) is compact, the condition (iii) is unnecessary.*

Except the last statement this follows from the comparison principle (Theorem 5.2.5) and $D \in S$ (resp. $E \in S$). If $\partial D_0$ is bounded, as observed in Chapter 4, there always exists an open set-theoretic supersolution $\hat{G}$ with bounded $\partial\hat{G}$ that satisfies $\hat{G}(0) \supset D_0$. For given open $G$ satisfying (i), (ii), the set $G \cap \hat{G}$ is a set-theoretic supersolution by the closedness under inf operation (CL). Since $\partial(G \cap \hat{G})$ is now bounded, $G \cap \hat{G}$ fulfills (iii) as well as (i), (ii). Thus $\cap\{G; G \in S\}$ is the same as a set even if (iii) in the definition of $S$ is not assumed. The proof for $E$ is symmetric so is omitted.

**Remark 5.2.12.** In Example 5.2.1 we gave three set-theoretic solutions of the curve shortening equation in $\mathbf{R}^2$ with the same initial data $D_0$. It turns out that the (open) level set solution with initial data $D_0$ is $D_2$ while the (closed) level set solution with initial data $\overline{D_0}$ is $\overline{D_1}$. Indeed, if we set

$$E_j = \{(x,y,t) \in \mathbf{R}^2 \times [0,\infty);\ |x - j^{-1}| > h(|y|,\ t + j^{-1})\},$$

then $\{E_j\}$ satisfies all properties of Definition 5.2.2 with $D = D_2$ since $\{\partial E_j(t)\}$ is a classical solution of the curve shortening equation so that it is a set-theoretic subsolution by Corollary 5.1.5. The uniform upper semicontinuity of the complement of $D_2$ is easy to check so $D_2$ is the level set solution with initial data $D_0$. The proof for $\overline{D_1}$ is similar. Theorem 5.2.11 says that $D_2$ is the minimal supersolution with $\overline{D_1}(0) \subset \overline{D_0}$ (satisfying the uniformly upper semicontinuity at time zero).

## 5.3 Barrier solutions

We shall discuss a relation of set-theoretic solutions to evolutions avoiding smooth evolutions.

**Definition 5.3.1.** Let $J$ be an interval in $[0, T)$. For an interval $I$ let $\mathcal{E}_I$ be a family of sets $G$ in $\mathbf{R}^N \times I$. For $T > 0$ we say that a set $B$ in $\mathcal{O} = \mathbf{R}^N \times J$ is a *barrier supersolution* in $\mathcal{O}$ associated to $\mathcal{E} = \{\mathcal{E}_I;\ I \subset J\}$ if $G \in \mathcal{E}_I$ with $I = [t_0, t_1) \subset J$ and $G(t_0) \subset B(t_0)$ always fulfills $G \subset B \cap (\mathbf{R}^N \times I)$. The set of all barrier supersolutions associated to $\mathcal{E}$ denotes Barr $(\mathcal{E})$.

By definition a restriction $G \cap \mathbf{R}^N \times J'$ for $J' \subset J$ is always a barrier supersolution in $\mathbf{R}^N \times J'$ if $G \in$ Barr $(\mathcal{E})$ in $\mathbf{R}^N \times J$.

For the surface evolution equation (5.1.1) there are several reasonable choices of $\mathcal{E}_I$. For example

$$
\begin{aligned}
\mathcal{E}_I^- = \{G;\ & G \text{ is a bounded closed set } \mathbf{R}^N \times I \text{ and } \{\partial G(t)\}_{t\in I} \text{ is a } C^{2,1} \text{ evolving} \\
& \text{hypersurface satisfying } V \le f(z,\ \mathbf{n},\ \nabla\mathbf{n}) \quad \text{on} \quad \partial G(t) \quad \text{for} \quad t \in \operatorname{Int} I\}, \\
\mathcal{E}_I = \{G \in & \mathcal{E}_I^-;\ V = f(z,\ \mathbf{n},\ \nabla\mathbf{n}) \quad \text{on} \quad \partial G(t) \quad \text{for} \quad t \in \operatorname{Int} I\}, \\
\mathcal{E}_I^\infty = \{G \in & \mathcal{E}_I;\ \{\partial G(t)\}_{t\in I} \quad \text{is a smooth evolving hypersurface}\}.
\end{aligned}
$$

By definition

$$\mathrm{Barr}(\mathcal{E}^-) \subset \mathrm{Barr}(\mathcal{E}) \subset \mathrm{Barr}(\mathcal{E}^\infty)$$

with $\mathcal{E}^- = \cup\mathcal{E}_I^-,\ \mathcal{E} = \cup\mathcal{E}_I,\ \mathcal{E}^\infty = \cup\mathcal{E}_I^\infty$. In many cases all these class becomes identical although in general the above inclusions may be strict. For general purpose the set Barr $(\mathcal{E}^-)$ is most useful so we mainly discuss the class Barr $(\mathcal{E}^-)$. We shall compare the notion of a barrier supersolution with a set-theoretic supersolution.

**Proposition 5.3.2.** *Assume (f1) concerning $f$ in (5.1.1). Let $J$ be an open interval in $(0, T)$. Let $B$ be an open set in $\mathcal{O} = \mathbf{R}^N \times J$.*

*(i) Assume (f2) concerning $f$ in (5.1.1). Assume that the comparison principle (CPS $t_0$) holds for (5.1.1) with $0 \le t_0 < T$. Then $B$ is a barrier supersolution in $\mathcal{O}$ associated to $\mathcal{E}^-$ if $B$ is a set-theoretic supersolution of (5.1.1) in $\mathcal{O}$ with the property that $(\mathcal{O}\backslash B)(t)$ is uniformly right upper semicontinuous on $J$, i.e.,*

$$\lim_{s\downarrow t}\ \sup\{\mathrm{dist}\ (x,\ (\mathcal{O}\backslash B)(t));\ x \notin B(s)\} = 0 \quad \text{for } t \in J$$

*(ii) If $B$ is a barrier supersolution in $\mathcal{O}$ associated to $\mathcal{E}^-$, then $B$ is a set-theoretic supersolution of (5.1.1) in $\mathcal{O}$. Assume that $f$ is uniformly continuous on $\mathbf{R}^N \times [0, T] \times K$ for*

*every compact set $K$ in $\mathbf{E}$. Then $(\mathcal{O}\backslash B)(t)$ is uniformly right upper semicontinuous at all $t \in J$. If $B$ is a barrier supersolution in $Z = \mathbf{R}^N \times [t_0, t_1]$, $(t_0, t_1) \subset [0, T)$ then $(Z\backslash B)(t)$ is uniformly right upper semicontinuous at all $t, t_0 \le t < t_1$ including $t = t_0$.*

It turns out that the comparison principle is unnecessary to prove the equivalence of barrier supersolutions and set-theoretic supersolution.

**Theorem 5.3.3.** *Assume (f1) and (f2) concerning $f$ in (5.1.1). An open set $B$ in $\mathcal{O} = \mathbf{R}^N \times J$ is a barrier supersolution associated to $\mathcal{E}^-$ if and only if $B$ is a set-theoretic supersolution of (5.1.1) in $\mathcal{O}$, where $J$ is an open interval in $(0, T)$.*

**Corollary 5.3.4.** *Assume (f1) and (CPS) and the uniform continuity of $f$ in Proposition 5.3.2 (ii). Let $D_0$ be an open set in $\mathbf{R}^N$ and let $D \subset Z = \mathbf{R}^N \times [0, T)$ be an open level set solution of (5.1.1) with initial data $D(0) = D_0$. Then*

$$D = \bigcap \{B : B \in \mathrm{Barr}(\mathcal{E}^-) \quad \text{and } B \text{ is open in } Z = \mathbf{R}^N \times [0, T) \text{ with} \quad B(0) \supset D_0\}.$$

The last indentity gives a characterization of a level set solution by minimal barrier containing $D_0$ at time zero. Since $D = \cap\{G;\ G \in S\}$ by Theorem 5.2.11, Corollary 5.3.4 follows from Proposition 5.3.2 and Theorem 5.3.3.

**Remark 5.3.5.** The condition of openness of $B$ in Theorem 5.3.3 and Corollary 5.3.4 is unnecessary thanks to the next lemma.

**Lemma 5.3.6.** *Assume (f1) concerning $f$ in (5.1.1). Let $B$ be a set in $\mathbf{R}^N \times J$, where $J$ is an interval in $[0, T)$. Then int $B \in$ Barr $(\mathcal{E}^-)$ if and only if $B \in$ Barr $(\mathcal{E}^-)$.*

We postpone the proof of Theorem 5.3.3 and Lemma 5.3.6 in §5.4.2.

*Proof of Proposition 5.3.2.* (i) Assume that $B$ is a set-theoretic supersolution of (5.1.1) with the right uniform upper semicontinuity. Let $G$ be in $\mathcal{E}_I^-$ with $[t_0, t_1) \subset J$. Since $G$ is a classical subsolution, it is a set-theoretic subsolution of (5.1.1) by Corollary 5.1.5; here the degenerate parabolicity of (5.1.1) is invoked. Since $G$ is bounded, $G(t)$ is right uniformly upper semicontinuous. By (CPS) if $G(t_0) \subset B(t_0)$, then $G(t) \subset B(t)$ for all $t \in I$. Thus $B$ is a barrier supersolution associated to $\mathcal{E}^-$.

(ii) Assume now that $B \in$ Barr $(\mathcal{E}^-)$. We shall prove that right uniform upper semitontinuity of $B(t)$. We may assume $J = [0, T)$. We recall existence of fundamental subsolutions which is proved essentially in Lemma 4.3.3 as a level set of $v^-$; note that condition (4.3.1) is unnecessary as pointed out in Remark 4.3.7. Here we use the degenerate ellipticity (f2) of $f$. Note that the uniform continuity of $f$ in $x$ is invoked to guarantee that $\eta$ is independent of $(x_0, t_0)$.

**(F) Fundamental subsolution** For sufficiently small $r > 0$, say $r < r_0$, there is $\eta = \eta(r) > 0$ such that for each $(x_0, t_0) \in Z = \mathbf{R}^N \times [0, T)$ there exists $G \in \mathcal{E}_I^-$ for

$I = [t_0,\ t_0 + \eta)$ with $G(t_0) = B_r(x_0)$ (the closed ball of radius $r$ centered at $x_0$) that satisfies $x_0 \in G(t) \subset B_r(x_0)$ for all $t \in I$.

Suppose that $(Z\backslash B)(t)$ were not uniformly right upper semicontinuous at $t_0 \in [0, T)$. Then there would exist $\varepsilon_0 > 0$ and a sequence $\{(x_j,\ t_j)\}_{j=1}^{\infty}$ with $x_j \notin B(t_j)$ $(t_j \downarrow t_0)$ that satisfies dist $(x_j,\ (Z\backslash B)(t_0)) \geq \varepsilon_0$ for large $j$, say $j \geq j_0$. We may assume that $\varepsilon_0 < r_0$. By (F) we take $r = \varepsilon_0$ and $G_j \in \mathcal{E}_I^-$ for $I = [t_0,\ t_0 + \eta)$ with $G(t_0) = B_r(x_j)$ so that $G_j(t) \ni x_j$ for $t \in I$. Since $G_j(t_0) \subset B(t_0)$ for $j \geq j_0$ and $B \in$ Barr $(\mathcal{E}^-)$, $G_j(t) \subset B(t)$ for $t \in I$. This contradicts $x_j \notin B(t_j)$. Thus $(Z\backslash B)(t)$ is uniformly right upper semicontinuous at each $t \in [0, T)$.

We shall prove that $B$ is a set-theoretic supersolution in $\mathcal{O} = \mathbf{R}^N \times J$ of (5.1.1), where $J$ is an open interval in $(0, T)$. We shall check the criterion given in Theorem 5.1.2 with Remark 5.1.4 (i). The left accessibility of $\mathcal{O}\backslash B$ follows from (F). Indeed, if not, there would exist a point $(x_0, t_0) \in \mathcal{O}\backslash B$ such that $B_r(x_0) \subset B(t)$ for $t_0 - \delta \leq t \leq t_0$ with some $\delta > 0$, $r > 0$. We may assume that $r < r_0$ and $\delta < \eta = \eta(r)$ for $r_0$, $\eta$ given in (F). By (F) there is $G \in \mathcal{E}_I^-$, $I = [t_0 - \delta,\ t_0 - \delta + \eta)$ that satisfies $G(t_0) = B_r(x_0)$, $G(t) \ni x_0$ for all $t \in I$. Since $B$ is a barrier supersolution with $B_r(x_0) \subset B(t_0 - \delta)$, we see $G(t) \subset B(t)$ for $t \in I$. Since $t_0 - \delta + \eta > t_0$, this contradicts $x_0 \notin B(t_0)$. Note that we only need a weaker version of (F) where $\eta$ may depends $x_0$. Since this version only needs the continuity (f1) and (f2) of $f$, we do not need the uniform continuity of $f$ to prove the left accessibility.

It remains to check a version of definition of supersolution corresponding to Theorem 5.1.2 (i). Let $\{S_t\}$ be a smoothly evolving hypersurface around $(x_0, t_0)$ and $\{S_t\}$ has only intersection with $(\mathcal{O}\backslash B)(t)$ at $x_0 \in \partial(\mathcal{O}\backslash B)(t_0)$ around $(x_0, t_0)$. By extending $\{S_t\}$ we may assume that there is a bounded closed set $G$ in $B \cap (\mathbf{R}^N \times I) \cup \{(x_0, t_0)\}$ with the property that $\partial G(t) = S_t$ is a smoothly evolving hypersurface for some small interval $I = [t_0 - \delta,\ t_0 + \delta)$, $\delta > 0$. Assume that $B$ does not satisfy the supersolution version of Theorem 5.1.2 (i). Then there would exist $\{S_t\}$ and $(x_0, t_0)$ satisfying above properties with

$$V_{S_t} < f(x,\ t,\ \mathbf{n}_{S_t},\ \nabla \mathbf{n}_{S_t}) \quad \text{at} \quad x = x_0,\ t = t_0.$$

By continuity of $f$ and $\{S_t\}$ we may assume for some $\varepsilon > 0$

$$V_{S_t} \leq f(x,\ t,\ \mathbf{n}_{S_t},\ \nabla \mathbf{n}_{S_t}) - \varepsilon \quad \text{on} \quad S_t \cap B_r(x_0)$$

for $t \in I = [t_0 - \delta,\ t_0 + \delta)$ by taking $\delta$ smaller if necessarily. We shall modify $S_t$ so that this type of inequality holds for on whole $S_t$ not necessarily near $x_0$. As we see in the next lemma, there is a closed set $\hat{G} \in \mathcal{E}_I^-$ such that $\hat{G}(t_0 - \delta) = G(t_0 - \delta)$ and $\hat{G} = G$ neighborhood of $(x_0, t_0)$ in $\mathcal{O}$ by taking $\delta$ smaller. Since $B \in$ Barr $(\mathcal{E}^-)$ and $\hat{G}(t_0 - \delta) \subset B(t_0 - \delta)$, we see $\hat{G}(t_0) \subset B(t_0)$. Since $\hat{G}(t_0) \ni x_0$ and $B$ is open, this conntradicts the choice of $x_0 : x_0 \notin B(t_0)$.

**Lemma 5.3.7.** (Modification) *Assume the continuity (f1) of $f$ in (5.1.1). Let $G$ be compact set in $\mathbf{R}^N \times I$ with $I = [t_0 - \delta,\ t_0 + \delta] \subset (0, T)$ for some $\delta > 0$. Assume that $S_t = (\partial G)(t)$ is a smoothly evolving hypersurface on $I$. If there is $\eta > 0$ that satisfies*

$$V_{S_t} \leq f(x,\ t,\ \mathbf{n}_{S_t},\ \nabla \mathbf{n}_{S_t}) - \eta \text{ on } S_t \tag{5.3.1}$$

*for all $x \in B_\delta(x_0),\ t \in I$, then there is $\delta_1 > 0$ ($\delta_1 < \delta$) and a compact set $\hat{G}$ in $\mathbf{R}^N \times J$ with $J = [t_0 - \delta_1,\ t_0 + \delta_1]$ with the property that $\hat{G}(t_0 - \delta_1) = G(t_0 - \delta_1)$ and $\hat{G} = G$ near $(x_0, t_0)$ and that $\hat{G} \in \mathcal{E}_J^-$.*

*Proof.* For $G$ there is a function $u \in C(\mathbf{R}^N \times I)$ such that

$$G = \{(x,t) \in \mathbf{R}^N \times I;\ u(x,t) \geq 0\}$$

and $u$ is smooth near $\partial G$ with $\nabla u \neq 0$. By taking $\delta > 0$ smaller we may assume that $G \subset U \times I$ with a bounded open set $U$ containing $B_\delta(x_0)$ and that $u \in C^\infty(U \times I)$ and $\nabla u \neq 0$ on $U \times I$. We rewrite our assumption (5.3.1) to get

$$u_t + F(z,\ \nabla u,\ \nabla^2 u) \leq -\eta|\nabla u| \quad \text{in} \quad B_\delta(x_0) \times I \cap \partial G.$$

Since $|\nabla u| > 0$ in $B_\delta(x_0) \times I$ and $F$ is continuous, there is a constant $\lambda > 0$ that satisfies

$$u_t + F(z,\ \nabla u,\ \nabla^2 u) \leq -\lambda \quad \text{in} \quad B_\delta(x_0) \times I \tag{5.3.2}$$

by taking $\delta$ smaller if necessary. We set

$$v(x,t) = u(x,t) - \mu\alpha(x)(t - t_1).$$

Here $\alpha \in C^\infty(\mathbf{R}^N)$ is taken so that $\alpha \equiv 1$ on $\mathbf{R}^N \backslash B_\delta(x_0)$ and $\alpha \equiv 0$ on $B_{\delta/2}(x_0)$ with $\alpha \geq 0$. The constant $t_1 \in (t_0 - \delta,\ t_0)$ and $\mu > 0$ are to be determined later. Since $U$ is bounded we take $\mu$ large enough so that

$$u_t + F(z,\ \nabla u,\ \nabla^2 u) \leq \mu \quad \text{in} \quad U \times I.$$

Since $\alpha \equiv 1$ on $\mathbf{R}^N \backslash B_\delta(x_0)$, this choice of $\mu$ yields

$$v_t + F(z,\ \nabla v,\ \nabla^2 v) \leq 0 \tag{5.3.3}$$

for $x \in \mathbf{R}^N \backslash B_\delta(x_0),\ t \in I$. Since $\alpha \equiv 0$ on $B_{\delta/2}(x_0)$, (5.3.3) is also valid for $x \in B_\delta(x_0)$ by (5.3.2). Taking $t_1 = t_0 - \delta_1$ close to $t_0$ we obtain (5.3.3) for $x \in B_\delta(x_0) \backslash B_{\delta/2}(x_0)$ for $t \in [t_1,\ t_0 + \delta_1)$ since (5.3.2) holds. We may assume that $\nabla v \neq 0$ on $U \times J,\ J = [t_1,\ t_0 + \delta_1]$ and that the zero level set is contained in $U$ by taking $\delta_1$ smaller. We now set

$$\hat{G} = \{(x,t) \in U \times J;\ v(x,t) \geq 0\}.$$

and observe that $\hat{G} \in \mathcal{E}_J^-$ by (5.3.3). From definition of $v$ it follows that $\hat{G} = G$ near $(x_0, t_0)$ and $\hat{G}(t_1) = G(t_1)$.

**Remark 5.3.8.** (Choice of $\mathcal{E}$) If the comparison principle holds for smooth sub- and supersolutions of (5.1.1) and the local smooth solution exists for every $C^\infty$ initial closed hypersurface (whose existence time interval may depend on smoothness of initial surface), then Barr ($\mathcal{E}^-$) = Barr ($\mathcal{E}^\infty$). Indeed for $B \notin$ Barr ($\mathcal{E}^-$) with $B \in$ Barr ($\mathcal{E}^\infty$) there is $G \in \mathcal{E}_I^-$ with some $I = [t_0,\ t_1)$ that satisfies $G(t_0) \subset B(t_0)$ but $G(t)\backslash B(t) \neq \emptyset$ for some $t,\ t_0 < t < t_1$. We may assume that $\{\partial G(t)\}$ is a smooth evolving hypersurface by approximation. Since $\overline{G}$ is compact, we see

$$t_* = \inf\{t \in I;\ G(t)\backslash B(t) \neq \phi\} > t_0.$$

By the local existence of solutions for each $\delta \geq 0$ there is $\eta > 0$ and $W^\delta \in \mathcal{E}_J^\infty$ with $J = [t_* - \delta,\ t_* - \delta + \eta)(\subset I)$ and $W(t_* - \delta) = G(t_* - \delta)$. By uniform regularity of $G(t)$ in $t$ we may assume that $\eta$ is independent of $\delta$ and take $\delta > 0$ so that $\eta - \delta > 0$. By comparison of smooth sub- and supersolutions, we see

$$G(t) \subset W^\delta(t) \quad \text{for} \quad t \in J.$$

Since $W^\delta(t_* - \delta) = G(t_* - \delta) \subset B(t - \delta)$ and $B \in$ Barr ($\mathcal{E}^\infty$) it follows that $G(t) \subset W^\delta(t) \subset B(t)$ for $t \in J$. This contradicts the definition of $t_*$ since $\eta - \delta > 0$.

**Remark 5.3.9.** (Subsolutions) There is of course a notion of barriers subsolution corresponding to subsolutions. Let $\mathcal{E}_I^+$ be the set defined by

$$\begin{aligned}\mathcal{E}_I^+ = \ & \{U \subset \mathbf{R}^N \times I; \quad U \quad \text{is a bounded open set and} \\ & \{\partial U(t)\}_{t\in I} \quad \text{is a } C^{2,1} \text{ evolving} \\ & \text{hypersurface satisfying} \\ & \qquad V \geq f(z,\ \mathbf{n},\ \nabla \mathbf{n}) \quad \text{on} \quad \partial U(t)\},\end{aligned}$$

where $\mathbf{n}$ is taken outward from $U(t)$. For an interval $J$ in $[0, T)$ a set $B$ in $\mathbf{R}^N \times J$ is called a barrier subsolution associated to $\mathcal{E}^+ = \{\mathcal{E}_I^+; I \subset J\}$ if $U \in \mathcal{E}_I^+$ with $I = [t_0, t_1) \subset J$ and $U(t_0) \supset B(t_0)$ always fulfills $U \supset B \cap (\mathbf{R}^N \times I)$. The set of all barrier subsolution associated to $\mathcal{E}^+$ denotes barr ($\mathcal{E}^+$). It is easy to obtain statements as in Proposition 5.3.2–Lemma 5.3.6 for barrier subsolutions by argument symmetric to the case of supersolutions. For example, the statement corresponding to Theorem 5.3.3 reads: a closed set $B$ in $\mathcal{O} = \mathbf{R}^N \times J$ ($J$: open interval) is a barrier subsolution if and only if $B$ is a set-theoretic subsolution of (5.1.1) in $\mathcal{O}$ under the assumptions (f1) and (f2). The statement corresponding to Lemma 5.3.6 reads: $B \in$ barr ($\mathcal{E}^+$) if and only if $\overline{B} \in$ barr ($\mathcal{E}^+$) under the assumption of (f1) and (f2). The right uniform upper semicontinuity of $\overline{B}$ follows if $B \in$ Barr ($\mathcal{E}^-$) provided that $f$ satisfies the uniform continuity in Proposition 5.3.2 (ii).

## 5.4 Consistency

We shall prove that our notion of a level set subsolution is consistent with classical subsolutions. As we observed in Corollary 5.1.5 a classical subsolution is always a set-theoretic subsolution so the question is whether or not there is an approximate family of set-theoretic subsolutions in Definition 5.2.2. We shall also prove Theorem 5.3.3 and Lemma 5.3.6.

### 5.4.1 Nested family of subsolutions

**Theorem 5.4.1.** *Assume the continuity (f1) of $f$ in (5.1.1). Let $F$ be as in (5.1.2). Let $G$ be in $\mathcal{E}_I^-$ with a closed interval $I \subset [0,T)$. Then there is a tubular neighborhood $W$ of $\partial G$ in $\mathbf{R}^N \times I$ and $u_1 \in C^{2,1}(W\backslash G) \cap C^1(W\backslash \text{int } G)$ (resp. $u_2 \in C^{2,1}$ (int $G \cap W$) $\cap C^1(G \cap W)$) that satisfies*

$$u_t + F(z,\ \nabla u,\ \nabla\nabla u) \leq 0 \quad \text{in} \quad W\backslash G \quad (\text{resp.} \quad \text{int } G \cap W) \tag{5.4.1}$$

*with $u_1 < 0$ in $W\backslash G$, $u_1 = 0$ on $\partial G$ (resp. $u_2 > 0$ in int $G \cap W, u_2 = 0$ on $\partial G$) and that $\nabla u_1 \neq 0$ in $W\backslash$ int $G$ (resp. $\nabla u_2 \neq 0$ in $G \cap W$.) The inequality in (5.4.1) may be replaced by the strict inequality.*

As level sets of $u_1$ or $u_2$ there is a nested family of classical subsolutions $G_s = \{(x,t) \in W;\ u_i(x,t) \geq s\}$ approximating $G$ from inside or outside of $G$.

**Theorem 5.4.2.** *Assume the same hypothesis of Theorem 5.4.1 concerning $f$ and $G$. Then there is a nested family $\{G_s\}$ in $\mathcal{E}_I^-$ for $s \in (-\delta,\ \delta)$ with some $\delta > 0$ with the property that*

*(i) $G_0 = G$, $G_s \subset$ int $G_\tau$ if $\tau < s$, $\tau, s \in (-\delta,\ \delta)$.*

*(ii) For each $\varepsilon > 0$ there is a $\delta_0 > 0$ that satisfies*

$$\begin{aligned}
&\sup\{\text{dist }(x,\ G(t));\ x \in G_s(t),\ 0 \leq s \leq \delta_0,\ t \in I\} < \varepsilon,\\
&\sup\{\text{dist }(x, (W\backslash \text{ int } G)(t));\ x \in G_s(t),\ -\delta_0 \leq s \leq 0,\ t \in I\} < \varepsilon.
\end{aligned}$$

*(iii) The strict inequality*

$$V < f(z, \mathbf{n}, \nabla\mathbf{n})$$

*is fulfilled on $\partial G_s$ if $s \neq 0$.*

*Proof of Theorem 5.4.1.* We first remark that the construction of $u_1$ satisfying (5.4.1) is very easy if $f$ is independent of the spatial variable $x$. In this case we take $u_1 = \text{sd} \wedge 0$ with $\text{sd}(x,t) = \text{sd}(x, G(t))$ and observe by Theorem 5.1.7 that $u_1$ is a subsolution of (5.1.2) in outside $G$. Since $u_1 \in C^{2,1}(W\backslash G)$ by regularity of $G$ (see. e.g. D. Gilbarg and N. Trudinger (1998)) for some tubular neighborhood $W$ of $\partial G$, $u_1$ is a function satisfying (5.4.1) in $W\backslash G$ in a classical sense. Unfortunately, if $f$ depends on $x$, $u_1$ is no longer a subsolution of (5.1.2); it is a subsolution of (5.1.4). We need to use another argument to construct $u_1$ for the general case as well as to construct $u_2$.

Since $\{\partial G(t)\}$ is a $C^{2,1}$ evolving hypersurface on $I$, there are a tubular neighborhood $W$ of $\partial G$ and $\varphi \in C^{2,1}(W)$ that satisfies $\nabla\varphi \neq 0$ in $\overline{W}$ with

$$G \cap W = \{(x,t) \in W;\ \varphi(x,t) \geq 0\}.$$

Since $G \in \mathcal{E}_I^-$, $\varphi$ satisfies

$$\varphi_t + F(z,\ \nabla\varphi,\ \nabla^2\varphi) \le 0 \quad \text{on} \quad \partial G.$$

Since we may assume that $\overline{W}$ is compact in $\mathbf{R}^N \times I$ and $\nabla\varphi \neq 0$ in $\overline{W}$,

$$\omega_0(\sigma) = \sup\{(\varphi_t + F(z,\ \nabla\varphi,\ \nabla^2\varphi))(x,t),\ (x,t) \in W,\ |\varphi(x,t)|\ \le \sigma\}$$

is nondecreasing and $\omega_0(\sigma) \to 0$ as $\sigma \to 0$. By Lemma 2.1.9 there is a modulus $\omega \in C^\infty(0,\ \infty) \cap C^0\ [0,\infty)$ with $\omega_0 \le \omega$ that satisfies

$$\varphi_t + F(z,\ \nabla\varphi,\ \nabla^2\varphi) \le \omega(|\varphi|) \quad \text{in} \quad W. \tag{5.4.2}$$

We may assume that $\omega(\sigma) \ge \sigma$. If we set

$$\theta(s) = \exp\left(-\int_s^1 \frac{d\sigma}{\omega(\sigma)}\right),\ \ s > 0$$

then $\theta \in C^1[0,\infty) \cap C^\infty(0,\infty)$ with $\theta'(s) > 0$ for $s > 0$. Since $F$ in (5.4.2) is geometric, from (5.4.2) it follows that the function $w = -\theta(-\varphi)(\in C^1(W\backslash \mathrm{int}\ G) \cap C^{2,1}(W\backslash G))$ solves

$$w_t + F(z,\ \nabla w,\ \nabla^2 w) \le \theta'(-\varphi)\omega(|\varphi|) \quad \text{in} \quad W\backslash G. \tag{5.4.3}$$

By the choice of $\theta$ we see

$$\theta'(-\varphi)\omega(|\varphi|) = -w \quad \text{in} \quad W\backslash \mathrm{int}\ G.$$

From (5.4.3) the function $u_1 = e^{\lambda t} w$ with $\lambda > 1$ solves

$$u_t + F(z,\ \nabla u,\ \nabla^2 u) \le e^{\lambda t}(\lambda - 1)w < 0 \quad \text{in} \quad W\backslash\ \mathrm{int}\ G.$$

Since the construction of $u_2$ is similar, so is omitted.

**Remark 5.4.3** Of course, statements similar to Theorems 5.4.1 and 5.4.2 holds for classical supersolutions although we do not state them here explicity.

## 5.4.2 Applications

**Theorem 5.4.4.** *Assume the continuity (f1) and the degenerate ellipticity (f2) of $f$ in (5.1.1). If $G \in \mathcal{E}_I^-$ with an interval $I = [t_0, t_1) \subset [0,T)$, then $G$ is a level set subsolution of (5.1.1) in $\mathbf{R}^N \times [t_0, t_1)$.*

This follows from Corollary 5.1.5 and Theorem 5.4.2 since $G$ is compact. Applying Corollary 5.2.10 we have:

**Corollary 5.4.5.** *Assume (f1) and (f2) concerning $f$ in (5.1.1). Let $G$ be a subset of $Z = \mathbf{R}^N \times [0,T)$. Assume that $G(t)$ is bounded. If $\partial G$ is a smoothly evolving hypersurface* *in* $[0, T)$ *and fulfills (5.1.1) on* $\partial G$ *for* $t > 0$, *where* $\mathbf{n}$ *directs outward to* $G$. *Then int* $G$ *and* $\overline{G}$ *are the level set solution with initial data* $G(0)$ *and* $\overline{G}(0)$, *respectively. In particular, for initial data* $G(0)$ *and* $\overline{G}(0)$ *an open and closed set-theoretic solution is unique, respcetively.*

*Proof of Lemma 5.3.6.* Suppose that int $B$ does not belong to Barr $(\mathcal{E}^-)$. Then there would exist $I = [t_0,\ t_1)$ and $G \in \mathcal{E}_I^-$ that satisfies $G(t_0) \subset \text{int } B(t_0)$, $G(t) \subset B(t)$ for $t \in I$ but $G(t_*) \cap \partial B(t_*) \neq \emptyset$ at some $t_* \in (t_0,\ t_1)$. Since dist $(G(t_0),\ \mathbf{R}^N \setminus (\text{int } B)(t_0)) = d_0 > 0$, by Theorem 5.4.2 there is $G_s \in \mathcal{E}_I^-$ that satisfies int $G_s \supset G$ and $G_s(t_0) \subset \text{int } B(t_0)$. Since int $G_s \supset G$, we see $G_s(t_*)$ contains a point outside $B(t_*)$. This contradicts $B \in$ Barr $(\mathcal{E}^-)$.

Suppose that $B$ does not belong to Barr $(\mathcal{E}^-)$. Then there would exist $I = [t_0, t_1)$ and $G \in \mathcal{E}_I^-$ that satisfies $G(t_0) \subset B(t_0)$ but $G(t_2) \setminus B(t_2) \neq \emptyset$ for some $t_2 \in I$. As before by Theorem 5.4.2 there is $G_s \in \mathcal{E}_I^-$ that satisfies int $G \supset G_s$ and $G_s(t_0) \subset \text{int } B(t_0)$ and $G_s(t_2) \setminus (\text{int } B)(t_2) \neq \emptyset$ which contradicts int $B \in$ Barr $(\mathcal{E}^-)$. $\square$

As another application of existence of a nested family of subsolutions we prove that a supersolution is always a barrier supersolution without assuming comparison principle.

*Proof of Theorem 5.3.3.* Suppose that $B$ were not a barrier supersolution. There would exist $G \in \mathcal{E}_I^-$ with some $I = [t_0, t_1)$ in $(0, T)$ that satisfies $G(t_0) \subset B(t_0)$ but $G(t) \setminus B(t) \neq \emptyset$ for some $t \in I$. We may assume that $I = [t_0, t_1]$ by taking $t_1 (> t_0)$ smaller. Since $G$ is compact, the time

$$t_* = \inf\{t \in I;\ G(t) \setminus B(t) \neq \emptyset\} \quad (< T)$$

is strictly larger than $t_0$. By Theorem 5.4.2 there is $\tilde{G} \in \mathcal{E}_I^-$ that satisfies

$$\begin{aligned} \tilde{G}(t_0) &\subset B(t_0),\ (\text{int } \tilde{G})(t) \supset G \cap (\mathbf{R}^N \times I) \\ V &< f(x,\ \mathbf{n},\ \nabla \mathbf{n}) \quad \text{on} \quad \partial \tilde{G}. \end{aligned} \tag{5.4.4}$$

Since $\tilde{G}$ is compact, we see

$$t_0 < s_* = \inf\{t \in I;\ \tilde{G}(t) \setminus B(t) \neq \emptyset\} < t_*.$$

Since $\chi_B$ is a supersolution in $\mathbf{R}^N \times (0, s_*]$ as proved in Chapter 3, there is a set-theoretic characterization corresponding to Theorem 5.1.2 with Remark 5.1.4 (iv). Since $\tilde{G}(t)$ touches $\partial B(t)$ at first time at $s_*$, $\partial \tilde{G}(t)$ is regarded as a 'test surface' at $t = s_*$. Since $B$ is a supersolution we must have

$$V \geq f(z,\ \mathbf{n},\ \nabla \mathbf{n}) \quad \text{on } \partial \tilde{G}(s_*) \cap \partial B(s_*)$$

which evidently contradicts (5.4.4). The proof of the converse is included in the proof of Proposition 5.3.2 (ii).

### 5.4.3 Relation among various solutions

We have defined several notions of solutions of (5.1.1) for a set in $\mathbf{R}^N \times J$ with an interval $J$ in $[0, T)$. We summarize the relation of these sets. We first assume that $J$ is open in $(0, T)$.

(i) a barrier supersolution $\Rightarrow$ a set-theoretic supersolution. To prove this implication we have used the existence (F) of fundamental subsolutions and the modification (Lemma 5.3.7) as presented in the proof of Proposition 5.3.2 (ii).

(ii) a set-theoretic supersolution $\Rightarrow$ a barrier supersolution. To prove this implication we have used the existence of a nested family of smooth subsolutions (Theorem 5.4.2) as presented in the proof of Theorem 5.3.3 in §5.4.2. In both (i) and (ii) we first assume the set is open but thanks to Lemma 5.3.6 such an assumption turns to be unnecessary. In the proof of Lemma 5.3.6 in §5.4.2 we have again used the existence of a nested family of smooth subsolutions. Of course, both (i) and (ii) are still valid if a subsolution replaces a supersolution.

(iii) Both implications (i) and (ii) can be extended to various settings although we only state a few examples without detailed proofs. From the proof one may take $J$ a semi-closed interval $[t_0, t_1)$ in both (i) and (ii). Also (i) and (ii) apply to boundary value problems with appropriate modifications.

(iv) Perron's method (cf. §2.4) is easy to establish for barrier solutions without using results in §2.4.

**Theorem 5.4.6** *Let Barr $(\mathcal{E}^-)$ (resp. barr $(\mathcal{E}^+)$) be the set of barrier supersolutions (resp. subsolutions) associated to $\mathcal{E}^-$ (resp. $\mathcal{E}^+$) in $\mathcal{O} = \mathbf{R}^N \times J$ with an open interval in $(0, T)$.*

*(i) (Closedness) Let $S$ be a subset in Barr $(\mathcal{E}^-)$. Then $\cap B_{B \in S} \in$ Barr $(\mathcal{E}^-)$.*

*(ii) (Minimality) Assume that $U \in \mathcal{E}_I^+$ is always a barrier supersolution associated to $\mathcal{E}^-$ in $\mathbf{R}^N \times$ int $I$. Assume that $B_0 \in$ Barr $(\mathcal{E}^-)$ and $B_1 \in$ barr $(\mathcal{E}^+)$ fulfills $B_0 \supset B_1$. Then*

$$D = \cap\{B;\ B \in \mathrm{Barr}(\mathcal{E}^-),\ B_1 \subset B \subset B_0\}$$

*is a barrier sub- and supersolution in $\mathcal{O}$ associated to $\mathcal{E}^+$ and $\mathcal{E}^-$ respcetively.*

*Proof.* (i) is trivial by definition. By (i) to see (ii) it suffices to prove that $D \in$ barr $(\mathcal{E}^+)$. If not, there is a interval $I = [t_0, t_1) \subset J$ and $U \in \mathcal{E}_I^+$ that satisfies $U(t_0) \supset D(t_0)$ but $D(t) \backslash U(t) \neq \emptyset$ for some $t \in I, t > t_0$. Since $B_1 \in$ barr $(\mathcal{E}^+)$, $U(t) \supset B_1(t)$ for all $t \in I$. As in (i) it is easy to see that

$$\tilde{D} = (D \cap U \cap (\mathbf{R}^N \times I)) \cap (D \cap (\mathbf{R}^N \times (J \backslash I)))$$

belongs to Barr $(\mathcal{E}^-)$ since $U$ is a barrier supersolution. Since $U(t) \supset B_1(t)$ for all $t \in I$, $\tilde{D} \supset B_1$. This contradicts the minimality of $D$. □

The assumption that $U \in \mathcal{E}_I^+$ is always a barrier subsolution is fulfilled if (f2) in (5.1.1) is assumed. So this assumption gives some parabolicity of the problem. Using implication of (i), (ii) our Theorem 5.4.6 establishes the Perron's method for set-theoretic solutions under (f1) and (f2), which also can be proved by applying results in §2.4 for characteristic functions.

(v) As observed in (iv) the notion of barrier solutions simplify several properties of set-theoretic solutions. The reader might be curious whether there is a notion of barrier solution corresponding to viscosity solution of

$$u_t + F(z, u, \nabla u, \nabla\nabla u) = 0 \tag{5.4.5}$$

in $\mathcal{O} = \mathbf{R}^N \times (0, T)$. At least for continuous $F$ we propose to say that $u$ is a barrier supersolution if $v < u$ on $\partial_p Q$ always implies $v < u$ in $\overline{Q}$ whenever $v \in C^{2,1}(\overline{Q})$ is a $C^{2,1}$ subsolution of (5.4.5) in $Q$ and $Q$ is of form $Q = D \times I$ with a bounded domain $D$ in $\mathbf{R}^N$ and an open interval $I$ in $(0, T)$. Our notion is localized but we still have equivalence results as in (i) and (ii). We leave the details to the reader.

We next compare level set solutions with other solutions. Let $J$ be a semi-closed interval $[t_0, t_1)$ in $[0, T)$.

(vi) a level set subsolution $\Rightarrow$ a set-theoretic subsolution in $\mathbf{R}^N \times (t_0, t_1)$ for an open set $D$ in $\mathbf{R}^N \times [t_0, t_1)$. However, the converse may not be true as explained below.

(vii) a level set solution $D$ with (open) initial data $D_0$ at $t = t_0$ is a minimal open set-theoretic supersolution $G$ whose initial data $G(t_0)$ contains $D_0$ (at least when $\partial D_0$ is bounded and for general $D_0$ the uniform upper semicontinuity of $G$ at $t = t_0$ is assumed) (Theorem 5.2.11). This is based on the comparison principle for set-theoretic sub- and supersolutions. A level set solution with given initial data $D_0$ is unique while the set-theoretic solution may not be unique as explained in Example 5.2.1 and Remark 5.2.12. In particular the converse of (vi) is not always true.

(viii) a generalized open evolution with initial data $D_0 \Rightarrow$ a level set solution with the same initial data (Theorem 5.2.8). By uniqueness of generalized open evolution (based on the comparison principle for (5.1.2)) the converse is also true.

## 5.5 Separation and comparison principle

We give an alternate way to prove the comparison principle (BCPS) for set-theoretic solutions based on local existence of classical solutions of (5.1.1). The key ingradients are separation lemma and the equivalence of barrier solutions and set-theoretic solutions.

**Lemma 5.5.1 (Separation).** *Let $E_0$ be a compact set contained in an open set $D_0$ in $\mathbf{R}^N$. Then there is a bounded open set $U_0$ with $C^{1+1}$ boundary (i.e. the unit normal vector field is Lipschitz) such that*
(i) $E_0 \subset U_0$, $\overline{U}_0 \subset D_0$

(ii) $\mathrm{dist}(E_0, \partial U_0) + \mathrm{dist}(\overline{U}_0, \partial D_0) = \mathrm{dist}(E_0, \partial D_0)$.

This lemma is due to T. Ilmanen (1993a). We do not give the proof here.

A typical local existence results we need is
(LE) Let $U_0$ be a bounded open set in $\mathbf{R}^N$ with $C^\infty$ boundary, where the unit normal is taken outward from $U_0$. Then for a given $t_0 \in [0,T)$ there are $I = [t_0, t_1)$ and $U \in \mathcal{E}_I^- \cap \mathcal{E}_I^+$ with $U(t_0) = U_0$ for some $t_1 (> t_0)$ depending on initial data $U_0$ only through a bound for the second fundamental form of $\partial U_0$.

We present a version of the comparison principle when $f$ of (5.1.1) does not depend on the spatial variable.

**Theorem 5.5.2.** *Assume that $f$ in (5.1.1) satisfies (f1) and is independent of $x \in \mathbf{R}^N$. Assume that (LE) holds. Then (BCPS) holds.*

*Proof.* Let $E$ and $D$ be set-theoretic sub- and supersolutions of (5.1.1) in $\mathbf{R}^N \times (0,T)$. We may assume that $D$ and $E$ are bounded since the case that the complements of $D$ and $E$ are bounded can be handled in a similar way. Assume that $E_0 = E^*(0)$ is contained in $D_0 = D_*(0)$. Our goal is to prove $E^* \subset D_*$. We may assume that $E$ is closed and $D$ is open in $\mathbf{R}^N \times [0,T)$.

We shall prove that

$$\operatorname{dist}(E(t),\ D^c(t)) \geq \operatorname{dist}(E_0,\ D_0^c) \quad \text{for} \quad t \in (0,T)$$

where $c$ denotes the complements in $\mathcal{O} = \mathbf{R}^N \times (0,T)$. (This evidently implies that $E \subset D$ but it is actually equivalent to (BCPS) since (5.1.1) is translation invariant in space.) We set

$$t_* = \sup\{t \in [0,T);\ \operatorname{dist}(E(\tau), D^c(\tau)) \geq \operatorname{dist}(E_0, D_0^c) \quad \text{for all} \quad 0 \leq \tau \leq t\}.$$

By left accessibility of $E$ and $D^c$ at $t_*$

$$\operatorname{dist}(E(t_*), D^c(t_*)) \geq \operatorname{dist}(E_0, D_0^c).$$

Assume that $t_* < T$. Then by the separation (Lemma 5.5.1) there is an open set $U_0$ with $C^{1+1}$ boundary such that

(i) $E(t_*) \subset U_0,\ \overline{U}_0 \subset D(t_*)$

(ii) $\operatorname{dist}(E(t_*), \partial U_0) +$ $\operatorname{dist}(\overline{U}_0, D^c(t_*)) = \operatorname{dist}(E(t_*), D^c(t_*))$.

Unfortunately, $U_0$ is not quite smooth to apply (LE). Let $U_{0\varepsilon}$ be an open set with smooth boundary satisfying (i) with $U_0$ replaced by $U_{0\varepsilon}$ and

(ii)' $\operatorname{dist}(E(t_*), \partial U_{0\varepsilon}) +$ dist $(\overline{U}_{0\varepsilon}, D^c(t_*))$

$$\geq \operatorname{dist}\ (E(t_*), D^c(t_*)) - \varepsilon;$$

the principal curvatures of $U_{0\varepsilon}$ is arranged to be bounded uniformly for small $\varepsilon > 0$. By (LE) there is $U_\varepsilon \in \mathcal{E}_I^- \cap \mathcal{E}_I^+$ with $U_\varepsilon(t_*) = U_{0\varepsilon}$ for some $I = [t_*, t_1)$ with $t_1 > t_*$ independent of $\varepsilon$.

Now we recall that $E \in \operatorname{barr}(\mathcal{E}^+)$ and $D \in \operatorname{Barr}(\mathcal{E}^-)$ by Theorem 5.3.3 as subsets of $\mathbf{R}^N \times (0,T)$. Since we consider $E^*$ and $D_*$ as subsets of $\mathbf{R}^N \times [0,T)$, it is easy to see that

$E^* \in$ barr $(\mathcal{E}^+), D_* \in \mathrm{Barr}(\mathcal{E}^-)$ as subsets of $\mathbf{R}^N \times [0,T)$. By this observation one can compare $D$ with $\mathcal{E}_I^-$ for $I$ starting zero. By definition of barrier solutions and translation invariance we see

$$\begin{aligned}&\mathrm{dist}(E(t), \partial U_\varepsilon(t)) \geq \mathrm{dist}(E(t_*), \partial U_{0\varepsilon})\\&\mathrm{dist}(\overline{U}_\varepsilon(t), D^c(t)) \geq \mathrm{dist}(\overline{U}_{0\varepsilon}, D^c(t_*))\end{aligned}$$

for $t \in I$. This together with (ii)' yields

$$\mathrm{dist}(E(t), D^c(t)) \geq \mathrm{dist}(E(t_*), D^c(t_*)) - \varepsilon,$$

Since $I$ is independent of $\varepsilon$, this yields

$$\mathrm{dist}(E(t), D^c(t)) \geq \mathrm{dist}(E(t_*), D^c(t_*))$$

for $t \in [t_*, t_1)$. This contradicts the definition of $t_*$ so $t_* = T$; the proof is now complete. $\square$

**Remark 5.5.3.** (i) In various situations a local existence of solutions for (5.1.1) has been proved. For the mean curvature flow equation it has been proved by G. Huisken (1984). Existence time $t_1$ has been estimated by

$$t_1 \geq t_0 + c_N/M^2,$$

where $M$ is a bound of the second fundamental form and $c_N$ depends only on the dimension. Our theorem apparently does not assume parabolicity (f2) but to establish (LE) we need some parabolicity. The statement for nonparabolic equations is of little importance since there are little chance to have sub- and supersolutions.

(ii) Theorem 5.5.2 together with Lemma 5.1.9 yields (CP (ii)) for the level set equations without using comparison results in Chapter 2. Note that our local existence (LE) does not require uniqueness of smooth solutions.

## 5.6 Notes and comments

A set-theoretic interpretation of solutions was introduced by H. M. Soner (1993) for the anisotropic curvature flow equation with no explicit dependence on the spatial variables and the time variable. However, his definition is based on (signed) distance functions instead of characteristic functions used in Definition 5.1.1. Our definition seems new although characteristic functions are often used to describe typical properties of generalized evolutions for example in G. Barles, H.M.Soner and P. Souganidis (1993). It turns out that our definition agrees with the one by distance functions as stated in Theorem 5.1.7. For the mean curvatuer flow equation K. Ishii and H. M. Soner (1996) called a set satisfying Theorem 5.1.2 (i) a *weak solution*. They compared weak subsolutions with set-theoretic subsolutions based on distance functions. In fact, based on results by H. M. Soner (1993)

(§7, §14) they stated Corollary 5.1.8 for the mean curvature flow equation. Our Theorem 5.1.2 gives a geometric interpretation of our set-theoretic solutions. Although Theorem 5.1.2 follows from Theorem 5.1.7 once Corollary 5.1.8 is proved, it is not stated in the literature as far as the author knows. We give a direct proof without appealing distance functions.

**Consistency.** It is clear that smooth solution $\Gamma \subset \mathbf{R}^N \times [0, T_0)$ is a set-theoretic solution of (5.1.1) thanks to Corollary 5.1.5. However, it is not very obvious that $\Gamma$ is also a level set solution or generalized interface evolution since one has to prove that there is no fattening. The consistency of smooth solution (with smooth initial data) with an interface evolution has been proved first by L. C. Evans and J. Spruck (1991) for the mean curvature flow equation and later by Y. Giga and S. Goto (1992b) for general equations (5.1.1) satisfying (f1) and (f2) when $F$ is independent of $x$ and grows linealy in $\nabla\mathbf{n}$. The proof can be extended for all (5.1.1) satisfying (f1) and (f2) without further restrictions. The major task is to construct sub and supersolution of the level set equation whose zero level set is the evolving surface by using distance function. We do not present the detail.

Our characterization of solutions of level set equation (Theorem 5.1.6) is important to bridge the set-theoretic approach to the level set approach. The proof of 'only if' part is standard and this implication is often used in the literature, for example, in G. Barles, H. M. Soner and P. Souganidis (1993). However, the converse seems to be unfamiliar.

Our Theorem 5.1.7 characterizes set-theoretic solutions by distance functions. The proof of 'only if' part has been proved by G. Barles, H. M. Soner and P. Souganidis (1993) by representing distance function by sup convolutions. The proof given here is direct and does not appeal to sup convolutions. The converse is contained in Theorem 5.1.6 when $F$ in (5.1.4) is independent of the spatial variable. Otherwise we need a little bit extra work as mentioned in the proof.

It is clear that the comparison principle for level set equations implies the comparison principle for set-theoretic solutions. We here note the converse is also easy to prove. We have given two kind of such statements (Lemma 5.1.9 and 5.1.10) for bounded sets and general sets. It seems that there is no literature containing both lemmas.

Nonuniquenss of set-theoretic solutions of $V = k$ with a given initial data was first observed by L. C. Evans and J. Spruck (1991), where they described this phenomena as 'fattening' of level sets. Example 5.2.1 is taken from H. M. Soner (1993). Several criteria on initial data for uniqueness were given in G. Barles, H. M. Soner and P. Souganidis (1993), H. M. Soner and P. Souganidis (1993), S. Altschuler, S. Angenent and Y. Giga (1995) described in §4.5. The latter two works show that axisymmetric evolution for the mean curvature flow is regular. Examples of nonuniqueness were given for various equations by several authors including Y. Giga, S. Goto and H. Ishii (1992), G. Barles, H. M. Soner and P. Souganidis (1993), G. Bellettini and M. Paolini (1994) and Y. Giga (1995b) even if initial data has smooth boundary. For the mean curvature flow equations with $N \geq 3$, examples of nonuniqueness for smooth initial data is also proved by S. Angenent, T. Ilmanen and J. J. L. Valazquez for $4 \leq N \leq 7, N = 8$ and numerically conjectured by S. Angenent, D. L. Chopp and T. Ilmanen (1995).

The notion of level set solution is introduced to describe generalized evolution constructed in Chapter 4 from the set-theoretic point of view.

The notion of barrier solutions was first introduced by T. Ilmanen (1993a) and by E. De Giorgi (1990), and developed by G. Bellettini and M. Paolini (1995) and L. Ambrosio and H. M. Soner (1996). These works include the statement that minimal barrier is equivalent to generalized open evolution (Corollary 5.3.4) for several equations. (For general geometric equations see the work of G. Bellettini and M. Novaga (1998)). However, the equivalence of a barrier supersolution and a set-theoretic supersolution seems to be new. This characterization provides a method to prove the comparison results for level set equations via local existence of classical solutions of surface evolution equations.

The notion of barrier solutions is also useful to prove convergence of internal layers of the Allen-Cahn equation to the mean curvature flow. In fact, G. Barles and P. E. Souganidis (1998) introduced a kind of barrier solutions to solve such problems. In particular, they proved the global-in-time convergence for the homogeneous Neumann problem which was only known for convex domains (M. Katsoulakis, G. T. Kossioris and R. Reitich (1995)).

**Approximation:** There are several ways to approximate solutions of level set equations or generalized interface evolution itself. We list several related articles (which applies the mean curvature flow equation) without detailed explanation.

(i) **Finite difference approximations.** M. G. Crandall and P. L. Lions (1996) proves the convergence of their schemes (see a paper by K. Deckelnick (2000) for further development). The stability of more intuitive scheme has been proved in Y.-G. Chen, Y. Giga, T. Hitaka and M. Honma (1994).

(ii) **Bence-Merriman-Osher scheme.** This is a kind of filter. The convergence has been proved by G. Barles and C. Georgelin (1995) and independently by L. C. Evans (1993). It is further developed by H. Ishii (1995) and by H. Ishii, G. E. Pires and P. E. Souganidis (1999).

(iii) **A variational scheme.** This is poposed by F. Almgren, J. E. Taylor and L. Wang (1993) and by I. Fonceca and M. A. Katsoulakis (1995). For further development see a paper by S. Luckhaus and T. Sturzenhecker (1995).

(iv) **Approximation by interacting particle systems.** This relates the description of macroscopic model and microscopic models. It has been srudied by M. Katsoulakis and P. E. Souganidis (1994).

The reader is referred to a review by C. M. Elliott (1996) and a book by F. Cao (2002?) for other approximations. Note that above approximation works for generalized evolution not necessarily smooth.

**Regularity.** There are several attempts to prove some regularity of generalized interface evolutions for the mean curvature flow equation. For example, L. C. Evans and J. Spruck (1992b) proved that the evolution fulfills the clearing-out lemma which is important for the Brakke flow. As an application they estimate the extinction time from above by using the area of initial data. An estimate from below is given by Y. Giga and K. Yama-uchi (1993). They also prove that if a part of interface evolution is given as the graph of a continuous function, the part of evolution is represented a graph-like solution of the equation. Such a solution has be proved to be regular by them. The proof of this reduction is simplified by M.-H. Giga and Y. Giga (2001).

Furthermore, L. C. Evans and J. Spruck (1995) prove that almost every level set of solutions of the level set mean curvature flow equation is a Brakke type flow. This is recently generalized by Y. Tonegawa (2000) for anisotropic curvature flow equations. For monotone evolutions the size of the singular sets of generalized interface evolution $\Gamma$ is well-estimated. In fact, B. White (2000) proved that the singular set of $\Gamma \subset \mathbf{R}^N \times [0, \infty)$ has the parabolic Hausdorff dimension at most $N-2$. This is optimal since to a torus shrinking to a ring has $N-2$ dimensional singularity.

We do not mention application of the level set method. Instead we give here two new recent books on image processings based on a level set method. One is by F. Guichard and J. M. Morel (2001?) and one is by F. Cao (2002?).

# References


K. Ahara and N. Ishimura (1993), On the mean curvature flow of "thin" doughnuts, In: *Lect. Notes Numer. Appl. Anal.,* (eds. K. Masuda, M. Mimura and T. Nishida) Kinokuniya, Tokyo, **12**, 1-33.

A. D. Alexandrov (1956), Uniqueness theorems for surfaces in the large I, *Vestrila Leningrad Univ. Math.*, **11**, 5-17.

S. M. Allen and J. W. Cahn (1979), A macroscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, *Acta. Metall.,* **27**, 1085-1095.

F. Almgren, J. E. Taylor and L. Wang (1993), Curvature-driven flows: a variational approach, *SIAM. J. Control Optim.* **31**, 387-438.

S. Altschuler, S. Angenent and Y. Giga (1995), Mean curvature flow through singularities for surfaces of rotation, *J. Geom. Anal.,* **5**, 293-358.

L. Alvarez, F. Guichard, P.-L. Lions and J. -M. Morel (1993), Axioms and fundamental equations of image processing, *Arch. Rational Mech. Anal.,* **123**, 199-257.

L. Ambrosio and H. M. Soner (1996), Level set approach to mean curvature flow in arbitrary codimension, *J. Differential Geometry,* **43**, 693-737.

K. Anada (2001), Construction of surfaces by harmonic mean curvature flows and nonuniqueness of their self similar solutions, *Calc. Var. Partial Differential Equations*, **12**, 109-116.

B. Andrews (1994), Contraction of convex hypersurfaces in Euclidean space, *Calc. Var. Partial Differential Equations,* **2**, 151-171.

Ben Andrews (1996), Contraction of convex hypersurfaces by their affine normal, *J. Differential Geometry*, **43**, 207-230.

Ben Andrews (1998), Evolving convex curves, *Calc. Var. Partial Differential Equations*, **7**, 315-371.

Ben Andrews (2000), Motion of hypersurfaces by Gauss curvature, *Pacific J. of Math.*, **195**, 1-34.

Ben Andrews (2002?), Non-convergence and instability in the asymptotic behaviour of curves evolving by curvature, *Commun. Analysis and Geometry*, to appear.

S. B. Angenent (1992), Shrinking doughnuts, In: *Nonlinear diffusion equations and their equilibrium states,* (eds. N. G. Lloyd, W. M. Ni, L. A. Peletier and J. Serrin), *Birkhaüser, Boston-Basel-Berlin,* **3**, 21-38.

S. Angenent, D. L. Chopp and T. Ilmanen (1995), A computed example of nonuniqueness of mean curvature flow in $R^3$, *Comm. Partial Differential Equations,* **20**, 1937-1958.

S. B. Angenent and M. E. Gurtin (1989), Multiphase thermomechanics with interfacial structure 2., Evolution of an isothermal interface, *Arch. Rational. Mech. Anal.,* **108**, 323-391.

S. B. Angenent, T. Ilmanen and J. J. L. Velásquez, Slow blowup and nonuniqueness of motion by mean curvature, in preparation.

S. B. Angenent, T. Ilmanen and J. J. L. Velásquez, Nonuniqueness of motion by mean curvature flow in dimension four through seven, in preparation.

S. B. Angenent and J. J. L. Velásquez (1997), Degenerate neckpinches in mean curvature flow, *J. reine angew. Math.*, **482**, 15-66.

M. Bardi and I. Capuzzo-Dolcetta (1997), Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Birkhäuser, Boston.

M. Bardi, M. G. Crandall, L. C. Evans, H. M. Soner and P. E. Souganidis (1997), Viscosity solutions and applications, Lecture Notes in Math., **1660**, Springer-Verlag, Berlin.

G. Barles (1985), Remark on a flame propagation model, *Report INRIA #464.*

G. Barles (1993), Fully nonlinear Neumann type boundary conditions for second-order elliptic and parabolic equations, *J. Differential Equations*, **106**, 90-106.

G. Barles (1994), Solutions de viscosité des équations de Hamilton-Jaocbi, Mathématiques & Applications 17, Springer-Verlag, Paris.

G. Barles (1999), Nonlinear Neumann boundary conditions for quasilinear degenerate elliptic equations and applications, *J. Differential Equations*, **154**, 191-224.

G. Barles and C. Georgelin (1995), A simple proof of convergence for an approximate scheme computing motions by mean curvature, *SIAM J. Numer. Anal.,* **32**, 484-500.

G. Barles and B. Perthame (1987), Discontinuous solutions of deterministic optimal stopping time problems, *RAIRO Modèl. Math. Anal. Num.* **21**, 557-579.

G. Barles and B. Perthame (1988), Exit time problems in optimal control and vanishing viscosity method, *SIAM J. Control. Optim.*, **26**, 1133-1148.

G. Barles, H. M. Soner and P. E. Souganidis (1993), Front propagation and phase field theory, *SIAM J. Control Optim.*, **31**, 439-469.

G. Barles and P. E. Souganidis (1998), A new approach to front propagation problems: theory and applications, *Arch. Rational Mech. Anal.,* **141**, 237-296.

G. Bellettini and M. Novaga (1998), Comparison results between minimal barriers and viscosity solutions for geometric evolutions, *Ann. Scuola Norm. Sup. Pisa Cl Sci.* (4) **26**, 97-131.

G. Bellettini and M. Paolini (1994), Two examples of fattening for the curvature flow with a driving force, *Atti Acad. Naz. Lincei Cl. Sc. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.,* **5**, 229-236.

G. Bellettini and M. Paolini (1995), Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature, *Rend. Accad. Naz. Sci. XL Mem. Mat. (5),* **19**, 43-67.

J. Bence, B. Merriman and S. Osher (1992), Diffusion motion generated by mean curvature, Computational Crystal Grower Workshop, AMS, 73-83.

K. A. Brakke (1978), The motion of a surface by its mean curvature, *Mathematical Notes, Princeton Univ. Press,* Princeton, New Jersey.

L. Bronsard and R. V. Kohn (1991), Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics, *J. Differential Equations*, **90**, 211-237.

L. A. Caffarelli (1989), Interior a priori estimates for solutions of fully nonlinear equations, *Ann. of Math.,* **130**, 189-213.

L. A. Caffarelli and X. Cabré (1995), Fully nonlinear elliptic equations, *American Math. Soc. Colloq. Pub.*, **43**, American Math. Soc., Providence, RI.

L. Caffarelli, L. Nirenberg and J. Spruck (1985), The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian, *Acta Math.*, **155**, 261-301.

L. Caffarelli, L. Nirenberg and J. Spruck (1988), Nonlinear second-order elliptic equations V. The Dirichlet problem for Weingarten hypersurfaces, *Comm. Pure Appl. Math.,* **41**, 47-70.

J. W. Cahn and D. W. Hoffman (1974), A vector thermodynamics for anisotropic surfaces-2. Curved and faceted surfaces, *Acta Metall.,* **22**, 1205-1214.

J. W. Cahn, J. E. Taylor and C. A. Handwerker (1991), Evolving crystal forms: Frank's characteristics revisited, Sir Charles Frank, OBE, ERS: An eightieth birthday tribute (eds. R. G. Chambers et al), Adam Hilger, Bristol, Philadelphia and New York, pp.88-118.

F. Camilli (1998), A stability property for the generalized mean curvature flow equation, *Adv. in Differential Equations*, **3**, 815-846.

F. Cao (2002?), Geometric curve ecolution and image processing, Lecture Notes, to appear.

X.-Y. Chen (1991), Dynamics of interfaces in reaction diffusion systems, *Hiroshima Math. J.*, **21**, 47-83.

Xinfu Chen (1992), Generation and propagation of interface for reaction-diffusion equations, *J. Differential Equations*, **96**, 116-141.

Xinfu Chen (1996), Global asymptotic limit of solutions of the Cahn-Hilliard equation, *J. Differential Geom.*, **44**, 262-311.

Y.-G. Chen, Y. Giga and S. Goto (1989), Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, *Proc. Japan Acad. Ser. A. Math. Sci.*, **65**, 207-210.

Y.-G. Chen, Y. Giga and S. Goto (1991a), Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, *J. Differential Geometry*, **33**, 749-786.

Y.-G. Chen, Y. Giga and S. Goto (1991b), Remarks on viscosity solutions for evolutions for evolution equations, Proc. Japan Acad. Ser. A. Math. Sci., **67**, (1991) 323-328.

Y.-G. Chen, Y. Giga and S. Goto (1991c), Analysis toward snow crystal growth, In: *Proceedings of International Symposium on Functional Analysis and Related Topics*(ed. S. Koshi), World Scientific, Singapore, 43-57.

Y.-G. Chen, Y. Giga, T. Hitaka and M. Honma (1994), A stable difference scheme for computing motion of level surfaces by the mean curvature, In: *Proc. of the third GARC symposium on pure and applied mathematics* (eds. D. Kim et al), Seoul National University, Seoul, 1-19.

A. A. Chernov (1984), Modern Crystallography III, Crystal Growth, Solid-State Sciences, Springer Verlag, **36**

D. L. Chopp (1994), Numerical computation of self-similar solutions for mean curvature flow, Experiment. Math., **1**, 1-15.

K.-S. Chou and X.-P. Zhu (1999a), Anisotropic flows for convex curves, *Duke Math. J.*, **97**, 579-619.

K.-S. Chou and X.-P. Zhu (1999b), A convexity theorem for a class of anisotropic flows of plane curves, *Indiana Univ. Math. J.*, **48**, 139-154.

K.-S. Chou and X.-P. Zhu (2001), The curve shortening problem, Chapman and Hall/CRC, Boca Raton.

B. Chow (1985), Deforming convex hypersurfaces by the $n$-th root of the Gaussian curvature, *J. Differential Geometry*, **23**, 117-138.

R. Courant and D. Hilbert (1962), Method of mathematical physics I, *Interscience*, John Wiley & Sons Inc., New York.

M. G. Crandall, L. C. Evans and P.-L. Lions (1984), Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **282**, 487-502.

M. G. Crandall, H. Ishii and P.-L. Lions (1992), User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.,* **27**, 1-67.

M. G. Crandall and P.-L. Lions (1981), Conditions d'unicité pour les solutions de Hamilton-Jacobi du premier ordre, *C.R. Acad. Sci. Paris, Sér I. Math.,* **292**, 183-186.

M. G. Crandall and P. L. Lions (1983), Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.,* **277**, 1-42.

M. G. Crandall and P. L. Lions (1996), Convergent difference schemes for nonlinear parabolic equations and mean curvature motion, *Numer. Math.*, **75**, 17-41.

M. G. Crandall and H. Ishii (1990), The maximum principle for semicontinuous functions, *Differential and Integral Equations*, **3**, 1001-1014.

B. Dacorogna and C. E. Pfister (1992), Wulff theorem and best constant in Sobolev inequality, *J. Math. Pures Appl.*, **(9)71**, 97-118.

E. De Giorgi (1990), Conjectures on flow by mean curvature, unpublished note.

K. Deckelnick (2000), Error bounds for a difference scheme approximating viscosity solutions of mean curvature flow, *Interfaces Free Bound.*, **2**, 117-142.

A. Dinghas (1944), Uber einen geometrischen Satz von Wulff für die Gleichgewichts form von Kristallen, *Z. Kristallogr., Mineral. Petrogr.*, **105**, 304-314.

C. Dohmen and Y. Giga (1994), Selfsimilar shrinking curves for anisotropic curvature flow equations, *Proc. Japan Acad.*, **70-7**, Ser.A, 252-255.

C. Dohmen, Y. Giga and N. Mizoguchi (1996), Existence of selfsimilar shrinking curves for anisotropic curvature flow equations, *Calc. Var. Partial Differential Equations*, **4**, 103-119.

P. de Mottoni and M. Schatzman (1995), Geometrical evolution of developed interfaces, *Trans. AMS.*, **347**, 1533-1589. announcement: Evolution geométrique d' interfaces, C. R. Acad. Sci. Paris, Sér. I. Math., **309** (1989), 453-458.

C. M. Elliott (1996), Approximation of curvature dependent interface motion, *Inst. Math. Appl. Conf. Ser. New Ser.*, **63**, 407-440.

C. M. Elliott and H. Garcke (1997), Existence results for diffusive surface motion laws, *Adv. Math. Sci. Appl.*, **7**, 467-490.

S.-I. Ei, M.-H. Sato and E. Yanagida (1996), Stability of stationary interfaces with contact angle in a generalized mean curvature flow, *Amer. J. Math.*, **118**, 653-687.

J. Escher, U. F. Mayer and G. Simonett (1998), The surface diffusion flow for immersed hypersurfaces, *SIAM J. Math. Anal.*, **29**, 1419-1433.

L. C. Evans (1993), Convergence of an algorithm for mean curvature motion, *Indiana Univ. Math. J.*, **42**, 533-557.

L. C. Evans, H. M. Soner and P. E. Souganidis (1992), Phase transitions and generalized motion by mean curvature, *Comm. Pure Appl. Math.*, **45**, 1097-1123.

L. C. Evans and P. E. Souganidis (1984), Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaccs equations, *Indiana Univ. Math. J.*, **33**, 773-797.

L. C. Evans and J. Spruck (1991), Motion of level sets by mean curvature.I, *J. Differential Geom.* **33**, 635-681.

L. C. Evans and J. Spruck (1992a), Motion of level sets by mean curvature.II, *Trans. Amer. Math. Soc.*, **330**, 321-332.

L. C. Evans and J. Spruck (1992b), Motion of level sets by mean curvature.III, *J. Geom. Anal.*, **2**, 121-150.

L. C. Evans and J. Spruck (1995), Motion of level sets by mean curvature.IV, *J. Geom. Anal.*, **5**, 77-114.

W. J. Firey (1974), Shapes of worn stones, *Mathematika*, **21**, 1-11.

W. H. Fleming and H. M. Soner (1993), Controlled Markov Processes and Viscosity Solutions, Applications of Mathematics, **25**, Springer-Verlag, New York.

I. Fonseca (1991), The Wulff theorem revisited, *Proc. Roy. Soc. London Ser. A.*, **432**, 125-145.

I. Fonseca and M. A. Katsoulakis (1995), Γ-convergence, minimizing movements and generalized mean curvature evolution, *Differential and Integral Equations*, **8**, 1619-1656.

I. Fonseca and S. Müller (1991), A uniqueness proof for the Wulff theorem, *Proc. Roy. Soc. Edinburgh A.*, **119**, 125-136.

T. Fukui and Y. Giga (1996), Motion of a graph by nonsmooth weighted curvature, In: *Proc. First World Congress of Nonlinear Analysis* (ed. V. Lakshmikantham), Walter de Gruyter, Berlin, vol I, 47-56.

M. E. Gage (1993), Evolving plane curves by curvature in relative geometries, *Duke Math. J.*, **72**, 441-466.

M. E. Gage and Y. Li (1994), Evolving plane curves by curvature in relative geometries II, *Duke Math. J.*, **75**, 79-98.

M. Gage and R. S. Hamilton (1986), The heat equation shrinking convex plane curves, *J. Differential Geometry,* **23**, 69-96.

M.-H. Giga and Y. Giga (1998a), Evolving graphs by singular weighted curvature, *Arch. Rational Mech. Anal.,* **141**, 117-198.

M.-H. Giga and Y. Giga (1998b), Motion driven by nonlocal curvature in the plane, In: Asymptotics in Nonlinear Diffusive Systems, (eds. Y. Nishiura et al.), Tohoku Math. Publ., **8**, 75-83.

M.-H. Giga and Y. Giga (1999), Stability for evolving graphs by nonlocal weighted curvature, *Commun. in Partial Differential Equations*, **24**, 109-184.

M.-H. Giga and Y. Giga (2000), Crystalline and level set flow – Convergence of a crystalline algorithm for a general anisotropic curvature flow in the plane, *Proc. FBP '99* (ed. N. Kenmochi), *Gakuto Internat. Ser. Math. Sci. Appl.*, **13**, 64-79, Gakuto, Tokyo.

M.-H. Giga and Y. Giga (2001), Generalized motion by nonlocal curvature in the plane, *Arch. Rational Mech. Anal.*, **159**, 295-333.

Y. Giga (1994), Motion of a graph by convexified energy, *Hokkaido Math. J.,* **23**, 185-212.

Y. Giga (1995a), A level set method for surface evolution equations, *Sūgaku*, **47**, 321-340; English traslation, *Sugaku Exposition*, **10** (1997), 217-241.

Y. Giga (1995b), Evolving curves with boundary conditions, In: *Curvature Flows and Related Topics* (eds. A. Damlamian, J. Spruck and A. Visintin), Mathematical Sciences and Applications, Gakuto, Tokyo, **5**, 99-109.

Y. Giga (2000), Anisotropic curvature effects in interface dynamics, Sūgaku **52**, 113-127; English translation, to appear.

Y. Giga and S. Goto (1992a), Motion of hypersurfaces and geometric equations, *J. Math. Soc. Japan,* **44**, 99-111.

Y. Giga and S. Goto (1992b), Geometric evolution of phase-boundaries, In: *On the evolution of phase-boundaries* (eds. M. Gurtin and G. McFadden), IMA vol. Math. Appl., Springer, **43**, 51-65.

Y. Giga, S. Goto and H. Ishii (1992), Global existence of weak solutions for interface equations coupled with diffusion equations, *SIAM J. Math. Anal.,* **23**, 821-835.

Y. Giga, S. Goto, H. Ishii and M.-H. Sato (1991), Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, *Indiana Univ. Math. J.,* **40**, 443-470.

Y. Giga and K. Ito (1998), On pinching of curves moved by surface diffusion, *Commun. Appl. Anal.*, **2**, 393-405.

Y. Giga and M.-H. Sato (1993), Neumann problem for singular degenerate parabolic equations, *Differential and Integral Equations*, **6**, 1217-1230.

Y. Giga and K. Yama-uchi (1993), On a lower bound for the extinction time of surfaces moved by mean curvature, *Calculus of Variations and Partial Differential Equations,* **1**, 417-428.

D. Gilbarg and N. S. Trudinger (1998), Elliptic partial differential equations of second order, Reviced third printing, Grundlehren der mathematischen Wissenschaften 224, Springer, Berlin.

P. M. Girão (1995), Convergence of a crystalline algorithm for the motion of a simple closed convex by weighted curvature, *SIAM J. Numer. Anal.,* **32**, 886-899.

P. M. Girão and R. V. Kohn (1994), Convergence of a crystalline algorithm for the heat equation in one dimension and for the motion of a graph by weighted curvature, *Numer. Math.,* **67**, 41-70.

P. M. Girão and R. V. Kohn (1996), The crystalline algorithm for computing motion by curvature, In: Variational Methods for Discontinous Structures, (eds: R. Serapioni and F. Tomarelli), Progress in Nonlinear Differential Equations and Applications, **25**, 7-18, Birkhäuser, Basel.

S. Goto (1994), Generalized motion of hypersurfaces whose growth speed depends superlinearly on the curvature tensor, *Differential and Integral Equations*, **7**, 323-343.

M. A. Grayson (1987), The heat equation shrinks embedded plane curves to round points, *J. Differential Geometry,* **26**, 285-314.

M. A. Grayson (1989), A short note on the evolution of a surface by its mean curvature, *Duke. Math. J.,* **58**, 555-558.

F. Guichard and J. M. Morel (2001?), Image analysis and P.D.E.'s, IPAM GBM Tutotial 2001, to appear.

M. E. Gurtin (1988a), Toward a nonequilibrium thermomechanics of two-phase materials, *Arch. Rational Mech. Anal.*, **100**, 275-312.

M. E. Gurtin (1988b), Multiphase thermomechanics with interfacial structure, I: Heat conduction and the capillary balance law, *Arch. Rational Mech. Anal.*, **104**, 195-221.

M. E. Gurtin (1993), Thermomechanics of Evolving Phase Boundaries in the Plane, Oxford, Clarendon Press.

M. E. Gurtin, H. M. Soner and P. E. Souganidis (1995), Anisotropic motion of an interface relaxed by the formulation of infinitesimal wrinkles, *J. Differential Equations* **119**, 54-108.

Q. Han and F.-H. Lin (1997), Elliptic partial differential equations, Courant lecture Notes in Math. **1**, NYU, New York.

G. Huisken (1984), Flow by mean curvature of convex surfaces into spheres, *J. Differential Geometry*, **20**, 237-266.

G. Huisken (1990), Asymptotic behaviour for singularities of the mean curvature flow, *J. Differential Geometry*, **31**, 285-299.

G. Huisken and T. Ilmanen (1997), The Riemannian Penrose inequality, *Internat. Math. Res. Notices*, **20**, 1045-1058.

G. Huisken and A. Polden (1999), Geometric evolution equations for hypersurfaces, *Calculus of variations and geometric evolution problems*, Lecture Note in Math., **1713**, 45-84 Springer, Berlin.

G. Huisken and C. Sinestrari (1999), Convexity estimates for mean curvature flow and singularities of mean convex surfaces, *Acta Math.,* **183**, 45-70.

T. Ilmanen (1992), Generalized flow of sets by mean curvature on a manifold, *Indiana Univ. Math. J.,* **41**, 671-705.

T. Ilmanen (1993a), The level set flow on a manifold, *Differential geometry; partial differential equations on manifolds*, Proc. Sympos. Pure Math., **54** Part I (eds. R. Greene and S.-T. Yau), 193-204 AMS, Providence RI, 1993.

T. Ilmanen (1993b), Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature, *J. Differential Geometry*, **38**, 417-461.

H. Ishii (1985), Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets, *Bull. Fac. Sci. Engrg. Chuo. Univ.,* **28**, 33-77.

H. Ishii (1987), Perron's method for Hamilton-Jacobi equations, *Duke Math. J.,* **55**, 369-384.

H. Ishii (1989a), On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDE's, *Comm. Pure Appl. Math.*, **42**, 15-45.

H. Ishii (1989b), A boundary value problem of the Dirichlet type of Hamilton-Jacobi equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **(4)16**, 105-135.

H. Ishii (1994), Viscosity solutions for nonlinear PDE, *Sūgaku* **46**, 144-157; English translation, *Sugaku Exposition* **9**(1996), 135-152.

H. Ishii (1996), Degenerate parabolic PDEs with discontinuities and generalized evolutions of surfaces, *Advances in Differential Equations*, **1**, 51-72.

H. Ishii (1995), A generalization of the Bence, Merriman and Osher algorithm of motion by mean curvature, In: *Curvature Flows and Related Topics* (eds. A. Damlamian, J. Spruck and A. Visintin), Mathematical Sciences and Applications **5**, Gakuto, Tokyo, 111-127.

H. Ishii and P.-L. Lions (1990), Viscosity solution of fully nonlinear second order elliptic partial differential equations, *J. Differential Equations*, **83**, 26-78.

H. Ishii, G. E. Pires and P. E. Souganidis (1999), Threshold dynamics type approximation schemes for propagating fronts, *J. Math. Soc. Japan*, **51**, 267-308.

H. Ishii and P. Souganidis (1995), Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor, *Tôhoku Math. J.*, **47**, 227-250.

K. Ishii and H.-M. Soner (1999), Regularity and convergence of crystalline motion, *SIAM J. Math. Anal.*, **30**, 19-37.

N. M. Ivochkina, S. I. Prokof'eva and G. V. Yakunina (1995), On a class of equations of Monge–Ampere type, *J. Math. Sci.*, **73**, 663-673.

R. Jensen (1988), The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations, *Arch. Rational Mech. Anal.*, **101**, 1-27.

T. Kato (1982), A short introduction to perturbation theory for linear operators, Springer, New York.

M. Katsoulakis, G. T. Kossioris and F. Reitich (1995), Generalized motion by mean curvature with Neumann conditions and the Allen-Cahn model for phase transitions, *J. Geom. Anal.*, **5**, 255-279.

M. A. Katsoulakis and P. E. Souganidis (1994), Interacting particle systems and generalized evolution of fronts, *Arch. Rational Mech. Anal.*, **127**, 133-157.

B. Kawohl and N. Kutev (2000), Comparison principle and Lipschitz regularity of viscosity solutions of some classes of nonlinear partial differential equations, *Funkcial. Ekvac.*, **43**, 241-253.

S. Kobayashi and K. Nomizu (1963), Foundations of Differential Geometry, vol I, John Wiley & Sons, New York.

S. Kobayashi and K. Nomizu (1969), Foundations of Differential Geometry, vol II, John Wiley & Sons, New York.

O. A. Ladyžhenskaja, V. Solonnikov and N. Ural'ceva (1968), Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs, *Amer. Math. Soc.,* Providence, **23**.

P. L. Lions (1982), Generalized solutions of Hamilton-Jacobi equations, Research Notes in Mathematics 69, Pitman, Boston, MA.

P. L. Lions (1983), Optimal control of diffusion process and Hamilton-Jacobi-Bellman equations, Part I: The dynamic programming principle and applications and Part II: Viscosity solutions and uniqueness, *Comm. Partial Differential Equations*, **8**, 1101-1174 and 1229-1276.

P. L. Lions (1984), Some recent results in the optimal control of diffusion processes. Stochastic analysis (Katata/Kyoto, 1982), 333-367, North-Holland Math. Library, 32, North-Holland, Amsterdam.

P. L. Lions (1985), Neumann type boundary conditions for Hamilton-Jacobi equations, *Duke Math. J.,* **52**, 793-820.

S. Luckhaus and T. Sturzenhecker (1995), Implicit time discretization for the mean curvature flow equation, *Calculus of Variations and Partial Differential Equations* **3**, 253-271.

A. Lunardi (1995), Analytic semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel-Boston-Berlin.

A. M. Meirmanov (1992), The Stefan problem, Walter de Gruyter.

D. S. Mitrinović (1970), Analytic inequalities, Springer-Verlag, Berlin.

F. Morgan (1993), Riemannian geometry, a beginner's guide Jones and Bartlett Publishers, Boston.

H. Müller-Krumbhaar, T. W. Burkhardt and D. M. Kroll (1977), A generalized kinetic equation for crystal growth, *J. Crystal Growth* **38**, 13-22.

W. W. Mullins (1956), Two-dimensional motions of idealized grain boundaries, *J. Appl. Phys.,* **27**, 900-904.

J. Oaks (1994), Singularities and self-intersections of curves evolving on surfaces, *Indiana Univ. Math. J.*, **43**, 959-981.

M. Ohnuma and K. Sato (1997), Singular degenerate parabolic equations with applications to the $p$-Laplace diffusion equation, Comm. Partial Differential Equations, **22**, 381-411.

M. Ohnuma and M.-H. Sato (1993), Singular degenerate parabolic equations with applications to geometric evolutions, *Differential and Integral Equations* **6**, 1265-1280.

T. Ohta, D. Jasnow and K. Kawasaki (1982), Universal scaling in the motion of random interfaces, *Physics Review Letters*, **49**, 1223-1226.

S. Osher and J. A. Sethian (1988), Fronts propagating with curvature dependent speed: Algorithms based on Hamilton-Jacobi formulations, *J. Comp. Phys.,* **79**, 12-49.

L. I. Rubinstein (1971), The Stefan problem, AMS, Providence, RI.

G. Sapiro and A. Tannenbaum (1994), On affine plane curve evolution, *J. Funct. Analysis*, **119**, 79-120.

M.-H. Sato (1990), Comparison principle for singular degenerate elliptic equations on unbounded domains, *Proc. Japan Acad. Ser. A*, **66**, 252-256.

M.-H. Sato (1994), Interface evolution with Neumann boundary condition, *Adv. Math. Sci. Appl.,* **4**, 249-264.

M.-H. Sato (1996), Capillary problem for singular degenerate parabolic equations on a half space, *Differential and Integral Equations*, **9**, 1213-1224.

R. Schneider (1993), Convex bodies: The Brunn-Minkowski theory, *Cambridge Univ. Press*, Cambridge.

J. A. Sethian (1996), Level set methods, Evolving interfaces in geometry, fluid mechanics comuputer vision, and materials science, Cambridge Monographs on Applied and Computational Mathematics, 3 Cambridge University Press, Cambridge.

L. Simon (1983), Lectures on geometric measure theory, Proceedings of the Centre for Mathematical Analysis, Australian National University, **3**.

K. Smoczyk (1994), The symmetric “doughnut” evolving by its mean curvature, *Hokkaido Math. J.,* **23**, 523-547.

H. M. Soner (1986), Optimal control with state-space constraint I, II. *SIAM J. Control Optim.,* **24**, 552-561, 1110-1122.

H. M. Soner (1993), Motion of a set by the curvature of its boundary, *J. Differential Equations*, **101**, 313-372.

H. M. Soner (1995), Convergence of the phase-field equations to the Mullins-Sekerka problem with kinetic undercooling, *Arch. Rational Mech. Anal.,* **131**, 139-197.

H. M. Soner (1997), Guinzburg-Landau equation and motion by mean curvature, I: Convergence, *J. Geom. Anal.*, **7**, 437-475.

H. M. Soner and P. E. Souganidis (1993), Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature, *Commun. in Partial Differential Equations*, **18**, 859-894.

J. E. Taylor (1978), Crystalline variational problems, *Bull. Amer. Math. Soc.,* **84**, 568-588.

J. E. Taylor (1991), Constructions and conjectures in crystalline nondifferential geometry, In: *Differential Geometry,* (eds. B. Lawson and K. Tanenblat), Proceedings of the Conference on Differential Geometry, Rio de Janeiro, Pitman Monographs surveys Pure Appl. Math., **52**, 321-336, Pitman, London.

J. E. Taylor (1992), Mean curvature and weighted mean curvature, *Acta Metall.,* **40**, 1475-1485.

J. Taylor, J. Cahn and C. Handwerker (1992), Geometric models of crystalline growth, *Acta Metall.,* **40**, 1443-1474.

Y. Tonegawa (2000), Some remarks on the level set flow by anisocropic curvature, *Calc. Var. Rational Differential Equations*, **10**, 101-118.

N. S. Trudinger (1990), The Dirichlet problem for the prescribed curvature equations, *Arch. Rational Mech. Anal.*, **111**, 153-179.

K. Tso (1985), Deforming a hypersurface by its Gauss-Kronecker curvature, *Comm. Pure Appl. Math.,* **38**, 867-882.

J. Urbas (1999), Self-similar solutions of Gauss curvature flows, *Contemporary Mathematics*, **226**, 157-172.

J. J. L. Velázquez (1994), Curvature blow-up in perturbations of minimal cones evolving by mean curvature flow, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.,* **21**, 595-628.

A. Visintin (1996), Models of phase transitions, *Progress in Nonlinear Differential Equations and their Applications*, **28**, Birkhäuser Boston, Inc., Boston, MA

L. Wang (1990), On the regularity theory of fully nonlinear parabolic equations, *Bull. Amer. Math. Soc.*, **22**, 107-114.

B. White (1998), The nature of singularities in mean curvature flow of mean-convex sets, preprint (http://math.stanford.edu/white).

B. White (2000), The size of the singular set in mean curvature flow of mean-convex sets, *J. Amer. Math. Soc.,* **13**, 665-695.

G. Wulff (1901), Zur Frage der und der Auflösung der Krystallflachen, *Zeit. Krystall. Min.*, **34**, 489-530.