The 11th Mathematical Society of
Japan - Seasonal Institute

- The Role of Metrics in the Theory of Partial Differential Equations -

Yoshikazu Giga, Hideo Kubo, Tohru Ozawa

Series #174. July, 2018

HOKKAIDO UNIVERSITY
TECHNICAL REPORT SERIES IN MATHEMATICS

#152 M. Hayashi, T. Nakazi and M. Yamada, 第 20 回関数空間セミナー, 89 pages. 2012.

#153 Y. Giga, S. Jimbo, G. Nakamura, T. Ozawa, T. Sakajo, H. Takaoka, Y. Tonegawa and K. Tsutaya, Proceedings of the 37th Sapporo Symposium on Partial Differential Equations, 81 pages. 2012.

#154 N. Hu, Doctoral thesis "Affine geometry of space curves and homogeneous surfaces", 69 pages. 2012.

#155 2013 代数幾何学シンポジウム, 127 pages. 2013.

#156 M. Hayashi, S. Miyajima, T. Nakazi, I. Saito and M. Yamada, 第 21 回関数空間セミナー, 90 pages. 2013.

#157 D. Suyama, T. Ito, M. Kuroda, Y. Goto, N. Teranishi, S. Futakuchi, T. Fuda and N. Miyagwa, 第 9 回数学総合若手研究集会, 344 pages. 2013.

#158 Y. Giga, S. Jimbo, H. Terao, K. Yamaguchi, Proceedings of the 6th Pacific RIM Conference on Mathematics 2013, 154 pages. 2013.

#159 Y. Giga, S. Jimbo, T. Ozawa, K. Tsutaya, Y. Tonegawa, H. Kubo, T. Sakajo, and H. Takaoka, Proceedings of the 38th Sapporo Symposium on Partial Differential Equations, 76 pages. 2013.

#160 M. Kuroda, Y. Goto, K.Sasaki, S. Futakuchi, D. Funakawa, T. Yamashita, and K. Wada, 第 10 回数学総合若手研究集会, 335 pages. 2014.

#161 S. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo, H. Takaoka, Y. Tonegawa, and K. Tsutaya, Proceedings of the 39th Sapporo Symposium on Partial Differential Equations, 147 pages. 2014.

#162 D. Funakawa, T. Kagaya, Y. Kabata, K. Sasaki, H. Takeda, Y. Chino, A. Tsuchida, T. Yamashita, and K. Wada, 第 11 回数学総合若手研究集会, 359 pages. 2015.

#163 S. Jimbo, S. Goto, Y. Kohsaka, H. Kubo, Y. Maekawa, and M. Ohnuma, Mathematics for Nonlinear Phenomena: Analysis and Computation - International Conference in honor of Professor Yoshikazu Giga on his 60th birthday- , 47 pages. 2015.

#164 S. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo, H. Takaoka, Y. Tonegawa and K. Tsutaya, Proceedings of the 40th Sapporo Symposium on Partial Differential Equations, 122 pages. 2015.

#165 A. Tsuchida, Y. Aikawa, K. Asahara, M. Abe, Y. Kabata, H. Saito, F. Nakamura and S. Honda, 第 12 回数学総合若手研究集会, 373 pages. 2016.

#166 S.-I. Ei, Y. Giga, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo, H. Takaoka, Y. Tonegawa and K. Tsutaya, Proceedings of the 41st Sapporo Symposium on Partial Differential Equations, 110 pages. 2016.

#167 Kubo, Hideo and Ozawa, Tohru and Takamura, Hiroyuki, Mathematical Analysis for Stability in Nonlinear Dynamics - in honor of Professor Vladimir Georgiev on his 60th birthday -, 79 pages. 2016.

#168 F. Nakamura, Y. Aikawa, K. Asahara, M. Abe, D. Komori, H. Saito, S. Handa, K. Fujisawa, S. Honda, A. Rodrguez Mulet, 第 13 回数学総合若手研究集会, 501 pages. 2017.

#169 N. Tanaka (K. Kiyohara, T. Morimoto, K. Yamaguchi, Eds.), Geometric Theory of Systems of Ordinary Differential Equations I, 177 pages. 2017.

#170 N. Tanaka (K. Kiyohara, T. Morimoto, K. Yamaguchi, Eds.), Geometric Theory of Systems of Ordinary Differential Equations II, 519 pages. 2017.

#171 S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, T. Ozawa, T. Sakajo, Y. Tonegawa, and K. Tsutaya, Proceedings of the 42nd Sapporo Symposium on Partial Differential Equations - In memory of Professor Taira Shirota - , 102 pages. 2017.

#172 Shigeru Kuroda, Takao Namiki, Akihiro Yamaguchi, Yutaka Yamaguchi, 複雑系数理の新展開：津田一郎教授退職記念研究集会報告集, 69 pages. 2018.

#173 S.Handa,D.Komori,K.Fujisawa,A.Rodriguez Mulet,M.Aoki,Y.kamijima,I.Fukuda,T.Yabu 第 14 回数学総合若手研究集会, 580 pages. 2018.

The 11th Mathematical Society of Japan
- Seasonal Institute

- The Role of Metrics in the Theory of Partial Differential Equations -

Edited by
Yoshikazu Giga, Hideo Kubo, Tohru Ozawa

Venue: Hokkaido University
July 2-6
Conference Hall
July 9-13
Frontier Research in Applied Sciences Building

Contents

Preface

We welcome all the participants to the conference: The 11th Mathematical Society of Japan (MSJ) Seasonal Institute "The Role of Metrics in the Theory of Partial Differential Equations", held for the period of July 2-13, 2018 at Hokkaido University. MSJ-Seasonal Institutes have been held annually since 2008, based on the financial support from Mathematical Society of Japan. Each year, the topic of the workshop is chosen by MSJ and leading mathematicians from all over the world are invited accordingly. This volume is intended as the proceedings of the event of 2018.

In this seasonal institute, various kinds of approaches to deal with partial differential equations will be delivered. We wish that this workshop would provide a chance to broaden your perspective on huge variety of subjects from the theory of PDE to harmonic analysis and geometric analysis beyond your own research subject, and to create new relationship among participants from different fields. This seasonal institute would be also expected to be an occasion to understand what kind of effect on the behavior of solutions the non-flat metric appearing in PDE on/toward a manifold, the evolution of surfaces, the weight on the measure, and so on, brings.

We hope you enjoy this seasonal institute as well as your stay in Sapporo.

Organizing Committee:
Hideo Kubo (Chair/Hokkaido University)
Yoshikazu Giga (The University of Tokyo)
Tohru Ozawa (Waseda University)

Acknowledgments

The organizing committee would like to extend its sincere gratitude to the following associations and organizations for their valuable support and commitment to make this event successful.

Sapporo International Communication Plaza Foundation
公益財団法人　札幌国際プラザ

Japan Society for the Promotion of Science
独立行政法人　日本学術振興会

Grant-in-aid for Scientific Research (S)　Grant Number JP26220702
Grant-in-aid for Scientific Research (A)　Grant Number JP18H03670
Grant-in-aid for Scientific Research (A)　Grant Number JP17H01091
Grant-in-aid for Scientific Research (B)　Grant Number JP18H01128
Grant-in-aid for Scientific Research (B)　Grant Number JP18H01132
Grant-in-aid for Scientific Research (C)　Grant Number JP18K03379

Japan Science and Technology Agency, CREST
科学技術振興構機

Grant Number JPMJCR14D3, Japan

Department of Mathematics,
Hokkaido University

北海道大学大学院理学研究院数学部門
北海道大学大学院理学院数学専攻
北海道大学理学部数学科

The Mathematical Society of Japan
一般社団法人　日本数学会

Steering Committee

Hideo Kubo (Hokkaido University), Chair
Mi-Ho Giga (The University of Tokyo),
Hirotoshi Kuroda (Hokkaido University),
Nao Hamamuki (Hokkaido University),

Program (July 2-6, 2018)

Venue： 学術交流会館 Hokkaido University Conference Hall

July 2, 2018 (Monday)

Course Lecture (Room: 小講堂 Conference Hall)

10:30-10:40	Opening
10:40-11:30	Sergiu Klainerman (Princeton University) On the mathematical theory of black holes, I
11:30-13:30	お昼休み（Break）
13:30-14:50	Poster Session (Venue: 第 1 会議室 Conference Room 1)

Session A (Room: 小講堂 Conference Hall)

15:00-15:50	Gerhard Huisken (Universität Tübingen / Mathematisches Forschungsinstitut Oberwolfach) Mean curvature flow with surgery of embedded mean-convex surfaces in 3-manifolds
16:00-16:50	Mu-Tao Wang (Columbia University) Quasilocal mass and isometric embedding

Session B (Room： 第 1 会議室 Conference Room 1)

15:00-15:50	Yi Zhou (Fudan University) Global Instability of the Multi-dimensional Plane Shocks for the Isothermal Euler Equations
16:00-16:50	Piero D'Ancona (Sapienza - Universitá di Roma) On the defocusing supercritical wave equation outside a ball

July 3, 2018 (Tuesday)

Course Lecture (Room: 小講堂　Conference Hall)

9:30-10:20	Mihalis Dafermos (Princeton University / University of Cambridge) Black hole interiors and the strong cosmic censorship conjecture in general relativity, Ⅰ
10:40-11:30	Sergiu Klainerman (Princeton University) On the mathematical theory of black holes, Ⅱ
11:30-13:30	お昼休み（Break）

Session A (Room: 小講堂　Conference Hall)

13:30-14:00	梶ヶ谷徹（東京電機大学）Toru Kajigaya (Tokyo Denki University) Hamiltonian stability for weighted measure and generalized Lagrangian mean curvature flow
14:10-14:40	棚橋典大（九州大学）Norihiro Tanahashi(Kyushu University) Separability of Maxwell equation in rotating black hole spacetime and its geometric aspects
15:00-15:50	Gilbert Weinstein (Ariel University) Harmonic maps with prescribed singularities and applications in general relativity
16:00-16:50	山田澄生（学習院大学）Sumio Yamada (Gakushuin University) Harmonic maps in general relativity

Session B (Room：第 1 会議室　Conference Room 1)

13:30-14:00	岡本葵（信州大学）Mamoru Okamoto (Shinshu University) Random data Cauchy problem for the energy-critical nonlinear Schrödinger equations
14:10-14:40	戍亥隆恭（大阪大学）Takahisa Inui (Osaka University) Global behavior of solutions to the energy critical nonlinear damped wave equation
15:00-15:50	伊藤健一（東京大学）Kenichi Ito (The University of Tokyo)

	Asymptotics of generalized eigenfunctions on manifold with Euclidean and/or hyperbolic ends
16:00-16:50	Wei-Min Wang (CNRS / Université de Cergy-Pontoise) Time quasi-periodic solutions to the nonlinear Klein-Gordon equations

July 4, 2018 (Wednesday)

Course Lecture (Room: 小講堂　Conference Hall)

9:30-10:20	Jean-Marc Delort (Université Paris 13) Long time existence results for Hamiltonian non-linear Klein-Gordon equations on some compact manifolds,　I
10:40-11:30	Mihalis Dafermos (Princeton University / University of Cambridge) Black hole interiors and the strong cosmic censorship conjecture in general relativity,　II
12:00-	Excursion

July 5, 2018 (Thursday)

Course Lecture (Room: 小講堂　Conference Hall)

9:30-10:20　Sergiu Klainerman (Princeton University)
On the mathematical theory of black holes, Ⅲ

10:40-11:30　Jean-Marc Delort (Université Paris 13)
Long time existence results for Hamiltonian non-linear Klein-Gordon equations on some compact manifolds, Ⅱ

11:30-13:30　お昼休み（Break）

13:30-14:50　Poster Session　**(Venue: ホール　Hall)**

Session A (Room: 小講堂　Conference Hall)

15:00-15:50　Vanessa Styles (University of Sussex)
Stability and error analysis for a diffuse interface approach to an advection-diffusion equation on a moving surface

16:00-16:50　Richard Tsai (The University of Texas at Austin)
Volumetric extensions of variational problems defined on surfaces

Session B (Room：第 1 会議室　Conference Room 1)

15:00-15:50　Joachim Krieger (École Polytechnique Fédérale de Lausanne)
On stability of blow up solutions for the critical co-rotational wave maps problem

16:00-16:50　片山聡一郎（大阪大学）Soichiro Katayama (Osaka University)
Global existence and the asymptotic behavior for systems of nonlinear wave equations violating the null condition

July 6, 2018 (Friday)

Course Lecture (Room: 小講堂　Conference Hall)

9:30-10:20　Mihalis Dafermos (Princeton University / University of Cambridge)
Black hole interiors and the strong cosmic censorship conjecture in general relativity, Ⅲ

10:40-11:30 Jean-Marc Delort (Université Paris 13)
Long time existence results for Hamiltonian non-linear Klein-Gordon equations on some compact manifolds, Ⅲ

11:30-13:30 お昼休み（Break）

Session A (Room: 小講堂　Conference Hall)

13:30-14:00 榊原航也（京都大学）Koya Sakakibara (Kyoto University)
Numerical analysis of one-harmonic equation with values in matrix Lie group

14:10-14:40 三浦達彦（東京大学）Tatsu-Hiko Miura (The University of Tokyo)
Singular limit problem for the Navier-Stokes equations in a curved thin domain

15:00-15:50 小俣正朗（金沢大学）Seiro Omata (Kanazawa University)
Free boundary problems of hyperbolic type with a focus on the free boundary conditions

16:00-16:50 Selim Esedoglu (University of Michigan)
Algorithms for motion of networks by weighted mean curvature

Session B (Room：第 1 会議室　Conference Room 1)

13:30-14:00 池田正弘（理化学研究所・慶應義塾大学）Masahiro Ikeda (RIKEN /Keio University)
Blow-up of solutions to semilinear wave equation with a scaling invariant critical damping

14:10-14:40 宮崎隼人（津山工業高等専門学校）Hayato Miyazaki (National Institute of Technology, Tsuyama College)
The initial value problem for the generalized KdV equation with low degree of non-linearity

15:00-15:50 Jacob Sterbenz (University of California San Diego)
Local energy decay and the vector field method

16:00-16:50 Ning-An Lai (Lishui University)
Weighted L^2-L^2 estimate for wave equation and its application

Program (July 9-13, 2018)

Venue： フロンティア応用科学研究棟

(Frontier Research in Applied Sciences Building)

<u>July 9, 2018 (Monday)</u>

Course Lecture (Room: レクチャーホール　Akira Suzuki Hall)

10:40-11:30	Felix Otto (Max Planck Institute for Mathematics in the Sciences, Leipzig) Nonlinear partial differential equations and interfacial motions as gradient flow, Ⅰ
11:30-13:30	お昼休み（Break）
13:30-14:50	Poster Session（**Venue: ホワイエ　Foyer**）

Session A (Room: レクチャーホール　Akira Suzuki Hall)

15:00-15:50	二木昭人（清華大学）Akito Futaki (Tsinghua University) Geometric nonlinear problems and GIT stability through moment maps
16:00-16:50	利根川吉廣（東京工業大学）Yoshihiro Tonegawa (Tokyo Institute of Technology) Existence theorem for Brakke flow

Session B (Room: セミナー室 2　Seminar Room 2)

15:00-15:50	高村博之（東北大学）Hiroyuki Takamura (Tohoku University) Multipliers on the wave-like blow-up for nonlinear damped wave equations
16:00-16:50	関口雄一郎（東邦大学）Yuichiro Sekiguchi (Toho University) Numerical relativity: generating space-time on computers

<u>July 10, 2018 (Tuesday)</u>

Course Lecture (Room: レクチャーホール　Akira Suzuki Hall)

9:30-10:20	Antonin Chambolle (CNRS / École Polytechnique) Anisotropic and crystalline mean curvature flows, Ⅰ
10:40-11:30	Felix Otto (Max Planck Institute for Mathematics in the Sciences, Leipzig) Nonlinear partial differential equations and interfacial motions as gradient flow, Ⅱ
11:30-13:30	お昼休み（Break）

Session A (Room: レクチャーホール　Akira Suzuki Hall)

13:30-14:00	本多正平（東北大学）Shouhei Honda (Tohoku University) Recent developments on metric measure spaces with Ricci bounds from below
14:10-14:40	Kelei Wang (Wuhan University) Simons cone and saddle solutions of the Allen-Cahn equation
15:00-15:50	芥川和雄（中央大学）Kazuo Akutagawa (Chuo University) Edge-cone Einstein metrics and Yamabe metrics
16:00-16:50	Matteo Novaga (Università di Pisa) Evolution by curvature of planar networks

Session B (Room: セミナー室 2　Seminar Room 2)

13:30-14:00	杉山裕介（滋賀県立大学）Yusuke Sugiyama (The University of Shiga Prefecture) Global existence and degeneracy in finite time for 1D quasilinear wave equations
14:10-14:40	佐々木多希子（明治大学）Takiko Sasaki (Meiji University) The blow-up curve of solutions for semilinear wave equations with Dirichlet boundary conditions in one space dimension
15:00-15:50	Karen Yagdjian (The University of Texas Rio Grande Valley)

	A novel integral transform approach to solving hyperbolic equations in the curved spacetimes
16:00-16:50	中村誠（山形大学）Makoto Nakamura (Yamagata University)
	On some effects of background metrics for several partial differential equations

July 11, 2018 (Wednesday)

Course Lecture (Room: レクチャーホール　Akira Suzuki Hall)

9:30-10:20	Charles M. Elliott (The University of Warwick)
	Variational, geometric and surface PDEs in cell biology, I
10:40-11:30	Antonin Chambolle (CNRS / École Polytechnique)
	Anisotropic and crystalline mean curvature flows, II
12:00-	Excursion

<u>July 12, 2018 (Thursday)</u>

Course Lecture (Room: レクチャーホール　Akira Suzuki Hall)

9:30-10:20　Felix Otto (Max Planck Institute for Mathematics in the Sciences, Leipzig)
Nonlinear partial differential equations and interfacial motions as gradient flow, Ⅲ

10:40-11:30　Charles M. Elliott (The University of Warwick)
Variational, geometric and surface PDEs in cell biology, Ⅱ

11:30-13:30　お昼休み（Break）

13:30-14:50　Poster Session（**Venue: ホワイエ　Foyer**）

Session A (Room: レクチャーホール　Akira Suzuki Hall)

15:00-15:50　小磯深幸（九州大学）Miyuki Koiso (Kyushu University)
Uniqueness problem for closed non-smooth hypersurfaces with constant anisotropic mean curvature

16:00-16:50　Neshan Wickramasekera (University of Cambridge)
Regularity of minimal and CMC stable hypersurfaces

Session B (Room: セミナー室 2　Seminar Room 2)

15:00-15:50　谷口雅治（岡山大学）Masaharu Taniguchi (Okayama University)
Multidimensional traveling fronts in reaction-diffusion equations

16:00-16:50　Elliott Ginder (Meiji University)
Approximation methods for oscillatory and constrained interfacial dynamics

July 13, 2018 (Friday)

Course Lecture (Room: レクチャーホール　Akira Suzuki Hall)

9:30-10:20　Antonin Chambolle (CNRS / École Polytechnique)
Anisotropic and crystalline mean curvature flows, Ⅲ

10:40-11:30　Charles M. Elliott (The University of Warwick)
Variational, geometric and surface PDEs in cell biology, Ⅲ

11:30-13:30　お昼休み（Break）

Session A (Room: レクチャーホール　Akira Suzuki Hall)

13:30-14:00　柳青（福岡大学）Qing Liu (Fukuoka University)
Large exponent behavior of power curvature flow and applications

14:10-15:00　高坂良史（神戸大学）Yoshihito Kohsaka (Kobe University)
On the dynamics for the volume-preserving geometric flows

15:10-16:00　石井克幸（神戸大学）Katsuyuki Ishii (Kobe University)
Convergence of a threshold-type algorithm for mean curvature flow

Session B (Room: セミナー室 2　Seminar Room 2)

13:30-14:00　Bronwyn Hajek (University of South Australia)
Analytical solutions of nonlinear reaction-diffusion equations in various coordinate systems

14:10-15:00　Petrus van Heijster (Queensland University of Technology)
Pinned localised solutions in a heterogeneous three-component FitzHugh-Nagumo model

15:10-16:00　Sanjeeva Balasuriya (The University of Adelaide)
Stochastic sensitivity of finite-time Lagrangian motion

16:00-16:10　Closing

The 11th Mathematical Society of Japan Seasonal Institute
Lecture Timetables
(As of June 5, 2018)

July 2-6 : Venue : Hokkaido University Conference Hall

Monday, July 2			Tuesday, July 3			Wednesday, July 4	Thursday, July 5			Friday, July 6		
9:30-10:20			9:30-10:20	M. Dafermos		J.-M. Delort	9:30-10:20	S. Klainerman		9:30-10:20	M. Dafermos	
10:40-11:30	S. Klainerman		10:40-11:30	S. Klainerman		M. Dafermos	10:40-11:30	J.-M. Delort		10:40-11:30	J.-M. Delort	
Lunch												
13:30-14:50	Poster Session		13:30-14:00	T. Kajigaya	M. Okamoto	Excursion	13:30-14:50	Poster Session		13:30-14:00	K. Sakakibara	M. Ikeda
			14:10-14:40	N. Tanahashi	T. Inui					14:10-14:40	T.-H. Miura	H. Miyazaki
15:00-15:50	G. Huisken	Y. Zhou	15:00-15:50	G. Weinstein	K. Ito		15:00-15:50	V. Styles	J. Krieger	15:00-15:50	S. Omata	J. Sterbenz
16:00-16:50	M.-T. Wang	P. D'Ancona	16:00-16:50	S. Yamada	W.-M. Wang		16:00-16:50	R. Tsai	S. Katayama	16:00-16:50	S. Esedoglu	N. A. Lai
Room :	Conference Hall	Conference Room 1		Conference Hall	Conference Room 1			Conference Hall	Conference Room 1		Conference Hall	Conference Room 1

July 9-13 : Venue : Frontier Research in Applied Sciences Building

Monday, July 9			Tuesday, July 10			Wednesday, July 11	Thursday, July 12			Friday, July 13		
9:30-10:20			9:30-10:20	A. Chambolle		C. M. Elliott	9:30-10:20	F. Otto		9:30-10:20	A. Chambolle	
10:40-11:30	F. Otto		10:40-11:30	F. Otto		A. Chambolle	10:40-11:30	C. M. Elliott		10:40-11:30	C. M. Elliott	
Lunch												
13:30-14:50	Poster Session		13:30-14:00	S. Honda	Y. Sugiyama	Excursion	13:30-14:50	Poster Session		13:30-14:00	Q. Liu	B. Hajek
			14:10-14:40	K. Wang	T. Sasaki					14:10-15:00	Y. Kohsaka	P. van Heijster
15:00-15:50	A. Futaki	H. Takamura	15:00-15:50	K. Akutagawa	K. Yagdjian		15:00-15:50	M. Koiso	M. Taniguchi	15:10-16:00	K. Ishii	S. Balasuriya
16:00-16:50	Y. Tonegawa	Y. Sekiguchi	16:00-16:50	M. Novaga	M. Nakamura		16:00-16:50	N. Wickramasekera	E. Ginder			
Room :	Lecture Hall (Akira Suzuki Hall)	Seminar Room 2		Lecture Hall (Akira Suzuki Hall)	Seminar Room 2			Lecture Hall (Akira Suzuki Hall)	Seminar Room 2		Lecture Hall (Akira Suzuki Hall)	Seminar Room 2

Abstracts
(Course Lectures)

S. Klainerman (July 2, 3, and 5)

M. Dafermos (July 3, 4, and 6)

J.-M. Delort (July 4, 5, and 6)

F. Otto (July 9, 10, and 12)

A. Chambolle (July 10, 11, and 13)

C. M. Elliott (July 11, 12, and 13)

On the Mathematical Theory of Black Holes
Sergiu Klainerman

Sergiu Klainerman

May 29, 2018

1 Basic Introduction

In my lectures [1] I restrict my attention to solutions of the Einstein vacuum equations,

$$\textbf{(EVE)} \qquad R_{\alpha\beta} = 0.$$

with $R_{\alpha\beta}$ the Ricci curvature of a four dimensional, Lorentzian manifold $(\mathcal{M}, g)$. An initial data set $(\Sigma_0, g_{(0)}, k_{(0)})$ consists of a 3 dimensional manifold Σ together with a complete Riemannian metric $g_{(0)}$ and a symmetric 2-tensor $k_{(0)}$ which verify a well known set of constraint equations. A Cauchy development of an initial data set is a globally hyperbolic space-time $(\mathcal{M}, g)$, verifying (EVE) and an embedding $i : \Sigma \longrightarrow \mathcal{M}$ such that $i_*(g_{(0)}), i_*(k_{(0)})$ are the first and second fundamental forms of $i(\Sigma_{(0)})$ in $\mathcal{M}$. A well known foundational result in GR associates a unique maximal, global hyperbolic, future development to all sufficiently regular initial data sets. I further restrict the discussion to asymptotically flat initial data sets, i.e. assume that outside a sufficiently large compact set K, $\Sigma_{(0)} \setminus K$ is diffeomorphic to the complement of the unit ball in $\mathbb{R}^3$ and admits a system of coordinates in which $g_{(0)}$ is asymptotically euclidean and $k_{(0)}$ vanishes at appropriate order at infinity.

(EVE) admits a remarkable two parameter family of explicit solutions, the Kerr spacetimes $\mathcal{K}(a, m)$, $0 \le a \le m$, which are stationary and axisymmetric. The modern understanding of a black hole is mostly based on these solutions.

[1] The general Einstein field equations connects the Ricci tensor of the space-time to the energy-momentum tensor of a matter-field. Most of the issues discussed in this proposal can be extended to various matter-fields.

2 Final State Conjecture

A far reaching conjecture in General Relativity, called the final state conjecture, states that the evolution of generic initial data sets for the Einstein vacuum field equations leads to solutions which behave, in the large, like a finite number of Kerr black holes, moving away from each other, plus a radiative decaying term.

Though the statement is consistent with both astrophysical observations and numerical simulations, it lacks a solid understanding, even at a basic heuristic level. The conjecture is regarded as the holy grail of mathematical General Relativity and is on a par, or higher, in depth and beauty with any other major open problem in Mathematics, such as the millennium Clay problems.

While still out of reach in full generality, the final state conjecture is related to a myriad of deep and challenging problems which are the focus of active research. For example, if true, the conjecture implies either that the initial data is too small to concentrate and thus must disperse to zero; "stability of Minkowski space"; or large enough to produce bound states, i.e. black holes; "problem of collapse". The fact the final states must be Kerr solutions implies, in particular, that any stationary solution of the equations must be a Kerr solution; "problem of rigidity" and that these must be stable under perturbations; "stability of Kerr". Moreover, singularities are, typically, not physically permissible; a statement postulating this is known as the "weak cosmic censorship conjecture". The final state conjecture also requires us to understand the theoretical underpinnings of interactions of black holes such as the in-spiraling detected by a historical 2015 LIGO experiment. Other fascinating mathematical problems concern the behavior of spacetime inside black holes, reach subject which I will not cover in my lectures.

3 Tests of Reality for Black Holes

In my lectures I will address the following problems, mentioned above, which I call mathematical tests of reality[2] for black holes.

1. RIGIDITY. Does the Kerr family $\mathcal{K}(a, m)$, $0 \le a \le m$, textit exhaust all possible vacuum black holes?
2. STABILITY. Is the Kerr family *stable* under arbitrary small perturbations?

[2]They all test, in some sense, the physical relevance of these mathematical solutions.

3. COLLAPSE. Can black holes form starting from *reasonable* initial data configurations? Formation of trapped surfaces.

Lecture 1. On the Reality of Black Holes

I will give an broad overview of the problems of Rigidity, Stability and Collapse and how they fit with regard to the Final State Conjecture.

Lecture 2. Rigidity of Black Holes.

The rigidity conjecture states that all asymptotically flat, stationary solutions of the Einstein vacuum equations are Kerr solutions i.e the remarkable explicit family of explicit solutions depending only on two parameters discovered by Kerr in 1963. A famous result of Hawking-Carter-Robinson solves the problem in the real analytic case. The realistic, non-analytic case turns out to be a lot more difficult and far richer mathematically. I will discuss a series of results obtained[3] in the last ten years which, though leave the main conjecture still open, establish rigidity for stationary spacetime which are close, in some precise geometric sense, to a Kerr solution.

.

Lecture 3. Problem of Collapse.

The famous singularity result of Penrose connects the presence of a trapped surfaces in the initial conditions to geodesic incompleteness. But how do trapped surfaces form in the first place? I will discuss recent important mathematical results which shed some light on this basic problem in the mathematical theory of gravitational collapse. I will concentrate on two results. The first, due to Christodoulou, illustrates the mathematical mechanism for formation of trapped surfaces in vacuum for sufficiently large, uniformly distributed characteristic initial data. The second, in collaboration with J. Luk and I. Rodnianski, dispense of uniformity by showing how one can produce a trapped surface starting with characteristic initial data heavily concentrated in a given angular direction.

Lectures 4-5. Stability.

The issue of the non linear stability of the Kerr family is one of the most pressing problems in mathematical GR today. Roughly the problem here is to show that all spacetime developments of initial data sets, sufficiently close to the initial data set of a Kerr spacetime, behave in the large like a (typically another) Kerr solution. This is not only a deep mathematical question but one with serious astrophysical implications. Indeed, if the

[3]In connection with my colleague A. Ionescu and S. Alexakis.

Kerr family would be unstable under perturbations, black holes would be nothing more than mathematical artifacts. Here is a more precise formulation of the conjecture.

Conjecture SKC. [*Stability of Kerr Conjecture*] *Vacuum initial data sets sufficiently close to Kerr initial data have a maximal development with complete future null infinity*[4] *and with domain of outer communications which approaches (globally) a nearby Kerr solution.*

So far the only space-time for which full nonlinear stability has been established is the Minkowski space, corresponding to $a = m = 0$.

Theorem 3.1 (Global Stability of Minkowski)**.** *Any asymptotically flat initial data set which is sufficiently close to the trivial one has a regular, complete, maximal development*[5]*.*

Here are, very schematically, the main ideas in the proof of stability of Minkowski space, ideas which are also relevant to the stability of the Kerr family.

(I) Perturbations radiate and decay *sufficiently fast* (just fast enough !).

(II) Interpret the Bianchi identities as a Maxwell like system. This is an effective, *invariant*, way to treat the hyperbolic character of the equations. It also provides an effective *conceptual linearization*[6] of the equations.

(III) Require four important PDE advances of late last century:

(i) Vectorfield approach to get decay based on *approximate* Killing and conformal Kiling symmetries of the equations.

(ii) Generalized energy estimates using both the Bianchi identities and the approximate Killing and conformal Killing vector fields.

(iii) The *null condition* identifies the deep mechanism for nonlinear stability, i.e. the specific structure of the nonlinear terms enables stability despite the low decay of the perturbations.

[4] This means, roughly, that observers which are far away from the black hole will not experience its effects.

[5] The complete result in [?] also provides very precise information about the decay of the curvature tensor along null and timelike directions as well as many other geometric information concerning the causal structure of the corresponding spacetime. Of particular interest are *peeling properties* i.e. the precise decay rates of various components of the curvature tensor along future null geodesics.

[6] We do not in fact linearize the Einstein equations

(iv) Complex boot-strap argument according to which one makes educated assumptions about the behavior of the space-time and then proceeds to show that they are in fact satisfied. This amounts to a *conceptual linearization*, i.e. a method by which the equations become, essentially, linear without actually linearizing them.

This fundamental and very difficult problem is the focus of very intense current research which I will try to bring in perspective. In addition to global nonlinear stability of the Minkowski space due to Christodoulou and myself as well the remarkable progress made in recent years on the problem of linear stability.

The simplest linear equation, relevant to the stability problem, is the linear wave equation on a Kerr background. Yet even in this case one encounters serious difficulties to prove boundedness and decay of solutions, for very reasonable, smooth, compactly supported, data. Below is a very short description of these.

1. *The problem of trapped null geodesics.* This concerns the existence of null geodesics[7] neither crossing the event horizon nor escaping to "null infinity", along which solutions can concentrate for arbitrary long times. This leads to degenerate Morawetz type estimates which require a very delicate analysis.

2. *The trapping properties of the horizon.* The horizon itself is ruled by null geodesics, which do not communicate with null infinity and can thus concentrate energy. This problem was solved by understanding the so called red-shift effect associated to the event horizon, which more than counteracts this type of trapping.

3. *The problem of superradiance.* This is essentially the failure of the stationary Killing field ∂_t to be everywhere timelike in the domain of outer communications, and thus, of the associated conserved energy to be positive. Note that this problem is absent in Schwarzschild and, in general, for axially symmetric solutions.

4. *Superposition problem.* This is problem of combining the estimates in the near region, close to the horizon, (including the ergoregion and trapping) with estimates in the asymptotic region, where the spacetime looks Minkowskian.

The full linear stability problem contains, in addition of these, many other difficulties which I will describe in my lectures. Chief among them is the problem of gauges. Without a well chosen gauge condition one cannot derive even the boundedness of solutions. I will discuss at length this specific problem which, of course, it is fundamental in tackling the nonlinear problem. Finally I will discuss a recent result, obtained in collaboration with

[7] In the Schwarzschild case, these geodesics are associated with the celebrated photon sphere $r = 3M$.

J. Szeftel, concerning the nonlinear stability of the Schwarzschild metric under axially symmetric, polarized, perturbations. The restriction is needed to make sure that the final state of evolution is again a Schwarzschild space. The result contains many new ideas which I hope to be able to convey in my final lecture.

Black hole interiors and the strong cosmic censorship conjectures in general relativity

Mihalis Dafermos[1,2]

[1] *University of Cambridge, Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*

[2] *Princeton University, Department of Mathematics, Fine Hall, Washington Road, Princeton, NJ 08544, United States of America*

These lectures will describe some recent developments concerning the **strong cosmic censorship** conjecture, one of the most basic problems of classical general relativity.

The conjecture, originally due to Roger Penrose [Pen74], naturally arises from a peculiar property of the celebrated Kerr solutions [Ker63] of the Einstein vacuum equations

$$\mathrm{Ric}(g) = 0. \tag{1}$$

Like with the more elementary Schwarzschild solutions, so too the Kerr solutions, thought of as the maximal Cauchy development of suitable initial data, are *geodesically incomplete*: all observers entering the black hole region live for only finite time within the solution. By Penrose's celebrated incompleteness theorem [Pen65], this ominous property is stable to pertubation of initial data. Unlike the Schwarzschild spacetime, however, in which these observers encounter a "ferocious" spacelike singularity, in Kerr, spacetime extends beyond a so-called *Cauchy horizon* [Haw67], beyond which the solution is no longer uniquely determined by initial data. Thus, general relativity cannot predict the future of incomplete observers falling into the black hole region of Kerr. *This is a spectacular failure of classical determinism!*

Strong cosmic censorship is the statement that the above pathological behaviour of Kerr should be non-generic, and thus, in any real physical situation, determinism survives. A modern mathematical formulation is as follows:

Conjecture. *For generic asymptotically flat initial data for the Einstein vacuum equations* (1)*, the maximal Cauchy development* $(\mathcal{M}, g)$ *is inextendible as a suitably regular Lorentzian manifold.*

The conjecture was not just motivated by wishful thinking! It was motivated by the fact there is a *blue-shift instability* [Pen68] associated to the Kerr Cauchy horizon, suggesting that solutions of the wave equation

$$\Box_g \psi = 0 \tag{2}$$

on a fixed Kerr background break down there.

To make the above conjecture precise, one must stipulate *in what sense* $(\mathcal{M}, g)$ is to be inextendible, that is to say, how singular must its boundary be, and thus, how definitive a statement of determinism one can obtain. The most satisfying formulation of the above conjecture would be if $(\mathcal{M}, g)$ were inextendible as a Lorentzian manifold with metric assumed to be merely continuous. This is known as the *C^0 formulation of strong cosmic censorship.* This would correspond to a very strong blow-up at the boundary of spacetime, strong enough to destroy any incomplete classical observers. A related statement is that spacetime should generically terminate in an everywhere-spacelike singularity. This is known as the *spacelike singularity conjecture.* Both these statements are true in the Schwarzschild case [Sbi18] and represented the conjectured prototype for generic singular boundaries of spacetime.

Unfortunately, it turns out that the above expectations do not appear to hold finally. In recent work with Jonathan Luk [DL17], we have shown the following theorem:

Theorem. *If the Kerr solution is stable in the exterior region of the black hole (as is widely expected to be the case!), then both the C^0 formulation of strong cosmic censorship, as well as the spacelike singularity conjecture, are in fact false.*

At first glance, the above result is surprising in view of the blue-shift instability discussed above. The above theorem should be thought of as a weak stability statement, which can be proven irrespectively of the possible loss of regularity at the Cauchy horizon due to the blue-shift. On the other hand, our results strongly suggest that an alternative version of strong cosmic censorship, due to Christodoulou [Chr09], may yet be true, and the correct picture of (at least part of) the singular boundary of spacetime is that of a *weak null singularity* [Luk18] replacing the smooth Cauchy horizon of Kerr.

In the lectures that follow, I intend to give an overview of the strong cosmic censorship conjecture culminating in a discussion of the above work. The lectures will not assume previous knowledge of the Kerr solution, and will begin with a discussion of its geometry and that of the closely related (but more elementary) Reissner–Nordström solution, as well as the problem of the linear wave equation (2) on fixed black hole interiors, which can be thought of as a model problem for study of the full non-linear (1). Though the setting of the above theorem may be unfamiliar, the proof involves many themes which have appeared widely in the theory of nonlinear wave equations of geometric type, including weighted energy estimates associated to vector fields, and null structure, as exhibited by writing the Einstein equations (1) in a double null gauge. Additional useful references, besides those mentioned above, include [Wal84, Chr99, DR13, Daf05, Daf14, Fra16, LO17a, LO17b, DSR17].

References

[Chr99] D. Christodoulou. On the global initial value problem and the issue of singularities. *Class. Quantum Grav.*, 16:A23–A35, 1999.

[Chr09] D. Christodoulou. *The formation of black holes in general relativity.* EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2009.

[Daf05] M. Dafermos. The interior of charged black holes and the problem of uniqueness in general relativity. *Comm. Pure Appl. Math.*, 58(4):445–504, 2005.

[Daf14] M. Dafermos. Black holes without spacelike singularities. *Commun. Math. Phys.*, 332(2):729–757, Dec 2014.

[DL17] M. Dafermos and J. Luk. The interior of dynamical vacuum black holes I: The C^0-stability of the Kerr Cauchy horizon. *arXiv:1710.01722, preprint*, 2017.

[DR13] M. Dafermos and I. Rodnianski. Lectures on black holes and linear waves. In *Evolution equations, Clay Mathematics Proceedings, Vol. 17*, pages 97–205. Amer. Math. Soc., Providence, RI, 2013.

[DSR17] M. Dafermos and Y. Shlapentokh-Rothman. Time-translation invariance of scattering maps and blue-shift instabilities on Kerr black hole spacetimes. *Commun. Math. Phys.*, 350(3):985–1016, 2017.

[Fra16] A. T. Franzen. Boundedness of massless scalar waves on Reissner–Nordström interior backgrounds. *Commun. Math. Phys.*, 343(2):601–650, 2016.

[Haw67] S. Hawking. The occurrence of singularities in cosmology. III. Causality and singularities. *Proc. Royal Soc. A*, 300(1461):187–201, 1967.

[Ker63] R. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, 11 (5):237–238, 1963.

[LO17a] J. Luk and S.-J. Oh. Proof of linear instability of the Reissner–Nordström Cauchy horizon under scalar perturbations. *Duke Math. J.*, 166(3):437–493, 2017.

[LO17b] J. Luk and S.-J. Oh. Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data I. The interior of the black hole region. *arXiv:1702.05715, preprint*, 2017.

[Luk18] J. Luk. Weak null singularities in general relativity. *J. Amer. Math. Soc.*, 31(1):1–63, 2018.

[Pen65] R. Penrose. Gravitational collapse and space-time singularities. *Phys. Rev. Lett.*, 14:57–59, Jan 1965.

[Pen68] R. Penrose. Structure of space-time. In C. deWitt and J. Wheeler, editors, *Battelle rencontres. 1967 lectures in mathematics and physics.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[Pen74] R. Penrose. Gravitational collapse. In C. Dewitt-Morette, editor, *Gravitational Radiation and Gravitational Collapse*, volume 64 of *IAU Symposium*, pages 82–91, 1974.

[Sbi18] J. Sbierski. The C_0-inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry. *J. Differential Geom.*, 108(2):319–378, 02 2018.

[Wal84] R. M. Wald. *General Relativity.* The University of Chicago Press, Chicago, 1984.

Long time existence results for Hamiltonian non-linear Klein-Gordon equations on some compact manifolds

Jean-Marc Delort
Université Paris 13,
Sorbonne Paris Cité, LAGA, CNRS (UMR 7539),
99, Avenue J.-B. Clément,
F-93430 Villetaneuse

1 Introduction

Consider X a compact riemannian manifold of dimension d. Let Δ be the Laplace operator on X, and if $m > 0$ is given, consider the semi-linear Klein-Gordon equation on X, given by

$$(\partial_t^2 - \Delta + m^2)w = w^{n+1} \tag{1}$$

where n is in $\mathbb{N}^*$. Take initial data $w|_{t=0} = \epsilon w_0$, $\partial_t w|_{t=0} = \epsilon w_1$, where (w_0, w_1) is in the unit ball of the space $H^{s+1}(X;\mathbb{R}) \times H^s(X;\mathbb{R})$ for some large s, and ϵ is small. Writing down the energy inequality associated to equation (1), one gets readily for any $t \geq 0$

$$\|w(t,\cdot)\|^2_{H^{s+1}} + \|\partial_t w(t,\cdot)\|^2_{H^s} \leq \|w(0,\cdot)\|^2_{H^{s+1}} + \|\partial_t w(0,\cdot)\|^2_{H^s} + C\int_0^t \|w(\tau,\cdot)\|^{n+2}_{H^{s+1}}\,d\tau, \tag{2}$$

as long as the solution exists. By a bootstrap argument, together with local existence theory, one deduces from (2) that, if s is large enough, and ϵ small enough, the solution to (1) exists over a time interval $]-T_\epsilon, T_\epsilon[$ with $T_\epsilon \geq c\epsilon^{-n}$. Moreover, on that time interval, one has for the solution an a priori estimate

$$\|w(t,\cdot)\|^2_{H^{s+1}} + \|\partial_t w(t,\cdot)\|^2_{H^s} = O(\epsilon^2). \tag{3}$$

The question we want to address is the following one:

Can we extend the solution, together with a bound of the form (3), over a longer time interval? More precisely, can we get for T_ϵ a lower bound $T_\epsilon \geq c\epsilon^{-An}$ for some $A > 1$? Under convenient assumptions, could we eventually get such a lower bound for any $A > 1$?

If instead of working on a compact manifold, we were considering the equation on $X = \mathbb{R}^d$, and if we were taking the initial data smooth, small, and *decaying* at infinity, we could use the dispersive effect of the equation, that implies that solutions of the linear evolution group, with such data, decay in L^∞ norm like $t^{-d/2}$ when t goes to infinity. This is the property that allows, in such a framework, to prove global existence in dimension larger or equal to three for instance. But when X is a compact manifold, we no longer have any dispersion, and must rely on other properties in order to extend solutions beyond the time $c\epsilon^{-n}$ given by local existence theory.

2 Normal forms

Our main tool will be normal forms, that have been introduced in this context by Shatah. Consider a model for equation (1), given by

$$(D_t - \Lambda_m)u = M(\underbrace{u,\ldots,u}_{p \text{ terms}}, \underbrace{\bar u,\ldots,\bar u}_{n+1-p \text{ terms}}) \tag{4}$$

where $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, $\Lambda_m = \sqrt{-\Delta + m^2}$, $M(u, \dots, \bar{u}) = (\Lambda_m^{-1}u)^p(\Lambda_m^{-1}\bar{u})^{n+1-p}$ and u is complex valued. Notice that if one defines $u = (D_t + \Lambda_m)w$, with w solution of (1), then it solves an equation just a little bit more general than (4), where M is a combination of monomials of the form $(\Lambda_m^{-1}u)^p(\Lambda_m^{-1}\bar{u})^{n+1-p}$, with p going from 0 to $n+1$. The idea of normal forms is to find a modification of u, of the form $v = u - \tilde{M}(u, \dots, \bar{u})$, such that v will solve an equation of the form $(D_t - \Lambda_m)v = O(v^{n'+1})$ for sme integer $n' > n$. In that way, one gets for v an energy estimate of the form

$$\|v(t,\cdot)\|_{H^s}^2 \leq \|v(0,\cdot)\|_{H^s}^2 + C\int_0^t \|v(\tau,\cdot)\|_{H^s}^{n'+2}\, d\tau \tag{5}$$

for $t \geq 0$, and deduces that if, at the initial time, $\|v(0,\cdot)\|_{H^s}^2 \leq C\epsilon^2$, then an estimate of the form $\|v(t,\cdot)\|_{H^s}^2 \leq 2C\epsilon^2$ will hold for $t \leq c\epsilon^{-n'}$ for some c. Since v is a small perturbation of u, the same type of bound holds for that function, which provides an extension of the time of existence of the solution u up to $c\epsilon^{-n'}$.

An equivalent procedure, instead of trying to modify the unknown u itself, is to introduce a modification of the energy $\Theta_s(u) = \|u\|_{H^s}^2$. Adding some correction to $\Theta_s(u)$, one defines $\tilde{\Theta}_s(u)$, of the same magnitude as $\Theta_s(u)$, in such a way that $\frac{d}{dt}[\tilde{\Theta}_s(u(t,\cdot))] = O(\|u(t,\cdot)\|^{n'+2})$, which will bring as well an estimate of the form (5).

In the case $n = 1$, corresponding to a quadratic nonlinearity in (4), such a normal form goes back to the work of Shatah [6]. If for instance X is the unit circle, so that we may use Fourier series, define from a bilinear map $M(u,u)$ another bilinear map by

$$\tilde{M}(u,u) = \sum_{k_1,k_2,k_3\in\mathbb{N}} F_m(k_1,k_2,k_3)^{-1}\Pi_{k_3}M(\Pi_{k_1}u, \Pi_{k_2}), \tag{6}$$

where F_m is to be defined and Π_k is the spectral projector associated to the eigenvalue k of $\sqrt{-\Delta}$. If u solves (4), with its right hand side replaced by $M(u,u)$, and if we set $v = u - \tilde{M}(u,u)$, one gets by a direct calculation

$$(D_t - \Lambda_m)v = \Lambda_m\tilde{M}(u,u) - \tilde{M}(\Lambda_m u, u) - \tilde{M}(u, \Lambda_m u) + M(u,u) + \text{ cubic terms.} \tag{7}$$

In order to get for v an equation involving only cubic terms in the right hand side, one has just to find $\tilde{M}$ so that

$$\tilde{M}(\Lambda_m u, u) + \tilde{M}(u, \Lambda_m u) - \Lambda_m\tilde{M}(u,u) = M(u,u). \tag{8}$$

If we make act on the above equation the spectral projector Π_{k_3}, and replace the first (resp. second) argument u by $\Pi_{k_1}u$ (resp. $\Pi_{k_2}u$), we see, using that $\Lambda_m\Pi_k = \sqrt{m^2+k^2}\Pi_k$, that (8) will be satisfied if we define $\tilde{M}$ by (6) with

$$F_m(k_1,k_2,k_3) = \sqrt{m^2+k_1^2} + \sqrt{m^2+k_2^2} - \sqrt{m^2+k_3^2}. \tag{9}$$

To conclude the proof, one has to check that this F_m does not vanish, and that the $\tilde{M}$ it defines is bounded on H^s for large enough s. This follows from the fact that when $\Pi_{k_3}\big((\Pi_{k_1}u)(\Pi_{k_2}u)\big) \not\equiv 0$, then one has a relation $\pm k_3 = \pm k_1 + \pm k_2$, and that for indices satisfying that relation, and for any m,

$$|F_m(k_1,k_2,k_3)| \geq c(1 + \min(k_1,k_2,k_3))^{-1}. \tag{10}$$

Our goal, following Bourgain [2], is to proceed in a similar way, for non linearities M that are no longer quadratic, but homogeneous of degree $n+1$ as in (4). The point is that in (9), we shall have $n+2$ terms instead of three, so that the right hand side of this equality will necessarily vanish for some values of the k_j. To circumvent that problem, we shall have to exclude some values of m, i.e. to take m outside a subset of zero measure. In that way, we shall be able to obtain in general only a weaker lower bound than (10), namely a lower bound in terms of some large negative power of *the largest* of the arguments of F_m (instead of the minimum – or the third largest – as in (10)). Because of that, a perturbation $\tilde{M}$ like (8) is no longer bounded from H^s to itself: some derivative losses appear. Though, we shall show that, for nonlinearities as in (4), we may partly compensate that loss in order to get an improvement of the time of existence of solutions for that problem, on an *arbitrary* compact manifold. We state this result in the next section. In the following one, we move to the problem of getting a lower bound for the time of existence in $c_N\epsilon^{-N}$ for *any* N. We are not able to do that on an arbitrary manifold, and instead will limit ourselves to *Zoll* manifolds. We shall consider both the case of Hamiltonian semi-linear and quasi-linear equations.

3 Semi-linear equations on a compact manifold

The first result we shall discuss is the following one, obtained in joint work with Rafik Imekraz [3]:

Theorem 1 *Let (X,g) be a compact riemannian manifold. There is $\mathcal{N}\subset]0,+\infty[$, a subset of zero measure, and for any m in $]0,+\infty[-\mathcal{N}$, there are $A>1$, $s_0>0$ and for any $s>s_0$, there are $c>0, K>0, \epsilon_0>0$ such that, for any (w_0,w_1) in the unit ball for $H^{s+1}(X,\mathbb{R})\times H^s(X,\mathbb{R})$, any $\epsilon\in]0,\epsilon_0[$, equation (1) with data $w|_{t=0}=\epsilon w_0$, $\partial_t w|_{t=0}=\epsilon w_1$, has a unique solution w in $C^0(]-T_\epsilon,T_\epsilon[,H^{s+1})\cap C^1(]-T_\epsilon,T_\epsilon[,H^s)$ with $T_\epsilon\geq c\epsilon^{-An}$. Moreover for any $t\in]-T_\epsilon,T_\epsilon[$, one has*

$$\|w(t,\cdot)\|_{H^{s+1}}+\|\partial_t w(t,\cdot)\|_{H^s}\leq K\epsilon. \tag{11}$$

The first step in the proof of the above theorem is to define a counterpart of the spectral projectors Π_k of the preceding section. One actually uses the Weyl asymptotics for the eigenvalues of $\sqrt{-\Delta}$ to check that, on any compact manifold of dimension d, one may construct a sequence $(I_k)_{k\in\mathbb{N}}$ of disjoint intervals satisfying the following properties:

$$\begin{gathered}\text{Length}(I_k)=O(1),\ \text{dist}(I_k,I_{k+1})\geq c(1+k)^{-(d-1)},\\ \text{dist}(I_k,0)\sim k,\ \#\big(I_k\cap\text{Spec}(\sqrt{-\Delta})\big)=O(k^{d-1}).\end{gathered} \tag{12}$$

Denote by Π_k the spectral projector associated to interval I_k. Let us give the idea of the proof on the model given by (4), assuming moreover that each interval I_k above contains just one eigenvalue ξ_k of $\sqrt{-\Delta}$. Proceeding as in section 2, but with the spectral projectors we just defined, we could try to write an equation for $v=u-\tilde{M}(u,\dots,\bar{u})$, where instead of (6), $\tilde{M}$ would be given by

$$\tilde{M}(u_1,\dots,u_{n+1})=\sum_{k_1,\dots,k_{n+2}\in\mathbb{N}}F_m(k_1,\dots,k_{n+2})^{-1}\Pi_{k_{n+2}}M(\Pi_{k_1}u_1,\dots,\Pi_{k_{n+1}}u_{n+1}) \tag{13}$$

with

$$F_m(k_1,\dots,k_{n+2})=\sum_{j=1}^{p}\sqrt{m^2+\xi_{k_j}^2}-\sum_{j=p+1}^{n+2}\sqrt{m^2+\xi_{k_j}^2}, \tag{14}$$

in order to eliminate the right hand side of (4), up to some remainders vanishing at higher order at zero. To proceed with the analogy, we would need two things: First, an estimate on multilinear expressions of the form

$$\Pi_{k_{n+2}}M(\Pi_{k_1}u_1,\dots,\Pi_{k_{n+1}}u_{n+1})$$

that would allow us to obtain Sobolev estimates on such multilinear expressions. Second, a bound that would replace (10). The first estimate is easy to prove, and may be found in Delort-Szeftel [5]. It says that there is ν and for any N a constant C such that, for any $k_1,\dots,k_{n+2}$ in $\mathbb{N}$, any $u_1,\dots,u_{n+1}$

$$\|\Pi_{k_{n+2}}M(\Pi_{k_1}u_1,\dots,\Pi_{k_{n+1}}u_{n+1})\|_{L^2}\leq\frac{(k_3^*)^{\nu+N}}{(k_1^*-k_2^*+k_3^*)^N}\prod_{j=1}^{n+1}\|u_j\|_{L^2} \tag{15}$$

where $k_1^* \geq k_2^* \geq k_3^*$ are the largest three among the integers $k_1+1,\dots,k_{n+2}+1$. It is easy to see that such an estimate implies that $u \to M(u,\dots,\bar{u})$ is bounded from H^s to H^s if s is large enough (with respect to ν). Concerning the second estimate, analogous to (10), it is clear that we have to impose some further assumptions. Actually, we notice first that if $n+2$ is even and $p = \frac{n+2}{2}$, expression (14) may vanish identically if $\{k_1,\dots,k_p\} = \{k_{p+1},\dots,k_{n+2}\}$. Even when we are not in this case, (14) may vanish for some values of m. Nevertheless, it is proved in [4] that one may find a zero measure subset $\mathcal{N}$ of $]0,+\infty[$, such that, if m is not in that subset, there are $c>0, N_0 \in \mathbb{N}$, such that for any integers $k_1,\dots k_{n+2}$, one has

$$|F_m(k_1,\dots,k_{n+2})| \geq c(1+k_1+\cdots+k_{n+2})^{-N_0}, \tag{16}$$

except when $p = \frac{n+2}{2}$ and $\{k_1,\dots,k_p\} = \{k_{p+1},\dots,k_{n+2}\}$. Notice that this condition is much less favorable that (10), as if we plug this lower bound in the general term of the right hand side of (13), and bound then the L^2 norm using estimate (15), we shall get a bound in terms of

$$\frac{(k_1^*)^{N_0}(k_3^*)^{\nu+N}}{(k_1^*-k_2^*+k_3^*)^N}\prod_{j=1}^{n+1}\|u_j\|_{L^2}. \tag{17}$$

The extra factor $(k_1^*)^{N_0}$ means that we lose N_0 derivatives, i.e. that $u \to \tilde{M}(u,\dots,\bar{u})$ sends H^s to H^{s-N_0}. This makes the whole construction useless. But one may notice that in the definition $M(u,\dots,\bar{u}) = (\Lambda_m^{-1}u)^p(\Lambda_m^{-1}\bar{u})^{n+1-p}$ of the right hand side of (4), we have the operators Λ_m^{-1} which make gain one derivative. Consequently, if we decompose $u = u_L + u_H$ where u_L is the low frequency part of u, given by $u_L = \sum_{k\leq\epsilon^{-\delta}}\Pi_k u$, then $\|\Lambda_m^{-1}u_H\|_{L^2} = O(\epsilon^\delta\|u\|_{L^2})$. Because of that, the contributions to the right hand side of (4) where one of the factors u has been replaced by u_H will have a Sobolev bound in $\epsilon^\delta\|u\|_{H^s}^{n+1}$. As ϵ is the size of u in H^s, this means that these terms behave like non linearities vanishing at order $n+1+\delta$ at zero, and thus do not play a role if one wants to get a priori estimates up to a time in $\epsilon^{-n-\delta}$. For the remaining contributions, i.e. those of the form $(\Lambda_m^{-1}u_L)^p(\Lambda_m^{-1}\bar{u_L})^{n+1-p}$, one tries to eliminate them through the construction of some $\tilde{M}$ as in (13). The fact that in (17) we have now $k_1^* \leq \epsilon^{-\delta}$, replaces the loss of N_0 derivatives by the loss of a factor $\epsilon^{-N_0\delta}$. Optimizing δ one gets a proof of the theorem, using the fact that, in the case when we cannot make a normal form, namely $p = \frac{n+2}{2}$ and $\{k_1,\dots,k_p\} = \{k_{p+1},\dots,k_{n+2}\}$, the contribution of the corresponding terms to an energy inequality is actually zero.

4 Hamiltonian nonlinear equations on Zoll manifolds

We would like now to obtain a better result than Theorem 1, namely to prove that solutions exist up to a time $T_\epsilon \geq c_N\epsilon^{-N}$ for any N, and satisfy (11) on such an interval. We are not able to prove such a result on any compact manifold, because of the loss of derivatives that appears in (17) and that comes from (16). We have to find a framework where the factor $(k_1^*)^{N_0}$ in (17) is replaced by $(k_3^*)^{N_0}$, that allows then to show that (13) is bounded from H^s to H^s for large enough s. In other words, we have to be able to improve the lower bound (16), in order to get in the right hand side a negative power of the *third* largest among $k_1,\dots,k_{n+2}$ instead of the largest one. We shall be able do so in the framework of Zoll manifolds.

Definition 2 *A Zoll manifold is a compact riemannian manifold over which the geodesic flow is periodic.*

The main example of Zoll manifolds are spheres. Zoll manifolds satisfy the following property, that actually characterizes them : If X is Zoll, the spectrum of $\sqrt{-\Delta}$ is contained in a union of disjoint intervals I_k, centered at an arithmetic progression αk, and of length $O(k^{-\sigma})$ for some $\sigma > 0$ (actually one may take $\sigma = 1$). In other words, while in the general case we could include the spectrum of $\sqrt{-\Delta}$ only in intervals separated by a negative power of k, we have now that the distance between two successive I_k's is of order one, while the length of these intervals is $O(k^{-\sigma})$. Using these properties, one may show that (16) may be improved to

$$|F_m(k_1,\dots,k_{n+2})| \geq c(k_3^*)^{-N_0}, \tag{18}$$

where k_3^* is the third largest among $k_1,\dots,k_{n+2}$. This property implies that if $\tilde{M}$ is defined by (13), it satisfies the same estimates as M in (15), with ν replaced by $\nu+N_0$, so that $u \to M(u,\dots,\bar{u})$ is bounded from H^s to itself. Then $v = u - \tilde{M}(u,\dots,\bar{u})$ is a small perturbation of u, and satisfies an equation of the form

$$(D_t - \Lambda_m)v = O(v^{2n+1}) \tag{19}$$

in the case when n is odd. This shows that one may obtain for v an estimate in $O(\epsilon)$ up to a time in ϵ^{-2n}, and thus extend the solution up to that time, in the case of a Zoll manifold.

Thought, this is still not the result we want, as our goal is to reach times in ϵ^{-N} for *any* N. The idea would be then to iterate the preceding operation. The difficulty comes from the fact that, as we already remarked, we get in any case perturbations that cannot be eliminated, corresponding to the case n even, $p = \frac{n+2}{2}$ and $\{k_1,\dots,k_p\} = \{k_{p+1},\dots,k_{n+2}\}$ in (13). When one considers equation (1) rewritten under the form (4), one may check by hand that these terms, that cannot be eliminated through normal forms, give a zero contribution to the energy. But if one iterates the construction, it is not really possible to check directly, at any order of the induction, that the terms that cannot be eliminated actually do not make grow the energy. One has instead to find a structure that is preserved along the iterations, and that ensures that this compatibility condition always holds. We shall use the *Hamiltonian* character of the equation, in order to prove (see Bambusi-Delort-Grébert-Szeftel [1]):

Theorem 3 *Let X be a Zoll manifold. Let P be a polynomial, vanishing at leat at order 2 at zero. There is a subset $\mathcal{N}$ of $]0,+\infty[$ of zero measure and, for any m in $]0,+\infty[-\mathcal{N}$, any N in $\mathbb{N}$, there is $s_0 > 0$ and for any $s > s_0$, there are $\epsilon_0 > 0, c > 0, C > 0$ such that, for any $\epsilon \in]0,\epsilon_0[$, any (w_0,w_1) in the unit ball of $H^{s+1}(X)\times H^s(X)$, one may find a unique w in $C^0(]-T_\epsilon,T_\epsilon[,H^{s+1}(X)) \cap C^1(]-T_\epsilon,T_\epsilon[,H^s(X))$ solution of*

$$\begin{aligned}(\partial_t^2 - \Delta + m^2)w &= P[w]\\ w|_{t=0} &= \epsilon w_0\\ \partial_t w|_{t=0} &= \epsilon w_1,\end{aligned} \tag{20}$$

where $T_\epsilon \geq c\epsilon^{-N}$ and

$$\sup_{]-T_\epsilon,T_\epsilon[} [\|w(t,\cdot)\|_{H^{s+1}} + \|\partial_t w(t,\cdot)\|_{H^s}] \leq C\epsilon.$$

To prove that theorem, one rewrites the problem under a Hamiltonian formulation. Introducing a complex unknown, as we did in (4), equation (20) may be reduced to

$$\partial_t u = i\nabla_{\bar{u}} G(u,\bar{u}) \tag{21}$$

where if $U = (u,\bar{u})$, $G(U) = \sum_{p=0}^P G_p(U)$ with

$$G_0(U) = \int_X \Lambda_m u\bar{u}\,dx,\ G_p(U) = \sum_{\ell=0}^{p+1}\int_X P_{p,\ell}(u,\bar{u})\,dx,\ p\geq 1 \tag{22}$$

where G_p is real valued and $P_{p,\ell}$ is a polynomial homogeneous of degree ℓ in u and $p+2-\ell$ in $\bar{u}$. One starts from a quadratic map $U \to \Theta_s(U)$ on H^s, equivalent to the square of the H^s norm, and one tries to construct from it a new map $U \to \tilde{\Theta}_s(U)$, that will be equivalent to $\Theta_s(U)$ for small U, and such that

$$\frac{d}{dt}\tilde{\Theta}_s(U(t,\cdot)) = O(\|U(t,\cdot)\|_{H^s}^{N+2}). \tag{23}$$

This is an estimate similar to (5), but with n replaced by an arbitrary N, that allows to obtain the conclusion of the theorem. The idea is to replace the normal forms method we used earlier by a Birkhoff one, looking for $\tilde{\Theta}_s$ under the form $\tilde{\Theta}_s(U) = \Theta_s \circ \chi(U)$, where χ is some canonical transformation, tangent to the identity at zero, such that (23) holds. If one defines the Poisson bracket between to C^1 functions F_1, F_2 defined on H^s by $\{F_1, F_2\} = i[D_u F_1 \nabla_{\bar{u}} F_2 - D_u F_2 \nabla_{\bar{u}} F_1]$, one gets when u solves (21),

$$\frac{d}{dt}\Theta_s \circ \chi(U(t,\cdot)) = \{\Theta_s \circ \chi, G\}(U(t,\cdot)) = \{\Theta_s, G \circ \chi^{-1}\} \circ \chi(U(t,\cdot)). \tag{24}$$

One has to construct χ in such a way that the last term will be $O(\|u\|_{H^s}^{N+2})$. Looking for χ in terms of a generating function $Q = \sum_p Q_p$, one is reduced to a sequence of *homological* equations

$$\{\Theta_s, -\{Q_p, G_0\} + H_p\} = 0,\ p \geq 1 \tag{25}$$

where Q_p is the unknown, H_p is given and G_0 is given by (22). It turns out that the resolution of (25) may be done using formulas like (13), (14). Because for Zoll manifolds we have a lower bound of the form (18), one may get a solution that does not lose derivatives. Moreover, in the case when (18) does not hold because F_m vanishes identically, the corresponding contribution to H_p in (25) is in the kernel of the map $\{\Theta_s, \cdot\}$, so it does not need to be eliminated.

The preceding theorem may be extended to Hamiltonian quasi-linear equations, namely when, in local coordinates,

$$P[w] = \frac{\partial f}{\partial z}\Big(w, x, \frac{\partial w}{\partial x}\Big) - \sum_{j=1}^{d} (\det g)^{-1/2} \frac{\partial}{\partial x_j}\Big[(\det g)^{1/2}\Big(\frac{\partial f}{\partial \xi_j}\Big)\Big(w, x, \frac{\partial w}{\partial x}\Big)\Big],$$

where $(z, \rho) \to f(z, \rho)$ is some smooth function on $\mathbb{R} \times T^*X$, polynomial in the fiber variable, vanishing at least at order three at $z = 0, \rho \in X$. The same statement as Theorem 3 holds then on spheres, for such quasi-linear nonlinearities. The proof relies on the same ideas as above, but is much more involved, because of the quasi-linear character of the problem.

Short Bibliography

[1] D. Bambusi, J.-M. Delort, B. Grébert, and J. Szeftel. Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds. *Comm. Pure Appl. Math.*, 60(11):1665–1690, 2007.

[2] J. Bourgain. Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations. *Geom. Funct. Anal.*, 6(2):201–230, 1996.

[3] J.-M. Delort and R. Imekraz. Long-time existence for the semilinear Klein-Gordon equation on a compact boundary-less Riemannian manifold. *Comm. Partial Differential Equations*, 42(3):388–416, 2017.

[4] J.-M. Delort and J. Szeftel. Long-time existence for small data nonlinear Klein-Gordon equations on tori and spheres. *Int. Math. Res. Not.*, (37):1897–1966, 2004.

[5] J.-M. Delort and J. Szeftel. Long-time existence for semi-linear Klein-Gordon equations with small Cauchy data on Zoll manifolds. *Amer. J. Math.*, 128(5):1187–1218, 2006.

[6] J. Shatah. Normal forms and quadratic nonlinear Klein-Gordon equations. *Comm. Pure Appl. Math.*, 38(5):685–696, 1985.

Nonlinear partial differential equations and interfacial motions as gradient flow

Felix Otto
Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig, Germany

In these three talks, we advocate uncovering and using gradient flow structures to gain a better understanding of evolutionary nonlinear partial differential equations and of interfacial motions. In particular, we advocate not to shy away from handling, even if just in a formal way, gradient flow structures in an infinite-dimensional Riemannian setting.

0.1. **General remarks.** Intuitively speaking, a gradient flow is a steepest descent in a landscape. Such a notion obviously relies on two elements: 1) a height function and 2) a notion of (horizontal) distances between points – without the latter, there is no concept of steepness. More mathematically speaking, a gradient flow is determined by two structures on a given space $\mathcal{M}$ (which we think of as a differentiable manifold): 1) A function E on $\mathcal{M}$, 2) a metric tensor g on $\mathcal{M}$, that is, a scalar product on the tangent space $T_\rho\mathcal{M}$ of $\mathcal{M}$ in any point $\rho \in \mathcal{M}$, providing $\mathcal{M}$ with a Riemannian structure. It is the metric tensor that converts the linear form $\mathrm{diff}_{|\rho}E$ on the tangent space (that is, a co-tangent vector) into the tangent vector $\mathrm{grad}_{|\rho}E$ by providing a canonical isomorphism (i. e. the Riesz isomorphism) between the dual $T_\rho\mathcal{M}^*$ of the tangent space and the tangent space itself. It is only the tangent vector field $\mathrm{grad}E$ that defines the dynamical system $\frac{d\rho}{dt} = -\mathrm{grad}_{|\rho}E$.

It is an immediate consequence that E is a Liapunov functional:

$$\frac{d}{dt}E(\rho) = -g_\rho(\mathrm{grad}_{|\rho}E, \mathrm{grad}_\rho E) = -g_\rho(\frac{d\rho}{dt}, \frac{d\rho}{dt}) \leq 0. \tag{1}$$

An important global object of the Riemannian space $(\mathcal{M}, g)$ is the induced distance $\mathrm{d}(\rho_0, \rho_1)$ between to points $\rho_0, \rho_1 \in \mathcal{M}$ defined as the length of the shortest connecting curve $[0,1] \ni s \mapsto \rho(s) \in \mathcal{M}$; it is more convenient to define its square as the minimal action:

$$\mathrm{d}^2(\rho_0, \rho_1) = \inf\{\int_0^1 g_\rho(\frac{d\rho}{ds}, \frac{d\rho}{ds})ds \,|\, \rho(0) = \rho_0, \rho(1) = \rho_1\}; \tag{2}$$

the Euler-Lagrange equation of this variational problem provides the geodesic equation.

Any gradient flow allows for a natural time discretization given by a sequence of variational problems: Given $\rho^{(n-1)} \in \mathcal{M}$ at time step $n-1$ (i. e. at time $t = (n-1)h$, where h denotes the time step size), $\rho^n \in \mathcal{M}$ at time step n minimizes

$$\frac{1}{2h}\mathrm{d}^2(\rho, \rho^{n-1}) + E(\rho), \tag{3}$$

which following de Giorgi is called minimizing movements scheme. In the Euclidean case, this discretization turns into the implicit Euler scheme. The merit of (3) consists in only appealing to the metric, not the differential structure. De Giorgi observed that gradient flows can be (formally) characterized by the following *in*equality

$$\begin{aligned} & E(\rho(T)) + \int_0^T \Big(\frac{1}{2} g_\rho(\mathrm{grad}_{|\rho} E, \mathrm{grad}_{|\rho} E) + \frac{1}{2} g_\rho(\frac{d\rho}{dt}, \frac{d\rho}{dt})\Big) dt \\ & \qquad \le E(\rho(0)) \end{aligned} \tag{4}$$

and developed tools to establish the convergence of (3) on this level, see Subsection 0.5.

Models from continuum mechanics in which frictional effects are so dominant that inertia is completely negligible typically have the structure of a gradient flow, which reflects their thermodynamic consistency. In our applications, $\mathcal{M}$ will be an (infinite-dimensional) configuration space, E will be a (free) energy functional, and g will encode the dissipation mechanisms. It is easy to see that for a gradient flow, at any given time t, the rate of change $\frac{d\rho}{dt}(t)$ is determined as the minimizer of

$$\frac{1}{2} g_{\rho(t)}(\delta\rho, \delta\rho) + \mathrm{diff} E_{|\rho(t)}.\delta\rho$$

among all $\delta\rho \in T_{\rho(t)}\mathcal{M}$. Here $g_{\rho(t)}(\delta\rho, \delta\rho)$ has the physical interpretation of infinitesimal dissipation generated by the infinitesimal ("virtual") change $\delta\rho$ of the configuration $\rho(t)$, and $\mathrm{diff} E_{|\rho(t)}.\delta\rho$ is the infinitesimal change δE in energy. The Euler-Lagrange equation of this quadratic variational problem is given by

$$g_{\rho(t)}(\delta\rho, \frac{d\rho}{dt}(t)) + \mathrm{diff} E_{|\rho(t)}.\delta\rho = 0 \tag{5}$$

for all $\delta\rho \in T_{\rho(t)}\mathcal{M}$. In the applied rheological literature, (5) is often used to build evolutionary models.

0.2. **Title and summary of the three talks.** The titles and plan for these three lectures is as follows:

A) The porous medium equation as a gradient flow: contraction in the Wasserstein metric. This is based on [3] and on joint work with Michael Westdickenberg [5].
B) Diffusive phase transitions as a gradient flows: upper bounds on coarsening rates. This is based on joint work with Robert V. Kohn [1] and, if time permits, on joint work with Christian Seis and Dejan Slepcev [4].
C) Flow by mean curvature as a gradient flow: convergence of the thresholding scheme. This is based on joint work with Tim Laux [2].

The cited works contains the further relevant references, which are only alluded to in this extended abstract.

From an abstract point of view, the energy landscape in A) is (geodesically) convex, while in B) and C) it is highly non-convex. As formally expected, the convexity of the energy landscape in A), which means that E is convex along geodesics, translates into contractivity of the gradient flow with respect to the induced distance function (2), which means that for two gradient flow trajectories u_0, u_1, $t \mapsto \mathrm{d}(u_0(t), u_1(t))$ is non-increasing. In B) more global properties of the energy landscape lead to global properties of the gradient flow evolution. More precisely, if the energy cannot decrease too fast as a function of the induced distance $\mathrm{d}(\cdot, u_0)$, cf. (2), to a reference configuration u_0, the energy cannot decrease too fast as a function of time if the initial data are sufficiently close to u_0. In C) we interpret a numerical scheme as a natural time discretization (3) of a gradient flow and use de Giorgi's characterization (4) to pass to the limit of vanishing time step size.

0.3. **The porous medium equation as gradient flow: contraction in the Wasserstein metric.** In the first talk, we interpret the porous medium equation

$$\partial_t \rho - \triangle \pi(\rho) = 0, \tag{6}$$

say on the torus $[0, L)^d$, which is a ubiquitous nonlinear (degenerate) parabolic equation, as the gradient flow with respect to

$$\begin{aligned} E(\rho) &:= \int e(\rho) dx, \\ g_\rho(\delta\rho_0, \delta\rho_1) &:= \int \nabla\phi_0 \cdot \nabla\phi_1 \rho dx \quad \text{where } -\nabla \cdot \rho \nabla \phi_i = \delta\rho_i. \end{aligned}$$

Here $\rho = \rho(t, x) \geq 0$ describes the density of a gas in a porous medium with an (internal) energy function $e(\rho)$ related to the "osmotic pressure" function $\pi(\rho)$ via $\pi(\rho) = \rho e'(\rho) - e(\rho)$.

An easy (formal) calculation shows that the metric tensor is of the form

$$\begin{aligned} g_\rho(\delta\rho, \delta\rho) &= \inf\{\int \frac{1}{\rho}|j|^2 dx \,|\, \delta\rho + \nabla \cdot j = 0\} \\ &= \inf\{\int \rho |u|^2 dx \,|\, \delta\rho + \nabla \cdot (\rho u) = 0\}, \end{aligned} \tag{7}$$

which allows for its interpretation as the (minimal) dissipation needed to generate the infinitesimal density variation $\delta\rho$ via a (virtual) velocity u. Formally, the gradient flow structure follows from rewriting (6) as $\partial_t \rho - \nabla \cdot \rho \nabla e'(\rho) = 0$, a couple of integrations by parts, and appealing to (5).

The first representation in (7) shows via the action formulation (2) that the induced distance is given by the convex (!) variational problem

$$\begin{aligned}&\mathrm{d}^2(\rho_0,\rho_1)\\&=\inf\{\int_0^1\int\frac{1}{\rho}|j|^2dxds\,|\,\partial_s\rho+\nabla\cdot j=0,\ \rho(0)=\rho_0,\rho(1)=\rho_1\},\end{aligned}$$

which is the Eulerian or Benamou-Brenier formulation of the Wasserstein distance. Hence, formally, the induced distance is given by the optimal transportation distance

$$\mathrm{d}^2(\rho_0,\rho_1)=W^2(\rho_0,\rho_1):=\min_\pi\int|x_0-x_1|^2\pi(dx_0dx_1), \tag{8}$$

where the infimum is taken over all non-negative measures π on the product space that have marginals ρ_0dx_0 and ρ_1dx_1. Minimizers π concentrate on the graph of the (sub-)gradient of a convex function so that geodesics are well-understood. McCann realized that the functional of the type of E is (geodesically) convex provided it is convex under the linear change of coordinate $x=A\hat{x}$ with symmetric and positive definite A. In terms of π this condition assumes the form

$$\rho\pi'(\rho)\geq(1-\frac{1}{d})\pi(\rho)\geq0\quad\text{for all }\rho\geq0. \tag{9}$$

The main result is

Theorem 1. *Suppose (9) holds and let ρ_0,ρ_1 be two solutions of (6). Then $W(\rho_0(t),\rho_1(t))$ is non-increasing.*

In fact, this result remains unchanged if the torus is replaced by a (finite-dimensional, compact and smooth) Riemannian manifold provided it has *non-negative Ricci curvature.*

In establishing this contraction result, we don't appeal to geodesics with their subtle regularity (especially in case of an underlying Riemannian manifold). In abstract Riemannian terms, instead of probing the convexity of the function E, that is, the positive semi-definiteness of its Hessian Hess$E$, as is usual along geodesics $s\mapsto\rho(s)$, that is,

$$\frac{D}{ds}\frac{d\rho}{ds}=0, \tag{10}$$

for which we have

$$\frac{d^2}{ds^2}E(\rho)=g_\rho(\frac{d\rho}{ds},\mathrm{Hess}E_{|\rho}\frac{d\rho}{ds}),$$

we probe it along infinitesimal variations $t\mapsto\delta\rho(t)$ of a gradient flow trajectory $t\mapsto\rho(t)$, that is,

$$\frac{D}{dt}\delta\rho+\mathrm{Hess}E_{|\rho}\delta\rho=0, \tag{11}$$

for which we have

$$\frac{d}{dt}\frac{1}{2}g_\rho(\delta\rho,\delta\rho) = -g_\rho(\delta\rho, \mathrm{Hess}E_{|\rho}\delta\rho). \tag{12}$$

Hence we replace (10), which has the character of a hyperbolic PDE, by (11), which has the character of a parabolic PDE, Moreover, (12) is directly monitoring contractivity, the property we are aiming for, on an infinitesimal level. On the concrete level of the porous medium equation, (12) turns into an explicit but tedious calculation, which thanks to its Riemannian interpretation can be well-organized.

The porous medium equation (6) has been extensively studied for many decades; its contractivity in L^1 and in $\dot{H}^{-1}$ (the dual to the homogeneous H^1-norm) are key elements of its classical well-posedness theory. While the contractivity in $\dot{H}^{-1}$ was assimilated to the (Euclidean) gradient flow structure with respect to this Hilbert space, the contractivity in the Wasserstein distance was only discovered alongside with the above (formal) Riemannian gradient flow structure. It is more physical and detects the (Ricci) curvature of the underlying space.

0.4. **Diffusive phase transitions as a gradient flows: upper bounds on coarsening rates.** In this second talk, we are interested in the interfacial motion often named after Mullins-Sekerka. It describes the evolution $t \mapsto \Omega(t) \subset [0,L)^d$ of an (open) subset of the torus $[0,L)^d$ in terms of the normal velocity V of its boundary $\partial\Omega$ via

$$V = [\frac{\partial p}{\partial \nu}],$$

where $[\frac{\partial p}{\partial \nu}]$ denotes the jump in the normal component of ∇p across $\partial\Omega$, where p is determined by Dirichlet problem

$$\triangle p = 0 \text{ in } [0,L)^d - \partial\Omega, \quad p = H \text{ on } \partial\Omega,$$

with H denoting the mean curvature of $\partial\Omega$. The linearization around a flat interface reveals that this is a third-order (and thus necessarily and obviously non-local) parabolic interfacial motion, for which there exists a classical local well-posedness theory.

On a formal level, it is easy to check that this evolution is a gradient flow. Here, the configurational space $\mathcal{M}$ is given by all sets Ω of a given volume (say $\frac{1}{2}L^d$ and thus of volume fraction $\frac{1}{2}$), so that the tangent space $T_\Omega\mathcal{M}$ can be identified with (virtual) normal velocities V (with $\int_{\partial\Omega} V = 0$). Function and tensor are then given by

$$\begin{aligned} &E(\Omega) = \mathcal{H}^{d-1}(\partial\Omega), \\ &g_\Omega(V_0,V_1) = \int \nabla\phi_0 \cdot \nabla\phi_1 dx \ \text{ where } \ -\triangle\phi_i = V_i\mathcal{H}^{d-1}\lfloor\partial\Omega, \end{aligned} \tag{13}$$

where the last equation is to be read distributionally.

Obviously, E has the interpretation of an interfacial energy between the two phases Ω and Ω^c. The physical interpretation of the metric tensor becomes more apparent when rewriting it as

$$g_\Omega(V,V) = \inf\{\int |j|^2 \mid \nabla \cdot j = V\mathcal{H}^{d-1}\lfloor\partial\Omega\}; \tag{14}$$

where j is interpreted as the (virtual) diffusion flux j in the bulk that generates the infinitesimal change V of the phase boundary. Hence this evolution describes a diffusive phase transition, were "atoms" that predominantly form phase Ω detach from it, diffuse through the complement Ω^c and re-attach (and vice versa for the other type of atom, "evaporation-recondensation"). Mathematically, it is known by asymptotic and (short-time) rigorous analysis to be the sharp-interface limit of the Cahn-Hilliard equation, which is the $\dot{H}^{-1}$-gradient flow of the (real) Ginzburg-Landau functional.

Mullins-Sekerka (in form of the underlying Cahn-Hilliard equation) is appreciated in chemical physics because it numerically reproduces the experimentally observed universality in demixing ("spinodal decomposition"): The typical length scale ℓ of the phase configuration Ω increases over time ("coarsening") in a statistically self-similar way (until it enters in conflict with the sample size which here is imposed through the side-length L of the torus). By a simple scaling argument, the coarsening exponent is predicted to be $\frac{1}{3}$, which is reproduced by numerical simulation and seen in experiments. This means that, at least generically, the interfacial energy per volume, $\frac{1}{L^d}E(\Omega(t))$, is expected to decrease like $\sim t^{-\frac{1}{3}}$. We present a rigorous argument that this is indeed the case on the level of a lower bound.

Our argument is based on the gradient flow structure. As one might guess from (14), the metric structure comes from the Euclidean $\dot{H}^{-1}$-structure restricted to characteristic functions $\chi_\Omega \in \{-\frac{1}{2}, \frac{1}{2}\}$ of sets Ω; we thus substitute the (non-explicit) induced distance $d(\Omega_0,\Omega_1)$ by the smaller distance $\|\chi_{\Omega_0} - \chi_{\Omega_1}\|_{\dot{H}^{-1}}$ in the ambient space ("arc vs. chord distance"). With respect to this distance, one has that E does not decrease faster than the inverse of $\|\chi_\Omega\|_{\dot{H}^{-1}}$, the latter being the chord distance to the well-mixed state:

$$E(\Omega)\|\chi_\Omega\|_{\dot{H}^{-1}} \geq c(d)(L^d)^{\frac{3}{2}}. \tag{15}$$

In fact, this statement is a consequence of the (scale-invariant) interpolation estimate $\|f\|_{L^{\frac{4}{3}}} \leq C(d)\|\nabla f\|_{L^1}^{\frac{1}{2}}\|f\|_{\dot{H}^{-1}}^{\frac{1}{2}}$ by Cohen-Dahmen-Daubechies-DeVore, which is of independent interest.

Equipped with this property of the energy landscape "in the large", we prove that provided the initial data are (intrinsically) close to being well-mixed, the demixing cannot proceed faster than expected, in the sense that the volume-averaged interfacial energy $\frac{1}{L^d}E(\Omega(t))$ cannot decrease faster than $t^{-\frac{1}{3}}$, at least in a time-averaged fashion.

Theorem 2. *For all $T \geq \|\chi_{\Omega(0)}\|_{\dot{H}^{-1}}^3$ we have*

$$\int_0^T \big(\frac{1}{L^d}E(\Omega(t))\big)^2 dt \geq c(d) \int_0^T \big(t^{-\frac{1}{3}}\big)^2 dt.$$

It turns out that this approach is flexible enough to treat various physically relevant coarsening phenomena. For instance, if the substances are (viscous) fluids, flow next to cross-diffusion becomes available as a coarsening mechanism, and in fact dissipates less energy by (inner) friction on sufficiently large scales. This is experimentally and heuristically known to lead to a cross-over in the coarsening exponent from $\frac{1}{3}$ to 1 ("Siggia's growth"). On the level of the gradient flow structure, this means that dissipation by outer friction, cf. (14), is replaced by dissipation by inner (Stokes) friction:

$$g_\Omega(V,V) = \inf\{\int |\nabla u|^2 \mid \nabla\cdot(\chi_\Omega u) = V\mathcal{H}^{d-1}\lfloor\partial\Omega,\ \ \nabla\cdot u = 0\}.$$

This of course changes the induced distance d to the well mixed state and thus also its proxy: $\|\chi_\Omega\|_{\dot{H}^{-1}}$ has to be replaced by a transportation distance, cf. (8), between the negative and the positive part of χ_Ω, however with a logarithmic instead of quadratic cost function. On the one hand this transportation distance is sufficiently weak to provide a lower bound on the (non-explicit) induced distance by an adaptation of the Crippa-DeLellis quantification of the DiPerna-Lions theory; on the other hand it is sufficiently strong to provide a lower bound on the energy landscape in the spirit of (15).

0.5. **Flow by mean curvature as a gradient flow: convergence of the thresholding scheme.** In the third and last talk, we are interested in flow by mean curvature, which as for Mullins-Sekerka we think of as an evolution $t \mapsto \Omega(t) \subset [0,L)^d$ of sets, which here is more directly curvature-driven:

$$2V = H$$

(the factor of two is for convenience). Again formally, this evolution is the gradient flow of the same functional as in (13), namely $E = \mathcal{H}^{d-1}(\partial\Omega) = \int |\nabla\chi_\Omega|$ but this time w. r. t. $2\times L^2(\partial\Omega)$ inner product

$$g_\Omega(V_0,V_1) := \int_{\partial\Omega} V_0 V_1 d\mathcal{H}^{d-1}. \tag{16}$$

This follows from the fact that the first (inner) variation of the area functional w. r. t. to a test vector field ξ is given by

$$\delta E(\chi,\xi) = \int_{\partial\Omega} H\nu\cdot\xi = \int_{\partial\Omega} \nabla^{tan}\cdot\xi = \int (\nabla\cdot\xi - \nu\cdot D\xi\nu)|\nabla\chi|, \tag{17}$$

where in the last expression, ν is the measure-theoretic normal $\frac{d\nabla\chi}{d|\nabla\chi|}$.

In view of this gradient flow structure of mean curvature flow (MCF), it is natural to study a minimizing movements (MM) scheme (3), which formulated in terms of the characteristic functions takes the form

$$\chi^n \text{ minimizes } \quad \frac{1}{2h}\mathrm{d}^2(\chi,\chi^{n-1}) + E(\chi).$$

Since the induced distance d for the tensor (16) degenerates (Mumford-Michor) Almgren-Taylor-Wang introduced the proxy $\mathrm{d}^2(\chi,\chi^0)$ $= 4\int_{\Omega\triangle\Omega^0}\mathrm{dist}(\cdot,\partial\Omega^0)$, and Luckhaus-Sturzenhecker established a convergence result for the piecewise constant interpolation χ_h of $\{\chi^n\}_n$ in time, under the assumptions that (for a subsequence $h\downarrow 0$)

$$\chi_h \to \chi \text{ in } L^1 \quad \text{and} \quad \int\int|\nabla\chi_h|dt \to \int\int|\nabla\chi|dt, \tag{18}$$

where the second assumption, as opposed to the first, does not follow from the natural a priori estimates and makes the convergence result a conditional one.

However, it is not known whether this limit satisfies the dissipation inequality corresponding to the first identity in (1), which here takes the form $\frac{d}{dt}2\mathcal{H}^{d-1}(\partial\Omega)+\int_{\partial\Omega}H^2\le 0$, and which in view of (17) can be weakly encoded by

$$\frac{d}{dt}\int|\nabla\chi| + \int(\nabla\cdot\xi-\nu\cdot D\xi\nu)|\nabla\chi| - \frac{1}{2}\int(\xi\cdot\nu)^2|\nabla\chi|\le 0 \tag{19}$$

distributionally in t for all test vector fields ξ.

We show that a computationally highly efficient and robust scheme, the thresholding scheme introduced by Merriman-Bence-Osher in '92, produces limits that satisfy even all localized versions of (19), cf. (21), which actually characterizes MCF as pointed out by Brakke. In doing so, we use the tools introduced by de Giorgi for MM in metric spaces, cf. (4), as presented by Ambrosio-Gigli-Savaré.

The thresholding scheme proceeds in two steps: First, the characteristic function χ^{n-1} is convolved by the centered Gaussian of variance h, ie $G_h(z) := \frac{1}{\sqrt{2\pi h}}\exp(-\frac{|z|^2}{2h})$. Second, χ^n is the characteristic function of the set where this convolution is larger than $\frac{1}{2}$:

$$\chi^{n-1} \rightsquigarrow u^n := G_h*\chi^{n-1} \rightsquigarrow \chi^n := I(u^n>\frac{1}{2}). \tag{20}$$

This scheme clearly preserves the comparison principle and thus its limits have been shown to satisfy MCF in the sense of viscosity solutions. Its main practical interest lies in its easy extension to the multi-phase case, which we can treat by the very same analysis, and which is a model for the coarsening of grain configurations in polycrystalline solids ("grain growth").

The localized dissipation inequality assumes the form of $\frac{d}{dt}2\int_{\partial\Omega}\zeta + \int_{\partial\Omega}(\zeta H^2 + \nabla\zeta\cdot\nu H) \le 0$ for any test function $\zeta \ge 0$. Brakke observed that this family of inequalities characterizes a smooth solution of MCF, and used it to introduce a weak notion of solution based on varifolds. On our BV-level, this takes the form of

$$\begin{aligned}\frac{d}{dt}\int\zeta|\nabla\chi| + \int(\nabla\cdot(\zeta\xi) - \nu\cdot D(\zeta\xi)\nu)|\nabla\chi| - \frac{1}{2}\int\zeta(\xi\cdot\nu)^2|\nabla\chi|\\ + \int(\triangle\zeta - \nu\cdot D^2\zeta\nu)|\nabla\chi| \le 0\end{aligned} \tag{21}$$

distributionally in t, for all test vector fields ξ and test functions $\zeta \ge 0$. It was observed by Esedoglu and the author that thresholding can be interpreted as a minimizing movements scheme. Indeed, (20) is equivalent to

$$\chi^n \text{ minimizes among } v = v(x) \in [0,1] \quad \frac{1}{2h}d_h^2(v,\chi^{n-1}) + E_h(v), \tag{22}$$

where $E_h(v) := \frac{1}{\sqrt{h}}\int(1-v)G_h * v$ Γ-converges to $c_0\int|\nabla\chi|$ (wrt to underlying L^1-convergence with the understanding that the value is $+\infty$ if the limit is not a $\{0,1\}$-valued function of bounded variation), and where the distance function is given by $\frac{1}{2}d_h^2(v,w) := \sqrt{h}\int(G_{\frac{h}{2}} * (v-w))^2$. It is this variational interpretation that allowed for an extension of thresholding to the multi-phase case with surface tensions that depend on the phases they separate. It also allowed for the first convergence result to a BV-type solution in the multi-phase case, however conditional like Luckhaus-Sturzenhecker, namely under condition (23):

Theorem 3. *Let $\chi^0 = \chi^0(x) \in \{0,1\}$ be st $\int|\nabla\chi^0| < \infty$. Suppose that for a subsequence $h \downarrow 0$ we have the analogue of (18), that is,*

$$\chi_h \to \chi \text{ in } L^1 \quad \text{and} \quad \int E_h(\chi_h)dt \to \int c_0\int|\nabla\chi|dt. \tag{23}$$

Then χ satisfies (21).

The proof relies on the fact that next to (22), thresholding satisfies a MM principle for every non-negative localizing (test) function $\zeta \ge 0$:

$$\chi^n \text{ minimizes} \quad \frac{1}{2h}\tilde{d}_h^2(v,\chi^{n-1}) + \tilde{E}_h(v,\chi^{n-1}),$$

where $\frac{1}{2}\tilde{d}_h^2(v,w) = \sqrt{h}\int\zeta(G_{\frac{h}{2}} * (v-w))^2$ is the localization of $\frac{1}{2}d_h^2(v,w)$ but $\tilde{E}_h(\cdot,\chi^{n-1})$ is a perturbed (next to localized) version of E_h:

$$\begin{aligned}\tilde{E}_h(v,\chi) := \frac{1}{\sqrt{h}}\int\big(\zeta(1-v)G_h * v\\ + (v-\chi)[\zeta, G_h*](1-\chi) + (v-\chi)[\zeta, G_{\frac{h}{2}}*]G_{\frac{h}{2}} * (v-\chi)\big).\end{aligned}$$

This MM principle allows to appeal to de Giorgi's tools for MM, yielding a discrete version of the dissipation inequality (4) in form of

$$\begin{aligned}&\tilde{E}_h(\chi^n,\chi^{n-1})-\tilde{E}_h(\chi^{n-1},\chi^{n-1})\\&+\frac{1}{2}|\partial\tilde{E}_h(\cdot,\chi^{n-1})|^2(\chi^n)+\frac{1}{2h}\int_{(n-1)h}^{nh}|\partial\tilde{E}_h(\cdot,\chi^{n-1})|^2(u_h(t))dt\le 0,\end{aligned}$$

where $|\partial E|(v):=\limsup_{w\to v}\frac{(E(v)-E(w))_+}{d(v,w)}$ denotes the metric slope of a functional E and $u_h((n-1)h+\tau):=\operatorname{argmin}\big(\frac{1}{2\tau}d^2(v,\chi^{n-1})+E(v)\big)$ denotes the variational interpolation of χ^{n-1} and χ^n in a MM scheme for metric d and functional E.

Thanks to these tools, the proof is essentially soft and relies on the following ingredients. The metric slope is bounded by below by the first variation:

$$\frac{1}{2}|\partial\tilde{E}_h(\cdot,\chi)|^2(v)\ge\delta\tilde{E}_h(\cdot,\chi)(v,\xi)-\sqrt{h}\int\zeta(G_{\frac{h}{2}}*(\xi\cdot\nabla v))^2.$$

For the first variation, the relation of $\tilde{E}_h$ to E_h simplifies asymptotically

$$\big|\delta\tilde{E}_h(\cdot,\chi)(v,\xi)-\delta E_h(v,\zeta\xi)\big|\le Ch^{\frac{1}{4}}\frac{d_h(v,\chi)}{h},$$

where C is a generic constant only depending on ζ and ξ. The increments of $\tilde{E}_h(v,\chi)$ in χ are related to the first variation of the squared metric d_h^2:

$$\begin{aligned}&\big|\frac{1}{h}(\tilde{E}_h(v,\chi)-\tilde{E}_h(v,v))+\frac{1}{2h}\delta d_h^2(\cdot,\chi)(v,\xi)\big|\\&\qquad\le C\big(\sum_{p=1}^{2}(h^{\frac{1}{4}}\frac{d_h(v,\chi)}{h})^p+h^{\frac{1}{2}}E_h(\chi)\big),\end{aligned}$$

which has to be combined with the Euler-Lagrange equation for (22). Elementary arguments show that both terms

$$\big|\delta E_h(v,\xi)-\frac{1}{\sqrt{h}}\int D\xi:(1-v)(G_h\mathrm{id}-hD^2G_h)*v\big|,$$

$$\big|\sqrt{h}\int\zeta(G_{\frac{h}{2}}*(\xi\cdot\nabla v))^2-\frac{1}{\sqrt{h}}\int\zeta\xi\otimes\xi:(1-v)hD^2G_h*v\big|$$

are estimated by $Ch^{\frac{1}{2}}E_h(v)$, so that the result relies on the assumption (23) only in the form that $v\to\chi$ in L^1 with $\int(1-v)G_h*v\to c_0\int|\nabla\chi|$ implies

$$\begin{aligned}&\frac{1}{\sqrt{h}}\int\big(\zeta(1-v)G_h*v+\sigma:(1-v)h^2D^2G_h*v\big)\\&\qquad\to c_0\int(\zeta+\nu\cdot\sigma\nu)|\nabla\chi|.\end{aligned}$$

References

[1] R. V. Kohn, F. Otto, *Upper bounds on coarsening rates*, Comm. Math. Phys. **229** (2002), no. 3, 375395.

[2] T. Laux, F. Otto, *Brakke's inequality for the thresholding scheme*, arXiv: 1708.03071.

[3] F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations **26** (2001), no. 1-2, 101174.

[4] F. Otto, C. Seis, D. Slepcev, *Crossover of the coarsening rates in demixing of binary viscous liquids*, Commun. Math. Sci. **11** (2013), no. 2, 441464.

[5] F. Otto, M. Westdickenberg, *Eulerian calculus for the contraction in the Wasserstein distance*, SIAM J. Math. Anal. **37** (2005), no. 4, 12271255.

Anisotropic and Crystalline mean curvature flows.

Antonin Chambolle

works with V. Caselles, M. Novaga,
M. Morini, M. Ponsiglione

1 Introduction: anisotropic perimeters

The series of three lectures will start with an introduction to perimeters defined by anisotropic norms, as follows:

$$P_\phi(E) = \int_{\partial E} \phi(\nu^E)\, d\mathcal{H}^{N-1}.$$

Here, $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff (surface) measure, $E \subset \mathbb{R}^N$ is a (smooth, or Caccioppoli set [18][1]), and ϕ a convex, even, one-homogeneous function, which is the "surface tension". It is useful to introduce the *polar*

$$\phi^\circ(x) = \sup_{\phi(\nu)\le 1} \nu \cdot x$$

which defines a metric in $\mathbb{R}^N$ (and P_ϕ can be then considered as the perimeter in this metric).

It is then known that the sets which minimize the perimeter P_ϕ for a given volume are the "Wulff shapes" [14], that is, a translate of a sublevel of ϕ°:

$$W_c = \{\phi^\circ \le c\}.$$

We will be particularly interested in the degenerate *crystalline case*, where W_c is a polytope. In general, the classical theories become difficult or fail to apply when ∂W_c loses its ellipticity and has flat regions (corresponding to singularities of ϕ).

The *anisotropic curvature* $\kappa_E^\phi(x)$ $(x \in \partial E)$ is then introduced as the first variation of the perimeter (obtained by computing appropriate inner variations of a set). The *crystalline curvature*, in the crystalline case, might be not well defined or multivalued, because of the singularities of the surface tension.

2 MCF as the "gradient flow" of a perimeter

In this context, one can introduce a "mean curvature flow" as a formal gradient flow of the perimeter [3]. It is usually written

$$V_n = -\kappa_E^\phi(x) \tag{1}$$

[1]The references in this abstract are mostly pointers to where essential results or references can be found and by no means exhaustive.

where V_n is the normal velocity of the evolving family of sets $E(t)$ and $\kappa_E^\phi(x)$ the ϕ-curvature of E at $x \in \partial E$.

We first will focus on the two-dimensional case, where most of the difficulties can be easily explained, and where easy ODE-based approaches can describe evolutions [23, 1].

In the smooth anisotropic cases (that is, in fact, when ϕ is smooth, rather than ϕ°) the standard "viscosity solutions" theory is the most natural approach to define generalized solutions for all time (including past possible singularities). We will recall the main definitions and relevant results [12] in this context (which have been partly extended to the crystalline case only recently [15, 16, 17]).

We will also introduce the issue of mobilities and possible forcing terms: namely, we could consider that the normal velocity V_n in (1) is replaced with the velocity along an "anisotropic normal vector field" such as the *Cahn-Hoffmann vector field* $\nabla\phi(\nu^E)$ (or a selection in the crystalline case), or a field depending on another metric ψ; or equivalently that the evolution is given by

$$V_n = \psi(\nu^E)(\kappa_E^\phi(x) + g(x,t)) \tag{2}$$

where $g(x,t)$ is a general (L^∞, Lipschitz in space) forcing term.

3 "Algorithms"

3.1 Almgren, Taylor, Wang

We will next introduce "algorithms" for deriving variational curvature flows, and in particular two celebrated methods. The first dates back to Almgren, Taylor and Wang [2] (see also Luckhaus-Sturzenhecker in the isotropic case [21]) and consists in introducing a substitute for the "L^2" distance of a set F to a previous set E:

$$\delta(F;E) = \int_F \operatorname{dist}(x,\partial E)dx,$$

and then define the evolution as a "minimizing movement" of the perimeter. Given $h > 0$ (the discrete time-step), and E^0, one lets for $n \geq 1$:

$$E^n \in \arg\min_E P_\phi(E) + \frac{1}{h}\delta(E;E^{n-1})$$

Then, $E_h(t) := E^{[t/h]}$. It is then shown in [2] that for smooth E^0 and ϕ, as $h \to 0$, $E_h(t)$ converges to the mean curvature flow starting from E_0, for small time, as well as many more results.

This construction (which can be actually implemented [6] but does not lead to particularly efficient methods) is useful for building evolutions [generalized or classical] as in [2].

3.2 Merriman, Bence, Osher

Another algorithmic approach for isotropic or anisotropic flows is due to Merriman, Bence and Osher [22]. It consists in building E^n from E^{n-1}, this time, by letting

$$E^n = \{\rho_h * \chi_{E^n} \geq \tfrac{1}{2}\}$$

where ρ_h is the heat kernel at time $h > 0$.

We will only mention rapidly this method which has not been as successful as the previous for building actual solutions (except very recently, see [20]), but is a much better numerical method, allowing for instance the efficient computation of mean curvature evolutions for partitions with many sets [13].

We will mention some attempts to extend this approach to anisotropic cases, including sometimes crystalline [10], see [19, 5].

4 Crystalline evolutions

We will mostly focus on the application of the previously mentioned schemes in the crystalline case. In particular, we will describe:

- how one can adapt the BMO scheme to the crystalline case;
- how the ATW scheme can be used to show existence of "classical" crystalline evolutions in 2D [11], with forcing; of (classical) convex evolutions in any dimension [4];
- how it can be used to show existence of generalised solutions in arbitrary dimensions [9, 7].

We will in particular describe the theory which we have developed in the past two years [9, 7, 8] to define generalised crystalline curvature flows, with forcing term and various mobilities. We will explain how we can show a comparison results and indicate two different approaches to existence.

4.1 A new definition for sub/superflows

Our existence and uniqueness result is based on the following characterization of subflows and superflows, which extends naturally to the crystalline case. We give it here in the simplest situation, for the motion (2) above with $\psi = \phi$ and $g = 0$. We have that $E(t)$ is a *superflow* (that is, evolving faster than (2)) if, in addition to some technical assumptions, the distance function $d(x,t) = \operatorname{dist}^{\phi^\circ}(x, E(t))$, defined by

$$\operatorname{dist}^{\phi^\circ}(x, E(t)) = \min_{y \in E(t)} \phi^\circ(x - y),$$

is a *supersolution* of the ϕ-total variation flow outside of E:

$$\partial_t d \geq \operatorname{div} \nabla\phi(\nabla d) \text{ in } (\mathbb{R}^N \times (0,T)) \setminus E.$$

We assume that this equation is *distributional* (it is well known that in the viscosity sense it is an appropriate definition of a viscosity superflow). This allows to extend the definition naturally to the crystalline case, where we say that E is a superflow if there exists $z \in \partial\phi(\nabla d)$ such that $\operatorname{div} z \in L^\infty(\{d > \delta\})$ for any $\delta > 0$ and

$$\partial_t d \geq \operatorname{div} z \text{ in } (\mathbb{R}^N \times (0,T)) \setminus E,$$

in the sense of distributions.

A *subflow* is defined as the complement of a superflow.

This definition is weaker than the viscosity solution: in particular it is difficult to assert, here, that an intersection of superflow is still a superflow. However, we can show that if ϕ is smooth enough, it is equivalent.

Next, we can show quite easily, using relatively standard parabolic comparison, that if at time zero a superflow is inside a subflow, then they stay at distance one from another.

Eventually, we use the iterative scheme in [2] to build actual flows (ie, space-time tubes which are both super- and subflows), showing existence and uniqueness for such crystalline motions. This is mostly the construction in [9].

4.2 Extensions

In [7], we show how to extend this construction to more general equations of the form (2) (with arbitrary even norm ψ and forcing term g). The idea is to replace the equation with

$$\partial_t d \geq \operatorname{div} z + g - Md$$

where $M \geq 0$ is a constant (related to the spatial Lipschitz constant of g) and d is now the ψ°-distance to $E(t)$.

This extension is not totally straightforward. In particular, the technical assumption $\operatorname{div} z \in L^\infty(\{d > \delta\})$ is needed for the comparison result. It turns out that this property cannot be true in general if ψ and ϕ do not have some sort of "compatibility" (namely, we need $\psi = \psi^0 + \epsilon\phi$ for some norm ψ^0 and some $\epsilon > 0$). In a first step, we show how to extend the previous construction in the case of compatible anisotropies.

Next, in the general case (a typical non-compatible case is when ϕ is crystalline and ψ the Euclidean norm), we cannot anymore hope to show existence and uniqueness in the previous sense. However, we are able to show a stability result which states that all approximations with compatible anisotropies converge to a unique flow, which turns out to be also the limit of the iterative scheme of [2]. Such uniqueness result has to be understood in an appropriate level-set sense.

References

[1] Fred Almgren and Jean E. Taylor. Flat flow is motion by crystalline curvature for curves with crystalline energies. *J. Differential Geom.*, 42(1):1–22, 1995.

[2] Fred Almgren, Jean E. Taylor, and Lihe Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31(2):387–438, 1993.

[3] G. Bellettini and M. Paolini. Anisotropic motion by mean curvature in the context of Finsler geometry. *Hokkaido Math. J.*, 25(3):537–566, 1996.

[4] Giovanni Bellettini, Vicent Caselles, Antonin Chambolle, and Matteo Novaga. Crystalline mean curvature flow of convex sets. *Arch. Ration. Mech. Anal.*, 179(1):109–152, 2006.

[5] Eric Bonnetier, Elie Bretin, and Antonin Chambolle. Consistency result for a non monotone scheme for anisotropic mean curvature flow. *Interfaces Free Bound.*, 14(1):1–35, 2012.

[6] Antonin Chambolle. An algorithm for mean curvature motion. *Interfaces Free Bound.*, 6(2):195–218, 2004.

[7] Antonin Chambolle, Massimiliano Morini, Matteo Novaga, and Marcello Ponsiglione. Existence and uniqueness for anisotropic and crystalline mean curvature flows. 2017.

[8] Antonin Chambolle, Massimiliano Morini, Matteo Novaga, and Marcello Ponsiglione. Generalized crystalline evolutions as limits of flows with smooth anisotropies. Preprint hal-01632948, 2017.

[9] Antonin Chambolle, Massimiliano Morini, and Marcello Ponsiglione. Existence and uniqueness for a crystalline mean curvature flow. *Comm. Pure Appl. Math*, 2016.

[10] Antonin Chambolle and Matteo Novaga. Convergence of an algorithm for the anisotropic and crystalline mean curvature flow. *SIAM J. Math. Anal.*, 37(6):1978–1987, 2006.

[11] Antonin Chambolle and Matteo Novaga. Existence and uniqueness for planar anisotropic and crystalline curvature flow. In L. Ambrosio, Y. Giga, P. Rybka, and Y. Tonegawa, editors, *Variational Methods for Evolving Objects*, volume 67 of *Advanced Studies in Pure Mathematics*, pages 87–113. Mathematical Society of Japan, 2015.

[12] Yun Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.

[13] Selim Esedolu and Felix Otto. Threshold dynamics for networks with arbitrary surface tensions. *Comm. Pure Appl. Math.*, 68(5):808–864, 2015.

[14] Irene Fonseca and Stefan Müller. A uniqueness proof for the Wulff theorem. *Proc. Roy. Soc. Edinburgh Sect. A*, 119(1-2):125–136, 1991.

[15] Mi-Ho Giga and Yoshikazu Giga. Evolving graphs by singular weighted curvature. *Arch. Rational Mech. Anal.*, 141(2):117–198, 1998.

[16] Yoshikazu Giga and Norbert Požár. A level set crystalline mean curvature flow of surfaces. Preprint, Hokkaido University, 2016.

[17] Yoshikazu Giga and Norbert Požár. Approximation of general facets by regular facets with respect to anisotropic total variation energies and its application to the crystalline mean curvature flow. Preprint arXiv:1702.05220, available at `https://arxiv.org/abs/1702.05220`, 2017.

[18] Enrico Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[19] Hitoshi Ishii, Gabriel E. Pires, and Panagiotis E. Souganidis. Threshold dynamics type approximation schemes for propagating fronts. *J. Math. Soc. Japan*, 51(2):267–308, 1999.

[20] Tim B. Laux and Felix Otto. Convergence of the thresholding scheme for multi-phase mean-curvature flow. Preprint, Max Planck Institute for Mathematics in the Sciences, 2015.

[21] Stephan Luckhaus and Thomas Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations*, 3(2):253–271, 1995.

[22] B. Merriman, J. K. Bence, and S. J. Osher. Diffusion generated motion by mean curvature. In J. E. Taylor, editor, *"Computational Crystal Growers Workshop"*, Selected Lectures in Math. Amer. Math. Soc., Providence, Rhode Island, 1992.

[23] Jean E. Taylor. Crystalline variational problems. *Bull. Amer. Math. Soc.*, 84(4):568–588, 1978.

Variational, Geometric and Surface PDEs in Cell Biology

Charlie Elliott

Motivation

The interplay of proteins and curvature of lipid bilayers is known to regulate cell morphology and a variety of cellular functions. Here, interplay not only means that proteins can induce curvature by shaping but also that the membrane curvature plays an active role in organizing membrane proteins. Moreover, microscopic causes, such as hydrophobic mismatch of proteins and amphiphilic lipids, may have macroscopic effects, such as budding or fission. For example, membrane remodelling during endocytosis is a consequence of the interplay between the elastic membrane and the actions of membrane proteins that can both sense and create membrane curvature. Coarse-grained molecular models using a particle-based approach are often used for simulation.

Our concern is with the macroscopic models. Pure continuum models are based on the fundamental Canham-Helfrich model of lipid membranes. The associated Canham-Helfrich energy is obtained by expansion of the bending energy with respect to the principal curvatures up to second order. The Canham-Helfrich energy is related to the Willmore functional in differential geometry. The model describes equilibrium and close to equilibrium properties of biological membranes. It has been modified so as to include the effect of different lipid components or large-scale protein assemblies. Lipids and proteins are then represented by concentrations. Their impact on the membrane morphology is modelled by concentration-dependent bending rigidities, phase-dependent spontaneous curvature, and a line energy associated with the phase boundaries. The Euler-Lagrange equations associated with these surface energy functionals usually are fourth-order, nonlinear (geometric) partial differential equations. These equations are typically coupled with nonlinear partial differential equations describing the evolution of lipids and proteins on the surface.

Receptor-ligand interactions are a primary mechanism by which cells sense and respond to their environment. Such processes constitute a fundamental part of many basic phenomena in cell biology such as proliferation, motility, the maintenance of structure or form, adhesion, cellular signalling etc. Often sub-systems of complex models are coupled surface bulk reaction diffusion systems.

Modelling the directional motility of cells is of great importance especially because of the role directed cell migration plays in several biological phenomena, such as embryonic development, cancer, tissue development and immune responses. Broadly speaking, the motile cycle of a cell consists of the following processes: polarization where the cell develops a front and a back through the redistribution of proteins and lipids within the cell, protrusion at the leading edge of the cell pushing the front of the cell outwards, and retraction of the rear of the cell towards the leading edge. Further details when added to the modelling yield more complexity. The deformation of the cell surface is linked to the dynamics of actin and other cell-resident proteins, and these dynamics must be coupled in a consistent way with the evolution of the cell surface.

Mathematical models, although complex in structure from the analysis and computational perspectives, are still simple from the biological perspective. In these lectures we focus on mathematical methodology for PDE models in complex evolving domains involving energetic surfaces which may be useful as more complex models are developed. The lectures rely on joint work with co-authors cited in the bibliography.

Modelling and computation

We emphasise the role of mathematical analysis in modelling **ABM** and computation **ABC**:-

- Analysis Based Modelling

  Mathematical modelling

  Due to the complexity of the biochemistry involved in signalling networks and the underlying geometry an integrated approach combining theoretical and computational mathematical studies with experimental and modelling efforts appears necessary. Developments in large scale computing, the well posedness and rigorous numerical analysis theory for nonlinear PDEs and computational algorithms numerical analysis allow the study of PDE systems in complex and evolving domains. Our work is focused on understanding mathematical models involving evolving complex domains defined by curved energetic surfaces on which there are surface processes, possibly coupled to bulk processes.

  Formulation of PDEs

  In our context, we need concepts from mechanics, biochemistry, biophysics and geometry in order to build systems of PDES. To begin we model the surface of a cell by $\Gamma(t)$, a moving curved n-dimensional hypersurface in $\mathbb{R}^{n+1}$. $\Omega(t)$ may denote a moving *bulk* domain in $\mathbb{R}^{n+1}$ with boundary (or part of the boundary) $\Gamma(t)$ which models the interior or exterior of the cell. We are led to the following types of initial-boundary value problems

  Geometric PDEs: Energetic unknown surfaces/interfaces in which Γ is determined by geometric quantities

  Free boundary(interface problems) in which Γ is to be determined

  Surface PDEs: PDEs on a stationary or evolving surface

  Coupled bulk surface PDEs Coupling equations for and on a surface to equations in the bulk

  Interface/free boundary on an unknown evolving surface

  Goals

  The use of the mathematical analysis in this context should bear in mind the following interelated goals:

  $$Simulation\ +\ Parameter\ \ identification\ +\ Prediction\ +\ Control$$

  Here we note the concept of the forward problem defined by an initial boundary value problem in which the parameters and data are determined. Frequently there are many unknown parameters which might have to be determined using the forward model and experimental observations within an inverse problem framework.

- Analysis Based Computation
  - *Development of numerical methods*: Each discretisation method depends on the mathematical formulation e.g. variational equation, level set, phase field, viscosity solutions etc.
  - *Analysis of discretisation* Frequently the numerical analysis is is related to the mathematical analysis of well posedness. For example, stability and existence is often approached using Galerkin approximations which also yield the possibility of finite element approximations and analysis.
  - *Implementation and analysis of solvers*: depends on discretisation and software requirements. This is the domain of scientific computing and has its own mathematical interest. It is not the focus of these lectures.

Topics of lectures

- PDEs on evolving domains

  We will present an abstract framework for treating the theory of well-posedness of solutions to abstract parabolic partial differential equations on evolving Hilbert spaces. The theory is applicable to variational formulations of PDEs on evolving spatial domains including moving hypersurfaces. We formulate an appropriate time derivative on evolving spaces called the material derivative and define a weak material derivative in analogy with the usual time derivative in fixed domain problems; our setting is abstract and not restricted to evolving domains or surfaces. Then we show well-posedness to a certain class of parabolic PDEs under some assumptions on the parabolic operator and the data. Specifically, we study in turn a surface heat equation, an equation posed on a bulk domain, a novel coupled bulk-surface system and an equation with a dynamic boundary condition. In order to prove the well-posedness, we make use of the abstract framework. See [1, 2, 6].

- Receptor ligand dynamics

  We discuss well posedness of for a reaction-diffusion system of coupled bulk- surface equations on a moving domain modelling receptor-ligand dynamics in cells. The nonlinear coupling between the three unknowns is through the Robin boundary condition for the bulk quantity yielding a flux onto the right hand sides of the two surface equations. We also consider a number of biologically relevant asymptotic limits of the model in the case of a stationary surface. We prove convergence to limiting problems which take the form of free boundary problems posed on the cell surface. See [16, 17, 3].

- Small deformations of Helfrich energy minimising surfaces with applications to biomembranes

  We introduce a mathematical model for small deformations induced by external forces of closed surfaces that are minimisers of Helfrich-type energies. Our model is suitable for the study of deformations of cell membranes induced by the cytoskeleton. We describe the deformation of the surface as a graph over the undeformed surface. A new Lagrangian and the associated Euler-Lagrange equations for the height function of the graph are derived. This is the natural generalisation of the well known linearisation in the Monge gauge for initially flat surfaces. We discuss energy perturbations of point constraints and point forces acting on the surface. We establish existence and uniqueness results for variational solutions on flat domains, spheres and tori. See [15, 14]

- ## Finite element analysis

  We describe the evolving surface finite-element method for numerically approximating partial differential equations on hypersurfaces which evolve with time. The key idea is based on approximating $\Gamma(t)$ by an evolving interpolated polyhedral surface consisting of a union of simplices. A finite element space of functions is then defined on the triangulated surface. This method derived jointly with Dziuk is the basis for the abstract approach to parabolic equations on moving domains. Geometric analysis may be used to generate nicely evolving meshes. See [7, 8, 10, 9, 13, 12, 11]

- ## Computational modelling of geometric biomembranes and cell motility

  Biomembranes consisting of multiple lipids may involve phase separation phenomena leading to coexisting domains of different lipid compositions. The modeling of such biomembranes involves an elastic or bending energy together with a line energy associated with the phase interfaces. This leads to a free boundary problem for the phase interface on the unknown equilibrium surface which minimizes an energy functional subject to volume and area constraints. We propose a surface phase field in which the line energy is approximated by a surface Ginzburg-Landau functional. The relaxation dynamics couple a nonlinear fourth order geometric evolution equation of Willmore flow type for the membrane with a surface Allen-Cahn equation describing the lateral decomposition. Numerical experiments exhibit convergence and the power of this new method for two component geometric biomembranes by computing equilibria such as dumbbells, discocytes and starfishes with lateral phase separation. See [18, 19, 20].

  We present a mathematical and a computational framework for the modelling of cell motility. The cell membrane is represented by an evolving surface, with the movement of the cell determined by the interaction of various forces that act normal to the surface. We consider external forces such as those that may arise owing to inhomogeneities in the medium and a pressure that constrains the enclosed volume, as well as internal forces that arise from the reaction of the cells' surface to stretching and bending. We also consider a protrusive force associated with a reaction?diffusion system posed on the cell membrane modlling cell polarization. The computational method is based on an evolving surface finite-element method. The general method can account for the large deformations that arise in cell motility and allows the simulation of cell migration in three dimensions. Illustrative applications include numerical simulations of a model for eukaryotic chemotaxis and a model for the persistent movement of keratocytes in two and three space dimensions. See [5, 21, 4].

BIBLIOGRAPHY

[1] A. Alphonse, C. M. Elliott, and B. Stinner. An abstract framework for parabolic pdes on evolving spaces. *Portugaliae Mathematica*, 71(1):1–46, 2015.

[2] A. Alphonse, C. M. Elliott, and B. Stinner. On some linear parabolic pdes on moving hypersurfaces. *Interfaces and Free Boundaries*, 17:157–187, 2015.

[3] A. Alphonse, C. M. Elliott, and J. Terra. A coupled ligand-receptor bulk-surface system on a moving domain: Well posedness, regularity, and convergence to equilibrium. *SIAM Journal on Mathematical Analysis*, 50:1544–1592, 2017.

[4] W. Croft, C. M. Elliott, G. Ladds, B. Stinner, C. Venkataraman, and C. Weston. Parameter identification problems in the modelling of cell motility. *Journal of Mathematical Biology*, 71:399–436, 2015.

[5] K Deckelnick, G Dziuk, and C M Elliott. Computation of geometric partial differential equations and mean curvature flow. *Acta Numerica*, 14:139–232, 2005.

[6] K. P. Deckelnick, C. M. Elliott, and Tatsu-Hiko Miura. Hamillton -Jacobi equations on an evolving surface. Preprint 7, Universitat Magdeburg, 2018.

[7] G. Dziuk and C. M. Elliott. Finite elements on evolving surfaces. *IMA Journal Numerical Analysis*, 25:385–407, 2007.

[8] G. Dziuk and C. M. Elliott. Fully discrete evolving surface finite element method. *SIAM J. Numer. Anal.*, 50(5):2677–2694, 2012.

[9] G. Dziuk and C. M. Elliott. Finite element methods for surface partial differential equations. *Acta Numerica*, 22:289–396, 2013.

[10] G. Dziuk and C. M. Elliott. L^2 estimates for the evolving surface finite element method. *Math. Comp.*, 82:1–24, 2013.

[11] C. M. Elliott and H. Fritz. On algorithms with good mesh properties for problems with moving boundaries based on the harmonic map heat flow and the DeTurck trick. *SMAI Journal of Computational Mathematics*, 2:141–176, 2016.

[12] C. M. Elliott and H. Fritz. On approximations of the curve shortening flow and of the mean curvature flow based on the DeTurck trick. *IMA Journal Numerical Analysis*, 37:543–603, 2017.

[13] C. M. Elliott, H. Fritz, and G. Hobbs. Second order splitting for a class of fourth order equations. arXiv:1711.04151, 2017.

[14] C. M. Elliott, H. Fritz, and G. Hobbs. Small deformations of Helfrich energy minimising surfaces with applications to biomembranes. *Math. Mod. Meth. Appl. Sci.*, 27:1547–1586, 2017.

[15] C. M. Elliott, C. Gräser, G. Hobbs, R. Kornhuber, and M.aren-Wanda Wolf. A variational approach to particles in lipid membranes. *Archive for Rational Mechanics and Analysis*, 222:1011–1075, 2016.

[16] C. M. Elliott and T. Ranner. Finite element analysis for a coupled bulk–surface partial differential equation. *IMA J. Numer. Anal.*, 33:377–402, 2013.

[17] C. M. Elliott, T. Ranner, and C. Venkataraman. Coupled bulk-surface free boundary problems arising from a mathematical model of receptor-ligand dynamics. *SIAM Journal on Mathematical Analysis*, 49:360–397, 2017.

[18] C. M. Elliott and B. Stinner. Modeling and computation of two phase geometric biomembranes using surface finite elements. *J Comput Phys*, 229:6585–6612, 2010.

[19] C. M. Elliott and B. Stinner. A surface phase field model for two-phase biological membranes. *SIAM Journal on Applied Mathematics*, 70:2904–2928, 2010.

[20] C. M. Elliott and B. Stinner. Computations of to phase biomembranes with phase dependent material parameters using surface finite elements. *Communications in Computational Physics*, 13(2):325–360, 2012.

[21] C. M. Elliott, B. Stinner, and C. Venkataraman. Modelling cell motility and chemotaxis with evolving surface finite elements. *Journal of the Royal Society Interface*, 9(76):3027–3044, 2012.

Abstracts
(Sessions)

Mean curvature flow with surgery of embedded mean-convex surfaces in 3-manifolds

Gerhard Huisken
Universät Tübingen /
/ Mathematisches Forschungsinstitut Oberwolfach

The lecture will describe joint work with Simon Brendle on the deformation of closed 2-dimensional embedded surfaces with positive mean curvature in a Riemannian ambient manifold in direction of their mean curvature vector. It is explained how all singularities of this quasi-linear parabolic system can be classified and quantitatively controlled by a priori estimates. We then show that the flow can be extended by surgeries beyond singularities until the area is exhausted or the surface converges to a minimal surface.

QUASI-LOCAL MASS AND ISOMETRIC EMBEDDING

MU-TAO WANG

One of the most well known problems in Riemannian geometry is whether the geometry described intrinsically by a Riemannian metric can be realized as the geometry of a submanifold of the familiar Euclidean space. In the case of surfaces in the ambient Euclidean three space, the theory went back to the nineteenth century. In 1927, Cohn-Vossen proved the rigidity of closed convex surfaces in this case. Weyl then conjectured that every closed surface with positive curvature should be isometrically embeddable into the Euclidean three space. Weyl's problem can be considered as an existence and uniqueness problem for a global elliptic equation and he himself contributed an important estimate of the second fundamental form for this problem. However, it was not until 1954 that Pogorelov and Nirenberg solved the Weyl problem for smooth metrics independently. The work of Pogorelov-Nirenberg was subsequently applied in an essential way in general relativity.

The question of defining and understanding the concept of quasi-local mass had been a central topic since the works of Hawking and Penrose. In fact, in 1979, Penrose announced that the first major unsolved problem in classical general relativity was the concept of quasi local mass. The problem of defining mass in general relativity can be traced back to Einstein. It was difficult because the theory of general relativity is a nonlinear theory where gravity is dictated by a tensor while Newtonian gravity is basically linear and depends on a scalar. Besides, there is no continuous symmetry to help. Another formulation for general relativity was proposed by Arnowit-Deser-Misner in 1962, in terms of the Hamilton-Jacobi theory of a spacelike hypersurface, called an initial data set. The dynamics of gravity is dictated by two geometric quantities: one is the induced metric and the other one is the second fundamental form.

Brown-York made the first important step towards the definition of quasi-local mass by the Hamiltonian approach. They assume the second fundamental form of the spacelike hypersurface is trivial, and discuss how to associate a mass to a closed space-like 2-surface in the initial three dimensional manifold. The definition is nice, but it depends on the three manifold that the surface bounds. The idea is to compare a quantity defined on the surface with a similar quantity defined on the surface in Euclidean space obtained by the isometric embedding theorem of Pogorelov- Nirenberg. Without knowing the works in GR, Shi and Tam proved that the mass of Brown York is positive. The gauge dependence of the Brown-York definition was later removed by Liu-Yau in the general case (without assuming the second fundamental form is trivial).

Date: May 9, 2018.

The author was partially supported by the National Science Foundation under grant DMS-1105483.

Soon after, it was realized that these definitions, as good as they are, fail an important criterion for the definition of quasi-local mass. Namely, for a surface in the Minkowski spacetime, as there is no matter field and no gravitation, the mass should be zero. A simple calculation shows that a surface in the light cone of the Minkowski spacetime has strictly positive Brow-York and Liu-Yau mass, unless it is a round sphere.

It is now clear that isometric embeddings into the Minkowski spacetime need to be taken into account in order to obtain a mass expression that satisfies the last criterion. By a simple degree of freedom counting, this yields a underdetermined problem, and an additional equation is needed to expect any type of uniqueness. On the other hand, the surface Hamiltonian expression relies on a definite choice of gauge choice, without which the definition remains ambiguous.

In 2009, the author and Yau come up with a variational approach to tackle this problem. Pogorelov-Nirenberg's isometric embedding theorem allowed us to identify the extra degree of freedom as the time-function. Returning to the derivation of Brown-York, we discovered that there is indeed a canonical choice of gauge with respect to each time function, to which a quasilocal energy (instead of mass) can be assigned. The choice of this gauge is justified by the positivity of the quasi-local energy, whose proof comprises ideas from Schoen-Yau, Shi-Tam, Bartnik, Witten, and Liu-Yau. We later realized the choice is closely related to a gravitational conservation law, which plays an important role in the study of the dynamics of the Einstein equation.

With this positivity, we minimize the energy with respect to different time functions and define the quasi-local mass as the energy at critical points. This is very much like the rest mass of special relativity which realizes the minimal energy seen among all observers. The Euler-Lagrange equation is a fourth order nonlinear equation of the time-function. Together with the isometric embedding equation, this gives a well-determined system of four equations for four unknowns. We do not yet have a general existence or uniqueness theorem for this system. However, when the quasi-local mass density is positive, the system is elliptic and the linearized equation can be solved. This is sufficient in studying many unsettled problems for isolated gravitating systems, on which most current study of general relativity focus. We expect more applications to come when this optimal isometric embedding is better understood.

More recently, we come up with the definition of quasi-local mass with reference to a static spacetime. The definition generalizes the quasilocal mass definition with Minkowski spacetime reference previously defined by Wang-Yau in 2009. I shall discuss properties and applications of these new definitions as well. This talk is based on joint work with Po-Ning Chen, Ye-Kai Wang, and Shing-Tung Yau.

Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027

Global Instability of the Multi-dimensional Plane Shocks for the Isothermal Euler Equations [1]

Yi Zhou

School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
e-mail: yizhou@fudan.edu.cn

This talk is concerned with the long time behavior of the generalized Riemann problem of isothermal compressible Euler system in two dimensions

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2)_x + (\rho uv)_y + \rho_x = 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2)_y + \rho_y = 0, \end{cases} \tag{0.1}$$

where ρ and (u, v) are the density and the velocity respectively, with the initial data

$$t = 0 : \begin{cases} \rho = \begin{cases} \rho_r + \varepsilon\rho_0(x, y), x > \varepsilon\Pi(y), \\ \rho_l + \varepsilon\rho_0(x, y), x < \varepsilon\Pi(y), \end{cases} \\ u = \begin{cases} u_r + \varepsilon u_0(x, y), x > \varepsilon\Pi(y), \\ u_l + \varepsilon u_0(x, y), x < \varepsilon\Pi(y), \end{cases} \\ v = \varepsilon v_0(x, y), \end{cases} \tag{0.2}$$

where ρ_r, ρ_l, u_r, u_l are constants, functions $\Pi(y) \in C_0^\infty(\mathbf{R})$, $\rho_0, u_0, v_0 \in C_0^\infty(\mathbf{R}^2)$ are compact supported and satisfy

$$supp\ \Pi(y) \subset \{y||y| \le 1\}$$

and

$$supp\ \rho_0, u_0, v_0 \subset B_1 = \{x : |x| \le 1\}.$$

[1] This talk is presented in The 11th Mathematical Society of Japan (MSJ) Seasonal Institute (SI)- The Role of Metrics in the Theory of Partial Differential Equations, Hokkaido University, Sapporo, Japan, July 2-13, 2018. This is a joint work with Ning-An Lai (Lishui Univ.) and Wei Xiang(City Univ. of Hong Kong).

By using test function method, blow-up result will be established for the fan-shaped wave structure solution, including two shocks and one contact discontinuity. What is more, we get the sharp upper bound of lifespan estimate$\big(T(\varepsilon) \leq C\varepsilon^{-2}\big)$, which coincides with that of the smooth solutions of the Cauchy problem of nonlinear wave equations in two space dimensions. This result is a complementary to the local existence of piecewise smooth solution of equations (0.1) with initial data (0.2), which was established by Chen and Li [1].

Denoting the coordinates as (x, y_1, y_2), we consider the compressible isothermal Euler system which is radial symmetrical to the variables (y_1, y_2). Let

$$v_1(t,x,y) = \frac{y_1}{r}v(t,x,r),\ v_2(t,x,y) = \frac{y_2}{r}v(t,x,r),$$

then the system can be written as

$$\begin{cases} (r\rho)_t + (r\rho u)_x + \partial_r(r\rho v) = 0, \\ (\rho u)_t + \big(\rho u^2\big)_x + \partial_r(\rho u v) + \rho_x = 0, \\ (r\rho v)_t + (r\rho u v)_x + \partial_r\left(r\rho v^2\right) + r\rho_r = 0, \end{cases} \tag{0.3}$$

where $r = \sqrt{y_1^2 + y_2^2}$. Assuming that we have local existence, then we can establish the similar blow-up result and lifespan is still the same as that of nonlinear wave equations in three space dimensions. We point out that Godin [2] proved global stability in $N(N \geq 5)$ space dimensions.

References

[1] S. Chen and D. Li, Cauchy problem with general discontinuous initial data along a smooth curve for 2-d Euler system. *J. Differential Equations*, **257** (2014), 1939-1988.

[2] P. Godin, Long time existence of a class of perturbation of planar shock fronts for second order hyperbolic conservation laws. *Duke Math. J.*, **60** (1990), 425-463.

On the defocusing supercritical wave equation outside a ball

Piero D'Ancona
Sapienza, Università di Roma
11th MSJ-SI
The Role of Metrics in the Theory of PDE
July 2 - 13, 2018
Hokkaido University, Sapporo

We consider a defocusing semilinear wave equation

$$\Box u + |u|^{p-1}u = 0, \qquad t \geq 0, \quad |x| > 1$$

on the exterior of the unit ball of $\mathbb{R}^n$, in dimension $n = 3$ or larger, with Dirichlet boundary conditions

In a first result, we prove that for (large) radial initial data a global bounded solution exists for arbitrary power p, and if a suitable weighted norm of the data is bounded, the solution decays at least like t^{-1} as $t \to +\infty$. Higher Sobolev regularity of the initial data propagates and Sobolev norms do not inflate.

In a second result, we prove a pointwise dispersive estimate for solutions to the exterior problem for linear wave equation with a decaying potential depending on both time and space

$$\Box u + V(t,x)u = 0.$$

The potential is assumed to decay both in time and in space, at a sufficiently fast rate. This result is based on the methods of [3] and [2].

In the main result, we prove that the radial solution previously constructed is stable for small perturbations of the initial data in a weighted high Sobolev norm of order $O(n)$ of Christodoulou type (studied in [1]), and for sufficiently high powers $p > O(n)$. This produces a fanily of global large solutions to the supercritical wave equation on the exterior of the ball, with arbitrarily high power nonlinearity.

References

[1] P. D'Ancona, V. Georgiev, and H. Kubo, Weighted decay estimates for the wave equation, *J. Differential Equations* 177 (2001) 146–208

[2] Kubo, Hideo, On point-wise decay estimates for the wave equation and their applications. In: Dispersive nonlinear problems in mathematical physics, Dept. Math., Seconda Univ. Napoli, Caserta, pp. 123–148, 2004.

[3] Shibata, Yoshihiro and Tsutsumi, Yoshio, On a global existence theorem of small amplitude solutions for nonlinear wave equations in an exterior domain. *Mathematische Zeitschrift*, 191 (1986) 165–199

HAMILTONIAN STABILITY FOR WEIGHTED MEASURE AND GENERALIZED LAGRANGIAN MEAN CURVATURE FLOW

TORU KAJIGAYA

This talk is based on a joint work with Keita Kunikawa (Tohoku University).

The *mean curvature flow (MCF)* is the gradient flow of the volume functional for a submanifold in a Riemannian manifold (M, g), and actively studied in the area of geometric analysis. When M admits a Kähler-Einstein structure (ω, J) with the compatible Kähler metric g and the initial submanifold L is a *Lagrangian submanifold*, which is a submanifold of half dimension of M and $\omega|_L \equiv 0$, it is known due to Smoczyk that the MCF of initial L preserves the Lagrangian condition, and we call the flow *Lagrangian mean curvature flow (LMCF)* for this particular case. The LMCF is deeply studied especially when M is a Calabi-Yau manifold (Ricci flat case), and there is a conjecture about the convergence of LMCF (see [2] for more precisely). On the other hand, as pointed out in [2], it is a natural question to ask whether we can establish a suitable analogue in a non-Ricci flat Kähler-Einstein manifold or more general Kähler manifold. This is a motivation of our work.

In this talk, we mainly concern a compact Kähler manifold of *positive* Ricci curvature. Namely, we assume that M is a Fano manifold (M, J) equipped with a Kähler form $\omega \in 2\pi c_1(M)$. For a Fano manifold M, it is well known that there exists an obstruction for the existence of Kähler-Einstein metric on M, and hence, it is natural to assume the Kähler form ω is not necessary Kähler-Einstein, but in $2\pi c_1(M)$. More generally, we consider a Kähler manifold (M^{2n}, ω, J) equipped with a Kähler form ω satisfying $\rho = C\omega + ndd^c f$ for some positive constant $C > 0$ and a function $f \in C^\infty(M)$, where ρ is the Ricci form of M. We remark that an LMCF under this kind of assumption for ω has already been considered in other contexts or more general framework (e.g. [1] and [6]). We also remark that a symplectic manifold satisfying the assumption for ω is called *monotone* and extensively studied in symplectic geometry.

Let L be a compact Lagrangian submanifold in (M, ω, J, g) with $\rho = C\omega + ndd^c f$. In this setting, it makes sense to consider a *modified* MCF for L in M, namely, the gradient flow of the *weighted* volume functional $\mathrm{Vol}_f(L) := \int_L e^{nf} dv_g$ for the potential function f. More precisely, we consider a family of immersions $F : L \times [0, T) \to M$ satisfying

$$\frac{\partial F}{\partial t} = K := H - n(\overline{\nabla} f)^\perp, \tag{1}$$

where H is the mean curvature vector of the immersion F. The modified vector K and the flow satisfying (1) are so called *generalized mean curvature vector* and *generalized Lagrangian mean curvature flow (GLMCF)* (cf. [1], [6]), respectively. Also, we call a stationary solution of (1), that is, a critical point of Vol_f *f-minimal Lagrangian.*

This kind of modification is useful in order to generalize several results in Kähler-Einstein manifolds or to deal with a symplectic geometric property of L in (M, ω) by using a metric. For instance, it was first proved by Behrndt [1] that the modified flow (1) preserves the Lagrangian condition, although the usual MCF does not preserve in general. Moreover, we show that the flow preserves the *exactness* of a Lagrangian submanifold, namely, if the generalized mean curvature form $\alpha_K := (i_K\omega)|_L$ of the initial L is exact 1-form, then the exactness of α_K preserves under the flow. In other words, this means that the flow generates a Hamiltonian deformation of L in (M, ω) whenever L is exact. We also show that the exact Lagrangian is actually equivalent to the notion of *monotone Lagrangian* in symplectic geometry. In particular, it turns out that any f-minimal Lagrangian is monotone.

Since the flow is a gradient flow of Vol_f and contained in a Hamiltonian isotopy class whenever the initial L is exact, we naively expect that the flow converges to an f-minimal Lagrangian which minimizes (at least locally) the weighted volume in its Hamiltonian isotopy class. Here, we give a precise definition of the local minimizer. This is a generalization of the notion introduced by Oh [5] in a Kähler-Einstein manifold: An f-minimal Lagrangian L is called *Hamiltonian f-stable* if $d^2/ds^2|_{s=0}\mathrm{Vol}_f(L_s) \geq 0$ for any Hamiltonian deformation L_s of $L = L_0$. We prove that a Hamiltonian f-stable Lagrangian naturally appears in a toric Fano manifold. For example, we consider a torus orbit in the weighted projective space $\mathbb{C}P^n_{\mathbf{a}}$ for $\mathbf{a} = (1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ equipped with a Kähler structure and a potential function $f_{\mathbf{a}}$ obtained by a Kähler reduction of $\mathbb{C}^{n+1}$. We naturally find an explicit example of $f_{\mathbf{a}}$-minimal Lagrangian torus orbit in $\mathbb{C}P^n_{\mathbf{a}}$ and prove that the torus is Hamiltonian $f_{\mathbf{a}}$-stable in $\mathbb{C}P^n_{\mathbf{a}}$. This generalizes the result by Oh [5] for the case of Clifford torus in $\mathbb{C}P^n$.

By the above observations, it is natural to ask whether the GLMCF of initial exact Lagrangian converges to a Hamiltonian f-stable f-minimal Lagrangian. In this talk, we confirm this question under some initial conditions. Namely, we generalize the result of Haozhao Li [4] in Kähler-Einstein manifolds, and prove that if the initial Lagrangian in a Kähler manifold with $\rho = C\omega + ndd^c f$ for $C > 0$ is exact and a small Hamiltonian deformation of an f-minimal Hamiltonian f-stable Lagrangian, then the GLMCF of initial L converges exponentially fast to an f-minimal Lagrangian submanifold.

References

[1] T. Behrndt, *Generalized Lagrangian mean curvature flow in Kähler manifolds that are almost Einstein*, Complex and Differential Geometry, Springer Proceedings in Mathematics **8** (2011), Springer-Verlag, Heidelberg, New York, 65–80.

[2] D. Joyce, *Conjectures on Bridgeland stability for Fukaya categories of Calabi-Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow.* EMS Surv. Math. Sci. 2 (2015), no. 1, 1–62.

[3] T. Kajigaya and K. Kunikawa, *Hamiltonian stability for weighted measure and generalized Lagrangian mean curvature flow*, J. Geom. Phys. 128C (2018) pp. 140-168.

[4] H. Li, *Convergence of Lagrangian mean curvature flow in Kähler-Einstein manifolds*, Math. Zeit. **271** (2012), no. 1, 313–342.

[5] Y.-G. Oh, *Second variation and stabilities of minimal lagrangian submanifolds in Kähler manifolds*, Invent. Math. **101** (1990), 501–519.

[6] K. Smoczyk and M.-T. Wang, *Generalized Lagrangian mean curvature flows in symplectic manifolds*, Asian J. Math. **15** (2011), no. 1, 129–140.

Department of mathematics, School of Engineering, Tokyo Denki University, 5 Senju Asahi-cho, Adachi-ku, Tokyo 120-8551, Japan

E-mail address: `kajigaya@mail.dendai.ac.jp`

Separability of Maxwell equation in rotating black hole spacetime and its geometric aspects

Tsuyoshi Houri[a], Norihiro Tanahashi[b], and Yukinori Yasui[c]

[a] *National Institute of Technology, Maizuru College, 234 Shiroya, Maizuru, Kyoto 625-8511, Japan*

[b] *Institute of Mathematics for Industry, University of Kyushu, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan*

[c] *Faculty of Science and Engineering, Setsunan University, 17-8 Ikeda-Nakamachi, Neyagawa, Osaka, 572-8508 Japan*

May 17, 2018

Black holes are basic objects in the gravitation theory and their various aspects have been studied intensively. Particularly perturbative dynamics of fields in black hole spacetime is attracting much interest since it governs astrophysical phenomena around black holes and also it shows intriguing mathematical properties. In this context, the separability of the perturbation equations plays a key role. The perturbation equations are given by complicated partial differential equations in general, and to analyze them one typically has to resort to elaborate numerical techniques. When the perturbation equations admit separation of variables, however, the equations can be reduced to a set of ordinary differential equations that may be solved analytically.

In this work we focus on spacetime with a rotating black hole, in which the perturbation equations become complicated due to the rotation of the spacetime. About the perturbations in this spacetime Teukolsky made a significant progress in his seminal work, which showed that the equations of motion for the gravitational and Maxwell field can be separated if they are expressed in the so-called Newman-Penrose formalism [1]. However, even in this formalism one of the dynamical variables of the Maxwell field could not be separated. Adding to that, this formalism works only in the four dimensions, and generalization to higher dimensions remained unaccomplished for a long time.

Recently a breakthrough about this problem was made by Lunin [2] by introducing bland-new variables to express the vector potential, which is the dynamical variables of the Maxwell field. In terms of these new variables, the perturbation equations can be separated for all the propagating modes, and also this formalism can be generalized to the higher-dimensional case straightforwardly. One problem of his approach is that the new variables are introduced in a relatively heuristic manner and it was unclear why this new definition works well.

In Lunin's work, symmetries of the rotating black hole spacetime played an important role. It is known that the rotating black hole spacetime has hidden symmetries signified by the existence of nontrivial symmetric tensor satisfying $\nabla_{(a}K_{bc)} = 0$. This tensor is called the Killing tensor, and it yields a constant of motion $K^{ab}p_a p_b$, where p^a is the momentum vector, that is preserved on a geodesic. This property persists even in the higher-dimensional version of the rotating black hole spacetime. In Lunin's work, this tensor was used to introduce special coordinates that helped separating the variables of the perturbation equations.

Frolov et al. made a step forward about clarifying the significance of the hidden symmetry of the spacetime in this problem [3]. They focused on the off-shell Kerr-NUT-AdS metric in $2n$ dimensions

$$g_{ab}dx^a dx^b = \sum_{\mu=1}^{n} \frac{U_\mu}{X_\mu} dx_\mu^2 + \sum_{\mu=1}^{n} \frac{X_\mu}{U_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 , \tag{1}$$

where U_μ, X_μ and $A_\mu^{(k)}$ are certain functions of x^μ. This metric is the most general one admitting the principal tensor h_{ab} which is an anti-symmetric generalization of the Killing tensor defined by

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a, \qquad \xi_a = \frac{1}{2n-1}\nabla^b h_{ba}, \tag{2}$$

where ∇_a is the covariant derivative (Levi-Civita connection) of g_{ab}. Then they showed that, by expressing the vector potential expressed with a scalar Z as

$$\mathsf{A}^a = B^{ab}\nabla_b Z, \qquad B^{ac}\left(g_{cb} - \beta h_{cb}\right) = \delta^a_b, \tag{3}$$

where β is a constant, the Maxwell equation in the Lorentz gauge $\nabla_a \mathsf{A}^a = \nabla_a \left(B^{ab}\nabla_b Z\right) = 0$ can be reduced to a single scalar equation given by

$$B^{ab}\nabla_b \left(\Box Z + 2\beta\xi_c B^{cd}\nabla_d Z\right) = 0, \tag{4}$$

where $\Box \equiv \nabla_a \nabla^a$ is the d'Alembertian operator. They also showed that these equations can be expressed using derivative operators whose commutators vanish, which makes it possible to solve the equations by separation of variables. This result suggests that the separability of the perturbation equation and the hidden symmetry of the spacetime are related with each other, although their precise correspondence still remains unclear.

In our work, we shed new light on this problem based on studies about the hidden symmetry of the spacetime and the separability. One technique useful here is the Eisenhart-Duval lift [4, 5], which supplements the original spacetime with additional dimensions as

$$ds^2 = \hat{g}_{AB}d\hat{x}^A d\hat{x}^B = g_{ab}dx^a dx^b + 2\mathcal{A}_a dx^a du + 2du dv - 2V du^2, \tag{5}$$

where g_{ab} is the original spacetime metric and du, dv correspond to the additional spacetime dimensions. By suitably choosing the metric functions $\mathcal{A}^a, V$, the derivative operator $\Box + 2\beta\xi_c B^{cd}\nabla_d$ in Eq. (4) can be expressed simply as the d'Alembertian $\hat{\Box} = \hat{\nabla}_A \hat{\nabla}^A$ in terms of the covariant derivative $\hat{\nabla}_A$ associated with $\hat{g}_{AB}$. Hence, with this technique the perturbation equation (4) can be reduced to a massless scalar equation by absorbing the additional term $2\beta\xi_c B^{cd}\nabla_d$ into the structure of the higher-dimensional spacetime.

It turns out that Eq. (5) falls into the standard form of the metric whose geodesics is completely integrable [6], hence it is associated with the Killing tensors $\hat{K}_{AB}$ corresponding to the constants of motion on those geodesics. Using these Killing tensors we can construct derivative operators $\hat{\nabla}_A \hat{K}^{AB}\hat{\nabla}_B$, and it can be shown that these operators commute with the d'Alembertian $\hat{\Box}$. It implies that $\hat{\nabla}_A \hat{K}^{AB}\hat{\nabla}_B$ act as symmetry operators that map solutions of Eq. (4) to other ones, which can be employed to achieve the separation of variables. We also have some clues that these derivative operators are related to the commuting derivative operators found in [3]. Focusing on this point we try to clarify the true origin of the separability, hoping that this study would be useful when we try to extend this technique to more general cases such as the gravitational perturbations.

References

[1] S. A. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972).

[2] O. Lunin, JHEP **1712**, 138 (2017) [arXiv:1708.06766 [hep-th]].

[3] P. Krtouš, V. P. Frolov and D. Kubizňák, arXiv:1803.02485 [hep-th].

[4] L. P. Eisenhart, Annals. Math. **30**, 211 (1928).

[5] C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, Phys. Rev. D **31**, 1841 (1985).

[6] S. Benenti, Progress in Mathematics **99**, 1 (1991).

HARMONIC MAPS WITH PRESCRIBED SINGULARITIES AND APPLICATIONS IN GENERAL RELATIVITY

GILBERT WEINSTEIN

1. Introduction to harmonic maps

Let (M, g) and (N, h) be Riemannian manifolds and let $\varphi\colon M \to N$ be a smooth map. Then $d\varphi\colon M \to TM \otimes \varphi^{-1}TN$ is a cross-section of the tangent bundle of M tensor-product with the pull back bundle of TN under φ. In local coordinates we have:

$$d\varphi = \frac{\partial \varphi^\mu}{\partial x^i}\, dx^i \otimes \frac{\partial}{\partial y^\mu}.$$

There is a canonical positive definite inner product on this bundle with the norm expressed in local coordinates:

$$|d\varphi|^2 = \frac{\partial \varphi^\mu}{\partial x^i} \frac{\partial \varphi^\nu}{\partial x^j}\, g^{ij}\, h_{\mu\nu}.$$

From this we can form an energy:

$$E_\Omega(\varphi) = \int_\Omega |d\varphi|^2\, dV_g$$

Definition 1. A map $\varphi\colon M \to N$ is *harmonic* if for each $\Omega \subset\subset M$ it is a critical point of E_Ω. The map φ is an *local energy minimizer* if for every $\Omega \subset\subset M$, $E_\Omega(\varphi) \le E_\Omega(\psi)$ for all ψ which agrees with φ on the boundary of Ω. It is an energy minimizer if the same is true with $\Omega = M$.

Clearly every local energy minimizer is a harmonic map. The Euler-Lagrange equations read

$$\tau^\mu(\varphi) = \Delta_g \varphi^\mu + \Gamma^\mu_{\alpha\beta}(\varphi)\, g(\nabla\varphi^\alpha, \nabla\varphi^\beta) = 0,$$

where $\Gamma^\mu_{\alpha\beta}(\varphi)$ are the Christoffel symbols of N evaluated along φ, and $\Delta_g = \operatorname{tr}\nabla^2$ is the Laplace-Beltrami operator on M. This forms a system of semi-linear elliptic PDEs on M with the nonlinearity being quadratic in the first derivatives of φ. For an arbitrary map φ, the left-hand side τ is a cross section of $\varphi^{-1}TN$ called the *tension* of φ.

Familiar examples of harmonic maps are the two cases: (i) $\dim N = 1$, then $\Gamma = 0$ and the harmonic map problem reduces to the linear Laplace equation; (ii) $\dim M = 1$, then the harmonic map problem becomes the ODE problem for geodesics. The first significant nonlinear PDE result concerning harmonic maps was obtained by Eells-Sampson [5]: *If the Riemann curvature of N satisfies $\kappa_N \le 0$, and $\varphi_0\colon M \to N$ is any smooth map, then there is a harmonic map φ homotopic to φ_0.* Without the assumption on κ_N, harmonic maps and even energy minizers can develop singularities, and the best one can do is control the size of the singular set [12].

2. Harmonic maps with prescribed singularities

We first consider the linear case. Let (M, g) be a Riemannian manifold and let $\Sigma \subset M$ be a submanifold of co-dimension ≥ 2. Let $\rho\colon \Sigma \to \mathbb{R}^+$ be a positive continuous function, and let $G(x, y)$ be the Green function for Δ_g on M, then

$$u(x) = \int_\Sigma \rho(y)\, G(x, y)\, dv_\Sigma(y).$$

is a harmonic function on $M \setminus \Sigma$ which blows up on Σ, $\Delta_g u = \rho\delta_\Sigma$. Now if γ is a geodesic in N, then $\gamma \circ u$ is a harmonic map into N which blows up on Σ. In this talk, we will outline a proof of a nonlinear analog of this construction, and discuss applications to general relativity. Before we state the result, we need two definitions:

Definition 2. A *model map* is a map $\varphi_0\colon M \setminus \Sigma \to N$ such that (i) $|\tau(\varphi_0)|$ is bounded; and (ii) $|\tau(\varphi_0)|$ decays "fast enough", in the sense that there is a function $v > 0$ with $v \to 0$ at infinity and $|\tau(\varphi_0)| \le -\Delta_g v$.

It turns out for instance that for $M = \mathbb{R}^3$ it is sufficient to require $|\tau(\varphi_0)| = O(r^{-2-\epsilon})$ with $\epsilon > 0$.

Definition 3. Two maps $\varphi_1, \varphi_2\colon M \setminus \Sigma \to N$ are *asymptotic* if $\operatorname{dist}_N(\varphi_1, \varphi_2)$ is (i) bounded; and (ii) falls off to 0 at infinity.

Part (i) of this definition is inspired by the analogous definition for geodesics [4].

Theorem 1. *Let $\varphi_0\colon M \setminus \Sigma \to N$ be a model map, and let either $-b^2 < \kappa_N < -a^2$, or N be a symmetric space of non-compact type, then there is a unique harmonic map $\varphi\colon M \setminus \Sigma \to N$ which is asymptotic to φ_0.*

3. Applications to general relativity

The applications to general relativity of the above rely on the following fact. It turns out that, under the hypothesis of *stationarity* and (D-3)-*axisymmetry*, i.e. admitting an isometric action of the group $\mathbb{R}\times U(1)^{D-3}$, the Einstein vacuum equations in D dimensions reduce to a harmonic map problem $\varphi\colon \mathbb{R}^3\setminus\Sigma\to N$ where $\Sigma\subset$ z-axis, and $N=SL(n,\mathbb{R})/SO(n)$, $n=D-2$ is a symmetric space of non-compact type. We will briefly review some of these applications. One of these will be discussed in more detailed in the talk of my colleague Sumio Yamada.

(1) Uniqueness of the Kerr solution as the only family of asymptotically flat stationary axisymmetric solutions of the 4-D Einstein equation with a connected apparent horizon [11]. In this case, the target is $N=SL(2,\mathbb{R})/SO(2)=\mathbb{H}^2$ the hyperbolic plane. In 4-D, this result can be generalized to the Einstein-Maxwell equations, in which case the target is $N=\mathbb{H}^2_{\mathbb{C}}$ the complex hyperbolic plane [10].
(2) Construction of multiple black hole metrics. This work originated in the linear (non-rotating) case with Bach-Weyl [1], and was pursued further in a series of papers with [13] the latest. Here too, it is possible to generalize to Einstein-Maxwell.
(3) Construction of multiple extreme black holes solutions. As discovered by Dain [3], these results are useful for obtaining inequalities such as $m\geq |J|$ in vacuum, or in electro-vacuum:
$$m^2\geq\frac{1}{2}\left(q^2+\sqrt{q^4+4J^2}\right),$$
since it turns out that the renormalized energy of the corresponding harmonic mpa bounds the total mass of any data set from below; see [2,6].
(4) Periodic (in z) solutions of the Einstein vacuum equations. The linear (non-rotating) case is discussed in [9]. The nonlinear case is still open.
(5) Construction of black hole solutions in 5-D vacuum gravity with all possible apparent horizon topologies: S^3, $S^1\times S^2$, and *lens space* $L(p,q)$. We constructed solutions which are asymptotically flat [7], and also solutions which are asymptotically locally Euclidean, i.e. whose spacelike infinity is modeled on $(a,\infty)\times L(p,q)$, as well as asymptotically Kaluza-Klein, i.e. with spacelike infinity modeled on $(a,\infty)\times S^1\times S^2$ [8]. As in (2), we are also able to construct multiple black hole solutions, provided the necessary topological obstructions are avoided.
(6) Generalization of the results in the previous item to higher-D. This is work in progress.

In cases (1), (2), (5), and (6), a conical singularity can potentially form when reconstructing the Einstein metric. This was first discussed in [1] and the authors related the angle deficiency to the attractive gravitational force. Some results were obtained in cases (2), and (5). Interestingly, this issue does not come up in (4). We will briefly discuss the conical singularity, as time allows.

References

[1] Rudolf Bach and Hermann Weyl, *Republication of: New solutions to Einstein's equations of gravitation. B. Explicit determination of static, axially symmetric fields. By Rudolf Bach. With a supplement on the static two-body problem. By H. Weyl.*, General Relativity and Gravitation **44** (2012Mar), no. 3, 817–832.
[2] Piotr T. Chruściel, Yanyan Li, and Gilbert Weinstein, *Mass and angular-momentum inequalities for axi-symmetric initial data sets. II. Angular momentum*, Ann. Physics **323** (2008), no. 10, 2591–2613. MR2454699 (2009m:83036b)
[3] Sergio Dain, *Proof of the angular momentum-mass inequality for axisymmetric black holes*, J. Differential Geom. **79** (2008), no. 1, 33–67. MR2414749 (2009j:83007)
[4] P. Eberlein and B. O'Neill, *Visibility manifolds.*, Pacific J. Math. **46** (1973), no. 1, 45–109.
[5] James Eells and J. H. Sampson, *Harmonic Mappings of Riemannian Manifolds*, American Journal of Mathematics **86** (1964), no. 1, 109–160.
[6] Marcus Khuri and Gilbert Weinstein, *The Positive Mass Theorem for Multiple Rotating Charged Black Holes*, Calculus of Variations and Partial Differential Equations **55** (April 2016), no. 2, 1–29, available at `1502.06290`.
[7] Marcus Khuri, Gilbert Weinstein, and Sumio Yamada, *Stationary Vacuum Black Holes in 5 Dimensions* (2017), available at `http://arxiv.org/abs/1711.05229`.
[8] ______, *Asymptotically Locally Euclidean/Kaluza-Klein Stationary Vacuum Black Holes in 5 Dimensions* (2018), available at `1802.02457`.
[9] D. Korotkin and H. Nicolai, *A Periodic analog of the Schwarzschild solution* (1994), available at `gr-qc/9403029`.
[10] P O Mazur, *Proof of uniqueness of the Kerr-Newman black hole solution*, Journal of Physics A: Mathematical and General **15** (1982), no. 10, 3173.
[11] D. C. Robinson, *Uniqueness of the Kerr Black Hole*, Phys. Rev. Lett. **34** (1975Apr), 905–906.
[12] Richard Schoen and Karen Uhlenbeck, *A regularity theory for harmonic maps*, J. Differential Geom. **17** (1982), no. 2, 307–335.
[13] Gilbert Weinstein, *Harmonic maps with prescribed singularities into Hadamard manifolds*, Math. Res. Lett. **3** (1996), no. 6, 835–844. MR1426540 (98b:58049)

HARMONIC MAPS IN GENERAL RELATIVITY

SUMIO YAMADA (GAKUSHUIN UNIVERSITY)

1. The Einstein equation as an elliptic system under symmetries

In the theory of general relativity, the spacetime is a Lorentzian n-manifold $(N^n, \bar{g})$ satisfying the Einstein equations

$$R_{ab} - \frac{1}{2} R\, \bar{g}_{ab} = T_{ab}$$

where R_{ab} is the Ricci curvature of the Lorentzian metric $\bar{g}$, $R_{\bar{g}}$ is the scalar curvature, and T is the energy-momentum-stress tensor of the matter fields. In this talk we assume T is identically zero, namely the spacetime is vacuum. The vacuum Einstein equation is then equivalent to the spacetime being Ricci-flat: $R_{ab} = 0$. The Einstein equation is the Euler-Lagrange equation for the Hilbert-Einstein functional;

$$\mathcal{H}(\bar{g}) = \int_N \{R_{\bar{g}} + L\}\, d\mu_{\bar{g}}$$

where L is the Lagrangian for non-gravitational fields, which is zero when vacuum.

The first exact solution to this seemingly very difficult system of nonlinear partial differential equations was found by Schwarzschild, only a few weeks after A. Einstein presented his theory in 1916. H. Weyl did not lose any time (1917) in rewriting down the Schwarzschild metric in a cylindical coordinate system, namely

$$ds^2 = -(1 - \frac{m}{r})dt^2 + (1 - \frac{m}{r})^{-1} dr^2 + r^2 d\omega^2_{S^2}.$$

as

$$ds^2 = -V^2 dt^2 + V^{-2}\{\rho^2 d\phi^2 + e^{2\sigma}(d\rho^2 + dz^2)\}.$$

When V is written as e^u, the vacuum Einstein equation $R_{ab} = 0$ becomes

$$\triangle_{\mathbb{R}^3} u = 0, \quad \sigma_z = 2\rho u_z u_\rho, \quad \sigma_\rho = \rho(u_\rho^2 - u_z^2)$$

and the harmonicity of u is the integrability condition for the quadrature $d\sigma = \frac{1}{2}\rho\,(\mathrm{Re}|\partial \mathrm{u}|^2)\,\mathrm{d}\xi \quad (\xi = \rho + \mathrm{iz})$, enabling to solve the pair of equation for σ. In other words, the Schwarzschild solution satisfying the vacuum Einstein equation with two Killing vector fields $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$, is reduced to finding the harmonic function u in ρ, z. The function u determines the $\bar{g}$-norm of $\frac{\partial}{\partial \phi}$. This approach has been generalized to characterize the Kerr metric in [2, 3, 17] We formulate this as follows: With the global coordinate system

$$[\mathbb{R} \times U(1)] \times \{(\rho, z) \mid \rho > 0,\ z \in \mathbb{R}\}$$

the Einstein metric is written as

$$\bar{g} = G_{ij}(\rho, z) dx^i \otimes dx^j + e^{2\sigma(\rho,z)}(d\rho^2 + dz^2)$$

where $\rho = \sqrt{|\det G_{ij}|}$ and determining the part G_{ij} of the metric restricted to the fiber of the isometry group, in this case the time symmetry and one axial symmetry, by solving an elliptic system, which in the case of 4 dimension, corresponds to a harmonic map from $\mathbb{R}^3 \setminus z$-axis into the hyperbolic space $\mathbb{H}^2 = SL(2, \mathbb{R})/SO(2)$. Such a representation of the Lorentzian metric is called the Weyl-Papapetrou coordinate system. It turns out ([7, 10]) that, under the hypothesis of *stationarity* and (D-3)-*axisymmetry*, i.e. admitting an isometric action of the group $\mathbb{R} \times U(1)^{D-3}$, the Einstein vacuum equations in D dimensions reduce to a harmonic map problem $\varphi\colon \mathbb{R}^3 \setminus \Sigma \to N$ where $\Sigma \subset z$-axis, and $N = SL(n, \mathbb{R})/SO(n)$, $n = D - 2$ is a symmetric space of non-compact type, of higher rank.

2. Bi-axisymmetric stationary spacetimes in 5 dimension

In [8, 9], we have obtained an existence theorem of 5-dimensional spacetimes, which is stationary, bi-axisymmetric. In addition to the natural generalization of the 4-dimensional Kerr metric [11] where the blackhole horizon is an S^3, the construction of black ring solution [12], whose black hole horizon is diffeormorphic to $S^2 \times S^1$, was a big breakthrough in the geometry of higher dimensional spacetime. One of the known obstructions to the topological type of the horizon is the result of Galloway-Schoen [13]: the horizon is of positive Yamabe-type. Under the additional condition of bi-axisymmetry, it is known [14] that the horizon is diffeomorphic to one of S^3, $S^2 \times S^1$ or lens spaces $L(p, q)$. The projective space $\mathbb{R}P^3$ is a lens space $L(2, 1)$.

Even though the no-hair theorem [15] of 4-dimensional stationary spacetime no longer holds, a uniqueness theorem for harmonic maps with prescribed singularities into a non-positively curved target $N = SL(n,\mathbb{R})/SO(n)$ still holds in higher dimensions. In particular, any bi-axially symmetric stationary vacuum solution is determined by a finite set of parameters. The parameter spaces of harmonic map is called the rods structures, corresponding the the asymptotic Dirichlet condition of the harmonic map, which determines the part G_{ij} of the Einstein metric restricted to the fiber of the 3-dimensional symmetry group $\mathbb{R} \times U(1)^2$ as described above. The rod structure, together with the angular momenta around the two axes, constitutes the rod data set $\mathcal{D}$ [6]. The rod structure encodes blow-up behaviors near the symmetry axes in the spacetime, and the resulting harmonic map can be regarded as a non-linear Green's function [19, 20]. Using the Weyl-Papapetrou coordinate system, we model the blow-up set $\Gamma = \cup\Gamma_l$ as a subset of the z-axis of the ρz-(right) half plane. In order to determine the physical relevance of a solution, define on each bounded axis rod Γ_l a function b_l to be the logarithm of the limiting ratio between the length of the closed orbit of the Killing field degenerating on Γ_l, and 2π times the radius from Γ_l to this orbit. It turns out that b_l is constant on Γ_l. The absence of a conical singularity on Γ_l is the *balancing condition* $b_l = 0$ [18].

We can now state the main theorem.

Theorem 1. *Given an admissible rod data set $\mathcal{D}$, there is a unique harmonic map $\varphi\colon \mathbb{R}^3 \setminus \Gamma \to SL(3,\mathbb{R})/SO(3)$ with prescribed singularities on Γ corresponding to $\mathcal{D}$. A well-behaved 5-dimensional asymptotically flat, stationary, bi-axially symmetric solution of the vacuum Einstein equations without degenerate horizons can be constructed from φ if and only if the resulting metric coefficients are sufficiently smooth across Γ and $b_l = 0$ on any bounded axis rod.*

The technical point one needs to overcome in proving the theorem is how to handle the geometry of higher rank symmetric space of non-compact type, where because of flats (in our case isometrically embedded $\mathbb{R}^2$ in $SL(3,\mathbb{R})/SO(3)$) there is less convexity available, which hinders the standard argument in proving the existence of harmonic maps.

This result enables us to construct black hole solutions in 5-dimensional vacuum gravity with all possible apparent horizon topologies: S^3, $S^1 \times S^2$, and *lens space* $L(p,q)$ as well as their combinations. We constructed solutions which are asymptotically flat [8], and also solutions which are asymptotically locally Euclidean, i.e. whose spacelike infinity is modeled on $(a,\infty) \times L(p,q)$, as well as asymptotically Kaluza-Klein, i.e. with spacelike infinity modeled on $(a,\infty) \times S^1 \times S^2$ [9].

References

[1] R. Bach and H. Weyl, Republication of: *New solutions to Einstein's equations of gravitation. B. Explicit determination of static, axially symmetric fields. By Rudolf Bach. With a supplement on the static two-body problem. By H. Weyl.*, General Relativity and Gravitation **44** (2012nMar), no. 3, 817–832.

[2] B. Carter *Black hole equilibrium states*, in Black Holes, C. DeWitt and B. S. DeWitt, eds., New York, Gordon and Breach, 1973.

[3] Ernst, F. J., New formulation of the gravitational field problem, Phys. Rev. Letters 167, 1968, pp.1175–1178.

[4] R. Emparan, and H. Reall, *Black holes in higher dimensions*, Living Rev. Rel., **11** (2008), no. 6. arXiv:0801.3471

[5] T. Harmark, *Stationary and axisymmetric solutions of higher-dimensional general relativity*, Phys. Rev. D, **70** (2004), 124002. arXiv:hep-th/0408141

[6] S. Hollands, and A. Ishibashi, *Black hole uniqueness theorems in higher dimensional spacetimes*, Class. Quantum Grav., **29** (2012), no. 16, 163001. arXiv:1206.1164

[7] D. Ida, A. Ishibashi, and T. Shiromizu, *Topology and uniqueness of higher dimensional black holes*, Prog. Theor. Phys. Suppl., **189** (2011), 52-92. arXiv:1105.3491

[8] M. Khuri, G. Weinstein, and S. Yamada, *Stationary vacuum black holes in 5 dimensions*, preprint, 2017. arXiv:1711.05229

[9] M. Khuri, G. Weinstein, and S. Yamada, *Asymptotically locally Euclidean/Kaluza-Klein stationary vacuum black holes in 5 dimensions*, to appear in Progress of Theoretical and Experimental Phyics 2018, arXiv:1802.02457 [hep- th].

[10] D. Maison, *Ehlers-Harrison-type transformations for Jordan's extended theory of gravitation*, Gen. Relativity Gravitation, **10** (1979), no. 8, 717-723.

[11] R. Myers, and M. Perry, *Black holes in higher dimensional space-times*, Ann. Physics, **172** (1986), no. 2, 304-347.

[12] R. Emparan, and H. Reall, *A rotating black ring in five dimensions*, Phys. Rev. Lett., **88** (2002), 101101. arXiv:hep-th/0110260.

[13] G. Galloway, and R. Schoen, *A generalization of Hawking's black hole topology theorem to higher dimensions*, Commun. Math. Phys., **266** (2006), no. 2, 571-576.

[14] T. Harmark, *Stationary and axisymmetric solutions of higher-dimensional general relativity*, Phys. Rev. D, **70** (2004), no. 12, 124002. arXiv:hep-th/0408141

[15] S. Hawking, *Black holes in general relativity*, Commun. Math. Phys., **25** (1972), 152-166.

[16] P. Li, and L.-F. Tam, *Uniqueness and regularity of proper harmonic maps*, Ann. of Math. **137** (1993), no. 1, 167-201.

[17] G. Weinstein, *On rotating black holes in equilibrium in general relativity*, Comm. Pure App. Math., **43** (1990), no. 7, 903-948.

[18] G. Weinstein, *On the force between rotating co-axial black holes*, Trans. Amer. Math. Soc., **343** (1994), no. 2, 899-906.

[19] G. Weinstein, *On the Dirichlet problem for harmonic maps with prescribed singularities*, Duke Math. J., **77** (1995), 135-165.

[20] G. Weinstein, *Harmonic maps with prescribed singularities into Hadamard manifolds*, Math. Res. Lett., **3** (1996), no. 6, 835-844.

Random data Cauchy problem for the energy critical nonlinear Schrödinger equations

Mamoru Okamoto*

1 Introduction

We consider the following Cauchy problem for the nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u = |u|^{\frac{4}{d-2}}u, \\ u|_{t=0} = \phi, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1}$$

where $u = u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is an unknown function and $d = 5, 6$. The equation (1) is invariant under the scaling transformation

$$u(t,x) \longmapsto u_\mu(t,x) = \mu^{\frac{d-2}{2}} u(\mu^2 t, \mu x) \quad (\mu > 0),$$

and hence our problem is energy critical. In addition, the Cauchy problem is well-posed in the energy space $H^1(\mathbb{R}^d)$ ([2], [6]).

In this talk, we consider the Cauchy problem in $H^s(\mathbb{R}^d)$ with randomized initial data for $s < 1$. The randomization is defined as follows:

Definition 1 ([1])**.** *Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be a cut-off function satisfying*

$$\operatorname{supp}\psi \subset [-1,1]^d, \quad \sum_{n \in \mathbb{Z}^d} \psi(\xi - n) = 1 \quad (\xi \in \mathbb{R}^d).$$

Then, we define the randomization of ϕ by

$$\phi^\omega := \sum_{n \in \mathbb{Z}^d} g_n(\omega) \mathcal{F}^{-1}[\psi(\cdot - n)\mathcal{F}[\phi]], \tag{2}$$

where $\{g_n\}$ is a sequence of independent complex valued Gaussian random variables in a probability space (Ω, P).

Theorem 2 ([5])**.** *Let $d = 5, 6$ and assume $1 - \frac{1}{d} < s < 1$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its randomization defined in (2). Then, for almost all $\omega \in \Omega$, there exists $T_\omega > 0$ and a solution $u = u^\omega \in C([-T_\omega, T_\omega]; H^s(\mathbb{R}^d))$ to (1) with $u(0) = \phi^\omega$.*

Bényi et al. [1] studied the case $d = 4$. Note that when $d = 5, 6$, the nonlinearity is no longer algebraic.

We also prove almost sure global existence.

*Division of Mathematics and Physics, Faculty of Engineering, Shinshu University
e-mail: m_okamoto@shinshu-u.ac.jp

Theorem 3 ([5])**.** *Let $d = 5, 6$ and assume*

$$\frac{63}{68} < s < 1 \quad (d = 5), \qquad \frac{20}{23} < s < 1 \quad (d = 6).$$

Given $\phi \in H^s(\mathbb{R}^d)$, let be ϕ^ω its randomization define in (2)*. Then, for almost all $\omega \in \Omega$, there exists a global-in-time solution $u = u^\omega \in C(\mathbb{R}; H^s(\mathbb{R}^d))$ to* (1) *with $u(0) = \phi^\omega$.*

Kellip et al. [4] proved almost sure global existence for $d = 4$ under the radial assumption. Dodson et al. [3] improved their result.

2 Outline of proof

Set $S(t) := e^{it\Delta}$. The randomization improves integrability in the following sense:

Lemma 4 ([1])**.** *Given finite $q, r \geq 2$, there exist constants $C, c > 0$ such that*

$$P\Big(\|S(t)\phi^\omega\|_{L_t^q L_x^r((-T,T)\times\mathbb{R}^d)} > \lambda\Big) \leq C \exp\bigg(- c\frac{\lambda^2}{T^{\frac{2}{q}}\|\phi\|_{L^2}^2}\bigg)$$

for all $T > 0$ and $\lambda > 0$.

Let $z(t) = z^\omega(t) := S(t)\phi^\omega$ denote the linear solution with ϕ^ω as initial data. If u is a solution to (1), then the residual term $v := u - z$ satisfies the following:

$$\begin{cases} i\partial_t v + \Delta v = \mathcal{N}(v + z^\omega), \\ u|_{t=0} = 0, \end{cases}$$

where $\mathcal{N}(u) := |u|^{\frac{4}{d-2}}u$. By using the contraction mapping theorem, we prove existence of a solution $v \in C([-T, T]; H^1(\mathbb{R}^d))$ to this perturbed equation, which shows Theorem 2.

For the proof of Theorem 3, the main task is to control the growth of the $H^1(\mathbb{R})$-norm for the residual part, namely

$$\sup_{t\in[-T,T]} \|v(t)\|_{H^1} \leq C(T).$$

References

[1] Á. Bényi, T. Oh, and O. Pocovnicu, *On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$, $d \geq 3$*, Trans. Amer. Math. Soc. Ser. B **2** (2015), 1–50.

[2] T. Cazenave and F. Weissler, *Some remarks on the nonlinear Schrödinger equation in the critical case,* Nonlinear semigroups, partial differential equations and attractors (Washington, DC, 1987), 18–29, Lecture Notes in Math., 139.

[3] B. Dodson, J. Luhrmann, and D. Mendelson, *Almost sure local well-posedness and scattering for the 4D cubic nonlinear Schrödinger equation*, arXiv:1802.03795.

[4] R. Killip, J. Murphy, and M. Vişan, *Almost sure scattering for the energy-critical NLS with radial data below $H^1(\mathbb{R}^4)$*, arXiv:1707.09051.

[5] T. Oh, M. Okamoto, and O. Pocovnicu, *On the probabilistic well-posedness of the nonlinear Schrödinger equations with non-algebraic nonlinearities,* arXiv:1708.01568.

[6] M. Vişan, *The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions,* Duke Math. J. **138** (2007), no. 2, 281–374.

GLOBAL BEHAVIOR OF SOLUTIONS TO THE ENERGY CRITICAL NONLINEAR DAMPED WAVE EQUATION

TAKAHISA INUI

1. Introduction

In this talk, we consider the global behavior of the solutions to the energy critical nonlinear damped wave equation:

$$\text{(NLDW)} \qquad \begin{cases} \partial_t^2 u - \Delta u + \partial_t u = |u|^{\frac{4}{d-2}} u, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ (u(0), \partial_t u(0)) = (u_0, u_1), & x \in \mathbb{R}^d, \end{cases}$$

where $d \geq 3$, (u_0, u_1) is a $\mathbb{C}^2$-valued given function, and u is an $\mathbb{C}$-valued unknown function.

Before discussing (NLDW), we shall investigate the linear equation:

$$(1.1) \qquad \begin{cases} \partial_t^2 \phi - \Delta \phi + \partial_t \phi = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ (\phi(0), \partial_t \phi(0)) = (\phi_0, \phi_1), & x \in \mathbb{R}^d, \end{cases}$$

where $d \in \mathbb{N}$. Matsumura [2] applied the Fourier transform to (1.1) and obtained the formula

$$\phi(t,x) = \mathcal{D}(t)(\phi_0 + \phi_1) + \partial_t \mathcal{D}(t) \phi_0,$$

where $\mathcal{D}(t)$ is defined by

$$\mathcal{D}(t) := e^{-\frac{t}{2}} \mathcal{F}^{-1} L(t,\xi) \mathcal{F}$$

with

$$L(t,\xi) := \begin{cases} \dfrac{\sinh(t\sqrt{1/4 - |\xi|^2})}{\sqrt{1/4 - |\xi|^2}} & \text{if } |\xi| < 1/2, \\[2ex] \dfrac{\sin(t\sqrt{|\xi|^2 - 1/4})}{\sqrt{|\xi|^2 - 1/4}} & \text{if } |\xi| > 1/2. \end{cases}$$

By this formula, Matsumura [2] proved the L^p-L^q type estimate:

$$\|\phi(t)\|_{L^p} \lesssim \langle t \rangle^{-\frac{d}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \|(\phi_0, \phi_1)\|_{L^q \times L^q} + e^{-\frac{t}{4}} \left(\|\phi_0\|_{H^{[\frac{d}{2}]+1}} + \|\phi_1\|_{H^{[\frac{d}{2}]}} \right),$$

where $1 \leq q \leq 2 \leq p \leq \infty$ and $[d/2]$ denotes the integer part of $d/2$. This L^p-L^q type estimate means that the solution of (1.1) behaves like the solution of the heat equation and the wave equation in some sense. More precisely, the low frequency part of the solution to the damped wave equation behaves like the solution of the heat equation and the high frequency part behaves like the solution of the wave equation but decays exponentially. Such L^p-L^q type estimates have been studied well (see for instance [1]). However, to investigate the energy critical nonlinear problem (NLDW), it is not enough to have L^p-L^q type estimates. We should obtain the space-time estimates of the solutions to (1.1).

2. Abstract of this talk

Recently, Watanbe obtained the space-time estimates, what we call the Strichartz estimates, of the solutions to the linear equation (1.1) when $d = 2, 3$ in [4]. In this talk, I introduce the Strichartz estimates in the higher dimensional case. The Strichartz estimates are also combinations of heat and wave. Applying the Strichartz estimates and the contraction mapping principle, we show the local existence of the solutions to (NLDW) in the energy space when $3 \leq d \leq 5$. Moreover, we prove that the global solution with a finite space-time norm decays to zero and we show existence of the finite time blow-up solutions. The decaying result, which of course comes from the damping, is obtained in the similar way to that of the corresponding nonlinear heat equation and the blow-up result is obtained by applying the method of [3].

References

[1] Ikeda M., Inui T., Okamoto M., Wakasugi Y.: L^p-L^q estimates for the damped wave equation and the critical exponent for the nonlinear problem with slowly decaying data, preprint, arXiv:1710.06538.

[2] Matsumura A.: On the asymptotic behavior of solutions of semi-linear wave equations, Publ. Res. Inst. Math. Sci. **12**, no. 1, 169–189 (1976/77).

[3] Ohta M.: Blowup of solutions of dissipative nonlinear wave equations, Hokkaido Math. J. **26**, no. 1, 115–124 (1997).

[4] Watanabe T.:, Strichartz type estimates for the damped wave equation and their application, RIMS Kôkyûroku Bessatsu, **B63**, 77–101 (2017).

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: `inui@math.sci.osaka-u.ac.jp`

Asymptotics of generalized eigenfunctions on manifold with Euclidean and/or hyperbolic ends

Kenichi ITO (University of Tokyo)

This talk is based on joint works [1], [2] with Erik Skibsted, Aarhus University. Let (M,g) be a connected and complete Riemannian manifold, and we study the Schrödinger operator on $\mathcal{H}=L^2(M)$:

$$H=H_0+V;\quad H_0=-\tfrac{1}{2}\Delta,$$

where Δ is the *Laplace–Beltrami operator*.

We call an open subset $E\subset M$ an *end*, if there is a diffeomorphism $\bar{E}\cong[2,\infty)\times S$ for some connected and closed manifold S. The metric g is said to be of *warped-product type* on E, if

$$g(r,\theta)=\mathrm{d}r\otimes\mathrm{d}r+f(r)h_{\alpha\beta}(\theta)\,\mathrm{d}\theta^\alpha\otimes\mathrm{d}\theta^\beta;\quad (r,\theta)\in(2,\infty)\times S$$

for some Riemannian metric h on S.

Assumption A. (M,g) has N number of ends E_n, $n=1,\dots,N$, such that

1. For each $n\in\{1,\dots,N\}$ the metric g is of warped-product type on E_n, and the associated scalar f_n is one of the following form:
   (a) $f_n(r)=r^\delta$ with $\delta>1$;
   (b) $f_n(r)=\exp(\kappa r^\delta)$ with $\kappa>0$, $0<\delta<1$;
   (c) $f_n(r)=\exp(\kappa r)$ with $\kappa>0$.
2. $M\setminus(E_1\cup\cdots\cup E_N)$ is compact.

We extend $r\in C^\infty(E_1\cup\cdots\cup E_N)$ onto M such that $r\in C^\infty(M)$ and $r\geq 1$ on M. Then we define the *effective potential* q as

$$q=V+\tfrac{1}{8}(\Delta r)^2.$$

Assumption B. There exists a splitting $q=q_1+q_2$ such that for some $\epsilon>1/2$

$$|\nabla q_1|+|\nabla^2 q_1|\leq Cr^{-1-\epsilon},\quad |q_2|\leq Cr^{-1-\epsilon}.$$

Define the *critical energy* as $\lambda_H = \limsup_{r\to\infty} q$. We denote the dyadic annuli by $\Omega_\nu = \{x \in M;\ 2^\nu \leq r(x) < 2^{\nu+1}\}$ for $\nu \geq 0$, and define the associated *Agmon–Hörmander spaces* as

$$\mathcal{B}^* = \Big\{\psi \in L^2_{\mathrm{loc}}(M);\ \|\psi\|_{\mathcal{B}^*} = \sup_{\nu\geq 0} 2^{-\nu/2}\|\chi_{\Omega_\nu}\psi\|_{\mathcal{H}} < \infty\Big\},$$
$$\mathcal{B}^*_0 = \Big\{\psi \in \mathcal{B}^*;\ \lim_{\nu\to\infty} 2^{-\nu/2}\|\chi_{\Omega_\nu}\psi\|_{\mathcal{H}} = 0\Big\}.$$

Let

$$\mathcal{G} = L^2(S_1) \oplus \cdots \oplus L^2(S_N), \quad \bar{E}_n \cong [2,\infty) \times S_n \ \text{ for } n = 1,\ldots,N. \tag{1}$$

For any $\lambda > \lambda_H$ and $\xi \in \mathcal{G}$ we introduce $\phi^\pm[\xi] \in \mathcal{B}^*$ by

$$\phi^\pm[\xi](r,\theta) = \frac{\eta(r)}{\sqrt[4]{2[\lambda - q_1(r,\theta)]f(r)^{d-1}}}\exp\left(\pm\mathrm{i}\int_{r_0}^{r}\sqrt{2[\lambda - q_1(s,\theta)]}\,\mathrm{d}s\right)\xi(\theta).$$

Here η is a smooth cut-off function such that $\eta = 1$ for large r.

Theorem. *Let $\lambda > \lambda_H$, and set $\mathcal{E}_\lambda = \{\phi \in \mathcal{B}^*;\ (H-\lambda)\phi = 0\}$.*

1. *One has $\mathcal{E}_\lambda \cap \mathcal{B}^*_0 = \emptyset$.*
2. *For any $\phi \in \mathcal{E}_\lambda$ there exist unique $\xi_\pm \in \mathcal{G}$ such that*
$$\phi - \phi^+[\xi_+] + \phi^-[\xi_-] \in \mathcal{B}^*_0. \tag{2}$$
3. *Conversely, for any $\xi_\pm \in \mathcal{G}$ there exist unique $\phi \in \mathcal{E}_\lambda$ and $\xi_\mp \in \mathcal{G}$, respectively, such that* (2) *holds.*

The *stationary scattering matrix* $S(\lambda)\colon \mathcal{G} \to \mathcal{G}$ is defined as $\xi_+ = S(\lambda)\xi_-$. According to (2), we can write $S(\lambda) = (S(\lambda)_{ij})_{i,j}$. By the last theorem for any $\xi_- \in \mathcal{G}$ there exists $\phi \in \mathcal{E}_\lambda$ such that

$$\phi - \phi^+[S(\lambda)\xi_-] + \phi^-[\xi_-] \in \mathcal{B}^*_0.$$

Let, e.g., $\xi_- = (\xi_{-,1}, 0, \ldots, 0)$. Then $(S(\lambda)\xi_-)_2, \ldots, (S(\lambda)\xi_-)_N$ are all non-zero due to $\mathcal{E}_\lambda \cap \mathcal{B}^*_0 = \emptyset$. This in general implies that $S(\lambda)_{ij}$ are injective for all $i \neq j$.

References

[1] K. Ito, E. Skibsted, Stationary scattering theory on mainifolds, I. Preprint 2016.

[2] K. Ito, E. Skibsted, Stationary scattering theory on mainifolds, II. Preprint 2016.

TIME QUASI-PERIODIC SOLUTIONS TO THE NONLINEAR KLEIN-GORDON EQUATIONS

W.-M. Wang

Abstract. We present time quasi-periodic solutions to the nonlinear Klein-Gordon equations on the torus in arbitrary dimensions. These solutions are, by construction, global in time. The additional difficulty, compared to nonlinear Schrödinger equations, is the *dense* spectrum of the wave operator. We show how number theory can help with this issue.

We consider real solutions to the nonlinear Klein-Gordon equation (NLKG) on the d-torus $\mathbb{T}^d = [0, 2\pi)^d$:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u + u^{p+1} = 0, \tag{1.1}$$

where $p \in \mathbb{N}$ and $p \geq 1$ is arbitrary; considered as a function on $\mathbb{R}^d$, u is periodic: $u(\cdot, x) = u(\cdot, x + 2j\pi)$, $x \in [0, 2\pi)^d$ for all $j \in \mathbb{Z}^d$.

We start from the linear, second order in time, equation:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u = 0, \tag{1.2}$$

and define the wave operator $D := \sqrt{-\Delta + 1}$. Using Fourier series, the spectrum of D, $\sigma(D) = \{\sqrt{|j|^2 + 1} | j \in \mathbb{Z}^d\}$, is *dense*, in the sense that the spacing of *non-equal* eigenvalues tends to 0, as $|j| \to \infty$, and degenerate for $d \geq 2$, which is an essential difficulty.

Denote $|j|^2$ by j^2, the solutions to (1.2) are linear combinations of cosine and sine functions of the form:

$$\cos(-(\sqrt{j^2 + 1})t + j \cdot x)$$

and

$$\sin(-(\sqrt{j^2 + 1})t + j \cdot x),$$

where $\cdot$ is the usual scalar product. These solutions are, generally speaking, quasi-periodic in time ("periodic" with several frequencies).

After the addition of the nonlinear terms, it is natural to seek solutions u to (1.1), which are close to appropriate solutions $u^{(0)}$ to the linear equation (1.2) and are quasi-periodic. We have the following result:

Theorem. *There is a set of global solutions to the NLKG in (1.1), which are quasi-periodic in time. These solutions are close to certain appropriate solutions to the linear equation (1.2), and are Gevrey in space and time.*

For more precise statements, see [W2]. The proof of the Theorem uses the method developed in [W1], with the additional input from number theory, namely spacing of non-equal square-roots, cf. [S]. It seems to be the only known result on quasi-periodic solutions to (1.1) in arbitrary dimensions. By contrast, the non-equal eigenvalue spacing for the Laplacian in NLS is bounded below by 1, and quasi-periodic solutions are known from the works in [B1, 2, EK, PP, W1].

Finally, we note that the search for special solutions to the NLKG in (1.1) has a long history, dating back from the 1978 paper of Rabinowitz on existence of periodic solutions to (1.1) in one dimension under Dirichlet boundary conditions, cf. [R]. Lastly, we note that when $p \geq \frac{4}{d-2}$ ($d \geq 3$), global solutions to (1.1) do not seem to be known in general.

References

[**B1**] J. Bourgain, *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations*, Ann. of Math. **148** (1998), 363-439.

[**B2**] J. Bourgain, *Green's function estimates for latttice Schrödinger operators and applications*, Ann. of Math. Studies **158** (2005), Princeton University Press.

[**EK**] L. H. Eliasson, S. E. Kuksin, *KAM for the nonlinear Schrödinger equation*, Ann. of Math. **172 (2)** (2010), 371-435.

[**PP**] C. Procesi, M. Procesi, *A KAM algorithm for the resonant non-linear Schrödinger equation*, Adv. in Math. **272** (2015), 399-470.

[**R**] P. H. Rabinowitz, *Free vibrations for a semi-linear wave equation*, Commun. Pure. Appl. Math. **31** (1978), 225-251.

[**S**] W. Schmidt, *Norm form equations*, Ann. of Math. **96** (1972), 526-551.

[**W1**] W.-M. Wang, *Energy supercritical nonlinear Schrödinger equations: Quasi-periodic solutions*, Duke Math. J. **165** (2016), no. 6, 1129-1192.

[**W2**] W.-M. Wang, *Quasi-periodic solutions for nonlinear Klein-Gordon equations*, arXiv1609.00309v2 (2017), 66pp.

CNRS and Department of Mathematics, Université Cergy-Pontoise, 95302 Cergy-Pontoise Cedex, France

E-mail address: wei-min.wang@math.cnrs.fr

Stability and error analysis for a diffuse interface approach to an advection-diffusion equation on a moving surface

Vanessa Styles (University of Sussex)

joint work with Klaus Deckelnick (University of Magdeburg)

In this talk we consider a finite element approach for solving the parabolic surface PDE equation

$$\begin{aligned} \partial_t^{\bullet} u + u\nabla_\Gamma \cdot \mathbf{v} - \Delta_\Gamma u &= f \qquad \text{on } S_T && (1)\\ u(\cdot,0) &= u_0 \qquad \text{on } \Gamma(0), && (2) \end{aligned}$$

which models advection and diffusion of a surface quantity u with $u(\cdot,t):\Gamma(t)\to\mathbb{R}$. Here, $\{\Gamma(t)\}_{t\in[0,T]}$ is a family of closed hypersurfaces in $\mathbb{R}^{n+1}(n=1,2)$ evolving in time, $S_T = \bigcup_{t\in(0,T)}\big(\Gamma(t)\times\{t\}\big)$ and $\mathbf{v}:\overline{S_T}\to\mathbb{R}^{n+1}$ denotes a given velocity field. Furthermore, ∇_Γ is the tangential gradient, $\Delta_\Gamma = \nabla_\Gamma\cdot\nabla_\Gamma$ the Laplace Beltrami operator and $\partial_t^{\bullet} = \partial_t + \mathbf{v}\cdot\nabla$ denotes the material derivative.

Parabolic surface PDEs of the form (1) have applications in fluid dynamics and materials science, such as the transport and diffusion of surfactants on a fluid/fluid interface, [6] or diffusion-induced grain boundary motion, [1].

We are concerned with the diffuse interface approach for solving (1), which was introduced in [4] for a stationary surface and in [5], [3] and [7] for evolving surfaces. The surface quantity u is extended to a bulk quantity satisfying a suitable parabolic PDE in a neighbourhood of $\Gamma(t)$ and the bulk equation is then localised to a thin layer of thickness ε with the help of a phase field function (see [2] for a corresponding convergence analysis). Since we are interested in using finite elements, the localised PDE needs to be written in a suitable variational form. Following [3] this is achieved with the help of a transport identity and results in a discretisation by linear finite elements in space and a backward Euler scheme in time. We give a detailed derivation along with an existence result for the discrete solution and then we derive conditions relating the interface width ε, the spatial grid size h and the time step τ which allow for a rigorous stability and error analysis. More precisely, we shall prove that the numerical solution is bounded uniformly in $L^\infty(L^2)$ and $L^2(H^1)$ over the diffuse interface and that it converges with respect to these norms both over the diffuse interface and on the sharp interface with an order $O(\varepsilon)$ provided that

$$h \le c_1\varepsilon,\ \tau \le c_2\varepsilon^2,$$

An advantage of our approach is that in the implementation the evolution of the hypersurfaces is easily incorporated by evaluating the phase field function. We shall employ a function with compact support, namely $\rho(x,t) := g(\frac{\phi(x,t)}{\varepsilon})$, where $\Gamma(t)$ is the zero level set of $\phi(\cdot,t)$ and

$$g(r) = \begin{cases} \cos^2(r), & |r| \le \frac{\pi}{2},\\ 0, & |r| > \frac{\pi}{2}. \end{cases}$$

In view of the evolution of the hypersurfaces the numerical scheme then naturally contains terms in which ρ is evaluated at different times. One of the main challenges in the analysis is to handle the corresponding differences, for which one has to bound integrals that are multiplied by a negative power of ε (arising from derivatives of ρ) as well as integrals that are not weighted with ρ. We shall deal with these difficulties by introducing an additional stabilisation term with extended support that is also used for proving the well–posedness of the scheme.

We conclude with some numerical tests for $n = 1$ and $n = 2$.

References

[1] Cahn, J.W., Fife, P., Penrose, O.: A phase field model for diffusion induced grain boundary motion. Acta Mater. **45**, 4397—4413 (1997).

[2] Elliott, C.M., Stinner, B.: Analysis of a diffuse interface approach to an advection diffusion equation on a moving surface. Math. Models Methods Appl. Sci. **19**, 787—802 (2009).

[3] Elliott, C.M., Stinner, B., Styles, V., Welford, R.: Numerical computation of advection and diffusion on evolving diffuse interfaces. IMA J. Numer. Anal. **31**, 786-–812 (2011)

[4] Rätz, A., Voigt, A.: PDE's on surfaces - a diffuse interface approach. Comm. Math. Sci. **4**, 575–590 (2006).

[5] Rätz, A., Voigt, A.: A diffuse-interface approximation for surface diffusion including adatoms, Nonlinearity **20**, 177–192 (2007).

[6] Stone, H.A.: A simple derivation of the time—dependent convective—diffusive equation for surfactant transport along a deforming interface. Phys. Fluids A **2**, 111—112 (1990).

[7] Teigen, K.E., Li, X., Lowengrub, J., Wang, F., Voigt, A.: A diffuse—interface approach for modeling transport, diffusion and adsorption/desorption of material quantities on a deformable interface. Commun. Math. Sci. **7**, 1009—1037 (2009).

Volumetric extensions of variational problems defined on surfaces

Richard Tsai

This work targets applications that use implicit or non-parametric representations of closed surfaces require numerical solutions of variational problems defined on these objects. Let Ω be a bounded open set with C^2 boundary Γ in $\mathbb{R}^3$. Let X_Γ be a suitable Hilbert space defined on Γ. We consider energy functionals of the form

$$I_\Gamma[u] = \int_\Gamma L(x, u(x), \nabla_\Gamma u(x))\, dS(x),$$

where $u \in X_\Gamma$, $\nabla_\Gamma u$ is the surface gradient and $S(x)$ is the surface area element. For sufficiently small $\epsilon > 0$, we define the ϵ-narrowband of Γ as

$$T_\epsilon = \{z \in \mathbb{R}^3 | \min_{x\in\Gamma} |z - x| < \epsilon\}.$$

The proposed approach relies on properly extending I_Γ to I_{T_ϵ}, which is a functional of functions defined on T_ϵ. The extension is defined such that the minimizers of I_Γ and I_{T_ϵ} should be equivalent in some sense. The point is that I_{T_ϵ} is generally easier to discretize on a variety of grid structures.

In [1], we showed that $I_{T_\epsilon}[v] = I_\Gamma[u]$, as long as v is the constant-along-surface-normal extension of u (we say that v is equivalent to u). Furthermore, for strictly convex and smooth I_Γ, the minimizer of the extended energy function I_{T_ϵ} is equivalent to that of I_Γ. This formulation allows development of simple finite difference and finite element methods without explicit parameterizations of Γ nor body fitted meshes.

In this talk, we present our on-going investigation on this approach for finding the minimizer of I_Γ via minimizing the system obtained from discretizing I_{T_ϵ}, particularly for the cases where I_Γ is not strictly convex and possible non-differentiable.

As an example, consider the energy function $\frac{1}{2}\int_\Gamma |\nabla_\Gamma u|^2 - f u\, dS(x)$ on the unit sphere, and

$$I_{T_\epsilon}[v] = \int_{T_\epsilon} K_\epsilon(r) \left(|\nabla v(z)|^2 - \frac{1}{r^2} v(z)\overline{f} \right) dz,$$

where $r = |z|$, $\overline{f}$ is the constant-along-surface-normal extension of f, and K_ϵ is an averaging kernel supported in $[-\epsilon, \epsilon]$. For any v, the integral $I_{T_\epsilon}[v]$ is approximated numerically by the composite Trapezoidal rule with the gradient of v approximated by centered finite differencing. To close the system, the "ghost nodes" in the system, i.e. those grid nodes in the finite difference stencil that lie outside of the ϵ-narrowband, are "pulled back" inside T_ϵ along the surface normals, and the associated ghost values defined by interpolation using nearby grid values. The discretized energy function takes the form

$$I_h[V] = \frac{1}{2} V^T A V - F^T V,$$

and the minimizer is simply $V^* = A^{-1}F$. It is noted that the matrix A is symmetric, semi-positive definite and the null space of A is generated by the vector $\mathbb{1}$. The symmetry of A is more stable on eigenvalue problems and CG iteration can be applied for solving linear system.

The choice of kernel K_ϵ, involving its smoothness, moments and how it decays near the end points of the support, has a few interesting consequences. It influences the order of approximation of the minimizers. Furthermore, our preliminary analysis shows that with the help of smooth kernel functions, it is possible use lower order (linear) approximation for the ghost values while achieving higher order accuracy in the computed minimizers.

We present an example involving the following obstacle problem:

$$\min_{\varphi_1 \le u \le \varphi_2} \frac{1}{2} \int_\Gamma |\nabla_\Gamma u|^2 dS(x), \tag{1}$$

where the obstacles are defined by two piecewise-constant functions ϕ_1 and ϕ_2. This problem is solved by a generalization of the approach introduced in [2] for the unconstrained minimization problem

$$\min_u \frac{1}{2} \int_\Gamma |\nabla_\Gamma u|^2 + \mu_1(\phi_1 - u)_+ + \mu_2(u - \phi_2)_+ dS(x), \tag{2}$$

where $(z)_+ := \max(z, 0)$, and μ_1 and μ_2 are suitable parameters. Of course, the above problem is first extended to an extended version involving integrals on T_ϵ, then Bregman iterative method is used to solve the discretized system of the extended problem. In the numerical experiment, we choose Γ to be the unit sphere with

$$\varphi_1(\vec{x}) := \begin{cases} 1, & |x_1 + x_2 + x_3| > 1.7 \\ -2, & \text{otherwise} \end{cases}, \quad \varphi_2(\vec{x}) := \begin{cases} -2, & |-x_1 - x_2 + x_3| > 1.7 \\ 1, & \text{otherwise.} \end{cases}$$

The result is shown in Figure 1.

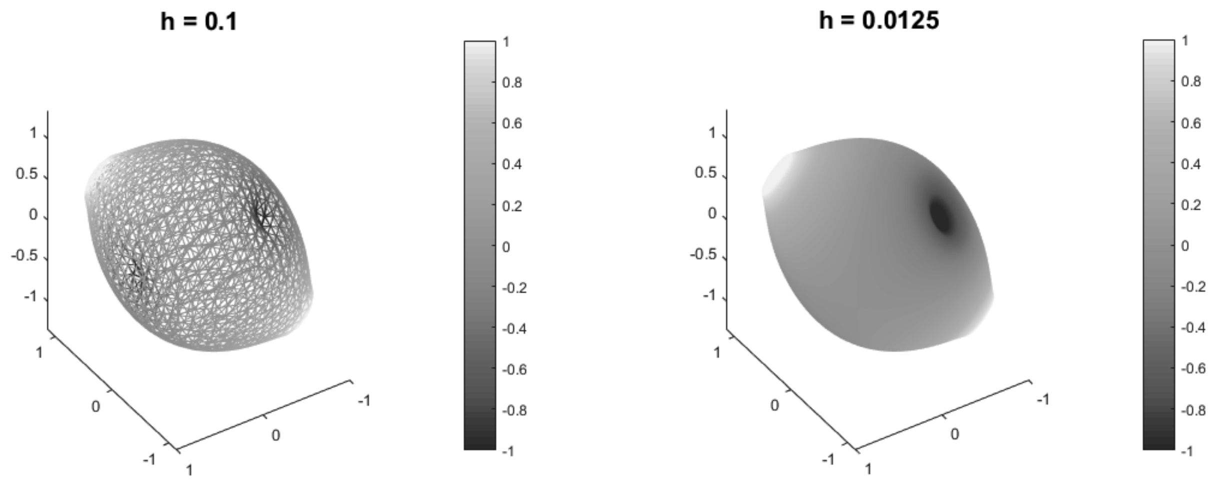


Figure 1: The surface is repesented by $r(x) = |x| + u(x)/4$ with x on the unit sphere in spherical coordinate and color shows value of u.

References

[1] J. Chu and R. Tsai Volumetric variational principles for a class of partial differential equations defined on surfaces and curves. *Res. Math. Sci.* 5 (19), 2018.

[2] G. Tran, H. Schaeffer, W.M. Feldman and S. Osher An L^1 Penalty Method for General Obstacle Problems. *SIAM Journal on Applied Mathematics*, 75(4), pp.1424-1444.

ON STABILITY OF BLOW UP SOLUTIONS FOR THE CRITICAL CO-ROTATIONAL WAVE MAPS PROBLEM

JOACHIM KRIEGER

Wave Maps are a natural nonlinear generalisation of the free wave equation, defined formally as critical points of the Lagrangian action functional

$$\mathcal{L}[u] := \frac{1}{2}\int_{\mathbb{R}^{n+1}} [-|u_t|^2 + |\nabla_x u|^2]\, dtdx$$

where $u : \mathbb{R}^{n+1} \longrightarrow M \hookrightarrow \mathbb{R}^N$ represents a map into a Riemannian manifold, which may be realised as a sub manifold of some $\mathbb{R}^N$. The functional $\mathcal{L}[u]$ is formally the same as the one leading to the standard free wave equation $\Box u = -u_{tt} + \triangle u = 0$, except that u here takes values in a (curved) Riemannian manifold.

The corresponding Euler Lagrange equations take the form of a system of semilinear wave equations of the form

$$\Box u^i + B^i_{jk}(u)\partial_\alpha u^j \partial^\alpha u^k = 0,\ \alpha = 0, 1, \ldots, n, \tag{0.1}$$

where we use the convention $\partial_0 = -\partial^0 = \partial_t$, while $\partial_j = \partial_{x_j}$, and throughout the Einstein summation convention is in use. Also, B^i_{jk} stands for the second fundamental form.

Of particular interest is the case $M = S^2 \hookrightarrow \mathbb{R}^3$, and spatial dimension $n = 2$, which is the so-called critical case of Wave Maps. In this case, the equations can be cast in the form

$$\Box u = -(-|u_t|^2 + |\nabla_x u|^2)u. \tag{0.2}$$

The key issue is to understand the global behaviour of solutions to the Cauchy problem, i. e. prescribing the data $u[0] = (u(0,\cdot), u_t(0,\cdot))$, give a qualitative and if possible even quantitative description of the solutions.

What makes the model (0.2) particularly interesting is the existence of a large class of finite energy static solutions, which are in fact the family of harmonic maps $Q : \mathbb{R}^2 \longrightarrow S^2$. These are known to play a pivotal role in the general global description of all solutions of this model, in the context of the conjectured soliton resolution of such Wave Maps. A particular instance of the latter phenomenon occurs during the formation of singularities for (0.2), where at least one harmonic bubble has to emerge in an energy concentration scenario.

In this talk, we are interested in a qualitative description of such blow up solutions. At this time, the only approach to such solutions is via a suitable symmetry reduction, reducing (0.2) to a co-rotational problem. In particular, a one co-rotational Wave Map $u : \mathbb{R}^{2+1} \longrightarrow S^2$ is one which, after passage to polar coordinates (r, θ) on $\mathbb{R}^2$ and spherical coordinates (ϕ, θ) on S^2, satisfies

$$u :\ (t, r, \theta) \longrightarrow (\phi(r, t), \theta).$$

The resulting wave maps problem in terms of ϕ then becomes

$$-\phi_{tt} + \phi_{rr} + \frac{1}{r}\phi_r = \frac{\sin(2\phi)}{2r^2}. \tag{0.3}$$

A special static solution of this problem, which corresponds to a minimum energy nontrivial harmonic map into S^2, is given by $\phi(r,t) = Q(r) = 2\arctan r$.

These solutions can be perturbed to lead to collapsing solutions. Specifically, the following result gives a continuum of such blow up solutions:

Theorem 0.1. *(K.-Schlag-Tataru* [2]*, Gao-K.* [1]*) Let $\nu > 0$, and $\lambda(t) = t^{-1-\nu}$. Then there is a solution $\phi(r,t)$ of* (0.3) *of the form*

$$\phi(r,t) = Q(\lambda(t)r) + \epsilon(t,r),\ t \in (0, t_0],$$

with $\epsilon \in H^{1+\nu-}$.

This theorem says nothing about any kind of stability properties of these solutions. Numerical simulations of P. Bizon from 2004 suggested that such solutions with $\nu > 0$ sufficiently small might in fact be stable under suitable generic perturbations.

The following provides a partial answer to this question:

Theorem 0.2. *(K. -Miao 2018* [3]*) There is $\nu_0 > 0$, such that for each $\nu_0 > \nu > 0$, there is a solution, as in the preceding theorem, which is stable under small co-rotational perturbations of regularity H^4.*

The proof of this result mostly avoids Morawetz and monotonicity type arguments, and is instead built on constructing a suitable parametrix for the linearisation of (0.3) around $Q(\lambda(t)r)$, which in turn is achieved by means of Fourier methods. We believe that such methods may have much broader applicability.

References

[1] C. Gao, J. Krieger *Optimal polynomial blow up range for critical wave maps.* Commun. Pure Appl. Anal. 14 (2015), no. 5, 1705–1741.

[2] J. Krieger, W. Schlag, D. Tataru *Renormalization and blow up for charge one equivariant critical wave maps*, Inventiones Mathematicae, vol. 171, p. 543–615, 2008.

[3] J. Krieger, S. Miao *On stability of blow up solutions for the critical co-rotational wave maps problem* preprint

Joachim Krieger

Bâtiment des Mathématiques, EPFL
Station 8, CH-1015 Lausanne, Switzerland
joachim.krieger@epfl.ch

GLOBAL EXISTENCE AND THE ASYMPTOTIC BEHAVIOR FOR SYSTEMS OF NONLINEAR WAVE EQUATIONS VIOLATING THE NULL CONDITION

SOICHIRO KATAYAMA

Let $u = u(t,x) = \big(u_i(t,x)\big)_{1\le i\le N}$ be an $\mathbb{R}^N$-valued unknown functions of $(t,x) \in (0,\infty)\times\mathbb{R}^n$ with $n = 2$ or 3. We consider the Cauchy problem for systems of nonlinear wave equations:

$$\Box u = F(u, \partial u, \partial^2 u), \qquad (t,x) \in (0,\infty)\times\mathbb{R}^n, \tag{1}$$

$$u(0,x) = \varepsilon f(x),\ (\partial_t u)(0,x) = \varepsilon g(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where $\Box = \partial_t^2 - \Delta = \partial_t^2 - \sum_{k=1}^n \partial_k^2$,

$$\partial u = (\partial_a u)_{0\le a\le n}, \quad \partial^2 u = (\partial_a\partial_b u)_{0\le a,b\le n}$$

with $\partial_0 = \partial_t$ and $\partial_k = \partial_{x_k}$ for $1 \le k \le n$. We assume that $f = (f_i)_{1\le i\le N}, g = (g_i)_{1\le i\le N} \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^N)$ and that ε is a positive and sufficiently small parameter. We suppose that each component F_i of $F = (F_i)_{1\le i\le n}$ is a homogeneous polynomial of degree p with respect to $(u, \partial u, \partial^2 u)$ with

$$p = \begin{cases} 3 & (n=2), \\ 2 & (n=3), \end{cases} \tag{3}$$

which is the critical power for the small data global existence. We also suppose that F is quasi-linear and diagonal in $\partial^2 u$, that is to say,

$$F_i(u, \partial u, \partial^2 u) = \sum_{a,b=0}^n G_i^{ab}(u,\partial u)\partial_a\partial_b u_i + H_i(u,\partial u), \quad i = 1,\dots,N, \tag{4}$$

where G_i^{ab} and H_i are homogeneous polynomials of degree $p-1$ and p, respectively, in $(u, \partial u)$.

We define the *reduced nonlinearity* $F^{\rm red}(\omega, X, Y, Z) = \big(F_i^{\rm red}(\omega, X, Y, Z)\big)_{1\le i\le N}$ with

$$F_i^{\rm red}(\omega, X, Y, Z) = F_i\big(X, (\omega_a Y)_{0\le a\le n}, (\omega_a\omega_b Z)_{0\le a,b\le n}\big)$$

for $\omega = (\omega_1,\dots,\omega_n) \in \mathbb{S}^{n-1}$ and $X, Y, Z \in \mathbb{R}^N$, where the right-hand side is obtained by substituting X, $\omega_a Y$, and $\omega_a\omega_b Z$ in place of u, $\partial_a u$, and $\partial_a\partial_b u$ of $F_i(u,\partial u,\partial^2 u)$, respectively, with $\omega_0 = -1$.

As the power p given by (3) is critical, we need some restriction on the nonlinearity F to obtain the small data global existence. The *null condition*, which was introduced by Klainerman and is given by

$$F^{\rm red}(\omega, X, Y, Z) \equiv 0, \quad \omega \in \mathbb{S}^{n-1},\ X, Y, Z \in \mathbb{R}^N, \tag{5}$$

is a famous sufficient condition for small data global existence, and the global solution under the null condition is asymptotically free, that is to say, the solution tends to a free solution u^+ to $\Box u^+ = 0$ in the energy norm as $t \to \infty$ (see [3, 15] for $n = 3$; [1, 5, 7, 10] for $n = 2$).

The null condition is sufficient, but not necessary for small data global existence, and the notion of the *weak null condition* was introduced by Lindblad-Rodnianski [17]: Writing $u(t,x) = |x|^{-(n-1)/2}U(t, |x|-t, x/|x|)$ with $U(t,\sigma,\omega) = \big(U_i(t,\sigma,\omega)\big)_{1\le i\le N}$, and neglecting terms which are expected to decay faster, we obtain the *reduced system*

$$\partial_t\partial_\sigma U(t,\sigma,\omega) = -\frac{1}{2t}F^{\rm red}(\omega, U, \partial_\sigma U, \partial_\sigma^2 U) \tag{6}$$

from (1). We say that the weak null condition is satisfied if the reduced system (6) admits a global solution U (with at most polynomial growth in t) for given small initial data at $t = 1$, say. One may conjecture that the weak null condition implies the small data global existence for the original PDE system (1), and also that the asymptotic behavior of $u(t,x)$ is given by $|x|^{(n-1)/2}U(t, |x|-t, x/|x|)$, where U is a solution to the reduced system (6).

The conjecture above is far from being solved, but for the semilinear case, namely the case where $G_i^{ab}(u, \partial u) \equiv 0$, the small data global existence and the asymptotic behavior were obtained under various additional structural conditions, which are stronger than the weak null condition, but considerably weaker than the null condition. The first one of such conditions is the essentially linear case that the reduced system is written as

$$\begin{cases} \partial_t \partial_\sigma V = -\dfrac{1}{2t} \left(\displaystyle\sum_{k=1}^{2} g^k(\omega, W, \partial_\sigma W) \partial_\sigma^k V + h(\omega, W, \partial_\sigma W) \right), \\ \partial_t \partial_\sigma W = 0 \end{cases} \tag{7}$$

with some g^k and h satisfying $h(\omega, W, 0) = 0$, where $U = (V, W) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ with $N = N_1 + N_2$ (the first equation can be viewed as a linear equation for V with variable coefficients). The second one is the case under the condition

$$Y \cdot F^{\mathrm{red}}(\omega, X, Y, Z) \geq 0, \quad \omega \in \mathbb{S}^{n-1},\ X, Y, Z \in \mathbb{R}^N, \tag{8}$$

which ensures the boundedness of $\partial_\sigma U$ (we also need to assume that $F = F(\partial u, \partial^2 u)$ in this case). For the semilinear systems, the small data global existence and the asymptotic behavior under the conditions (7) or (8) were obtained in [2, 6, 8, 9, 10, 11, 12, 13, 14, 16]. It should be noted that a various kind of the asymptotic behavior can be observed under these weaker conditions: growing up of the energy to infinity as $t \to \infty$; non-asymptotically-free solutions with bounded energy; convergence to asymptotically free solutions with strongly restricted asymptotic data.

In this talk, we would like to consider the quasi-linear systems whose reduced ones have the form (7). We will obtain the small data global existence and investigate the asymptotic behavior for such systems. The main result is an extension of the previous ones in [2, 4, 8, 9].

References

[1] S. Alinhac, *The null condition for quasilinear wave equations in two space dimensions, II*, American J. Math. **123** (2000), 1071–1101.

[2] S. Alinhac, *Semilinear hyperbolic systems with blowup at infinity*, Indiana Univ. Math. J. **55** (2006), 1209–1232.

[3] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), 267–282.

[4] K. Hidano and K. Yokoyama, *Global existence for a system of quasi-linear wave equations in 3D satisfying the weak null condition*, to appear in IMRN.

[5] A. Hoshiga, *The initial value problems for quasi-linear wave equations in two space dimensions with small data*, Adv. Math. Sci. Appl. **5** (1995), 67–89.

[6] A. Hoshiga, *The existence of the global solutions to semilinear wave equations with a class of cubic nonlinearities in 2-dimensional space,* Hokkaido Math. J. **37** (2008), 669–688.

[7] S. Katayama, *Global existence for systems of nonlinear wave equations in two space dimensions, II*, Publ. RIMS, Kyoto Univ. **31** (1995), 645–665.

[8] S. Katayama, *Global existence and asymptotic behavior of solutions to systems of semilinear wave equations in two space dimensions*, Hokkaido J. Math. **37** , Special Issue "Nonlinear Wave Equations" (2008), 689–714.

[9] S. Katayama, *Asymptotic pointwise behavior for systems of semilinear wave equations in three space dimensions*, J. Hyperbolic Differ. Equ. **9** (2012), 263–323.

[10] S. Katayama, Global Solutions and the Asymptotic Behavior for Nonlinear Wave Equations with Small Initial Data, MSJ Memoirs Vol. **36**, Math. Soc. Japan, Tokyo, 2017.

[11] S. Katayama and H. Kubo, *Asymptotic behavior of solutions to semilinear systems of wave equations*, Indiana Univ. Math. J. **57** (2008), 377–400.

[12] S. Katayama, T. Matoba and H. Sunagawa, *Semilinear hyperbolic systems violating the null condition*, Math. Ann. **31** (2015), 275–312.

[13] S. Katayama, A. Matsumura and H. Sunagawa, *Energy decay for systems of semilinear wave equations with dissipative structure in two space dimensions*, NoDEA **22** (2015), 601–628.

[14] S. Katayama, D. Murotani and H. Sunagawa, *The energy decay and asymptotics for a class of semilinear wave equations in two space dimensions*, J. Evol. Equ. **12** (2012), 891–916.

[15] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Lectures in Appl. Math. **23** (1986), 293–326.

[16] H. Kubo, *Asymptotic behavior of solutions to semilinear wave equations with dissipative structure,* Discrete Contin. Dynam. Systems, Supplement Volume 2007 (2007), 602–613.

[17] H. Lindblad and I. Rodnianski, *The weak null condition for Einstein's equations*, C. R. Math. Acad. Sci. Paris **336** (2003), 901–906.

Department of Mathematics, Graduate School of Science, Osaka University, 1-1 Machikaneyama-cho, Toyonaka, Osaka 560-0043, Japan

Numerical analysis of one-harmonic equation with values in matrix Lie group

Koya Sakakibara

Graduate School of Science, Kyoto University
RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program (iTHEMS)
ksakaki@math.kyoto-u.ac.jp

In this lecture, we consider a gradient flow (1-harmonic flow) of a total variation functional

$$E(u) := \int_{\Omega} |\nabla u| \, \mathrm{d}x$$

for $u\colon \Omega \to G$, where Ω is a bounded region in $\mathbb{R}^D$ ($D \geq 1$) with Lipschitz boundary. G is a matrix Lie group, and it forms a differentiable manifold. Therefore ∇ means the differential of a map between two manifolds.

The above constrained one-harmonic flow has application in several fields. First application appears in [7], where they consider the case that $G = S^{n-1}$ ($(n-1)$-dimensional sphere) for color image denoising. As for the case that $G = SO(3)$, this flow is important prototype of the continuum model for a time evolution of grain boundaries in a crystal, proposed in [6, 5].

In spite of its applicability, the mathematical analysis to the manifold-constrained gradient flow is still developing. One difficulty in mathematical analysis is that there are several choices for the distance to measure jumps of the function u. For example, in [3], they choose the ambient distance, that is, the metric of the Euclidean space in which the manifold G is embedded. On the other hand, in [1, 2], the authors adopt the intrinsic distance of G to measure the jumps. These two choices deduce the different notions of a solution of constrained gradient flow and these do not coincide. We adopt the definition in [3] as the notion of the solution, so that we measure the jump by the ambient distance. The aim of this lecture is to construct the gradient flow governed by the spatially discretized version of the total variation flow, efficient numerical scheme to compute it, and establish error estimate of our numerical scheme.

We briefly summarize the numerical scheme below. Let $\Omega_\Delta := \{\Omega_\alpha\}_{\alpha\in\Delta}$ be a cuboid decomposition of Ω. Based on this decomposition, we propose the following numerical scheme.

Let $\tau > 0$ be a time step, and an initial data $u_0 \in H_\Delta(G)$ is given. We then define a sequence $\{u_\tau^n\}_{n=0}^{N(\tau)}$ in $H_\Delta(G)$ as follows:

1. $u_\tau^0 := u_0$.
2. $u_\tau^{n+1} = \mathrm{Exp}_{u_\tau^n} U_\tau^{n+1}$ for $n = 0, 1, \ldots, N(\tau) - 1$, where $U_\tau^{n+1} \in \underset{U \in H_\Delta(u_\tau^n)}{\operatorname{argmin}} \Phi^\tau_{\Delta,\mathrm{loc}}(u_\tau^n; U)$.

Here $N(\tau)$ denotes the maximal number of iterations, a minimal integer greater than T/τ.

In the above algorithm, we adopt the following notations.

$$
\begin{aligned}
H_\Delta(G) &:= \left\{ \sum_{\alpha\in\Delta} u^\alpha 1_{\Omega_\alpha} \;\middle|\; u^\alpha \in G,\ \alpha\in\Delta \right\}, \quad \|u\|_{H_\Delta} := \sqrt{\sum_{\alpha\in\Delta} \|u^\alpha\|_{\mathrm{F}}^2 \mathcal{L}^D(\Omega_\alpha)}, \quad u \in H_\Delta(G),\\
H_\Delta(u) &:= \left\{ \sum_{\alpha\in\Delta} u^\alpha X^\alpha 1_{\Omega_\alpha} \;\middle|\; X^\alpha \in \mathfrak{g},\ \alpha\in\Delta \right\}, \quad u \in H_\Delta(G),\\
\mathrm{Exp}_u(U) &:= \sum_{\alpha\in\Delta} u^\alpha \exp X^\alpha, \quad u \in H_\Delta(G),\ U = \sum_{\alpha\in\Delta} u^\alpha X^\alpha 1_{\Omega_\alpha} \in H_\Delta(u),\\
\Phi^\tau_{\Delta,\mathrm{loc}}(U;u) &:= \Phi_\Delta(u+U) + \frac{1}{2\tau}\|U\|^2_{H_\Delta}, \quad (u,U) \in H_\Delta(G)\times H_\Delta(u),\\
\Phi_\Delta(u) &:= \sum_{\alpha\sim\beta} \|u^\alpha - u^\beta\|_{\mathrm{F}} \mathcal{H}^{D-1}(\partial\Omega_\alpha \cap \partial\Omega_\beta), \quad u \in H_\Delta(G),
\end{aligned}
$$

where $\alpha \sim \beta$ means that $\partial\Omega_\alpha \cap \partial\Omega_\beta$, $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm, and $\mathfrak{g} = T_e(G)$ is the Lie algebra of G, the tangent space of G at the identity $e \in G$.

Results of convergence analysis are summarized as in the following theorem.

Theorem . *Let $I := [0,T)$, $\tau > 0$, and $u_0 \in H_\Delta(G)$. Let $u \in C(I; H_\Delta(G))$ be a solution of spatial discretized gradient system, and $u_\tau \in C(I; H_\Delta(G))$ be the Rothe interpolation (piecewise affine interpolation) of a time-discretized solution $\{u^n_\tau\}_{n=0}^{N(\tau)}$ generated by our numerical scheme. Then there exist positive constants C, and D such that the following estimate holds:*

$$\|u_\tau - u\|^2_{H_\Delta} \le t e^{tC} D\tau.$$

I will show our numerical scheme in detail, convergence analysis, and numerical results in my lecture. The content of this lecture is based on a joint work with Prof. Yoshikazu Giga (The University of Tokyo), Mr. Kazutoshi Taguchi (The University of Tokyo), and Dr. Masaaki Uesaka (Hokkaido University).

References

[1] Giacomelli, L.; Mazón, J. M.; Moll, S. The 1-harmonic flow with values into $\mathbb{S}^1$. *SIAM J. Math. Anal.* 45 (2013), no. 3, 1723–1740.

[2] Giacomelli, L.; Mazón, J. M.; Moll, S. The 1-harmonic flow with values in a hyperoctant of the N-sphere. *Anal. PDE* 7 (2014), no. 3, 627–671.

[3] Giga, Y.; Kobayashi, R. On constrained equations with singular diffusivity. *Methods Appl. Anal.* 10 (2003), no. 2, 253–277.

[4] Giga, Y.; Kuroda, H.; Yamazaki, N. An existence result for a discretized constrained gradient system of total variation flow in color image processing. *Interdiscip. Inform. Sci.* 11 (2005), no. 2, 199–204.

[5] Kobayashi, R.: Warren, J. A.; Modeling the fomation and dynamics of polycrystals in 3D. *Phys. A* 356 (2005), no. 1, 127–132.

[6] Kobayashi, R.; Warren, J. A., Carter W. C.; A continuum model of grain boundaries. *Phys. D* 140 (2000), no. 1–2, 141–150.

[7] Tang, B.; Sapiro, G.; Caselles, G. Color image enhancement via chromaticity diffusion. *IEEE Trans. Image Process.* 10 (2001), no. 10, 701–707.

Singular limit problem for the Navier–Stokes equations in a curved thin domain

Tatsu-Hiko Miura*
Graduate School of Mathematical Sciences, the University of Tokyo
e-mail: thmiura@ms.u-tokyo.ac.jp

1 Introduction and notations

Let Ω_ε be a curved thin domain in $\mathbb{R}^3$ with very small width of order $\varepsilon > 0$. We consider the Navier–Stokes equations with Navier's slip boundary conditions

$$\begin{cases} \partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \nu \Delta u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon, \quad \operatorname{div} u^\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon \times (0,\infty), \\ u^\varepsilon \cdot n_\varepsilon = 0, \quad [\sigma n_\varepsilon]_{\tan} + \gamma_\varepsilon u^\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \times (0,\infty), \\ u^\varepsilon|_{t=0} = u_0^\varepsilon & \text{in} \quad \Omega_\varepsilon. \end{cases} \tag{1}$$

Here Γ_ε is the boundary of Ω_ε with unit outward normal vector n_ε and $\nu > 0$ and $\gamma_\varepsilon > 0$ are the viscosity and friction coefficients, respectively. Also, $\sigma := 2\nu D(u^\varepsilon) - p^\varepsilon I_3$ denotes the stress tensor with $D(u^\varepsilon) := \{\nabla u^\varepsilon + (\nabla u^\varepsilon)^T\}/2$ and $[\sigma n_\varepsilon]_{\tan}$ is the tangential component of the stress vector on Γ_ε given by $[\sigma n_\varepsilon]_{\tan} := (I_3 - n_\varepsilon \otimes n_\varepsilon)\sigma n_\varepsilon = 2\nu(I_3 - n_\varepsilon \otimes n_\varepsilon) D(u^\varepsilon) n_\varepsilon$.

A main subject of this talk is a singular limit problem for (1) when the thin domain Ω_ε degenerates into a closed surface Γ as $\varepsilon \to 0$. More precisely, under suitable assumptions we prove that the average in the normal direction of Γ of a solution to (1) converges in an appropriate function space on Γ and the limit is a unique solution to surface Navier–Stokes type equations. We give our recent results in a restricted case.

To state our results, let us introduce several notations on a closed surface and a curved thin domain. Let Γ be a two-dimensional closed (i.e. compact and without boundary), connected, oriented, and smooth surface in $\mathbb{R}^3$. By n we denote the unit outward normal vector field of Γ and set $P := I_3 - n \otimes n$. For $\eta \colon \Gamma \to \mathbb{R}$ and $v \colon \Gamma \to \mathbb{R}^3$ we define the tangential gradient of η and the surface divergence of v as $\nabla_\Gamma q := P \nabla \eta$ and $\operatorname{div}_\Gamma v := \operatorname{tr}[P \nabla v]$. We also set

$$H^1(\Gamma) := \{\eta \in L^2(\Gamma) \mid \nabla_\Gamma \eta \in L^2(\Gamma)^3\}, \quad H^1(\Gamma, T\Gamma) := \{v \in H^1(\Gamma)^3 \mid v \cdot n = 0 \text{ on } \Gamma\},$$
$$L^2_{g\sigma}(\Gamma, T\Gamma) := \{v \in L^2(\Gamma)^3 \mid v \cdot n = 0, \operatorname{div}_\Gamma(gv) = 0 \text{ on } \Gamma\}$$

and write $H^{-1}(\Gamma, T\Gamma)$ for the dual space of $H^1(\Gamma, T\Gamma)$ (via the $L^2(\Gamma)$-inner product). Let g_0 and g_1 be smooth functions on Γ such that $g := g_1 - g_1$ is bounded from below by a positive constant on Γ. For a sufficiently small $\varepsilon > 0$ we define a curved thin domain in $\mathbb{R}^3$ by

$$\Omega_\varepsilon := \{y + rn(y) \mid y \in \Gamma, \, \varepsilon g_0(y) < r < \varepsilon g_1(y)\}.$$

By $\mathbb{P}_\varepsilon$ and A_ε we denote the Helmholtz–Leray projection from $L^2(\Omega_\varepsilon)^3$ onto the solenoidal space $L^2_\sigma(\Omega_\varepsilon)$ and the Stokes operator on $L^2_\sigma(\Omega_\varepsilon)$ associated with slip boundary conditions. For a vector field u on Ω_ε we define the tangential average of u by

$$M_\tau u(y) := \frac{1}{\varepsilon g(y)} \int_{\varepsilon g_0(y)}^{\varepsilon g_1(y)} P(y) u(y + rn(y)) \, dr, \quad y \in \Gamma.$$

*The work of the author was supported by Grant-in-Aid for JSPS Fellows No. 16J02664 and the Program for Leading Graduate Schools, MEXT, Japan.

2 Main results

First we give our result on the global existence of a strong solution to (1) and several estimates for the strong solution without proof.

Theorem 1. *Suppose that there exists $c > 0$ such that $c^{-1}\varepsilon \le \gamma_\varepsilon \le c\varepsilon$ for all $\varepsilon > 0$. Then there exist $\varepsilon_0, c_0 \in (0,1)$ such that the following statement holds: for $\varepsilon \in (0,\varepsilon_0)$ and $\alpha, \beta \ge 0$ if given data $u \in D(A_\varepsilon^{1/2})$ and $f^\varepsilon \in L^\infty(0,\infty;L^2(\Omega_\varepsilon)^3)$ satisfies*

$$\begin{aligned} \|A_\varepsilon^{1/2}u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + \|\mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon))}^2 &\le c_0\varepsilon^{-1+\alpha}, \\ \|M_\tau u_0^\varepsilon\|_{L^2(\Gamma)}^2 + \|M_\tau \mathbb{P}_\varepsilon f^\varepsilon\|_{L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))}^2 &\le c_0\varepsilon^{-1+\beta}, \end{aligned} \tag{2}$$

then there exists a global strong solution $u^\varepsilon \in C([0,\infty);D(A_\varepsilon^{1/2})) \cap L^2_{loc}([0,\infty);D(A_\varepsilon))$ to the Navier–Stokes equations (1) that satisfies

$$\begin{gathered} \|u^\varepsilon(t)\|_{L^2(\Omega_\varepsilon)}^2 \le c_1(\varepsilon^{1+\alpha}+\varepsilon^\beta), \quad \int_0^t \|u^\varepsilon(s)\|_{H^1(\Omega_\varepsilon)}^2\,ds \le c_1(1+t)(\varepsilon^{1+\alpha}+\varepsilon^\beta), \\ \|u^\varepsilon(t)\|_{H^1(\Omega_\varepsilon)}^2 \le c_1(\varepsilon^{-1+\alpha}+\varepsilon^{-1+\beta}), \quad \int_0^t \|u^\varepsilon(s)\|_{H^2(\Omega_\varepsilon)}^2\,ds \le c_1(1+t)(\varepsilon^{-1+\alpha}+\varepsilon^{-1+\beta}) \end{gathered}$$

for all $t \ge 0$, where $c_1 > 0$ is a constant independent of ε.

Based on the above result we show the convergence of the average of a strong solution and characterize the limit as a unique solution to surface Navier–Stokes type equations.

Theorem 2. *Under the same assumption on γ_ε as in Theorem 1, let ε_0 and c_0 be the constants given in Theorem 1. For $\varepsilon \in (0,\varepsilon_0)$ let $u_0^\varepsilon \in D(A_\varepsilon^{1/2})$ and $f^\varepsilon \in L^\infty(0,\infty;L^2(\Omega_\varepsilon)^3)$. Suppose that there exist $\alpha > 0$, $\gamma > 0$, $v_0 \in L^2(\Gamma)^3$, and $f \in L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma))$ such that the inequalities (2) are satisfied with $\beta = 1$ for all $\varepsilon \in (0,\varepsilon_0)$ and*

$$\begin{gathered} \lim_{\varepsilon\to0}\frac{\gamma_\varepsilon}{\varepsilon} = \gamma, \quad \lim_{\varepsilon\to0} M_\tau u_0^\varepsilon = v_0 \quad \text{weakly in} \quad L^2(\Gamma)^3, \\ \lim_{\varepsilon\to0} M_\tau \mathbb{P}_\varepsilon f^\varepsilon = f \quad \text{weakly-}\star \text{ in} \quad L^\infty(0,\infty;H^{-1}(\Gamma,T\Gamma)). \end{gathered}$$

For $\varepsilon \in (0,\varepsilon_0)$ let u^ε be the global strong solution to (1) given in Theorem 1. Then

$$\lim_{\varepsilon\to0} Mu^\varepsilon\cdot n = 0 \quad \text{strongly in} \quad C([0,\infty);L^2(\Gamma)).$$

Moreover, there exists $v \in C([0,\infty);L^2_{g\sigma}(\Gamma,T\Gamma)) \cap L^2_{loc}([0,\infty);H^1(\Gamma,T\Gamma))$ such that

$$\lim_{\varepsilon\to0} M_\tau u^\varepsilon = v \quad \text{strongly in } L^2(0,T;L^2(\Gamma,T\Gamma)) \text{ and weakly in } L^2(0,T;H^1(\Gamma,T\Gamma))$$

for all $T > 0$ and v is a unique weak solution to the equations

$$\begin{cases} \partial_t v + \overline{\nabla}_v v - \dfrac{2\nu}{g}\left(P\mathrm{div}_\Gamma[gD_\Gamma(v)] - \dfrac{1}{g}(\nabla_\Gamma g\otimes\nabla_\Gamma g)v\right) + \dfrac{2\gamma}{g}v + \nabla_\Gamma q = f & \text{on} \quad \Gamma\times(0,\infty), \\ \mathrm{div}_\Gamma(gv) = 0 & \text{on} \quad \Gamma\times(0,\infty), \\ v|_{t=0} = v_0 & \text{on} \quad \Gamma. \end{cases}$$

Here $\overline{\nabla}_v v := P(v\cdot\nabla)v$ is the covariant derivative of v along itself and for $i = 1,2,3$ the i-th component of $\mathrm{div}_\Gamma[gD_\Gamma(v)]$ is given by

$$D_\Gamma(v) := P\left(\frac{\nabla v + (\nabla v)^T}{2}\right)P, \quad \left(\mathrm{div}_\Gamma[gD_\Gamma(v)]\right)_i := \mathrm{div}_\Gamma(gD_\Gamma(v)e_i),$$

where $\{e_1,e_2,e_3\}$ is the standard basis of $\mathbb{R}^3$.

Free boundary problems of hyperbolic type with a focus on the free boundary conditions

OMATA, Seiro
Kanazawa University omata@se.kanazawa-u.ac.jp

1 Tape peeling energy

In this, we consider the problem of removing an adhesive tape from a solid plane. The tape is described by the graph of scalar function u. The simplist setting is as follows: (1) Strech an adhesive tape uniformly (with a constant tension force S), (2) attach it to a solid plane, and finally (3) lift the tape up by its edge. In this way, the tape gradually peels off from the plane.

We also assume that the adhesion force has a potential, i.e. the enegy of the peeling process. As depicted in the following picture, we can determine the angle at the free boundary by finding the equilibrium between the resulting tension force and the the maximum adhesion force in the vertical direction. We call this force $\tilde{Q}$. The equilibrium condition is $S\sin\theta = \tilde{Q}$, then the force which we need to peel the tape off is $S(1-\cos\theta)$. It is proportional to the area on which the tape is peeled off.

Consequently, the energy can be expressed in the following way:

$$\begin{aligned}&\int_{\{u>0\}} S\big((\sqrt{1+|\nabla u|^2}-1)+(1-\sqrt{1-(\tilde{Q}/S)^2})\big)dx \\ &\quad = S\int_\Omega \big(\sqrt{1+|\nabla u|^2}+(1-\sqrt{1-(\tilde{Q}/S)^2})\chi_{u>0}\big)dx - S|\Omega|,\end{aligned}$$

and the stationary state is described by the minimizer of the above funcitonal.

1.1 Approximation by the Dirichlet integral

Here we assume that the contact angle θ is small enough. In this case, we can approximate the energy by the Dirichlet integral:

$$I(u) := \frac{S}{2}\int_\Omega (|\nabla u|^2 + Q^2\chi_{u>0})dx \tag{1}$$

which is introduced by H.W.Alt, L.A.Caffarelli and A.Friedman [1]. Here, $Q^2 = (\tilde{Q}/S)^2$ in this approximation. They show that the minimizer of this functional satisfies

$$\Delta u = 0 \quad \text{in } \{u>0\}, \qquad |\nabla u| = Q \quad \text{on } \partial\{u>0\}.$$

2 A time dependent problem

In this section, we consdier a time dependent problem. To this end we introduce a Lagrangean and an action integral for this phenomenon. In the usual manner, the action can be expressed as

$$J(u) = \int_0^T \int_\Omega \Big((u_t)^2\chi_{u>0} - |\nabla u|^2 - Q^2\chi_{u>0}\Big)dxdt, \tag{2}$$

where T is a positive constant.

By the first variation, we have

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad \Omega \times (0,T) \cap \{u > 0\}, \\ |\nabla u|^2 - (u_t)^2 = Q^2 & \text{on} \quad \Omega \times (0,T) \cap \partial\{u > 0\}. \end{cases} \tag{3}$$

For this problem, we have shown the existence of a strong solution in the one dimensional case.

Theorem 1. *(K.Kikuchi, S.Omata [2]) There exist unique solution to* (3)*, under some compatibility conditions.*

By the above consideration, we can write the (degenerate) hyperbolic type equation as

$$\chi_{\overline{u>0}} u_{tt} - \Delta u = -\frac{Q^2}{|Du|} \mathscr{H}^n \llcorner \partial\{u > 0\}, \tag{4}$$

where $Du = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial t})$. We understand that when $u > 0$, u satisfies the wave equation and when $u < 0$, $\Delta u = 0$ a.e. $t \in (0,T)$. By the maximum principle for the Laplace operator, we can say on $u \leq 0$, u is identically zero, $u \equiv 0$. Thus we see that when $u < 0$, u satisfies the wave equation. If we write equation (4) in a weak form, we have

$$-\int_0^T \int_\Omega \left(\chi_{\overline{u>0}}\, \zeta\right)_t u_t \, dx \, dt + \int_0^T \int_\Omega \nabla u \cdot \nabla \zeta \, dx \, dt = -\int_{\partial\{u>0\}} \frac{Q^2}{|Du|} \zeta \, d\mathscr{H}^n.$$

We split the integral on the right-hand side into four parts, $\Omega \times (0,T) \cap \{u > 0\}$, $\Omega \times (0,T) \cap \partial\{u > 0\}$, $\Omega \times (0,T) \cap \partial\{u = 0\}^\circ$ and $\Omega \times (0,T) \cap \{u = 0\}^\circ$. For each, we integrate by parts, and we conclude that except for $\Omega \times (0,T) \cap \partial\{u > 0\}$, all the integrals vanish.

Since $\chi_{\overline{u>0}} = 1$ on $\partial\{u > 0\}$, the right-hand side is

$$\begin{aligned} &\iint_{\Omega \times (0,T) \cap \partial\{u>0\}} \left(-\chi_{\overline{u>0}} \zeta u_t \frac{-u_t}{|Du|} + \zeta \nabla u \cdot \frac{-\nabla u}{|Du|} \right) d\mathscr{H}^n \\ &= \iint_{\Omega \times (0,T) \cap \partial\{u>0\}} \left(\zeta \frac{u_t^2}{|Du|} + \zeta \frac{-|\nabla u|^2}{|Du|} \right) d\mathscr{H}^n. \end{aligned}$$

We formally recover $|\nabla u|^2 - u_t^2 = Q^2$.

References

[1] H.W.Alt and L.A.Caffarelli, Existence and regularity for a minimum problem with free boundary, J. Reine Angew. Math., **325** (1981), 105-144.

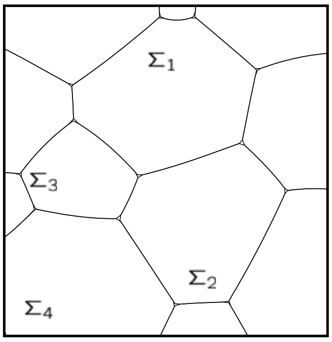
[2] K. Kikuchi and S. Omata, A free boundary problem for a one dimensional hyperbolic equation, Adv. Math. Sci. Appl. **9** No.2 (1999) 775-786.

[3] A. Tachikawa, A variational approach to constructing weak solutions of semilinear hyperbolic systems, Adv. Math. Sci. Appl. **4** (1994), 93-103.

Algorithms for Motion of Networks by Weighted Mean Curvature

Selim Esedoḡlu

Motion by weighted mean curvature for networks of surfaces arises in a variety of applications, such as the dynamics of foam and the evolution of microstructure in polycrystalline materials. It is steepest descent (gradient flow) for an energy: the sum of the areas of the surfaces constituting the network. Mathematically, such a network is described as the union of boundaries $\cup_{i=1}^{N} \partial\Sigma_i$ of sets $\mathbf{\Sigma} = (\Sigma_1, \dots, \Sigma_N)$ (also called *phases*) that partition a domain $D \subset \mathbb{R}^3$ (typically a periodic box) without overlaps or vacuum:



$$
\begin{gathered}
D = \bigcup_{j=1}^{N} \Sigma_j, \text{ and} \\
\Sigma_i \cap \Sigma_j = (\partial\Sigma_i) \cap (\partial\Sigma_j) \text{ for } i \neq j.
\end{gathered}
\tag{1}
$$

The cost function has the form:

$$
E(\mathbf{\Sigma}) = \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \int_{(\partial\Sigma_i)\cap(\partial\Sigma_j)} \sigma_{i,j}(n_{i,j}(x))\, dS(x) \tag{2}
$$

Here, dS is the surface area element, $n_{i,j}(x)$ for $x \in (\partial\Sigma_i) \cap (\partial\Sigma_j)$ denotes the unit normal from Σ_i to Σ_j, and the continuous, even functions $\sigma_{i,j} : \mathbb{S}^2 \to \mathbb{R}^+$ are the *surface tensions* associated with the interfaces.

Away from junctions (where three or more phases meet), normal speed of interfaces under L^2 gradient flow of energy (2) is given by

$$
\begin{aligned}
v_\perp(x) = \mu_{i,j}(n_{i,j}(x)) \Big(&\big(\partial^2_{s_1}\sigma_{i,j}(n_{i,j}(x)) + \sigma_{i,j}(n_{i,j}(x))\big)\kappa_1(x) \\
&+ \big(\partial^2_{s_2}\sigma_{i,j}(n_{i,j}(x)) + \sigma_{i,j}(n_{i,j}(x))\big)\kappa_2(x)\Big)
\end{aligned}
\tag{3}
$$

for an $x \in (\partial\Sigma_i) \cap (\partial\Sigma_j)$. Here, κ_1 and κ_2 are the two principal curvatures, and ∂_{s_i} denotes differentiation along the great circle on $\mathbb{S}^2$ that passes through $n(x)$ and has as its tangent the i-th principal curvature direction. The additional factor $\mu_{i,j}$ is known as the *mobility* associated with the interface $(\partial\Sigma_i) \cap (\partial\Sigma_j)$, and may also be normal dependent: $\mu_{i,j} : \mathbb{S}^2 \to \mathbb{R}^+$.

In addition to (3), the *Herring angle condition* is required along triple junctions: At a junction formed by the meeting of the three phases Σ_i, Σ_j , and Σ_k , this condition reads

$$
\begin{aligned}
(\ell \times n_{i,j})\sigma_{i,j}(n_{i,j}) + (\ell \times n_{j,k})\sigma_{j,k}(n_{j,k}) + (\ell \times n_{k,i})\sigma_{k,i}(n_{k,i}) \\
+ n_{j,i}\sigma'_{i,j}(n_{i,j}) + n_{k,j}\sigma'_{j,k}(n_{j,k}) + n_{i,k}\sigma'_{k,i}(n_{k,i}) = 0
\end{aligned}
\tag{4}
$$

where $\ell = n_{j,k} \times n_{i,j}$ is a unit vector tangent to the triple junction, and $\sigma'_{i,j}(n_{i,j})$ denotes derivative of $\sigma_{i,j}$ taken on $\mathbb{S}^2$ in the direction of the vector $\ell \times n_{i,j}$.

Merriman, Bence, and Osher (MBO) proposed [3] a remarkably elegant algorithm for dynamics (3) & (4) in the special case that all surface tensions and mobilities satisfy $\sigma_{i,j}(n) = \mu_{i,j}(n) = 1$ for all n and $i \neq j$; the motion law (3) then simply stipulates that each interface in the network moves by its mean curvature $v_\perp(x) = \kappa_1(x) + \kappa_2(x)$, and the junction condition (4) reduces to stating that all angles formed at triple junctions are 120°. Called *threshold dynamics*, the MBO algorithm generates a discrete in time approximation to this simplified special case of the flow, starting from an initial partition $\mathbf{\Sigma^0} = (\Sigma_1^0, \ldots, \Sigma_N^0)$, merely by alternating the two simple operations of convolution and thresholding, as follows:

Algorithm: (MBO'92): Given a time step size $\delta t > 0$, alternate the following steps:

1. Convolution:
$$\psi_i^k = G_t * \mathbf{1}_{\Sigma_i^k}. \tag{5}$$
2. Redistribution:
$$\Sigma_i^{k+1} = \left\{ x \, : \, \psi_i^k(x) \geq \max_{j \neq i} \psi_j^k(x) \right\}. \tag{6}$$

where $G_t(x) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \exp\left(-\frac{|x|^2}{4t}\right)$ and $t > 0$ is the time step size.

I will describe an extension of the MBO scheme, in joint work with Matt Jacobs and Pengbo Zhang [1], to the full generality of model (3) & (4), while preserving its simplicity and efficiency. This builds on earlier joint work [2] with Felix Otto on a variational formulation of the MBO scheme (5) & (6) that elucidates the path to its various generalizations. Anisotropic motions of the form (3) are achieved by replacing the Gaussian in step (5) of the original scheme by kernels that need not be radially symmetric. An essential novelty is the ability to *bake* the desired possibly anisotropic surface tension $\sigma : \mathbb{S}^2 \to \mathbb{R}^+$ and the desired possibly anisotropic mobility $\mu : \mathbb{S}^2 \to \mathbb{R}^+$ into the convolution kernel, which is new even in the two -phase setting. We describe simple and explicit kernel constructions that can be used in the fully multiphase, anisotropic setting, where $\binom{N}{2}$ possibly distinct anisotropic surface tension & mobility pairs $\big(\sigma_{i,j}(n)\,,\,\mu_{i,j}(n)\big)$ may be specified.

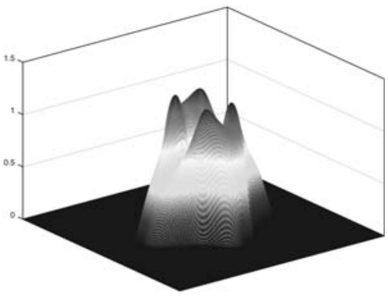


Figure 1: Convolution kernel that replaces the Gaussian G_t in step (5) of the MBO scheme if 2D anisotropic flow with surface tension $\sigma(x_1, x_2) = \sqrt{\frac{1}{4}x_1^2 + x_2^2} + \sqrt{x_1^2 + \frac{1}{4}x_2^2}$ and mobility $\mu(x_1, x_2) = \frac{2x_1^2 + 3x_2^2}{4\sqrt{x_1^2 + x_2^2}}$ is desired. The construction can be used in the multi-phase setting, in 2D or 3D.

References

[1] S. Esedoḡlu, M. Jacobs, P. Zhang. *Kernels with prescribed surface tension & mobility for threshold dynamics schemes.* Journal of Computational Physics. **337** (2017), pp. 62–83.

[2] S. Esedoḡlu, F. Otto. *Threshold dynamics for networks with arbitrary surface tensions.* Communications on Pure and Applied Mathematics. **68** (2015), pp. 808–864.

[3] B. Merriman, J. K. Bence, S. J. Osher. *Motion of multiple junctions: a level set approach.* Journal of Computational Physics. **112** (1994), pp. 334–363.

BLOW-UP OF SOLUTIONS TO SEMILINEAR WAVE EQUATION WITH A SCALING INVARIANT CRITICAL DAMPING

Masahiro Ikeda
Center for Advanced Intelligence Project, Riken, Department of Mathematics, Keio University

1. Introduction

I would like to introduce our recent results[1] [4, 5], which give blow-up results and upper estimates of lifespan for small data to semilinear wave equations with a scaling invariant damping. The Cauchy problem to semilinear wave equations with a space-dependent critical damping is

$$(\mathrm{DW}_{V_0}) \qquad \begin{cases} \partial_t^2 u - \Delta u + \frac{V_0}{|x|}\partial_t u = |u|^p, & (t,x) \in (0,T) \times \mathbb{R}^N, \\ (u, \partial_t u)|_{t=0} = \varepsilon(u_0, u_1). & x \in \mathbb{R}^N. \end{cases}$$

Here $N \geq 3$, $(u_0, u_1) \in H^2 \times H^1$ is a given function satisfying supp $(u_0, u_1) \subset \{|x| \leq R_0\}$ for some R_0, which denotes the shape of the data, $\varepsilon > 0$ is a (small) parameter, which denotes the size of the data, $V_0 \geq 0$ and $p > 1$ are constant, $u \in C([0,T) : H^2) \cap C^1([0,T) : H^1)$ is an unknown function,

$$T = T(\varepsilon) := \sup\{T \in (0,\infty] : \exists \text{ sol } u \text{ to } (\mathrm{DW}_{V_0}) \text{ on } [0,T)\}$$

denotes the maximal existence time of the function u.

Local well-posedness in $H^2 \times H^1$ to (DW_{V_0}) for arbitrary data is proved via a contraction argument with the Hardy inequality $\||x|^{-1}f\|_{L^2} \lesssim \|\nabla f\|_{L^2}$ due to $N \geq 3$ and the Sobolev inequality, in the case $V_0 \geq 0$ and $p < \frac{N-2}{N-4}$ ($p < \infty$, when $N = 3, 4$).

When $V_0 = 0$, the Strauss conjecture and the sharp estimate of lifespan are verified:

$$\begin{cases} p > p_S(N) \Longrightarrow \forall (u_0,u_1), \exists \varepsilon_0 > 0 \text{ s.t } \forall \varepsilon \in [0, \varepsilon_0],\ T = \infty, \\ p \leq p_S(N) \Longrightarrow \exists (u_0,u_1), \exists \varepsilon_0 > 0 \text{ s.t } \forall \varepsilon \in (0, \varepsilon_0], T < \infty, \end{cases}$$

$$T \sim \begin{cases} C\varepsilon^{-\frac{2p(p-1)}{\gamma(N;p)}}, & \text{if } p < p_S(N), \\ \exp(C\varepsilon^{-p(p-1)}), & \text{if } p = p_S(N), \end{cases}$$

where $\gamma(n;p) := 2 + (n+1)p - (n-1)p^2$, $p_S(n) :=$ (the positive root of $\gamma(n;p) = 0$), for $n \geq 2$, $p_S(1) := \infty$ and C is independent of ε.

The following result means the extension of the above blow-up results to $V_0 \geq 0$:

Theorem 1.1 ([4]). *Let $V_0 < \frac{(N-1)^2}{N+1}$, $\frac{N}{N-1} < p \leq p_S(N+V_0)$, $u_0, u_1 \geq 0$ with $u_0 + u_1 \neq 0$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $T < \infty$. And the following estimate holds:*

$$T \leq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_S(N+V_0), \\ C_\delta \varepsilon^{-\frac{2p(p-1)}{\gamma(N+V_0;p)} - \delta} & \text{if } \max\left(p_S(N+2+V_0), \frac{2}{N-1-V_0}\right) \leq p < p_S(N+V_0), \end{cases}$$

where C, C_δ are independent of ε and $\delta > 0$ can be taken arbitrarily small.

Remark 1.1. *1. The proof of the theorem is based on a test-function method with properties of Gauss' hypergeometric functions initiated by Zhou-Han [7] ($V_0 = 0$).*
2. Zhou-Han [7] uses a time decay lower estimate of solutions to the free wave equation proved by Yordanov-Zhang '06, though our argument does not need such estimate.

[1]These are joint works with Motohiro Sobajima in Tokyo University of Science

Our method is applicable to the case of the scaling invariant time-dependent damping:

$$(\mathrm{DW}_\mu) \qquad \begin{cases} \partial_t^2 v - \Delta v + \frac{\mu}{1+t}\partial_t v = |v|^p, & (t,x) \in (0,T) \times \mathbb{R}^N, \\ (v, \partial_t v)|_{t=0} = \varepsilon(v_0, v_1), & x \in \mathbb{R}^N. \end{cases}$$

Here $N \geq 1$, $(v_0, v_1) \in H^1 \times L^2$ is a given function satisfying supp $(v_0, v_1) \subset \{|x| \leq r_0\}$ for some $r_0 < 1$, $\mu \geq 0$ is constant, $v \in C([0,T) : H^1) \cap C^1([0,T) : L^2)$ is an unknown function.

By the similar argument as in Theorem 1.1, we get the following blow-up result to (DW_μ):

Theorem 1.2. *Let $N \in \mathbb{N}$, $0 < \mu < \frac{N^2+N+2}{N+2}$, $1 + \frac{2}{N} < p \leq p_S(N+\mu)$ and $v_0, v_1 \geq 0$ with $v_0 + v_1 \neq 0$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $T < \infty$. And the following estimate holds:*

$$T \leq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & \text{if } p = p_S(N+\mu), \\ C_\delta \varepsilon^{-\frac{2p(p-1)}{\gamma(N+\mu;p)} - \delta} & \text{if } \exists p_0(N,\mu) \leq p < p_S(N+\mu). \end{cases}$$

Here $C, C_\delta > 0$ are independent of ε, and $\delta > 0$ can be taken arbitrarily small.

Theorem 1.2 extends the results obtained in [6, 3]. Small data global existence to (DW_μ) is proved in some μ, p in [1, 2]. The relation between the previous results and ours is as follows:

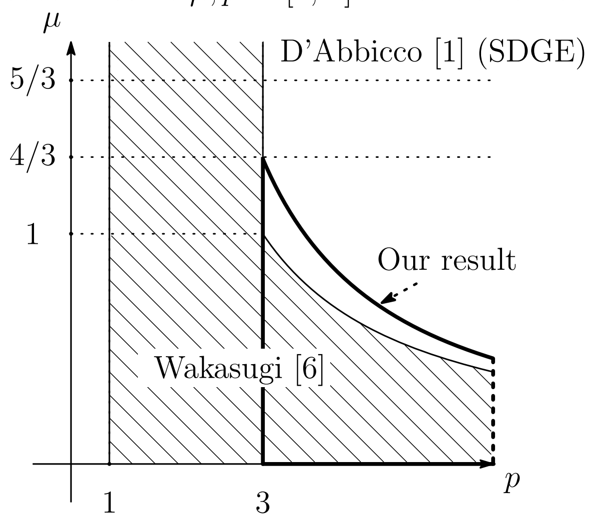


Figure 1: the case $N = 1$

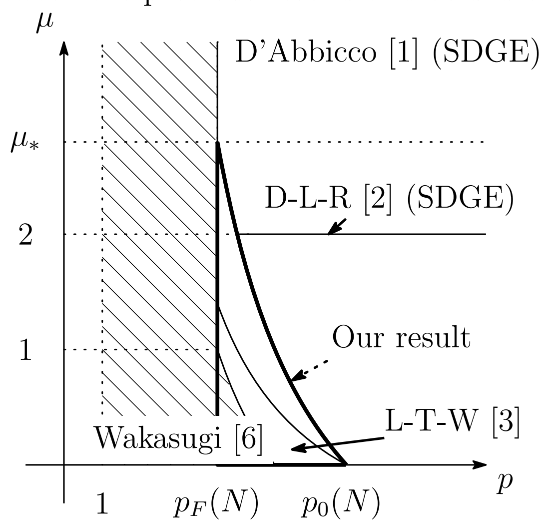


Figure 3: the case $N \geq 3$

References

[1] M. D'Abbicco, *The threshold of effective damping for semilinear wave equations*, Math. Methods, Appl. Sci., **38** (2015), 1032–1045.

[2] M. D'Abbicco, S. Lucente, M. Reissig, *A shift in the Strauss exponent for semilinear wave equations with a not effective damping*, J. Diff. Equs., **259** (2015), 5040–5073.

[3] N.A. Lai, H. Takamura, K. Wakasa, *Blow-up for semilinear wave equations with the scaling invariant damping and super-Fujita exponent*, J. Diff. Equs., **263** (2017), 5377–5394.

[4] M. Ikeda, M. Sobajima, *Life-span of blowup solutions to semilinear wave equation with space-dependent critical damping*, submitted, arXiv:1709.04401.

[5] M. Ikeda, M. Sobajima, *Life-span of solutions to semilinear wave equation with time-dependent critical damping for specially localized initial data*, to appear in Mathematische Annalen.

[6] Y. Wakasugi, *Critical exponent for the semilinear wave equation with scale invariant dampint*, Fourier analysis, 375-390, Trends Math., Birkhauser/Springer, Cham, 2014.

[7] Y. Zhou, W. Han, *Life-span of solutions to critical semilinear wave equations*, Commun.P.D.E. **39** (2014), 439–451.

The initial value problem for the generalized KdV equation with low degree of non-linearity

Hayato Miyazaki*
(National Institute of Technology, Tsuyama College)

This talk is based on a joint work with Gustavo Ponce (UCSB) and Felipe Linares (IMPA).

1 Introduction

In this talk, we consider the initial value problem for the generalized KdV type equation

$$\text{(GK)} \qquad \begin{cases} \partial_t u + \partial_x^3 u \pm |u|^\alpha \partial_x u = 0, \ (t,x) \in \mathbb{R}^2, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}, \end{cases}$$

where $u = u(t,x)$ is a $\mathbb{R}$-valued unknown function and $\alpha \in (0,1)$. Formally, the solution of (GK) has three conserved quantities

$$I_1(u) = \int_{-\infty}^{\infty} u(t,x)\,dx, \quad I_2(u) = \int_{-\infty}^{\infty} u^2(t,x)\,dx,$$
$$I_3(u) = \int_{-\infty}^{\infty} ((\partial_x u)^2 \mp \frac{2}{(\alpha+1)(\alpha+2)}|u|^{\alpha+2})(t,x)\,dx.$$

The equation (GK) is a lower nonlinearity version of the generalized KdV equation

$$\text{(g-KdV)} \qquad \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \ k \in \mathbb{Z}^+.$$

The initial value problem and the periodic boundary value problem associated to (g-KdV) have been extensively studied (for instance, sharp local and global well-posedness, stability of special solutions and existence of blow-up solutions, see [4] §7-8). However, the lower nonlinearity version of (g-KdV) have not been studied as far as we know. Because, roughly speaking, the nonlinearity of (GK) is non-Lipschitz in the Sobolev spaces $H^s(\mathbb{R}) = (1 - \partial_x^2)^{-\frac{s}{2}} L^2(\mathbb{R})$ or the weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(\langle x \rangle^r\,dx)$ for all s, $r \in \mathbb{R}$, so that the local well-posedness can not be established in these spaces.

2 Main results

To state main results, we set $\langle x \rangle = (1+|x|^2)^{\frac{1}{2}}$ and $[x]$ denotes the greatest integer less than or equal to x for any $x \in \mathbb{R}$. Our first goal is to establish the local well-posedness of (GK).

Theorem 2.1 ([3]). *Fix $m = \left[\frac{1}{\alpha}\right] + 1$. Let $s \in \mathbb{Z}^+$ satisfy $s \geqslant 2m+4$. Assume that*

$$\text{(2.1)} \qquad u_0 \in H^s(\mathbb{R}),\ \langle x \rangle^m u_0 \in L^\infty(\mathbb{R}),\ \langle x \rangle^m \partial_x^j u_0 \in L^2(\mathbb{R}),\ j = 1,2,3,4,$$

with $\|u_0\|_{H^s} + \|\langle x \rangle^m u_0\|_{L^\infty} + \sum_{j=1}^4 \|\langle x \rangle^m \partial_x^j u_0\|_{L^2} =: \delta$, and

$$\text{(2.2)} \qquad \inf_{x \in \mathbb{R}} \langle x \rangle^m |u_0(x)| =: \lambda > 0.$$

*624-1 Numa, Tsuyama city, Okayama, 708-0824, Japan
Email: miyazaki@tsuyama.kosen-ac.jp
Website: http://www.tsuyama-ct.ac.jp/miyazaki/
H.M. was supported by the Overseas Research Fellowship Program by National Institute of Technology.

Then there exists $T = T(\alpha;\delta;s;\lambda) > 0$ *such that* (GK) *has a unique local solution*

$$u \in C([0,T], H^s(\mathbb{R})), \quad \langle x\rangle^m \partial_x^j u \in C([0,T], L^2(\mathbb{R})), \quad j = 1,2,3,4 \tag{2.3}$$

with

$$\langle x\rangle^m u \in C([0,T], L^\infty(\mathbb{R})), \quad \partial_x^{s+1} u \in L^\infty(\mathbb{R}, L^2([0,T])), \tag{2.4}$$

and $\sup_{0\leqslant t\leqslant T} \|\langle x\rangle^m (u(t) - u_0)\|_{L^\infty} \leqslant \frac{\lambda}{2}$.

Moreover, the map $u_0 \mapsto u(t)$ *is continuous in the following sense: For any compact* $I \subset [0,T]$*, there exists a neighborhood* V *of* u_0 *satisfying* (2.1) *and* (2.2) *such that the map is Lipschitz continuous from* V *into the class defined by* (2.3) *and* (2.4).

Remark 2.2. The assumption (2.2) inspired by [1] comes from that $|u|^\alpha$ is not regular enough (only C^α for $\alpha \in (0,1)$). In the proof of Theorem 2.1, we also apply the Kato smoothing effect as in [5], in order to remove derivative loss of the nonlinearity.

Remark 2.3. Theorem 2.1 can be extended to the case u is a $\mathbb{C}$-valued unknown function.

Remark 2.4. A typical data satisfying the hypotheses in Theorem 2.1 is $u_0(x) = \frac{2\lambda e^{i\theta}}{\langle x\rangle^m} + \varphi(x)$ for some $\theta \in \mathbb{R}$ and some $\varphi \in \mathcal{S}(\mathbb{R})$ with $\|\langle x\rangle^m \varphi\|_{L^\infty} \leqslant \lambda$.

Remark 2.5. We observe that (real) traveling wave solutions (positive, even and radially decreasing) with speed $c > 0$

$$\phi_{c,\alpha}(x,t) = c^{\frac{1}{\alpha}} \phi(\sqrt{c}(x - ct)), \ \phi(x) = \left(\frac{(\alpha+1)(\alpha+2)}{2} \operatorname{sech}\left(\frac{\alpha x}{2}\right)\right)^{\frac{2}{\alpha}}$$

are not in the class of solutions provided by Theorem 2.1.

Our second result is concerned with the propagation of regularity in the right hand side of the data for positive times of real solutions in the class provided by Theorem 2.1. The result tells us that this regularity moves with infinite speed to its left as time evolves. This is extension of [2] into the case $\alpha \in (0,1)$.

Theorem 2.6 ([3])**.** *In addition to the assumptions of Theorem 2.1, suppose that there exist* $l \in \mathbb{Z}^+$ *and* $x_0 \in \mathbb{R}$ *such that*

$$u_0\Big|_{(x_0,\infty)} \in H^{s+l}((x_0,\infty)).$$

Then, it holds that

$$\sup_{0\leqslant t\leqslant T} \int_{x_0+\varepsilon-vt}^{\infty} (\partial_x^{s+j} u(t,x))^2 dx < c^* := c^*\left(\varepsilon; v; \|u_0\|_{H^{s+l}((x_0,\infty))}; l; s\right)$$

and

$$\int_0^T \int_{x_0+\varepsilon-vt}^{x_0+R-vt} (\partial_x^{s+l+1} u(t,x))^2 dxdt < c^{**} := c^{**}\left(\varepsilon; R; v; \|u_0\|_{H^{s+l}((x_0,\infty))}; l; s\right)$$

for any $\varepsilon > 0$, $v > 0$, $R > 0$ *and* $j = 1, \cdots, l$.

Remark 2.7. The proof is based on weighted energy estimates performed in the differential equation (GK) for which we need to have $\mathbb{R}$-valued solutions.

References

[1] T. Cazenave and I. Naumkin, Commun. Contemp. Math. **19** (2017), no. 2, 1650038, 20.

[2] P. Isaza, F. Linares, and G. Ponce, Comm. Partial Differential Equations **40** (2015), no. 7, 1336–1364.

[3] F. Linares, H. Miyazaki, and G. Ponce, preprint, available at `arXiv:1802.07345`.

[4] F. Linares and G. Ponce, Universitext, Springer, New York, 2015.

[5] F. Linares, G. Ponce, and G. N Santos, preprint, available at `arXiv:1712.00663`.

LOCAL ENERGY DECAY AND THE VECTOR FIELD METHOD

JACOB STERBENZ

ABSTRACT. We discuss some recent work on stability of solutions to linear and nonlinear wave equations on asymptotically flat space times. This is joint work with Jason Metcalfe, Jesus Oliver, and Daniel Tataru.

1. INTRODUCTION

Recall that for the scalar wave equation $\Box = -\partial_t^2 + \sum_{i=1}^n \partial_i^2$ on Minkowski space $\mathbb{R}^{n+1}$, solutions to the initial value problem:

$$\Box\phi \ = \ 0 \ , \qquad \phi(0,x) \ = \ f(x) \ , \quad \partial_t\phi(0,x) \ = \ g(x) \ , \tag{1}$$

exhibit the conservation of energy:

$$E(t) \ = \ E(0) \ , \qquad E(t) \ = \ \frac{1}{2}\int_{\mathbb{R}^n} (\phi_t^2 + |\nabla_x\phi|^2)(t,x)dx \ , \tag{2}$$

assuming $(f,g) \in \dot{H}^1_x \times L^2_x$.

Solutions to (1) also exhibit transport type behavior. This can be seen by taking a geometrical optics ansatz $\phi_\lambda = a_\lambda e^{i\lambda u}$, where u solves eikonal equation $|u_t| = |\nabla_x u|$, and to first order a_λ solves the amplified transport equation $La_\lambda = (\frac{1}{2}\Box u)a_\lambda$. Here L is the null vector field $L = u_t\partial_t - \nabla_x u\cdot\nabla_x$ whose integral curves are always rays of the form $\gamma(s) = (s, x_0 + \omega s)$ for various directions $\omega \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. This implies that any initially localized solution to (1) will eventually move energy away from spatially compact sets such as $|x| \leq R$. In particular one can show that localizing the energy (2) leads to a spacetime bound:

$$\int_0^\infty \int_{|x|\leq R} (\phi_t^2 + |\nabla_x\phi|^2)(t,x)dxdt \ \leq \ C(R)E(0) \ . \tag{3}$$

Estimate (3) is essentially optimal if one considers only energy class initial data. On $\mathbb{R}^{3+1}$ this can be seen directly by taking an infinite sum of incoming spherical wave packets of the form $\phi = \sum_i a_i\phi_0(t-i,x)$, where $\phi_0 = |x|^{-1}(\chi(t-|x|) - \chi(t+|x|))$, and where $\chi(s)$ is a smooth cutoff on $\frac{1}{2} \leq s \leq 2$ and $\sum_i a_i^2 = 1$.

On the other had if one restricts to a single wave packet in the previous example the bounds are much better than (3). In fact for reasonably smooth initial data, again on $\mathbb{R}^{3+1}$, one has the explicit solution formula:

$$\phi(t,x) \ = \ \frac{1}{4\pi t}\int_{|y-x|=t} g(y)dS + \frac{1}{4\pi t^2}\int_{|y-x|=t} \big(f + (y-x)\cdot\nabla f\big)(y)dS \ . \tag{4}$$

In particular when $f,g \in \mathcal{S}(\mathbb{R}^n)$ are Schwartz class functions we see the solution exhibits the behavior:

$$|\phi(t,x)| \ \leq \ C(N,f,g)(1+|t|+|x|)^{-1}(1+\big|t-|x|\big|)^{-N} \ . \tag{5}$$

for every $N > 0$.

A robust way to interpolate between estimates (3) and (5) is given by Klainerman's vector field method, see [8] and [9]. One starts with the observation that the vector fields $\mathbb{L} = \{\partial_\alpha, x^j\partial_i - x^i\partial_j, t\partial_i + x^i\partial_t, t\partial_t + x\cdot\nabla_x\}$ satisfy $[\Box, X] = c(X)\Box$ for a constant $c(X)$ depending on X. Thus, if $\Box\phi = 0$, one has the conserved energy (2) for $X^\alpha\phi$, where X^α is some finite product of elements in $\mathbb{L}$. Via rescaled Sobolev embeddings this gives:

$$|\nabla\phi(t,x)| \ \leq \ C(f,g)(1+|t|+|x|)^{-1}(1+\big|t-|x|\big|)^{-\frac{1}{2}} \ , \tag{6}$$

where $C(f,g)$ is the initial energy of $X^\alpha\phi$ for $|\alpha| \leq 2$.

Now consider the wave equation $\Box_g = \nabla^\alpha\nabla_\alpha$ on a curved space time $(\mathcal{M}, g)$, where g is some Lorentzian metric. Unfortunately in this case there is no useful analog of (4) for studying global problems. Instead one must work with the equation $\Box_g$ more directly.

In order the get good decay estimates one must first of put some asymptotic conditions on $(\mathcal{M}, g)$. For many metrics of interest one has the asymptotic flatness condition $|\partial^\alpha(g-m)| \lesssim (1+|x|)^{-|\alpha|-\gamma}$, for some $\gamma > 0$ where m is the Minkowski metric.

Where uniform decay estimates are concerned, it turns out that generic asymptotic conditions are only the beginning, and much more detailed local properties of $(\mathcal{M}, g)$ must be known. This can be seen by again looking at the geometrical optics ansatz $\phi_\lambda = a_\lambda e^{i\lambda u}$ where $La_\lambda = (\frac{1}{2}\Box u)a_\lambda$ (to first order) and $L = -\nabla u$ is the null generator of $u = const.$ Such approximate solutions show that the long time behavior of null geodesics in $(\mathcal{M}, g)$ will determine the local decay of solutions to $\Box_g \phi = 0$, at least in the high frequency regime. Now there are mainly two cases to consider.

If every null geodesic of $(\mathcal{M}, g)$ leaves any spatially compact set in a finite amount of time, then under general asymptotic conditions estimates such as (3) continue to hold. For example see the works [7], [2], [17], and the results described below.

On the other hand if $(\mathcal{M}, g)$ happens to have spatially trapped null geodesics, it can be shown the precise estimate (3) fails. However, for may of the metrics which arise in applications one can often recover a weakened version[1] such as:

$$\int_0^\infty \int_{|x|\leq R} \phi^2(t,x)dxdt \ \leq \ C(R)E(0) \ . \tag{7}$$

For example bounds of this type have been established for stationary black holes in a number of works such as [5], [20], [21], and recently [6] which covers all subextremal Kerr metrics.

With some additional structure estimates such as (7) are enough to bootstrap local errors generated by the commutators $[\Box_g, X]$ when X are suitably chosen vector fields. This ultimately leads to decay estimates like (6) for both linear and nonlinear waves on a variety of interesting backgrounds. See for example the works [14], [15], [12], [13], [18], and the results described below.

2. Results on local energy decay

Here we discuss the results of [16] which is joint work with Jason Metcalfe and Daniel Tataru. The goal of this project is to provide general conditions under which estimate (3) holds for solutions to the perturbed wave equation:

$$P(t,x,D) = \Box_{A,g} + V(t,x), \qquad \Box_{A,g} = (D_\alpha + A_\alpha)g^{\alpha\beta}(D_\beta + A_\beta) \tag{8}$$

where A and V are complex valued, and $D_\alpha = (1/i)\partial_\alpha$. The metric g and the coefficients A, V are assumed to satisfy optimal asymptotic flatness condition of the form

$$\| (g-m, A, V) \|_{AF} < \infty \tag{9}$$

where $m = (-1,1,1,1)$ and:

$$\| (h,A,V) \|_{AF} \ = \ \| h \|_2 + \| \langle x\rangle A \|_1 + \| \langle x\rangle^2 V \|_0 \ , \qquad \| q \|_k \ = \ \sum_{|\alpha|\leq k} \|\langle x\rangle^{|\alpha|}\partial^\alpha q\|_{l^1 L^\infty} \ .$$

The key additional assumptions we make here is that g is nontrapping for null geodesics, and that (g, A, V) is either *stationary* (i.e. time independent), or in certain cases that $\partial_t(g, A, V)$ is sufficiently small in the AF norm. For the strongest theorems to hold we also need to enforce a reality condition on (A, V) which implies that P is (almost) symmetric.

To measure local energy decay in an optimal way we use the dyadic norms of Metcalfe-Tataru:

$$\| u \|_{LE} = \sup_{k>0} 2^{-\frac{1}{2}k} \| u \|_{L^2(|x|\approx 2^k)}, \quad \| u \|_{LE^1} = \| (\partial u, \langle x\rangle^{-1}u) \|_{LE}, \quad \| F \|_{LE^*} = \sum_{k>0} 2^{\frac{1}{2}k} \| F \|_{L^2(|x|\approx 2^k)} \ .$$

Then we say that the *local energy decay condition holds for P* if:

$$\| u \|_{LE^1[0,T]} + \| \partial u \|_{L^\infty L^2[0,T]} \ \lesssim \ \| \partial u(0) \|_{L^2} + \| Pu \|_{LE^* + L^1L^2[0,T]}. \tag{10}$$

With this notation one of the main results of [16] reads as follows:

---

[1]Roughly speaking one needs to take into account the amplification factor $\frac{1}{2}\Box u$ in the transport equation for a_λ. This measures the spreading of nearby null geodesics, and reasonable local energy decay can still be established if it is negative along trapped null geodesics.

Theorem 2.1. *a) Let P be stationary, nontrapping and asymptotically flat. Then local energy decay holds for P if and only if P has no negative eigenvalues, real resonances, or zero eigenvalues/resonances.*

b) Let P be stationary, symmetric, nontrapping and asymptotically flat. Then local energy decay holds for P if and only if P has no negative eigenvalues or zero eigenvalues/resonances.

The various obstructions described in the above theorem are:

i) An "eigenvalue" corresponds to an L^2 solution ϕ_ω to the reduced wave equation $P_\omega = P|_{D_t=\omega}$ for some $\omega \in \mathbb{C}$. We say ω is "negative" if $\Im(\omega) < 0$. Note that even when P_ω is symmetric, the eigenvalue problem here is quadratic, so there is no reason for ω to be real. Negative eigenvalues give rise to highly localized finite energy solutions $e^{i\omega t}\phi_\omega$ with exponentially growing local energy forward in time.

ii) A "real resonance" is a solution to $P_\omega \phi_\omega = 0$ with $\omega \in \mathbb{R} \setminus \{0\}$, which satisfies the outgoing Sommerfeld condition:
$$2^{-j/2}\|(\partial_r + i\omega)\phi_\omega\|_{L^2(|x|\approx 2^j)} \to 0.$$
After multiplication of $e^{i\omega t}\phi_\omega$ by a cutoff $\chi(t-r)$ supported where $t - r > 0$, these correspond to approximate waves which spread in a timelike subset of $|x| \le t$. They are the most subtle obstruction to the local energy decay estimate.

iii) Finally, by a "zero eigenvalues/resonance" we simply mean a stationary solution to $P_0\phi_0 = 0$ with finite LE^1 norm when $0 \le t \le 1$. These are associated with a root space at zero energy which leads to finite energy solutions with no time oscillation which grow at some algebraic rate in time.

The importance of the symmetry assumption in part b) of Theorem 2.1 is that it automatically eliminates real (nonzero) resonances through an energy bootstrap argument. Each of i)–iii) represents a distinct obstruction to the dispersive estimate (10), and by Theorem these are the only possible obstructions.

In [16] we also prove a version of of Theorem 2.1 part b) which includes the case when $\| \partial_t(g, A, V) \|_{AF} \le \epsilon$. Moreover this can be done in the context where one drops the assumption of no negative eigenvalues:

Theorem 2.2. *Let P be symmetric, slowly varying, nontrapping, asymptotically flat, and satisfying the following uniform estimate at zero energy:*
$$\|u\|_{\dot H^1} \le K_0 \|P_0 u\|_{\dot H^{-1}}.$$
Then there is a direct sum decomposition of the energy space into invariant subspaces
$$\mathcal{E} = S^0 + S^+ + S^-,$$
with S^+ and S^- of equal finite dimension, so that the wave flow on the subspaces S^+, S^0 and S^- is as follows:

a) Solutions in S^+ are spatially localized and grow exponentially forward in time.

b) Solutions in S^0 are uniformly bounded and satisfy local energy decay.

c) Solutions in S^- are spatially localized and decay exponentially forward in time.

One can think of this result as a linear stable manifold theorem for symmetric adiabatic operators $P(t, x, D)$ which holds whenever there is a spectral gap around zero energy. This provides a quantitative version of the spectral theory for Klein-Gordon equations established in [10], and also an infinite dimensional analog of the classical "roughness theory" of nonautonomous ODE [4].

The key technical step in all of the above theorems is to prove a local energy decay estimate for solutions to $P\phi = F$, where ϕ is assumed to be compactly supported in time. This restriction allows one to truncate the time frequency variable into low, medium, and high frequencies which can each be analyzed separately. One then removes the time truncation with various limiting arguments depending on the context.

3. Results on the vector field method

Here we discuss [19] which is joint work with Jesus Oliver. In this project we start by *assuming* a weaker version of (10) which is:

$$\text{(11a)} \quad \sup_{0\le t\le T} \|\partial\phi(t)\|_{H^s_x} + \|\phi\|_{WLE^s[0,T]} \ \le C_s\big(\|\partial\phi(0)\|_{H^s_x} + \|\Box_g\phi\|_{(WLE^{*,s}+L^1_t H^s_x)[0,T]}\big)\ ,$$

$$\text{(11b)} \quad \sup_{0\le t\le T} \|\partial\phi(t)\|_{H^s_x} + \|\phi\|_{LE^s[0,T]} \ \le C_s\big(\|\partial\phi(0)\|_{H^s_x} + \|(\phi,\partial_t\phi)\|_{L^2\times H^s(r<R_0)[0,T]} + \|\Box_g\phi\|_{LE^{*,s}[0,T]}\big)\ ,$$

where the norms are defined as follows:

$$\| \phi \|_{LE^s[0,T]} = \sum_{|J|\leq s} \big(\| \langle x\rangle^{-1}\partial^J\phi \|_{LE[0,T]} + \| \partial\partial^J\phi \|_{LE[0,T]} \big) \ , \qquad \| F \|_{LE^{*,s}[0,T]} = \sum_{|J|\leq s} \| \partial^J F \|_{LE^*[0,T]} \ ,$$

$$\|\phi\|_{WLE^s[0,T]} = \sum_{|J|\leq s} \big(\|\langle x\rangle^{-1}\partial^J\phi\|_{LE[0,T]} + \|\partial\partial^J\phi\|_{LE(r>R_0)[0,T]} \big) \ ,$$

$$\| F \|_{WLE^{*,s}[0,T]} = \sum_{|J|\leq s} \big(\| \partial^J F \|_{LE^*[0,T]} + \| \partial\partial^J F \|_{L^2(r<R_0)[0,T]} \big) \ .$$

Here R_0 is some sufficiently large radius which in applications is fixed so the exterior $|x| > R_0$ does not contain trapped null rays. Note that the s index in LE^s is shifted here by one unit compared to (10) to make the notation more uniform. In this case the space time $(\mathcal{M}, g)$ may only defined in the spatial variable for $|x| \geq r_0$ where r_0 may not be zero.

The other main consideration of [19] is we replace the condition (9) with the weaker assumption that there exists an outgoing optical function $u = u(t, x)$ satisfying the two conditions:

I) (Weak Asymptotic Flatness) We have that (u, x) forms a uniform set of coordinates in the sense that $C^{-1} < u_t < C$, and u has the symbol bounds:

$$\| (\tau_-\partial_t)^i(\tau_x\tau_0\partial_x)^J(\partial_t u - 1, \partial_i u + \omega^i) \|_{\ell^1_r L^\infty[0,\infty)} \ < \ \infty \ , \qquad \text{for all} \quad (i, J) \in \mathbb{N}\times\mathbb{N}^4 \ , \tag{13}$$

where $\tau_x = \langle x\rangle$, $\tau_- = \langle u\rangle$, $\tau_+ = \langle (t,x)\rangle$, $\tau_0 = \tau_-\tau_+^{-1}$, and where $\omega^i = x^i\tau_x^{-1}$.

II) (Outgoing Radiation Condition) First define symbol classes $\mathcal{Z}^k$:

$$|\!|\!| q |\!|\!|_{k,N} \ := \sum_{i+|J|\leq N} \| \tau_0^{-k}(\tau_-\partial^b_u)^i(\tau_x\partial^b_x)^J q \|_{\ell^1_r L^\infty} + \sum_{i+|J|\leq N} \| \tau_0^{-k}(\tau_-\partial^b_u)^i(\tau_x\partial^b_x)^J q \|_{\ell^1_u\ell^1_r L^\infty(\frac{1}{2}t<r<2t)} \ .$$

Here $\partial^b = (\partial^b_u, \partial^b_x)$ denotes the (u, x) coordinate derivatives. Then for the inverse metric $g^{\alpha\beta}$ in (u, x) coordinates one has the symbol bounds:

$$g^{\alpha\beta} - h^{\alpha\beta} \ \in \ \mathcal{Z}^0 \ , \qquad \sqrt{|g|}g^{iu} + \omega^i \ \in \ \mathcal{Z}^{\frac{1}{2}} \ , \qquad g^{ui} - \omega^i\omega_j g^{uj} \ \in \ \mathcal{Z}^1 \ , \qquad g^{uu} \ \in \ \mathcal{Z}^2 \ , \tag{14}$$

where $h^{uu} = 0$, $h^{ui} = -\omega^i$, and $h^{ij} = \delta^{ij}$ is the Minkowski metric in Bondi coordinates (u, x).

Note that in the regions $|x| < (1 - \epsilon)t$ and $|x| > (1 + \epsilon)t$ for $\epsilon > 0$ the asymptotic bounds (14) are consistent with (9). On the other hand in the region $\big|t - |x|\big| \leq \epsilon t$ these conditions are consistent with the asymptotics of the metric g in the stability of Minkowski space, see [3], [1], and [11].

The main result of [19] is to show that by assuming (13) and (14) the estimates (11a) and (11b) imply uniform decay rates for both linear and nonlinear problems. On the linear side we have.

Theorem 3.1 (Peeling Estimates for the Inhomogeneous Wave Equation)**.** *Given any $k \geq 1$ there exists an integer $N = N(k)$ depending (linearly) on k such that if $\Box_g\phi = 0$ one has:*

$$\sum_{i+|J|\leq k} \| \tau_+^{\frac{3}{2}}\tau_0^{\frac{1}{2}}(\tau_-\partial^b_u)^i(\tau_x\partial^b_x)^J\phi \|_{L^\infty[0,T]} \leq \ C_k \sum_{|J|\leq N} \| \tau_x(\tau_x\partial_x)^J\partial\phi(0) \|_{L^2_x} \ . \tag{15}$$

Estimates similar to (15) are shown to hold for solutions to the inhomogeneous wave equation $\Box_g\phi = F$, with optimal norms on F. This leads to a nonlinear result:

Theorem 3.2 (Peeling Estimates for Solutions to Null Form Systems)**.** *Suppose quadratic forms $\mathcal{N}$ are given which satisfy:*

$$|(\tau_-\partial^b_u)^i(\tau_x\partial^b_x)^J\partial^k_\phi\mathcal{N}^{\alpha\beta}| \ \leq \ c_{i,J,k}(|\phi|) \ , \qquad |(\tau_-\partial^b_u)^i(\tau_x\partial^b_x)^J\partial^k_\phi\mathcal{N}^{uu}| \ \leq \ c_{i,J,k}(|\phi|)\tau_0 \ .$$

Let ϕ be a solution to the system of semilinear equations:

$$\Box_g\phi \ = \ \mathcal{N}^{\alpha\beta}(t, x, \phi)\partial_\alpha\phi\partial_\beta\phi \ ,$$

Then for sufficiently small and well localized initial data the solution is global and enjoys the peeling estimates:

$$\sum_{i+|J|\leq k} \| \tau_+^{\frac{3}{2}}\tau_0^{\frac{1}{2}}(\tau_-\partial^b_u)^i(\tau_x\partial^b_x)^J\phi \|_{L^\infty[0,T]} \ \leq \ C_k\epsilon \ .$$

The proof of these two theorems uses a graduated series of norms $\| \cdot \|_{S^a}$ which are, roughly speaking, the local energy decay norm LE^s, with additional weighs of $\tau_+^a \tau_0$ for $\partial\phi$ and τ_+^a for $\partial_x^b \phi$. Estimates in the space S^a come at the expense of local errors which are roughly $\| \tau_+^{a-1} X^\alpha \phi \|_{L^2(r\leq R)[0,T]}$ for $X \in \widetilde{\mathbb{L}} = \mathbb{L} \backslash \{t\partial_i + x^i \partial_t\}$. Such errors can be fed back into the local energy decay estimate (11a) to close the argument.

References

[1] Bieri, Lydia *An extension of the stability theorem of the Minkowski space in general relativity.* J. Differential Geom. **86** (2010), no. 1, 17-70.

[2] J.-F. Bony, D. Häfner, *The semilinear wave equation on asymptotically Euclidean manifolds.* Comm. Partial Differential Equations **35** (2010), no. 1, 23-67.

[3] Christodoulou, Demetrios; Klainerman, Sergiu **The global nonlinear stability of the Minkowski space.** Princeton Mathematical Series, 41. Princeton University Press, Princeton, NJ, 1993. x+514 pp.

[4] Coppel, W. A. *Dichotomies in stability theory.* Lecture Notes in Mathematics, Vol. 629. Springer-Verlag, Berlin-New York, 1978.

[5] Dafermos, M., Rodnianski, I. *The red-shift effect and radiation decay on black hole spacetimes.* Commun. Pure Appl. Math. **62** (7), 859-919

[6] Dafermos, Mihalis; Rodnianski, Igor; Shlapentokh-Rothman, Yakov *Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < M$.* Ann. of Math. (2) **183** (2016), no. 3, 787-913.

[7] Keel, Markus; Smith, Hart F.; Sogge, Christopher D. *Almost global existence for some semilinear wave equations.* Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. **87** (2002), 265-279.

[8] Klainerman, Sergiu *Uniform decay estimates and the Lorentz invariance of the classical wave equation.* Comm. Pure Appl. Math. **38** (1985), no. 3, 321-332.

[9] Klainerman, S. *The null condition and global existence to nonlinear wave equations.* Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), 293-326, Lectures in Appl. Math., 23, Amer. Math. Soc., Providence, RI, 1986.

[10] Langer, Heinz; Najman, Branko; Tretter, Christiane *Spectral theory of the Klein-Gordon equation in Pontryagin spaces.* Comm. Math. Phys. **267** (2006), no. 1, 159-180.

[11] Lindblad, Hans *On the asymptotic behavior of solutions to the Einstein vacuum equations in wave coordinates.* Comm. Math. Phys. 353 (2017), no. 1, 135-184.

[12] Luk, Jonathan *A vector field method approach to improved decay for solutions to the wave equation on a slowly rotating Kerr black hole.* Anal. PDE **5** (2012), no. 3, 553-625.

[13] Luk, Jonathan *The null condition and global existence for nonlinear wave equations on slowly rotating Kerr spacetimes.* J. Eur. Math. Soc. (JEMS) **15** (2013), no. 5, 1629-1700.

[14] Metcalfe, Jason; Sogge, Christopher D. *Hyperbolic trapped rays and global existence of quasilinear wave equations.* Invent. Math. **159** (2005), no. 1, 75-117.

[15] Metcalfe, Jason; Sogge, Christopher D. *Global existence of null-form wave equations in exterior domains.* Math. Z. **256** (2007), no. 3, 521-549.

[16] J. Metcalfe; J. Sterbenz; D. Tataru *Local energy decay for scalar fields on time dependent non-trapping backgrounds.* to appear in AJM.

[17] J. Metcalfe, D. Tataru, *Global parametrices and dispersive estimates for variable coefficient wave equations.* Math. Ann. 353 (2012), no. 4, 1183-1237.

[18] Moschidis, Georgios *The r^p-weighted energy method of Dafermos and Rodnianski in general asymptotically flat spacetimes and applications*, Ann. PDE (2016) 2:6.

[19] J. Oliver; J. Sterbenz *A Vector Field Method for Radiating Black Hole Spacetimes.* `arXiv:1705.10714`

[20] Marzuola, Jeremy; Metcalfe, Jason; Tataru, Daniel; Tohaneanu, Mihai *Strichartz estimates on Schwarzschild black hole backgrounds.* Comm. Math. Phys. **293** (2010), no. 1, 37-83.

[21] Tataru, Daniel; Tohaneanu, Mihai *A local energy estimate on Kerr black hole backgrounds.* Int. Math. Res. Not. IMRN 2011, no. 2, 248-292.

Department of Mathematics, University of California, San Diego (UCSD) 9500 Gilman Drive, Dept 0112 La Jolla, CA 92093-0112 USA

E-mail address: `jsterben@math.ucsd.edu`

Weighted $L^2 - L^2$ estimate for wave equation and its application [1]

Ning-An Lai

Institute of Nonlinear Analysis and Department of Mathematics, Lishui University, Lishui 323000, China.

School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
e-mail: hyayue@gmail.com

we are devoted to establishing a weighted $L^2 - L^2$ estimate for the linear inhomogeneous wave equation with zero initial data in 3-D

$$\partial_t^2\phi - \Delta\phi = F(t,x), \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^3. \tag{0.1}$$

$$\begin{cases} \partial_t^2\phi - \Delta\phi = F(t,x), & (t,x) \in \mathbf{R}_+ \times \mathbf{R}^3, \\ \phi(0,x) = 0, \quad \phi_t(0,x) = 0, & x \in \mathbf{R}^3. \end{cases} \tag{0.2}$$

For this purpose, we introduce a Morawetz multiplier

$$\begin{aligned} X =&(t+2+r)^s(\partial_t + \partial_r) + (t+2-r)^s(\partial_t - \partial_r) \\ &+ \frac{1}{r}\Big((t+2+r)^s - (t+2-r)^s\Big) \\ =&(t+2+r)^s\Big(\partial_t + \partial_r + \frac{1}{r}\Big) + (t+2-r)^s\Big(\partial_t - \partial_r - \frac{1}{r}\Big). \end{aligned} \tag{0.3}$$

By using this multiplier, we establish the following weighted $L^2 - L^2$ inequality for (0.2)

$$\begin{aligned} &\sup_t \left(\left\|(t+2-r)^{s/2}\nabla_{t,x}\phi\right\|_{L^2(\mathbf{R}^3)} + \left\|(t+2-r)^{s/2}\left(\frac{\phi}{r}\right)\right\|_{L^2(\mathbf{R}^3)} \right) \\ \le& C_\delta \left\|(t+2+r)^{s/2}(t+r-2)^{1/2+\delta}F\right\|_{L^2(\mathbf{R}_+\times\mathbf{R}^3)}, \end{aligned} \tag{0.4}$$

[1] This talk is presented in The 11th Mathematical Society of Japan (MSJ) Seasonal Institute (SI)- The Role of Metrics in the Theory of Partial Differential Equations, Hokkaido University, Sapporo, Japan, July 2-13, 2018. The result is supported by Natural Science Foundation of Zhejiang Province(LY18A010008), NSFC(11501273, 11726612, 11726611, 11771194, 11771359), Chinese Postdoctoral Science Foundation(2017M620128), the Scientific Research Foundation of the First-Class Discipline of Zhejiang Province (B)(201601).

for any constant $\delta > 0$ and $1 < s < 2$. And C_δ denotes a positive constant depending on δ.

The weighted $L^2 - L^2$ inequality (0.4) can be used to provide a new proof of global existence for the Cauchy problem

$$\begin{cases} \partial_t^2\phi - \Delta\phi = |\phi|^p, \quad (t,x) \text{ in } \mathbf{R}_+ \times \mathbf{R}^3, \\ \phi(0,x) = \varepsilon f(x), \quad \phi_t(0,x) = \varepsilon g(x), \quad x \in \mathbf{R}^3 \end{cases} \tag{0.5}$$

for $1 + \sqrt{2} < p < 3$, which has been proved by [2] and [3]. Also, by combining strong Huygen's principle for wave equations in 3-D, we may prove global existence for the Cauchy problem

$$\begin{cases} \partial_t^2\Phi - \Delta\Phi + \dfrac{2}{1+t}\Phi_t = |\Phi|^p, \quad (t,x) \text{ in } \mathbf{R}_+ \times \mathbf{R}^3, \\ \Phi(0,x) = \varepsilon f(x), \quad \Phi_t(0,x) = \varepsilon g(x), \quad x \in \mathbf{R}^3, \end{cases} \tag{0.6}$$

for $p_c(5) < p < 3$, which was proved by [1] under radial symmetrical assumption. Here $p_c(n)(n \geq 2)$ denotes the positive root of the following quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

References

[1] M.D'Abbicco, S.Lucente and M.Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping, *J. Differential Equations*, **259** (2015), 5040-5073.

[2] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.*, **28** (1979), 235-268.

[3] H. Lindblad and C. D. Sogge, Long-time existence for small amplitude semilinear wave equations, *Amer. J. Math.*, **118** (1996), 1047-1135.

GEOMETRIC NONLINEAR PROBLEMS AND GIT STABILITY THROUGH MOMENT MAPS

AKITO FUTAKI (TSINGHUA UNIVERSITY, BEIJING)

Let (Z, Ω) be a symplectic manifold, i.e. Ω is a closed non-degenerate 2-form on Z. Here the 2-form Ω is said to be non-degererenerate if $i(v)\Omega = 0$ for a tangent vector v implies $v = 0$. Let K be a Lie group acting on Z preserving Ω and $\mathfrak{k} = \mathrm{Lie}(K)$ the Lie algebra of K. The moment map μ for the action of K is the K-equivariant map $\mu : Z \to k^*$ such that, for any $X \in \mathfrak{k}$, if we define the smooth function μ_X on Z by

$$\langle \mu(p), X \rangle = \mu_X(p), \qquad p \in Z$$

then μ_X is a Hamiltonian function for $X^\sharp$, that is,

$$i(X^\sharp)\Omega = -d\mu_X$$

where $X^\sharp$ is the vector field on Z induced by X. When Z is a Kähler manifold and K acts on Z isometrically, and thus the complexification $G = K^c$ of K acts on Z holomorphically, Kemph-Ness theorem says that for any $p \in Z$ the orbit $G \cdot p$ is GIT stable if and only if μ has a zero in $G \cdot p$. Here GIT is an abbreviation of Geometric Invariant Theory, which is a theory due to Mumford that considering the orbit space of stable and semi-stable ones will give us a good moduli space of algebraic objects.

The purpose of this talk is about the philosophy that if a geometric nonlinear PDE problem is set up in the moment map picture then the the existence of the solution to the PDE should be related to GIT stability in algebraic geometry. Here moment map picture of the PDE problem is meant by a picture that the relevant function space has an infinite dimensional symplectic manifold structure with Hamiltonian group action and the solutions of the PDE are zero set of the moment map.

The typical successful examples are
(i) Kobayashi-Hitchin correspondence of Hermitian-Einstein metrics on stable bundles in the sense of Mumford and Takamoto. [2], [11].
(ii) Kähler-Einstein metrics on K-stable Fano manifolds. [1], [10].

The examples we expect this philosophy can be applied are
(a) Kähler metrics of constant scalar curvature on K-stable polarized manifolds (M, L). Here $L \to M$ is a holomorphic line bundle with positive first Chern class $c_1(L)$ so that $c_1(L)$ can be considered as the space of Kähler forms. [3]
(b) Conformally Kähler, Einstein-Maxwell metrics. [5], [6], [8].
(c) Kähler metrics with constant Cahen-Gutt moment map, which is related to closedness of Fedosov's star product. [7].

In all these examples, using the derivative of the norm squared moment map $\|\mu\|^2$, one obtains two obstructions to the existence of the PDE. One is a Lie algebra character $\mathrm{Fut} : \mathfrak{g} \to \mathbf{R}$ of the Lie algebra $\mathfrak{g}$ of the holomorphic vector fields

Date: April 1, 2018.

on M, typically given in [4] for Kähler-Einstein metrics on Fano manifolds, and a structure theorem of the same Lie algebra $\mathfrak{g}$, [9]. The Lie algebra character Fut is used to define "K-stability". The conjecture is the equivalence of the existence of the solution to the PDE and K-stability.

For a Kähler metric g, the scalar curvature $\mathrm{Scal}(g)$ of g is given by $\mathrm{Scal}(g) = -\Delta_g \log\det g$. But if we choose a Kähler metric $\tilde{g}$ so that the Kähler form of $\tilde{g}$ belongs to the same cohomology class as the Kähler form of g, $\tilde{g}$ is given by $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \overline{z^j}}$ for a smooth function φ where $z^1, \cdots, z^m$ are local holomorphic coordinates. Thus in (a), finding a constant scalar curvature metric in a fixed Kähler class is reduced to solving

$$-\Delta_{\tilde{g}} \log\det(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \overline{z^j}}) = \text{constant}.$$

The PDEs for (b) and (c) include the scalar curvature, and the problem is reduced to solving PDEs similar to the above.

A compact Kähler manifold of complex dimension m is said to toric if it has an effective Hamiltonian and isometric action of real m-dimensional torus. Then it has a moment map. On the moment map image, the scalar curvature is expressed as

$$\mathrm{Scal} = -\sum_{i,j} \phi^{ij}{}_{,ij}$$

where ϕ is the symplectic potential, which is the Legendre transform of the Kähler potential on the moment map image, ϕ^{ij} is the inverse of the Hessian (ϕ_{ij}) and $,ij$ denotes the covariant derivative. Thus on compact toric Kähler manifolds, PDE's involving the scalar curvature are reduced to equations involving $\sum_{i,j} \phi^{ij}{}_{,ij}$. The K-stability condition in (a) is expressed in terms of certain integral over the moment map image, see [3]. For $m = 2$, Donaldson has completed the proof of (a).

References

[1] X.-X. Chen, S.K. Donaldson and S. Sun: Kähler-Einstein metrics on Fano manifolds, III: limits as cone angle approaches 2π and completion of the main proof. J. AMS. 28(2015), 235–278.

[2] S.K. Donaldson : Anti-self-dual Yang-Mills connections on complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. 50(1985), 1-26.

[3] S.K. Donaldson : Scalar curvature and stability of toric varieties, J. Differential Geometry, 62(2002), 289-349.

[4] A. Futaki : An obstruction to the existence of Einstein Kähler metrics. Invent. Math. 73 (1983), no. 3, 437–443.

[5] A. Futaki and H. Ono : Volume minimization and Conformally Kähler, Einstein-Maxwell geometry, preprint, to appear in J. Math. Soc. Japan. arXiv:1706.07953.

[6] A. Futaki and H. Ono : Conformally Einstein-Maxwell Kähler metrics and the structure of the automorphisms, preprint, arXiv1708.01958.

[7] A. Futaki and H. Ono : Cahen-Gutt moment map, closed Fedosov's star product and structure of the automorphism group, preprint, arXiv1802.10292.

[8] A. Futaki and H. Ono : On the existence problem of Einstein-Maxwell Kähler metrics, preprint, arXiv1803.06801.

[9] Y. Matsushima : Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kaehlérienne, Nagoya Math. J., **11**, 145-150 (1957).

[10] G. Tian: K-Stability and Kähler-Einstein Metrics. Comm. Pure Appl. Math. 68.7 (2015): 1085-1156. Corrigendum : 2082–2083.

[11] K. Uhlenbeck and S.-T.Yau : On the existence of hermitian Yang-Mills connections on stable bundles over compact Kähler manifolds, Comm. Pure Appl. Math. 39(1986), 257-293. Comm. Pure Appl. Math. 42(1989), 703–707.

EXISTENCE THEOREM FOR BRAKKE FLOW

YOSHIHIRO TONEGAWA

A family of n-dimensional surfaces $\{\Gamma(t)\}_{t\geq 0}$ in $\mathbb{R}^{n+1}$ is called the mean curvature flow (abbreviated by MCF) if the velocity is equal to its mean curvature at each point and time. Given a smooth surface Γ_0, one can find a smoothly moving MCF starting from Γ_0 until some singularities such as vanishing or pinching occur. The presence of singularities necessitates a weak formulation of MCF, and there have been intensive research dealing with this aspect in the last few decades. Among such attempts, Brakke started a theory of MCF ("Brakke flow") inclusive of singularities in the framework of geometric measure theory in his seminal book [1]. In particular, he developed a general existence theory of MCF: starting from any integral varifold (with a minor technical restriction) of any codimension, he showed that there exists a family of varifolds satisfying the motion law of MCF in a weak sense and existing for all time. One major concern for the validity of his existence theorem is that the proof does not guarantee the non-triviality of the solution when Γ_0 is not a smooth surface. In [3], we rectify this point of non-triviality by introducing a few framework and also modifying Brakke's original argument. The main existence theorem of [3] may be stated roughly as follows.

Theorem 1. *Suppose that $\Gamma_0 \subset \mathbb{R}^{n+1}$ is a closed countably n-rectifiable set whose complement $\mathbb{R}^{n+1} \setminus \Gamma_0$ equals $\cup_{i=1}^{N} E_{0,i}$, where $E_{0,1}, \ldots, E_{0,N} \subset \mathbb{R}^{n+1}$ are mutually disjoint non-empty open sets and $N \geq 2$. Assume that the n-dimensional Hausdorff measure of Γ_0 is finite or grows at most exponentially near infinity. Then, for each $i = 1, \ldots, N$, there exists a family of open sets $\{E_i(t)\}_{t\geq 0}$ with $E_i(0) = E_{0,i}$ such that $E_1(t), \ldots, E_N(t)$ are mutually disjoint for each $t \geq 0$ and $\Gamma(t) := \cup_{i=1}^{N} \partial E_i(t)$ coincides with the space-time support of a nontrivial Brakke flow starting from Γ_0. Each $E_i(t)$ moves continuously in time with respect to the Lebesgue measure.*

We may regard each $E_i(t) \subset \mathbb{R}^{n+1}$ as a region of "i-th phase" at time t, and $\Gamma(t)$ as the "phase boundaries" which move by the mean curvature in a generalized sense. Some of $E_i(t)$ shrink and vanish, and some may grow and may even occupy the whole $\mathbb{R}^{n+1}$ in finite time. Note that the continuity of $E_i(t)$ guarantees that $\Gamma(t) \neq \emptyset$ at least for a short initial time interval, and $\Gamma(t) \neq \emptyset$ unless $E_i(t) = \mathbb{R}^{n+1}$ for some i.

Notion not stated clearly in Theorem 1 is that of Brakke flow, which is as follows. For simplicity, assume Γ_0 has a finite n-dimensional Hausdorff measure, $\mathcal{H}^n(\Gamma_0) < \infty$.

Definition 1. *A Brakke flow starting from Γ_0 is a family of n-varifolds $\{V_t\}_{t\geq 0}$ satisfying the following.*

(1) $V_0 = |\Gamma_0|$ *=unit density varifold induced from Γ_0.*
(2) *For $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$, V_t is an integral varifold with L^2 generalized mean curvature vector $h(\cdot, V_t)$.*
(3) $\|V_t\|(\mathbb{R}^n)$ *is decreasing in t and* $\int_0^\infty \int_{\mathbb{R}^{n+1}} |h(\cdot, V_t)|^2 \, d\|V_t\| dt \leq \mathcal{H}^n(\Gamma_0)$.
(4) *For any $0 \leq t_1 < t_2 < \infty$ and $\phi \in C_c^1(\mathbb{R}^{n+1} \times \mathbb{R}^+; \mathbb{R}^+)$, we have*

$$\|V_t\|(\phi(\cdot,t))\Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^{n+1}} \{\nabla\phi(\cdot,t) + \phi(\cdot,t)h(\cdot,V_t)\}\cdot h(\cdot,V_t) + \frac{\partial \phi}{\partial t}(\cdot,t)\, d\|V_t\| dt.$$

Here, $\|V\|$ is the weight measure of V.

The property (4) is a weak form of the motion law of MCF. The property (2) allows a possibility of having a higher (≥ 2) multiplicity representing a "folding" of surfaces (whether it happens or not is not clear). When the multiplicity stays 1 for a.e. $t > 0$, we say that the flow is a unit density flow. Almost everywhere regularity of unit density flow for general Brakke flow has been studied originally by Brakke and is recently completed by [2, 5, 4]. If the flow is a limit of smooth MCF, White's reguralty theory [6] gives also the almost everywhere regularity. Theorem 1 includes as a part of theorem the existence of $\{V_t\}_{t\geq 0}$ satisfying the Definition 1. We may then define a Radon measure μ on $\mathbb{R}^{n+1} \times \mathbb{R}^+$ by $d\mu = d\|V_t\|dt$. The claim of Theorem 1 is that $\{x \in \mathbb{R}^{n+1} : (x,t) \in \operatorname{spt}\mu\} = \Gamma(t)$ for all $t > 0$.

In my talk, I will mostly concentrate on how one actually constructs the time-discrete approximate MCF.

References

[1] K. A. Brakke, *The motion of a surface by its mean curvature*, Mathematical Notes, **20** Princeton University Press, 1978.

[2] K. Kasai, Y. Tonegawa, *A general regularity theory for weak mean curvature flow*, Calc. Var. Partial Differential Equations **50** (2014) no. 1, 1–68.

[3] L. Kim, Y. Tonegawa, *On the mean curvature flow of grain boundaries*, Annales de l'Institut Fourier (Grenoble) **66** (2017) no. 1, 43–142.

[4] A. Lahiri, *Regularity of the Brakke flow*, Dissertation, Freie Universität Berlin, 2014.

[5] Y. Tonegawa, *A second derivative Hölder estimate for weak mean curvature flow*, Advances in Calculus of Variations **7** (2014) no. 1, 91–138.

[6] B. White, *A local regularity theorem for classical mean curvature flows*, Ann. of Math. (2) **161** (2005), no. 3, 1487–1519.

Multipliers on the wave-like blow-up for nonlinear damped wave equations [1]

Hiroyuki Takamura

Department of Complex and Intelligent Systems, Faculty of Systems Information Science, Future University Hakodate,
116-2 Kamedanakano-cho, Hakodate, Hokkaido 041-8655, Japan.
e-mail: takamura@fun.ac.jp

Current affiliation:

Mathematical Institute, Tohoku University,
6-3, Aramaki Aza-Aoba, Aoba-ku, Sendai 980-8578, Japan.
e-mail: hiroyuki.takamura.a1@tohoku.ac.jp

We are interested in the critical exponent $p_c(n)$ of the following problem.

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^\beta} u_t = |u|^p & \text{in} \quad \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x), u_t(x,0) = \varepsilon g(x), & \end{cases} \tag{1}$$

where $\mu > 0$, $\beta \in \mathbf{R}$. $\varepsilon > 0$ is a "small" parameter. For simplicity, we assume that $f \in H^1(\mathbf{R}^n)$, $g \in L^2(\mathbf{R}^n)$ with compact support and $n \geq 2$. The "critical" means that, for the energy solution, we have the global-in-time existence when $p_c(n) < p$ and the blow-up in finite time when $1 < p \leq p_c(n)$.

It is known that there is no such a $p_c(n)$ when $\beta < -1$ (overdamping case), namely we have the global existence for any $p > 1$. For $-1 \leq \beta < 1$ (effective damping), we know that $p_c(n) = p_F(n)$, where

$$p_F(n) := 1 + \frac{2}{n}$$

is so-called Fujita exponent which is the critical exponent for $u_t - \Delta u = u^p$. For $\beta = 1$ (scaling invariant damping), there is a conjecture such that

$$\begin{cases} \mu \geq \mu_0(n) & \Longrightarrow \quad p_c(n) = p_F(n) \quad \text{(heat-like)}, \\ 0 < \mu < \mu_0(n) & \Longrightarrow \quad p_c(n) = p_S(n+\mu) \quad \text{(wave-like)}, \end{cases} \tag{2}$$

[1] This talk is presented in the 11th MSJ-SI, Hokkaido University, Sapporo, Japan, July 9-10, 2018. All the results in this talk are based on joint works with Ning-An Lai (Lishui Univ. China), which are partially supported by the Grant-in-Aid for Scientific Research (B)(No.18H01132) and (C)(No.15K04964), Japan Society for the Promotion of Science.

where

$$\mu_0(n) := \frac{n^2+n+2}{n+2} \quad \text{and} \quad p_S(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}$$

is so-called Strauss exponent which is the critical exponent for $u_{tt} - \Delta u = |u|^p$. We note that

$$p_F(n) < p_S(n) \quad \text{and that} \quad 0 < \mu < \mu_0(n) \Longleftrightarrow p_F(n) < p_S(n+\mu).$$

The existence part of the conjecture (2) is still open. All the references appear in Introduction of [1].

In this talk, I will introduce you a new conjecture for $\beta > 1$ (scattering damping) that

$$p_c(n) = p_S(n) \quad \text{for all} \quad \mu > 0. \tag{3}$$

Making use of a multiplier $\exp\Big(\mu(1+t)^{1-\beta}/(1-\beta)\Big)$ for a differential equation or inequality of

$$\frac{d}{dt}\int_{\mathbf{R}^n} u(x,t)dx,$$

we succeed to prove the subcritical part of (3). Also the lifespan estimate will be discussed. All results in this talk are from [1]. I would like to show you also that this multiplier is useful for another nonlinearity $|u_t|^p$ (see [2]) and their combined nonlinearity $|u_t|^p + |u|^q$ (see [3]). If I have enough time more, I will introduce another multiplier which makes us to consider a "negative" mass term in the equation.

References

[1] N.-A.Lai and H.Takamura, *Blow-up for semilinear damped wave equations with subcritical exponent in the scattering case*, Nonlinear Analysis, TMA, **168(1)**, 222-237 (2018).

[2] N.-A.Lai and H.Takamura, *Nonexistence of global solutions of nonlinear wave equations with weak time-dependent damping related to Glassey conjecture*, arXiv:1711.07591, to appear in Differential and Integral equations.

[3] N.-A.Lai and H.Takamura, *Nonexistence of global solutions of wave equations with weak time-dependent damping and combined nonlinearity*, arXiv:1802.10273.

Numerical relativity : generating space-time on computers

Yuichiro Sekiguchi (Toho University)

1 Introduction

Einstein ' s theory of general relativity contains the rules of gravity and provides the basis for modern theories of astrophysics and cosmology. The fundamental equations, Einstein's equations, are among the most complicated seen in (classical) physics. For many years, a number of efforts have been made to develop mathematical techniques to solve them. Despite more than 70 years of intense studies, only a handful of special solutions for Einstein's equations which are relevant for astrophysics and cosmology have been presented. The theory remains largely untested yet, except in the weak-field, slow velocity regime. Also, many of the dynamical phenomena thought to occur in nature have not been studied in detail. With the advent of supercomputers, however, the situation has been drastically changed: We are now able to tackle these complicated equations numerically and explore astrophysically realistic, high-velocity, strong-field systems. This is the field of *numerical relativity*, which is reviewed in my talk together with recent applications.

2 Numerical Relativity : a brief summary

Here we list a brief summary of characteristic feature of numerical relativity. We will use metric signature $(- \ + \ + \ +)$, Einstein's summation convention, and the geometrical unit with $c = G = 1$, where c and G are the speed of light and the gravitational constant, respectively.

- **Einstein ′ s equations as a Cauchy problem**: Einstein's equations in their covariant form,

$$G_{ab} = 8\pi T_{ab}, \tag{1}$$

where G_{ab} and T_{ab} are Einstein tensor and stress-energy-momentum tensor, contain complicated mixed derivatives in time and space. Thus, we must decompose the spacetime into time and space ($3 + 1$ decomposition) and rewrite Einstein's equations as a Cauchy problem [1].

- **No absolute background spacetime**: In general relativity, the notion of time and space is not absolute but relative (general covariance) so that the time and spatial axes must be specified. We have four degrees of freedom in specifying the $3 + 1$ decomposition: the lapse function, α which determines the time slice Σ_t with unit normal vector n^a; and shift vector, β^a which controls the spatial direction of the time axis vector $t^a = \alpha n^a + \beta^a$. Then, the spacetime metric, g_{ab}, is decomposed in terms of lapse, shift and the induced metric on a slice $\gamma_{ab} = g_{ab} + n_a n_b$:

$$ds^2 = -(\alpha dt)^2 + \gamma_{ij}(\beta^i dt + dx^i)(\beta^j dt + dx^j). \tag{2}$$

Based on geometric insight and numerical experimentations, there have been several good gauge conditions for treating important astrophysical problems [2].

- **Einstein′ s equations as a constrained system**: Taking parallel and perpendicular projecting of Einstein's equations with respect to n^a, we have a $3 + 1$ decomposition of Einstein's equations

$$R + \mathrm{tr}K^2 - K_{ab}K^{ab} = 16\pi E \qquad \text{(Hamiltonian constraint)}, \tag{3}$$

$$D_b K^b_a - D_a \mathrm{tr}K = 16\pi P_a \qquad \text{(Momentum constraint)}, \tag{4}$$

$$(\partial_t - \mathcal{L}_\beta) K_{ab} = -D_a D_b \alpha \left[R_{ab} + \mathrm{tr}K K_{ab} - 2K_{ac}K^c_b\right] - 4\pi\alpha \left(2S_{ab} - \gamma_{ab}(\mathrm{tr}S - E)\right), \tag{5}$$

where $E = n^a n^b T_{ab}$, $P_a = -n^b \gamma_a^c T_{bc}$, and $S_{ab} = \gamma_a^c \gamma_b^d T_{cd}$. R, R_{ab}, and D_a are Ricci scalar, Ricci tensor, and covariant derivative with respect to γ_{ab}. $\mathcal{L}_\beta$ is Lie derivative with respect to the shift vector β^a. K_{ab} is the extrinsic curvature of the slice Σ_t:

$$(\partial_t - \mathcal{L}_\beta)\,\gamma_{ab} = -2\alpha K_{ab}. \tag{6}$$

These equations are known to as ADM (Arnowitt-Deser-Misner) system [3]. It is important to note that ADM system is a constrained system which consists of 4 constraint equations (Hamiltonian and momentum) and 12 evolution equations for K_{ab} and γ_{ab}.

- **Construction of initial data**: Before starting simulations, we must specify gravitational fields on the initial slice that are compatible with the constraints. The constraint equations are coupled nonlinear elliptic-type equations and it is computationally expensive to solve them numerically. Also, we need astrophysically realistic configuration of the gravitational and matter fields. Construction of such initial data is one of the important issues in numerical relativity [4].

- **Reformulating the ADM system**: Mathematically, it is guaranteed that the constraint conditions are always satisfied in the later time provided that the evolution equations are solved correctly and the constraint conditions are satisfied initially. However, it is not possible to solve the equations without numerical error in numerical simulations. Then, violation of constraints increases with time and simulations break down eventually in ADM system. This is because the ADM system is only weakly hyperbolic [5]. So we need a reformulation to obtain strongly or symmetric hyperbolic system. Among such systems, the so-called BSSN (Baumgarte-Shapiro-Shibata-Nakamura) system [6], which is strongly hyperbolic, outworks.

- **Matter fields**: Besides gravity, we should solve the equation of motion of 'matter' field:

$$\nabla_a T^{ab} = 0. \tag{7}$$

Relativistic hydrodynamics is particularly important in many astrophysical systems. Magnetic fields also play an important role. In many astrophysical applications, the gas is highly ionized and a very excellent conductor of current. Then the ideal magneto-hydrodynamics should be applied to treat them. When photons and neutrinos are involved, we must include a radiation stress-energy tensor, which is determined by solving equations of radiation-hydrodynamics. In many astrophysical phenomena shocks are formed in general. Classical finite difference schemes based on Taylor expansion present deficiency in dealing with the shock discontinuities and we need so-called high-resolution shock-capturing schemes [7], based on the conservation laws and characteristic structure of the system.

References

[1] Y. Chouet-Bruhat " General Relativity and the Einstein Equations " (Oxford University Press, 2009)

[2] E. Gourgoulhon "3+1 Formalism in General Relativity: Bases of Numerical Relativity" (Springer, 2012); T. W. Baungarte and S. L. Shapiro, "Numerical Relativity: Solving Einstein's Equations on the Computer" (Cambridge University Press, 2010)

[3] R. Arnowitt, S. Deser, and C. W. Misner " The dynamics of general relativity ", in " Gravitation : An Introduction to Current Research ", ed. L. Witten (Wiley, 1962); J. K. York, " Kinematics and Dynamics of General Relativity " , in " Sources of Gravitational Radiation " , ed. L. Smarr, (Cambridge University Press, 1979).

[4] G. Cook, Living Reviews in Relativity 3 (2000) 5; E. Gourgoulhon. Journal of Physics: Conference Series 91 (2007) 012001

[5] M. Alcubierre, " Introduction to 3+1 Numerical Relativity " (Oxford University Press, 2008)

[6] M. Shibata and T. Nakamura, Physical Review D 52, 5428 (1995); T. W. Baumgarte and S. L. Shapiro, Physical Review D 59, 024007 (1999).

[7] J. A. Font, Living Reviews in Relativity 11, 7 (2008).

Recent developments on metric measure spaces with Ricci bounds from below

Shouhei Honda *

We say that a triple $(X, \mathsf{d}, \mathfrak{m})$ is a *metric measure space* if (X, d) is a complete separable metric measure space and $\mathfrak{m}$ is a Borel measure on X with $\operatorname{supp}\mathfrak{m} = X$. It makes sense that $(X, \mathsf{d}, \mathfrak{m})$ has a lower bound $K \in \mathbb{R}$ on Ricci curvature and an upper bound $N \in [1, \infty]$ on dimension, so called $\mathrm{RCD}(K, N)$ condition (see [LV09, St06, AGS14b, G15a]). The study of such spaces, $\mathrm{RCD}(K, N)$ spaces, are quickly, widely developed. In particular most of parts in Cheeger-Colding theory [CC96, CC97, CC00a, CC00b] are already generalized to the RCD setting (c.f. [BS18, G13, MN14]).

In this talk I will introduce several recent developments including [AH16, AH17, AHT18, AHPT18] which show Γ-convergence of p-Cheeger energies, that of BV-energies, Weyl's law and so on for such spaces.

For example the following will be explained;

Theorem 0.1. *[AH16] Let us denote by $\mathcal{M} := \mathcal{M}(K, N, D)$ the set of all isometry classes of $\mathrm{RCD}(K, N)$ spaces with the diameter at most D. We define a function F on $\mathcal{M} \times [1, \infty]$ by*

$$F\left((X, \mathsf{d}, \mathfrak{m}), p\right) := \begin{cases} 2(\operatorname{diam} X)^{-1} & \text{if } p = \infty, \\ (\lambda_{1,p}(X, \mathsf{d}, \mathfrak{m}))^{1/p} & \text{if } p \in (1, \infty), \\ h(X, \mathsf{d}, \mathfrak{m}) & \text{if } p = 1, \end{cases}$$

where $\operatorname{diam}, \lambda_{1,p}, h$ denote the diameter, the first positive eigenvalue of p-Laplacian, the Cheeger constant of $(X, \mathsf{d}, \mathfrak{m})$, respectively. Then F is continuous.

One of the fundamental problems in RCD theory is to find continuous functions on $\mathcal{M}$ as many as possible. It is important that the theorem provides uncountably many such examples.

As a corollary we obtain;

Corollary 0.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $\mathrm{RCD}(0, N)$ space. Then*

$$(\lambda_{1,p}(X, \mathsf{d}, \mathfrak{m}))^{1/p} \overset{N}{\asymp} h(X, \mathsf{d}, \mathfrak{m}) \overset{N}{\asymp} (\operatorname{diam} X)^{-1}$$

for all $p \in (1, \infty)$, where $a \overset{N}{\asymp} b$ means that there exists $C := C(N) > 1$ such that $C^{-1}a \le b \le Ca$.

This corollary is a kind of isoperimetric inequalities and it is new even for smooth Riemannian manifolds (M^n, g, vol) with the Riemannian volume.

*Tohoku University, shonda@m.tohoku.ac.jp

References

[AGS14b] L. Ambrosio, N. Gigli, G. Savaré: *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Math. J. **163** (2014), 1405–1490.

[AH16] L. Ambrosio, S. Honda: *New stability results for sequences of metric measure spaces with uniform Ricci bounds from below*, in Measure Theory in Non-Smooth Spaces, 1–51, De Gruyter Open, Warsaw, 2017.

[AH17] L. Ambrosio, S. Honda: *Local spectral convergence in* $\mathrm{RCD}^*(K,N)$ *spaces*, ArXiv preprint: 1703.04939 (2017), to appear in Nonlinear Anal.

[AHT18] L. Ambrosio, S. Honda, D. Tewodrose: *Short-time behavior of the heat kernel and Weyl's law on* $\mathrm{RCD}^*(K,N)$*-spaces*, Ann. Glob. Anal. Geom. **53**(1) (2018), 97-119.

[AHPT18] L. Ambrosio, S. Honda, J. Portegies, D. Tewodrose: *Embedding of* $\mathrm{RCD}^*(K,N)$ *spaces in* L^2 *via eigenfunctions*, preprint.

[BS18] E. Brué, D. Semola: *Constancy of dimension for* $\mathrm{RCD}^*(K,N)$ *spaces via regularity of Lagrangian flows.* ArXiv preprint: 1803.04387 (2018).

[CC96] J. Cheeger, T. H. Colding: *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Annals of Math., Second Series **144**(1) (1996), 189–237.

[CC97] J. Cheeger, T. H. Colding: *On the structure of spaces with Ricci curvature bounded below. I.*, J. Differential Geom. **46**(3) (1997), 406–480.

[CC00a] J. Cheeger, T. H. Colding: *On the structure of spaces with Ricci curvature bounded below. II.*, J. Differential Geom. **54** (2000), 13–35.

[CC00b] J. Cheeger, T. H. Colding: *On the structure of spaces with Ricci curvature bounded below. III.*, J. Differential Geom. **54** (2000), 37–74.

[G13] N. Gigli: *The splitting theorem in non-smooth context*, ArXiv preprint: 1302.5555 (2013).

[G15a] N. Gigli: *On the differential structure of metric measure spaces and applications*, Mem. Am. Math. Soc. **236** (2015), no. 1113.

[LV09] J. Lott, C. Villani: *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. **169** (2009), 903–991.

[MN14] A. Mondino, A. Naber: *Structure Theory of Metric-Measure Spaces with Lower Ricci Curvature Bounds I*, to appear in JEMS, ArXiv preprint: 1405.2222 (2014).

[St06] K.-T. Sturm: *On the geometry of metric measure spaces, I and II*, Acta Math. **196** (2006), 65–131 and 133–177.

SIMONS CONE AND SADDLE SOLUTIONS OF THE ALLEN-CAHN EQUATION

KELEI WANG
(A JOINT WORK WITH YONG LIU AND JUNCHENG WEI)

There is an intricate connection bewteen the Allen-Cahn equation

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n, \quad |u| < 1, \tag{0.1}$$

and minimal surfaces. De Giorgi [6] conjectured that for $n \leq 8$, if a solution to (0.1) is monotone in one direction, then its level sets are all hyperplanes. De Giorgi conjecture is parallel to the Bernstein conjecture in minimal surface theory, which states that if $F : \mathbb{R}^n \to \mathbb{R}$ is a solution to the minimal surface equation, then F must be a linear function in its variables.

The Bernstein conjecture has been shown to be true for $n \leq 7$, see [11]. The famous Bombieri-De Giorgi-Giusti minimal graph (see [2]) gives a counter-example for $n \geq 8$. As for the De Giorgi conjecture, it has been proven to be true for $n = 2$ (Ghoussoub-Gui [8]), $n = 3$ (Ambrosio-Cabré [1]), and for $4 \leq n \leq 8$ under an additional limiting condition (Savin [10]). Counterexamples for $n \geq 9$ have been constructed in [7].

Savin [10] also proved that if u is a global minimizer in $\mathbb{R}^n$ and $n \leq 7$, then u is one dimensional. While the monotone solutions of Del Pino-Kowalczyk-Wei provides examples of nontrivial global minimizers in dimension $n \geq 9$, it is still not known whether there are nontrivial global minimizers for $n = 8$.

It is known that in $\mathbb{R}^8$, there is a minimal cone with one singularity at the origin which minimizes the area, called Simons cone. It is given explicitly by:

$$\left\{x_1^2 + ... + x_4^2 = x_5^2 + ... + x_8^2\right\}.$$

The minimality (area-minimizing property) of this cone is proved in [2].

It turns out there are analogous objects as Simons cone in the theory of Allen-Cahn equation. They are the so-called saddle-shaped solutions, which are solutions in $\mathbb{R}^{2m}$ of (0.1) vanishing exactly on the Simons cone (Cabré-Terra [3, 4] and Cabré [5]). In [3, 4] it is proved that for $2 \leq m \leq 3$, saddle solutions are unstable, while for $m \geq 7$, they are stable ([5]). It is also conjectured in [5] that for $m \geq 4$, the saddle solution should be a global minimizer.

Our main result is (see [9] for more details) For any $m \geq 4$, there exists a global minimizer $U(a)$ of the Allen-Cahn equation (0.1) indexed by $a \in \mathbb{R}$.

If one could show that this family of solutions U_a depends continuously on a and is ordered, then the saddle solution will be a global minimizer. However, our variational construction gives no information on the ordering and continuity in a.

References

[1] L. Ambrosio; X. Cabré, Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi.*J. Amer. Math. Soc.* 13 (2000), no. 4, 725–739.
[2] E. Bombieri; De Giorgi; Giusti, Minimal cones and the Bernstein problem. *Invent. Math.* 7 (1969), 243–268.

[3] X. Cabré; J. Terra, Saddle-shaped solutions of bistable diffusion equations in all of $\mathbb{R}^{2m}$. *J. Eur. Math. Soc. (JEMS)* 11 (2009), no. 4, 819–843.
[4] X. Cabré; J. Terra, Qualitative properties of saddle-shaped solutions to bistable diffusion equations. *Comm. Partial Differential Equations* 35 (2010), no. 11, 1923–1957.
[5] X. Cabré, Uniqueness and stability of saddle-shaped solutions to the Allen-Cahn equation. *J. Math. Pures Appl.* (9) 98 (2012), no. 3, 239–256.
[6] E. De Giorgi, Convergence problems for functionals and operators, Proc. Int. Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), 131188, Pitagora, Bologna (1979).
[7] M. del Pino; M. Kowalczyk; J. Wei, On De Giorgi's conjecture in dimension $N \geq 9$. *Ann. of Math.* (2) 174 (2011), no. 3, 1485–1569.
[8] N. Ghoussoub; C. Gui, On a conjecture of De Giorgi and some related problems.*Math. Ann.* 311 (1998), no. 3, 481–491.
[9] Y. Liu; K. Wang; J. Wei, Global minimizers of Allen-Cahn equation in dimensions $n \geq 8$. J. Math. Pure Appl. 108 (2017), no. 6, 818-840.
[10] O. Savin, Regularity of flat level sets in phase transitions. *Ann. of Math.* (2) 169 (2009), no. 1, 41–78.
[11] J. Simons, Minimal varieties in riemannian manifolds. *Ann. of Math.* (2) 88(1968), 62–105.

EDGE-CONE EINSTEIN METRICS AND YAMABE METRICS

KAZUO AKUTAGAWA*

Abstract. For any closed smooth conformal n-manifold (M, C) $(n \geq 3)$, the Yamabe problem is always solvable (cf. [8], [10]), that is, a minimizer $\check{g} \in C$ (i.e, a *Yamabe metric*) of the functional I below always exists:

$$I \ : \ C \to \mathbb{R}, \quad g \mapsto \frac{\int_M R_g d\mu_g}{V_g^{(n-2)/n}} .$$

Here, $R_g, d\mu_g$ and V_g denote respectively the scalar curvature and the volume measure of g, and the volume of (M, g).

Contrary to the smooth case, the Yamabe is not always solvable for the singular case, even for compact orbifolds (with finitely many singularities). The class of *almost Riemannian metric-measure spaces* is one of appropriate wide classes of singular spaces to consider the Yamabe problem (cf. [2]), which includes orbifolds, conic manifolds, simple edge-spaces, more over iterated edge-spaces. The Aubin's inequality for Yamabe constants can be generalized to the one for Yamabe constants $Y(X, [g])$ of almost Riemannian metric-measure space (X, g) as

$$Y(X, [g]) \leq Y_\ell(X, [g]) \ \cdots (*) .$$

Here, $Y(X, [g])$ is called the *local Yamabe constant* of $(X, [g])$, which is a generalization of the Yamabe constant $Y(S^n, [g_{\mathbb{S}}])$ of the standard round n-sphere $(S^n, g_{\mathbb{S}})$. In [2], we have proved that, when $Y(X, [g]) < Y_\ell(X, [g])$, a *singular* Yamabe metric always exists. However, Viaclovsky [11] has proved that some compact 4-orbifolds satisfying the equality in $(*)$ do not admit orbifold Yamabe metrics. His proof is a nice modification of the proof of Obata's Theorem for Einstein metrics.

In this talk, we show a second example of singular spaces which have no Yamabe metrics.

Definition. (cf. [6]) The standard round metric $g_{\mathbb{S}} = g_{S^n}$ of constant curvature 1 on the n-sphere S^n can be written as a doubly warped product

$$g_{\mathbb{S}} = dr^2 + \sin^2 r d\theta^2 + \cos^2 r \cdot g_{S^{n-2}} =: h_1$$

on $S^n - (S^{n-2} \sqcup S^1) = (0, \frac{\pi}{2}) \times S^1 \times S^{n-2} \ni (r, \theta, x)$. For each $\beta > 0$ $(\beta \neq 1)$, we define the *standard edge-cone metric* h_β *of cone angle* $2\pi\beta$ *on* (S^n, S^{n-2}) by

$$h_\beta := dr^2 + \beta^2 \sin^2 r d\theta^2 + \cos^2 r \cdot g_{S^{n-2}} .$$

Since each h_β is locally isomorphic to $h_1 = g_{\mathbb{S}}$, h_β is of constant curvature 1, and hence it is an edge-cone Einstein metric.

Our main result is the following:

Theorem ([4]). For any $\beta \geq 2$, there is no (edge-cone) Yamabe metric on $(S^n, [h_\beta])$.

Remark. (1) Our proof depends on a branched covering version of *Aubin's Lemma for finite coverings* [7], [5], which is quite different from Viaclovsky's one. We also note that the dimension of the singularities of Viaclovsky's example is 0. On the other

Date: April, 2018.

* supported in part by the Grants-in-Aid for Scientific Research (B), Japan Society for the Promotion of Science, No. 18H01117.

hand, the one of our example is $n-2$, which is the highest one among the dimensions of singularities.
(2) We conjecture that, for any β $(1<\beta<2)$, the same conclusion as that in Theorem B still holds.

Outline of Proof of Theorem. Let $P:(S^n,h_\beta)\to(S^n,h_{\beta/2})$ be the natural branched double covering along S^{n-2}. One can check that the Aubin type Lemma still holds for the branched covering P, that is,

$$Y(S^n,[h_\beta])>Y(S^n,[h_{\beta/2}])$$

provided that
(1) the existence of a Yamabe metric $u^{4/(n-2)}\cdot h_\beta$ $(u>0)$ on $(S^n,[h_\beta])$,
(2) the following equality holds:

$$\int_{S^n} u\,\Delta_{h_\beta}u\,d\mu_{h_\beta}=-\int_{S^n}|\nabla u|^2 d\mu_{h_\beta}.$$

Suppose that the assertion (1) holds. Set $r(p):=\mathrm{dist}_{h_\beta}(p,S^{n-2})$ for $p\in S^n$. By the regularity result of solutions for the Yamabe-type equations in [2, 3], on each ε-open neighborhood $U_\varepsilon:=U_\varepsilon(S^{n-2})$

$$0<c\le u\le C,\qquad \partial_r u(r,\theta,x)=O(r^{\frac{1}{\beta}-1})\quad\text{as}\quad r\searrow 0.$$

(The regularity also implies that the Yamabe metric is an edge-cone metric.) Hence,

$$\int_{S^n-U_\varepsilon} u\,\Delta_{h_\beta}u\,d\mu_{h_\beta}+\int_{S^n-U_\varepsilon}|\nabla u|^2 d\mu_{h_\beta}=\int_{\partial U_\varepsilon} u\,\partial_r u\,d\sigma_{h_\beta}=O(\varepsilon^{\frac{1}{\beta}})\searrow 0$$

as $\varepsilon\searrow 0$. Then, the assertion (2) holds. On the other hand, if $\beta/2\ge 1$, Mondello's result [9] (cf. [1]) implies that

$$Y(S^n,[h_\beta])=Y(S^n,[g_S])=Y(S^n,[h_{\beta/2}]).$$

This contradicts the previous strict inequality. $\square$

References

[1] K. Akutagawa, *Constant scalar curvature metrics on the edge-cone sphere*, in preparation.
[2] K. Akutagawa, G. Carron and R. Mazzeo, *The Yamabe problem on stratified spaces*, Geom. Funct. Anal. **24** (2014), 1039–1079.
[3] K. Akutagawa, G. Carron and R. Mazzeo, *The Hölder regularity of solutions for Schrödinger operators on stratified spaces*, J. Funct. Anal. **269** (2015), 815-840.
[4] K. Akutagawa and I. Mondello, *Edge-cone Einstein metrics and the Yamabe invariant*, in preparation.
[5] K. Akutagawa and A. Neves, *3-manifolds with Yamabe invariant greater than that of* $\mathbb{RP}^3$, J. Differential Geom. **75** (2007), 13–37.
[6] M. Atiyah and C. LeBrun, *Curvature, cones and characteristic numbers*, Math. Proc. Cambridge Phil. Soc. **155** (2013), 13–37.
[7] T. Aubin, *The scalar curvature*, Differential Geometry and Relativity, Math. Phys. and Appl. Math. **3**, 5–18, Reidel, Dordrecht, 1976.
[8] J. Lee and T. H. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), 37–91.
[9] I. Mondello, *The local Yamabe constant of Einstein stratified spaces*, Ann. Inst. H. Poincaré Anal. Non linéaire **34**, no. 1, 249–275.
[10] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Lecture Notes in Math. **1365**, Springer, 1989, pp. 120–154.
[11] J. Viaclovsky, *Monopole metrics and the orbifold Yamabe problem*, Ann. Inst. Fourier (Grenoble) **60** (2010), 2503–2543.

E-mail address: akutagawa@math.chuo-u.ac.jp

Department of Mathematics, Chuo University, Tokyo 112-8551, Japan

Evolution by curvature of planar networks

Matteo Novaga, University of Pisa

Abstract

We give an overview of the state–of–the–art of the motion by curvature of planar networks collecting classical, recent and new results.

A strong motivation to studying this flow is the analysis of models of two–dimensional multiphase systems where the problem of the dynamics of the interfaces between different phases arises naturally. As an example, the model where the energy of a configuration is simply the total length of the interfaces has proven to be useful to model the growth of grain boundaries in the ideal case of a pure metal.

Another motivation is more theoretical: the evolution of such a network of curves is the simplest example of motion by mean curvature of a set which is essentially singular. Despite the mean curvature flow of a smooth submanifold has been extensively studied and deeply, even if not completely, understood, the evolution of generalized possibly singular submanifolds (for instance varifolds) presents a series of different challenge that the classical results cannot solve.

To treat the case of the motion by curvature of non–regular sets many generalized (and weak) definitions of the flow has been introduced in the literature. In his seminal work K. Brakke proved existence of a global (very) weak solution, in a geometric measure theory context, called “Brakke flow”. Unfortunately it is not granted that his solution is not trivial as the empty set is always a solution of the Brakke flow. Recently this issue was overcome by L. Kim and Y. Tonegawa in the case of the evolution of grain boundaries in $\mathbb{R}^n$. They proved a global existence theorem and they also show that there exists a finite family of open sets which move continuously with respect to the Lebesgue measure and whose boundaries coincide with the space-time support of the mean curvature flow. For a global existence result in any codimension and with special regularity properties, obtained adapting the elliptic regularization scheme of Ilmanen. Despite these recent improvements Brakke’s definition is apparently too weak, if one is interested in a detailed description of the flow. Another definition of evolution is based on *minimizing movements*: an implicit variational scheme introduced by F. Almgren, J. Taylor and L. Wang. The functional to be minimized in this time–discrete variational scheme is a perturbation of the length functional. Hence at every step the minima automatically satisfy the Herring condition namely at the junctions the sum of the unit tangent vectors of the concurring curves is zero. A very recent conditional compactness theorem by T. Laux and F. Otto give a limit flow sharing such a property at almost every time. Another possibility is the “level set” approach by L. C. Evans and J. Spruck or, alternatively, by Y. G. Chen, Y. Giga and S. Goto.

Notice that all these approaches provide a globally defined evolution. Unfortunately they also have in common a not negligible issue: the solutions are not unique.

In our context the definition of the flow is the first problem one has to face, due to the contrast between the intrinsic singular nature of a network and the reasonable desire to have something as “smooth” as possible. Our choice is suggested by the variational nature of the problem. The flow can be regarded as the gradient flow in the space of networks of the Length

functional (the sum of the lengths of all the curves of the network). At every internal point of the curves of the network (not at the end–points of the curves) the normal velocity that induces the steepest descent of the energy with respect to the L^2–inner product is the curvature vector of the curve. It must anyway be said that the ”space of networks” does not share a natural linear structure and such a “gradient” is not well defined at the multiple junctions. At the multi–points the unit tangent vectors of the concurring curves have to sum up to zero: this is only information at the junctions that comes from understanding the motion as a gradient flow. From the energetic point of view it is natural to expect that configurations with multi–points of order greater than three, being unstable for the length functional, should be present only in the initial network or that they should appear only at some discrete set of times during the flow. This property is also suggested by numerical simulations and physical experiments. We call *regular* a network that presents only triple junctions where the unit tangent vectors of the three concurrent curves form angles of 120 degrees. Following the “energetic” and experimental motivations we simply impose such regularity condition in the definition of a curvature flow for every positive time (at the initial time it could fail). If the initial network is regular and smooth enough, we will see that this definition leads to an almost satisfactory, “classical” short time existence theory.

When instead the initial datum is a non–regular network (namely a network with multi–points of order greater than three of triple junction not satisfying the Herring condition), several difficulties arise. Consider for instance the simplest example of a network described by two curves crossing each other forming a 4–point. There is definitely more than one good candidate for the flow: the concurrent curves can separate in two pairs of curves moving independently, or they can stay attached at the 4–point (but it is not clear how the angles should behave). It could also be allowed the possible “birth” of new multi–points from such a single one. One would like to have a definition of the flow that on one hand takes into account all the just describe plausible behaviors, but on the other hand that get uniqueness of the motion at least for “generic” initial data and ”generically” force the network by a instantaneous regularization to have only triple junctions satisfying the Herring condition, with the possible exception of some discrete set of times. One may expect that some sort of parabolic regularization could play a role. For instance if only two curves meet at a multi–point, it can be shown that the two curves become instantaneously a single smooth curve moving by curvature. If at the junctions concur more that three curves at the moment there is not complete answer to the behavior of the flow.

We underline that even starting with an initial regular network, in the long time behavior analysis one has to deal with non–regular networks. Indeed during the flow some of the triple junctions could “collide” along a “collapsing” curve of the network, when the length of the latter goes to zero (hence modifying the topological structure of the network). If this is the case, one has to ”restart” the evolution with an initial datum which has *multi–points* of order higher than three.

Global existence and degeneracy in finite time for 1D quasilinear wave equations

YUUSUKE SUGIYAMA*
The University of Shiga Prefecture

In this talk, we consider the Cauchy problem of the following quasilinear wave equation:

$$\begin{cases} \partial_t^2 u = \partial_x((1+u)^{2a}\partial_x u), & (t,x) \in (0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $u(t,x)$ is an unknown real valued function and $a > 0$. The wave equation (1) arises in the theory of elasticity and elasto-plasticity.

Throughout this talk, we always assume that there exists a constant $c_0 > 0$ such that

$$1 + u_0(x) \geq c_0 \tag{2}$$

for all $x \in \mathbb{R}$. This assumption enables us to regard the equation in (1) as a strictly hyperbolic equation near $t = 0$. By the standard local existence theorem for strictly hyperbolic equations, the local solution of (1) with smooth initial data uniquely exists until the one of the following two phenomena occurs. The first one is the blow-up:

$$\overline{\lim_{t \nearrow T^*}} (\|\partial_t u(t)\|_{L^\infty} + \|\partial_x u(t)\|_{L^\infty}) = \infty.$$

The second is the degeneracy of the equation:

$$\lim_{t \nearrow T^*} \inf_{(s,x)\in[0,t]\times\mathbb{R}} 1 + u(s,x) = 0.$$

When the equation degenerates, the standard local existence theorem does not work since the equation in (1) loses the strictly hyperbolicity. The aim of this talk is to obtain a sufficient condition for the occurrence of the degeneracy of the equation in finite time.

Theorem 1. Let $(u_0, u_1) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$. Suppose that the initial data (u_0, u_1) satisfy that (2),

$$u_1(x) \pm (1 + u_0(x))^a \partial_x u_0(x) \leq 0 \quad \text{for all} \quad x \in \mathbb{R} \tag{3}$$

and

$$\int_{\mathbb{R}} u_1(x)dx < \frac{-2}{a+1}. \tag{4}$$

Then there exists $T^* > 0$ such that a local unique solution $u \in C([0,T^*); H^3(\mathbb{R})) \cap C^1([0,T^*); H^2(\mathbb{R}))$ of (1) exists and the degeneracy occurs at T^*.

The equation in (1) is formally equivalent to the following 2×2 hyperbolic conservation system:

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} - \partial_x \begin{pmatrix} V \\ \dfrac{(1+U)^{2a+1}}{2a+1} \end{pmatrix} = 0$$

where $U(t,x) = u(t,x)$, $V(t,x) = \int_{-\infty}^x \partial_t u(t,y)dy$ and u is a solution to the equation in (1). The global existence of solution to more general 2×2 conservation systems has been known (e.g. Johnson [1] and Yamaguchi and Nishida [5]). By applying the global existence theorem, we can show the following proposition.

*e-mail:sugiyama.y@e.usp.ac.jp

Proposition 2. Let $(u_0, u_1) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$. Suppose that the initial data (u_0, u_1) satisfy that (2), (3) and

$$\int_{\mathbb{R}} u_1(x)dx > \frac{-2}{a+1}.$$

Then (1) has a global unique solution $u \in C([0,\infty); H^3(\mathbb{R})) \cap C^1([0,\infty); H^2(\mathbb{R}))$ satisfying that there exists a constant $c_1 > 0$ such that

$$1 + u(t,x) \geq c_1$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}$.

This proposition and our main theorem say that $\frac{-2}{a+1}$ is a threshold of $\int_{\mathbb{R}} u_1(x)dx$ separating the global existence of solutions (such that the equation does not degenerate) and the degeneracy of the equation under the assumption of (3). The assumption (3) means that Riemann invariants are decreasing functions with x at $t = 0$. The decreasing property of the Riemann invariants conserves for $t \geq 0$. If (3) is not satisfied, solutions can blow up in finite time.

We give a known result on the degeneracy of the following parameterized equation:

$$\partial_t^2 u = (1+u)^{2a} \partial_x^2 u + a\lambda(1+u)^{2a-1}(\partial_x u)^2, \tag{5}$$

where $0 \leq \lambda \leq 2$. We consider this equation with initial data (u_0, u_1). When $\lambda = 0, 1$ or 2, the equation of (5) has some physical backgrounds. (5) with $\lambda = 2$ is the equation in (1). If $\lambda \neq 2$, then the equation of (5) does not have the structure of conservation system. However, the decreasing property of the Riemann invariant conserves for $t \geq 0$, which is a motivation to consider the equation (5). In [2, 4], the author has shown that the equation of (5) with $0 \leq \lambda < 2$ can degenerate regardless of $\int_{\mathbb{R}} u_1(x)dx$ as follows:

Theorem 3. Let $0 \leq \lambda < 2$, $(u_0, u_1) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$ and $u_1 \not\equiv 0$. Suppose that the initial data (u_0, u_1) satisfy that (2) and (3) Then there exists $T^* > 0$ such that a solution $u \in C([0,T^*); H^3(\mathbb{R})) \cap C^1([0,T^*); H^2(\mathbb{R}))$ of (5) exists uniquely and that the degeneracy occurs at T^*.

Our proof of Theorems 1 and 3 are based on a contradiction argument for the function $F(t) = -\int_{\mathbb{R}} u(t,x) - u_0(x)dx$. The idea for the proof is to divide the real line $\mathbb{R}$ into three parts by using the plus and minus characteristic curves in the estimate of $F(t)$.

References

[1] J. L. Johnson, Global continuous solutions of hyperbolic systems of quasilinear equations, Bull. Amer. Math. Soc., 73 (1967), pp. 639-641.

[2] Y. Sugiyama, Global existence of solutions to some quasilinear wave equation in one space dimension, Differential Integral Equations, 6 (2013), pp. 487-504.

[3] Y. Sugiyama, Degeneracy in finite time of 1D quasilinear wave equations, SIAM J. Math. Anal., 48 (2016) 847-870.

[4] Y. Sugiyama, Degeneracy in finite time of 1D quasilinear wave equations II, Evolution Equations and Control Theory, 6 (2017) 615-628.

[5] M. Yamaguchi and T. Nishida, On some global solution for quasilinear hyperbolic equations, Funkcial. Ekvac., 11 (1968), pp. 51-57.

The blow-up curve of solutions for semilinear wave equations with Dirichlet boundary conditions in one space dimension

Takiko Sasaki*
School of Science and Technology, Meiji University

This talk is based on the joint work with Professor Tetsuya Ishiwata.

We consider the nonlinear wave equation

$$\begin{cases} (\partial_t^2-\partial_x^2)u = 2^p|\partial_t u|^{p-1}\partial_t u & x\geq 0, \quad t\geq 0, \\ \partial_t u(0,t)=0 & t\geq 0, \\ u(x,0)=u_0(x), \quad \partial_t u(x,0)=u_1(x) & x\geq 0, \end{cases} \tag{1}$$

where $u=u(x,t)$ is a real-valued unknown function. In this talk, we assume that p is a positive constant such that $p=2,3$ or $p\geq 4$.

We introduce $\phi=\phi(x,t)$ and $\psi=\psi(x,t)$ defined by

$$\phi=\partial_t u+\partial_x u, \quad \psi=\partial_t u-\partial_x u.$$

Then, we can rewrite (1) as follows

$$\begin{cases} (\partial_t-\partial_x)\phi = |\phi+\psi|^{p-1}(\phi+\psi) & x\geq 0, \quad t\geq 0, \\ (\partial_t+\partial_x)\psi = |\phi+\psi|^{p-1}(\phi+\psi) & x\geq 0, \quad t\geq 0, \\ (\phi+\psi)(0,t)=0 & t\geq 0, \\ \phi(x,0)=f(x), \quad \psi(x,0)=g(x) & x\geq 0, \end{cases} \tag{2}$$

where $f(x)=u_1(x)+\partial_x u_0(x)$ and $g(x)=u_1(x)-\partial_x u_0(x)$.

Let Q^*, R^*, S^* and T^* be positive constants and

$$B_{R_1,R_2}=\{x\in\mathbb{R} \mid R_1<x<R_2\}, \quad K_-(x_0,t_0)=\{(x,t)\in\mathbb{R}^2 \mid x,t,>0, \ |x_0-x|<t_0-t\},$$

$$K_1=\bigcup_{x\in B_{Q^*+T^*,Q^*+T^*+R^*}} K_-(x,T^*), \quad K_2=K_-(0,S^*), \quad K=K_1\cup K_2.$$

where R_1, R_2, x_0 and t_0 are nonnegative constants. In what follows, we assume $Q^*+T^*< S^*<Q^*+2T^*$.

We then consider the following function

$$T(x)=\sup\{t\in(0,T^*) \mid \ |\partial_t u(x,t)|<\infty\} \quad \text{for} \quad x\in B_{0,Q^*+T^*+R^*}.$$

The function $T(x)$ is the blow-up time of the classical solution of (1) at x.

The blow-up curve has been studied from the viewpoints of its differentiability and singularity. For example, the differentiability of the blow-up curve was proved by Caffarelli and Friedman [CF] if the initial conditions are large and smooth enough. On the other hand, Merle and Zaag [MZ] showed that the blow-up curve has singular points under suitable initial conditions.

*Email : takiko@meiji.ac.jp

However, it is not clear for the author why the blow-up curve has singular points. In this talk, we consider when the blow-up curve has singular points for (1) and we limit the discussion to the case $\partial_t u$ is an odd function. Particular, we show

$$T'(x) \downarrow -1 \quad \text{as} \quad x \downarrow 0$$

under suitable initial conditions.

To state main results, we will make the following assumptions:

$$\begin{cases} \dfrac{d}{dt}\tilde{\phi} = |\tilde{\phi}+\tilde{\psi}|^{p-1}(\tilde{\phi}+\tilde{\psi}) & t \ge 0, \\ \dfrac{d}{dt}\tilde{\psi} = |\tilde{\phi}+\tilde{\psi}|^{p-1}(\tilde{\phi}+\tilde{\psi}) & t \ge 0, \\ \tilde{\phi}(0) = \gamma, \quad \tilde{\psi}(0) = 0. \end{cases}$$

where $\tilde{\phi} = \tilde{\phi}(t)$ and $\tilde{\psi} = \tilde{\psi}(t)$ are real-valued unknown functions and γ is a positive constant.

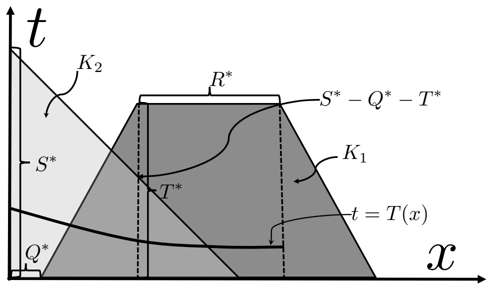


Figure 1. The dependence domain.

Then, there exists a positive constant $\tilde{T}$ depending only on γ such that

$$(\tilde{\phi}+\tilde{\psi})(t) \to \infty \quad \text{as} \quad t \to \tilde{T}.$$

We make the following assumptions.

(A1) $\tilde{T} < S^* - Q^* - T^*$.

(A2) $\begin{cases} f \ge \gamma, \quad g \ge -\gamma \text{ in } B_{0,Q^*+R^*+2T^*}, \\ g \ge 0 \text{ in } B_{Q^*,Q^*+R^*+2T^*}. \end{cases}$

(A3) $f, g \in C^4(B_{0,Q^*+R^*+2T^*})$.

(A4) $\exists \varepsilon_0 > 0$ such that

$$2\gamma^p \ge (1+\varepsilon_0)(|\partial_x f| + |\partial_x g|) \quad \text{in } B_{Q^*,Q^*+R^*+2T^*}.$$

(A5.1) $\partial_x f \ge 0,\ \partial_x g > 0$ in B_{0,S^*}.

(A5.2) $\partial_x^2 f,\ \partial_x^2(f+g) \ge 0$ in B_{0,S^*}.

(A5.3) $\partial_x^3 f,\ \partial_x^3 g \ge 0$ in B_{0,S^*}.

(A6) $\partial_x(f-g) \ge 0$ in B_{0,S^*}.

We now state the main results of this paper.

Theorem 1. *Let Q^*, R^*, S^* and T^* be arbitrary positive constants. Assume that $Q^t * +T^* < S^* < Q^*+2T^*$ and* (A1)–(A6) *hold. Then, there exists a unique $C^1(B_{0,Q^*+R^*+T^*})$ function T such that $0 < T(x) < S^*$ $(x \in B_{0,Q^*+R^*+T^*})$ and a unique $(C^{3,1}(\Omega))^2$ solution (ϕ,ψ) of* (2) *satisfying*

$$\phi(x,t),\ \psi(x,t) \to \infty \quad \text{as} \quad t \to T(x)$$

for any $x \in B_{0,Q^*+R^*+T^*}$, where $\Omega = \left\{(x,t) \in \mathbb{R}^2 \mid x \in B_{0,Q^*+R^*+T^*},\ 0 < t < T(x)\right\}$.

Moreover,

$$T'(x) \downarrow -1 \quad \text{as} \quad x \downarrow 0.$$

References

[CF] L. A. Caffarelli and A. Friedman, Trans. Amer. Math. Soc., **297** (1986), 223–241.

[MZ] F. Merle and H. Zaag, Amer. J. Math., **134** (2012), 581–648.

A novel integral transform approach to solving hyperbolic equations in the curved space-times

Karen Yagdjian

School of Mathematical and Statistical Sciences,
University of Texas RGV, 1201 W. University Drive,
Edinburg, TX 78539, USA

Abstract

In the talk we will present the integral transform that allows us to construct solutions of the hyperbolic partial differential equations with variable coefficients via solutions of a simpler equation. This integral transform is aimed to generate representation formulas for the solutions to the equation

$$u_{tt} - a^2(t)A(x,\partial_x)u - M^2u = f, \qquad t \in (0,T),\ x \in \Omega\,. \tag{1}$$

where $M \in \mathbb{C}$ and $A(x,\partial_x)$ is elliptic operator $A(x,\partial_x) = \sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha$, with smooth coefficients $a_\alpha(x) \in C^\infty(\Omega)$ in the domain in $\Omega \subseteq \mathbb{R}^n$. For the given smooth function $f = f(x,t)$ defined in $\Omega \times [0,T]$ consider a solution $w = w(x,t;b)$ of the problem

$$\begin{cases} w_{tt} - A(x,\partial_x)w = 0, & t \in (0,T_1),\ \ x \in \Omega, \\ w(x,0;b) = f(x,b), & w_t(x,0;b) = 0, \quad x \in \Omega, \end{cases} \tag{2}$$

with the parameter $b \in (0,T)$ and $0 < T_1 \le \infty$. We define the operator $\mathcal{E}\mathcal{E}_A$ by $w = \mathcal{E}\mathcal{E}_A[f]$. Next we introduce the integral operator

$$\mathcal{K}\,:\, w \longmapsto u,$$

which maps function $w = w(x,t;b)$ into solution u of the equation (1). The integral operator $\mathcal{K}$ is defined by

$$u(x,t) \;=\; \mathcal{K}[w](x,t) := \int_0^t db \int_0^{|\phi(t)-\phi(b)|} K(t;r,b;M)w(x,r;b)\,dr, \quad x \in \Omega,\ t \in (0,T).$$

Here $\phi(t) = \int_0^t a(\tau)\,d\tau$ is a distance function produced by $a = a(t)$, $M \in \mathbb{C}$. Integral transform $\mathcal{K}$ is applicable to the distributions and fundamental solutions as well. In fact, $u = u(x,t)$ takes initial values $u(x,0) = 0,\ u_t(x,0) = 0, \quad x \in \Omega$. This operator generates solutions of different well-known equations. A class of operators for which we have obtained representation formulas for the solutions includes $a(t) = t^\ell,\ \ell \in \mathbb{R},\ a(t) = e^{\pm t}$. If $\mathcal{E}\mathcal{E}_A$ is a resolving operator of the problem (2), then, by applying integral transform, we obtain the solution operator $u = \mathcal{G}_A[f] = \mathcal{K} \circ \mathcal{E}\mathcal{E}_A[f]$.

Klein-Gordon equation in the de Sitter space-time. Here $a(t) = e^{-t}$. This is the first important application of the our integral transform approach. Define a *chronological future* $D_+(x_0,t_0)$ and a *chronological past* $D_-(x_0,t_0)$ of the point (x_0,t_0), $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$: $D_\pm(x_0,t_0) := \{(x,t) \in \mathbb{R}^{n+1}\,;\, |x - x_0| \le \pm(e^{-t_0} - e^{-t})\,\}$. Then, for $(x_0,t_0) \in \mathbb{R}^n \times \mathbb{R}$, $M \in \mathbb{C}$, we define the function

$$\begin{aligned} E(x,t;x_0,t_0;M) \;\; := \;\; & 4^{-M}e^{M(t_0+t)}\Big((e^{-t_0}+e^{-t})^2 - (x-x_0)^2\Big)^{M-\frac{1}{2}} \\ & \times F\Big(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(e^{-t_0}-e^{-t})^2 - (x-x_0)^2}{(e^{-t_0}+e^{-t})^2 - (x-x_0)^2}\Big), \end{aligned}$$

where $(x,t) \in D_+(x_0,t_0) \cup D_-(x_0,t_0)$ and $F\big(a,b;c;\zeta\big)$ is the hypergeometric function. In fact, E depends on $r^2 = (x-x_0)^2$, that is $E(x,t;x_0,t_0;M) = E(r,t;0,t_0;M)$. For this equation we set

$K(t;r,b;M) = E(r,t;0,b;M)$. We note that the operator $A(x,\partial_x)$ is of arbitrary order, that is, the equation of (2) can be an evolution equation, not necessarily hyperbolic.
The Huygens' principle. In [8] it is proved that the value $m = \sqrt{n^2-1}/2$ is the only value of the physical mass m, such that the equation $\Phi_{tt}+n\Phi_t-e^{-2t}\Delta\Phi+m^2\Phi=0$, obeys the Huygens' principle, $n \geq 3$ is an odd number. Moreover, it is also proved in [8] that two Klein-Gordon equations obey the incomplete (that is the second initial datum vanishes) Huygens' principle if and only if $n = 3$ and $m_1 = 0$, $m_2 = \sqrt{2}$. In fact, Paul Ehrenfest in 1918 raised the question: "Why has our space just three dimensions?". In quantum field theory the interval $(0, \sqrt{2})$ is known as the so-called *Higuchi bound*. In the talk a special attention will be given also to the global in time existence of self-interacting scalar field in the de Sitter universe, to the Higuchi bound of the quantum field theory, and the Klein-Gordon equations with the Higgs potential.
The Einstein-de Sitter space-time, the generalized Tricomi equation. For this equation $M = 0$ and $a^2(t) = t^\ell$, $\ell \in \mathbb{C}$. We define the integral operator with the kernel

$$K(t;r,b) := c_\ell \left((\phi(t)+\phi(b))^2 - r^2\right)^{-\gamma} F\Big(\gamma,\gamma;1;\frac{(\phi(t)-\phi(b))^2-r^2}{(\phi(t)+\phi(b))^2-r^2}\Big),$$

with the distance function $\phi(t) = 2t^{(\ell+2)/2}/(\ell+2)$ and the numbers $\gamma := \ell/(2(\ell+2))$, $\ell \in \mathbb{C}\setminus\{-2\}$, and $c_\ell = (\ell+2)^{-\frac{\ell}{\ell+2}}\, 4^{\frac{\ell}{\ell+2}}$. There are four important examples of equations which are amenable to the integral transform approach, when $\ell = 3, 1, -1, -4/3$; those are the small disturbance equations for the perturbation velocity potential of a two-dimensional near sonic uniform flow of dense gases in a physical plane, the Tricomi equation, the equation of waves in the radiation dominated universe and in the EdeS space-time, respectively.

References

[1] Galstian, A., Kinoshita, T., Yagdjian,K.: A note on wave equation in Einstein and de Sitter space-time. J. Math. Phys. **51** (5), 052501 (2010)

[2] Galstian, A., Yagdjian, K.: Global solutions for semilinear Klein-Gordon equations in FLRW spacetimes. Nonlinear Anal. **113**, 339–356 (2015)

[3] Galstian, A., Yagdjian, K.: Global in time existence of self-interacting scalar field in de Sitter spacetimes. Nonlinear Anal. Real World Appl. 34 (2017), 110-139.

[4] Yagdjian, K.: Global existence for the n-dimensional semilinear Tricomi-type equations. Comm. Partial Diff. Equations, **31**, 907-944 (2006)

[5] Yagdjian, K., Galstian A.: Fundamental Solutions for the Klein-Gordon Equation in de Sitter Spacetime. Comm. Math. Phys., **285**, 293-344 (2009)

[6] Yagdjian, K.: Global existence of the scalar field in de Sitter spacetime. J. Math. Anal. Appl. **396** (1), 323–344 (2012)

[7] Yagdjian, K.: On the global solutions of the Higgs boson equation. Comm. Partial Differential Equations **37** (3), 447–478 (2012)

[8] Yagdjian, K.: Huygens' Principle for the Klein-Gordon equation in the de Sitter spacetime. J. Math. Phys. **54** (9), 091503 (2013)

[9] Yagdjian, K.: Integral transform approach to solving Klein-Gordon equation with variable coefficients. Mathematische Nachrichten, **288** (17-18), 2129-2152 (2015)

[10] Yagdjian, K.: Integral transform approach to generalized Tricomi equations. *J. Differential Equations* **259** (2015) 5927–5981.

[11] Yagdjian, K.: Integral transform approach to time-dependent partial differential equations. Mathematical analysis, probability and applications–plenary lectures, Springer Proc. Math. Stat., 177, Springer, 281–336 (2016)

On some effects of background metrics for several partial differential equations

Makoto Nakamura (Yamagata University) *

We denote the spatial dimension by $n \geq 1$, the Planck constant by $\hbar := h/2\pi$, the mass by $m > 0$, and the speed of light $c > 0$. Let $\sigma \in \mathbb{R}$, $a_0 > 0$, $a_1 \in \mathbb{R}$. We put $T_0 := \infty$ when $(1+\sigma)a_1 \geq 0$, $T_0 := -2a_0/n(1+\sigma)a_1 (> 0)$ when $(1+\sigma)a_1 < 0$. We define a scale-function $a(t)$ for $t \in [0, T_0)$ by

$$a(t) := \begin{cases} a_0 \left(1 + \frac{n(1+\sigma)a_1 t}{2a_0}\right)^{2/n(1+\sigma)} & \text{if } \sigma \neq -1, \\ a_0 \exp\left(\frac{a_1 t}{a_0}\right) & \text{if } \sigma = -1. \end{cases} \tag{1}$$

Let $\lambda \in \mathbb{C}$, $1 \leq p < \infty$, $-\pi/2 < \omega \leq \pi/2$, and $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$. We define a weight function $w(t) := (a_0/a(t))^{n/2}$. We consider the equation

$$\pm i \frac{2m}{\hbar} \partial_t u + \frac{1}{a^2 e^{2i\omega}} \Delta u - \frac{\lambda}{e^{2i\omega}} |uw|^{p-1} u = 0. \tag{2}$$

We use a change of variable $s = s(t) := \int_0^t a(\tau)^{-2} d\tau$. For $0 \leq \mu_0 < n/2$ and $0 < S \leq S_0$, we consider the Cauchy problem given by

$$\begin{cases} \pm i \frac{2m}{\hbar} \partial_s u(s,x) + \frac{1}{e^{2i\omega}} \Delta u(s,x) - \frac{\lambda a(s)^2}{e^{2i\omega}} \left(|uw|^{p-1} u\right)(s,x) = 0, \\ u(0,\cdot) = u_0(\cdot) \in H^{\mu_0}(\mathbb{R}^n) \end{cases} \tag{3}$$

for $(s,x) \in [0,S) \times \mathbb{R}^n$.

Let us consider the well-posedness of (3). For any real numbers $2 \leq q \leq \infty$ and $2 \leq r < \infty$, we say that the pair (q,r) is admissible if it satisfies $1/r + 2/nq = 1/2$. For $\mu_0 \geq 0$ and two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$, we define a function space

$$X^{\mu_0}([0,S)) := \{u \in C([0,S), H^{\mu_0}(\mathbb{R}^n));\ \max_{\mu=0,\mu_0} \|u\|_{X^\mu([0,S))} < \infty\}$$

with a metric $d(u,v) := \|u - v\|_{X^0([0,S))}$ for u, $v \in X^{\mu_0}([0,S))$, where

$$\|u\|_{X^\mu([0,S))} := \begin{cases} \|u\|_{L^\infty((0,S),L^2(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S),L^{r_j}(\mathbb{R}^n))} & \text{if } \mu = 0, \\ \|u\|_{L^\infty((0,S),\dot{H}^\mu(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S),\dot{B}^\mu_{r_j 2}(\mathbb{R}^n))} & \text{if } \mu > 0. \end{cases}$$

*Faculty of Science, Yamagata University, Kojirakawa-machi 1-4-12, Yamagata 990-8560, JAPAN. E-mail: nakamura@sci.kj.yamagata-u.ac.jp

We note that the scaling critical number of p for (3) is $p(\mu_0) := 1 + 4/(n - 2\mu_0)$ when $a(\cdot) = 1$. For $\sigma \neq -1$, we put

$$p_1(\mu_0) := 1 + \frac{4}{n - 2\mu_0} \cdot \left(1 + \frac{4}{n - 2\mu_0} \cdot \frac{2\mu_0}{n(1+\sigma)}\right)^{-1}.$$

Theorem 1. *Let $n \geq 1$, $\lambda \in \mathbb{C}$, $0 \leq \mu_0 < n/2$, and $1 \leq p \leq p(\mu_0)$. Let ω satisfies $0 \leq \pm\omega \leq \pi/2$ and $\omega \neq -\pi/2$. Assume $\mu_0 < p$ if p is not an odd number. There exist two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$ with the following properties.*

(1) (Local solutions.) For any $u_0 \in H^{\mu_0}(\mathbb{R}^n)$, there exist $S > 0$ with $S \leq S_0$ and a unique local solution u of (3) in $X^{\mu_0}([0, S))$. Here, S depends only on the norm $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ when $p < p(\mu_0)$, while S depends on the profile of u_0 when $p = p(\mu_0)$. The solutions depend on the initial data continuously.

(2) (Small global solutions.) Assume that one of the following conditions from (i) to (vi) holds: (i) $\mu_0 = 0$, $p = p(0)$, (ii) $\mu_0 > 0$, $p = p(\mu_0)$, $a_1 \geq 0$, (iii) $1 < p < p(\mu_0)$, $a_1 > 0$, $\sigma < -1$, (iv) $1 < p < p_1(\mu_0)$, $a_1 < 0$, $\sigma > -1$, (v) $p_1(\mu_0) < p < p(\mu_0)$, $a_1 > 0$, $\sigma > -1$, (vi) $\mu_0 > 0$, $1 < p < p(\mu_0)$, $a_1 > 0$, $\sigma = -1$. If $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ is sufficiently small, then the solution u obtained in (1) is a global solution, namely, $S = S_0$.

Corollary 2. *Let $\mu_0 = 0$ or $\mu_0 = 1$. Let $\lambda > 0$. Let $1 \leq p < 1 + 4/n$ when $\mu_0 = 0$. Let $1 \leq p < 1 + 4/(n-2)$ and $a_1(p - 1 - 4/n) \geq 0$ when $\mu_0 = 1$. For any $u_0 \in H^{\mu_0}(\mathbb{R}^n)$, the local solution u given by (1) in Theorem 1 is a global solution.*

Corollary 3. *Let $\mu_0 = 1$, $\lambda < 0$, $a_1 \geq 0$ and $1 \leq p < 1 + 4/n$. Let $\omega = 0$ or $\omega = \pi/2$. For any $u_0 \in H^1(\mathbb{R}^n)$, the local solution u given by (1) in Theorem 1 is a global solution.*

Corollary 4. *Let $\mu_0 = 1$ and $\lambda < 0$. Let $\omega \neq 0, \pi/2$. Put $p_0 := 2/(\sin 2\omega)^2 - 1$. Let $p_0 < p \leq 1 + 4/(n-2)$. Let $a_1(p - 1 - 4/n) \leq 0$ and $S_0 = \infty$. For any $u_0 \in H^1(\mathbb{R}^n)$ with negative energy*

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla u_0(x)|^2 + \frac{\lambda a_0^2 |u_0(x)|^{p+1}}{p+1} dx < 0, \tag{4}$$

the solution u given by (1) in Theorem 1 blows up in finite time.

Corollary 5. *Let $\mu_0 = 1$ and $\lambda < 0$. Let $\omega = 0$ or $\omega = \pi/2$. Let $1 + 4/n \leq p \leq 1 + 4/(n-2)$. Let $a_1 \leq 0$ and $S_0 = \infty$. For any $u_0 \in H^1(\mathbb{R}^n)$ which satisfies $\||x|u_0(x)\|_{L^2_x(\mathbb{R}^n)} < \infty$ and (4), the solution u given by (1) in Theorem 1 blows up in finite time.*

Uniqueness problem for closed non-smooth hypersurfaces with constant anisotropic mean curvature

Miyuki Koiso

(Institute of Mathematics for Industry, Kyushu University, Japan)

We discuss a variational problem for piecewise-smooth hypersurfaces in the $(n+1)$-dimensional euclidean space $\mathbb{R}^{n+1}$. Our energy functional is an anisotropic surface energy which is a natural generalization of area for surfaces. An anisotropic surface energy is the integral of an energy density that depends on the surface normal over the considered surface, which was introduced to model the surface tension of a small crystal. The minimizer of such an energy among all closed surfaces enclosing the same 3-dimensional volume is unique and it is (up to rescaling) so-called the Wulff shape. The Wulff shape and equilibrium surfaces of this energy are not smooth in general. In this talk, we give uniqueness and non-uniqueness results for closed (not necessarily smooth) equilibrium hypersurfaces and applications to anisotropic mean curvature flow.

Let $\gamma : S^n \to \mathbb{R}_{>0}$ be a positive continuous function on the unit sphere $S^n = \{\nu \in \mathbb{R}^{n+1} \mid \|\nu\| = 1\}$ in $\mathbb{R}^{n+1}$. The boundary W_γ of the convex set

$$\tilde{W}[\gamma] := \cap_{\nu \in S^n} \Big\{ X \in \mathbb{R}^{n+1} \mid \langle X, \nu \rangle \leq \gamma(\nu) \Big\} \tag{1}$$

is called the Wulff shape for γ, where $\langle \ , \ \rangle$ means the standard inner product in $\mathbb{R}^{n+1}$. Let us give two simple examples: For $\gamma \equiv 1$, W_γ is S^n. For $\gamma(\nu) = \gamma(\nu_1, \cdots, \nu_{n+1}) = \sum_{i=1}^{n+1} |\nu_i|$, W_γ is the cube $\{x = (x_1, \cdots, x_{n+1}) \in \mathbf{R}^{n+1} \mid \max\{|\mathrm{x}_1|, \cdots, |\mathrm{x}_{\mathrm{n+1}}|\} = 1\}$.

The homogeneous extension $\overline{\gamma} : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ of γ is defined by

$$\overline{\gamma}(rX) = r\gamma(X), \qquad \forall X \in S^n, \ \forall r \geq 0.$$

If $\overline{\gamma}$ is convex and satisfies $\overline{\gamma}(-X) = \overline{\gamma}(X)$, $\overline{\gamma}$ defines a norm on $\mathbb{R}^{n+1}$, and the unit sphere $\{Y \in \mathbb{R}^{n+1} \,|\, \overline{\gamma}^*(Y) = 1\}$ of the dual norm $\overline{\gamma}^*(Y) = \sup\{\langle Y, Z\rangle \,|\, \overline{\gamma}(Z) \leq 1\}$ coincides with W_γ.

Definition 1. A continuous map $\gamma : S^n \to \mathbb{R}_{>0}$ is called a convex integrand (or simply, convex) if its homogeneous extension $\overline{\gamma} : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ is a convex function.

We formulate our piecewise-smooth hypersurfaces as follows. Let $M = \cup_{i=1}^k M_i$ be an n-dimensional oriented connected compact C^∞ manifold, and $X : M \to \mathbb{R}^{n+1}$ be a piecewise-C^3 immersion. This means that X is continuous on M and it is a C^3 immersion on each M_i, where M_i is an n-dimensional submanifold of M with smooth boundary. We sometimes say that X is a piecewise-smooth hypersurface. Also, we sometimes identify X with $X(M)$.

The author is supported in part by JSPS KAKENHI Grant Number JP18H04487.
E-mail address: koiso@math.kyushu-u.ac.jp

Denote by $\mathcal{S}$ the set of singularities of X. Here a singularity of X is a point in M at which X is not an immersion. Let $\nu : M \setminus \mathcal{S} \to S^n$ be the "outward-pointing" unit normal vector field along $X|_{M\setminus\mathcal{S}}$. We can think that ν is defined on each M_i. An anisotropic energy $\mathcal{F}_\gamma(X)$ of X is defined as follows.

$$\mathcal{F}_\gamma(X) := \sum_{i=1}^{k} \int_{M_i} \gamma(\nu)\, dM,$$

where dM is the n-dimensional volume form of M induced by X. In the special case where $\gamma \equiv 1$, $\mathcal{F}_\gamma(X)$ is the usual n-dimensional volume of the hypersurface X.

The $(n+1)$-dimensional volume $V(X)$ enclosed by X is given by $V(X) := \frac{1}{n+1}\sum_{i=1}^{k} \int_{M_i} \langle X, \nu\rangle\, dM$. The anisotropic mean curvature Λ of X at each regular point is defined as

$$\Lambda := \frac{1}{n}(-\mathrm{div}_M D\gamma + nH\gamma),$$

where $D\gamma$ is the gradient of γ and H is the mean curvature of X. If $f : D(\subset \mathbb{R}^2) \to \mathbb{R}$ is a C^∞ function and $X : D \to \mathbb{R}^3$ is a graph $X(u_1, u_2) = (u_1, u_2, f(u_1, u_2))$ of f, then

$$\Lambda = \frac{1}{2}\sum_{i,j=1,2} \overline{\gamma}_{x_i x_j}\Big|_{(-Df,1)} f_{u_i u_j}, \quad f_i := f_{u_i}, \quad f_{ij} := f_{u_i u_j}, \quad Df := (f_1, f_2).$$

Hence, if $\overline{\gamma}$ is strictly convex, then the equation "$\Lambda \equiv$ constant" is elliptic.

X is a critical point of $\mathcal{F}_\gamma$ for all $(n+1)$-dimensional volume-preserving variations if and only if $\Lambda \equiv$ constant and X satisfies a certain condition about ν at the singularities. If X satisfies these conditions, we call X a closed CAMC (constant anisotropic mean curvature) hypersurface. A closed CAMC hypersurface X is said to be stable if the second variation of $\mathcal{F}_\gamma$ for any $(n+1)$-dimensional volume-preserving variation is nonnegative. X is said to be embedded if it does not have self-intersections.

The following uniqueness and non-uniqueness results for closed CAMC hypersurfaces hold.

Theorem 1. (Koiso [1], [2]). *Assume $\gamma : S^n \to \mathbb{R}^+$ is of C^3 and it is the convex integrand of its Wulff shape W. Let X be a closed CAMC hypersurface in $\mathbb{R}^{n+1}$ for γ. If X is either stable or embedded, then it is (up to translation and homothety) the Wulff shape W_γ.*

Theorem 2. (Koiso [2]). *There exists a C^3 function $\gamma : S^n \to \mathbb{R}^+$ that is not a convex integrand such that there exists a closed embedded CAMC hypersurface for γ which is not (any homothety and translation of) the Wulff shape W_γ.*

Let us give some remarks about the regularity of the Wulff shape W_γ and CAMC hypersurfaces. When W_γ is a smooth strictly convex hypersurface (which is equivalent to the condition that the matrix $D^2\gamma + \gamma 1$ is positive-definite at each point in S^n), any closed CAMC hypersurface X is also smooth, and the conclusions in Theorem 1 were known. In fact, if X satisfies either one of the following conditions (i)-(iii), it is a homothety of the Wulff shape: (i) X is stable (B. Palmer 1998). (ii) X is embedded (Y. He-H. Li-H. Ma-J. Ge 2009). (iii) $n = 2$ and the genus of M is zero (M. Koiso-B. Palmer 2010, N. Ando 2012).

In this talk, we also discuss the case when γ is not differentiable.

References

[1] M. Koiso, Uniqueness of stable closed non-smooth hypersurfaces with constant anisotropic mean curvature, preprint.

[2] M. Koiso, Uniqueness and non-uniqueness of closed embedded non-smooth hypersurfaces with constant anisotropic mean curvature, in preparation.

Regularity of minimal and CMC stable hypersurfaces

Neshan Wickramasekera

In the late 1970's and early 1980's, a fundamental regularity and compactness theory for uniformly mass bounded stable minimal hypersurfaces was established by the work of Schoen–Simon–Yau [SSY75] (in low dimensions) and of Schoen–Simon [SS81] (in general dimensions). The methods employed in these works required the a priori assumption that the singular sets are sufficiently small (and completely absent in [SSY75]), and yielded curvature estimates for the hypersurfaces and sharp bounds (in general dimensions) on the size of their singular set. This work implied embeddedness (away from a small singular set) of the weak minimal hypersurfaces (stationary integral n-varifolds) in compact $(n+1)$-dimensional Riemannian manifolds whose existence had been established by a deep min-max theory for the area functional developed by Almgren in the 1960's ([Alm65]) and strengthened by Pitts in the 1970's ([Pit77]).

In a series of works in the past several years, beginning with the speaker's 2014 work [Wic14] on stable minimal hypersurfaces followed by its recent extension [BW18] (joint with C. Bellettini) to CMC hypersurfaces, and the work [BCW18] (joint with O. Chodosh and C. Bellettini) establishing curvature estimates for weakly stable CMC hypersurfaces, the Schoen–Simon–Yau and Schoen–Simon theories have been strengthened and extended. This work considers a general class of n-dimensional hypersurfaces V, namely, codimension 1 integral n-varifolds whose generalized mean curvature is locally summable to a power $> n$. It follows from the Allard regularity theory that the C^1 embedded part $\mathrm{reg}_1\, V$ of the support of such a varifold V is dense, but examples show that the singular set $\mathrm{spt}\, V \setminus \mathrm{reg}_1\, V$ need not have zero n-dimensional measure (even when the generalized mean curvature is locally bounded). The work [BW18] assumes that any orientable portion of $\mathrm{reg}_1\, V$ is stationary with respect to the mass functional for volume preserving deformations. This hypothesis implies that the C^2 immersed part of $\mathrm{spt}\, V$ is a classical constant-mean-curvature (CMC) immersion. The work further assumes that this C^2 immersed part is weakly stable, i.e. stable (as an immersion) with respect to volume preserving deformations.

The main new discovery in [Wic14], [BW18] is that for varifolds V satisfying the above variational hypotheses, in contrast to a smallness hypothesis on the entire singular set as in Schoen–Simon–Yau and Schoen–Simon theories, two considerably less stringent *structural conditions* (described below) are sufficient (and necessary) to give the same sharp size control on their singular sets. Specifically, the work shows that if V satisfies these structural conditions, then except on a closed set Σ (to be thought of as the genuine singular set) of Hausdorff dimension at most $n-7$, $\mathrm{spt}\, V$ is a CMC hypersurface of class C^2 (and hence is smooth or real analytic depending on whether the ambient metric is smooth or real analytic) and is quasi-embedded, i.e. locally is either a single embedded disk or precisely two embedded disks intersecting tangentially with each piece lying on one side of the other. Furthermore, the work establishes compactness of any family of varifolds as above satisfying additionally a locally uniform mass bound and a uniform mean curvature bound.

The first structural condition is that V has no *classical singularities*; a classical singularity is a point p near which the support of V is the union of three or more embedded $C^{1,\alpha}$ hypersurfaces-with-boundary for some $\alpha \in (0,1)$ that meet only along a common free boundary containing p. This is the only structural condition necessary in the special case of minimal hypersurfaces considered in [Wic14], i.e. when the stationarity and stability conditions as above hold for deformations that are not required to be volume preserving. The second structural condition, which is necessary for the more general CMC case (in addition to the no-classical-singularities hypothesis), requires that every *touching singularity*—which by definition is a point p near which $\operatorname{spt} V$ is the union of two distinct $C^{1,\alpha}$ embedded hypersurfaces for some $\alpha \in (0,1)$ that meet at p tangentially—has a neighborhood in which the density of V is almost everywhere (with respect to the n-dimensional Hausdorff measure) less than the density at p. By the De Giorgi structure theory, this conditions is automatically satisfied in the important special case when the hypersurface is the boundary of a Caccioppoli set.

The advantage of these structural conditions is that they are local conditions involving only the parts of the varifold consisting of $C^{1,\alpha}$ regular pieces coming together in a regular fashion. In fact in the above results, somewhat surprisingly *all* hypotheses except for summability of the generalized mean curvature to a sufficiently high power concern only the regular parts of the varifold, making them easy to check in principle.

Among the applications of this work is a considerably more efficient PDE theoretic alternative to the Almgren–Pitts existence theory for minimal hypersurfaces. This new approach, a result of the combined work of Guaraco ([Gua18]), Hutchinson–Tonegawa ([HT00]), Tonegawa ([Ton05]) and Tonegawa–Wickramasekera ([TW12]), constructs the minimal hypersurface as a weak (i.e. varifold) limit of level sets of a sequence of solutions to singularly perturbed elliptic Allen–Cahn equations. It uses a standard PDE mountain pass lemma in place of the varifold min-max construction of Almgren and Pitts. The full strength of the sharp regularity theory of [Wic14] is used to establish regularity of the varifold limit of the level sets.

In this lecture we shall discuss these recent developments and outline some future directions.

References

[Alm65] F. J. Almgren Jr. *Theory of varifolds,* Mimeographed notes, Princeton University (1965).

[BW18] C. Bellettini, N. Wickramasekera. *Stable CMC integral vasrifolds of codimension 1: regularity and compactness*, arXiv 2018.

[BCW18] C. Bellettini, O. Chodosh, N. Wickramasekera. *Curvature estimates and sheeting theorems for weakly stable CMC hypersurfaces*, arXiv 2018.

[Gua18] M. Guaraco. *Min-max for phase transitions and the existence of embedded minimal hypersurfaces,* Jour. Diff. Geom. 108 (2018), 91–133.

[HT00] J. E. Hutchinson, Y. Tonegawa. *Convergence of phase interfaces in the van der Waals–Cahn–Hilliard theory,* Calc. Var. 10 (2000), no. 1, 49–84.

[Pit77] J. Pitts. *Existence and regularity of minimal surfaces on Riemannian manifolds,* Princeton University press (1977).

[SS81] R. Schoen, L. Simon *Regularity of stable minimal hypersurfaces* Comm. Pure Appl. Math. 34 (1981), 741–797

[SSY75] R. Schoen, L. Simon, S. T. Yau. *Curvature estimates for minimal hypersurfaces,* Acta Math. 134 (1975), 275–288.

[Ton05] Y. Tonegawa. *On stable critical points for a singular perturbation problem,* Comm. Anal. Geom. 13 (2005), 439–459.

[TW12] Y. Tonegawa, N. Wickramasekera. *Stable phase interfaces in the Van der Waals–Cahn–Hilliard theory,* J. Reine Angew. Math. 668 (2012), 191–210.

[Wic14] N. Wickramasekera. *A general regularity theory for stable codimension 1 integral varifolds,* Ann. of Math. 179 (2014), 843–1007.

Multidimensional traveling fronts in reaction-diffusion equations

Masaharu Taniguchi*

Department of Mathematics, Faculty of Science, Okayama University
3-1-1, Tsushimanaka, Kita-ku, Okayama City, 700-8530, JAPAN

We study the Allen–Cahn equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + f(u) && \boldsymbol{x} \in \mathbb{R}^N, t > 0, \\ &u(\boldsymbol{x}, 0) = u_0 && \boldsymbol{x} \in \mathbb{R}^N. \end{aligned} \tag{1}$$

Here $\Delta = \sum_{j=1}^{N} D_{jj}$ with $D_j = \partial/\partial x_j$ and $D_{jj} = (\partial/\partial x_j)^2$ for $1 \le j \le N$. Now $N \ge 3$ is a given integer, and u_0 is a given bounded and uniformly continuous function from $\mathbb{R}^N$ to $\mathbb{R}$.

(A1) $f \in C^1[-1, 1]$ satisfies $f(1) = 0$, $f(-1) = 0$, $f'(1) < 0$, $f'(-1) < 0$, $\int_{-1}^{1} f(s)ds > 0$.

(A2) For $a_* \in (-1, 1)$, one has $f(s) < 0$ if $s \in (-1, -a_*)$, and $f(s) > 0$ if $s \in (-a_*, 1)$.

There exists a one-dimensional traveling front Φ with speed $k > 0$ and it is unique up to translation. The Allen–Cahn equation by a moving coordinate system with speed c is given by

$$\begin{aligned} &(D_t - \Delta - cD_N)\, w - f(w) = 0 && \boldsymbol{x} \in \mathbb{R}^N, t > 0, \\ &w(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) && \boldsymbol{x} \in \mathbb{R}^N. \end{aligned} \tag{2}$$

We assume $c > k$. We denote the solution of (2) by $w(\boldsymbol{x}, t; u_0)$. Two-dimensional V-form fronts are studied by [3, 1]. An axially symmetric traveling front solution U is studied by [1, 6]. Traveling fronts of pyramidal shapes and convex polyhedral shapes are studied by [4, 2, 5]. Let a surface be the boundary of a convex compact set in $\mathbb{R}^{N-1}$ and assume that all principal curvatures are positive everywhere. For any positive-valued function $g \in C^2(S^{N-2})$ let $D_g = \left\{ r\boldsymbol{\xi} \,|\, 0 \le r < g(\boldsymbol{\xi}), \boldsymbol{\xi} \in S^{N-2} \right\}$ and let $C_g = \partial D_g = \{ g(\boldsymbol{\xi})\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in S^{N-2} \}$. Then, if all principal curvatures of C_g are positive at every point of C_g, $\overline{D_g}$ is a strictly convex compact set in $\mathbb{R}^{N-1}$. Let $\mathcal{G}$ be given by

$$\left\{ g \in C^2(S^{N-2}) \,|\, g > 0, \text{all principal curvature of } C_g \text{ are positive at every point of } C_g \right\}.$$

For any $g \in \mathcal{G}$ and $a \ge 0$ we define $g_1 = \tau_a g$ by

$$C_{g_1} = \left\{ \boldsymbol{z} \in \mathbb{R}^{N-1} \backslash D_g \mid \operatorname{dist}(\boldsymbol{z}, C_g) = a \right\}.$$

Then τ_a becomes a mapping in $\mathcal{G}$. We define an equivalence relation $g_1 \sim g_2$ if and only if one has either $g_1 = \tau_a g_2$ or $g_2 = \tau_a g_1$ for some $a \ge 0$.

*taniguchi-m@okayama-u.ac.jp

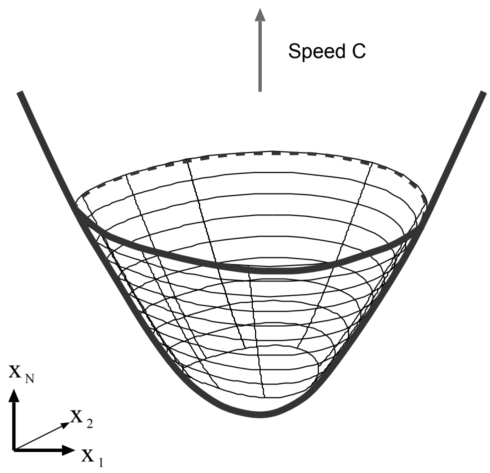


Figure 1: The graph of a level set of $\widetilde{U}$.

Theorem 1 *Assume $c > k$. For any given $g \in \mathcal{G}$, there exists a unique solution $\widetilde{U}$ to*

$$\left(-\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - c\frac{\partial}{\partial x_N}\right)\widetilde{U} - f(\widetilde{U}) = 0 \quad \text{in } \mathbb{R}^N, \tag{3}$$

$$\lim_{s\to\infty} \sup_{|\boldsymbol{x}|\geq s} \left|\widetilde{U}(\boldsymbol{x}) - \min_{\boldsymbol{\xi}\in S^{N-2}} U(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N)\right| = 0. \tag{4}$$

Here $\boldsymbol{x} = (\boldsymbol{x}', x_N)$. Let $\widetilde{U}_j$ be the solution to (3)–(4) associated with $g_j \in \mathcal{G}$ for $j = 1, 2$, respectively. Then one has

$$\widetilde{U}_2(x_1, \ldots, x_{N-1}, x_N) = \widetilde{U}_1(x_1, \ldots, x_{N-1}, x_N - \zeta) \tag{5}$$

for some $\zeta \in \mathbb{R}$ if and only if $g_1 \sim g_2$.

References

[1] F. Hamel, R. Monneau and J.-M. Roquejoffre, *Existence and qualitative properties of multidimensional conical bistable fronts*, Discrete Contin. Dyn. Syst., **13**, No. 4 (2005), 1069–1096.

[2] Y. Kurokawa and M. Taniguchi, *Multi-dimensional pyramidal traveling fronts in the Allen–Cahn equations*, Proc. Roy. Soc. Edinburgh Sect. A, **141** (2011), 1031–1054.

[3] H. Ninomiya and M. Taniguchi, *Existence and global stability of traveling curved fronts in the Allen–Cahn equations*, J. Differential Equations, **213**, No. 1 (2005), 204–233.

[4] M. Taniguchi, *Traveling fronts of pyramidal shapes in the Allen–Cahn equations*, SIAM J. Math. Anal., **39**, No. 1 (2007), 319–344.

[5] M. Taniguchi, *Multi-dimensional traveling fronts in bistable reaction-diffusion equations*, Discrete Contin. Dyn. Syst., **32** (2012), 1011–1046.

[6] M. Taniguchi, *An $(N-1)$-dimensional convex compact set gives an N-dimensional traveling front in the Allen–Cahn equation*, SIAM J. Math. Anal., **47**, No. 1 (2015), 455–476.

[7] M. Taniguchi, *Convex compact sets in $\mathbb{R}^{N-1}$ give traveling fronts of cooperation-diffusion systems in $\mathbb{R}^N$*, J. Differential Equations, **260** (2016), 4301-4338.

[8] M. Taniguchi, "*Traveling front solutions in bistable reaction-diffusion equations*," submitted to MSJ Memoirs, Mathematical Society of Japan.

Approximation methods for oscillatory and constrained interfacial dynamics

Elliott Ginder[1]
School of Interdisciplinary Mathematical Sciences
Meiji University, Japan

Karel Svadlenka
Department of Mathematics
Kyoto University, Japan

Abstract

We investigate the use of *threshold dynamics* for approximating the motion of oscillating interfaces. Interfaces are encoded within the signed distance vector field and, instead of utilizing a diffusion process to evolve the curve, are propagated by the wave operator. Initial conditions are selected to accumulate velocities over each time step and prescribe acceleration to the target motions.

Keywords: curvature dependent acceleration, threshold dynamics, interfacial motion

1 Introduction

We consider an approximation method for interfacial motions governed by the following equation:

$$\alpha V' + \beta V = -\kappa \boldsymbol{n}, \tag{1}$$

where V' and V denote the normal acceleration and velocity of the interface, respectively, and $\boldsymbol{n}$ is the outer normal vector along the interface. Here κ refers to the mean curvature of the interface, and both α and β are nonnegative parameters. In addition to an initial configuration, the interface is accompanied by an initial velocity field, which is assumed to act normal to the interface.

2 Threshold dynamics

Let $\tau > 0$ be a small time step and M be a nonnegative integer. Then the approximation method that we consider evolves an interface by repeating the following, for $k = 1, ..., M$:

(1) Evolve the interface by solving the wave equation for a time τ :

$$\begin{cases} u_{tt} = c^2 \Delta u, & \text{in } (0,\tau)\times\Omega, \\ u(0,\boldsymbol{x}) = a((2z_\epsilon^k(\boldsymbol{x}) - z_\epsilon^{k-1}(\boldsymbol{x})), & \text{in } \Omega, \\ u_t(0,\boldsymbol{x}) = -bz_\epsilon^k, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial(0,\tau)\times\Omega \end{cases} \tag{2}$$

where the initial conditions are constructed using the signed distance vector field (SDVF):

$$z_\epsilon^k(x) = \sum_{i=1}^{N} \boldsymbol{p}_i \chi_{\{d_i(x)>\epsilon/2\}} + \frac{1}{\epsilon}\left(\frac{\epsilon}{2} + d_i(x)\right) \boldsymbol{p}_i \chi_{\{-\epsilon/2\le d_i(x)\le\epsilon/2\}}. \tag{3}$$

[1]This research was supported by the JSPS Grant-in-aid for Young Scientists (B), 15KT0099.

Here $\epsilon > 0$ is an interpolation parameter, χ_E denotes the characteristic function of the set E, and $d_i(x)$ denotes the signed distance to the boundary of phase i at location x (taken positive inside the phase region, and negative outside):

$$d_i(x) = \begin{cases} \inf_{y \in \partial P_i^k} ||x-y|| & \text{if } x \in P_i^k, \\ -\inf_{y \in \partial P_i^k} ||x-y|| & \text{otherwise.} \end{cases}$$

Each $\boldsymbol{p}_i$ denotes the i^{th} coordinate vector of a *regular simplex* in $\mathbf{R}^{N-1}$, $i = 1, ..., N$ where N denotes the number of phases.

(2) Determine the location of the interface after its evolution. The location of the i^{th} phase can be obtained through the following process:

$$P_i^k = \{\boldsymbol{x} \in \Omega \mid \langle u(k\tau, x), \boldsymbol{p}_i \rangle = \max_{j=1,...,N} \langle u(k\tau, x), \boldsymbol{p}_j \rangle \}, \quad \Gamma_i^k = \partial P_i^k.$$

We remark that, in the two phase setting, this step is equivalent to finding the zero level set of the solution to (2) (figure 1B shows an application of the SDVF in 3D).

(3) Reconstruct the signed distance functions and the signed distance vector field.

For appropriate choices of a, b, c, one can show that, upon repeating the above process for $k = 1, 2, ...$, the sequence of interfaces $\{\Gamma_i^k\}_{k \in \mathbb{N}}$ approximates evolution by (1) with order $O(\tau)$.

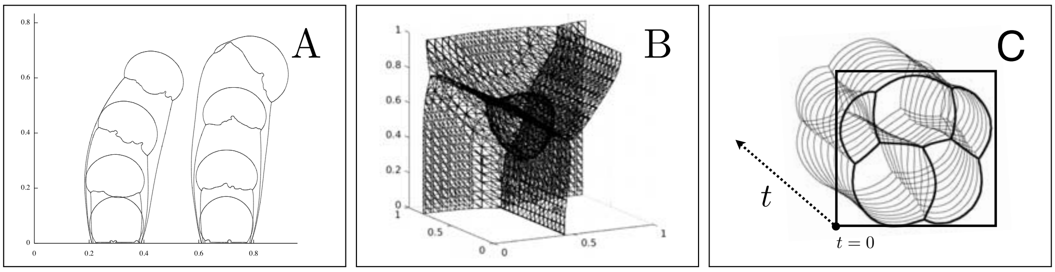


Figure 1: (Left) Vapor-Liquid-Solid simulation. (Center) Interfaces within the SDVF, in 3D. (Right) An oscillatory, multiphase volume preserving motion.

We will show applications of the SDVF related to vapor-liquid solid simulations and, using minimizing movements, to computing motion by multiphase volume preserving hyperbolic mean curvature flow (see figure 1A and 1C).

References

[1] S. Esedoglu, S. Ruuth, R. Tsai. *Diffusion generated motion using signed distance functions.* J. Comp. Phys., **229** (2010) 1017-1042.

[2] M. Gurtin. *A hyperbolic theory for the evolution of plane curves.* SIAM J. Math. Anal., 22 (3), 575-586.

[3] E. Ginder, K. Svadlenka. *Wave-type threshold dynamics and the hyperbolic mean curvature flow* Japan Journal of Industrial and Applied Mathematics, 33 No. 2 (2016), 501-523.

[4] B. Merriman, J. Bence, S. Osher. *Motion of multiple junctions: a level set approach.* J. Comp. Phys., **112** (1994) 334-363.

[5] G. Orlandi. *Time-like minimal surfaces in Minkowski space.* Hokkaido University Technical Report Series in Mathematics, (2013) 69-71.

LARGE EXPONENT BEHAVIOR OF POWER CURVATURE FLOW AND APPLICATIONS

Qing Liu
Department of Applied Mathematics, Fukuoka University
Email: qingliu@fukuoka-u.ac.jp

Abstract. Motivated by applications in image processing, we study asymptotic behavior for the level set equation of power curvature flow as the exponent tends to infinity. More precisely, we consider the level set equation

$$u_t - |\nabla u| \left| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right|^{\alpha-1} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty) \tag{1}$$

with a Lipschitz initial value u_0 in $\mathbb{R}^n$. Here $\alpha > 0$ is a given exponent and the power function is understood as $\kappa^\alpha = |\kappa|^{\alpha-1}\kappa$ to maintain the parabolicity of the operator. It is well known [3] that for any given $\alpha > 0$ there exists a unique viscosity solution u^α of the initial value problem associated to (1). We are particularly interested in the limit behavior of u^α as $\alpha \to \infty$, which has important applications in image denoising [1].

If u_0 satisfies

$$\begin{array}{ll} \text{(Quasiconvexity)} & \{x \in \mathbb{R}^n : u_0(x) \le c\} \quad \text{is convex for any } c \in \mathbb{R} \text{ and} \\ \text{(Coercivity)} & \displaystyle\inf_{|x|\ge R} u_0(x) \to \infty \quad \text{as } R \to \infty, \end{array}$$

the limit equation can be characterized as the following stationary obstacle problem involving 1-Laplacian:

$$\min\left\{ -\operatorname{div}\left(\frac{\nabla U}{|\nabla U|}\right) + 1,\ U - u_0 \right\} = 0 \quad \text{in } \mathbb{R}^n. \tag{2}$$

In [5], we discuss various properties of the obstacle problem and show that $u^\alpha(\cdot, t) \to V$ locally uniformly in $\mathbb{R}^n$ as $\alpha \to \infty$ for any $t > 0$, where V is the minimal supersolution of (2) .

The situation is much more complicated when the assumption on quasiconvexity of u_0 is dropped. In this case, for simplicity we only talk about the following simplified equation in one space dimension:

$$u_t - |u_{xx}|^{\alpha-1} u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0,\infty) \tag{3}$$

with a Lipschitz initial value u_0 that is not necessarily convex in $\mathbb{R}$.

When u_0 is compactly supported in $\mathbb{R}$, we show that the unique solution u^α of (3) converges locally uniformly in $\mathbb{R} \times (0, \infty)$ as $\alpha \to \infty$ to V, where the limit V again depends only on the space variable and can be characterized via the following analysis of initial layer.

Assuming first that u_0 is of class $C^{1,1}(\mathbb{R})$ and applying the scaling

$$U^\alpha(x,t) = u^\alpha\left(x, \frac{t^\alpha}{\alpha}\right) \qquad \text{for } (x,t) \in \mathbb{R} \times (\tau, \infty),$$

one can show that $U^\alpha \to U$ locally uniformly in $\mathbb{R} \times [\tau, \infty)$ as $\alpha \to \infty$, where U solves at least formally a fully nonlinear parabolic equation

$$U_t(x,t) = F(t, U_{xx}(x,t)) \quad \text{in } \mathbb{R} \times (\tau, \infty), \tag{4}$$

where $F : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a discontinuous operator given by

$$F(t,z) = \begin{cases} 0 & \text{if } |z| < 1/t, \\ \operatorname{sgn}(z)/t & \text{if } |z| = 1/t, \\ \infty & \text{if } z > 1/t, \\ -\infty & \text{if } z < -1/t. \end{cases}$$

It turns out [4] that the large exponent limit V of the solution to (3) coincides with $U(\cdot, 1)$. We justify the convergence of U^α to U by proving a comparison principle for the Cauchy problem for (4). The convergence of u^α in $\mathbb{R} \times (0, \infty)$ for a general Lipschitz initial value u_0 can be obtained by approximation.

We finally discuss another application of the large exponent analysis for (3), which is closely related to a math model established in [2] to describe unstable sandpiles.

References

[1] F. Cao. *Geometric curve evolution and image processing*, volume 1805 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.

[2] L. C. Evans, M. Feldman, and R. F. Gariepy. Fast/slow diffusion and collapsing sandpiles. *J. Differential Equations*, 137(1):166–209, 1997.

[3] H. Ishii and P. Souganidis. Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor. *Tohoku Math. J. (2)*, 47(2):227–250, 1995.

[4] Q. Liu. Large exponent behavior for a power-type diffusion equation and applications. *in preparation.*

[5] Q. Liu and N. Yamada. An obstacle problem arising in large exponent limit of power mean curvature flow equation. *preprint.*

On the dynamics for the volume-preserving geometric flows

Yoshihito Kohsaka

Graduate School of Maritime Sciences, Kobe University

A dynamics of curves or surfaces governed by the volume-preserving geometric flows will be studied. Let $\Gamma(t)$ be a evolving curve or surface with time t. As the volume-preserving geometric flows, we will consider the volume-preserving mean curvature flow and the surface diffusion equation:

$$V = H - H_{av}, \tag{1}$$

$$V = -\Delta_{\Gamma(t)} H. \tag{2}$$

Here V and H are the normal velocity and the mean curvature of $\Gamma(t)$, respectively. In our sign convention, the mean curvature H for spheres with outer unit normal is negative. Moreover, H_{av} is the mean value of H given by

$$H_{av} = \frac{\displaystyle\int_{\Gamma(t)} H \, dS_t}{\displaystyle\int_{\Gamma(t)} dS_t}$$

and Δ_Γ is the Laplace-Beltrami operator on $\Gamma(t)$.

(1) and (2) are represented as the gradient flow for the area functional $\mathcal{A}[\Gamma(t)]$. Indeed, setting

$$L_0^2(\Gamma(t)) := \left\{ \varphi \in L^2(\Gamma(t)) \,\middle|\, \int_{\Gamma(t)} \varphi \, dS_t = 0 \right\},$$

$$H^{-1}(\Gamma(t)) := \left\{ \varphi \in [H^1(\Gamma(t))]^* \,\middle|\, \langle \varphi, 1 \rangle = 0 \right\},$$

where $[H^1(\Gamma(t))]^*$ is the dual space of $H^1(\Gamma(t))$ and $\langle \cdot\,, \cdot \rangle$ is the duality pairing of $[H^1(\Gamma(t))]^*$ and $H^1(\Gamma(t))$, the volume-preserving mean curvature flow (1) is the gradient flow for $\mathcal{A}[\Gamma(t)]$ in $L_0^2(\Gamma(t))$ and the surface diffusion equation (2) is that in $H^{-1}(\Gamma(t))$ (see [8]). Note that if the normal velocity V satisfies

$$\int_{\Gamma(t)} V \, dS_t = 0,$$

then it follows that

$$\frac{d}{dt}\mathcal{V}[\Gamma(t)] = \int_{\Gamma(t)} V \, dS_t = 0,$$

where $\mathcal{V}[\Gamma(t)]$ is the volume of the domain enclosed by $\Gamma(t)$. This means that both of the flows have a variational structure that the area $\mathcal{A}[\Gamma(t)]$ decreases with time t whereas the volume $\mathcal{V}[\Gamma(t)]$ is preserved. We also remark that (1) is described as the 2nd order nonlinear parabolic PDE with a nonlocal term and (2) is characterized as the 4th order nonlinear PDE.

When we analyze the motion of curves or surfaces governed by (1) or (2), we will try to find the steady states or particular solutions as traveling waves and self-similar solutions and study their stability (cf. [1, 2, 3, 4, 5, 6, 7]). In this talk, we will focus on one of their topics.

References

[1] T. Asai and Y. Giga, *On self-similar solutions to the surface diffusion flow equations with contact angle boundary conditions*, Interfaces Free Bound., 16 (2014), no. 4, 539–573.

[2] H. Garcke, K. Ito, and Y. Kohsaka, *Linearized stability analysis of stationary solutions for surface diffusion with boundary conditions*, SIAM J. Math. Anal., 36 (2005), no. 4, 1031–1056.

[3] H. Garcke, K. Ito, and Y. Kohsaka, *Surface diffusion with triple junctions: a stability criterion for stationary solutions*, Adv. Differential Equations, 15 (2010), no. 5-6, 437–472.

[4] Y. Kohsaka, *Stability analysis of Delaunay surfaces as steady states for the surface diffusion equation*, Geometric properties for parabolic and elliptic PDE's, Springer Proc. Math. Stat., 176, Springer, 2016, 121–148.

[5] Y. Kohsaka, *On the criteria for the stability of unduloids. Regularity and singularity for partial differential equations with conservation laws*, RIMS Kôkyûroku Bessatsu, B63 (2017), 169–194.

[6] J. Kanel, A. Novick-Cohen, A. Vilenkin, *Coupled surface and grain boundary motion: a travelling wave solution*, Nonlinear Anal., 59 (2004), no. 8, 1267–1292.

[7] M. Shimojo and T. Kagaya, *Exponential stability of a traveling wave for an area preserving curvature motion with two endpoints moving freely on a line*, Asymptot. Anal., 96 (2016), no. 2, 109–134.

[8] J. E. Taylor and J. W. Cahn, *Linking anisotropic sharp and diffuse surface motion laws via gradient flows*, J. Stat. Phys., 77 (1994), no.1-2, 183–197.

Convergence of a threshold-type algorithm for mean curvature flow

Katsuyuki Ishii
Graduate School of Maritime Sciences, Kobe University

In this talk I would like to consider a threshold-type algorithm for curvature-dependent motions. We say a family $\{\Gamma(t)\}_{t\geq 0}$ of hypersurfaces in $\mathbb{R}^N$ is a curvature-dependent motion (CDM for short) if $\Gamma(t)$ moves by the sollowing equation:

$$V = \kappa + \langle \mathbf{b}, \mathbf{n} \rangle + g \quad \text{on } \Gamma(t),\ t > 0. \tag{1}$$

Here $\mathbf{n}$ is the inner unit normal vector of $\Gamma(t)$, V is the velocity of $\Gamma(t)$ in the direction of $\mathbf{n}$, κ denotes the sum of all the principal curvatures of $\Gamma(t)$, $\mathbf{b}$ is a given vector field and g is a given forcing term. In particular in the case $\mathbf{b} \equiv 0$ and $g \equiv 0$, the above motion is often called a mean curvature flow (MCF). The (CDM) arises from various fields such as geometry, material science, phase transitions, image processing and so on. From the viewpoints fo these applications, the numerical computations have been studied by many people.

In this talk we treat the following algorithm: Let C_0 be a compact set in $\mathbb{R}^N$ and fix a time step $h > 0$. For $k = 0, 1, 2, \ldots$, set $\mathbf{b}_k(t, x) := \mathbf{b}(t+kh, x)$ and $g_k(t, x) := g(t+kh, x)$. Let $w_0 = w_0(t, x)$ be a solution of the initial value problem for the linear parabolic equation with $k = 0$:

$$L_k[w] + g_k = 0 \quad \text{in } (0, h) \times \mathbb{R}^N,\ L_k[w] := w_t - \Delta w + \langle \mathbf{b}_k, Dw \rangle, \tag{2}$$

$$w(0, x) = d(x, C_k) \quad \text{for } x \in \mathbb{R}^N, \tag{3}$$

where $d(x, E)$ is the signed distance function to ∂E defined by

$$d(x, E) := \begin{cases} \operatorname{dist}(x, \partial E) & \text{for } x \in E, \\ -\operatorname{dist}(x, \partial E) & \text{for } x \notin E, \end{cases} \tag{4}$$

for each closed subset $E(\neq \emptyset)$ of $\mathbb{R}^N$. We then set

$$C_1 := \{w_0(h, \cdot) \geq 0\} (:= \{x \in \mathbb{R}^N \mid w_0(h, x) \geq 0\}). \tag{5}$$

Let w_1 be a unique solution of (2) - (3) with $k = 1$. Again we define C_2 as the set in (5) with w_1 replacing w_0. Repeating this process, we have a sequence $\{C_k\}_{k=0}^{+\infty}$ of compact subsets of $\mathbb{R}^N$. We set

$$C^h(t) := C_k \quad \text{for } t \in [kh, (k+1)h),\ k = 0, 1, 2, \ldots \tag{6}$$

Letting $h \to 0$, we formally obtain a limit flow $\{C(t)\}_{t\geq 0}$ of compact sets in $\mathbb{R}^N$ and observe that $\partial C(t)$ moves by (1) starting from $\partial C(0) = \partial C_0$.

The above algorithm was numerically studied by e.g., [9], [5], [10] and so on. On the other hand, our algorithm is regarded as a variant of the Bence - Merriman - Osher (BMO for short) algorithmand that by Chambolle. (cf. [2] and [3]). Many people studied their algorithms and generalizations (e.g., [6], [1], [4], etc.).

The purpose of this talk is to present the convergence of the algorithm to the generalized CDM and its optimal rate to the smooth and compact CDM. The strategies of their proofs are the level-set approach and suitable constructions of sub- and super-solutions of the level-set equations of CDM.

This talk is based on my joint works with Professor Masato Kimura and Mr. Takahiro Izumi ([8] and [7]).

References

[1] G. Barles and C. Georgelin. A simple proof of convergence for an approximation scheme for computing motion by mean curvature. *SIAM J. Numer. Anal.*, **32** (1995), 484–500.

[2] J. Bence, B. Merriman, and S. Osher. Diffusion generated motion by mean curvature. in “Computational Crystal Growers Workshop”, J. Taylor ed. *Selected Lectures in Math., Amer. Math. Soc., Province*, 1992.

[3] A. Chambolle. An algorithm for mean curvature motion. *Interfaces Free Bound.*, **6** (2004), 195–218.

[4] A. Chambolle and M. Novaga. Approximation of the anisotropic mean curvature flow. *Math. Models Methods Appl. Sci.*, **6** (2007), 833–844.

[5] S. Esedoḡlu, S. J. Ruuth, and R. Tsai. Diffusion generated motion using the signed distance function. *J. Comp. Phys.*, **229** (2010), 1017–1042.

[6] L. C. Evans. Convergence of an algorithm for mean curvature motion. *Indiana Univ. Math. J.*, **42** (1993), 533–557.

[7] K. Ishii and T. Izumi. Remarks on the convergence of an algorithm for curvature-dependent motions of hypersurfaces. *Discrete and Continuous Dynamical Systems, Ser. A*, **38** (2018), 1103–1125.

[8] K. Ishii and M. Kimura. Convergence of a threshold-type algorithm using the signed distance function. *Interfaces Free Bound.*, **18** (2016), 479–522.

[9] M. Kimura and H. Notsu. A level set method using the signed distance function. *Japan J. Indust. Appl. Math.*, **19** (2002), 415–446.

[10] R. Z. Mohammad and K. Švadlenka. Multiphase volume-preserving interface motion via localized signed distance vector scheme. *Discrete and Continuous Dynamical Systems, Series S.*, **8** (2015), 969–988.

Analytical solutions of nonlinear reaction-diffusion equations in various coordinate systems

Bronwyn H Bradshaw-Hajek

Phenomics and Bioinformatics Research Centre, School of Information Technology and Mathematical Sciences, University of South Australia, South Australia, Australia

Reaction-diffusion equations with a nonlinear source are used widely to model many different systems and processes. They are particularly common in biology, for example in the areas of population dynamics [1], cell proliferation [2], predator-prey systems [3], and the transmission of nerve impulses [4], but they are also used to model chemical reactions [5], flow through porous media [6], and heat transfer [7]. The reaction-diffusion equations typically take the form

$$\frac{\partial u}{\partial t} = D\nabla^2 u + R(u)$$

where D is a constant diffusion coefficient, $R(u)$ is a (often nonlinear) source term and $u(\mathbf{x}, t)$ represents the quantity under consideration, for example, a cell population density [2] or a gene frequency [8]. In some situations, this equation has been generalised to include nonlinear diffusion,

$$\frac{\partial u}{\partial t} = \nabla \cdot [D(u)\nabla u] + R(u) \tag{1}$$

for example, in [9] where cell diffusion is dependent on contact interactions.

When modelling the types of systems and processes described above, the model equations are often solved in one (Cartesian) dimension, so that ∇^2 is the usual Laplacian operator in one dimension, $\nabla^2 = \partial^2/\partial x^2$, and the solution describes the behaviour of the system on a line or across a plane (for example, a planar wave). However, in some situations, the system or process being studied is better represented in two or three dimensions, or in a different coordinate system (for example, polar, cylindrical or spherical coordinates). Here, we use the nonclassical symmetry technique [10] to construct solutions for a number of example problems in different geometries.

Using the nonclassical symmetry technique, Arrigo and Hill [11] and Goard and Broadbridge [12] found that when the nonlinear reaction and diffusion terms satisfy a particular relationship, the temporal and spatial dependence of (1) may be separated. The result was shown to hold in one and two Cartesian dimensions. The required relationship between the reaction and diffusion terms can be expressed as

$$R(u) = \left(\frac{A}{D(u)} + \kappa\right) \int_{u^*}^{u} D(u')\, du', \tag{2}$$

where A and κ are constants arising from the nonclassical symmetry analysis, and u^* is chosen as one of the zeros of $R(u)$. Use of the transformation enables reduction of equation (1) to

$$\nabla^2\Psi(\mathbf{x}) + \kappa\Psi(\mathbf{x}),$$

that is, the Helmholtz ($\kappa > 0$), modified Helmholtz ($\kappa < 0$) or Laplace equation ($\kappa = 0$). The solution to equation (1) may then be found by calculating the following integral and inverting to find $u(\mathbf{x}, t)$,

$$\mu(u) = \int_{u^*}^{u} D(u')\ du' = \mathrm{e}^{At}\Psi(\mathbf{x}) \tag{3}$$

where $\mu(u)$ is known as the Kirchhoff variable.

The transformation (3) described above can be shown to be valid in n-dimensions [5], and in any orthogonal coordinate system. In particular, we can construct analytic solutions for Arrhenius reaction-diffusion on a disk, flow in a cylindrical section of porous media with a web of plant roots, spherically symmetric cell invasion in a sphere, and ion diffusion on the surface of a sphere. Solutions for these problems can be written in terms of Bessel functions, spherical harmonics, and hypergeometric functions.

This is joint work with Prof Phil Broadbridge, LaTrobe University.

References

[1] P. Broadbridge, B.H. Bradshaw-Hajek, Exact solutions for logistic reaction-diffusion equations in biology, Z. Agnew. Math. Phys., 67 (2016) 93

[2] M.J.Simpson, A. Merrifield, K.A. Landman, B.D Hughes, Simulating invasion with cellular automata: Connecting cell-scale and population scale properties, Phys. Rev. E 76 (2007) 021918

[3] N.F. Britton, Reaction-diffusion equations and their applications to biology, Academic Press, 1986

[4] A.C. Scott, The electrophysics of a nerve fiber, Reviews of Modern Physics 47 (1975) 487533

[5] P. Broadbridge, B.H. Bradshaw-Hajek, D. Triadis, Exact nonclassical symmetry solutions of Arrhenius reaction-diffusion, P. Roy. Soc. Lond. A Mat., 471 (2015) 20150580

[6] P. Broadbridge, E. Daly, J. Goard, Exact solutions of the Richards equation with nonlinear plant-root extraction, Water Resources Research, 53 (2017) 9679-9691

[7] R. J. Moitsheki, B. H. Bradshaw-Hajek, Symmetry analysis of a heat conduction model for heat transfer in a longitudinal rectangular fin of a heterogeneous material, Commun. Nonlinear. Sci. Numer. Simulat., 18 (2013) 2374-2387

[8] R.A. Fisher, The wave of advance of advantageous genes, Annals of Eugenics, 7 (1937) 355-369.

[9] A.E. Fernando, K.A. Landman, M.J. Simpson, Nonlinear diffusion and exclusion processes with contact interactions, Phys. Rev. E, 81 (2010) 011903

[10] G.W. Bluman, J.D. Cole, General similarity solution of the heat equation, J. Math. Mech., 18 (1969) 1025-1042.

[11] D.J. Arrigo, J.M. Hill, Nonclassical symmetries for nonlinear diffusion and absorption, Stud. Appl. Math., 94 (1995) 21-39

[12] J.M. Goard, P. Broadbridge, Nonclassical symmetry analysis of nonlinear reaction-diffusion equations in two spatial dimensions, Nonlinear Anal.-Theor., 26 (1996) 735-754

Pinned localised solutions in a heterogeneous three-component FitzHugh–Nagumo model

Peter van Heijster

In collaboration with: A. Doelman, C.-N. Chen, T.J. Kaper, Y. Nishiura, T. Teramoto, K.-I. Ueda, F. Xie

April 30, 2018

Abstract

During this talk I will take you on a journey that started nine years ago here at Hokkaido University. I visited the lab of Prof. Nishiura during the last stage of my PhD project to work on a project related to the pinning of localised solutions by a small defect, or heterogeneity, in a three-component singularly perturbed FitzHugh–Nagumo model. In nondimensionlised version, this model is given by

$$\begin{cases} U_t = \varepsilon^2 U_{xx} + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma(x))\,, \\ \tau V_t = V_{xx} + U - V\,, \\ \theta W_t = D^2 W_{xx} + U - W\,, \end{cases} \tag{1}$$

with a small jump-type heterogeneity, or defect, at $x = 0$:

$$\gamma(x) = \begin{cases} \gamma_1\,, & \text{for } x \le 0\,, \\ \gamma_2\,, & \text{for } x > 0\,. \end{cases} \tag{2}$$

Here, $0 < \varepsilon \ll 1; D > 1; \tau, \theta > 0; (x,t) \in \mathbb{R} \times \mathbb{R}^+; \alpha, \beta, \gamma_{1,2} \in \mathbb{R}$ and all parameters are *a priori* assumed to be $\mathscr{O}(1)$ with respect to ε. After some numerical exploration, it was soon discovered that the defect – for identical system parameters – can pin localised solutions in two different ways: at the defect (see the left two panels of Fig. 1) and away from the defect (see the right two panels of Fig. 1).

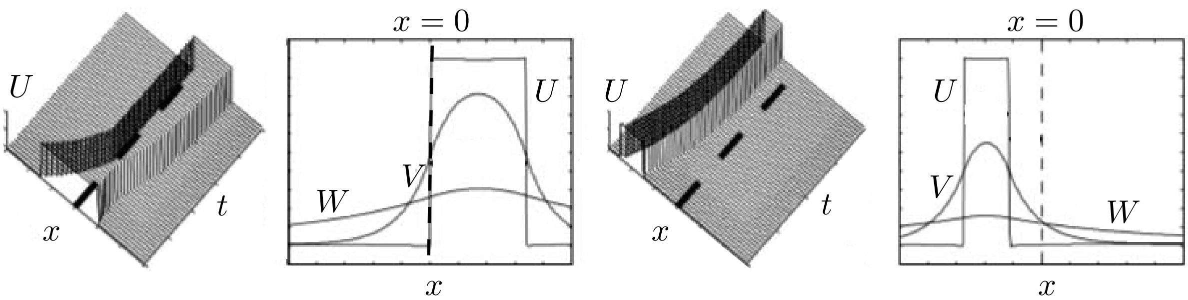


FIGURE 1: The coexistence of pinned pulse solutions supported by (1) pinned at the defect (left two panels) and away from the defect (right two panels). Adapted from Fig. 5 of [4].

Pinning at the defect

We (Doelman, Kaper, Nishiura, Ueda, PvH) started by analysing the pinning of the simplest localised structures – 1-front solutions and pulse (or 2-front) solutions – at the defect. By using geometric singular perturbation techniques (GSPT), we first showed that – under the right parameter conditions – the heterogeneity can pin a 1-front solution at the heterogeneity. Observe that in the homogeneous setting (i.e. $\gamma_1 = \gamma_2$) the associated 1-front solution travels with constant nonzero speed and the defect thus pins the travelling front solutions. We also established the stability of this heterogeneous pinned 1-front solution. Next, we analysed the pinning of 1-pulse solutions at the defect and derived conditions for their existence, see also Fig. 1. Observe that the associated homogeneous model supports – due to translation invariance and under the right parameter conditions – a 1-parameter family of stationary pulse solutions. The defect breaks this translation invariance and pinpoints the pulse. The main results are summarized in the theorem below:

Theorem 1 (reworded from [4])**.** *Let ε small enough. Then* (1) *supports*

- *a stable defect front solution pinned at the defect if and only if $\gamma_2 < 0 < \gamma_1$.*
- *a defect pulse solution pinned which its front at the heterogeneity if and only if $\gamma_1 > \gamma_2$ and there exists a $x^* > 0$ solving*

$$f(2x^*) = \gamma_2\,, \quad \text{with} \quad f(2x^*) := \alpha e^{-2x^*} + \beta e^{-2x^*/D}\,.$$

  Moreover, the width of the defect pulse solution is to leading order $2x^*$.

- *a defect pulse solution pinned which its back at the heterogeneity if and only if $\gamma_2 > \gamma_1$ and there exists a $x^* > 0$ solving*

$$f(2x^*) = \gamma_1\,, \quad \text{with} \quad f(2x^*) := \alpha e^{-2x^*} + \beta e^{-2x^*/D}\,.$$

  Moreover, the width of the defect pulse solution is to leading order $2x^*$.

Pinning away from the defect

Understanding the pinning of the localised structures away from the defect turned out to be significantly more complicated. Heuristically, this is because we have to understand how the defect interacts weakly with the localised solution that is pinned $\mathscr{O}(1)$-away from the defect and detailed higher order computations are thus needed. It was not until 2016 before we (Doelman, Xie, PvH) obtained the first results. By using a geometric method, we showed – under quite strict conditions on the system parameters – that these defect front and pulse solutions pinned away from the defect exist [3].

However, the results of [3] did not give any information regarding the pinning distance of a localised structure, i.e. the distance of the localised structure to the defect, nor about its stability. To study these, we (Chen, Nishiura, Teramoto, PvH) developed a new analytical technique by combining GSPT with an action functional approach[1]. In [2], we first benchmarked the method by deriving known results for the homogeneous version of (1) with the new method, and, in addition we also obtained several new results for the homogeneous model. Next, we extended the methodology of [2] to study the defect front and pulse solutions supported by (1) and pinned away from the defect. We obtained the following results:

Theorem 2 (reworded from [1])**.** *Let ε be small enough and let τ and θ be bounded by some $\mathscr{O}(1)$-constant.*

- *If $\gamma_1 = \varepsilon\tilde{\gamma}_1$, with $\tilde{\gamma}_1 = \Theta(1)$, $(\alpha, \beta, D, \gamma_2)$ are $\Theta(1)$ such that $\tilde{\gamma}_1 - \gamma_2 = \Theta(1)$ and if there exist an $x_d > 0$ solving*

$$2\tilde{\gamma}_1 - \frac{1}{2}\gamma_2 f(x_d) = 0\,, \quad \text{with} \quad f(x_d) := \alpha e^{-x_d} + \beta e^{-x_d/D}\,,$$

  then (1) *supports a defect front solution that is pinned to the left of the defect and that has leading order pinning distance x_d. The defect front solution is stable if and only if*

$$\gamma_2 g(x_d) > 0\,, \quad \text{with} \quad g(x_d) := \alpha e^{-x_d} + \frac{\beta}{D} e^{-x_d/D}\,.$$

- *If $(\alpha, \beta, D, \gamma_1, \gamma_2)$ are $\Theta(1)$ such that $\gamma_1 - \gamma_2 = \Theta(1)$ and if there exist an $x^* > 0$ and $x_d > 0$ solving*

$$f(2x^*) = \gamma_2\,, \qquad f(x_d) = f(x_d + 2x^*),$$

  then (1) *supports a defect pulse solution that is pinned to the right of the defect and that has with leading order width $2x^*$ and leading order pinning distance x_d. The defect pulse solution is stable if and only if*

$$g(2x^*) > 0\,, \quad \text{and} \quad (\gamma_2 - \gamma_1)(g(x_d) - g(x_d + 2x^*)) > 0\,.$$

- *By the symmetries of the system, similar results can be derived for defect front solutions pinned to the right of the defect and defect pulse solutions pinned to the left of the defect.*

Combining the results of Thm. 1 and 2 gives a broad picture of the pinned localised solutions supported by (1).

References

[1] P. van Heijster, C.-N. Chen, Y. Nishiura, T. Teramoto, Pinned solutions in a heterogeneous three-component FitzHugh–Nagumo model, submitted (2017).

[2] P. van Heijster, C.-N. Chen, Y. Nishiura, T. Teramoto, Localized patterns in a three-component FitzHugh–Nagumo model revisited via an action functional, *Journal of Dynamics and Differential Equations* (2016).

[3] A. Doelman, P. van Heijster, F. Xie, A geometric approach to stationary defect solutions in one space dimension, *SIAM Journal on Applied Dynamical Systems* **15**, 655-712 (2016).

[4] P. van Heijster, A. Doelman, T.J. Kaper, Y. Nishiura, K.-I. Ueda, Pinned fronts in heterogeneous media of jump type, *Nonlinearity* **24**, 127-157 (2011).

[1] Notably, the discussions leading to the development of this technique also started in Japan during the 4th MSJ-SI conference: *Nonlinear Dynamics in Partial Differential Equations* held in Kyushu in 2011.

Stochastic sensitivity of finite-time Lagrangian motion

Sanjeeva Balasuriya (University of Adelaide)

Physical, biological or chemical mixing occurring in our oceans and atmosphere can have profound implications to our environment. Physical properties such as heat or kinetic energy impact the nature of the velocity fields. Biological quantities such as plankton density in the oceans, or spore distribution by winds, strongly influence biodiversity. Chemicals such as ozone in the Antarctic Polar Vortex, oil spills in the ocean, or pollutants emitted from factories can result in environmental protection and degradation. In all these cases, some quantity gets dispersed in the environment as a result of advection by the prevailing oceanic/atmospheric velocities, in addition to undergoing diffusion. If this quantity can be thought of as an evolving probability density ρ of particles, then it can be modelled by the Fokker-Planck, or forward Kolmogorov, equation [6]

$$\frac{\partial}{\partial t}\rho(x,t) + \nabla \cdot [\rho(x,t)v(x,t)] = \frac{\varepsilon^2}{2}\nabla \cdot \nabla \cdot \left[\rho(x,t)\sigma(x,t)\sigma^\top(x,t)\right] , \qquad (1)$$

in which ε is a small parameter (quantifying the magnitude of the diffusion), $x \in \mathbb{R}^n$ (typically $n = 2$ or 3 for the applications cited), $t \in \mathbb{R}$ is time, v is the unsteady velocity field, and σ is an $n \times n$ diffusion matrix. The usual interpretation is that ρ is the unknown, everything else is specified or known, and an initial condition $\rho(x, 0)$ is given.

However, in typical applications in oceanic/atmospheric motion, the velocity field is only known over a finite-time, because it is based on data, imputed for example from sea-surface height measurements gathered from satellites [5]. Then, *in the absence of diffusion*, particle trajectories obey

$$\frac{dx}{dt} = v(x,t) \quad ; \quad t \in [0,T] \, . \qquad (2)$$

Understanding important structures associated with (2) (e.g., oceanic hot-spots, barriers between coherently evolving structures) falls within the framework of 'Lagrangian Coherent Structures' (LCSs) [4, 1]. While there are connections to well-established dynamical systems concepts such as invariant manifolds, the nonautonomous and finite-time nature of (2) provide theoretical difficulties; hence the development of a large class of heuristic methods for LCS extraction [3].

The analysis of (2) is extended in this presentation to include small stochasticity in the form of an Itô stochastic differential equation

$$\mathrm{d}x = v(x,t)\mathrm{d}t + \varepsilon\, \sigma(x,t)\mathrm{d}W_t \quad ; \quad t \in [0,T] \quad ; \quad |\varepsilon| \ll 1 \, , \qquad (3)$$

where W_t is Brownian motion. The rationale for this is that, in almost all situations in using (2) for coherent structure identification, the velocity field v is obtained from data, and hence contains (hopefully small) uncertainties. The spatio-temporal dependence of these (e.g., cloud cover over certain areas at certain times when taking satellite measurements) is accommodated for by allowing σ to be (x, t) dependent; the presence of x-dependence means this is 'multiplicative noise.' The Itô stochastic differential equation (3) and the Fokker-Planck equation (1) are equivalent in that the density field ρ resulting from (3) evolves according to (1) [6]. Thus, rather than considering the mixing/diffusion of the density through the partial differential equation (1), the idea is to obtain as much information as possible through the analysis of (3). An advantage is that the implementation of (3) does not rely on boundary conditions, unlike in (1).

This presentation addresses three aspects of (3), all in relation to $n = 2$, and to general (non-divergence-free) v:

1. Can one assign an uncertainty to the evolution of a curve from time 0 to T? In other words, can the 'fattening' of the curve be quantified [2]?
2. Stable/unstable manifolds for (2) in infinite time form flow barriers. Can the inclusion of the stochasticity via (3) be characterised as a calculable flow barrier 'region' in which cross-barrier mixing occurs? [This part in collaboration with Georg Gottwald, University of Sydney]
3. Is it possible to come up with a measure to explicitly quantify the time-T spread associated with choosing Dirac-delta initial conditions to the Fokker-Planck equation? This relates to the important question of determining the sensitivity of Lagrangian trajectories of (2) towards velocity uncertainties.

References

[1] Balasuriya, S. *Barriers and transport in unsteady flows: A Melnikov approach*, SIAM Press, Philadelphia (2016).

[2] Balasuriya, S. Stochastic uncertainty of advected curves in finite-time unsteady flows, *Phys. Rev. E* **95**, 062201 (2017).

[3] Balasuriya, S., Ouellette, N. & Rypina I. Generalized Lagrangian coherent structures, *Phys. D*, in press (2018).

[4] Haller, G. Lagrangian coherent structures, *Annu. Rev. Fluid Mech.* **47**, 137–162 (2015).

[5] Samelson, R. Lagrangian motion, coherent structures, and lines of persistent material strain, *Annu. Rev. Mar. Sci.* **5**, 137–163 (2013).

[6] Schuss, Z. Singular perturbation methods in stochastic differential equations of mathematical physics, *SIAM Rev.* **22**, 119–155 (1980).