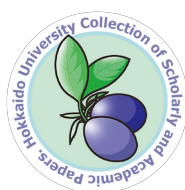




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CM PERIODS, CM REGULATORS AND HYPERGEOMETRIC FUNCTIONS, I

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ABSTRACT. We prove the Gross–Deligne conjecture on CM periods for motives associated with H^2 of certain surfaces fibered over the projective line. Then we prove for the same motives a formula which expresses the K_1 -regulators in terms of hypergeometric functions ${}_3F_2$, and obtain a new example of non-trivial regulators.

1. INTRODUCTION

Periods and regulators of a motive over a number field are very important invariants, whose arithmetic significance can be seen from their conjectural relations with values of the L -function at integers. Such conjectures include those of Birch–Swinnerton-Dyer, Deligne, Bloch, Beilinson and Bloch–Kato. If the motive has complex multiplication (CM) by a number field, especially by an abelian field, those invariants take a special form.

If A is an abelian variety with CM by a subfield of the N th cyclotomic field, its periods are written in terms of values of the gamma function at $\frac{1}{N}\mathbb{Z}$. When A is an elliptic curve, the formula is due to Lerch [16] and was rediscovered by Chowla–Selberg [8]. Gross [14] gave a geometric proof of a generalization of the formula and proposed a conjecture for any motivic Hodge–de Rham structure with CM by an abelian field, whose precise form was given by Deligne. Using Shimura’s monomial relation [24], Anderson [1] proved the formula for CM abelian varieties by reducing to the case of Fermat curves.

In this paper, we study a surface X fibered over $\mathbb{P}^1$ (t -line) with the general fiber defined by

$$y^p = x^a(1-x)^b(t^l - x)^{p-b}$$

where l and p are distinct prime numbers. It admits an action of μ_{lp} and its second cohomology modulo the image of classes supported at singular fibers gives a Hodge–de Rham structure $H = (H_{\mathrm{dR}}, H_B)$ with multiplication by $K := \mathbb{Q}(\mu_{pl})$ (see Sect. 2.2). We shall prove that H_B is one-dimensional over K (Theorem 4.12). For each embedding $\chi: K \hookrightarrow \mathbb{C}$, let H^χ be the eigencomponent. We shall determine its period and the Hodge type independently, and prove the Gross–Deligne conjecture.

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Theorem 1.1 (Period formula, see Theorem 5.4). *For each $\chi: K \hookrightarrow \mathbb{C}$, let $\chi(\zeta_p) = \zeta_p^n$, $\chi(\zeta_l) = \zeta_l^m$ and put $\alpha = \{\frac{na}{p}\}$, $\beta = \{\frac{nb}{p}\}$, $\mu = \{\frac{m}{l}\}$. Then we have*

$$\text{Per}(H^X) \sim_{K' \times} B(\beta, \mu) B(1 - \beta, \beta - \alpha + \mu)$$

where $K' := \mathbb{Q}(\mu_{2lp})$, and the Gross–Deligne conjecture holds.

On the other hand, regulators of the Fermat curve of degree N are written in terms of values at 1 of hypergeometric functions ${}_3F_2$ with parameters in $\frac{1}{N}\mathbb{Z}$ [19]. The conjectural relation with L -values is verified for some cases in [20], [21]. Recall that the beta function is related with the value at 1 of Gauss’ hypergeometric function ${}_2F_1$. It is also suggestive that the classical polylogarithm can be written as

$$\text{Li}_k(x) = x \cdot {}_{k+1}F_k \left(\begin{matrix} 1, 1, \dots, 1 \\ 2, \dots, 2 \end{matrix}; x \right),$$

and hence special values of Dirichlet L -functions are written in terms of ${}_{k+1}F_k$ -values.

For the surface X , we consider the Beilinson regulator [7] from the motivic cohomology to the Deligne cohomology

$$r_{\mathcal{D}}: H_{\mathcal{M}}^3(X, \mathbb{Q}(2)) \rightarrow H_{\mathcal{D}}^3(X_{\mathbb{C}}, \mathbb{Q}(2)).$$

In terms of algebraic K -theory, $H_{\mathcal{M}}^3(X, \mathbb{Q}(2)) = (K_1(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(2)}$ (the second eigenspace for the Adams operations). Let Z_1 be the union of fibers over μ_l and consider the image of $H_{\mathcal{M}, Z_1}^3(X, \mathbb{Q}(2)) \rightarrow H_{\mathcal{M}}^3(X, \mathbb{Q}(2))$. The Deligne cohomology can be regarded as functionals on $F^1 H_{\text{dR}}^2(X)$ up to periods, and we restrict them to $F^1 H_{\text{dR}}$.

Theorem 1.2 (Regulator formula, see Theorem 6.5). *Let χ be an embedding such that $H_{\text{dR}}^X \subset F^1 H_{\text{dR}}$. Then, for any $z \in H_{\mathcal{M}, Z_1}^3(X, \mathbb{Q}(2))$ and $\omega \in H_{\text{dR}}^X$, we have*

$$r_{\mathcal{D}}(z)(\omega) \sim_{K \times} B(1 - \alpha, \beta) \cdot {}_3F_2 \left(\begin{matrix} 1 - \alpha, \beta, \beta - \alpha + \mu \\ 1 - \alpha + \beta, \beta - \alpha + \mu + 1 \end{matrix}; 1 \right)$$

where α, β, μ are as before.

Moreover, we shall show the non-vanishing of the regulator image under a mild assumption (Theorem 6.6).

Regarding these examples, it is tempting to ask if the regulators and hence the L -values of a motive with CM by an abelian field can be written in terms of values of ${}_{k+1}F_k$, with k depending on the weight. In a forthcoming paper [4], we shall study more general fibrations of varieties over $\mathbb{P}^1$ with multiplication by a number field whose relative H^1 has a special type of monodromy.

Concerning the period conjecture, there is a result of Maillot–Roessler [17] using Arakelov theory on the absolute value of the period. Recently, Fresán [13] proved the formula for the alternating product of the determinants for any smooth projective variety with a finite order automorphism by reducing to a result of Saito–Terasoma [23]. Since we prove $\dim_K H_B = 1$ and $H^1(X) = H^3(X) = 0$, the Gross–Deligne conjecture for our H follows from Fresán’s result. We need, however, our precise computations for the study of regulators.

Our method is quite different from the previous works mentioned above. A crucial step is to compute explicitly Deligne’s canonical extension $\mathcal{H}_e$ of the Gauss–Manin connection on the relative first de Rham cohomology. Our fibration is

smooth outside $D := \{0, \infty\} \cup \mu_l$ and there is a connection

$$\nabla: \mathcal{H}_e \rightarrow \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{H}_e.$$

We shall describe it explicitly and determine the Hodge structure of H . The 1-periods of the fiber are Gauss hypergeometric functions ${}_2F_1$. By the integral representation of Euler type, the 2-periods of X are firstly written in terms of ${}_3F_2$ -values, which then turn out to be ${}_2F_1$ -values. The conjecture follows by comparing these computations.

It is more delicate in general to compute the regulators of given motivic elements, even for a fibration of curves. Here we use a new technique of the first author. It was originally written in the unpublished paper [3], but he decided to include it in the appendix of the present paper. Via the canonical extension, we shall represent elements of $F^1 H_{\text{dR}}$ by certain rational 2-forms. Then the regulators are expressed as integrals of those rational forms over Lefschetz thimbles, which are again written in terms of ${}_3F_2$ -values.

This paper proceeds as follows. In Sect. 2, we fix the setting and compute the 1-periods of the fiber and 2-periods of X . In Sect. 3, we determine the Gauss–Manin connection and the canonical extension. In Sect. 4, we determine the Hodge structure and show that H_B is one-dimensional over K . In Sect. 5, we give a basis of $F^1 H_{\text{dR}}$ and verify the Gross–Deligne conjecture. In Sect. 6, we prove the regulator formula and discuss the non-vanishing. The appendix (Sect. 7) due to the first author provides the technique to compute the regulators.

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Notations. Throughout this paper, $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For each positive integer N , μ_N denotes the group of N th roots of unity and we put $\zeta_N = e^{2\pi i/N}$. For a real number x , we write $x = [x] + \{x\}$ with $[x] \in \mathbb{Z}$, $0 \leq \{x\} < 1$, and put $\lceil x \rceil = -\lfloor -x \rfloor$. For $\alpha \in \mathbb{C}$ and an integer $n \geq 0$, $(\alpha)_n = \prod_{i=0}^{n-1} (\alpha + i)$ is the Pochhammer symbol and the generalized hypergeometric function is defined by

$${}_pF_q \left(\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix}; x \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p (\alpha_i)_n}{\prod_{j=1}^q (\beta_j)_n} \frac{x^n}{n!}.$$

We often drop the subscripts from ${}_pF_q$. It converges at $x = 1$ when $\text{Re}(\sum_j \beta_j - \sum_i \alpha_i) > 0$. We use the standard notation for the product of Γ -values

$$\Gamma \left(\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \right) = \frac{\prod_{i=1}^p \Gamma(\alpha_i)}{\prod_{j=1}^q \Gamma(\beta_j)}.$$

For a variety X over $\overline{\mathbb{Q}}$, $H_{\text{dR}}^n(X) = H_{\text{dR}}^n(X/\overline{\mathbb{Q}})$ denotes the algebraic de Rham cohomology and $H^n(X, \mathbb{Q})$ denotes the Betti cohomology of the analytic manifold $X(\mathbb{C})$, or the associated mixed Hodge structure.

2. PRELIMINARIES

2.1. The setting. Let p, l be distinct prime numbers and a, b, c be integers with $0 < a, b, c < p$ (we shall soon assume that $b + c = p$). We define a fibration of curves $f: X \rightarrow \mathbb{P}^1$ as follows. Let $g: Y \rightarrow \mathbb{P}^1$ be a proper flat morphism over $\overline{\mathbb{Q}}$ whose fiber Y_t at $t \in \mathbb{P}^1$ is the normalization of the curve defined by

$$y^p = x^a(1-x)^b(t-x)^c.$$

Then, g is smooth outside $\{0, 1, \infty\}$ and by the Riemann–Hurwitz formula, the genus of the generic fiber is $p - 1$. The fiber Y_1 is a union of $\mathbb{P}^1$ intersecting transversally with each other. We have an automorphism σ of order p of Y over $\mathbb{P}^1$ defined by

$$\sigma(x, y) = (x, \zeta_p^{-1}y).$$

Let $g^{(l)}: Y^{(l)} \rightarrow \mathbb{P}^1$ be the base change of g by the morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1; t \mapsto t^l$. The action of σ extends naturally to $Y^{(l)}$. On the other hand, the automorphism

$$\tau(t) = \zeta_l t$$

of $\mathbb{P}^1$ induces an automorphism τ of $Y^{(l)}$ over Y . There is a desingularization X of $Y^{(l)}$ such that σ and τ extend to automorphisms of X respectively over $\mathbb{P}^1$ and Y (for example, if one takes a sequence of blow-ups only at the singular points, then σ and τ extend automatically). As a result, we obtain a fibration $f: X \rightarrow \mathbb{P}^1$ of curves in the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y^{(l)} & \longrightarrow & Y \\ & \searrow f & \downarrow g^{(l)} & \square & \downarrow g \\ & & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

and for $t \notin \{0, \infty\} \cup \mu_l$, the fiber X_t is isomorphic to Y_{t^l} .

2.2. CM Hodge–de Rham structures. A *Hodge–de Rham structure* is a quadruple $H = (H_{\text{dR}}, H_B, \iota, F^\bullet)$ consisting of

- a finite-dimensional $\overline{\mathbb{Q}}$ -vector space H_{dR} ,
- a finite-dimensional $\mathbb{Q}$ -vector space H_B ,
- an isomorphism $\iota: H_{\text{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \rightarrow H_B \otimes_{\mathbb{Q}} \mathbb{C}$,
- a descending filtration $F^\bullet H_{\text{dR}}$ which induces a Hodge structure on H_B via ι .

For a proper smooth variety X over $\overline{\mathbb{Q}}$, its n th de Rham and Betti cohomology groups, the comparison isomorphism and the Hodge filtration define a Hodge–de Rham structure $H^n(X)$.

Let K be a finite extension of $\mathbb{Q}$. We say that H admits a K -multiplication if we are given K -actions on H_{dR} and H_B which are compatible with ι and $F^\bullet$. Moreover, we say that H has CM by K if $\dim_K H_B = 1$. For each embedding $\chi: K \hookrightarrow \mathbb{C}$, let $H_{\text{dR}}^\chi, H_B^\chi := (H_B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\chi$ denote the subspace on which K acts as the multiplication via χ . If $\dim_K H_B = 1$, then these subspaces are 1-dimensional over $\overline{\mathbb{Q}}$. Choosing any bases $\omega_{\text{dR}} \in H_{\text{dR}}^\chi$ and $\omega_B \in H_B^\chi$, we define the *period* $\text{Per}(H^\chi) \in \mathbb{C}^\times$ by

$$\iota(\omega_{\text{dR}}) = \text{Per}(H^\chi)\omega_B.$$

By the ambiguity of the choices, $\text{Per}(H^\chi)$ is only well-defined up to $\overline{\mathbb{Q}}^\times$. If $(H_{\text{dR}}, F^\bullet)$ is already defined over K , the period is well-defined up to $K^\times$.

Let X be as in Sect. 2.1 and let

$$Z = X \times_{\mathbb{P}^1} (\{0, \infty\} \cup \mu_l)$$

be the union of the singular fibers. Note that Z is stable under the actions of σ and τ . Put

$$R = \mathbb{Q}[\sigma, \tau], \quad K = \mathbb{Q}(\mu_{lp})$$

and regard K as an R -algebra by $\sigma \mapsto \zeta_p$, $\tau \mapsto \zeta_l$. The Hodge–de Rham structure we consider in this paper is

$$H := \text{Coker}(H_Z^2(X) \rightarrow H^2(X)) \otimes_R K.$$

It admits a K -multiplication and we shall show that $\text{rank}_K H_B = 1$ (Theorem 4.12). An embedding $\chi: K \hookrightarrow \mathbb{C}$ is identified with an element $h \in (\mathbb{Z}/lp\mathbb{Z})^\times$ such that $\chi(\zeta_{lp}) = \zeta_{lp}^h$. If

$$\text{Coker}(H_Z^2(X) \rightarrow H^2(X)) = \bigoplus_{m \in \mathbb{Z}/l\mathbb{Z}, n \in \mathbb{Z}/p\mathbb{Z}} H^{(m,n)}$$

denotes the decomposition into the eigenspaces on which τ (resp. σ) acts by ζ_l^m (resp. ζ_p^n), we have

$$H = \bigoplus_{m \neq 0, n \neq 0} H^{(m,n)}.$$

2.3. Periods of the fiber. For $n = 1, \dots, p-1$ and integers i, j, k , put a rational 1-form on Y_t by

$$\omega_n^{ijk} = \frac{x^i(1-x)^j(t-x)^k}{y^n} dx.$$

Then, we have

$$\sigma^* \omega_n^{ijk} = \zeta_p^n \omega_n^{ijk}. \quad (2.1)$$

Let $0 < t < 1$, and δ_0 be a path on Y_t from $(0, 0)$ to $(t, 0)$ defined by

$$x = ts, \quad y = \sqrt[p]{x^a(1-x)^b(t-x)^c}.$$

Let δ_1 be a path on Y_t from $(t, 0)$ to $(1, 0)$ defined by

$$x = t + (1-t)s, \quad y = \varepsilon^c \sqrt[p]{x^a(1-x)^b(x-t)^c}$$

where we put

$$\varepsilon = \begin{cases} i & \text{if } p = 2, \\ -1 & \text{if } p \text{ is odd.} \end{cases}$$

If we put

$$\kappa_m = (1 - \sigma)_* \delta_m \quad (m = 0, 1),$$

these define 1-cycles on Y_t , and we have

$$\int_{\kappa_m} \omega_n^{ijk} = \int_{\delta_m} (1 - \sigma)^* \omega_n^{ijk} = (1 - \zeta_p^n) \int_{\delta_m} \omega_n^{ijk}. \quad (2.2)$$

Lemma 2.1. *Fix integers $i, j, k \geq 0$. For $n = 1, \dots, p-1$, put*

$$\alpha = \frac{na}{p} - i, \quad \beta = \frac{nb}{p} - j, \quad \gamma = \frac{nc}{p} - k.$$

Then we have

$$\begin{aligned} \int_{\delta_0} \omega_n^{ijk} &= B(1-\alpha, 1-\gamma) \cdot t^{1-\alpha-\gamma} F\left(\begin{matrix} 1-\alpha, \beta \\ 2-\alpha-\gamma \end{matrix}; t\right), \\ \int_{\delta_1} \omega_n^{ijk} &= \varepsilon^{p\gamma} B(1-\beta, 1-\gamma) \cdot (1-t)^{1-\beta-\gamma} F\left(\begin{matrix} \alpha, 1-\beta \\ 2-\beta-\gamma \end{matrix}; 1-t\right). \end{aligned}$$

Proof. The first equality follows directly from Euler's integral representation of the Gauss hypergeometric function ${}_2F_1$:

$$B(b, c-b) \cdot F\left(\begin{matrix} a, b \\ c \end{matrix}; t\right) = \int_0^1 (1-tx)^{-a} x^{b-1} (1-x)^{c-b-1} dx$$

(let $a = \beta$, $b = 1-\alpha$, $c = 2-\alpha-\gamma$). The second one follows from the same formula and the transformation formula

$$F\left(\begin{matrix} a, c-b \\ c \end{matrix}; 1-\frac{1}{t}\right) = t^a \cdot F\left(\begin{matrix} a, b \\ c \end{matrix}; 1-t\right).$$

□

2.4. Cohomology of the fiber. We have decompositions

$$\begin{aligned} H^1(Y_t, \mathbb{C}) &= \bigoplus_{n=1}^{p-1} H^1(Y_t, \mathbb{C})^{(n)}, \\ H_1(Y_t, \mathbb{Q}(\mu_p)) &= \bigoplus_{n=1}^{p-1} H_1(Y_t, \mathbb{Q}(\mu_p))^{(n)}, \end{aligned}$$

where $^{(n)}$ denotes the subspace on which σ^* (resp. σ_*) acts as the multiplication by ζ_p^n . Note that $H^1(Y_t, \mathbb{C})^{(0)} = 0$ since Y_t/μ_p is a rational curve. The natural pairing induces a non-degenerate pairing

$$H^1(Y_t, \mathbb{C})^{(n)} \otimes H_1(Y_t, \mathbb{Q}(\zeta_p))^{(n)} \rightarrow \mathbb{C}.$$

We shall give bases of these spaces under a certain assumption.

Lemma 2.2. *Let $n = 1, \dots, p-1$ and $i, j, k \geq 0$ be integers.*

- (i) *If $p \nmid a+b+c$, then ω_n^{ijk} is a differential form of the second kind.*
- (ii) *Moreover, ω_n^{ijk} is holomorphic if and only if*

$$\begin{aligned} i &\geq \frac{na+1}{p} - 1, \quad j \geq \frac{nb+1}{p} - 1, \quad k \geq \frac{nc+1}{p} - 1, \\ i+j+k &\leq \frac{n(a+b+c)-1}{p} - 1. \end{aligned}$$

Proof. See [2] (18) (but see loc. cit. (13) for the correct sign in the fourth inequality). □

From now on, we assume:

$$b+c=p.$$

Then the condition $p \nmid a + b + c$ is automatically satisfied. By Lemma 2.2, ω_n^{ijk} is holomorphic if and only if

$$i = \left\lfloor \frac{na+1}{p} \right\rfloor - 1, \quad j = \left\lfloor \frac{nb+1}{p} \right\rfloor - 1, \quad k = \left\lfloor \frac{nc+1}{p} \right\rfloor - 1,$$

and we write this ω_n^{ijk} simply as ω_n . The α, β, γ in Lemma 2.1 become

$$\alpha = \left\{ \frac{na}{p} \right\}, \quad \beta = \left\{ \frac{nb}{p} \right\}, \quad \gamma = \left\{ \frac{nc}{p} \right\} = 1 - \beta.$$

In particular, $0 < \alpha, \beta, \gamma < 1$. Though these depend on n , we shall suppress it from the notation. By Lemma 2.1, we have

$$\begin{aligned} \int_{\delta_0} \omega_n &= B(1 - \alpha, \beta) \cdot t^{\beta - \alpha} F \left(\begin{matrix} 1 - \alpha, \beta \\ 1 - \alpha + \beta \end{matrix}; t \right), \\ \int_{\delta_1} \omega_n &= -\varepsilon^{p\beta} B(1 - \beta, \beta) \cdot F \left(\begin{matrix} \alpha, 1 - \beta \\ 1 \end{matrix}; 1 - t \right). \end{aligned} \quad (2.3)$$

For each n , let i, j, k be as above and put

$$\eta_n = \omega_n^{i, j+1, k}.$$

Then, β is replaced with $\beta - 1$ in Lemma 2.1 and we obtain

$$\begin{aligned} \int_{\delta_0} \eta_n &= B(1 - \alpha, \beta) \cdot t^{\beta - \alpha} F \left(\begin{matrix} 1 - \alpha, \beta - 1 \\ 1 - \alpha + \beta \end{matrix}; t \right), \\ \int_{\delta_1} \eta_n &= -\varepsilon^{p\beta} B(1 - \beta, \beta) \cdot (1 - \beta)(1 - t) F \left(\begin{matrix} \alpha, 2 - \beta \\ 2 \end{matrix}; 1 - t \right). \end{aligned} \quad (2.4)$$

Here we used $B(2 - \beta, \beta) = (1 - \beta)B(1 - \beta, \beta)$.

Proposition 2.3. *Let $n = 1, \dots, p-1$ and $0 < t < 1$. Then, $\{\omega_n, \eta_n\}$ is a basis of $H^1(Y_t, \mathbb{C})^{(n)}$.*

Proof. By (2.1), (2.2), (2.3) and (2.4), ω_n, η_n are non-trivial elements of $H^1(Y_t, \mathbb{C})^{(n)}$. Since ω_n is holomorphic and η_n is not, they are linearly independent. Since $\dim H^1(Y_t, \mathbb{C}) = 2(p-1)$, the proposition follows. $\square$

Proposition 2.4. *Let $n = 1, \dots, p-1$ and $0 < t < 1$.*

- (i) *The projections of κ_0, κ_1 form a basis of $H_1(Y_t, \mathbb{Q}(\mu_p))^{(n)}$.*
- (ii) *As a $\mathbb{Q}[\sigma]$ -module, $H_1(Y_t, \mathbb{Q})$ is generated by κ_0 and κ_1 .*

Proof. Put the period matrix

$$M_n(t) = \begin{pmatrix} \int_{\kappa_0} \omega_n & \int_{\kappa_0} \eta_n \\ \int_{\kappa_1} \omega_n & \int_{\kappa_1} \eta_n \end{pmatrix}.$$

It suffices to show that $\det M_n(t) \neq 0$. Since $\prod_{n=1}^{p-1} \det M_n(t)$ is constant, it coincides with its limit as $t \rightarrow 1$. Hence the proposition follows from the lemma below. $\square$

Lemma 2.5. *We have*

$$\lim_{t \rightarrow 1} \det M_n(t) = \varepsilon^{p\beta} (1 - \zeta_p^n)^2 \cdot \frac{B(\beta, 1 - \beta)}{1 - \alpha}.$$

Proof. By (2.2), (2.3), (2.4), we have

$$\det M_n(t) = -\varepsilon^{p\beta}(1-\zeta_p^n)^2 B(1-\alpha, \beta) B(1-\beta, \beta) t^{\beta-\alpha} \\ \times \det \begin{pmatrix} F\left(\begin{smallmatrix} 1-\alpha, \beta \\ 1-\alpha+\beta \end{smallmatrix}; t\right) & F\left(\begin{smallmatrix} 1-\alpha, \beta-1 \\ 1-\alpha+\beta \end{smallmatrix}; t\right) \\ F\left(\begin{smallmatrix} \alpha, 1-\beta \\ 1 \end{smallmatrix}; 1-t\right) & (1-\beta)(1-t)F\left(\begin{smallmatrix} \alpha, 2-\beta \\ 2 \end{smallmatrix}; 1-t\right) \end{pmatrix}.$$

Firstly, we have

$$\lim_{t \rightarrow 1} (1-t)F\left(\begin{smallmatrix} 1-\alpha, \beta \\ 1-\alpha+\beta \end{smallmatrix}; t\right) = 0.$$

This follows from the transformation formula (cf. [12], p. 74 (2))

$$F\left(\begin{smallmatrix} 1-\alpha, \beta \\ 1-\alpha+\beta \end{smallmatrix}; t\right) = \frac{1}{B(1-\alpha, \beta)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n (\beta)_n}{(n!)^2} (k_n - \log(1-t))(1-t)^n, \\ k_n := 2\psi(n+1) - \psi(1-\alpha+n) - \psi(\beta+n)$$

where $\psi(t) = \Gamma'(t)/\Gamma(t)$ is the digamma function. On the other hand, by Euler's formula, we have

$$F\left(\begin{smallmatrix} 1-\alpha, \beta-1 \\ 1-\alpha+\beta \end{smallmatrix}; 1\right) = \Gamma\left(\begin{smallmatrix} 1-\alpha+\beta \\ 2-\alpha, \beta \end{smallmatrix}\right) = \frac{1}{(1-\alpha)B(1-\alpha, \beta)}.$$

Hence the lemma follows. $\square$

2.5. Periods of X . Now we consider the fibration $f: X \rightarrow \mathbb{P}^1$. Recall that $X_t \simeq Y_{t^i}$. By abuse of notation, for each $s = 0, 1$, let δ_s (resp. κ_s) be the path (resp. loop) on X_t which corresponds to the one on Y_{t^i} defined in §2.3. For each s , let Δ_s be the 2-simplex obtained by sweeping δ_s along $0 \leq t \leq 1$. Since δ_s is vanishing as $t \rightarrow s$, the Lefschetz thimble $(1-\sigma)_* \Delta_s$ has boundary on the fiber X_{1-s} . We shall use $(1-\sigma)_* \Delta_1$ (resp. $(1-\sigma)_* \Delta_0$) to compute the periods (resp. regulators). Again by abuse of notation, let ω_n denote the pull-back to X of the rational 1-form ω_n on Y defined in §2.4. For $n = 1, \dots, p-1$ and an integer m , define rational 2-forms on X by

$$\omega_{m,n} = t^m \frac{dt}{t} \wedge \omega_n, \quad \eta_{m,n} = t^m \frac{dt}{t} \wedge \eta_n.$$

We have evidently,

$$(\tau^i \sigma^j)^* \omega_{m,n} = \zeta_l^{mi} \zeta_p^{nj} \omega_{m,n}, \quad (\tau^i \sigma^j)^* \eta_{m,n} = \zeta_l^{mi} \zeta_p^{nj} \eta_{m,n}.$$

Proposition 2.6. *Let $n = 1, \dots, p-1$ and $\alpha = \{\frac{na}{p}\}$, $\beta = \{\frac{nb}{p}\}$ as before. For an integer m , put $\mu = m/l$.*

(i) *If $\mu > \alpha - \beta$, then we have*

$$\int_{\Delta_1} \omega_{m,n} = -\frac{\varepsilon^{p\beta}}{l} \cdot B(\beta, \mu) B(1-\beta, \beta - \alpha + \mu), \\ \int_{\Delta_1} \eta_{m,n} = -\frac{\varepsilon^{p\beta}(1-\beta)}{l(1-\alpha+\mu)} \cdot B(\beta, \mu) B(1-\beta, \beta - \alpha + \mu).$$

(ii) *We have*

$$\int_{\Delta_0} \omega_{m,n} = \frac{B(1-\alpha, \beta)}{l(\beta - \alpha + \mu)} \cdot F\left(\begin{smallmatrix} 1-\alpha, \beta, \beta - \alpha + \mu \\ 1-\alpha+\beta, \beta - \alpha + \mu + 1 \end{smallmatrix}; 1\right), \\ \int_{\Delta_0} \eta_{m,n} = \frac{B(1-\alpha, \beta)}{l(\beta - \alpha + \mu)} \cdot F\left(\begin{smallmatrix} 1-\alpha, \beta-1, \beta - \alpha + \mu \\ 1-\alpha+\beta, \beta - \alpha + \mu + 1 \end{smallmatrix}; 1\right).$$

Proof. Recall the integral representation of ${}_3F_2$ (cf. [25, (4.1.2)]):

$$\Gamma\left(\begin{matrix} c, e-c \\ e \end{matrix}\right) F\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; t\right) = \int_0^1 F\left(\begin{matrix} a, b \\ d \end{matrix}; tx\right) x^{c-1} (1-x)^{e-c-1} dx.$$

By (2.3), we have

$$\begin{aligned} \int_{\Delta_1} \omega_{m,n} &= -\varepsilon^{p\beta} B(\beta, 1-\beta) \int_0^1 F\left(\begin{matrix} \alpha, 1-\beta \\ 1 \end{matrix}; 1-t^l\right) t^{m-1} dt \\ &= -\varepsilon^{p\beta} \frac{B(\beta, 1-\beta)}{l} \int_0^1 F\left(\begin{matrix} \alpha, 1-\beta \\ 1 \end{matrix}; 1-t\right) t^{\mu-1} dt \\ &= -\varepsilon^{p\beta} \frac{B(\beta, 1-\beta)}{l} \int_0^1 F\left(\begin{matrix} \alpha, 1-\beta \\ 1 \end{matrix}; t\right) (1-t)^{\mu-1} dt \\ &= -\varepsilon^{p\beta} \frac{B(\beta, 1-\beta)}{l\mu} F\left(\begin{matrix} \alpha, 1-\beta, 1 \\ 1, \mu+1 \end{matrix}; 1\right) \\ &= -\varepsilon^{p\beta} \frac{B(\beta, 1-\beta)}{l\mu} F\left(\begin{matrix} \alpha, 1-\beta \\ \mu+1 \end{matrix}; 1\right), \end{aligned}$$

which converges by the assumption. Using Euler's formula

$$F\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \Gamma\left(\begin{matrix} c, c-a-b \\ c-a, c-b \end{matrix}\right) \quad (\operatorname{Re}(c-a-b) > 0)$$

and the functional equations

$$\Gamma(x+1) = x\Gamma(x), \quad B(x, y) = \Gamma\left(\begin{matrix} x, y \\ x+y \end{matrix}\right),$$

we obtain the first equality of (i). The others follow similarly, using (2.4) for $\eta_{m,n}$. $\square$

3. CANONICAL EXTENSION

In this section, we compute the Gauss–Manin connection of the fibration and determine its canonical extension to $\mathbb{P}^1$.

3.1. Gauss–Manin connection. Let us start with the fibration $g: Y \rightarrow \mathbb{P}^1$; for a while, t denotes the coordinate of the base scheme of g . Put

$$T = \mathbb{P}^1 \setminus \{0, 1, \infty\}, \quad Y_T = Y \times_{\mathbb{P}^1} T.$$

Then the restriction $g: Y_T \rightarrow T$ is smooth. Put

$$\mathcal{H} = R^1 g_* \Omega_{Y_T/T}^\bullet, \quad \Omega_T^1 = \Omega_{T/\overline{\mathbb{Q}}}^1,$$

and let

$$\nabla: \mathcal{H} \rightarrow \Omega_T^1 \otimes \mathcal{H}$$

be the Gauss–Manin connection. For each $n = 1, \dots, p-1$, let $\mathcal{H}^{(n)} \subset \mathcal{H}$ be the subbundle on which σ^* acts as the multiplication by ζ_p^n . Then $\mathcal{H}^{(n)}$ is locally generated by ω_n, η_n as defined in Sect. 2.4, and the Hodge filtration $F^1 \mathcal{H}^{(n)}$ is generated by ω_n .

Proposition 3.1. *For $n = 1, \dots, p-1$, the Gauss–Manin connection*

$$\nabla: \mathcal{H}^{(n)} \rightarrow \Omega_T^1 \otimes \mathcal{H}^{(n)}$$

is given by

$$(\nabla \omega_n, \nabla \eta_n) = \frac{dt}{t} \otimes (\omega_n, \eta_n) \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\alpha \end{pmatrix} \begin{pmatrix} -1 & -1 \\ (1-t)^{-1} & 1 \end{pmatrix}$$

where we put $\alpha = \{\frac{na}{p}\}$, $\beta = \{\frac{nb}{p}\}$ as before.

Proof. We use the following standard derivation relations among Gauss hypergeometric functions (cf. [25], (1.4.1.1), (1.4.1.6)):

$$\frac{d}{dt} F \left(\begin{matrix} a, b \\ c \end{matrix}; t \right) = \frac{ab}{c} F \left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; t \right), \quad (3.1)$$

$$\frac{d}{dt} \left(t^{c-1} F \left(\begin{matrix} a, b \\ c \end{matrix}; t \right) \right) = (c-1)t^{c-2} F \left(\begin{matrix} a, b \\ c-1 \end{matrix}; t \right). \quad (3.2)$$

We also use the following contiguous relations (cf. loc. cit. (1.4.1), (1.4.3), (1.4.5), (1.4.9), (1.4.13)):

$$(c-2a+(a-b)t)F + a(1-t)F[a+1] = (c-a)F[a-1], \quad (3.3)$$

$$(c-a-b)F + a(1-t)F[a+1] = (c-b)F[b-1], \quad (3.4)$$

$$(c-a-1)F + aF[a+1] = (c-1)F[c-1], \quad (3.5)$$

$$(a-1+(1+b-c)t)F + (c-a)F[a-1] = (c-1)(1-t)F[c-1], \quad (3.6)$$

$$c(1-t)F + (c-a)tF[c+1] = cF[b-1]. \quad (3.7)$$

Here, $F = F \left(\begin{matrix} a, b \\ c \end{matrix}; t \right)$ and the notation $F[a+1]$ for example means $F \left(\begin{matrix} a+1, b \\ c \end{matrix}; t \right)$.

We are reduced to show:

$$t \frac{d}{dt} M_n(t) = M_n(t) \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\alpha \end{pmatrix} \begin{pmatrix} -1 & -1 \\ (1-t)^{-1} & 1 \end{pmatrix}. \quad (3.8)$$

We prove this for each row vector. For the first row vector, put

$$(f(t), g(t)) = \left(t^{\beta-\alpha} F \left(\begin{matrix} 1-\alpha, \beta \\ 1-\alpha+\beta \end{matrix}; t \right), t^{\beta-\alpha} F \left(\begin{matrix} 1-\alpha, \beta-1 \\ 1-\alpha+\beta \end{matrix}; t \right) \right).$$

Firstly, consider the case $\alpha \neq \beta$. By (3.2), we have

$$t \frac{d}{dt} (f(t), g(t)) = \left((\beta-\alpha)t^{\beta-\alpha} F \left(\begin{matrix} 1-\alpha, \beta \\ -\alpha+\beta \end{matrix}; t \right), (\beta-\alpha)t^{\beta-\alpha} F \left(\begin{matrix} 1-\alpha, \beta-1 \\ -\alpha+\beta \end{matrix}; t \right) \right).$$

Applying (3.6) to $F \left(\begin{matrix} \beta, 1-\alpha \\ 1-\alpha+\beta \end{matrix}; t \right)$, we obtain

$$t \frac{d}{dt} f(t) = -(1-\beta)f(t) + (1-\alpha)(1-t)^{-1}g(t).$$

Applying (3.5) to $F \left(\begin{matrix} \beta-1, 1-\alpha \\ 1-\alpha+\beta \end{matrix}; t \right)$, we obtain

$$t \frac{d}{dt} g(t) = -(1-\beta)f(t) + (1-\alpha)g(t).$$

Hence we are done. Now consider the case $\alpha = \beta$. Then

$$(f(t), g(t)) = \left(F \left(\begin{matrix} 1-\alpha, \alpha \\ 1 \end{matrix}; t \right), F \left(\begin{matrix} 1-\alpha, \alpha-1 \\ 1 \end{matrix}; t \right) \right).$$

By (3.1), we have

$$\frac{d}{dt}(f(t), g(t)) = \left((1-\alpha)\alpha F\left(\begin{matrix} 2-\alpha, 1+\alpha \\ 2 \end{matrix}; t\right), -(1-\alpha)^2 F\left(\begin{matrix} 2-\alpha, \alpha \\ 2 \end{matrix}; t\right) \right).$$

Applying (3.7) to $F\left(\begin{matrix} 2-\alpha, 1+\alpha \\ 1 \end{matrix}; t\right)$, we have

$$t \frac{d}{dt} f(t) = \alpha(1-t) F\left(\begin{matrix} 2-\alpha, 1+\alpha \\ 1 \end{matrix}; t\right) - \alpha F\left(\begin{matrix} 2-\alpha, \alpha \\ 1 \end{matrix}; t\right). \quad (3.9)$$

Applying (3.4) to $F\left(\begin{matrix} 1-\alpha, 1+\alpha \\ 1 \end{matrix}; t\right)$, we have

$$(1-\alpha)(1-t) F\left(\begin{matrix} 2-\alpha, 1+\alpha \\ 1 \end{matrix}; t\right) = F\left(\begin{matrix} 1-\alpha, 1+\alpha \\ 1 \end{matrix}; t\right) - \alpha f(t). \quad (3.10)$$

Applying (3.3) to $F\left(\begin{matrix} \alpha, 1-\alpha \\ 1 \end{matrix}; t\right)$, we have

$$\alpha(1-t) F\left(\begin{matrix} 1-\alpha, 1+\alpha \\ 1 \end{matrix}; t\right) = (2\alpha-1)(1-t)f(t) + (1-\alpha)g(t). \quad (3.11)$$

Applying (3.4) to $F\left(\begin{matrix} 1-\alpha, \alpha \\ 1 \end{matrix}; t\right)$, we have

$$(1-t) F\left(\begin{matrix} 2-\alpha, \alpha \\ 1 \end{matrix}; t\right) = g(t). \quad (3.12)$$

Combining (3.9)–(3.12), we obtain

$$t \frac{d}{dt} f(t) = (1-\alpha) (-f(t) + (1-t)^{-1} g(t)).$$

Applying (3.7) to $F\left(\begin{matrix} \alpha, 2-\alpha \\ 1 \end{matrix}; t\right)$, we have

$$\begin{aligned} t \frac{d}{dt} g(t) &= (1-\alpha) \left(-F\left(\begin{matrix} 1-\alpha, \alpha \\ 1 \end{matrix}; t\right) + (1-t) F\left(\begin{matrix} 2-\alpha, \alpha \\ 1 \end{matrix}; t\right) \right) \\ &\stackrel{(3.12)}{=} (1-\alpha) (-f(t) + g(t)). \end{aligned}$$

In both cases $\alpha \neq \beta$ and $\alpha = \beta$, we have proved (3.8) for the first row vector.

For the second row vector, put

$$(u(t), v(t)) = \left(F\left(\begin{matrix} \alpha, 1-\beta \\ 1 \end{matrix}; 1-t\right), (1-\beta)(1-t) F\left(\begin{matrix} \alpha, 2-\beta \\ 2 \end{matrix}; 1-t\right) \right).$$

Then, by (3.1) and (3.2), we have

$$\frac{d}{dt}(u(t), v(t)) = -(1-\beta) \left(\alpha F\left(\begin{matrix} \alpha+1, 2-\beta \\ 2 \end{matrix}; 1-t\right), F\left(\begin{matrix} \alpha, 2-\beta \\ 1 \end{matrix}; 1-t\right) \right).$$

Applying (3.7) to $F\left(\begin{matrix} \alpha, 2-\beta \\ 1 \end{matrix}; 1-t\right)$, we obtain

$$t \frac{d}{dt} v(t) = -(1-\beta)u(t) + (1-\alpha)v(t). \quad (3.13)$$

Applying (3.4) to $F\left(\begin{matrix} \alpha, 2-\beta \\ 2 \end{matrix}; 1-t\right)$, we have

$$t \frac{d}{dt} u(t) = (\beta-\alpha)(1-t)^{-1}v(t) - (1-\beta)\beta \cdot F\left(\begin{matrix} \alpha, 1-\beta \\ 2 \end{matrix}; 1-t\right). \quad (3.14)$$

Applying (3.6) to $F\left(\begin{smallmatrix} 2-\beta, \alpha \\ 2 \end{smallmatrix}; 1-t\right)$, we have

$$(1-\beta)\beta \cdot F\left(\begin{smallmatrix} \alpha, 1-\beta \\ 2 \end{smallmatrix}; 1-t\right) = (-(1-\beta)(1-t)^{-1} + 1-\alpha) v(t) - t \frac{d}{dt} v(t) \\ \stackrel{(3.13)}{=} (1-\beta) (u(t) - (1-t)^{-1} v(t)). \quad (3.15)$$

Combining (3.14) and (3.15), we obtain

$$t \frac{d}{dt} u(t) = -(1-\beta)u(t) + (1-\alpha)(1-t)^{-1}v(t).$$

Hence we have proved (3.8) for the second row vector. $\square$

3.2. Canonical extension. Now, we return to the fibration $f: X \rightarrow \mathbb{P}^1$, and from now on, t denotes the coordinate of the base scheme of f . Put

$$D = \{0, \infty\} \cup \mu_l, \quad T = \mathbb{P}^1 \setminus D, \quad U = X \times_{\mathbb{P}^1} T, \quad \mathcal{H} = R^1 f_* \Omega_{U/T}^\bullet,$$

and let $\nabla: \mathcal{H} \rightarrow \Omega_T^1 \otimes \mathcal{H}$ be the Gauss–Manin connection. The following is immediate from Proposition 3.1.

Proposition 3.2. *For $n = 1, \dots, p-1$, the Gauss–Manin connection*

$$\nabla: \mathcal{H}^{(n)} \rightarrow \Omega_T^1 \otimes \mathcal{H}^{(n)}$$

is given by

$$\begin{aligned} (\nabla \omega_n, \nabla \eta_n) &= l \frac{dt}{t} \otimes (\omega_n, \eta_n) \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\alpha \end{pmatrix} \begin{pmatrix} -1 & -1 \\ \frac{1}{1-t} & 1 \end{pmatrix} \\ &= l \frac{ds}{s} \otimes (\omega_n, \eta_n) \begin{pmatrix} 1-\beta & 0 \\ 0 & 1-\alpha \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{s^l}{1-s^l} & -1 \end{pmatrix} \end{aligned}$$

where $s = 1/t$.

Let $j: T \rightarrow \mathbb{P}^1$ denote the embedding. Let $\Omega_{\mathbb{P}^1}^1(\log D)$ be the sheaf of differentials on $\mathbb{P}^1$ with logarithmic poles along D . Then the *canonical extension*

$$\nabla: \mathcal{H}_e \rightarrow \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{H}_e$$

of Deligne [9, 5.1] is defined to be the unique sub-bundle of $j_* \mathcal{H}$ satisfying the following properties:

- (i) $\nabla(\mathcal{H}_e) \subset \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{H}_e$,
- (ii) For each $t \in D$, all the eigenvalues of $\text{Res}_t(\nabla)$ lie in the interval $[0, 1)$, where $\text{Res}_t(\nabla)$ denotes the residue at t of the connection matrix.

In fact, we have $\mathcal{H}_e = R^1 f_* \Omega_{X/\mathbb{P}^1}^\bullet(\log Z)$ (recall $Z = X \times_{\mathbb{P}^1} (\{0, \infty\} \cup \mu_l)$) by Steenbrink [26, (2.18), (2.20)]. This is determined as follows.

Proposition 3.3. *For $n = 1, \dots, p-1$, local bases of $\mathcal{H}_e^{(n)}$ at $t \in D$ are given as follows:*

$$\begin{aligned} \mathcal{H}_e^{(n)}|_0 &= \begin{cases} \langle \omega_n - \eta_n, t^{[(\alpha-\beta)l]}((1-\beta)\omega_n - (1-\alpha)\eta_n) \rangle & \text{if } \alpha \neq \beta, \\ \langle \omega_n, \eta_n \rangle & \text{if } \alpha = \beta, \end{cases} \\ \mathcal{H}_e^{(n)}|_\infty &= \begin{cases} \langle t^{[(1-\beta)l]}((1-\alpha-\beta)\omega_n + (1-\alpha)t^{-l}\eta_n), t^{[\alpha l]-l}\eta_n \rangle & \text{if } \alpha + \beta \neq 1, \\ \langle t^{[\alpha l]}\omega_n, t^{[\alpha l]-l}\eta_n \rangle & \text{if } \alpha + \beta = 1, \end{cases} \\ \mathcal{H}_e^{(n)}|_\zeta &= \langle \omega_n, \eta_n \rangle \quad (\zeta \in \mu_l). \end{aligned}$$

The residue matrices with respect to these bases are:

$$\begin{aligned} \text{Res}_0(\nabla) &= \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & \{(\beta - \alpha)l\} \end{pmatrix} & \text{if } \alpha \neq \beta, \\ l(1 - \alpha) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} & \text{if } \alpha = \beta, \end{cases} \\ \text{Res}_\infty(\nabla) &= \begin{cases} \begin{pmatrix} \{(1 - \beta)l\} & 0 \\ 0 & \{\alpha l\} \end{pmatrix} & \text{if } \alpha + \beta \neq 1, \\ \begin{pmatrix} \{\alpha l\} & 0 \\ (\alpha - 1)l & \{\alpha l\} \end{pmatrix} & \text{if } \alpha + \beta = 1, \end{cases} \\ \text{Res}_\zeta(\nabla) &= -(1 - \alpha) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Proof. Let A be the matrix of the connection from Proposition 3.2. For each $t \in D$, we shall find a matrix P with coefficients in local sections of $j_*\mathcal{O}_U$ such that $(\omega_n, \eta_n)P$ is a local basis of $\mathcal{H}_e$ at t . The connection matrix with respect to this basis is given by the gauge transformation

$$A_P := P^{-1}AP + P^{-1}P'$$

where $P' = \frac{d}{dt}P$. For $t = 0$, we let

$$P = \begin{pmatrix} 1 & 1 - \beta \\ -1 & -(1 - \alpha) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t^{\lceil(\alpha - \beta)l\rceil} \end{pmatrix}$$

if $\alpha \neq \beta$, and $P = I$ (the unit matrix) if $\alpha = \beta$. For $t = \zeta \in \mu_l$, we let $P = I$. Finally for $t = \infty$, we let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & t^{-l} \end{pmatrix} \begin{pmatrix} 1 - \alpha - \beta & 0 \\ 1 - \alpha & 1 \end{pmatrix} \begin{pmatrix} t^{\lfloor(1 - \beta)l\rfloor} & 0 \\ 0 & t^{\lfloor\alpha l\rfloor} \end{pmatrix}$$

if $\alpha + \beta \neq 1$, and

$$P = \begin{pmatrix} t^{\lfloor\alpha l\rfloor} & 0 \\ 0 & t^{\lfloor\alpha l\rfloor - l} \end{pmatrix}$$

if $\alpha + \beta = 1$. Then, one verifies that A_P satisfies the desired properties and its residue is given as stated. $\square$

To see the Hodge filtration, we rewrite the above bases as follows.

Corollary 3.4. *Let $n = 1, \dots, p - 1$.*

$$\begin{aligned} \mathcal{H}_e^{(n)}|_{t=0} &= \begin{cases} \langle \omega_n, t^{-\lfloor(\beta - \alpha)l\rfloor}((1 - \beta)\omega_n - (1 - \alpha)\eta_n) \rangle & \text{if } \alpha \leq \beta, \\ \langle t^{\lceil(\alpha - \beta)l\rceil}\omega_n, \omega_n - \eta_n \rangle & \text{if } \alpha > \beta. \end{cases} \\ \mathcal{H}_e^{(n)}|_{t=\infty} &= \begin{cases} \langle t^{\lfloor(1 - \beta)l\rfloor}\omega_n, t^{\lfloor\alpha l\rfloor - l}\eta_n \rangle & \text{if } \lfloor\alpha l\rfloor \geq \lfloor(1 - \beta)l\rfloor, \\ \langle t^{\lfloor\alpha l\rfloor}\omega_n, t^{\lfloor(1 - \beta)l\rfloor}((1 - \alpha - \beta)\omega_n + (1 - \alpha)t^{-l}\eta_n) \rangle & \text{if } \lfloor\alpha l\rfloor < \lfloor(1 - \beta)l\rfloor. \end{cases} \\ \mathcal{H}_e^{(n)}|_{t=\zeta} &= \langle \omega_n, \eta_n \rangle \quad (\zeta \in \mu_l). \end{aligned}$$

Write $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$ and define $F^1\mathcal{H}_e = \mathcal{H}_e \cap j_*(F^1\mathcal{H})$. Then, we have immediately:

Corollary 3.5. *Let $n = 1, \dots, p - 1$.*

- (i) We have $F^1 \mathcal{H}_e^{(n)} = \mathcal{O}(i)t^j \omega_n$ with
- $$(i, j) = \begin{cases} (\lfloor (1-\beta)l \rfloor, 0) & \text{if } \lfloor \alpha l \rfloor \geq \lfloor (1-\beta)l \rfloor, \alpha \leq \beta, \\ (\lfloor (1-\beta)l \rfloor - \lceil (\alpha-\beta)l \rceil, \lceil (\alpha-\beta)l \rceil) & \text{if } \lfloor \alpha l \rfloor \geq \lfloor (1-\beta)l \rfloor, \alpha > \beta, \\ (\lfloor \alpha l \rfloor, 0) & \text{if } \lfloor \alpha l \rfloor < \lfloor (1-\beta)l \rfloor, \alpha \leq \beta, \\ (\lfloor \alpha l \rfloor - \lceil (\alpha-\beta)l \rceil, \lceil (\alpha-\beta)l \rceil) & \text{if } \lfloor \alpha l \rfloor < \lfloor (1-\beta)l \rfloor, \alpha > \beta. \end{cases}$$
- (ii) According as the four cases as above, we have
- $$\mathrm{Gr}_F^0 \mathcal{H}_e^{(n)} = \begin{cases} \mathcal{O}(-\lceil (1-\alpha)l \rceil + \lfloor (\beta-\alpha)l \rfloor) t^{-\lfloor (\beta-\alpha)l \rfloor} ((1-\beta)\omega_n - (1-\alpha)\eta_n), \\ \mathcal{O}(-\lceil (1-\alpha)l \rceil) (\omega_n - \eta_n), \\ \mathcal{O}(\lfloor (\beta-\alpha)l \rfloor - \lceil \beta l \rceil) t^{-\lfloor (\beta-\alpha)l \rfloor} ((1-\alpha-\beta)t^l \omega_n - (1-\beta)\omega_n + (1-\alpha)\eta_n), \\ \mathcal{O}(-\lceil \beta l \rceil) ((1-\alpha-\beta)t^l \omega_n - (1-\alpha)(\omega_n - \eta_n)). \end{cases}$$

Here, by abuse of notation, the images of ω_n, η_n in $\mathrm{Gr}_F^1 \mathcal{H}_e^{(n)}$ are denoted by the same letters.

Corollary 3.6. *For each $\zeta \in \mu_l$, X_ζ is a normal crossing divisor in X with rational irreducible components.*

Proof. By Proposition 3.3, the local monodromy of $H^1(X_t, \mathbb{Q})$ at $t = \zeta$ is unipotent, hence X_ζ is normal crossing (see [22, Theorem 1]). By the Clemens–Schmid exact sequence (see [18, §4 (a)]), $H^1(X_\zeta, \mathbb{Q})$ is the kernel of the log local monodromy $N: H^1(X_t, \mathbb{Q}) \rightarrow H^1(X_t, \mathbb{Q})$. The cohomology group $H^1(X_t, \mathbb{Q})$ carries a limiting mixed Hodge structure and N is a morphism of mixed Hodge structures of type $(-1, -1)$. Since $\mathrm{rank} N = \frac{1}{2} \dim H^1(X_t, \mathbb{Q})$ by Proposition 3.3, we have $\mathrm{Gr}_1^W H^1(X_t, \mathbb{Q}) = 0$ and $W_0 H^1(X_t, \mathbb{Q}) = \mathrm{Ker}(N)$. Hence $H^1(X_\zeta)$ is of pure weight 0, and all the irreducible components of X_ζ are rational. $\square$

4. HODGE NUMBERS

In this section, we determine the Hodge numbers of the eigencomponents of our H and prove that it has CM by K , i.e. $\dim_K H_B = 1$.

4.1. Localization sequence. Let the notations be as in Sect. 3.2 and put $Z = X \setminus U$. We have the localization sequence

$$H_Z^2(X) \rightarrow H^2(X) \rightarrow H^2(U) \rightarrow H_Z^3(X) \rightarrow H^3(X)$$

both for the de Rham and Betti cohomologies. Let $\langle Z \rangle$ denote the image of the first map. Recall that we defined in Sect. 2.2 the Hodge–de Rham structure

$$H = H^2(X) / \langle Z \rangle \otimes_R K.$$

Proposition 4.1. $H^1(X) = H^3(X) = 0$.

Proof. By Poincaré duality, it suffices to show $H^1(X, \mathbb{Q}) = 0$. Since $H^1(X, \mathbb{Q}) \hookrightarrow W_1 H^1(U, \mathbb{Q})$, where $W_\bullet$ denotes the weight filtration, it suffices to show the vanishing of the latter. By the Leray spectral sequence, we have an exact sequence

$$0 \rightarrow H^1(T, \mathbb{Q}) \rightarrow H^1(U, \mathbb{Q}) \rightarrow H^0(T, R^1 f_* \mathbb{Q}) \rightarrow 0.$$

By the computation of $\mathrm{Res}_\infty(\nabla)$ in Proposition 3.3, for $n = 1, \dots, p-1$, the local monodromy around $t = \infty$ of $H^1(X_t, \mathbb{C})^{(n)}$ does not have 1 as an eigenvalue. Hence we have $H^0(T, R^1 f_* \mathbb{Q}) = 0$ (recall that $H^1(X_t, \mathbb{C})^{(0)} = 0$). Since $H^1(T, \mathbb{Q})$ is of weight 2, we have $W_1 H^1(U, \mathbb{Q}) = 0$. $\square$

As a result, we have an exact sequence on the de Rham side [15, Chapter II, Theorem (3.3) and Proposition (3.4)]

$$0 \rightarrow H_{\text{dR}}^2(X)/\langle Z \rangle \rightarrow H_{\text{dR}}^2(U) \xrightarrow{\partial} H_1^{\text{dR}}(Z) \rightarrow 0.$$

The middle term is described by the canonical extension as follows. The Leray spectral sequence yields an exact sequence

$$0 \rightarrow H^1(T, \mathcal{H}) \rightarrow H_{\text{dR}}^2(U) \rightarrow H^0(T, R^2 f_* \Omega_{U/T}^\bullet) \rightarrow 0.$$

Since σ^* acts on $R^2 f_* \Omega_{U/T}^\bullet$ trivially, we have $H^1(T, \mathcal{H}^{(n)}) \simeq H_{\text{dR}}^2(U)^{(n)}$ for $n = 1, \dots, p-1$. Put a complex of sheaves on $\mathbb{P}^1$ as

$$\mathcal{E} = [\mathcal{H}_e \xrightarrow{\nabla} \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{H}_e].$$

Then, the map of complexes

$$\begin{array}{ccc} \mathcal{H}_e & \longrightarrow & \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{H}_e \\ \downarrow & & \downarrow \\ j_* \mathcal{H} & \longrightarrow & j_*(\Omega_T^1 \otimes \mathcal{H}) \end{array}$$

induces an isomorphism

$$H^1(\mathbb{P}^1, \mathcal{E}) \simeq H^1(T, \mathcal{H}),$$

and the first group carries a mixed Hodge structure [27, Theorem (4.1)] and its Hodge filtration is given as follows [loc. cit., (4.10)]:

$$\begin{aligned} F^0 H^1(\mathbb{P}^1, \mathcal{E}) &= H^1(\mathbb{P}^1, \mathcal{E}), \\ F^1 H^1(\mathbb{P}^1, \mathcal{E}) &= H^1(\mathbb{P}^1, F^1 \mathcal{H}_e \rightarrow \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{H}_e), \\ F^2 H^1(\mathbb{P}^1, \mathcal{E}) &= H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes F^1 \mathcal{H}_e). \end{aligned} \tag{4.1}$$

It follows:

$$\begin{aligned} \text{Gr}_F^0 H^1(\mathbb{P}^1, \mathcal{E}) &= H^1(\mathbb{P}^1, \text{Gr}_F^0 \mathcal{H}_e), \\ \text{Gr}_F^1 H^1(\mathbb{P}^1, \mathcal{E}) &= \text{Coker} \left(H^0(\mathbb{P}^1, F^1 \mathcal{H}_e) \xrightarrow{\bar{\nabla}} H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes \text{Gr}_F^0 \mathcal{H}_e) \right) \end{aligned} \tag{4.2}$$

where $\bar{\nabla}$ is the map induced from the composition of ∇ and the projection $\mathcal{H}_e \rightarrow \text{Gr}_F^0 \mathcal{H}_e$.

4.2. Residues. For each $t \in D$, let

$$\partial_t: H_{\text{dR}}^2(U) \rightarrow H_1^{\text{dR}}(X_t)$$

be the t -component of the coboundary map ∂ . Let $N_t \subset \mathcal{H}_{e,t}$ be the image of the composite

$$\Gamma(U_t, \mathcal{H}_e) \xrightarrow{\nabla} \Gamma(U_t, \Omega_{\mathbb{P}^1}^1(\log t) \otimes \mathcal{H}_e) \xrightarrow{\text{Res}_t} \mathcal{H}_{e,t},$$

where U_t is a small open neighborhood of t . Then, it is not difficult to show that the diagram

$$\begin{array}{ccc} H^1(\mathbb{P}^1, \mathcal{E}) & \xrightarrow{\subset} & H_{\text{dR}}^2(U) \\ \downarrow \text{Res}_t & & \downarrow \partial_t \\ \mathcal{H}_{e,t}/N_t & \xrightarrow{\simeq} & H_1^{\text{dR}}(X_t) \end{array}$$

commutes where the lower map is an isomorphism. The following is immediate from Proposition 3.3.

Proposition 4.2. *For $n = 1, \dots, p-1$, we have*

$$\begin{aligned} N_0^{(n)} &= \langle t^{\lceil(\alpha-\beta)l\rceil}((1-\beta)\omega_n - (1-\alpha)\eta_n) \rangle, \\ N_\infty^{(n)} &= \mathcal{H}_{e,\infty}, \\ N_\zeta^{(n)} &= \langle \eta_n \rangle \quad (\zeta \in \mu_l). \end{aligned}$$

Therefore, we have

$$\dim H_1^{\text{dR}}(X_t)^{(n)} = \begin{cases} 1 & \text{if } t = 0 \text{ or } t \in \mu_l, \\ 0 & \text{if } t = \infty. \end{cases}$$

Later, we shall use the following.

Lemma 4.3. *Let $n = 1, \dots, p-1$.*

- (i) *If $\alpha \leq \beta$, then $t^m \omega_n|_{t=0} \in N_0^{(n)}$ if $m > 0$, and $\notin N_0^{(n)}$ if $m = 0$.*
- (ii) *If $\alpha > \beta$, then $t^m \omega_n|_{t=0} \in N_0^{(n)}$ if $m \geq \lceil(\alpha-\beta)l\rceil$.*

Proof. By Corollary 3.4 and Proposition 4.2, this is trivial except when $\alpha > \beta$ and $m = \lceil(\alpha-\beta)l\rceil$. In this case, we have

$$t^m \omega_n|_{t=0} = t^m \omega_n|_0 + \frac{1-\alpha}{\alpha-\beta} t^m (\omega_n - \eta_n)|_{t=0} = \frac{t^m ((1-\beta)\omega_n - (1-\alpha)\eta_n)|_{t=0}}{\alpha-\beta} \in N_0^{(n)}.$$

□

4.3. Hodge numbers. For each $n = 1, \dots, p-1$, we obtained an exact sequence

$$0 \rightarrow (H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)} \rightarrow H^1(\mathbb{P}^1, \mathcal{E}^{(n)}) \xrightarrow{\text{Res}} \mathcal{H}_{e,0}^{(n)}/N_0^{(n)} \oplus \bigoplus_{\zeta \in \mu_l} \mathcal{H}_{e,\zeta}^{(n)}/N_\zeta^{(n)} \rightarrow 0. \quad (4.3)$$

Firstly, we give a basis of F^2 . By (4.1), we have an embedding

$$\iota: F^2(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)} \hookrightarrow \Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes F^1 \mathcal{H}_e^{(n)}).$$

By this, we identify $F^2(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ with the elements of the right-hand side having trivial residues. Recall the rational 2-forms

$$\omega_{m,n} = t^m \frac{dt}{t} \otimes \omega_n.$$

Proposition 4.4. *For each $n = 1, \dots, p-1$, a basis of $F^2(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ is given by $\{\omega_{m,n} \mid m \in I_n^2\}$ where*

$$I_n^2 := \{m \mid \max\{1, \lceil(\alpha-\beta)l\rceil\} \leq m \leq \min\{\lfloor \alpha l \rfloor, \lfloor (1-\beta)l \rfloor\}\}.$$

In particular,

$$\dim F^2(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)} = \min\{\lfloor \alpha l \rfloor, \lfloor (1-\beta)l \rfloor\} - \max\{0, \lfloor (\alpha-\beta)l \rfloor\}.$$

Proof. Let $F^1 \mathcal{H}_e^{(n)} = \mathcal{O}(i) t^j \omega_n$ be as in Corollary 3.5 (i). One easily sees that a basis of $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes F^1 \mathcal{H}_e^{(n)})$ is given by

$$\omega_{m,n} \ (j \leq m \leq i+j), \quad t^j \frac{dt}{t-\zeta} \otimes \omega_n \ (\zeta \in \mu_l).$$

For the first type, the residues at $\zeta \in \mu_l$ are trivial. By Lemma 4.3, $\text{Res}_0(\omega_{m,n}) = t^m \omega_n$ is trivial for $m \geq j$ unless $\alpha \leq \beta$ and $m = 0$. For the second type, it has trivial residues except at ζ and

$$\text{Res}_\zeta \left(t^j \frac{dt}{t - \zeta} \otimes \omega_n \right) = t^j \omega_n,$$

which is non-trivial by Proposition 4.2. These show that a basis of $F^2(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ is given by $\omega_{m,n}$ with $j \leq m \leq i+j$ and $m \neq j = 0$ if $\alpha \leq \beta$. Hence the proposition follows from Corollary 3.5 (i). $\square$

Since $(\mathcal{H}_{e,0}/N_0)^{(n)}$ and $(\mathcal{H}_{e,\zeta}/N_\zeta)^{(n)}$ are all 1-dimensional, the above proof implies the following.

Corollary 4.5. *For $n = 1, \dots, p-1$, we have*

$$\text{Res} \left(F^2 H^1(\mathbb{P}^1, \mathcal{E}^{(n)}) \right) = \begin{cases} (\mathcal{H}_{e,0}/N_0)^{(n)} \oplus \bigoplus_{\zeta \in \mu_l} (\mathcal{H}_{e,\zeta}/N_\zeta)^{(n)} & \text{if } \alpha \leq \beta, \\ \bigoplus_{\zeta \in \mu_l} (\mathcal{H}_{e,\zeta}/N_\zeta)^{(n)} & \text{if } \alpha > \beta. \end{cases}$$

Corollary 4.6. *Suppose that $p < l$. Then we have $F^2(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)} \neq 0$ for any $n = 1, \dots, p-1$.*

Proof. Since $\alpha, 1-\beta \geq 1/p$, we have $l\alpha, l(1-\beta) > 1$. Since $\beta \geq 1/p$ and $\alpha \leq 1-1/p$, we have $(\alpha - \beta)l < \alpha l - 1, (1 - \beta)l - 1$. Hence we have $I_n^2 \neq \emptyset$. $\square$

Now, we determine the other Hodge numbers.

Lemma 4.7. *Let $n = 1, \dots, p-1$.*

(i) *If $\alpha \leq \beta$, then we have*

$$\text{Gr}_F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)} = \text{Gr}_F^1 H^1(\mathbb{P}^1, \mathcal{E}^{(n)}).$$

(ii) *If $\alpha > \beta$, then we have an exact sequence*

$$0 \rightarrow \text{Gr}_F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)} \rightarrow \text{Gr}_F^1 H^1(\mathbb{P}^1, \mathcal{E}^{(n)}) \xrightarrow{\text{Res}_0} (\mathcal{H}_{e,0}/N_0)^{(n)} \rightarrow 0.$$

Proof. By (4.3) and Corollary 4.5, we are left to show the non-triviality of Res_0 in the case (ii). If $\lfloor \alpha l \rfloor \geq \lfloor (1 - \beta)l \rfloor$, consider

$$\frac{dt}{t(1-t^l)} \otimes (\omega_n - \eta_n).$$

By Corollary 3.5 (ii), this is an element of $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes \text{Gr}_F^0 \mathcal{H}_e^{(n)})$. Its residue at 0 is $\omega_n - \eta_n \not\equiv 0 \pmod{N_0}$ by Proposition 4.2. If $\lfloor \alpha l \rfloor < \lfloor (1 - \beta)l \rfloor$, consider similarly

$$\frac{dt}{t(1-t^l)} \otimes ((1 - \alpha - \beta)t^l \omega_n - (1 - \alpha)(\omega - \eta_n)),$$

whose residue at 0 is $-(1 - \alpha)(\omega_n - \eta_n) \not\equiv 0 \pmod{N_0}$. $\square$

Proposition 4.8. *For each $n = 1, \dots, p-1$, we have*

$$\dim \text{Gr}_F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)} = \lfloor \alpha l \rfloor - \lfloor (1 - \beta)l \rfloor + \lfloor \alpha - \beta \rfloor l.$$

Proof. First, we show that the map

$$\bar{\nabla}: H^0(\mathbb{P}^1, F^1 \mathcal{H}_e^{(n)}) \rightarrow H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathrm{Gr}_F^0 \mathcal{H}_e^{(n)})$$

is injective. Let $F^1 \mathcal{H}_e^{(n)} = \mathcal{O}(i)t^j \omega_n$ as in Corollary 3.5 (i). Then, $H^0(\mathbb{P}^1, F^1 \mathcal{H}_e^{(n)})$ has a basis $\{\omega_{m,n} \mid j \leq m \leq i+j\}$, and

$$\nabla \omega_{m,n} = \frac{dt}{t} t^m \left\{ (m - l(1 - \beta)) \omega_n + \frac{l(1 - \alpha)}{1 - t^l} \eta_n \right\} \equiv l(1 - \alpha) \frac{dt}{t(1 - t^l)} t^m \eta_n \neq 0$$

modulo $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes F^1 \mathcal{H}_e^{(n)})$. Since $0 \leq i < l$ in every case, $\omega_{m,n}$ belong to different eigenspaces with respect to the τ -action. Hence the non-vanishing implies the injectivity.

By Corollary 3.5 (ii), we have $\mathrm{Gr}_F^0 \mathcal{H}_e^{(n)} \simeq \mathcal{O}(k)$ where

$$k := \begin{cases} -\lceil(1 - \alpha)l\rceil + \lfloor(\beta - \alpha)l\rfloor & \text{if } \lfloor\alpha l\rfloor \geq \lfloor(1 - \beta)l\rfloor, \alpha \leq \beta, \\ -\lceil(1 - \alpha)l\rceil & \text{if } \lfloor\alpha l\rfloor \geq \lfloor(1 - \beta)l\rfloor, \alpha > \beta, \\ \lfloor(\beta - \alpha)l\rfloor - \lceil\beta l\rceil & \text{if } \lfloor\alpha l\rfloor < \lfloor(1 - \beta)l\rfloor, \alpha \leq \beta, \\ -\lceil\beta l\rceil & \text{if } \lfloor\alpha l\rfloor < \lfloor(1 - \beta)l\rfloor, \alpha > \beta. \end{cases}$$

Note that $k < 0$ in any case. One sees that $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{O}(k))$ has a basis

$$\frac{t^m}{1 - t^l} \frac{dt}{t} \otimes \omega_n \quad (0 \leq m \leq l + k).$$

By (4.2) and the above injectivity, we have

$$\begin{aligned} \dim \mathrm{Gr}_F^1 H^1(\mathbb{P}^1, \mathcal{E}^{(n)}) &= \dim H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{O}(k)) - \dim H^0(\mathbb{P}^1, \mathcal{O}(i)) \\ &= (l + k + 1) - (i + 1) = l + k - i. \end{aligned}$$

By Corollary 3.5 (i) and Lemma 4.7, we obtain the desired formula. $\square$

Corollary 4.9. *Assume that $p < l$ and $p > 2$ when $a = b$. Then we have*

$$\mathrm{Gr}_F^1(H_{\mathrm{dR}}^2(X)/\langle Z \rangle)^{(n)} \neq 0$$

for any $n = 1, \dots, p - 1$.

Proof. If $a \neq b$, then $\lfloor |\alpha - \beta|l \rfloor \geq \lfloor \frac{l}{p} \rfloor \geq 1$. If $a = b$, then $\alpha \neq 1 - \alpha$ since $p > 2$, and hence $\lfloor \lfloor \alpha l \rfloor - \lfloor (1 - \alpha)l \rfloor \rfloor \geq 1$. $\square$

Proposition 4.10. *For each $n = 1, \dots, p - 1$, we have*

$$\dim \mathrm{Gr}_F^0(H_{\mathrm{dR}}^2(X)/\langle Z \rangle)^{(n)} = \min\{\lfloor(1 - \alpha)l\rfloor, \lfloor\beta l\rfloor\} - \max\{0, \lfloor(\beta - \alpha)l\rfloor\}.$$

Proof. By (4.2), Corollary 4.5 and Lemma 4.7, we have

$$\mathrm{Gr}_F^0(H_{\mathrm{dR}}^2(X)/\langle Z \rangle)^{(n)} = H^1(\mathbb{P}^1, \mathrm{Gr}_F^0 \mathcal{H}_e^{(n)}) = H^1(\mathbb{P}^1, \mathcal{O}(k)),$$

where k is as in the proof of Proposition 4.8. Since $k < 0$, we have

$$\dim H^1(\mathbb{P}^1, \mathcal{O}(k)) = \dim H^0(\mathbb{P}^1, \mathcal{O}(-k - 2)) = -k - 1.$$

Hence the proposition follows. $\square$

Remark 4.11. In fact, Proposition 4.10 is equivalent to the dimension formula in Proposition 4.4. Note that the complex conjugation switches n (resp. α, β) and $p - n$ (resp. $1 - \alpha, 1 - \beta$).

Theorem 4.12. *The Hodge-de Rham structure $H = (H^2(X)/\langle Z \rangle) \otimes_R K$ has CM by K , i.e. $\dim_K H_B = 1$.*

Proof. Combining Propositions 4.4, 4.8 and 4.10, one verifies that

$$\dim(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)} = l - 1$$

for each $n = 1, \dots, p - 1$. It follows that

$$\dim_{\mathbb{Q}} H_B \leq (l - 1)(p - 1) = [K : \mathbb{Q}].$$

It remains to show that $H \neq 0$, for which it suffices to show that τ is not the identity on $H_{\text{dR}}^2(X)/\langle Z \rangle$. If $p < l$, this follows from Proposition 4.4 and Corollary 4.6. The general case follows from Proposition 5.2 below. $\square$

5. PERIODS

We compute the periods of our H and verify the Gross–Deligne conjecture, for which it will suffice to consider $F^1 H_{\text{dR}}$.

5.1. Basis of $F^1 H_{\text{dR}}$. Recall that, by (4.3), we can identify $F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ with the elements of $F^1 H^1(\mathbb{P}^1, \mathcal{E}^{(n)})$ having trivial residues. Furthermore, they are identified with rational 2-forms by the following lemma. Put $T_1 = \mathbb{P}^1 \setminus \{0, \infty\}$.

Lemma 5.1. *For each $n = 1, \dots, p - 1$, there is a natural injection*

$$\iota: F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)} \hookrightarrow \Gamma(T_1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes F^1 \mathcal{H}_e^{(n)}).$$

Proof. By (4.1) and (4.3), it suffices to show the existence of an injection

$$H^1(\mathbb{P}^1, F^1 \mathcal{E}^{(n)}) \hookrightarrow \Gamma(T_1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes F^1 \mathcal{H}_e^{(n)})$$

where we put $F^1 \mathcal{E} = [F^1 \mathcal{H}_e \rightarrow \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{H}_e]$. Consider the commutative diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & \Omega_{\mathbb{P}^1}^1(\log D) \otimes F^1 \mathcal{H}_e^{(n)} \\ & & \downarrow \\ F^1 \mathcal{H}_e^{(n)} & \xrightarrow{\nabla} & \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{H}_e^{(n)} \\ \downarrow = & & \downarrow \\ F^1 \mathcal{H}_e^{(n)} & \xrightarrow{\overline{\nabla}} & \Omega_{\mathbb{P}^1}^1(\log D) \otimes \text{Gr}_F^0 \mathcal{H}_e^{(n)} \\ & & \downarrow \\ & & 0 \end{array}$$

where the right vertical sequence is exact. By Proposition 3.3, $\overline{\nabla}$ is an isomorphism on T_1 . Therefore, we have an isomorphism

$$\Gamma(T_1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes F^1 \mathcal{H}_e^{(n)}) \xrightarrow{\sim} H^1(T_1, F^1 \mathcal{E}^{(n)}).$$

It remains to show the injectivity of $H^1(\mathbb{P}^1, F^1 \mathcal{E}^{(n)}) \rightarrow H^1(T_1, F^1 \mathcal{E}^{(n)})$. This follows from the fact that $H^1(\mathbb{P}^1, F^1 \mathcal{E}) \rightarrow H^1(\mathbb{P}^1, \mathcal{E})$ is injective and $H^1(\mathbb{P}^1, \mathcal{E}) \rightarrow H^1(T_1, \mathcal{E})$ is an isomorphism. $\square$

Under the identification via ι , we have the following.

Proposition 5.2. *For each $n = 1, \dots, p-1$, a basis of $F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ is given by $\{\omega_{m,n} \mid m \in I_n^1\}$ where*

$$I_n^1 := \begin{cases} \{-(\beta - \alpha)l, \dots, -1\} \cup \{1, \dots, \max\{\lfloor \alpha l \rfloor, \lfloor (1 - \beta)l \rfloor\}\} & \text{if } \alpha < \beta, \\ \{1, \dots, \max\{\lfloor \alpha l \rfloor, \lfloor (1 - \beta)l \rfloor\}\} & \text{if } \alpha \geq \beta. \end{cases}$$

Recall that $\alpha = \{\frac{na}{p}\}$, $\beta = \{\frac{nb}{p}\}$.

Proof. It is a routine to verify that $|I_n^1| = \dim F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ using Propositions 4.4 and 4.8. Therefore, it suffices to show that $\omega_{m,n} \in F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ if $m \in I_n^1$. We construct Čech cocycles representing elements of $H^1(\mathbb{P}^1, F^1\mathcal{E}^{(n)})$ with trivial residues which correspond to $\omega_{m,n}$. Take a covering $\mathbb{P}^1 = U_0 \cup U_\infty$ where $U_0 := \mathbb{P}^1 \setminus \{\infty\}$, $U_\infty := \mathbb{P}^1 \setminus \{0\}$; note that $T_1 = U_0 \cap U_\infty$. A Čech cocycle in this case is a triple

$$(\psi, \varphi_0, \varphi_\infty) \in \Gamma(T_1, F^1\mathcal{H}_e^{(n)}) \oplus \bigoplus_{t=0, \infty} \Gamma(U_t, \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{H}_e^{(n)})$$

satisfying $\nabla\psi = \varphi_0|_{T_1} - \varphi_\infty|_{T_1}$. We construct such cocycles in four ways. By Proposition 3.2, we have

$$\begin{aligned} & l^{-1}\nabla(t^m\omega_n) \\ &= (\mu - 1 + \beta)\omega_{m,n} + \frac{1 - \alpha}{1 - t^l}\eta_{m,n} \end{aligned} \quad (5.1)$$

$$= \left(\mu - \alpha - \frac{1 - \alpha - \beta}{1 - t^l}\right)\omega_{m,n} + \frac{t^l}{1 - t^l} \left((1 - \alpha - \beta)\omega_{m,n} + \frac{1 - \alpha}{t^l}\eta_{m,n}\right) \quad (5.2)$$

$$= \left(\mu + (1 - \beta)\frac{t^l}{1 - t^l}\right)\omega_{m,n} - \frac{1}{1 - t^l}((1 - \beta)\omega_{m,n} - (1 - \alpha)\eta_{m,n}) \quad (5.3)$$

$$= \left(\mu - \alpha + \beta + (1 - \alpha)\frac{1 - t^l}{1 - t^l}\right)\omega_{m,n} - \frac{1 - \alpha}{1 - t^l}(\omega_{m,n} - \eta_{m,n}). \quad (5.4)$$

Put

$$j = \max\{0, \lceil(\alpha - \beta)l\rceil\}, \quad k = \min\{\lfloor \alpha l \rfloor, \lfloor (1 - \beta)l \rfloor\}.$$

(i) Suppose that $\lfloor \alpha l \rfloor \geq \lfloor (1 - \beta)l \rfloor$. Let $\psi = l^{-1}t^m\omega_n$,

$$\varphi_0 = (\mu - 1 + \beta)\omega_{m,n}, \quad \varphi_\infty = -\frac{1 - \alpha}{1 - t^l}\eta_{m,n}.$$

By (5.1) and Corollary 3.4, these define a cocycle if $j \leq m \leq \lfloor \alpha l \rfloor$. By Proposition 4.2, it has no residues unless $m = 0$, hence defines an element of $F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ if

$$j \leq m \leq \lfloor \alpha l \rfloor, \quad m \neq 0.$$

(ii) Suppose that $\lfloor \alpha l \rfloor < \lfloor (1 - \beta)l \rfloor$. Then, by (5.2) and Corollary 3.4, $\psi = l^{-1}t^m\omega_n$,

$$\varphi_0 = \left(\mu - \alpha - \frac{1 - \alpha - \beta}{1 - t^l}\right)\omega_{m,n},$$

$$\varphi_\infty = -\frac{t^l}{1 - t^l}((1 - \alpha - \beta)\omega_{m,n} + (1 - \alpha)t^{-l}\eta_{m,n})$$

define a cocycle if $j \leq m \leq \lfloor (1 - \beta)l \rfloor$. To kill the residues, we use Lemma 5.3 below. Then, by letting

$$\varphi_0 = (\mu - \alpha)\omega_{m,n}, \quad \varphi_\infty = (1 - \alpha - \beta)\omega_{m,n} - \frac{1 - \alpha}{1 - t^l}\eta_{m,n},$$

we obtain an element of $F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ for

$$j \leq m \leq \lfloor (1 - \beta)l \rfloor, \quad m \neq 0.$$

(iii) Suppose that $\alpha \leq \beta$. Then, by (5.3) and Corollary 3.4, $\psi = -l^{-1}t^m\omega_n$,

$$\varphi_0 = \frac{1}{1-t^l}((1-\beta)\omega_{m,n} - (1-\alpha)\eta_{m,n}), \quad \varphi_\infty = \left(\mu + (1-\beta)\frac{t^l}{1-t^l}\right)\omega_{m,n}$$

define a cocycle if

$$-\lfloor (\beta - \alpha)l \rfloor \leq m \leq k.$$

If $m < 0$, we can kill the residues using Lemma 5.3, and

$$\varphi_0 = (1-\beta)\omega_{m,n} - \frac{1-\alpha}{1-t^l}\eta_{m,n}, \quad \varphi_\infty = \mu\omega_{m,n}$$

define an element of $F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ for

$$-\lfloor (\beta - \alpha)l \rfloor \leq m < 0.$$

(iv) Finally, suppose that $\alpha > \beta$. Then, by (5.4) and Corollary 3.4, $-l^{-1}t^m\omega_n$,

$$\varphi_0 = \frac{1-\alpha}{1-t^l}(\omega_{m,n} - \eta_{m,n}), \quad \varphi_\infty = \left(\mu - \alpha + \beta + (1-\alpha)\frac{t^l}{1-t^l}\right)\omega_{m,n}$$

define a cocycle if $0 \leq m \leq k$. If $m \neq 0$, we can use Lemma 5.3 to kill the residues and

$$\varphi_0 = (1-\alpha)\omega_{m,n} - \frac{1-\alpha}{1-t^l}\eta_{m,n}, \quad \varphi_\infty = (\mu - 1 + \beta)\omega_{m,n}$$

define an element of $F^1(H_{\text{dR}}^2(X)/\langle Z \rangle)^{(n)}$ for

$$0 < m \leq k.$$

Combining (iii) and (i) (or (ii)), we obtain the first case of the proposition. For the second case, combine (iv) and (i) (or (ii)), just noting that $k \geq j-1 = \lfloor (\alpha-\beta)l \rfloor$. $\square$

Lemma 5.3. *If $j \leq m < l$, $m \neq 0$, then*

$$\frac{1}{1-t^l} \otimes \omega_{m,n} \in \Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes \mathcal{H}_e^{(n)}),$$

and it has trivial residues at $t = 0, \infty$.

Proof. This is immediate from Corollary 3.4 and Lemma 4.3. $\square$

5.2. Period formula. We prove the period formula which verifies the conjecture of Gross–Deligne [14, Sect. 4] (but see Remark 5.6 below). We identify an embedding $\chi: K \hookrightarrow \mathbb{C}$ with the element $h \in (\mathbb{Z}/lp\mathbb{Z})^\times$ such that $\chi(\zeta_{lp}) = \zeta_{lp}^h$, and write $H^{(h)}$ instead of H^χ . For each $h \in (\mathbb{Z}/lp\mathbb{Z})^\times$, let $(p(h), 2 - p(h))$ be the Hodge type of $H^{(h)}$. Put $K' = \mathbb{Q}(\mu_{2lp})$ ($K = K'$ if lp is odd).

Theorem 5.4. *Define a function $\varepsilon: \mathbb{Z}/lp\mathbb{Z} \rightarrow \mathbb{Z}$ by*

$$\varepsilon(i) = \begin{cases} 1 & \text{if } i \equiv lb, p, l(p-b), l(b-a) + p \pmod{lp}, \\ -1 & \text{if } i \equiv lb + p, l(p-a) + p \pmod{lp}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any $h \in (\mathbb{Z}/lp\mathbb{Z})^\times$, we have

$$p(h) = \sum_{i \in \mathbb{Z}/lp\mathbb{Z}} \varepsilon(i) \left\{ -\frac{hi}{lp} \right\}$$

and

$$\text{Per}(H^{(h)}) \sim_{K' \times} \prod_{i \in \mathbb{Z}/lp\mathbb{Z}} \Gamma\left(\left\{\frac{hi}{lp}\right\}\right)^{\varepsilon(i)}.$$

Proof. For real numbers x, y with $0 < x, y < 1$, $x + y \neq 1$, put

$$\delta(x, y) := \{-x\} + \{-y\} - \{-(x+y)\} = \begin{cases} 1 & \text{if } x+y < 1, \\ 0 & \text{if } x+y > 1. \end{cases}$$

Then we have

$$\varphi(h) := \sum_i \varepsilon(i) \left\{-\frac{hi}{lp}\right\} = \delta(\beta, \mu) + \delta(1-\beta, \{\beta-\alpha+\mu\}),$$

where we put $\alpha = \{ha/p\}$, $\beta = \{hb/p\}$, $\mu = \{h/l\}$. Firstly, we have $\varphi(h) = 2$ if and only if

$$\beta + \mu < 1, \quad 1 - \beta + \{\beta - \alpha + \mu\} < 1.$$

Letting $m = l\mu$, the first condition becomes $m < (1-\beta)l$, i.e. $m \leq \lfloor (1-\beta)l \rfloor$. Similarly, the second condition is equivalent to

$$(\alpha \leq \beta, m < \alpha l) \quad \text{or} \quad (\alpha > \beta, (\alpha - \beta)l < m < \alpha l).$$

Comparing with Proposition 4.4, we have $p(h) = 2$ if and only if $\varphi(h) = 2$. Secondly, since

$$p(h) + p(-h) = \varphi(h) + \varphi(-h) = 2,$$

we have $p(h) = 0$ if and only of $\varphi(h) = 0$. Since $p(h), \varphi(h) \in \{0, 1, 2\}$, we have $p(h) = \varphi(h)$ for any h .

For the second statement, we compute the periods over the 2-cycle

$$(1-\tau)_*(1-\sigma)_*\Delta_1.$$

Since $(1-\zeta_l)(1-\zeta_p)$ is invertible in K , it reduces to the periods over Δ_1 (Proposition 2.6 (i)). Consider firstly the two cases:

- (i) $\alpha \leq \beta$ and $p(h) \geq 1$,
- (ii) $\alpha > \beta$ and $p(h) = 2$.

In both cases, by Propositions 4.4 and 5.2, $H^{(h)}$ is generated by $\omega_{m,n}$ satisfying $\lceil (\alpha - \beta)l \rceil \leq m$, which is equivalent to $\alpha - \beta < \mu := m/l$. This is the assumption of Proposition 2.6 (i) and we obtain the desired formula.

The other cases are reduced to the above ones. If we replace χ with χ^{-1} , then h (resp. $\alpha, \beta, p(h)$) is replaced with $-h$ (resp. $1-\alpha, 1-\beta, 2-p(h)$). By Lemma 5.5 below, the cup-product $H^2(X) \otimes H^2(X) \rightarrow \mathbb{Q}(-2)$ induces an auto-duality on H , under which H^χ is dual to $H^{\chi^{-1}}$. Hence we have

$$\text{Per}(H^{(h)}) \cdot \text{Per}(H^{(-h)}) \sim_{K \times} (2\pi i)^2.$$

On the other hand, recall the reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \sim_{K' \times} 2\pi i$$

for any $x \in \frac{1}{lp}\mathbb{Z} \setminus \mathbb{Z}$. Therefore, the case where $\alpha \leq \beta$ and $p(h) = 0$ (resp. $\alpha > \beta$ and $p(h) \geq 1$) is equivalent to the case (ii) (resp. (i)). $\square$

Lemma 5.5. *Put $H^2(X)_Z = \text{Ker}(H^2(X) \rightarrow H^2(Z))$. Then, the composition*

$$H^2(X)_Z \hookrightarrow H^2(X) \twoheadrightarrow H^2(X)/\langle Z \rangle$$

induces an isomorphism of Hodge-de Rham structures $H^2(X)_Z \otimes_R K \simeq H$.

Proof. This follows from the fact that the kernel of the composite

$$H_Z^2(X, \mathbb{C}) \rightarrow H^2(X, \mathbb{C}) \rightarrow H^2(Z, \mathbb{C})$$

is one-dimensional by Zariski's lemma (cf. [6, III, (8.2)]). $\square$

Remark 5.6. Our definition of ε is slightly different from [14]; $\varepsilon(i)$ here is $\varepsilon(-i)$ of loc. cit., where Gross looks at the values $\Gamma(1 - \{hi/lp\})^{\varepsilon(i)}$. The former conforms to the definition of the Stickelberger element as

$$\sum_{h \in (\mathbb{Z}/N\mathbb{Z})^\times} \left\{ -\frac{h}{N} \right\} \sigma_h^{-1},$$

where $\sigma_h \in \text{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ sends an N th root of unity to its h th power.

6. REGULATORS

After explaining the regulator map we consider, we prove Theorem 1.2 from the introduction and its consequences on the non-vanishing.

6.1. Formulation. The Deligne cohomology of $X_{\mathbb{C}} := X \times_{\text{Spec } \overline{\mathbb{Q}}} \text{Spec } \mathbb{C}$ with coefficients in $\mathbb{Q}(2)$ is defined to be the hypercohomology of the complex

$$\mathbb{Q}(2) \rightarrow \mathcal{O}_{X_{\mathbb{C}}} \rightarrow \Omega_{X_{\mathbb{C}}/\mathbb{C}}^1$$

where $\mathbb{Q}(2) := (2\pi i)^2 \mathbb{Q}$ is placed in degree 0. Consider the Beilinson regulator map [7] from the motivic cohomology

$$r_{\mathcal{D}}: H_{\mathcal{M}}^3(X, \mathbb{Q}(2)) \rightarrow H_{\mathcal{D}}^3(X_{\mathbb{C}}, \mathbb{Q}(2)).$$

We have a natural isomorphism

$$H_{\mathcal{D}}^3(X_{\mathbb{C}}, \mathbb{Q}(2)) \simeq H^2(X, \mathbb{C})/(F^2 + H^2(X, \mathbb{Q}(2))),$$

and the Carlson isomorphism

$$H^2(X, \mathbb{C})/(F^2 + H^2(X, \mathbb{Q}(2))) \simeq \text{Ext}_{\text{MHS}}^1(\mathbb{Q}, H^2(X, \mathbb{Q}(2))).$$

Here, MHS denotes the abelian category of $\mathbb{Q}$ -mixed Hodge structures. By Poincaré duality $H^2(X, \mathbb{Q}(2)) \simeq H_2(X, \mathbb{Q})$, we obtain an identification

$$H_{\mathcal{D}}^3(X_{\mathbb{C}}, \mathbb{Q}(2)) \simeq \text{Ext}_{\text{MHS}}^1(\mathbb{Q}, H_2(X, \mathbb{Q})).$$

Let $Z \subset X$ be as before and consider the regulator map

$$r_{\mathcal{D}, Z}: H_{\mathcal{M}, Z}^3(X, \mathbb{Q}(2)) \rightarrow H_{\mathcal{D}, Z}^3(X, \mathbb{Q}(2)) \simeq H_1(Z, \mathbb{Q})$$

from the motivic cohomology supported on Z . Since $H_1(X, \mathbb{Q}) = 0$ by Proposition 4.1, we have an exact sequence of mixed Hodge structures

$$H_2(Z, \mathbb{Q}) \rightarrow H_2(X, \mathbb{Q}) \rightarrow H_2(X, Z; \mathbb{Q}) \xrightarrow{\partial} H_1(Z, \mathbb{Q}) \rightarrow 0.$$

If we denote the image of the first map by $\langle Z \rangle$, we have the connecting homomorphism

$$\rho: H_1(Z, \mathbb{Q}) \cap H^{0,0} \rightarrow \text{Ext}_{\text{MHS}}^1(\mathbb{Q}, H_2(X, \mathbb{Q})/\langle Z \rangle),$$

where $H^{0,0}$ denotes the Hodge $(0,0)$ -component of $H_1(Z, \mathbb{C})$. By the lemma and the remark below, ρ describes the restriction of $r_{\mathcal{D}}$ to the image of $H_{\mathcal{M}, Z}^3(X, \mathbb{Q}(2))$.

Lemma 6.1. *The diagram below is commutative up to sign:*

$$\begin{array}{ccccc}
H_{\mathcal{M},Z}^3(X, \mathbb{Q}(2)) & \xrightarrow{r_{\mathcal{D},Z}} & H_1(Z, \mathbb{Q}) \cap H^{0,0} & \xrightarrow{\rho} & \text{Ext}_{\text{MHS}}^1(\mathbb{Q}, H_2(X, \mathbb{Q})/\langle Z \rangle) \\
\downarrow & & & & \uparrow \\
H_{\mathcal{M}}^3(X, \mathbb{Q}(2)) & \xrightarrow{r_{\mathcal{D}}} & H_{\mathcal{D}}^3(X_{\mathbb{C}}, \mathbb{Q}(2)) & \xrightarrow{\simeq} & \text{Ext}_{\text{MHS}}^1(\mathbb{Q}, H_2(X, \mathbb{Q}))
\end{array}$$

where the vertical maps are the natural ones.

Proof. See [5], Theorem 11.2. $\square$

Remark 6.2. The right vertical arrow is surjective since $\text{Ext}_{\text{MHS}}^2 = 0$. Its kernel is topologically generated by decomposable elements, i.e. the image of

$$(\text{CH}_1(Z) \otimes \overline{\mathbb{Q}}^{\times}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_{\mathcal{M},Z}^3(X, \mathbb{Q}(2)).$$

Also, it is not difficult to show that $r_{\mathcal{D},Z}$ is surjective.

6.2. Regulator formula. Now, we regard the extension classes as functionals (up to period functionals). Let $H^2(X)_Z = \text{Ker}(H^2(X) \rightarrow H^2(Z))$ as before. Since $H^2(X, \mathbb{Q})_Z \simeq (H_2(X, \mathbb{Q})/\langle Z \rangle)^*$, we have

$$\text{Ext}_{\text{MHS}}^1(\mathbb{Q}, H_2(X, \mathbb{Q})/\langle Z \rangle) \simeq (F^1 H^2(X, \mathbb{C})_Z)^* / \text{Image } H_2(X, \mathbb{Q})$$

where $*$ denotes the $\mathbb{C}$ -linear dual. By Lemma 5.5, ρ induces a map

$$\rho: (H_1(Z, \mathbb{Q}) \cap H^{0,0}) \otimes_R K \rightarrow (F^1 H_{\mathbb{C}})^* / H_B^{\vee}$$

where $H_{\mathbb{C}} := H_B \otimes_{\mathbb{Q}} \mathbb{C}$ and $H^{\vee}$ denotes the dual Hodge–de Rham structure of H .

Put $Z_1 = \bigsqcup_{\zeta \in \mu_l} X_{\zeta}$. We shall describe the restriction of ρ to $H_1(Z_1, \mathbb{Q}) \otimes_R K$. Recall that $H_1(Z_1, \mathbb{Q}) \subset H^{0,0}$ (Corollary 3.6). We have in fact the following.

Lemma 6.3. *We have an isomorphism*

$$H_1(Z_1, \mathbb{Q}) \otimes_R K \xrightarrow{\simeq} H_1(Z, \mathbb{Q}) \otimes_R K.$$

Proof. By Proposition 4.2, τ acts trivially on $H_1(X_0, \mathbb{Q})$ and $H_1(X_{\infty}, \mathbb{Q}) = 0$. $\square$

Let $(1 - \sigma)_* \Delta_0 \in H_2(X, Z_1; \mathbb{Q})$ be the Lefschetz thimble defined in Sect. 2.5, and let $H_2(X, Z_1; \mathbb{Q})_{\text{Lef}} \subset H_2(X, Z_1; \mathbb{Q})$ denote the R -submodule generated by this element.

Lemma 6.4. *The restriction of the boundary map*

$$\partial: H_2(X, Z_1; \mathbb{Q})_{\text{Lef}} \otimes_R K \rightarrow H_1(Z_1, \mathbb{Q}) \otimes_R K$$

is surjective and $H_1(Z_1, \mathbb{Q}) \otimes_R K$ is one-dimensional over K .

Proof. By Proposition 4.2, $\dim_{\mathbb{Q}} H_1(X_{\zeta}, \mathbb{Q}) = p - 1$ for $\zeta \in \mu_l$. Since τ permutes the components of Z_1 , $H_1(Z_1, \mathbb{Q}) \otimes_R K$ is one-dimensional over K . Whereas κ_0 and κ_1 generate $H_1(X_t, \mathbb{Q})$ (Proposition 2.4 (ii)), κ_1 vanishes as $t \rightarrow 1$ by definition. Therefore κ_0 does not vanish, i.e. $\partial((1 - \sigma)_* \Delta_0)$ is non-trivial in $H_1(X_1, \mathbb{Q})$, hence so is in $H_1(Z_1, \mathbb{Q}) \otimes_R K$. $\square$

Now we state our main theorem. For $x \in K$, let x_* (resp. x^*) denote its action on homology (resp. cohomology). Since $1 - \zeta_p$ is invertible in K , we write

$$((1 - \zeta_p)^{-1})_* (1 - \sigma)_* \Delta_0 \in H_1(X, Z_1; \mathbb{Q}) \otimes_R K$$

simply as Δ_0 . For each m and n , define an embedding $\chi_{m,n}: K \hookrightarrow \mathbb{C}$ by

$$\chi_{m,n}(\zeta_l) = \zeta_l^m, \quad \chi_{m,n}(\zeta_p) = \zeta_p^n.$$

Theorem 6.5. *Let $\gamma \in H_1(Z_1, \mathbb{Q}) \otimes_R K$ and take $x \in K$ such that $\gamma = x_* \partial \Delta_0$. Let $\{\omega_{m,n} \mid n = 1, \dots, p-1, m \in I_n^1\}$ be the basis of $F^1 H_{\text{dR}}$ given in Proposition 5.2. Then we have*

$$\rho(\gamma)(\omega_{m,n}) = \chi_{m,n}(x) \frac{B(1-\alpha, \beta)}{l(\beta-\alpha+\mu)} \cdot F\left(\begin{matrix} 1-\alpha, \beta, \beta-\alpha+\mu \\ 1-\alpha+\beta, \beta-\alpha+\mu+1 \end{matrix}; 1\right)$$

where $\alpha = \{\frac{na}{p}\}$, $\beta = \{\frac{nb}{p}\}$, $\mu = \frac{m}{l}$.

Proof. We apply Theorem 7.3 of the appendix to our situation where $D = Z_1$ and $X^\circ = X \setminus (X_0 \cup X_\infty)$ (see the proof of Lemma 5.1). Note that $H_{\mathbb{C}} \simeq H_{\text{dR}}^2(X_{\mathbb{C}})_0 \otimes_R K$ by Lemma 5.5 since τ acts trivially on $H_{\text{dR}}^2(e(\mathbb{P}_{\mathbb{C}}^1))$ (see Sect. 7.2 for the notations).

Put $\Gamma = (1-\tau)_*(1-\sigma)_*\Delta_0$. Since $\Gamma \in H_2(X, Z_1; \mathbb{Q})$ does not necessarily comes from $H_2(X^\circ, Z_1; \mathbb{Q})$, we take a detour. Let Γ' be the Lefschetz thimble given by sweeping $(1-\sigma)_*\delta_0$ along the path $\kappa_1 + \kappa_2 + \kappa_3$ in $T \setminus \{0, \infty\}$, where κ_1 is the line segment from ζ to $\varepsilon\zeta$ ($\varepsilon > 0$), κ_2 is the arc from $\varepsilon\zeta$ to ε and κ_3 is the line segment from ε to 1. Then $\Gamma' \in H_2(X^\circ, Z_1; \mathbb{Q})$ and $\gamma := \partial(\Gamma) = \partial(\Gamma')$. Theorem 7.3 yields

$$\rho(\gamma)(\omega_{m,n}) = \int_{\Gamma'} \omega_{m,n}.$$

The right integral is computed similarly as Proposition 2.6 (ii), and letting $\varepsilon \rightarrow 0$ we obtain the theorem for $x = (1-\zeta_l)(1-\zeta_p)$. The general case follows by the cyclicity of $H_1(Z_1, \mathbb{Q}) \otimes_R K$. $\square$

6.3. Non-vanishing. We prove the non-vanishing of ρ under a mild assumption. The situation is different depending whether $a+b=p$ or not.

If $a+b \neq p$, the regulator does not vanish even in the Deligne cohomology with $\mathbb{R}$ -coefficients, or equivalently, the extension group of $\mathbb{R}$ -mixed Hodge structures

$$\text{Ext}_{\mathbb{R}\text{MHS}}^1(\mathbb{R}, H_{\mathbb{R}}) \simeq (F^1 H_{\mathbb{C}})^* / H_{\mathbb{R}}^\vee,$$

where $H_{\mathbb{R}} = H_B \otimes_{\mathbb{Q}} \mathbb{R}$, $H_{\mathbb{C}} = H_B \otimes_{\mathbb{Q}} \mathbb{C}$. Note that

$$\dim_{\mathbb{R}}(F^1 H_{\mathbb{C}})^* / H_{\mathbb{R}}^\vee = \dim_{\overline{\mathbb{Q}}} \text{Gr}_F^1 H_{\text{dR}}.$$

Let

$$\rho_{\mathbb{R}}: H_1(Z_1, \mathbb{Q}) \otimes_R K \rightarrow (F^1 H_{\mathbb{C}})^* / H_{\mathbb{R}}^\vee$$

be the composition of ρ and the natural surjection.

Theorem 6.6. *Suppose that $p < l$ and $a+b \neq p$ (so $p > 2$). Then $\rho_{\mathbb{R}}$ is non-trivial. In particular,*

$$\dim_{\mathbb{Q}} \rho_{\mathbb{R}}(H_1(Z_1, \mathbb{Q}) \otimes_R K) = (l-1)(p-1).$$

Proof. By restricting the functionals to $F^1 H_{\mathbb{R}} := F^1 H_{\mathbb{C}} \cap H_{\mathbb{R}}$ and taking the imaginary part, we obtain a $K \cap \mathbb{R}$ -linear map

$$\rho'_{\mathbb{R}}: H_1(Z_1, \mathbb{Q}) \otimes_R K \rightarrow \text{Hom}(F^1 H_{\mathbb{R}}, i\mathbb{R}).$$

For each $n = 1, \dots, p-1$, we have $\alpha \neq 1-\beta$ by the assumption. Hence $|\alpha - (1-\beta)| \geq 1/p > 1/l$ and there exists an m satisfying

$$\min\{\lfloor \alpha l \rfloor, \lfloor (1-\beta)l \rfloor\} < m \leq \max\{\lfloor \alpha l \rfloor, \lfloor (1-\beta)l \rfloor\}. \quad (6.1)$$

Then we have $\omega_{m,n} \in \mathrm{Gr}_F^1 H_{\mathrm{dR}}$ by Propositions 4.4 and 5.2. Since $m > \lfloor (\alpha - \beta)l \rfloor$, we have $\mu := m/l > \alpha - \beta$, hence we can apply Proposition 2.6 (i) to compute the period:

$$\Omega_{m,n} := \int_{\Delta_1} \omega_{m,n} = -\frac{(-1)^{p\beta}}{l} B(\beta, \mu) B(1 - \beta, \beta - \alpha + \mu).$$

Put a normalization

$$\tilde{\omega}_{m,n} = \Omega_{m,n}^{-1} \omega_{m,n}.$$

Then we have

$$\int_{x_* \Delta_1} \tilde{\omega}_{m,n} = \int_{\Delta_1} x^* \tilde{\omega}_{m,n} = \chi_{m,n}(x)$$

for any $x \in K$. If we let $n' = p - n$, $\alpha' = \{n'a/p\} = 1 - \alpha$, $\beta' = \{n'b/p\} = 1 - \beta$, $m' = l - m$ and $\mu' = \{m'/l\} = 1 - \mu$, then these satisfy the assumption (6.1). Hence $\tilde{\omega}_{m',n'}$ is defined and we have

$$\int_{x_* \Delta_1} \tilde{\omega}_{m',n'} = \overline{\chi_{m,n}(x)}$$

for any $x \in K$. Since $H_B^\vee$ is generated as a K -module by

$$((1 - \zeta_l)^{-1} (1 - \zeta_p)^{-1})_* (1 - \tau)_* (1 - \sigma)_* \Delta_1,$$

which we simply denote Δ_1 as before, we have $\overline{\tilde{\omega}_{m,n}} = \tilde{\omega}_{m',n'}$ and hence

$$\tilde{\omega}_{m,n} + \tilde{\omega}_{m',n'} \in F^1 H_{\mathbb{R}}.$$

Define the regulator as

$$R_{m,n} := \int_{\Delta_0} \omega_{m,n} = \frac{B(1 - \alpha, \beta)}{l(\beta - \alpha + \mu)} \cdot F \left(\begin{matrix} 1 - \alpha, \beta, \beta - \alpha + \mu \\ 1 - \alpha + \beta, \beta - \alpha + \mu + 1 \end{matrix}; 1 \right).$$

By Theorem 6.5, for any $\gamma \in H_1(Z_1, \mathbb{Q})$ corresponding to $x \in K$ as in loc. cit., we have

$$\begin{aligned} \rho'_\mathbb{R}(\gamma)(\tilde{\omega}_{m,n} + \tilde{\omega}_{m',n'}) &= \mathrm{Im} \left(\chi_{m,n}(x) \Omega_{m,n}^{-1} R_{m,n} + \overline{\chi_{m,n}(x)} \Omega_{m',n'}^{-1} R_{m',n'} \right) \\ &= \mathrm{Im}(\chi_{m,n}(x)) \left(\Omega_{m,n}^{-1} R_{m,n} - \Omega_{m',n'}^{-1} R_{m',n'} \right). \end{aligned}$$

Since $\Omega_{m,n} \Omega_{m',n'} < 0$ and $R_{m,n}, R_{m',n'} > 0$, the above does not vanish for $x \in K \setminus \mathbb{R}$. Hence $\rho_\mathbb{R}$ is non-trivial. Since $\rho_\mathbb{R}$ is K -linear, the second assertion follows. $\square$

The non-vanishing of ρ is a more subtle problem. For the case $a + b = p$, we have the following criterion.

Proposition 6.7. *Let p, l be distinct prime numbers and suppose that $a + b = p$. If ρ is trivial, then there exists an $x \in K$ such that*

$$R_{m,n} = \chi_{m,n}(x) \Omega_{m,n}$$

for any $n = 1, \dots, p - 1$ and $m \in I_n^1$ such that $\frac{m}{l} > \{\frac{na}{p}\} - \{\frac{nb}{p}\}$.

Proof. Let $\gamma = \partial \Delta_0$ and suppose that $\rho(\gamma) = 0$. Since $H_B^\vee$ is generated by Δ_1 over K , there exists an $x \in K$ such that $\rho(\gamma)$ is represented by the functional $\int_{x_* \Delta_1}$. If m, n are as in the statement, then

$$\int_{x_* \Delta_1} \omega_{m,n} = \int_{\Delta_1} x^* \omega_{m,n} = \chi_{m,n}(x) \Omega_{m,n}$$

by the definition. Hence the proposition follows. $\square$

Example 6.8. If $p = 2$, then $\alpha = \beta = 1/2$ and Y is nothing but the Legendre family of elliptic curves. By Proposition 4.8, we have $\mathrm{Gr}_F^1 H_{\mathrm{dR}} = 0$ and the Deligne cohomology with $\mathbb{R}$ -coefficients is trivial. Since the condition $\frac{m}{l} > \{\frac{na}{p}\} - \{\frac{nb}{p}\}$ ($= 0$) is automatically satisfied, Proposition 6.7 is in fact an equivalence. If $l = 3$ for example, then ρ is trivial if and only if

$$\sqrt{3} \left(\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})} \right)^2 \cdot F \left(\frac{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}}{1, \frac{4}{3}}; 1 \right) \in \mathbb{Q}.$$

Here we used $\mathbb{Q}(\zeta_3) \cap i\mathbb{R} = \sqrt{3}i\mathbb{Q}$.

7. APPENDIX: FIBRATION OF CURVES AND EXTENSION OF MOTIVES (BY M. ASAKURA)

In this appendix, we develop a technique to compute regulators for a fibration of curves and motivic elements constructed from degenerating fibers, which is used in the proof of the regulator formula (Theorem 6.5).

7.1. Relative cohomology. Let V be a quasi-projective smooth surface over $\mathbb{C}$. Let $D \subset V$ be a chain of curves. Let $\pi: \tilde{D} \rightarrow D$ be the normalization and $\Sigma \subset D$ be the set of singular points. Let $s: \tilde{\Sigma} := \pi^{-1}(\Sigma) \hookrightarrow \tilde{D}$ be the inclusion. There is an exact sequence

$$0 \rightarrow \mathcal{O}_D \xrightarrow{\pi^*} \mathcal{O}_{\tilde{D}} \xrightarrow{s^*} \mathbb{C}_{\tilde{\Sigma}}/\mathbb{C}_{\Sigma} \rightarrow 0,$$

where $\mathbb{C}_{\tilde{\Sigma}} = \mathrm{Maps}(\tilde{\Sigma}, \mathbb{C}) = \mathrm{Hom}(\mathbb{Z}\tilde{\Sigma}, \mathbb{C})$ and π^*, s^* are the pull-backs. For a smooth manifold M , let $\mathcal{A}^q(M)$ denote the space of smooth differential q -forms on M with coefficients in $\mathbb{C}$. We define $\mathcal{A}^\bullet(D)$ to be the mapping fiber of $s^*: \mathcal{A}^\bullet(\tilde{D}) \rightarrow \mathbb{C}_{\tilde{\Sigma}}/\mathbb{C}_{\Sigma}$:

$$\mathcal{A}^0(\tilde{D}) \xrightarrow{s^* \oplus d} \mathbb{C}_{\tilde{\Sigma}}/\mathbb{C}_{\Sigma} \oplus \mathcal{A}^1(\tilde{D}) \xrightarrow{0 \oplus d} \mathcal{A}^2(\tilde{D}),$$

where the first term is placed in degree 0. Then

$$H_{\mathrm{dR}}^q(D) = H^q(\mathcal{A}^\bullet(D))$$

is the de Rham cohomology of D , which fits into the exact sequence

$$\cdots \rightarrow H_{\mathrm{dR}}^0(\tilde{D}) \rightarrow \mathbb{C}_{\tilde{\Sigma}}/\mathbb{C}_{\Sigma} \rightarrow H_{\mathrm{dR}}^1(D) \rightarrow H_{\mathrm{dR}}^1(\tilde{D}) \rightarrow \cdots.$$

We have the natural pairing

$$\langle \cdot, \cdot \rangle_D: H_1(D, \mathbb{Z}) \otimes H_{\mathrm{dR}}^1(D) \rightarrow \mathbb{C}, \quad \gamma \otimes z \mapsto \int_{\gamma} \eta - c(\partial(\pi^{-1}\gamma)),$$

where z is represented by $(c, \eta) \in \mathbb{C}_{\tilde{\Sigma}}/\mathbb{C}_{\Sigma} \oplus \mathcal{A}^1(\tilde{D})$ with $d\eta = 0$ and ∂ denotes the boundary of homology cycles.

We define $\mathcal{A}^\bullet(V, D)$ to be the mapping fiber of $\tilde{i}^*: \mathcal{A}^\bullet(V) \rightarrow \mathcal{A}^\bullet(\tilde{D})$, the pull-back by $\tilde{i}: \tilde{D} \rightarrow V$:

$$\mathcal{A}^0(V) \xrightarrow{\mathcal{D}_0} \mathcal{A}^0(\tilde{D}) \oplus \mathcal{A}^1(V) \xrightarrow{\mathcal{D}_1} \mathbb{C}_{\tilde{\Sigma}}/\mathbb{C}_{\Sigma} \oplus \mathcal{A}^1(\tilde{D}) \oplus \mathcal{A}^2(V) \xrightarrow{\mathcal{D}_2} \cdots.$$

Then the relative de Rham cohomology is defined by

$$H_{\mathrm{dR}}^q(V, D) = H^q(\mathcal{A}^\bullet(V, D))$$

and fits into the exact sequence

$$\cdots \rightarrow H_{\mathrm{dR}}^{q-1}(D) \rightarrow H_{\mathrm{dR}}^q(V, D) \rightarrow H_{\mathrm{dR}}^q(V) \rightarrow H_{\mathrm{dR}}^q(D) \rightarrow \cdots. \quad (7.1)$$

An element of $H_{\text{dR}}^2(V, D)$ is represented by

$$(c, \eta, \omega) \in \mathbb{C}_{\Sigma}/\mathbb{C}_{\Sigma} \oplus \mathcal{A}^1(\tilde{D}) \oplus \mathcal{A}^2(V) \quad (7.2)$$

which satisfies $\tilde{i}^*\omega = d\eta$ and $d\omega = 0$. The natural pairing

$$\langle \cdot, \cdot \rangle_{V,D}: H_2(V, D; \mathbb{Z}) \otimes H_{\text{dR}}^2(V, D) \rightarrow \mathbb{C}$$

is given by

$$\langle \Gamma, z \rangle_{V,D} = \int_{\Gamma} \omega - \langle \partial \Gamma, (c, \eta) \rangle_D = \int_{\Gamma} \omega - \int_{\partial \Gamma} \eta + c(\partial(\pi^{-1}(\partial \Gamma))).$$

The complexes $\mathcal{A}^\bullet(V)$ and $\mathcal{A}^\bullet(D)$ are canonically equipped with Hodge and weight filtrations, and then $(\mathbb{Q}_V, \mathcal{A}^\bullet(V), F^\bullet, W_\bullet)$ and $(\mathbb{Q}_D, \mathcal{A}^\bullet(D), F^\bullet, W_\bullet)$ become cohomological mixed Hodge complexes in the sense of [11, (8.1.2)]. The Hodge and weight filtrations on $\mathcal{A}^\bullet(V, D)$ are induced from them and the data $(\mathbb{Q}_{V,D}, \mathcal{A}^\bullet(V, D), F^\bullet, W_\bullet)$ becomes a cohomological mixed Hodge complex as well. Hence we have an exact sequence

$$\cdots \rightarrow H^{q-1}(D, \mathbb{Q}) \rightarrow H^q(V, D; \mathbb{Q}) \rightarrow H^q(V, \mathbb{Q}) \rightarrow H^q(D, \mathbb{Q}) \rightarrow \cdots$$

of mixed Hodge structures which is compatible with (7.1). Taking its dual, we obtain an exact sequence

$$0 \rightarrow H_2(V, \mathbb{Q})/H_2(D) \rightarrow H_2(V, D; \mathbb{Q}) \xrightarrow{\partial} H_1(D, \mathbb{Q}) \rightarrow H_1(V, \mathbb{Q}).$$

Since $H_1(V, \mathbb{Q}) \cap H^{0,0} = 0$, we obtain the coboundary map

$$\rho_{V,D}: H_1(D, \mathbb{Q}) \cap H^{0,0} \rightarrow \text{Ext}_{\text{MHS}}^1(\mathbb{Q}, H_2(V, \mathbb{Q})/H_2(D)) \quad (7.3)$$

to the extension group of mixed Hodge structures. If we put

$$H_{\text{dR}}^2(V)_D := \text{Ker}[H_{\text{dR}}^2(V) \rightarrow H_{\text{dR}}^2(D)],$$

then we have the Carlson isomorphism

$$\text{Ext}_{\text{MHS}}^1(\mathbb{Q}, H_2(V, \mathbb{Q})/H_2(D)) \simeq \text{Coker}[H_2(V, \mathbb{Q}) \rightarrow (F^1 H_{\text{dR}}^2(V)_D)^*], \quad (7.4)$$

where $*$ denotes the $\mathbb{C}$ -linear dual and the map is the natural pairing. Under this identification, the map $\rho_{V,D}$ is described as follows. For $\gamma \in H_1(D, \mathbb{Q}) \cap H^{0,0}$, take a $\Gamma \in H_2(V, D; \mathbb{Q})$ such that $\partial(\Gamma) = \gamma$. Then we have

$$\rho_{V,D}(\gamma) = \left[\omega \mapsto \langle \Gamma, \omega_{V,D} \rangle_{V,D} \right], \quad (7.5)$$

where $\omega_{V,D} \in F^1 H_{\text{dR}}^2(V, D)$ is a lifting of ω , on which the pairing does not depend.

7.2. Rational forms. For a given ω , it is usually complicated to compute an analytic lifting $\omega_{V,D}$ explicitly. In the following situation, we shall be able to associate a *rational* 2-form via Deligne's canonical extension, which gives a simple expression of $\rho_{V,D}$.

Let C be a projective smooth curve over $\mathbb{C}$ and $f: X \rightarrow C$ be a fibration of curves with connected general fiber which admits a section $e: C \rightarrow X$. From now on, we use the algebraic de Rham cohomology groups (see [15]) and identify them with the analytic ones in the previous paragraph. For a Zariski open set $S \subset C$, let $V = f^{-1}(S)$ and put

$$H_{\text{dR}}^2(V)_0 = \text{Ker} \left[H_{\text{dR}}^2(V) \rightarrow \prod_{s \in S} H_{\text{dR}}^2(f^{-1}(s)) \times H_{\text{dR}}^2(e(S)) \right],$$

$$H_{\mathrm{dR}}^2(V, D)_0 = \mathrm{Ker} [H_{\mathrm{dR}}^2(V, D) \rightarrow H_{\mathrm{dR}}^2(V)/H_{\mathrm{dR}}^2(V)_0].$$

Then we have an exact sequence of mixed Hodge structures

$$H_{\mathrm{dR}}^1(V) \rightarrow H_{\mathrm{dR}}^1(D) \rightarrow H_{\mathrm{dR}}^2(V, D)_0 \rightarrow H_{\mathrm{dR}}^2(V)_0 \rightarrow 0. \quad (7.6)$$

The arrows are strictly compatible with the Hodge and weight filtrations. In particular, $F^1 H_{\mathrm{dR}}^2(V, D)_0 \rightarrow F^1 H_{\mathrm{dR}}^2(V)_0$ is surjective. Later, we shall use the following.

Lemma 7.1. *Let $g: V' \rightarrow V$ be a birational transformation which is an isomorphism outside D and put $D' = g^{-1}(D)$. Then the pull-back g^* induces isomorphisms $H_{\mathrm{dR}}^2(V)_0 \simeq H_{\mathrm{dR}}^2(V')_0$ and $H_{\mathrm{dR}}^2(V, D)_0 \simeq H_{\mathrm{dR}}^2(V', D')_0$.*

Proof. By (7.6) it is enough to show isomorphisms

$$H_{\mathrm{dR}}^1(V) \simeq H_{\mathrm{dR}}^1(V'), \quad H_{\mathrm{dR}}^1(D) \simeq H_{\mathrm{dR}}^1(D'), \quad H_{\mathrm{dR}}^2(V)_0 \simeq H_{\mathrm{dR}}^2(V')_0.$$

The first one is an easy exercise. Let X' be a smooth compactification of V' such that $X' \setminus D' \simeq X \setminus D$ and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} H_{\mathrm{dR}}^2(X') & \xrightarrow{a^2} & H_{\mathrm{dR}}^2(X' \setminus D') & \longrightarrow & H_1^{\mathrm{dR}}(D') & \longrightarrow & H_{\mathrm{dR}}^3(X') & \xrightarrow{a^3} & H_{\mathrm{dR}}^3(X' \setminus D') \\ g_* \downarrow & & \parallel & & g_* \downarrow & & g_* \downarrow \simeq & & \parallel \\ H_{\mathrm{dR}}^2(X) & \xrightarrow{b^2} & H_{\mathrm{dR}}^2(X \setminus D) & \longrightarrow & H_1^{\mathrm{dR}}(D) & \longrightarrow & H_{\mathrm{dR}}^3(X) & \xrightarrow{b^3} & H_{\mathrm{dR}}^3(X \setminus D). \end{array}$$

The second isomorphism follows from the fact that

$$\mathrm{Image}(a^n) = \mathrm{Image}(b^n) = W_n H_{\mathrm{dR}}^n(X \setminus D).$$

The last isomorphism follows from the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\mathrm{dR}}^2(V)_0 & \longrightarrow & H_{\mathrm{dR}}^2(V \setminus D)_0 & \longrightarrow & H_1^{\mathrm{dR}}(D) \\ & & g^* \downarrow & & \parallel & & g^* \downarrow \simeq \\ 0 & \longrightarrow & H_{\mathrm{dR}}^2(V')_0 & \longrightarrow & H_{\mathrm{dR}}^2(V' \setminus D')_0 & \longrightarrow & H_1^{\mathrm{dR}}(D') \end{array}$$

with exact rows. □

Now, fix a Zariski open set $S \subset C$ such that $U := f^{-1}(S) \rightarrow S$ is smooth. Put $T = C \setminus S$ and $Z = X \setminus U$. Let

$$\nabla: \mathcal{H}_e \rightarrow \Omega_C^1(\log T) \otimes \mathcal{H}_e$$

be the Deligne canonical extension of the Gauss–Manin connection ($\mathcal{H} = R^1 f_* \Omega_{U/S}^\bullet, \nabla$). Put $F^1 \mathcal{H}_e = j_* F^1 \mathcal{H} \cap \mathcal{H}_e$, where $j: S \hookrightarrow C$ and $\mathrm{Gr}_F^0 \mathcal{H}_e = \mathcal{H}_e / F^1 \mathcal{H}_e$. Let

$$\overline{\nabla}: F^1 \mathcal{H}_e \rightarrow \Omega_C^1(\log T) \otimes \mathrm{Gr}_F^0 \mathcal{H}_e$$

be the $\mathcal{O}_C$ -linear map induced from ∇ . In what follows, we assume the following:

(*) The map $\overline{\nabla}$ is generically bijective.

Let $C^\circ \subset C$ be a Zariski open set on which $\overline{\nabla}$ is bijective and put $X^\circ := f^{-1}(C^\circ)$. Note that $S \not\subset C^\circ$ in general and $X^\circ \rightarrow C^\circ$ is not necessarily smooth. Then the

commutative diagram

$$\begin{array}{ccc}
& & 0 \\
& & \downarrow \\
& & \Omega_C^1(\log T) \otimes F^1 \mathcal{H}_e \\
& & \downarrow \\
F^1 \mathcal{H}_e & \xrightarrow{\nabla} & \Omega_C^1(\log T) \otimes \mathcal{H}_e \\
\downarrow = & & \downarrow \\
F^1 \mathcal{H}_e & \xrightarrow{\bar{\nabla}} & \Omega_C^1(\log T) \otimes \mathrm{Gr}_F^0 \mathcal{H}_e \\
& & \downarrow \\
& & 0
\end{array}$$

induces an isomorphism

$$\Lambda^\circ := \Gamma(C^\circ, \Omega_C^1(\log T) \otimes F^1 \mathcal{H}_e) \xrightarrow{\sim} H^1(C^\circ, F^1 \mathcal{H}_e \rightarrow \Omega_C^1(\log T) \otimes \mathcal{H}_e).$$

Note that $\Lambda^\circ \subset \Gamma(X^\circ, \Omega_X^2(\log Z))$.

Lemma 7.2. *There are natural injections*

$$F^1 H_{\mathrm{dR}}^2(X)_0 \hookrightarrow F^1 H_{\mathrm{dR}}^2(U)_0 \hookrightarrow \Lambda^\circ.$$

Proof. The first injectivity follows from Zariski's lemma (cf. [6, III, (8.2)]). Since

$$H_{\mathrm{dR}}^2(U)_0 \simeq H^1(S, \mathcal{H} \rightarrow \Omega_S^1 \otimes \mathcal{H}) \simeq H^1(C, \mathcal{H}_e \rightarrow \Omega_C^1(\log T) \otimes \mathcal{H}_e)$$

and

$$F^1 H^1(S, \mathcal{H} \rightarrow \Omega_S^1 \otimes \mathcal{H}) = H^1(C, F^1 \mathcal{H}_e \rightarrow \Omega_C^1(\log T) \otimes \mathcal{H}_e)$$

(cf. [27, §5]), the second injectivity follows from that of $F^1 H_{\mathrm{dR}}^2(U)_0 \rightarrow F^1 H_{\mathrm{dR}}^2(U \cap X^\circ)_0$. $\square$

Define $\Lambda(X) \subset \Lambda(U) \subset \Lambda^\circ$ to be the images of $F^1 H_{\mathrm{dR}}^2(X)_0$, $F^1 H_{\mathrm{dR}}^2(U)_0$, respectively. By the commutative diagram

$$\begin{array}{ccccc}
F^1 H_{\mathrm{dR}}^2(X)_0 & \longrightarrow & F^1 H_{\mathrm{dR}}^2(U)_0 & \longrightarrow & H_1^{\mathrm{dR}}(Z) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \\
\Lambda(X) & \longrightarrow & \Lambda(U) & \longrightarrow & H^0(X^\circ, \Omega_X^2(\log Z)/\Omega_X^2)
\end{array}$$

we have $\Lambda(X) \subset \Gamma(X^\circ, \Omega_X^2)$. For any cohomology class $\omega \in F^1 H_{\mathrm{dR}}^2(X)_0$, let $\omega^\circ \in \Lambda(X)$ denote the corresponding rational 2-form.

7.3. Main result. Now, let $D \subset X^\circ$ be a finite union of fibers. We give a description of the map

$$\rho_{X,D}: H_1(D, \mathbb{Q}) \cap H^{0,0} \rightarrow \mathrm{Coker} [H_2(X, \mathbb{Q}) \rightarrow (F^1 H_{\mathrm{dR}}^2(X)_0)^*]$$

induced from (7.3), (7.4) and the inclusion $F^1 H_{\mathrm{dR}}^2(X)_0 \subset F^1 H_{\mathrm{dR}}^2(X)_D$. Note that this factors through $\rho_{X^\circ, D}$. We regard an element $\eta \in \Lambda^\circ$ as an element of $\mathcal{A}^2(X^\circ)$. For the dimension reasons, we have $\tilde{i}^* \eta = 0$ and $d\eta = 0$. Hence $(0, 0, \eta)$ as in (7.2)

defines a cohomology class $\widehat{\eta} \in H_{\text{dR}}^2(X^\circ, D)$. Note that $\widehat{\eta}$ does not necessarily belong to F^1 . For any $\omega \in F^1 H_{\text{dR}}^2(X)_0$, write $\widehat{\omega}$ instead of $\widehat{\omega^\circ}$.

Theorem 7.3.

- (i) For any $\omega \in F^1 H_{\text{dR}}^2(X)_0$, we have $\widehat{\omega} \in F^1 H_{\text{dR}}^2(X^\circ, D)_0$ and it lifts $\omega|_{X^\circ}$.
- (ii) For any $\gamma \in H_1(D, \mathbb{Q}) \cap H^{0,0}$, choose $\Gamma \in H_2(X^\circ, D)$ such that $\partial(\Gamma) = \gamma$. Then we have

$$\rho_{X,D}(\gamma) = \left[\omega \mapsto \int_{\Gamma} \omega^\circ \right].$$

Proof. By (7.5), the assertion (ii) follows immediately from (i). By Lemma 7.1, we may assume that D_{red} and Z_{red} are divisors with normal crossings. It suffices to prove the case where $D = f^{-1}(P)$, $P \in C^\circ$.

For a Zariski sheaf $\mathcal{F}$, let $(\check{C}^\bullet(\mathcal{F}), \delta)$ denote its Čech complex. Firstly, $H_{\text{dR}}^2(X)$ is given by the cohomology in the middle of the complex

$$\check{C}^1(\mathcal{O}_X) \times \check{C}^0(\Omega_X^1) \xrightarrow{\mathcal{D}_1} \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\Omega_X^1) \times \check{C}^0(\Omega_X^2) \xrightarrow{\mathcal{D}_2} \check{C}^3(\mathcal{O}_X) \times \check{C}^2(\Omega_X^1) \times \check{C}^1(\Omega_X^2).$$

A description of $H_{\text{dR}}^2(U) = H^2(X, \Omega_X^\bullet(\log Z))$ is given similarly. Finally, $H_{\text{dR}}^2(X, D)$ is given by the complex

$$\begin{aligned} \check{C}^1(\mathcal{O}_X) \times \check{C}^0(\mathcal{O}_{\widetilde{D}} \oplus \Omega_X^1) &\xrightarrow{\mathcal{D}_3} \check{C}^2(\mathcal{O}_X) \times \check{C}^1(\mathcal{O}_{\widetilde{D}} \oplus \Omega_X^1) \times \check{C}^0(\mathcal{O}_{\widetilde{D}}/\mathcal{O}_\Sigma \oplus \Omega_{\widetilde{D}}^1 \oplus \Omega_X^2) \\ &\xrightarrow{\mathcal{D}_4} \check{C}^3(\mathcal{O}_X) \times \check{C}^2(\mathcal{O}_{\widetilde{D}} \oplus \Omega_X^1) \times \check{C}^1(\mathcal{O}_{\widetilde{D}}/\mathcal{O}_\Sigma \oplus \Omega_{\widetilde{D}}^1 \oplus \Omega_X^2). \end{aligned}$$

Let $\omega \in F^1 H_{\text{dR}}^2(X)_0$ and take its representative $z = (0) \times (\alpha_{ij}) \times (\beta_i) \in \text{Ker}(\mathcal{D}_2)$. Since $\omega \in F^1 H_{\text{dR}}^2(X)_D$, there exists $(\epsilon_i) \in \check{C}^0(\Omega_{\widetilde{D}}^1)$ such that $\alpha_{ij}|_{\widetilde{D}} = \epsilon_j - \epsilon_i$. If we put

$$z_{X,D} = (0) \times (0, \alpha_{ij}) \times (0, \epsilon_i, \beta_i),$$

then $z_{X,D} \in \text{Ker}(\mathcal{D}_4)$. By the definition of the Hodge filtration, it represents a class $\omega_{X,D} \in F^1 H_{\text{dR}}^2(X, D)$ which lifts ω . Let $\omega_{X,D}|_{X^\circ}$ be its image in $H_{\text{dR}}^2(X^\circ, D)$.

Let $\widehat{\omega} \in H_{\text{dR}}^2(X^\circ, D)$ be the class of the Čech cocycle

$$\widehat{z} := (0) \times (0, 0) \times (0, 0, \omega^\circ).$$

The group $H^1(C^\circ, F^1 \mathcal{H}_e \rightarrow \Omega_C^1(\log T) \otimes \mathcal{H}_e)$ is given by the complex

$$\begin{aligned} \check{C}^0(F^1 \mathcal{H}_e|_{C^\circ}) &\xrightarrow{\mathcal{D}_5} \check{C}^1(F^1 \mathcal{H}_e|_{C^\circ}) \times \check{C}^0(\Omega_C^1(\log T) \otimes \mathcal{H}_e|_{C^\circ}) \\ &\xrightarrow{\mathcal{D}_6} \check{C}^2(F^1 \mathcal{H}_e|_{C^\circ}) \times \check{C}^1(\Omega_C^1(\log T) \otimes \mathcal{H}_e|_{C^\circ}). \end{aligned}$$

By the definition of ω° , there exists $y = (\nu_i) \in \check{C}^0(F^1 \mathcal{H}_e|_{C^\circ})$ such that $\mathcal{D}_5(y) = (\alpha_{ij}) \times (\beta_i) - (0) \times (\omega^\circ)$, i.e. $\nu_j - \nu_i = \alpha_{ij}$, $d\nu_i = \beta_i - \omega^\circ$. Hence we have

$$z_{X,D}|_{X^\circ} - \widehat{z} = (0) \times (0, \nu_j - \nu_i) \times (0, \epsilon_i, d\nu_i).$$

It is clear that this vanishes in $H_{\text{dR}}^2(X^\circ)$, hence $\widehat{\omega}$ lifts $\omega|_{X^\circ}$.

We are left to show that the class of $\widehat{\omega}$ lies in F^1 . Let V be a sufficiently small neighborhood of D so that we have an exact sequence

$$0 \rightarrow \Omega_V^1 \rightarrow \Omega_V^1(\log D) \xrightarrow{\text{Res}} \widetilde{i}_* \mathcal{O}_{\widetilde{D}} \rightarrow 0.$$

Since $H_{\text{dR}}^2(X^\circ, D)/F^1 \rightarrow H_{\text{dR}}^2(V, D)/F^1$ is injective, it suffices to show the claim after restricting to V . Since $\text{Res}(\nu_j) - \text{Res}(\nu_i) = \text{Res}(\alpha_{ij}) = 0$, $(\text{Res}(\nu_i))$ defines a class $e \in H^0(\widetilde{D}, \mathcal{O}_{\widetilde{D}})$. Consider the composite

$$H^0(\widetilde{D}, \mathcal{O}_{\widetilde{D}}) \xrightarrow{\delta} H^1(V, \Omega_V^1) \xrightarrow{\widetilde{i}^*} H^1(\widetilde{D}, \Omega_{\widetilde{D}}^1) \simeq H_{\text{dR}}^2(\widetilde{D}),$$

where δ is the connecting map. Then $(\tilde{i}^* \circ \delta)(e)$ is represented by $(\alpha_{ij}|_{\tilde{D}}) \in \check{C}^1(\Omega_{\tilde{D}})$. Therefore, under the above isomorphism, $(\tilde{i}^* \circ \delta)(e)$ corresponds to $\tilde{i}^*(\omega) = 0$. Let $t \in \mathcal{O}_{C,P}$ be a uniformizer at P . By Zariski's lemma (cf. [6, III, (8.2)]), $\text{Ker}(\tilde{i}^* \circ \delta)$ is one-dimensional and generated by $\text{Res}(\frac{dt}{t})$. Hence there exists a constant c such that

$$\theta_i := \nu_i - c \frac{dt}{t}$$

has no pole along D . By replacing ν_i with θ_i and taking $\varepsilon_i = \theta_i|_{\tilde{D}}$, we see that $\omega_{X,D}|_V - \hat{\omega}|_V$ is in the image of $F^1 H_{\text{dR}}^1(V) \rightarrow H_{\text{dR}}^2(V, D)$. Hence we obtain $\hat{\omega} \in F^1$ and the proof is complete. $\square$

REFERENCES

- [1] G. W. Anderson, Logarithmic derivatives of Dirichlet L -functions and the periods of Abelian varieties, *Compositio Math.* **45**, Fasc. 3 (1982), 315-332.
- [2] N. Archinard, Hypergeometric abelian varieties, *Canad. J. Math.* **55** (2003), 897-932.
- [3] M. Asakura, A formula for Beilinson's regulator map on K_1 of a fibration of curves having a totally degenerate semistable fiber, unpublished, arXiv:1310.2810.
- [4] M. Asakura and N. Otsubo, CM periods, CM regulators and hypergeometric functions, II, preprint, arXiv:1503.08894.
- [5] M. Asakura and K. Sato, Chern classes and Riemann-Roch theorem for cohomology without homotopy invariance, preprint, arXiv:1301.5829.
- [6] W. Barth, K. Hulek, C. Peters and A. Van de Ven, *Compact complex surfaces* (2nd ed.), Springer-Verlag, Berlin, 2004.
- [7] A. A. Beilinson, Higher regulators and values of L -functions, *J. Soviet Math.* **30** (1985), 2036-2070.
- [8] S. Chowla and A. Selberg, On Epstein's zeta-function, *J. reine angew. Math.* **227** (1967), 86-110.
- [9] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lect. Notes Math. **163**, Springer, 1970.
- [10] P. Deligne, Théorie de Hodge II, *Publ. Math. IHES* **40** (1971), 5-57.
- [11] P. Deligne, Théorie de Hodge III, *Publ. Math. IHES* **44** (1974), 5-77.
- [12] A. Erdélyi et al. ed., *Higher transcendental functions*, Vol. 1, California Inst. Tech, 1981.
- [13] J. Fresán, Periods of Hodge structures and special values of the gamma function, *Invent. math.* **208** (2017), 247-282.
- [14] B. H. Gross (with an appendix by D. E. Rohrlich), On the periods of Abelian integrals and a formula of Chowla-Selberg, *Invent. Math.* **45** (1978), 193-211.
- [15] R. Hartshorne, On the de Rham cohomology of algebraic varieties, *Publ. Math. IHES* **45** (1975), 5-99.
- [16] M. Lerch, Sur quelques formules relatives au nombre des classes, *Bull. Sci. Math.* **21** (1897), prem. partie, 290-304.
- [17] V. Maillot and D. Roessler, On the periods of motives with complex multiplication and a conjecture of Gross-Deligne, *Ann. Math.* **160** (2004), 727-754.
- [18] D. R. Morrison, The Clemens-Schmid exact sequence and applications, *Topics in Transcendental Algebraic Geometry* (P. Griffiths, ed.), Ann. Math. Studies **106**, Princeton Univ. Press, Princeton, 1984, 101-119.
- [19] N. Otsubo, On the regulator of Fermat motives and generalized hypergeometric functions, *J. reine angew. Math.* **660** (2011), 27-82.
- [20] N. Otsubo, Certain values of Hecke L -functions and generalized hypergeometric functions, *J. Number Theory* **131** (2011), 648-660.
- [21] N. Otsubo, On special values of Jacobi-sum Hecke L -functions, *Exper. Math.* **24** (2015), no. 2, 247-259.
- [22] T. Saito, Vanishing cycles and geometry of curves over a discrete valuation ring, *Amer. J. Math.* **109**, No. 6 (1987), 1043-1085.
- [23] T. Saito and T. Terasoma, Determinant of period integrals, *J. Amer. Math. Soc.* **10** (1997), 865-937.

- [24] G. Shimura, Automorphic forms and periods of abelian varieties, J. Math. Soc. Japan **31** (1979), 561-592.
- [25] L. J. Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, Cambridge 1966.
- [26] J. Steenbrink, Limits of Hodge structures. Invent. Math. **31** (1976), 229-257.
- [27] J. Steenbrink and S. Zucker, Variation of mixed Hodge structure. I, Invent. Math. **80** (1985), 489-542.

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