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Author(s)	Aihara, Yuta
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Integration by Parts Formulae for a General Class of Functions in Boson Fock Spaces

Yuta Aihara

Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan

Abstract: A functional directional differential operator on Boson Fock spaces in the Q-space representation with a Gelfand's triple is considered. Then, we derive an integration by parts formula on that differential operator by employing the notion of strongly continuous one parameter unitary group.

Keywords: functional directional differential operator, integration by parts formula, Gelfand's triple, Boson Fock space, Q-space representation, strongly continuous one parameter unitary group.

MSC-class: 47B38

I. INTRODUCTION

Integration by parts formulae play important roles in analysis of finite dimensional manifolds. Hence it is important to derive integration by parts formulae for foundations of analysis of infinite dimensional spaces. For this subject, there have been studies [5,7]. We have such formulae for functions of finitely many random variables on infinite dimensional spaces, which form dense linear subspaces of L^2 -spaces. But there might be general classes of functions to which integration by parts formulae can be applied, not those of finitely many random variables. The purpose of this paper is to derive integration by parts formulae for such general functions.

The outline of this paper is as follows. In Section II, we review some fundamental facts related to functional directional differential operators on Boson Fock spaces in the Q-space representation with a Gelfand's triple. In Section III, we consider the image measure of a probability measure by the translation in locally convex spaces and introduce a strongly continuous one parameter unitary group on Boson Fock spaces employing that measure. In Section IV, by applying results in Section III, we derive integration by parts formulae for a general class of functions on infinite dimensional spaces.

II. PRELIMINARIES

Let $\mathcal{H}$ be a real separable Hilbert space, $\mathcal{E}$ be a real Hausdorff locally convex space with $\bar{\mathcal{E}} = \mathcal{H}$, and ι be a continuous embedding from $\mathcal{E}$ to $\mathcal{H}$. Then, ${}^t\iota$, the transpose of ι , is injective and regarded as an embedding from $\mathcal{H}$ to $\mathcal{E}'$. Thus, we have the Gelfand's triple $(\mathcal{E}, \mathcal{H}, \mathcal{E}')$ such that $\mathcal{E} \subset \mathcal{H} \subset \mathcal{E}'$, and the canonical bilinear form $\phi(f)$ on $\mathcal{E}' \times \mathcal{E}$ ($\phi \in \mathcal{E}'$, $f \in \mathcal{E}$) satisfies

$$\phi(f) = \langle \phi, f \rangle_{\mathcal{H}}, \quad \phi \in \mathcal{H}, \quad f \in \mathcal{E}. \quad (2.1)$$

Let $\mathcal{B}$ be a σ -field generated by $\{\phi(f) | f \in \mathcal{E}\}$ and μ be a probability measure on $(\mathcal{E}', \mathcal{B})$ such that

$$\int_{\mathcal{E}'} e^{i\phi(f)} d\mu(\phi) = e^{-\frac{\|f\|_{\mathcal{H}}^2}{4}}, \quad f \in \mathcal{E}. \quad (2.2)$$

Then, we have

$$\int_{\mathcal{E}'} \phi(f)^2 d\mu(\phi) = \frac{\|f\|^2}{2}, \quad f \in \mathcal{E}. \quad (2.3)$$

We denote by $L^2(\mathcal{E}', d\mu; \mathbb{R})$ the real Hilbert space of all real valued $F \in L^2(\mathcal{E}', d\mu)$. Hence, the operator $f \mapsto \phi(f)$ from $\mathcal{E}$ to $L^2(\mathcal{E}', d\mu; \mathbb{R})$ is continuous linear and extends to the continuous linear operator T from $\mathcal{H}$ to $L^2(\mathcal{E}', d\mu; \mathbb{R})$ such that

$$\int_{\mathcal{E}'} T(f)^2 d\mu = \frac{\|f\|^2}{2}, \quad f \in \mathcal{H}. \quad (2.4)$$

For all $f \in \mathcal{H}$ and $\phi \in \mathcal{E}'$ we define $\phi(f)$ by

$$\phi(f) := T(f)(\phi).$$

Then, we have

$$\int_{\mathcal{E}'} \phi(f)^2 d\mu(\phi) = \frac{\|f\|^2}{2}, \quad \int_{\mathcal{E}'} e^{i\phi(f)} d\mu(\phi) = e^{-\frac{\|f\|_{\mathcal{H}}^2}{4}}, \quad f \in \mathcal{H}. \quad (2.5)$$

PROPOSITION 2.1. *For all $f \in \mathcal{H}$,*

$$\phi(f) = \langle \phi, f \rangle_{\mathcal{H}}, \quad \mu\text{-a.e. } \phi \in \mathcal{E}'. \quad (2.6)$$

Proof. Let $f \in \mathcal{H}$. Since $\bar{\mathcal{E}} = \mathcal{H}$, there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{E}$ such that $f_n \rightarrow f$ as $n \rightarrow \infty$. Since

$$T(f_n) \longrightarrow T(f), \quad \text{as } n \rightarrow \infty,$$

taking a subsequence of $\{f_n\}_{n \in \mathbb{N}}$, we can assume that

$$\phi(f_n) \longrightarrow \phi(f), \quad \mu\text{-a.e. } \phi \in \mathcal{E}', \quad \text{as } n \rightarrow \infty.$$

Since for all $n \in \mathbb{N}$ and $\phi \in \mathcal{H}$,

$$\phi(f_n) = \langle \phi, f_n \rangle_{\mathcal{H}},$$

we have

$$\phi(f) = \langle \phi, f \rangle_{\mathcal{H}}, \quad \mu\text{-a.e. } \phi \in \mathcal{H}.$$

□

We denote by $\mathcal{H}_{\mathbb{C}}$ the complexification of $\mathcal{H}$ and define a complex Hilbert space $\mathcal{F}_b(\mathcal{H}_{\mathbb{C}})$, which is called the Boson Fock space over $\mathcal{H}_{\mathbb{C}}$, by

$$\begin{aligned} \mathcal{F}_b(\mathcal{H}_{\mathbb{C}}) &:= \bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathcal{H}_{\mathbb{C}} \\ &= \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \mid \psi^{(n)} \in \bigotimes_s^n \mathcal{H}_{\mathbb{C}}, n \geq 0, \sum_{n=0}^{\infty} \|\psi^{(n)}\|_{\bigotimes_s^n \mathcal{H}_{\mathbb{C}}}^2 < \infty \right\}, \end{aligned}$$

where $\bigotimes_s^n \mathcal{H}_{\mathbb{C}}$ is the n -fold symmetric tensor product of $\mathcal{H}_{\mathbb{C}}$ with $\bigotimes_s^0 \mathcal{H}_{\mathbb{C}} := \mathbb{C}$. For all $f \in \mathcal{H}_{\mathbb{C}}$, we denote by $a(f)$ the Boson annihilation operator in $\mathcal{F}_b(\mathcal{H}_{\mathbb{C}})$ with test vector f , which is the unique densely defined closed linear operator on $\mathcal{F}_b(\mathcal{H}_{\mathbb{C}})$ such that its adjoint $a(f)^*$ takes the following form :

$$(a(f)^*\psi)^{(0)} = 0, \quad (a(f)^*\psi)^{(n)} = \sqrt{n} S_n(f \otimes \psi^{(n-1)}), \quad n \geq 1, \quad \psi \in D(a(f)^*),$$

where S_n denotes the symmetrization operator (symmetrizer) on $\bigotimes^n \mathcal{H}_{\mathbb{C}}$. The operator $a(f)^*$ is called the creation operator with test vector f .

For all $f_1, \dots, f_n \in \mathcal{H}$, we denote by $:\phi(f_1) \cdots \phi(f_n):$ the Wick product of the random variables $\phi(f_1), \dots, \phi(f_n)$, which obeys the following recursion relations :

$$:\phi(f_1): = \phi(f_1),$$

$$\begin{aligned} :\phi(f_1) \cdots \phi(f_n): &= \phi(f_1) : \phi(f_2) \cdots \phi(f_n) : \\ &- \frac{1}{2} \sum_{j=2}^n \langle f_1, f_j \rangle : \phi(f_2) \cdots \widehat{\phi(f_j)} \cdots \phi(f_n) : \quad n \geq 2, \end{aligned}$$

where $\widehat{\phi(f_j)}$ indicates the omission of $\phi(f_j)$. We define a vector $\Omega \in \mathcal{F}_b(\mathcal{H}_{\mathbb{C}})$ by

$$\Omega := \{1, 0, 0, \dots\},$$

which is called the Fock vacuum in $\mathcal{F}_b(\mathcal{H}_{\mathbb{C}})$. The following fact is well known (e.g. [5,8]):

THEOREM 2.2. *There exists a unique unitary operator U from $\mathcal{F}_b(\mathcal{H}_\mathbb{C})$ to $L^2(\mathcal{E}', d\mu)$ such that*

$$U\Omega = 1,$$

$$U(a(f_1)^* \cdots a(f_n)^* \Omega) = (\sqrt{2})^n : \phi(f_1) \cdots \phi(f_n) :, \quad f_1, \dots, f_n \in \mathcal{H}$$

For all $f \in \mathcal{H}$, we define $\pi(f)$ by

$$\pi(f) := U \left(\frac{i}{\sqrt{2}} \overline{(a(f)^* - a(f))} \right) U^{-1}.$$

Then, for all $f \in \mathcal{H}$, $\pi(f)$ is self-adjoint.

For all $f \in \mathcal{H}$, we define D_f , called the functional directional derivative along f , by

$$D_f := i\pi(f) + \phi(f).$$

We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all polynomials of n variables. We define $\mathcal{P}(\mathcal{E}')$ by

$$\mathcal{P}(\mathcal{E}') := \{F(\phi(f_1), \dots, \phi(f_n)) | F \in \mathcal{P}(\mathbb{R}^n), f_j \in \mathcal{E}, j = 1, \dots, n, n \in \mathbb{N}\}.$$

Then, as is well known, for all $f \in \mathcal{H}$, D_f is a closable linear operator acting in $L^2(\mathcal{E}', d\mu)$ and the following proposition holds.

PROPOSITION 2.3. [5] *For all $g \in \mathcal{H}$, $\mathcal{P}(\mathcal{E}')$ is a core of $\pi(g)$ and for all $F \in \mathcal{P}(\mathbb{R}^n)$ and $f_j \in \mathcal{H}, j = 1, \dots, n$,*

$$D_g F(\phi(f_1), \dots, \phi(f_n)) = \sum_{j=1}^n \langle f_j, g \rangle (\partial_j F)(\phi(f_1), \dots, \phi(f_n)). \quad (2.7)$$

III. A Strongly Continuous One Parameter Unitary Group

For all $g \in \mathcal{H}$, we define the translation τ_g acting in $\mathcal{E}'$ by

$$\tau_g(\phi) := \phi + g, \quad \phi \in \mathcal{E}'.$$

Then, we have the following proposition.

PROPOSITION 3.1. (1) *For all $g \in \mathcal{H}$ and $B \in \mathcal{B}$, $g + B \in \mathcal{B}$.*

(2) *For all $g \in \mathcal{H}$ and $B \subset \mathcal{E}'$, $\tau_g^{-1}(B) \in \mathcal{B}$ if and only if $B \in \mathcal{B}$.*

Proof. (1) Let $A \subset \mathbb{R}$ be a Borel set and $f \in \mathcal{E}$. It is enough to show that $g + T(f)^{-1}(A) \in \mathcal{B}$. Since

$$g + T(f)^{-1}(A) = T(f)^{-1}(\langle g, f \rangle + A)$$

and $\langle g, f \rangle + A$ is a Borel set, we have $g + T(f)^{-1}(A) \in \mathcal{B}$.

(2) Since $\tau_g^{-1}(B) = -g + B$, by (1), we have the conclusion. $\square$

We denote by μ_g the image measure of μ by τ_g . Then, noticing Proposition 3.1, we see the following proposition.

PROPOSITION 3.2. *Let F be a measurable function on $\mathcal{E}'$. Then,*

$$\begin{aligned} \int_{\mathcal{E}'} |F| d\mu_g < \infty &\iff \int_{\mathcal{E}'} |F| \circ \tau_g d\mu < \infty \\ &= \int_{\mathcal{E}'} |F(\phi + g)| d\mu(\phi) \end{aligned}$$

If

$$\int_{\mathcal{E}'} |F| d\mu_g < \infty,$$

then,

$$\int_{\mathcal{E}'} F d\mu_g = \int_{\mathcal{E}'} F \circ \tau_g d\mu \quad (3.8)$$

$$= \int_{\mathcal{E}'} F(\phi + g) d\mu(\phi) \quad (3.9)$$

PROPOSITION 3.3. *Let F be a bounded measurable function on $\mathcal{E}'$. Then, the mapping S from $\mathcal{H}$ to $L^1(\mathcal{E}', d\mu)$ defined by*

$$S(g)(\phi) = F(\phi) e^{\phi(g)}, \quad g \in \mathcal{H}$$

is continuous.

Proof. Let $g_1, g_2 \in \mathcal{H}$. Then,

$$\begin{aligned} &\left| \int_{\mathcal{E}'} F(\phi) e^{\phi(g_1)} d\mu(\phi) - \int_{\mathcal{E}'} F(\phi) e^{\phi(g_2)} d\mu(\phi) \right| \\ &\leq \|F\|_\infty \int_{\mathcal{E}'} |e^{\phi(g_1)} - e^{\phi(g_2)}| d\mu(\phi) \\ &\leq \|F\|_\infty \int_{\mathcal{E}'} (e^{\phi(g_1)} + e^{\phi(g_2)}) |\phi(g_1) - \phi(g_2)| d\mu(\phi) \\ &\leq \|F\|_\infty \left(\int_{\mathcal{E}'} (e^{\phi(g_1)} + e^{\phi(g_2)})^2 d\mu(\phi) \right)^{1/2} \left(\int_{\mathcal{E}'} |\phi(g_1) - \phi(g_2)|^2 d\mu(\phi) \right)^{1/2} \\ &\leq \sqrt{2} \|F\|_\infty \left(\int_{\mathcal{E}'} (e^{2\phi(g_1)} + e^{2\phi(g_2)}) d\mu(\phi) \right)^{1/2} \frac{1}{\sqrt{2}} \|g_1 - g_2\| \\ &= \|F\|_\infty \left(\int_{\mathcal{E}'} (e^{2\phi(g_1)} + e^{2\phi(g_2)}) d\mu(\phi) \right)^{1/2} \|g_1 - g_2\| \end{aligned}$$

On the other hand, for all $g \in \mathcal{H}$,

$$\int_{\mathcal{E}'} e^{\phi(g)} d\mu(\phi) = e^{\frac{\|g\|^2}{4}}.$$

Hence, we have

$$\left| \int_{\mathcal{E}'} F(\phi) e^{\phi(g_1)} d\mu(\phi) - \int_{\mathcal{E}'} F(\phi) e^{\phi(g_2)} d\mu(\phi) \right| \leq \|F\|_\infty \sqrt{e^{\|g_1\|^2} + e^{\|g_2\|^2}} \|g_1 - g_2\|.$$

Then, we have the conclusion. $\square$

PROPOSITION 3.4. *Let $\{e_n\}_{n \in \mathbb{N}}$ be a CONS of $\mathcal{H}$ such that for all $n \in \mathbb{N}$, $e_n \in \mathcal{E}$, G be a bounded Borel measurable function on $\mathbb{R}^n$, and F be a measurable function on $\mathcal{E}'$ such that*

$$F(\phi) = G(\phi(e_1), \dots, \phi(e_n)), \quad \phi \in \mathcal{E}'.$$

Then,

$$\int_{\mathcal{E}'} F(\phi + g) d\mu(\phi) = \int_{\mathcal{E}'} F(\phi) e^{-\|g\|^2 + 2\phi(g)} d\mu(\phi). \quad (3.10)$$

Proof.

$$\begin{aligned} \int_{\mathcal{E}'} F(\phi + g) d\mu(\phi) &= \int_{\mathcal{E}'} G(\phi(e_1) + \langle g, e_1 \rangle, \dots, \phi(e_n) + \langle g, e_n \rangle) \\ &= \int_{\mathbb{R}^n} G(x_1 + \langle g, e_1 \rangle, \dots, x_n + \langle g, e_n \rangle) \frac{e^{-\sum_{j=1}^n x_j^2}}{(\sqrt{\pi})^n} dx \\ &= \int_{\mathbb{R}^n} G(x_1, \dots, x_n) \frac{e^{-\sum_{j=1}^n (x_j - \langle g, e_j \rangle)^2}}{(\sqrt{\pi})^n} dx. \end{aligned}$$

Hence, for all $m \in \mathbb{N}$ with $n \leq m$,

$$\int_{\mathcal{E}'} F(\phi + g) d\mu(\phi) = \int_{\mathcal{E}'} F(\phi) e^{-\sum_{j=1}^m \langle g, e_j \rangle^2} e^{2\phi(\sum_{j=1}^m \langle g, e_j \rangle e_j)} d\mu(\phi)$$

By Proposition 3.3,

$$\begin{aligned} &\int_{\mathcal{E}'} F(\phi) e^{-\sum_{j=1}^m \langle g, e_j \rangle^2} e^{2\phi(\sum_{j=1}^m \langle g, e_j \rangle e_j)} d\mu(\phi) \\ &\longrightarrow \int_{\mathcal{E}'} F(\phi) e^{-\|g\|^2 + 2\phi(g)} d\mu(\phi), \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Hence, we have

$$\int_{\mathcal{E}'} F(\phi + g) d\mu(\phi) = \int_{\mathcal{E}'} F(\phi) e^{-\|g\|^2 + 2\phi(g)} d\mu(\phi).$$

$\square$

PROPOSITION 3.5. *Suppose that there exists a CONS $\{e_n\}_{n \in \mathbb{N}}$ of $\mathcal{H}$ such that for all $n \in \mathbb{N}$, $e_n \in \mathcal{E}$ and for all $f \in \mathcal{E}$, there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}$ such that for all $\phi \in \mathcal{E}'$,*

$$\phi(f) = \sum_{n=1}^{\infty} a_n \phi(e_n).$$

Then, for all $g \in \mathcal{H}$,

$$d\mu_g(\phi) = e^{-\|g\|^2 + 2\phi(g)} d\mu(\phi). \quad (3.11)$$

Moreover, if F be a measurable function on $\mathcal{E}'$ such that

$$\int_{\mathcal{E}'} |F| d\mu_g < \infty,$$

then,

$$\int_{\mathcal{E}'} F d\mu_g = \int_{\mathcal{E}'} F(\phi + g) d\mu(\phi) \quad (3.12)$$

$$= \int_{\mathcal{E}'} F(\phi) e^{-\|g\|^2 + 2\phi(g)} d\mu(\phi). \quad (3.13)$$

Proof. By the present assumption, $\mathcal{B}$ is generated by $\{\phi(e_n) | n \in \mathbb{N}\}$. Then, by Proposition 3.2, Proposition 3.4, and Hopf's extension theorem, we have the conclusion. $\square$

Remark. The measure μ is not a Borel measure. *Example.* Let A be a strictly positive operator acting in $\mathcal{H}$. Suppose that there exists a constant $\gamma > 0$ such that $A^{-\gamma}$ is a trace class operator. We denote by $\{\mathcal{H}_s(A)\}_{s \in \mathbb{R}}$ the Hilbert scale associated with A . We set

$$\mathcal{E} := \mathcal{H}_\gamma(A).$$

Then, the embedding mapping of $\mathcal{H}$ into

$$\mathcal{E}' = \mathcal{H}_{-\gamma}(A)$$

is Hilbert-Schmidt. Hence, by Minlos' theorem [5,9], there exists a unique probability measure μ on $(\mathcal{E}', \mathcal{B})$ such that the σ -field $\mathcal{B}$ is generated by $\{\phi(f) | f \in \mathcal{E}\}$ and

$$\int_{\mathcal{E}'} e^{i\phi(f)} d\mu(\phi) = e^{\frac{-\|f\|^2}{4}}, \quad f \in \mathcal{E}.$$

Then, $(\mathcal{E}, \mathcal{H})$ satisfies the assumption of Proposition 3.5 [1,2,3,4]. Throughout this paper, we assume the following.

Assumption I. (absolute continuity of μ_g) For all $g \in \mathcal{H}$ and $B \in \mathcal{B}$ such that $\mu(B) = 0$, $\mu(g + B) = 0$.

Under the assumption of Proposition 3.5, Assumption I is satisfied. Absolute continuity of infinite dimensional measures is well known in different contexts (e.g. [11,12,13]).

PROPOSITION 3.6. For all $f \in \mathcal{H}$,

$$(\phi + g)(f) = \phi(f) + \langle g, f \rangle, \quad \mu - a.e. \quad \phi \in \mathcal{E}'. \quad (3.14)$$

Proof. There exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{E}$ such that $f_n \rightarrow f$ as $n \rightarrow \infty$. Then, we have

$$\phi(f_n) \longrightarrow \phi(f) \text{ as } n \rightarrow \infty \text{ in } L^2(\mathcal{E}', d\mu).$$

By taking a subsequence of $\{f_n\}_{n \in \mathbb{N}}$, we can assume that

$$\phi(f_n) \longrightarrow \phi(f), \text{ as } n \rightarrow \infty, \mu - a.e. \phi \in \mathcal{E}'.$$

On the other hand, for all $n \in \mathbb{N}$,

$$(\phi + g)(f_n) = \phi(f_n) + \langle g, f_n \rangle, \quad \phi \in \mathcal{E}'.$$

Since

$$\begin{aligned} (\phi + g)(f_n) &\longrightarrow (\phi + g)(f), \\ \phi(f_n) + \langle g, f_n \rangle &\longrightarrow \phi(f) + \langle g, f \rangle, \text{ as } n \rightarrow \infty, \mu - a.e. \phi \in \mathcal{E}'. \end{aligned}$$

Hence, we have the conclusion. $\square$

Let X be a linear space and $D \subset X$. We denote by $\mathcal{L}(D)$ the linear hull of D . The following fact is well known (e.g.[5]):

PROPOSITION 3.7. $\mathcal{L}(\{e^{i\phi(f)} | f \in \mathcal{E}\})$ is dense in $L^2(\mathcal{E}', d\mu)$.

Let $\{e_n\}_{n \in \mathbb{N}}$ be a CONS of $\mathcal{H}$. We denote by $\mathcal{B}_b(\mathbb{R}^n)$ the set of bounded Borel measurable functions on $\mathbb{R}^n$ and define $\mathcal{B}_{\{e_n\}_{n \in \mathbb{N}}}(\mathcal{E}')$ by

$$\mathcal{B}_{\{e_n\}_{n \in \mathbb{N}}}(\mathcal{E}') := \{F(\phi(e_1), \dots, \phi(e_n)) | F \in \mathcal{B}_b(\mathbb{R}^n), n \in \mathbb{N}\}$$

Then, we have the following proposition.

PROPOSITION 3.8. $\mathcal{B}_{\{e_n\}_{n \in \mathbb{N}}}(\mathcal{E}')$ is dense in $L^2(\mathcal{E}', d\mu)$.

Proof. Let $f \in \mathcal{H}$. For all $n \in \mathbb{N}$, we set

$$f_n := \sum_{j=1}^n \langle f, e_j \rangle e_j.$$

Then,

$$\begin{aligned} \int_{\mathcal{E}'} |e^{i\phi(f_n)} - e^{i\phi(f)}|^2 d\mu(\phi) &\leq \int_{\mathcal{E}'} |\phi(f_n) - \phi(f)|^2 d\mu(\phi) \\ &= \frac{1}{2} \|f_n - f\|^2 \longrightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

On the other hand,

$$e^{i\phi(f_n)} = e^{i \sum_{j=1}^n \langle f, e_j \rangle \phi(e_j)} = \prod_{j=1}^n e^{i \langle f, e_j \rangle \phi(e_j)} \in \mathcal{B}_{\{e_n\}_{n \in \mathbb{N}}}(\mathcal{E}')$$

Hence, by Proposition 3.7, we have the conclusion. $\square$

PROPOSITION 3.9. For all $g \in \mathcal{H}$, there exists a unique unitary operator U_g on $L^2(\mathcal{E}', d\mu)$ such that for all $F \in L^2(\mathcal{E}', d\mu)$,

$$(U_g F)(\phi) = F(\phi + g) e^{-\frac{\|g\|^2}{2} - \phi(g)}, \quad \mu - \text{a.e. } \phi \in \mathcal{E}'. \quad (3.15)$$

Proof. There exists a CONS $\{e_n\}_{n \in \mathbb{N}}$ of $\mathcal{H}$ such that for all $n \in \mathbb{N}$, $e_n \in \mathcal{E}$. We define U_g , a linear operator from $\mathcal{B}_{\{e_n\}_{n \in \mathbb{N}}}(\mathcal{E}')$ to $L^2(\mathcal{E}', d\mu)$, by

$$(U_g F)(\phi) := F(\phi + g) e^{-\frac{\|g\|^2}{2} - \phi(g)}, \quad F \in \mathcal{B}_{\{e_n\}_{n \in \mathbb{N}}}(\mathcal{E}'), \quad \phi \in \mathcal{E}'.$$

By Proposition 3.4, we have

$$\int_{\mathcal{E}'} |F(\phi)|^2 d\mu(\phi) = \int_{\mathcal{E}'} |F(\phi + g)|^2 e^{-\|g\|^2 - 2\phi(g)} d\mu(\phi).$$

Hence, we have

$$\|F\|_{L^2(\mathcal{E}', d\mu)} = \|U_g F\|_{L^2(\mathcal{E}', d\mu)}.$$

Then, by Proposition 3.8, U_g extends uniquely to an isometry on $L^2(\mathcal{E}', d\mu)$.

To show that U_g is unitary, let $F \in L^2(\mathcal{E}', d\mu)$. Then, there exists a sequence $\{F_n\}_{n \in \mathbb{N}}$ in $\mathcal{B}_{\{e_n\}_{n \in \mathbb{N}}}(\mathcal{E}')$ such that

$$F_n \longrightarrow F \text{ in } L^2(\mathcal{E}', d\mu), \text{ as } n \rightarrow \infty.$$

Since

$$U_g F_n \longrightarrow U_g F \text{ in } L^2(\mathcal{E}', d\mu), \text{ as } n \rightarrow \infty,$$

taking a subsequence of $\{F_n\}_{n \in \mathbb{N}}$, one can assume that

$$F_n(\phi) \longrightarrow F(\phi), \quad (U_g F_n)(\phi) \longrightarrow (U_g F)(\phi) \text{ as } n \rightarrow \infty, \quad \mu - \text{a.e. } \phi \in \mathcal{E}'.$$

On the other hand, for all $n \in \mathbb{N}$,

$$(U_g F_n)(\phi) = F_n(\phi + g) e^{-\frac{\|g\|^2}{2} - \phi(g)}, \quad \mu - \text{a.e. } \phi \in \mathcal{E}'.$$

Since

$$F_n(\phi + g) \longrightarrow F(\phi + g) \text{ as } n \rightarrow \infty, \quad \mu - \text{a.e. } \phi \in \mathcal{E}',$$

we have

$$(U_g F)(\phi) = F(\phi + g) e^{-\frac{\|g\|^2}{2} - \phi(g)}, \quad \mu - \text{a.e. } \phi \in \mathcal{E}'.$$

Hence, for all $F \in L^2(\mathcal{E}', d\mu)$, we have

$$\begin{aligned} \int_{\mathcal{E}'} |F(\phi)|^2 d\mu(\phi) &= \int_{\mathcal{E}'} |F(\phi + g)|^2 e^{-\|g\|^2 - 2\phi(g)} d\mu(\phi), \\ \int_{\mathcal{E}'} |F(\phi)|^2 d\mu(\phi) &= \int_{\mathcal{E}'} |F(\phi - g)|^2 e^{-\|g\|^2 + 2\phi(g)} d\mu(\phi). \end{aligned}$$

Let $G \in L^2(\mathcal{E}', d\mu)$. We set

$$F(\phi) := G(\phi - g)e^{-\frac{\|g\|^2}{2} + \phi(g)}, \quad \phi \in \mathcal{E}'.$$

Then, $F \in L^2(\mathcal{E}', d\mu)$, and by Proposition 3.6,

$$U_g F = G.$$

Hence, U_g is a unitary operator. □

PROPOSITION 3.10. *For all $g \in \mathcal{H}$,*

$$d\mu_g(\phi) = e^{-\|g\|^2 + 2\phi(g)} d\mu(\phi). \quad (3.16)$$

Moreover, if F is a measurable function on $\mathcal{E}'$ such that

$$\int_{\mathcal{E}'} |F| d\mu_g < \infty,$$

then,

$$\int_{\mathcal{E}'} F d\mu_g = \int_{\mathcal{E}'} F(\phi + g) d\mu(\phi) \quad (3.17)$$

$$= \int_{\mathcal{E}'} F(\phi) e^{-\|g\|^2 + 2\phi(g)} d\mu(\phi). \quad (3.18)$$

Proof. By Proposition 3.9, for all $F \in L^2(\mathcal{E}', d\mu)$,

$$\int_{\mathcal{E}'} |F(\phi)|^2 d\mu(\phi) = \int_{\mathcal{E}'} |F(\phi - g)|^2 e^{-\|g\|^2 + 2\phi(g)} d\mu(\phi).$$

Let $B \in \mathcal{B}$. We denote by φ_B the characteristic function of B . Then,

$$\int_{\mathcal{E}'} \varphi_B(\phi + g) d\mu(\phi) = \mu(-g + B) < \infty.$$

Hence, we have

$$\mu_g(B) = \int_{\mathcal{E}'} \varphi_B(\phi + g) d\mu(\phi) = \int_{\mathcal{E}'} \varphi_B(\phi) e^{-\|g\|^2 + 2\phi(g)} d\mu(\phi).$$

Then, we have the conclusion. □

For all $g \in \mathcal{H}$ and $t \in \mathbb{R}$, we define $U_g(t)$, a linear operator on $L^2(\mathcal{E}', d\mu)$, by

$$U_g(t) := U_{tg}.$$

Then, by Proposition 3.9, for all $g \in \mathcal{H}$ and $t \in \mathbb{R}$, $U_g(t)$ is a unitary operator on $L^2(\mathcal{E}', d\mu)$.

In what follows, we consider $\mathcal{E}'$ equipped with a Hausdorff locally convex topology such that for all $f \in \mathcal{E}$, $T(f)$ is continuous. We denote by $\mathcal{C}_b(\mathcal{E}')$ the set of bounded continuous measurable functions on $\mathcal{E}'$. Then, by Proposition 3.7, we have the following proposition.

PROPOSITION 3.11. $\mathcal{C}_b(\mathcal{E}')$ is dense in $L^2(\mathcal{E}', d\mu)$.

Proof. We have

$$\mathcal{L}(\{e^{i\phi(f)}|f \in \mathcal{E}\}) \subset \mathcal{C}_b(\mathcal{E}').$$

Hence, by Proposition 3.7, we have the conclusion. $\square$

The following proposition is well known in functional analysis.

PROPOSITION 3.12. Let X, Y be Banach spaces, D be a dense subset of X , S be a bounded linear operator from X to Y , and $\{S_\lambda\}_{\lambda \in \Lambda}$ be a net in the Banach space of the bounded linear operators from X to Y . If

$$\sup_{\lambda \in \Lambda} \|S_\lambda\| < \infty,$$

and for all $x \in D$,

$$S_\lambda(x) \longrightarrow S(x),$$

then, for all $x \in X$,

$$S_\lambda(x) \longrightarrow S(x).$$

Proof. Let $x \in X$. Then, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in D such that

$$x_n \longrightarrow x \text{ as } n \rightarrow \infty \text{ in } X.$$

For all $n \in \mathbb{N}$ and $\lambda \in \Lambda$,

$$\begin{aligned} \|S_\lambda(x) - S(x)\| &\leq \|S_\lambda(x) - S_\lambda(x_n)\| + \|S_\lambda(x_n) - S(x_n)\| + \|S(x_n) - S(x)\| \\ &\leq \sup_{\lambda \in \Lambda} \|S_\lambda\| \|x_n - x\| + \|S_\lambda(x_n) - S(x_n)\| + \|S\| \|x_n - x\|. \end{aligned}$$

Since, for all $n \in \mathbb{N}$,

$$\|S_\lambda(x_n) - S(x_n)\| \longrightarrow 0 \text{ in } \lambda,$$

for all $n \in \mathbb{N}$,

$$\overline{\lim}_\lambda \|S_\lambda(x) - S(x)\| \leq \sup_{\lambda \in \Lambda} \|S_\lambda\| \|x_n - x\| + \|S\| \|x_n - x\|.$$

Hence, we have

$$\overline{\lim}_\lambda \|S_\lambda(x) - S(x)\| = 0.$$

Then, we have

$$S_\lambda(x) \longrightarrow S(x).$$

$\square$

THEOREM 3.13. For all $g \in \mathcal{H}$, $\{U_g(t)\}_{t \in \mathbb{R}}$ is a strongly continuous one parameter unitary group on $L^2(\mathcal{E}', d\mu)$.

Proof. By Proposition 3.6, for all $s, t \in \mathbb{R}$,

$$U_g(s)U_g(t) = U_g(s+t).$$

Let $F \in \mathcal{C}_b(\mathcal{E}')$. Then, for all $\phi \in \mathcal{E}'$,

$$(U_g(t)F)(\phi) = F(\phi + tg)e^{-\frac{t^2\|g\|^2}{2} - t\phi(g)} \longrightarrow F(\phi) \text{ as } t \rightarrow 0.$$

Let $\delta > 0$. Then, for all $t \in \mathbb{R}$ with $|t| \leq \delta$,

$$|F(\phi + tg)e^{-\frac{t^2\|g\|^2}{2} - t\phi(g)}| \leq \|F\|_\infty e^{\delta|\phi(g)|}.$$

Since

$$\int_{\mathcal{E}'} e^{2\delta|\phi(g)|} d\mu(\phi) \leq 2e^{\delta^2\|g\|^2} < \infty,$$

by the dominated convergence theorem, we have

$$U_g(t)F \longrightarrow F \text{ in } L^2(\mathcal{E}', d\mu), \text{ as } t \rightarrow 0.$$

Hence, by Proposition 3.11 and Proposition 3.12, for all $F \in L^2(\mathcal{E}', d\mu)$,

$$U_g(t)F \longrightarrow F \text{ in } L^2(\mathcal{E}', d\mu), \text{ as } t \rightarrow 0.$$

Then, we have the conclusion. □

IV. Integration by Parts Formulae in $L^2(\mathcal{E}', d\mu)$

By Stone's theorem [10], we have the following proposition.

PROPOSITION 4.1. *For all $g \in \mathcal{H}$, there exists a unique self-adjoint operator A_g such that for all $t \in \mathbb{R}$,*

$$U_g(t) = e^{itA_g}. \tag{4.19}$$

Moreover, for all $F \in L^2(\mathcal{E}', d\mu)$,

$$F \in D(A_g) \iff \lim_{t \rightarrow 0} \frac{U_g(t)F - F}{t} \text{ exists in } L^2(\mathcal{E}', d\mu),$$

and for all $F \in D(A_g)$,

$$A_g F = -i \lim_{t \rightarrow 0} \frac{U_g(t)F - F}{t} \text{ in } L^2(\mathcal{E}', d\mu). \tag{4.20}$$

In this section, we employ a notion of differentiation in locally convex spaces. There are some definitions of differentiation in locally convex spaces. In this paper, we consider the notion of Silva differentiation [6], which is proper to asymptotic analysis in infinite dimensional spaces (e.g.[2]). Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be a family of measurable functions. Then, $\sup_{\lambda \in \Lambda} |F_\lambda(\phi)|$, a function on $\mathcal{E}'$, might not be measurable. So we consider the following definition.

DEFINITION 4.2. We say that $\sup_{\lambda \in \Lambda} |F_\lambda(\phi)| \in L^2(\mathcal{E}', d\mu)$ if there exists a positive valued function $G \in L^2(\mathcal{E}', d\mu)$ such that

$$\sup_{\lambda \in \Lambda} |F_\lambda(\phi)| \leq G(\phi), \quad \mu - \text{a.e. } \phi \in \mathcal{E}'.$$

Similarly, We say that $\sup_{\lambda \in \Lambda} |F_\lambda(\phi)| \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu)$ if there exists a positive valued function $G \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu)$ such that

$$\sup_{\lambda \in \Lambda} |F_\lambda(\phi)| \leq G(\phi), \quad \mu - \text{a.e. } \phi \in \mathcal{E}'.$$

We introduce a linear subspace of $L^2(\mathcal{E}', d\mu)$.

DEFINITION 4.3. Let $g \in \mathcal{H}$ and F be a measurable function on $\mathcal{E}'$. We say that $F \in \mathcal{D}_g$ if F is Silva differentiable and for all $\delta > 0$,

$$\begin{aligned} \sup_{|t| \leq \delta} |F(\phi + tg)| e^{\delta|\phi(g)|}, \quad \sup_{|t| \leq \delta} |F(\phi + tg)| \phi(g) e^{\delta|\phi(g)|} &\in L^2(\mathcal{E}', d\mu), \\ \sup_{|t| \leq \delta} |F'(\phi + tg)(g)| e^{\delta|\phi(g)|} &\in L^2(\mathcal{E}', d\mu). \end{aligned}$$

We introduce another linear subspace of $L^2(\mathcal{E}', d\mu)$.

DEFINITION 4.4. Let $g \in \mathcal{H}$ and F be a measurable function on $\mathcal{E}'$. We say that $F \in \tilde{\mathcal{D}}_g$ if F is Silva differentiable and for all $\delta > 0$,

$$\sup_{|t| \leq \delta} |F(\phi + tg)|, \quad \sup_{|t| \leq \delta} |F'(\phi + tg)(g)| \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu).$$

PROPOSITION 4.5. For all $g \in \mathcal{H}$, $\tilde{\mathcal{D}}_g \subset \mathcal{D}_g$.

Proof. By the fact that for all $g \in \mathcal{H}$,

$$\phi(g), \quad e^{|\phi(g)|} \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu),$$

we have the conclusion. □

The following proposition is well known (e.g.[6]):

PROPOSITION 4.6. Let F, G be Silva differentiable functions. Then, for all $g \in \mathcal{H}$ and $\phi \in \mathcal{E}'$,

$$(FG)'(\phi)(g) = F'(\phi)(g)G(\phi) + F(\phi)G'(\phi)(g). \quad (4.21)$$

PROPOSITION 4.7. Let $g \in \mathcal{H}$. If $F, G \in \tilde{\mathcal{D}}_g$, then $FG \in \tilde{\mathcal{D}}_g$.

Proof. Let $\delta > 0$ and $F, G \in \widetilde{\mathcal{D}}_g$. Then,

$$\sup_{|t| \leq \delta} |F(\phi + tg)G(\phi + tg)| \leq \sup_{|t| \leq \delta} |F(\phi + tg)| \sup_{|t| \leq \delta} |G(\phi + tg)|$$

Since

$$\sup_{|t| \leq \delta} |F(\phi + tg)|, \sup_{|t| \leq \delta} |G(\phi + tg)| \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu),$$

we have

$$\sup_{|t| \leq \delta} |F(\phi + tg)G(\phi + tg)| \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu).$$

By Proposition 4.6, we have

$$\sup_{|t| \leq \delta} |(FG)'(\phi + tg)(g)| \leq \sup_{|t| \leq \delta} |F'(\phi + tg)(g)| \sup_{|t| \leq \delta} |G(\phi + tg)| + \sup_{|t| \leq \delta} |F(\phi + tg)| \sup_{|t| \leq \delta} |G'(\phi + tg)(g)|.$$

Since

$$\sup_{|t| \leq \delta} |F'(\phi + tg)(g)|, \sup_{|t| \leq \delta} |G'(\phi + tg)(g)| \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu),$$

we have

$$\sup_{|t| \leq \delta} |(FG)'(\phi + tg)(g)| \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu).$$

□

PROPOSITION 4.8. Let $g \in \mathcal{H}$. Then, $\mathcal{D}_g \subset D(A_g)$. Moreover, for all $F \in \mathcal{D}_g$,

$$(A_g F)(\phi) = -iF'(\phi)(g) + i\phi(g)F(\phi), \quad \mu - a.e. \quad \phi \in \mathcal{E}'. \quad (4.22)$$

Proof. Let $F \in \mathcal{D}_g$. For all $\phi \in \mathcal{E}'$, we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{(U_g(t)F)(\phi) - F(\phi)}{t} &= \left. \frac{d}{dt} F(\phi + tg) e^{-\frac{t^2 \|g\|^2}{2} - t\phi(g)} \right|_{t=0} \\ &= F'(\phi)(g) - \phi(g)F(\phi). \end{aligned}$$

Let $\delta > 0$. For all $t \in \mathbb{R}$ with $|t| \leq \delta$, by the mean-value theorem,

$$\begin{aligned} \left| \frac{(U_g(t)F)(\phi) - F(\phi)}{t} \right| &\leq \sup_{|t| \leq \delta} \left| \frac{d}{dt} (U_g(t)F)(\phi) \right| \\ &= \sup_{|t| \leq \delta} |F'(\phi + tg)(g) + (-t\|g\|^2 - \phi(g))F(\phi + tg)| e^{-\frac{t^2 \|g\|^2}{2} - t\phi(g)} \\ &\leq \sup_{|t| \leq \delta} |F'(\phi + tg)(g)| e^{\delta|\phi(g)|} + \delta\|g\|^2 \sup_{|t| \leq \delta} |F(\phi + tg)| e^{\delta|\phi(g)|} \\ &\quad + \sup_{|t| \leq \delta} |F(\phi + tg)| |\phi(g)| e^{\delta|\phi(g)|}. \end{aligned}$$

Then, since $F \in \mathcal{D}_g$, by the dominated convergence theorem,

$$\left(\lim_{t \rightarrow 0} \frac{U_g(t)F - F}{t} \right) (\phi) = F'(\phi)(g) - \phi(g)F(\phi), \quad \mu\text{-a.e. } \phi \in \mathcal{E}'.$$

Hence, by Proposition 4.1, we have $F \in D(A_g)$ and

$$(A_g F)(\phi) = -iF'(\phi)(g) + i\phi(g)F(\phi), \quad \mu\text{-a.e. } \phi \in \mathcal{E}'.$$

□

PROPOSITION 4.9. *Let $g \in \mathcal{H}$. Then, $\mathcal{P}(\mathcal{E}') \subset \widetilde{\mathcal{D}}_g$. Moreover, for all $F \in \mathcal{P}(\mathcal{E}')$,*

$$(A_g F)(\phi) = -iF'(\phi)(g) + i\phi(g)F(\phi), \quad \mu\text{-a.e. } \phi \in \mathcal{E}'. \quad (4.23)$$

Proof. Let $\delta > 0$ and $f \in \mathcal{E}$. We set

$$F(\phi) := \phi(f), \quad \phi \in \mathcal{E}'.$$

Then, we have

$$\begin{aligned} \sup_{|t| \leq \delta} |F(\phi + tg)| &= \sup_{|t| \leq \delta} |(\phi + tg)(f)| \\ &\leq \sup_{|t| \leq \delta} |\phi(f) + t\langle g, f \rangle| \\ &\leq |\phi(f)| + \delta |\langle g, f \rangle|. \end{aligned}$$

Since $|\phi(f)| \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu)$,

$$\sup_{|t| \leq \delta} |F(\phi + tg)| \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu).$$

On the other hand,

$$\sup_{|t| \leq \delta} |F'(\phi + tg)(g)| = |\langle g, f \rangle| \in \bigcap_{p \in \mathbb{N}} L^p(\mathcal{E}', d\mu).$$

Then, by Proposition 4.7 and Proposition 4.8, we have the conclusion. □

The following proposition is well known.

PROPOSITION 4.10. [6] *Let X, Y, Z be locally convex spaces, F be a differentiable mapping from X to Y , and G be a differentiable mapping from Y to Z . Then,*

$$(G \circ F)'(x) = G'(F(x)) \circ F'(x), \quad x \in X. \quad (4.24)$$

PROPOSITION 4.11. *For all $g \in \mathcal{H}$ and $F \in \mathcal{P}(\mathcal{E}')$,*

$$(D_g F)(\phi) = F'(\phi)(g), \quad \mu\text{-a.e. } \phi \in \mathcal{E}'. \quad (4.25)$$

Proof. By Proposition 2.3 and Proposition 4.10, we have the conclusion. $\square$

PROPOSITION 4.12. *For all $g \in \mathcal{H}$,*

$$A_g = \pi(g). \quad (4.26)$$

Proof. By Proposition 4.9 and Proposition 4.11, we have

$$\pi(g)|_{\mathcal{P}(\mathcal{E}')} \subset A_g.$$

Since $\mathcal{P}(\mathcal{E}')$ is a core of $\pi(g)$,

$$\pi(g) \subset A_g.$$

By the fact that $\pi(g)$ and A_g is self-adjoint, we have

$$A_g = \pi(g).$$

$\square$

By Proposition 4.1 and Proposition 4.12, we have the following theorem.

THEOREM 4.13. *For all $g \in \mathcal{H}$ and $t \in \mathbb{R}$,*

$$U_g(t) = e^{it\pi(g)}. \quad (4.27)$$

Moreover, for all $F \in L^2(\mathcal{E}', d\mu)$,

$$F \in D(\pi(g)) \iff \lim_{t \rightarrow 0} \frac{U_g(t)F - F}{t} \text{ exists in } L^2(\mathcal{E}', d\mu),$$

and for all $F \in D(\pi(g))$,

$$\pi(g)F = -i \lim_{t \rightarrow 0} \frac{U_g(t)F - F}{t} \text{ in } L^2(\mathcal{E}', d\mu). \quad (4.28)$$

THEOREM 4.14. *Let $g \in \mathcal{H}$. Then, $\mathcal{D}_g \subset D(\pi(g))$. Moreover, for all $F \in \mathcal{D}_g$,*

$$(D_g F)(\phi) = F'(\phi)(g), \quad \mu - a.e. \quad \phi \in \mathcal{E}'. \quad (4.29)$$

Proof. By Proposition 4.8 and Proposition 4.12, $\mathcal{D}_g \subset D(\pi(g))$ and

$$(\pi(g)F)(\phi) = -iF'(\phi)(g) + i\phi(g)F(\phi), \quad \mu - a.e. \quad \phi \in \mathcal{E}'.$$

By the assumption that $F \in \mathcal{D}_g$,

$$\phi(g)F(\phi) \in L^2(\mathcal{E}', d\mu).$$

Hence, we have

$$\begin{aligned} F'(\phi)(g) &= i\pi(g)F + \phi(g)F(\phi) \\ &= (D_g F)(\phi), \quad \mu - a.e. \quad \phi \in \mathcal{E}'. \end{aligned}$$

Then, we have the conclusion. $\square$

By Proposition 4.6, Proposition 4.7, and Theorem 4.14, we have the following corollary.

COROLLARY 4.15. *Let $g \in \mathcal{H}$. Then, for all $F, G \in \widetilde{\mathcal{D}}_g$,*

$$D_g(FG) = (D_g F)G + F(D_g G). \quad (4.30)$$

PROPOSITION 4.16. *Let $g \in \mathcal{H}$. Then, for all $F \in D(D_g)$, $\overline{F} \in D(D_g)$ and*

$$\overline{D_g F} = D_g \overline{F}. \quad (4.31)$$

Proof. Let $F \in D(D_g)$. Then, by Theorem 4.13, $\overline{F} \in D(\pi(g))$ and

$$\pi(g)\overline{F} = -\overline{\pi(g)F}.$$

On the otherhand, we have

$$\phi(g)\overline{F} \in L^2(\mathcal{E}', d\mu).$$

Hence, we have

$$\overline{D_g F} = D_g \overline{F}.$$

□

PROPOSITION 4.17. *Let $g \in \mathcal{H}$. For all $F, G \in D(D_g)$,*

$$\int_{\mathcal{E}'} (D_g F)G d\mu = \int_{\mathcal{E}'} F(\phi)(2\phi(g)G(\phi) - (D_g G)(\phi))d\mu(\phi). \quad (4.32)$$

Proof. Since $\pi(g)$ is self-adjoint,

$$\langle \pi(g)F, G \rangle_{L^2(\mathcal{E}', d\mu)} = \langle F, \pi(g)G \rangle_{L^2(\mathcal{E}', d\mu)}.$$

By Proposition 4.16,

$$\begin{aligned} \langle -iD_g F + i\phi(g)F, G \rangle &= \langle F, -iD_g G + i\phi(g)G \rangle \\ \iff \int_{\mathcal{E}'} (D_g \overline{F})G d\mu - \int_{\mathcal{E}'} \phi(g)\overline{F(\phi)}G(\phi)d\mu(\phi) &= - \int_{\mathcal{E}'} \overline{F}(D_g G)d\mu + \int_{\mathcal{E}'} \phi(g)\overline{F(\phi)}G(\phi)d\mu(\phi). \end{aligned}$$

Hence, we have

$$\int_{\mathcal{E}'} (D_g \overline{F})G d\mu = \int_{\mathcal{E}'} \overline{F(\phi)}(2\phi(g)G(\phi) - (D_g G)(\phi))d\mu(\phi).$$

Then, we have the conclusion. □

By Proposition 4.14 and Theorem 4.17, we have an integration by parts formula for the class $\mathcal{D}_g$.

THEOREM 4.18. Let $g \in \mathcal{H}$. Then, for all $F, G \in \mathcal{D}_g$,

$$\int_{\mathcal{E}'} F'(\phi)(g)G(\phi)d\mu(\phi) = \int_{\mathcal{E}'} F(\phi)(2\phi(g)G(\phi) - G'(\phi)(g))d\mu(\phi). \quad (4.33)$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. We define $\mathcal{S}(\mathcal{E}')$ and $\exp(\mathcal{E}')$ by

$$\mathcal{S}(\mathcal{E}') := \{F(\phi(f_1), \dots, \phi(f_n)) | F \in \mathcal{S}(\mathbb{R}^n), f_1, \dots, f_n \in \mathcal{E}, n \in \mathbb{N}\},$$

$$\exp(\mathcal{E}') := \mathcal{L}(\{e^{\phi(f)} | f \in \mathcal{E}\}).$$

Then, we rediscover the following integration by parts formula.

PROPOSITION 4.19. Let $g \in \mathcal{H}$. Then, $F, G \in \mathcal{D}(\mathcal{E}') \cup \mathcal{S}(\mathcal{E}') \cup \exp(\mathcal{E}')$,

$$\int_{\mathcal{E}'} (D_g F)G d\mu = \int_{\mathcal{E}'} F(\phi)(2\phi(g)G(\phi) - (D_g G)(\phi))d\mu(\phi). \quad (4.34)$$

Theorem 4.18 straightforwardly extends to the following.

THEOREM 4.20. Let $g \in \mathcal{H}$. Then, for all $F \in D(\overline{D_g}|_{\mathcal{D}_g})$ and $G \in \mathcal{D}_g$,

$$\int_{\mathcal{E}'} (\overline{D_g} F)G d\mu = \int_{\mathcal{E}'} F(\phi)(2\phi(g)G(\phi) - (D_g G)(\phi))d\mu(\phi). \quad (4.35)$$

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