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Inequivalent Representation of Canonical Commutation Relations in Relation to Casimir Effect

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Abstract

It is shown that an irreducible representation of the CCR over a dense subspace of a boson Fock space is associated with a quantum system whose space configuration may give rise to Casimir effect in the context of a quantum scalar field and that it is inequivalent to the Fock representation of the same CCR. A quantum scalar field is constructed from the representation. A new feature of the analysis is to treat a *singular* Bogoliubov transformation, which is different from the usual bosonic Bogoliubov transformation and from which the inequivalent irreducible representation of the CCR is constructed.

Keywords: Casimir effect, canonical commutation relations, inequivalent representation, boson Fock space, singular Bogoliubov transformation, quantum scalar field.

Mathematics Subject Classification 2010: 81R10, 47L60.

1 Introduction

In 1948 Casimir [9] predicted theoretically that two parallel perfectly conducting plates facing each other in the vacuum, even if there is no net charge on each of them, have an attractive force, the Casimir force, whose magnitude is given by $(\hbar c \pi^2 / 240) a^{-4}$ per unit surface area with a being the distance between the two plates, where $\hbar$ is the reduced Planck constant and c is the speed of light. This effect, called the Casimir effect, was experimentally confirmed by Sparnaay [28] and Lamoreaux [21].

The derivation of the Casimir force by Casimir [9] is based on calculating the difference in zero-point energies with a regularization procedure (a renormalization) between two different configurations in a cubic cavity of volume L^3 with $L > 0$ the length of a side

of the cubic, bounded by perfectly conducting walls; the one is the configuration where a perfectly conducting square plate with side L is placed in this cavity parallel to the xy face with a distance $0 < a \ll L$ from the xy face and the other is the configuration where such a square plate is placed in the cavity at a very large distance, say $L/2$, from the xy face. This derivation suggests that the Casimir effect is an effect due to the vacuum fluctuation of the quantum electromagnetic field, which has become a conventional interpretation of the Casimir effect (see, e.g., [20, §3-2-4]).

The physical notion of the Casimir effect has been extended to various geometric configurations of perfectly conducting bodies as well as to the case where the system under consideration is at finite temperature. There have so far been an enormous number of physics articles treating Casimir effects in the extended sense too (here we do not enumerate them). Concerning mathematical studies on Casimir effect, Hergeden [16, 17] presented a systematic and comprehensive treatment of Casimir effect (in a broadened sense) *without using zero-point energies*, taking the view-point of algebraic quantum field theory (see also [18]). This kind of approach to Casimir effect has been taken also by Dappiaggi et al [10]. We remark that zero-point energies appearing in the standard heuristic canonical quantization of a classical field, which are usually divergent and mathematically meaningless unless any regularization (renormalization) is made (in physics literatures, of course, such renormalizations are discussed), are not used in modern mathematically rigorous quantum field theory (e.g., [4, 5, 7, 11, 15, 29]).

A Casimir effect may be regarded as one of the “characteristic” quantum phenomena which appear in relation to macroscopic objects (e.g., the two parallel plates in the case of the Casimir effect described above; cf. [16] where an effect of Casimir type is understood as a “quantum backreaction”) . On the other hand, according to some studies so far done, it is inferred that *inequivalent* irreducible representations of canonical commutation relations (CCR) or canonical anti-commutation relations (CAR) are associated with such quantum phenomena (see, e.g., [1] (the Aharonov-Bohm effect), [2] (the boson masses), [3] (the masses of Dirac particles), [6, 13] (Bose-Einstein condensation)), where “inequivalence” means the inequivalence to the Fock representation of the CCR or the CAR over a relevant inner product space or mutual inequivalences of representations of CCR or CAR indexed by a set. The moral of these facts may be that the Universe uses inequivalent representations of CCR or CAR in relation to characteristic quantum phenomena. From this point of view, one may conjecture that, to each of Casimir effects in the extended sense, there corresponds to an inequivalent irreducible representation of CCR or CAR. In the present paper, we consider the Casimir effect in the case of a quantum scalar field in view of representation theory of CCR and show that, in this case, the conjecture just mentioned is right.

As heuristically argued in [14], the approach to the Casimir effect in terms of representations of CCR may make it possible to understand the Casimir effect without invoking zero-point energies. This kind of idea has been extensively developed in [16, 17] in the case of *regularized* boundary conditions for the quantum system under consideration. Although there are arguments on un-physicality of *sharp* (non-regular) boundary conditions in the context of Casimir effect [12, 16, 17], we take the sharp Dirichlet boundary condition in our analysis in the present paper; we want to see what kind of structures (which may be “singular”) are involved in this case in view of operator theory and representation

theory of CCR.

The outline of the present paper is as follows. In Section 2 we first recall the definition of representation of the CCR over a complex inner product space and the definition of irreducibility of a set $\mathfrak{A}$ of (not necessarily bounded) linear operators on a Hilbert space. Moreover, we give a lemma which states a necessary and sufficient condition for $\mathfrak{A}$ to be irreducible.

Section 3 is devoted to some aspects in representations of CCR on the abstract boson Fock space $\mathcal{F}_b(\mathcal{H})$ over a complex Hilbert space $\mathcal{H}$ (see Subsection 3.1). We define the Fock representation $\pi_F(\mathcal{D})$ of the CCR over a dense subspace $\mathcal{D}$ in $\mathcal{H}$, which is irreducible, and prove some related facts. Moreover, we introduce a *singular* Bogoliubov transformation where the relevant operators on the one-particle Hilbert space $\mathcal{H}$ to define it are *unbounded*. This may be a new feature. We emphasize that the standard theory of the bosonic Bogoliubov transformation (e.g., [8, 11, 19, 22, 25, 27]) cannot be applied to the singular Bogoliubov transformation. One needs a careful analysis on it, because unbounded operators are involved in it. We prove a theorem on inequivalence (to any direct sum representation of $\pi_F(\mathcal{D})$) of a representation of CCR defined by a singular Bogoliubov transformation (Theorem 3.7).

After these preparations of abstract theories, we start our analysis on Casimir effect from Section 4. In Section 4, we construct a free quantum scalar field on a d -dimensional finite box¹

$$\Lambda := (0, L)^{d-1} \times (0, L_d) = \{\mathbf{x} = (x_1, \dots, x_d) | x_1, \dots, x_{d-1} \in (0, L), x_d \in (0, L_d)\}, \quad (1.1)$$

with $d \geq 2$, $L > 0$ and $L_d > 0$. We use the Dirichlet Laplacian² Δ_D acting in $L^2(\Lambda)$ to define a one-particle Hamiltonian³

$$h := (-\Delta_D + m^2)^{1/2}, \quad (1.2)$$

where $m \geq 0$ is the mass of one boson (note that $-\Delta_D$ is a non-negative self-adjoint operator on $L^2(\Lambda)$ and hence the operator h , defined via functional calculus, is self-adjoint and non-negative). Then the Hamiltonian of the free quantum scalar field to be constructed is given by $d\Gamma(h)$, the second quantization operator of h acting in the boson Fock space $\mathcal{F}_b(L^2(\Lambda))$ over $L^2(\Lambda)$ (see Subsection 3.1). It is shown that the quantum scalar field $\phi(t, f)$ constructed with $t \in \mathbb{R}$, the time parameter, and $f \in L^2_{\text{real}}(\Lambda)$, the set of real-valued functions in $L^2(\Lambda)$, obeys the free Klein-Gordon equation with mass m in the sense of operator valued functional in f

$$\frac{d^2}{dt^2} \phi(t, f) \Psi - \phi(t, \Delta_D f) \Psi + m^2 \phi(t, f) \Psi = 0 \quad (1.3)$$

for all $f \in D(\Delta_D)$ (the domain of Δ_D) and $\Psi \in \mathcal{F}_0(L^2(\Lambda))$, the finite particle subspace of $\mathcal{F}_b(L^2(\Lambda))$ (see Subsection 3.1), where differentiation in t is taken in the strong sense.

¹The space dimension $d \geq 2$ is taken to be arbitrary for mathematical generality and to understand in what aspects the theory considered depends on or is independent of d .

²One can consider also Laplacians with other boundary conditions. But, in the present paper, we treat only the Dirichlet Laplacian for simplicity.

³In what follows, we use the physical unit system where $\hbar = 1$ and $c = 1$.

In Section 5, we consider the case where a perfectly conducting wall (plate)

$$W_a := \{\mathbf{x} \in \Lambda | x_d = a\} \quad (1.4)$$

perpendicular to the x_d -axis is put in Λ with $0 < a < L_d$. In this case Λ is decomposed as follows:

$$\Lambda = \Lambda_1 \cup W_a \cup \Lambda_2, \quad (1.5)$$

where

$$\Lambda_1 := \{\mathbf{x} \in \Lambda | 0 < x_d < a\}, \quad \Lambda_2 := \{\mathbf{x} \in \Lambda | a < x_d < L_d\}. \quad (1.6)$$

This decomposition of the box Λ induces the orthogonal decomposition

$$L^2(\Lambda) = L^2(\Lambda_1) \oplus L^2(\Lambda_2) \quad (1.7)$$

of the Hilbert space $L^2(\Lambda)$. But one should note that the Dirichlet Laplacian Δ_D cannot be reduced by $L^2(\Lambda_\ell)$ ($\ell = 1, 2$). This suggests that the placement of the wall W_a into Λ may give rise to physics different from that of the system without W_a even if no other interactions exist. This may be a mathematical origin of the Casimir effect in the present context.

Using the Dirichlet Laplacian Δ_ℓ for Λ_ℓ ($\ell = 1, 2$), we introduce a non-negative self-adjoint operator

$$h_{a,\ell} := (-\Delta_\ell + m^2)^{1/2} \quad (1.8)$$

on $L^2(\Lambda_\ell)$ and define

$$h_a := h_{a,1} \oplus h_{a,2} \quad (1.9)$$

relative to the decomposition (1.7). We investigate relations between h and h_a . In particular, we prove that $h^{-1/2}h_a^{1/2}$ is unbounded (Lemma 5.5). We define a singular Bogoliubov transformation $(a(\cdot), a(\cdot)^*) \mapsto (b(\cdot), b(\cdot)^*)$ in terms of $h^{\pm 1/2}$ and $h_a^{\pm 1/2}$, where $a(\cdot)$ denotes the annihilation operator on $\mathcal{F}_b(L^2(\Lambda))$ and $a(\cdot)^*$ (resp. $b(\cdot)^*$) is the adjoint of $a(\cdot)$ (resp. $b(\cdot)$). It is shown that there exists a dense subspace $\mathcal{E}_0$ (see (5.22)) such that the triple

$$\pi_b(\mathcal{E}_0) := (\mathcal{F}_b(L^2(\Lambda)), \mathcal{F}_0(L^2(\Lambda)), \{b(f), b(f)^* | f \in \mathcal{E}_0\}) \quad (1.10)$$

is a representation of the CCR over $\mathcal{E}_0$. We construct a quantum scalar field $\phi_b(t, f)$ with $f \in L^2_{\text{real}}(\Lambda)$ in terms of $b(\cdot)$ and $b(\cdot)^*$; this quantum field satisfies the same initial conditions as those of $\phi(t, f)$:

$$\phi(0, f) = \phi_b(0, f), \quad \pi(0, f) = \pi_b(0, f), \quad f \in D(h^{1/2}) \cap D(h_a) \cap L^2_{\text{real}}(\Lambda), \quad (1.11)$$

on $\mathcal{F}_0(L^2(\Lambda))$, where $\pi(t, f)$ (resp. $\pi_b(t, f)$) is the canonical conjugate momentum field of $\phi(t, f)$ (resp. $\phi_b(t, f)$), and (1.3) holds with ϕ , Δ_D and f replaced by ϕ_b , $\Delta_1 \oplus \Delta_2$ and $f \in D(\Delta_1 \oplus \Delta_2)$ respectively. We show that $\phi_b(t, f)$ is different from $\phi(t, f)$ as operator-valued mapping with variable (t, f) . Thus, in the case where W_a exists, one has a dynamics different from the dynamics given by $\phi(t, f)$ even if the time-zero fields are same and no other interactions exist there.

In the last section we prove that $\pi_b(\mathcal{E}_0)$ is an irreducible representation of the CCR over $\mathcal{E}_0$ which is inequivalent to the Fock representation $\pi_F(\mathcal{E}_0)$ of the CCR over $\mathcal{E}_0$. This is one of the main results in the present paper.

2 Representations of the CCR over a Complex Inner Product Space

For a linear operator A on a Hilbert space $\mathcal{X}$, we denote by $D(A)$ the domain of A . If A is densely defined (i.e., $D(A)$ is dense in $\mathcal{X}$), then A has its adjoint; we denote it by A^* .

Let $\mathcal{F}$ be a complex Hilbert space and $\mathcal{D}$ be a dense subspace of $\mathcal{F}$. Let $\mathcal{V}$ be a complex inner product space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ (linear in the second variable and antilinear in the first) and norm $\|\cdot\|_{\mathcal{V}}$ (we sometimes omit the subscript $\mathcal{V}$ in $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ and $\|\cdot\|_{\mathcal{V}}$ if there is no danger of confusion). Suppose that, for each $f \in \mathcal{V}$, a densely defined closed linear operator $C(f)$ on $\mathcal{F}$ is given. Then the triple $(\mathcal{F}, \mathcal{D}, \{C(f), C(f)^* | f \in \mathcal{V}\})$ is called a representation of the CCR over $\mathcal{V}$ if the following (i)–(iii) hold:

- (i) (domain invariance) For all $f \in \mathcal{V}$, $\mathcal{D} \subset D(C(f)) \cap D(C(f)^*)$, $C(f)\mathcal{D} \subset \mathcal{D}$, $C(f)^*\mathcal{D} \subset \mathcal{D}$.
- (ii) (anti-linearity in test vectors) For all $f, g \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$ (the set of complex numbers), $C(\alpha f + \beta g) = \alpha^* C(f) + \beta^* C(g)$ on $\mathcal{D}$, where, for $z \in \mathbb{C}$, z^* denotes the complex conjugate of z .
- (iii) (CCR over $\mathcal{V}$) For all $f, g \in \mathcal{V}$,

$$[C(f), C(g)^*] = \langle f, g \rangle_{\mathcal{V}}, \quad [C(f), C(g)] = 0 \quad \text{on } \mathcal{D},$$

where, for linear operators A and B on a Hilbert space, $[A, B] := AB - BA$ (the commutator of A and B).

Two representations $(\mathcal{F}, \mathcal{D}, \{C(f), C(f)^* | f \in \mathcal{V}\})$ and $(\mathcal{F}', \mathcal{D}', \{C'(f), C'(f)^* | f \in \mathcal{V}\})$ of the CCR over $\mathcal{V}$ are said to be equivalent if there exists a unitary operator $U : \mathcal{F} \rightarrow \mathcal{F}'$ such that, for all $f \in \mathcal{V}$, $UC(f)U^{-1} = C'(f)$.⁴

For two linear operators A and B on a Hilbert space $\mathcal{X}$, the symbol $A \subset B$ means that B is an extension of A , i.e., $D(A) \subset D(B)$ and $A\Psi = B\Psi$, $\forall \Psi \in D(A)$.

As usual, we denote by $\mathfrak{B}(\mathcal{X})$ the Banach space of everywhere defined bounded linear operators on $\mathcal{X}$.

For a set $\mathfrak{A}$ of linear operators on a Hilbert space $\mathcal{X}$, the set

$$\mathfrak{A}' := \{T \in \mathfrak{B}(\mathcal{X}) | TA \subset AT, \forall A \in \mathfrak{A}\} \subset \mathfrak{B}(\mathcal{X})$$

is called the strong commutant of $\mathfrak{A}$.

The set $\mathfrak{A}$ is said to be reducible if there is a non-trivial closed subspace $\mathcal{M}$ of $\mathcal{X}$ ($\mathcal{M} \neq \{0\}$, $\mathcal{X}$) such that every $A \in \mathfrak{A}$ is reduced by $\mathcal{M}$ (i.e., $P_{\mathcal{M}}A \subset AP_{\mathcal{M}}$, where $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$).

The set $\mathfrak{A}$ is said to be irreducible if it is not reducible.

A representation $(\mathcal{F}, \mathcal{D}, \{C(f), C(f)^* | f \in \mathcal{V}\})$ of the CCR over $\mathcal{V}$ is said to be reducible (resp. irreducible) if the set $\{C(f), C(f)^* | f \in \mathcal{V}\}$ is reducible (resp. irreducible).

The following fact is well known (see, e.g., [4, Proposition 5.9]):

⁴In this case, it follows that $UC(f)^*U^{-1} = C'(f)^*$, $f \in \mathcal{V}$.

Lemma 2.1 *Let $\mathfrak{A}$ be a set of linear operators on $\mathcal{X}$.*

- (i) *If $\mathfrak{A}' = \mathbb{C}I := \{\alpha I | \alpha \in \mathbb{C}\}$ (I denotes identity), then $\mathfrak{A}$ is irreducible.*
- (ii) *If $\mathfrak{A}$ is an irreducible set of densely defined linear operators on $\mathcal{X}$ and $*$ -invariant (i.e., $A \in \mathfrak{A} \implies A^* \in \mathfrak{A}$), then $\mathfrak{A}' = \mathbb{C}I$.*

3 Representations of CCR in Boson Fock Space

3.1 Boson Fock space

We first recall some basic objects and facts in the theory of abstract boson Fock space (for further details, see, e.g., [4, Chapter 5], [23, §VIII.10], [24, §X.7]).

Let $\mathcal{H}$ be a complex Hilbert space and, for each $n \in \mathbb{N}$ (the set of natural numbers), $\otimes_s^n \mathcal{H}$ be the n -fold symmetric tensor product Hilbert space of $\mathcal{H}$ with convention $\otimes_s^0 \mathcal{H} := \mathbb{C}$. Then the boson Fock space over $\mathcal{H}$ is the direct sum Hilbert space defined by

$$\begin{aligned} \mathcal{F}_b(\mathcal{H}) &:= \oplus_{n=0}^{\infty} \otimes_s^n \mathcal{H} \\ &= \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \Psi^{(n)} \in \otimes_s^n \mathcal{H}, n \geq 0, \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty \right\}. \end{aligned}$$

The subspace

$$\begin{aligned} \mathcal{F}_0(\mathcal{H}) &:= \{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \Psi^{(n)} \in \otimes_s^n \mathcal{H}, n \geq 0 \text{ and there exists an } n_0 \in \mathbb{N} \text{ such that} \\ &\quad \text{for all } n \geq n_0, \Psi^{(n)} = 0 \} \end{aligned}$$

is dense in $\mathcal{F}_b(\mathcal{H})$. This subspace is called the finite particle subspace of $\mathcal{F}_b(\mathcal{H})$.

We denote by $A(f)$ the annihilation operator with test vector $f \in \mathcal{H}$ on $\mathcal{F}_b(\mathcal{H})$, which is the unique densely defined closed linear operator on $\mathcal{F}_b(\mathcal{H})$ such that its adjoint $A(f)^*$, called the creation operator with test vector f , takes the following form: for all $\Psi \in D(A(f)^*)$, $(A(f)^*\Psi)^{(0)} = 0$ and

$$(A(f)^*\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1,$$

where S_n is the symmetrization operator on the n -fold tensor product $\otimes^n \mathcal{H}$ of $\mathcal{H}$. For all $f \in \mathcal{H}$, $\mathcal{F}_0(\mathcal{H}) \subset D(A(f)) \cap D(A(f)^*)$ and $A(f)$ and $A(f)^*$ leave $\mathcal{F}_0(\mathcal{H})$ invariant. Moreover, $\{A(f), A(f)^* | f \in \mathcal{H}\}$ satisfies the CCR over $\mathcal{H}$:

$$[A(f), A(g)^*] = \langle f, g \rangle_{\mathcal{H}}, \quad [A(f), A(g)] = 0, \quad [A(f)^*, A(g)^*] = 0 \quad (f, g \in \mathcal{H}) \quad (3.1)$$

on $\mathcal{F}_0(\mathcal{H})$.

Let

$$\Omega_{\mathcal{H}} = \{1, 0, 0, \dots\} \in \mathcal{F}_b(\mathcal{H}) \quad (3.2)$$

be the Fock vacuum in $\mathcal{F}_b(\mathcal{H})$. Then

$$A(f)\Omega_{\mathcal{H}} = 0, \quad f \in \mathcal{H}. \quad (3.3)$$

Let T be a densely defined closable linear operator on $\mathcal{H}$ and $\bar{T}$ be the closure of T . For each $n \in \mathbb{N}$, we denote by $T^{(n)}$ the closure of

$$\left(\sum_{j=1}^n I \otimes \cdots \otimes I \otimes \overbrace{\bar{T}}^{j\text{th}} \otimes I \cdots \otimes I \right) \upharpoonright \hat{\otimes}_s^n D(\bar{T}),$$

where $\hat{\otimes}_s^n D(\bar{T})$ denotes the n -fold algebraic symmetric tensor product of $D(\bar{T})$. Let $T^{(0)} := 0$ acting on $\mathbb{C}$. Then the direct sum operator

$$d\Gamma(T) := \oplus_{n=0}^{\infty} T^{(n)}$$

on $\mathcal{F}_b(\mathcal{H})$ is called the second quantization operator of T , which is densely defined closed.

If T is self-adjoint, then so is $d\Gamma(T)$ and

$$e^{itd\Gamma(T)} A(f) e^{-itd\Gamma(T)} = A(e^{itT} f), \quad f \in \mathcal{H}, t \in \mathbb{R}. \quad (3.4)$$

See, e.g., [4, Lemma 5.21].

3.2 Fock representation of CCR and related facts

Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$. Then, by the facts stated in the paragraph containing (3.1),

$$\pi_F(\mathcal{D}) := (\mathcal{F}_b(\mathcal{H}), \mathcal{F}_0(\mathcal{H}), \{A(f), A(f)^* | f \in \mathcal{D}\}) \quad (3.5)$$

is a representation of the CCR over $\mathcal{D}$. Moreover, one can show that $\pi_F(\mathcal{D})$ is irreducible (see, e.g., [4, Theorem 5.14]). Hence one has the following fact (well known in the case $\mathcal{D} = \mathcal{H}$):

Proposition 3.1 *Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$. Then $\pi_F(\mathcal{D})$ is an irreducible representation of the CCR over $\mathcal{D}$.*

The representation $\pi_F(\mathcal{D})$ is called the Fock representation of the CCR over $\mathcal{D}$.

We say that a densely defined closable linear operator A on a Hilbert space is Hilbert-Schmidt if the closure $\bar{A}$ is Hilbert-Schmidt.

The next proposition plays a basic role in the theory of representations of CCR in boson Fock spaces:

Proposition 3.2 *Assume that $\mathcal{H}$ is separable. Let S and T be (not necessarily bounded) linear operators on $\mathcal{H}$ such that $D(S)$ and $D(T)$ include a dense subspace $\mathcal{D}$ in $\mathcal{H}$ with property $\overline{T\mathcal{D}} = \mathcal{H}$ ($T\mathcal{D} := \{Tf | f \in \mathcal{D}\}$ and $\overline{T\mathcal{D}}$ is the closure of $T\mathcal{D}$) and $T_{\mathcal{D}} := T \upharpoonright \mathcal{D}$, the restriction of T to $\mathcal{D}$, is injective. Let C be a conjugation on $\mathcal{H}$ (i.e., C is an anti-linear mapping on $\mathcal{H}$ such that $C^2 = I$ and $\|Cf\| = \|f\|$, $f \in \mathcal{H}$). Suppose that there exists a non-zero vector $\Omega \in \mathcal{F}_b(\mathcal{H})$ such that, for all $\Psi \in \mathcal{F}_0(\mathcal{H})$ and $f \in \mathcal{D}$,*

$$\langle A(Tf)^* \Psi, \Omega \rangle = - \langle A(CSf) \Psi, \Omega \rangle. \quad (3.6)$$

Then $ST_{\mathcal{D}}^{-1}$ is Hilbert-Schmidt.

Proof. Let $\mathcal{F} := T\mathcal{D}$. Then, by the present assumption, $\mathcal{F}$ is dense in $\mathcal{H}$. By (3.6), we have for all $g \in \mathcal{F}$

$$\langle A(g)^*\Psi, \Omega \rangle = -\langle A(CST_{\mathcal{D}}^{-1}g)\Psi, \Omega \rangle.$$

Hence, by [2, Proposition 3.3]⁵, $ST_{\mathcal{D}}^{-1}$ is Hilbert-Schmidt. ■

Lemma 3.3 *Let X and Y be (not necessarily bounded) linear operators on $\mathcal{H}$ such that there exists a dense subspace $\mathcal{D} \subset D(X) \cap D(Y)$ and the following equation holds:*

$$\langle Xf, Xg \rangle - \langle Yf, Yg \rangle = \langle f, g \rangle, \quad f, g \in \mathcal{D}. \quad (3.7)$$

Let $X_{\mathcal{D}} := X \upharpoonright \mathcal{D}$. Then $X_{\mathcal{D}}$ is injective and $X_{\mathcal{D}}^{-1}$ is bounded with $\|X_{\mathcal{D}}^{-1}\| \leq 1$.

Proof. It follows from (3.7) that, for all $f \in \mathcal{D}$,

$$\|X_{\mathcal{D}}f\|^2 = \|f\|^2 + \|Yf\|^2 \geq \|f\|^2,$$

which implies the desired result. ■

The following proposition is new (to the author's best knowledge) and play a crucial role in the present paper.

Proposition 3.4 *Assume that $\mathcal{H}$ is separable. Let X and Y be as in Lemma 3.3 and suppose that X is unbounded and $X\mathcal{D}$ is dense in $\mathcal{H}$. Let Ω be a vector in $\mathcal{F}_b(\mathcal{H})$ satisfying*

$$\langle A(Xf)^*\Psi, \Omega \rangle = -\langle A(CYf)\Psi, \Omega \rangle, \quad \Psi \in \mathcal{F}_0(\mathcal{H}), f \in \mathcal{D}. \quad (3.8)$$

Then $\Omega = 0$.

Proof. We prove the proposition by reductio ad absurdum. Suppose that there existed a non-zero vector $\Omega \in \mathcal{F}_b(\mathcal{H})$ such that (3.8) holds. Since $\text{Ran } X_{\mathcal{D}}$ is dense by the present assumption and $X_{\mathcal{D}}$ is injective by Lemma 3.3, it follows from Proposition 3.2 that $K := YX_{\mathcal{D}}^{-1}$ is Hilbert-Schmidt. Hence $\overline{K}^* \overline{K}$ is trace class. Using (3.7) and a limiting argument, one can show that

$$\overline{K}^* \overline{K} = I - \overline{L}^* \overline{L}, \quad (3.9)$$

where $L := X_{\mathcal{D}}^{-1}$. For a linear operator A , we denote by $\sigma(A)$ the spectrum of A . Since $\overline{K}^* \overline{K}$ is a non-negative trace class operator, it is a non-negative compact operator. Hence $\sigma(\overline{K}^* \overline{K}) \setminus \{0\} = \{\lambda_n\}_{n=1}^N$, where $N < \infty$ or $N = \infty$ and λ_n is a positive eigenvalue of $\overline{K}^* \overline{K}$ with a finite multiplicity; if $N = \infty$, then $\lim_{n \rightarrow \infty} \lambda_n = 0 \cdots (*)$. Hence $\sigma(\overline{L}^* \overline{L}) \setminus \{1\} = \{1 - \lambda_n\}_{n=1}^N$.

Suppose that there existed a constant $\gamma > 0$ such that $\overline{L}^* \overline{L} \geq \gamma$. Then, for all $g \in \text{Ran } X_{\mathcal{D}}$, $\|X_{\mathcal{D}}^{-1}g\|^2 \geq \gamma\|g\|^2$. Hence $\|X_{\mathcal{D}}f\|^2 \leq \gamma^{-1}\|f\|^2$, $f \in \mathcal{D}$. But this contradicts the unboundedness of X . Hence $\inf \sigma(\overline{L}^* \overline{L}) = 0$. Hence $N = \infty$ and there exists a subsequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k \rightarrow \infty} (1 - \lambda_{n_k}) = 0$, i.e., $\lim_{k \rightarrow \infty} \lambda_{n_k} = 1$. But this contradicts (*). Thus we arrive at a contradiction. ■

⁵In the cited proposition, it is assumed that an operator L satisfying $\langle A(g)^*\Psi, \Omega \rangle = \langle A(CLg)\Psi, \Omega \rangle$ is bounded. But, as is seen from the proof, it is not necessary.

3.3 Singular Bogoliubov transformation and representation of CCR

Let T and S be densely defined (*not necessarily bounded*) linear operators on $\mathcal{H}$ such that there exists a dense subspace $\mathcal{D} \subset D(T) \cap D(S)$ and the following equations hold:

$$\langle Tf, Tg \rangle - \langle Sf, Sg \rangle = \langle f, g \rangle, \quad (3.10)$$

$$\langle Tf, CSg \rangle = \langle Sf, CTg \rangle, \quad f, g \in \mathcal{D}, \quad (3.11)$$

where C is a conjugation on $\mathcal{H}$. Then, for each $f \in \mathcal{D}$, the operator

$$B_0(f) := A(Tf) + A(CSf)^*$$

is a densely defined linear operator with $D(B_0(f)) \supset \mathcal{F}_0(\mathcal{H})$ and $B_0(f)^*$ is densely defined with $D(B_0(f)^*) \supset \mathcal{F}_0(\mathcal{H})$. Hence $B_0(f)$ is closable. Therefore one can define a densely defined closed operator $B(f)$ on $\mathcal{F}_b(\mathcal{H})$ by

$$B(f) := \overline{B_0(f)}. \quad (3.12)$$

It is obvious that $B(f)$ and $B(f)^*$ leave $\mathcal{F}_0(\mathcal{H})$ invariant. Moreover, using (3.10) and (3.11), one can easily show that $B(\cdot)$ and $B(\cdot)^*$ satisfy the CCR over $\mathcal{D}$ on $\mathcal{F}_0(\mathcal{H})$:

$$[B(f), B(g)^*] = \langle f, g \rangle, \quad [B(f), B(g)] = 0, \quad [B(f)^*, B(g)^*] = 0, \quad f, g \in \mathcal{D}. \quad (3.13)$$

Therefore

$$\pi_B(\mathcal{D}) := (\mathcal{F}_b(\mathcal{H}), \mathcal{F}_0(\mathcal{H}), \{B(f), B(f)^* | f \in \mathcal{D}\}) \quad (3.14)$$

is a representation of the CCR over $\mathcal{D}$.

The correspondence $T_B: (A(\cdot), A(\cdot)^*) \mapsto (B(\cdot), B(\cdot)^*)$ is a generalization of the standard bosonic Bogoliubov transformation (see, e.g., [8, 11, 19, 22, 25, 27]) in the sense that S or T may be unbounded and T_B is not necessarily invertible. This may be a new feature.

Equation (3.10) implies that T is bounded if and only if S is bounded. Based on this property, we say that the Bogoliubov transformation T_B is *singular* if T or S is unbounded (then both T and S are unbounded).

We are interested in conditions under which $\pi_B(\mathcal{D})$ is inequivalent to any direct sum representation of the Fock representation $\pi_F(\mathcal{D})$. On this aspect, we consider two cases separately.

Theorem 3.5 *Assume that $\mathcal{H}$ is separable. Suppose that T is bounded and S is not Hilbert-Schmidt. Then $\pi_B(\mathcal{D})$ is inequivalent to any direct sum representations of the Fock representation $\pi_F(\mathcal{D})$. In particular, if $\pi_B(\mathcal{D})$ is irreducible, then $\pi_B(\mathcal{D})$ is inequivalent to $\pi_F(\mathcal{D})$.*

Proof. Suppose that $\pi_B(\mathcal{D})$ were equivalent to a direct sum representation $\oplus_{n=1}^N \pi_F(\mathcal{D})$ of $\pi_F(\mathcal{D})$ with $N < \infty$ or $N = \infty$. Then there exists a unitary operator U from $\mathcal{F}_b(\mathcal{H})$ to $\oplus_{n=1}^N \mathcal{F}_b(\mathcal{H})$ such that $UB(f)U^{-1} = \oplus_{n=1}^N A(f)$, $f \in \mathcal{D}$. The vector $\Omega := U^{-1} \oplus_{n=1}^N$

$n^{-1}\Omega_{\mathcal{H}}$ is in $\mathcal{F}_b(\mathcal{H})$ (note that, in the case $N = \infty$, $\sum_{n=1}^{\infty} \|n^{-1}\Omega_{\mathcal{H}}\|^2 = \sum_{n=1}^{\infty} n^{-2} < \infty$). Then $\|\Omega\| = (\sum_{n=1}^N n^{-2})^{1/2} \neq 0$ and, by (3.3), $B(f)\Omega = 0$. This implies (3.6). Hence, by Proposition 3.2, $ST_{\mathcal{D}}^{-1}$ is Hilbert-Schmidt. In the present assumption, $T_{\mathcal{D}}$ is bounded. Hence $S_{\mathcal{D}} = (ST_{\mathcal{D}}^{-1})T_{\mathcal{D}}$ is Hilbert-Schmidt. But this is a contradiction. Thus the desired result follows. $\blacksquare$

Remark 3.6 Theorem 3.5 is only a slight generalization of a well known fact in the theory of the standard bosonic Bogoliubov transformation (see, e.g., the papers cited above) in that $\mathcal{D}$ is not equal to $\mathcal{H}$ and inequivalence is about direct sum representations of the Fock representation $\pi_F(\mathcal{D})$.

The following theorem is the main result in the abstract framework of the present paper.

Theorem 3.7 *Assume that $\mathcal{H}$ is separable. Suppose that T is unbounded and $T\mathcal{D}$ is dense in $\mathcal{H}$. Then $\pi_B(\mathcal{D})$ is inequivalent to any direct sum representation of the Fock representation $\pi_F(\mathcal{D})$. In particular, if $\pi_B(\mathcal{D})$ is irreducible, then $\pi_B(\mathcal{D})$ is inequivalent to $\pi_F(\mathcal{D})$.*

Proof. Suppose that $\pi_B(\mathcal{D})$ were equivalent to a direct sum representation of $\pi_F(\mathcal{D})$. Then, by the proof of the preceding theorem, there exists a non-zero vector $\Omega \in \cap_{f \in \mathcal{D}} D(B(f))$ such that $B(f)\Omega = 0$. Then (3.8) holds with $X = T$ and $Y = S$. But this contradicts Proposition 3.4. $\blacksquare$

4 A Free Quantum Scalar Field on a Finite Box

In this section we construct a free neutral quantum scalar field on the box Λ given by (1.1) with the Dirichlet boundary condition. We denote by Δ_D the Dirichlet Laplacian for Λ on $L^2(\Lambda)$ (see, e.g., [26, p.263] for the definition of Δ_D). The operator $-\Delta_D$ is a non-negative self-adjoint operator. Let

$$\Gamma := \left\{ \mathbf{k} = (k_1, \dots, k_d) \mid k_j \in \frac{\pi}{L}\mathbb{N}, j = 1, \dots, d-1, k_d \in \frac{\pi}{L_d}\mathbb{N} \right\}$$

where $\alpha\mathbb{N} := \{\alpha n \mid n \in \mathbb{N}\}$ ($\alpha \in \mathbb{R}$). It is well known that the spectrum $\sigma(-\Delta_D)$ of $-\Delta_D$ is purely discrete with

$$\sigma(-\Delta_D) = \{\mathbf{k}^2 \mid \mathbf{k} \in \Gamma\}.$$

An eigenvector of $-\Delta_D$ with eigenvalue $\mathbf{k}^2$ is given by

$$\varphi_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^d \varphi_{k_j}(x_j), \quad \mathbf{x} = (x_1, \dots, x_d) \in \Lambda \quad (4.1)$$

with

$$\varphi_{k_j}(x_j) := \sqrt{\frac{2}{L}} \sin(k_j x_j), \quad x_j \in (0, L), \quad j = 1, \dots, d-1, \quad (4.2)$$

$$\varphi_{k_d}(x_d) := \sqrt{\frac{2}{L_d}} \sin(k_d x_d), \quad x_d \in (0, L_d). \quad (4.3)$$

It is well known that $\{\varphi_{\mathbf{k}} | \mathbf{k} \in \Gamma\}$ is a complete orthonormal system (CONS) of $L^2(\Lambda)$. We have

$$(-\Delta_D + m^2)\varphi_{\mathbf{k}} = \omega(\mathbf{k})^2 \varphi_{\mathbf{k}}, \quad (4.4)$$

where

$$\omega(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m^2}. \quad (4.5)$$

Hence, by functional calculus, we have

$$h\varphi_{\mathbf{k}} = \omega(\mathbf{k})\varphi_{\mathbf{k}}, \quad \mathbf{k} \in \Gamma, \quad (4.6)$$

where h is defined by (1.2). Hence the spectrum of h is purely discrete with

$$\sigma(h) = \{\omega(\mathbf{k}) | \mathbf{k} \in \Gamma\}. \quad (4.7)$$

Note that

$$0 < \omega(\mathbf{k})^{-1} \leq \frac{1}{\sqrt{\left(\frac{\pi}{L}\right)^2 (d-1) + \left(\frac{\pi}{L_d}\right)^2}}, \quad \mathbf{k} \in \Gamma. \quad (4.8)$$

Hence $h^{-1} \in \mathfrak{B}(L^2(\Lambda))$.

We set

$$\mathcal{H}_1 := \ell^2(\Gamma) := \left\{ u : \Gamma \rightarrow \mathbb{C} \mid \sum_{\mathbf{k} \in \Gamma} |u(\mathbf{k})|^2 < \infty \right\},$$

which is a Hilbert space with inner product

$$\langle u, v \rangle := \sum_{\mathbf{k} \in \Gamma} u(\mathbf{k})^* v(\mathbf{k}), \quad u, v \in \mathcal{H}_1.$$

For each $f \in L^2(\Lambda)$, we define a function f_φ on Γ by

$$f_\varphi(\mathbf{k}) := \langle \varphi_{\mathbf{k}}, f \rangle_{L^2(\Lambda)}, \quad \mathbf{k} \in \Gamma.$$

It follows from the completeness of the orthonormal system $\{\varphi_{\mathbf{k}} | \mathbf{k} \in \Gamma\}$ in $L^2(\Lambda)$ that $f_\varphi \in \mathcal{H}_1$ and

$$\|f_\varphi\| = \|f\|.$$

Since $\{\varphi_{\mathbf{k}} | \mathbf{k} \in \Gamma\}$ is a CONS of $L^2(\Lambda)$, the operator $\mathcal{F}_D : L^2(\Lambda) \rightarrow \mathcal{H}_1$ defined by

$$\mathcal{F}_D f := f_\varphi, \quad f \in L^2(\Lambda)$$

is unitary. It is easy to see that

$$\mathcal{F}_D^{-1}u = \sum_{\mathbf{k} \in \Gamma} u(\mathbf{k})\varphi_{\mathbf{k}}, \quad u \in \mathcal{H}_1$$

in the sense of $L^2(\Lambda)$ -convergence. It follows that

$$\mathcal{F}_D h \mathcal{F}_D^{-1} = \omega,$$

where the right hand side denotes the multiplication operator by the function ω on Γ .

For all $\alpha \in \mathbb{R}$ and $f \in D(h^\alpha)$, we have

$$h^\alpha f = \sum_{\mathbf{k} \in \Gamma} \omega(\mathbf{k})^\alpha f_\varphi(\mathbf{k})\varphi_{\mathbf{k}}.$$

in the sense of $L^2(\Lambda)$ -convergence. Hence

$$h^\alpha D(h^\alpha) \cap L_{\text{real}}(\Lambda) \subset L_{\text{real}}^2(\Lambda), \quad (4.9)$$

where

$$L_{\text{real}}^2(\Lambda) := \{f \in L^2(\Lambda) | f^* = f\},$$

the real Hilbert space of real elements in $L^2(\Lambda)$.

We now define a free quantum scalar field on Λ . We denote by $a(f)$ ($f \in L^2(\Lambda)$) by the annihilation operator with test vector f on the boson Fock space $\mathcal{F}_b(L^2(\Lambda))$ over $L^2(\Lambda)$.

Then, for each $f \in L_{\text{real}}^2(\Lambda)$ and $g \in D(h^{1/2}) \cap L_{\text{real}}^2(\Lambda)$, we define

$$\phi(f) := \frac{1}{\sqrt{2}}(a(h^{-1/2}f)^* + a(h^{-1/2}f)) \quad (4.10)$$

and

$$\pi(g) := \frac{i}{\sqrt{2}}(a(h^{1/2}g)^* - a(h^{1/2}g)). \quad (4.11)$$

These operators satisfy the Heisenberg CCR

$$\begin{aligned} [\phi(f), \pi(g)] &= i \langle f, g \rangle, \\ [\phi(f), \phi(f')] &= 0, \quad [\pi(g), \pi(g')] = 0, \quad f, f' \in L_{\text{real}}^2(\Lambda), \quad g, g' \in D(h^{1/2}) \cap L_{\text{real}}^2(\Lambda) \end{aligned}$$

on $\mathcal{F}_0(L^2(\Lambda))$, the finite particle subspace of $\mathcal{F}_b(L^2(\Lambda))$, and are used as time-zero fields in constructing quantum scalar fields on Λ .

Let

$$H := d\Gamma(h),$$

the second quantization operator of h . Then we define the time- t field by

$$\phi(t, f) := e^{itH} \phi(f) e^{-itH}, \quad t \in \mathbb{R}. \quad (4.12)$$

It follows from (3.4) and the commutativity of $h^{-1/2}$ with e^{ith} that

$$\phi(t, f) = \phi(e^{ith}f), \quad f \in L_{\text{real}}^2(\Lambda), \quad (4.13)$$

Let $f \in D(h^2) \cap L_{\text{real}}^2(\Lambda)$. Then, for all $\Psi \in \mathcal{F}_0(L^2(\Lambda))$, $\phi(t, f)\Psi$ is twice strongly differentiable in t and

$$\frac{d}{dt}\phi(t, f)\Psi = \pi(t, f)\Psi, \quad (4.14)$$

$$\frac{d^2}{dt^2}\phi(t, f)\Psi = -\frac{1}{\sqrt{2}}(a(h^{-1/2}e^{ith}h^2f)^* + a(h^{-1/2}e^{ith}h^2f))\Psi, \quad (4.15)$$

where

$$\pi(t, g) := e^{itH}\pi(g)e^{-itH} = \pi(e^{ith}g), \quad g \in D(h^{1/2}) \cap L_{\text{real}}^2(\Lambda). \quad (4.16)$$

Equation (4.14) shows that $\pi(t, f)$ is the canonical conjugate momentum field of $\phi(t, f)$. Note that

$$h^2f = -\Delta_D f + m^2f.$$

Hence we obtain (1.3). This means that $\phi(t, f)$ is the free neutral quantum Klein-Gordon field with mass m on $\mathbb{R} \times \Lambda$ with the Dirichlet boundary condition and H is the Hamiltonian of it.

5 A Quantum Scalar Field on Λ with a Partition

We next consider the case where a perfectly conducting wall W_a (see (1.4)) is placed in Λ as a partition. In what follows, we assume the following:

Assumption (a) $L^2/L_d^2 \in \mathbb{Q}$ (the set of rational numbers) and $a^2/L_d^2 \notin \mathbb{Q}$

5.1 One-particle Hamiltonians with Dirichlet boundary conditions and related facts

Let Λ_1 and Λ_2 be given by (1.6) and Δ_ℓ be the Dirichlet Laplacian for Λ_ℓ ($\ell = 1, 2$). For each $\mathbf{k} \in \Gamma$ and $\ell = 1, 2$, we define functions $\psi_{\mathbf{k}}^{(\ell)}$ on Λ_ℓ as follows:

$$\psi_{\mathbf{k}}^{(1)}(\mathbf{x}) := \left(\prod_{j=1}^{d-1} \varphi_{k_j}(x_j) \right) \psi_{k_d}^{(1)}(x_d), \quad \mathbf{x} \in \Lambda_1, \quad (5.1)$$

$$\psi_{\mathbf{k}}^{(2)}(\mathbf{x}) := \left(\prod_{j=1}^{d-1} \varphi_{k_j}(x_j) \right) \psi_{k_d}^{(2)}(x_d), \quad \mathbf{x} \in \Lambda_2, \quad (5.2)$$

where

$$\psi_{k_d}^{(1)}(x_d) := \sqrt{\frac{2}{a}} \sin \frac{L_d k_d x_d}{a}, \quad x_d \in (0, a), \quad (5.3)$$

$$\psi_{k_d}^{(2)}(x_d) := \sqrt{\frac{2}{L-a}} \sin \frac{L_d k_d (x_d - a)}{L_d - a}, \quad x_d \in (a, L_d). \quad (5.4)$$

Then

$$(-\Delta_\ell + m^2)\psi_{\mathbf{k}}^{(\ell)} = \omega_\ell(\mathbf{k})^2\psi_{\mathbf{k}}^{(\ell)}, \quad \mathbf{k} \in \Gamma, \quad (5.5)$$

where

$$\begin{aligned} \omega_1(\mathbf{k}) &:= \sqrt{k_1^2 + \cdots + k_{d-1}^2 + (L_d k_d/a)^2 + m^2}, \\ \omega_2(\mathbf{k}) &:= \sqrt{k_1^2 + \cdots + k_{d-1}^2 + (L_d k_d/(L_d - a))^2 + m^2}. \end{aligned}$$

The set $\{\psi_{\mathbf{k}}^{(\ell)}\}_{\mathbf{k} \in \Gamma}$ is a CONS of $L^2(\Lambda_\ell)$.

By (5.5) and functional calculus, we have

$$h_{a,\ell}\psi_{\mathbf{k}}^{(\ell)} = \omega_\ell(\mathbf{k})\psi_{\mathbf{k}}^{(\ell)}, \quad (5.6)$$

where $h_{a,\ell}$ is defined by (1.8).

By (1.7), each $f \in L^2(\Lambda)$ is written as

$$f = f^{(1)} + f^{(2)}$$

or $f = (f^{(1)}, f^{(2)})$, where

$$f^{(\ell)} := \chi_{\Lambda_\ell} f \in L^2(\Lambda_\ell)$$

with χ_{Λ_ℓ} being the characteristic function of Λ_ℓ .

The direct sum operator

$$-\Delta_{12} := (-\Delta_1) \oplus (-\Delta_2)$$

is a non-negative self-adjoint operator on $L^2(\Lambda)$. By functional calculus, we have

$$(-\Delta_{12})^{1/2} = (-\Delta_1)^{1/2} \oplus (-\Delta_2)^{1/2}. \quad (5.7)$$

Lemma 5.1 $D((-\Delta_{12})^{1/2}) \subset D((-\Delta_D)^{1/2})$ and, for all $f \in D((-\Delta_{12})^{1/2})$,

$$\|(-\Delta_D)^{1/2} f\| = \|(-\Delta_{12})^{1/2} f\|. \quad (5.8)$$

In particular, $(-\Delta_D)^{1/2}(-\Delta_{12})^{-1/2}$ is in $\mathfrak{B}(L^2(\Lambda))$.

Proof. Let $f = f^{(1)} + f^{(2)} \in D((-\Delta_{12})^{1/2})$. Then, by the definition of Δ_ℓ , there exists a sequence $\{f_n^{(\ell)}\}_{n=1}^\infty$ in $C_0^\infty(\Lambda_\ell)$ such that $f_n^{(\ell)} \rightarrow f^{(\ell)}$ ($n \rightarrow \infty$) and

$$\|(-\Delta_\ell)^{1/2} f^{(\ell)}\|^2 = \lim_{n \rightarrow \infty} \sum_{j=1}^d \|\partial_j f_n^{(\ell)}\|^2,$$

where $\partial_j := \partial/\partial x_j$. Hence, letting $f_n := f_n^{(1)} + f_n^{(2)}$, we see that $f_n \in C_0^\infty(\Lambda)$ and $f_n \rightarrow f$ ($n \rightarrow \infty$),

$$\lim_{n \rightarrow \infty} \sum_{j=1}^d \|\partial_j f_n\|^2 = \|(-\Delta_1)^{1/2} f^{(1)}\|^2 + \|(-\Delta_2)^{1/2} f^{(2)}\|^2 = \|(-\Delta_{12})^{1/2} f\|^2.$$

Therefore $f \in D((-\Delta_D)^{1/2})$ (hence $D((-\Delta_{12})^{1/2}) \subset D((-\Delta_D)^{1/2})$) and (5.8) holds. $\blacksquare$

We next study relations between h and h_a .

Lemma 5.2

- (i) $D(h_a) \subset D(h)$ and hh_a^{-1} is in $\mathfrak{B}(L^2(\Lambda))$.
- (ii) $D(h_a^{1/2}) \subset D(h^{1/2})$ and $h^{1/2}h_a^{-1/2}$ is in $\mathfrak{B}(L^2(\Lambda))$.

Proof. (i) It follows from functional calculus that

$$D(h) = D((-\Delta_D)^{1/2}), \quad D(h_a) = D((-\Delta_{12})^{1/2}).$$

Hence, by Lemma 5.1, $D(h_a) \subset D(h)$. Hence $D(hh_a^{-1}) = L^2(\Lambda)$ and $hh_a^{-1} \in \mathfrak{B}(L^2(\Lambda))$.⁶

(ii) By part (i), h is h_a -bounded. Since h and h_a are non-negative self-adjoint operators, it follows from a general theorem ([24, Theorem X.18(a)]) that $h^{1/2}$ is $h_a^{1/2}$ -bounded. Hence $D(h_a^{1/2}) \subset D(h^{1/2})$ and $h^{1/2}h_a^{-1/2}$ is bounded with $D(h^{1/2}h_a^{-1/2}) = L^2(\Lambda)$. ■

For convenience, we extend each eigenfunction $\psi_{\mathbf{k}}^{(\ell)}$ of $h_{a,\ell}$ ($\mathbf{k} \in \Gamma$) to a function on Λ in the following way:

$$\tilde{\psi}_{\mathbf{k}}^{(\ell)}(\mathbf{x}) := \begin{cases} \psi_{\mathbf{k}}^{(\ell)}(\mathbf{x}) & \text{if } \mathbf{x} \in \Lambda_\ell \\ 0 & \text{if } \mathbf{x} \in \Lambda \setminus \Lambda_\ell \end{cases}. \quad (5.9)$$

Similarly we denote by $\tilde{\psi}_{k_d}^{(\ell)}$ ($\ell = 1, 2$) the extensions of the functions $\psi_{k_d}^{(1)}$ and $\psi_{k_d}^{(2)}$ to $(0, L_d)$.

By functional calculus, we have for all $\alpha > 0$

$$h_a^\alpha \tilde{\psi}_{\mathbf{k}}^{(\ell)} = \omega_\ell(\mathbf{k})^\alpha \tilde{\psi}_{\mathbf{k}}^{(\ell)}, \quad \ell = 1, 2, \mathbf{k} \in \Gamma. \quad (5.10)$$

For each $(\mathbf{k}, \mathbf{p}) \in \Gamma \times \Gamma$ and $\ell = 1, 2$, we define

$$\gamma_{\mathbf{k}\mathbf{p}}^{(\ell)} := \langle \varphi_{\mathbf{k}}, \tilde{\psi}_{\mathbf{p}}^{(\ell)} \rangle. \quad (5.11)$$

It follows from (4.1), (5.1) and (5.2) that

$$\gamma_{\mathbf{k}\mathbf{p}}^{(\ell)} = \left(\prod_{j=1}^{d-1} \delta_{k_j p_j} \right) \langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(\ell)} \rangle. \quad (5.12)$$

We set

$$c_1 := \frac{L_d}{a}, \quad c_2 := \frac{L_d}{L_d - a}. \quad (5.13)$$

Lemma 5.3 For all $k_d, p_d \in (\pi/L_d)\mathbb{N}$,

$$\langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(1)} \rangle = \frac{2}{\sqrt{L_d a}} \frac{(-1)^{L_d p_d} c_1 p_d \sin(ak_d)}{k_d^2 - c_1^2 p_d^2}, \quad (5.14)$$

$$\langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(2)} \rangle = -\frac{2}{\sqrt{L_d(L_d - a)}} \frac{c_2 p_d \sin(ak_d)}{k_d^2 - c_2^2 p_d^2}. \quad (5.15)$$

⁶The following fact is well known (which follows from the closed graph theorem): let A and B be closed linear operators on a Banach space $\mathcal{X}$. Suppose that $D(A) \subset D(B)$ and A is bijective. Then BA^{-1} is in $\mathfrak{B}(\mathcal{X})$.

Proof. By (4.3) and (5.3), we have

$$\left\langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(1)} \right\rangle = \frac{2}{\sqrt{L_d a}} \int_0^a \sin(k_d x_d) \sin(c_1 p_d x_d) dx_d.$$

Then, by direct computations, we obtain (5.14). Similarly one can prove (5.15). $\blacksquare$

Lemma 5.4 $D(h_a^{1/2}) \subsetneq D(h^{1/2})$.

Proof. By functional calculus, for all $\mathbf{k} \in \Gamma$, $\varphi_{\mathbf{k}} \in D(h^{1/2})$ with

$$h^{1/2} \varphi_{\mathbf{k}} = \omega(\mathbf{k})^{1/2} \varphi_{\mathbf{k}}.$$

It is sufficient to show that, for some $\mathbf{k}_0 \in \Gamma$, $\varphi_{\mathbf{k}_0} \notin D(h_a^{1/2})$. Let $\mathbf{k} \in \Gamma$ be such that $\varphi_{\mathbf{k}} \in D(h_a^{1/2})$. Then, by the fact that $\{\tilde{\psi}_{\mathbf{p}}^{(\ell)} | \ell = 1, 2, \mathbf{p} \in \Gamma\}$ is a CONS of $L^2(\Lambda)$, we have

$$s_{\mathbf{k}} := \sum_{\ell=1}^2 \sum_{\mathbf{p} \in \Gamma} \left| \left\langle \tilde{\psi}_{\mathbf{p}}^{(\ell)}, h_a^{1/2} \varphi_{\mathbf{k}} \right\rangle \right|^2 = \|h_a^{1/2} \varphi_{\mathbf{k}}\|^2 < \infty.$$

By the symmetry of $h_a^{1/2}$ and (5.10), we have

$$\left\langle \tilde{\psi}_{\mathbf{p}}^{(\ell)}, h_a^{1/2} \varphi_{\mathbf{k}} \right\rangle = \left\langle h_a^{1/2} \tilde{\psi}_{\mathbf{p}}^{(\ell)}, \varphi_{\mathbf{k}} \right\rangle = \omega_{\ell}(\mathbf{p})^{1/2} \gamma_{\mathbf{k}\mathbf{p}}^{(\ell)}.$$

Hence

$$s_{\mathbf{k}} = \sum_{\ell=1}^2 \sum_{\mathbf{p} \in \Gamma} \omega_{\ell}(\mathbf{p}) \left| \gamma_{\mathbf{k}\mathbf{p}}^{(\ell)} \right|^2.$$

Using (5.12), we have

$$s_{\mathbf{k}} = \sum_{\ell=1}^2 \sum_{p_d \in (\pi/L_d)\mathbb{N}} \omega_{\ell}(\tilde{\mathbf{k}}, p_d) \left| \left\langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(\ell)} \right\rangle \right|^2,$$

where

$$\tilde{\mathbf{k}} := (k_1, \dots, k_{d-1}) \in \left(\frac{\pi}{L} \mathbb{N} \right)^{d-1}.$$

By Lemma 5.3, we have

$$\lim_{p_d \rightarrow \infty} \left| p_d \left\langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(1)} \right\rangle \right|^2 = \frac{4 \sin^2(ak_d)}{L_d a c_1^2}. \quad \lim_{p_d \rightarrow \infty} \left| p_d \left\langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(2)} \right\rangle \right|^2 = \frac{4 \sin^2(ak_d)}{L_d (L_d - a) c_2^2}.$$

Now let $\mathbf{k}_0 := (\tilde{\mathbf{k}}, k_d)$ with $\tilde{\mathbf{k}} \in ((\pi/L)\mathbb{N})^{d-1}$ arbitrary and $k_d = \pi/L_d$. Then $0 < ak_d < \pi$. Hence $\sin ak_d \neq 0$. Therefore there exists a constant $C > 0$ such that

$$\left| \left\langle \varphi_{k_d}, \tilde{\psi}_{p_d}^{(\ell)} \right\rangle \right|^2 \geq \frac{C}{c_{\ell} p_d^2}.$$

Note that

$$\omega_\ell(\tilde{\mathbf{k}}, p_d) \geq c_\ell p_d.$$

Hence

$$s_{\mathbf{k}_0} \geq C \sum_{p_d \in (\pi/L_d)\mathbb{N}} \frac{1}{p_d} = \infty.$$

Thus $\varphi_{\mathbf{k}_0} \notin D(h_a^{1/2})$. ■

The following fact shows a singular nature of the pair (h, h_a) of one-particle Hamiltonians:

Lemma 5.5 *The operator $h^{-1/2}h_a^{1/2}$ is unbounded.*

Proof. Let $T := h^{-1/2}h_a^{1/2}$. We prove the unboundedness of T by reductio ad absurdum. Suppose that T were bounded. Since $D(T) = D(h_a^{1/2})$, T is densely defined. Hence it is closable and the closure $\overline{T}$ is in $\mathfrak{B}(L^2(\Lambda))$. We have $T^* = (\overline{T})^*$. Hence $D(T^*) = L^2(\Lambda)$. Since $h^{-1/2}$ is bounded with $D(h^{-1/2}) = L^2(\Lambda)$, it follows that $T^* = (h_a^{1/2})^*(h^{-1/2})^* = h_a^{1/2}h^{-1/2}$. This implies that $D(h^{1/2}) \subset D(h_a^{1/2})$ and $D(T^*) = h^{1/2}D(h_a^{1/2})$. But this contradicts Lemma 5.4. ■

Lemma 5.6 *Let*

$$S_\pm := \frac{1}{2}(h^{-1/2}h_a^{1/2} \pm h^{1/2}h_a^{-1/2}). \quad (5.16)$$

Then $S_\pm$ are unbounded.

Proof. This follows from Lemma 5.2(ii) and Lemma 5.5. ■

We note that

$$\mathcal{D}_a := D(S_+) = D(S_-) = D(h_a^{1/2}).$$

Lemma 5.7 *For all $f, g \in \mathcal{D}_a$, the following equations hold:*

$$\langle S_+f, S_+g \rangle - \langle S_-f, S_-g \rangle = \langle f, g \rangle, \quad (5.17)$$

$$\langle S_+f, S_-g \rangle = \langle S_-f, S_+g \rangle. \quad (5.18)$$

Proof. These equations follow from direct computations. ■

Remark 5.8 Equations of type (5.17) and (5.18) hold for operators in a general class: let A and B be injective (not necessarily bounded) symmetric operators on a Hilbert space $\mathcal{H}$ and define

$$T_\pm := \frac{1}{2}(A^{-1}B \pm AB^{-1}).$$

Then, for all $f, g \in D(T_+) \cap D(T_-)$, (5.17) and (5.18) hold with $S_\pm$ replaced by $T_\pm$.

Lemma 5.9 *The range $\text{Ran } S_+$ of S_+ is dense in $L^2(\Lambda)$.*

Proof. It is sufficient to show that the orthogonal complement $(\text{Ran } S_+)^\perp$ of $\text{Ran } S_+$ is $\{0\}$. Let $g \in (\text{Ran } S_+)^\perp$. Then, for all $f \in D(h_a^{1/2})$, $\langle S_+ f, g \rangle = 0$, which implies that $\langle h_a^{1/2} f, h^{-1/2} g \rangle = -\langle Gf, g \rangle$ with $G := h^{1/2} h_a^{-1/2}$. By Lemma 5.2(ii), G is bounded. Hence $\langle Gf, g \rangle = \langle f, G^* g \rangle$. Hence $h^{-1/2} g \in D(h_a^{1/2})$ and $h_a^{1/2} h^{-1/2} g = -G^* g$. Hence $g = -GG^* g$, which implies that $\|g\|^2 = -\|G^* g\|^2 \leq 0$. Therefore $g = 0$. Thus $(\text{Ran } S_+)^\perp = \{0\}$. ■

5.2 A singular Bogoliubov transformation and a representation of the CCR over a dense subspace

We denote by C_Λ the complex conjugation on $L^2(\Lambda)$:

$$C_\Lambda f := f^*, \quad f \in L^2(\Lambda).$$

It follows from (4.9) that

$$C_\Lambda h^{\pm 1/2} \subset h^{\pm 1/2} C_\Lambda.$$

Similarly one can show that

$$C_\Lambda h_a^{\pm 1/2} \subset h_a^{\pm 1/2} C_\Lambda.$$

Hence

$$C_\Lambda S_\pm \subset S_\pm C_\Lambda.$$

We define

$$b(f) := \overline{a(S_+ f) + a(C_\Lambda S_- f)^*}, \quad f \in \mathcal{D}_a. \quad (5.19)$$

Lemma 5.7 implies the following commutation relations: for all $f, g \in \mathcal{D}_a$,

$$[b(f), b(g)^*] = \langle f, g \rangle, \quad (5.20)$$

$$[b(f), b(g)] = 0, \quad [b(f)^*, b(g)^*] = 0 \quad (5.21)$$

on $\mathcal{F}_0(L^2(\Lambda))$. Hence the correspondence $(a(\cdot), a(\cdot)^*) \mapsto (b(\cdot), b(\cdot)^*)$ is a Bogoliubov transformation. By Lemma 5.6, this is a singular Bogoliubov transformation.

For a subset $\mathcal{D}$ of a vector space, $\text{l.h.}\mathcal{D}$ denotes the subspace algebraically spanned by all the vectors in $\mathcal{D}$. It is obvious that the subspace

$$\mathcal{E}_0 := \text{l.h.}\{\tilde{\psi}_{\mathbf{k}}^{(\ell)} | \mathbf{k} \in \Gamma, \ell = 1, 2\} \quad (5.22)$$

is dense in $L^2(\Lambda)$. It follows from functional calculus that, for all $\alpha > 0$, $\mathcal{E}_0 \subset D(h_a^\alpha)$ with

$$h_a^\alpha \tilde{\psi}_{\mathbf{k}}^{(\ell)} = \omega_\ell(\mathbf{k})^\alpha \tilde{\psi}_{\mathbf{k}}^{(\ell)}, \quad \mathbf{k} \in \Gamma, \ell = 1, 2.$$

Hence

$$h_a^\alpha \mathcal{E}_0 \subset \mathcal{E}_0. \quad (5.23)$$

By (5.20) and (5.21), we obtain the following lemma:

Lemma 5.10 *The triple*

$$\pi_b(\mathcal{E}_0) := (\mathcal{F}_b(L^2(\Lambda)), \mathcal{F}_0(L^2(\Lambda)), \{b(f), b(f)^* | f \in \mathcal{E}_0\}).$$

is a representation of the CCR over $\mathcal{E}_0$.

Remark 5.11 Lemma 5.10 holds also in the case where $\mathcal{E}_0$ is replaced by $\mathcal{D}_a$.

5.3 A quantum scalar field on Λ with W_a

To define a quantum field which may describe the dynamics of the system with the wall W_a , we introduce the following operators:

$$\phi_b(f) := \frac{1}{\sqrt{2}}(b(h_a^{-1/2}f)^* + b(h_a^{-1/2}f)), \quad f \in L^2_{\text{real}}(\Lambda), \quad (5.24)$$

$$\pi_b(g) := \frac{i}{\sqrt{2}}(b(h_a^{1/2}g)^* - b(h_a^{1/2}g)), \quad g \in D(h_a) \cap L^2_{\text{real}}(\Lambda). \quad (5.25)$$

Lemma 5.12 For all $f \in D(h^{1/2}) \cap D(h_a) \cap L^2_{\text{real}}(\Lambda)$,

$$\phi_b(f) = \phi(f), \quad \pi_b(f) = \pi(f) \quad \text{on } \mathcal{F}_0(L^2(\Lambda)). \quad (5.26)$$

Proof. For all $\Psi \in \mathcal{F}_0(L^2(\Lambda))$, we have

$$\phi_b(f)\Psi = \frac{1}{\sqrt{2}}(a((S_+h_a^{-1/2} + S_-h_a^{-1/2})f) + a((S_+h_a^{-1/2} + S_-h_a^{-1/2})f)^*)\Psi.$$

But

$$(S_+h_a^{-1/2} + S_-h_a^{-1/2})f = h^{-1/2}f.$$

Hence the first equation in (5.26) holds. Similarly, using the equality

$$(S_+h_a^{1/2} - S_-h_a^{1/2})f = h^{1/2}f,$$

one can prove the second equation in (5.26). ■

Remark 5.13 Heuristically the origin of the operator $b(f)$ is in requiring (5.26), i.e., the condition that the time-zero fields of the quantum field of the system with the wall W_a coincide with those of the free quantum field $\phi(t, \cdot)$ of the system without the wall W_a on a suitable dense subspace of $\mathcal{F}_b(\Lambda)$.

For each $t \in \mathbb{R}$, we define

$$\phi_b(t, f) := \phi_b(e^{ith_a}f), \quad f \in L^2_{\text{real}}(\Lambda), \quad (5.27)$$

$$\pi_b(t, g) := \pi_b(e^{ith_a}g), \quad g \in D(h_a^{1/2}) \cap L^2_{\text{real}}(\Lambda). \quad (5.28)$$

By (5.26), we have

$$\phi_b(0, f) = \phi(f), \quad \pi_b(0, g) = \pi(g) \quad \text{on } \mathcal{F}_0(L^2(\Lambda)). \quad (5.29)$$

As in the case of $\phi(t, \cdot)$ and $\pi(t, \cdot)$, one can show that $\phi_b(t, f)$ with $f \in D(\Delta_{12})$ obeys the free Klein-Gordon equation with mass m

$$\frac{d^2}{dt^2}\phi_b(t, f) + \phi_b(t, (-\Delta_{12} + m^2)f) = 0$$

and

$$\frac{d}{dt}\phi_b(t, f) = \pi_b(t, f)$$

on $\mathcal{F}_0(L^2(\Lambda))$, where differentiation in t is taken in the strong sense. By these facts and (5.29), $\phi(t, \cdot)$ may be a quantum field in the system with the wall W_a , which should be compared with the free field $\phi(t, \cdot)$.

To represent $\phi_b(t, \cdot)$ and $\pi_b(t, \cdot)$ in terms of $a(\cdot)^\#$, we introduce the following operators:

$$K(t) := h^{-1/2} \cos(th_a) + ih^{1/2} h_a^{-1} \sin(th_a), \quad (5.30)$$

$$L(t) := h^{1/2} \cos(th_a) + ih^{-1/2} h_a \sin(th_a), \quad t \in \mathbb{R}. \quad (5.31)$$

By direct computations, one can show that

$$\phi_b(t, f) = \frac{1}{\sqrt{2}}(a(K(t)f)^* + a(K(t)f)), \quad f \in L_{\text{real}}^2(\Lambda), \quad (5.32)$$

$$\pi_b(t, g) = \frac{i}{\sqrt{2}}(a(L(t)g)^* - a(L(t)g)), \quad g \in D(h_a) \cap L_{\text{real}}^2(\Lambda) \quad (5.33)$$

on $\mathcal{F}_0(\mathcal{H}_1)$.

We define operator-valued mappings ϕ_0 and $\phi_{b,0}$ from $\mathbb{R} \times L_{\text{real}}^2(\Lambda)$ to the set of linear operators on $\mathcal{F}_b(L^2(\Lambda))$ by

$$\phi_0(t, f) := \phi(t, f) \upharpoonright \mathcal{F}_0(L^2(\Lambda)), \quad (5.34)$$

$$\phi_{b,0}(t, f) := \phi_b(t, f) \upharpoonright \mathcal{F}_0(L^2(\Lambda)), \quad (t, f) \in \mathbb{R} \times L_{\text{real}}^2(\Lambda). \quad (5.35)$$

The following proposition shows that $\phi_{b,0}$ describes a dynamics different from that of ϕ_0 :

Proposition 5.14

$$\phi_0 \neq \phi_{b,0}. \quad (5.36)$$

Proof. Let Ω_0 be the Fock vacuum in $\mathcal{F}_b(L^2(\Lambda))$: $\Omega_0 := \Omega_{L^2(\Lambda)}$. Then, for all $(t, f) \in \mathbb{R} \times L_{\text{real}}^2(\Lambda)$,

$$\phi_{b,0}(t, f)\Omega_0 = \left\{0, \frac{1}{\sqrt{2}}K(t)f, 0, 0, \dots\right\}, \quad \phi_0(t, f)\Omega_0 = \left\{0, \frac{1}{\sqrt{2}}h^{-1/2}e^{ith}f, 0, 0, \dots\right\}.$$

Suppose that, for all $(t, f) \in (\mathbb{R} \setminus \{0\}) \times L_{\text{real}}^2(\Lambda)$, $\phi_{b,0}(t, f)\Omega_0 = \phi_0(t, f)\Omega_0$. Then $K(t) = h^{-1/2}e^{ith}$, $\forall t \in \mathbb{R} \setminus \{0\}$. Hence, in particular, $K(t)\tilde{\psi}_{\mathbf{k}}^{(\ell)} = h^{-1/2}e^{ith}\tilde{\psi}_{\mathbf{k}}^{(\ell)}$ for all $t \in \mathbb{R} \setminus \{0\}$ and $\mathbf{k} \in \Gamma$. Hence, for all $g \in D(h^2)$,

$$\langle g, K(t)\tilde{\psi}_{\mathbf{k}}^{(\ell)} \rangle = \langle e^{ith}g, f_{\mathbf{k}} \rangle, \quad (5.37)$$

where $f_{\mathbf{k}} := h^{-1/2}\tilde{\psi}_{\mathbf{k}}^{(\ell)}$. We have

$$K(t)\tilde{\psi}_{\mathbf{k}}^{(\ell)} = \cos(t\omega_{\ell}(\mathbf{k}))h^{-1/2}\tilde{\psi}_{\mathbf{k}}^{(\ell)} + i\omega_{\ell}(\mathbf{k})^{-1}\sin(t\omega_{\ell}(\mathbf{k}))h^{1/2}\tilde{\psi}_{\mathbf{k}}^{(\ell)}.$$

Differentiating the both sides of (5.37) twice in t and then taking the limit $t \rightarrow 0$, we obtain

$$\langle g, \omega_{\ell}(\mathbf{k})^2 f_{\mathbf{k}} \rangle = \langle h^2 g, f_{\mathbf{k}} \rangle.$$

This implies that $f_{\mathbf{k}} \in D(h^2)$ and $h^2 f_{\mathbf{k}} = \omega_{\ell}(\mathbf{k})^2 f_{\mathbf{k}}$. Hence $h f_{\mathbf{k}} = \omega_{\ell}(\mathbf{k}) f_{\mathbf{k}}$. Therefore $\omega_{\ell}(\mathbf{k})$ is an eigenvalue of h . But the eigenvalues of h are given by $\{\omega(\mathbf{p}) | \mathbf{p} \in \Gamma\}$. It follows from Assumption (a) that $\omega_1(\mathbf{k}) \notin \{\omega(\mathbf{p}) | \mathbf{p} \in \Gamma\}$. Thus we arrive at a contradiction. Thus (5.36) holds. $\blacksquare$

Remark 5.15 Unfortunately we have been unable to make it clear if there is a self-adjoint operator (a Hamiltonian) H_a on $\mathcal{F}_b(L^2(\Lambda))$ such that, for all $t \in \mathbb{R}$,

$$e^{itH_a} \phi_b(0, f) e^{-itH_a} = \phi_b(t, f), \quad e^{itH_a} \pi_b(0, f) e^{-itH_a} = \pi_b(t, f)$$

for all f in a suitable dense subspace of $L^2_{\text{real}}(\Lambda)$. This problem is left for future study.

6 Inequivalence of $\pi_b(\mathcal{E}_0)$ to the Fock Representation of the CCR over $\mathcal{E}_0$ on $\mathcal{F}_b(L^2(\Lambda))$

The Fock representation of the CCR over $\mathcal{E}_0$ on $\mathcal{F}_b(L^2(\Lambda))$ is given by

$$\pi_F(\mathcal{E}_0) := (\mathcal{F}_b(L^2(\Lambda)), \mathcal{F}_0(L^2(\Lambda)), \{a(f), a(f)^* | f \in \mathcal{E}_0\}).$$

Theorem 6.1 *The representation $\pi_b(\mathcal{E}_0)$ of the CCR over $\mathcal{E}_0$ is irreducible and inequivalent to the Fock representation $\pi_F(\mathcal{E}_0)$.*

Proof. We first prove the irreducibility of $\pi_b(\mathcal{E}_0)$. Let $T \in \{b(f), b(f)^* | f \in \mathcal{E}_0\}'$. Then, by (5.24), (5.25) and (5.23), $T \in \{\phi_b(f), \pi_b(f) | f \in \mathcal{E}_0 \cap L^2_{\text{real}}(\Lambda)\}'$. Then, by (5.26), we obtain

$$T \in \{\phi(f) \upharpoonright \mathcal{F}_0(L^2(\Lambda)), \pi(f) \upharpoonright \mathcal{F}_0(L^2(\Lambda)) | f \in \mathcal{E}_0 \cap L^2_{\text{real}}(\Lambda)\}',$$

where we have used the fact that $\mathcal{E}_0 \subset D(h_a^{1/2}) \subset D(h^{1/2})$. Recall that $\mathcal{F}_0(L^2(\Lambda))$ is a core for $\phi(f)$ and $\pi(f)$. Hence it follows from a limiting argument that

$$T \in \{\phi(f), \pi(f) | f \in \mathcal{E}_0 \cap L^2_{\text{real}}(\Lambda)\}'.$$

But the right hand side is $\mathbb{C}I$ (essentially due to [24, p.232. Lemma 1]; for a direct proof, see [4, p.289, Example 5.17]). Hence $\{b(f), b(f)^* | f \in \mathcal{E}_0\}' = \mathbb{C}I$. Thus $\{b(f), b(f)^* | f \in \mathcal{E}_0\}$ is irreducible.

We next show that $\pi_b(\mathcal{E}_0)$ is inequivalent to $\pi_F(\mathcal{E}_0)$. By Lemmas 5.6–5.9 and the irreducibility of $\pi_b(\mathcal{E}_0)$ proved in the preceding paragraph, we can apply Theorem 3.7 to conclude that $\pi_b(\mathcal{E}_0)$ is inequivalent to $\pi_F(\mathcal{E}_0)$. $\blacksquare$

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