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On boundary detachment phenomena for the total variation flow with dynamic boundary conditions

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Abstract

We combine the total variation flow suitable for crystal modeling and image analysis with the dynamic boundary conditions. We analyze the behavior of facets at the parts of the boundary where these conditions are imposed. We devote particular attention to the radially symmetric data. We observe that the boundary layer detachment actually can happen at concave parts of the boundary.

Key words: total variation flow, facet evolution, dynamic boundary conditions, boundary layer detachment

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1 Introduction

We consider the total variation flow with the dynamic boundary condition, possibly mixed with the Neumann boundary condition, which can be formally written as follows,

$$\begin{aligned}
u_t &= \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) && \text{for } (x, t) \in \Omega \times (0, T) =: Q_T; \\
\tau v_t &= -\frac{\nabla u}{|\nabla u|} \cdot \nu && \text{for } (y, t) \in \Gamma \times (0, T) =: S_T; \\
\frac{\partial u}{\partial \nu} &= 0 && \text{for } (y, t) \in (\partial\Omega \setminus \Gamma) \times (0, T); \\
u(x, 0) &= u_0(x) && \text{for } x \in \Omega; \\
v(y, 0) &= v_0(y) && \text{for } y \in \Gamma.
\end{aligned} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded spatial domain of dimension $N \in \mathbb{N}$, and when $N > 1$, the boundary $\partial\Omega$ is supposed to be sufficiently smooth. Moreover, $\Gamma \subset \partial\Omega$, possibly $\Gamma = \partial\Omega$ is a part of the boundary with positive $\mathcal{H}^{N-1}$ -measure. The outer normal to $\partial\Omega$ at x is denoted by $\nu(x)$.

It is well-known that the total variation flow leads to the creation of facets, i.e. persistent flat parts of solutions. Here, we study the interactions of facets at their junction with the boundary at Γ , where the dynamic boundary conditions are specified.

Even though the total variation flow with the Dirichlet boundary conditions was studied by a number of authors, see [4, 13, 41, 42], the details of the boundary behavior were not extensively discussed. In particular this applies to the evolution of facets touching the boundary. It is worth emphasizing that the authors of [4, 13] invested a lot of effort into finding the correct notion of the solution. This is particularly true for [13], where quite general time-dependent Dirichlet boundary data are considered.

It is worth noticing that it is known, see [5, 41, 42], that in general, the Dirichlet boundary data may not be attained in the sense of trace.

We are interested in a phenomenon, which was studied in [43] for the Dirichlet problem of graphs evolving by the mean curvature. The authors showed there that a boundary layer may detach from the solution in the bulk. This phenomenon is attributed to the lack of uniform parabolicity of the mean curvature flow for graphs. Obviously, this lack of uniform parabolicity occurs here too.

It is worth mentioning that the problem of the loss of the boundary conditions was studied by a number of authors in the context of viscosity solutions to parabolic equations with nonlinearities involving gradient of solution. A good example of such research is [45], where also a historical account is presented. However, the nature of the phenomenon studied in [45] is different from what we study here. In case of first order Hamilton-Jacobi equations we refer the reader to [21] for earlier study on unattainability of the boundary condition.

The non-attainment of boundary condition is a common problem for the steady states of the total variation flow. They are better known as solutions to the least gradient problem. Special geometric restrictions must be imposed on the domain Ω as well as on the boundary datum f to ensure attainment, see [35, 40, 50].

Here, we are observing a similar phenomenon of the boundary layer detachment. We introduce a family of evolution problems with dynamic boundary condition indexed by parameter

$\tau \in (0, \infty)$. By formally taking the limit as $\tau \rightarrow 0$ we recover the Neumann data, while the limit $\tau \rightarrow \infty$ yields the Dirichlet boundary conditions. However, a rigorous statement is outside the scope of this paper.

The study of the dynamic boundary conditions has a long history. In the early days, solvability of uniformly parabolic equations with dynamic boundary conditions was discussed by Escher, [23]. For fully nonlinear (possibly degenerate) parabolic equations, Barles established a quite general comparison result in [10, Sect. II] and [11, Sect. 3] for a general nonlinear dynamic boundary condition. The mean curvature flow for a level set, under a dynamic boundary condition is discussed in Giga-Hamamuki [31], which is not included in papers of Barles, [10, 11].

For further development in case of uniformly parabolic problems we refer to [20, 26, 51]. More recently this topic was studied in [16–18, 27]. It is worth emphasizing that the dynamic boundary conditions were studied in relation to Stefan problem, cf. [1], Allen–Cahn type equations [16, 18, 26, 49] or Cahn–Hilliard equations, [17, 25, 27].

Finally, we comment on the physical background of our system (1.1). The dynamic boundary conditions kindred to this study are found in the previous works of Stefan problems, e.g. [1, 47], and in particular, our dynamic boundary condition can be characterized as a singular limit of transmitted parabolic problems studied in [47]. Meanwhile, the singular diffusion as in (1.1) is associated with a phase transition model of mesoscale, which was proposed by Visintin [52, Chapter 6, page 176]. In view of these, our system (1.1) can be regarded as a basic problem for a mesoscale phase transition model, that takes into account interactive phase-exchanges reproduced by the dynamic boundary condition.

Our goal in this paper is to study instances of occurrence of the “boundary layer detachment phenomenon” in the case of the total variation flow under the dynamic boundary condition on a part of the boundary called Γ . More precisely, we investigate the evolution of the persistent facets touching Γ , such facets are called *calibrable*.

Moreover, if the solution is continuous at points of Γ , i.e. the facet moves with the same velocity as the boundary value, then we call such a calibrable facet *coherent*.

If a facet does not touch the boundary, its calibrability is well studied, especially when the facet is convex as well as the solution. In fact, the calibrability of a facet F is equivalent to saying that F is a Cheeger set, i.e. F minimizes the Cheeger quotient, $\lambda = |\partial F|/|F|$, among all subsets. Moreover, it is the same as saying that the inward mean curvature of F , κ , is dominated by the Cheeger quotient λ , see [2, 12].

Under some technical conditions we show that facet F is calibrable and coherent if the sum of inward principal curvature of $\partial\Omega$ near the intersection of F and Γ is greater than $-1/\tau$. In one dimensional case $N = 1$, we show all facets are calibrable and coherent. However, in $N = 2$, if one considers annuli, the facet touching the inner circle may not be coherent and boundary detachment phenomenon actually occurs. In order to derive these results, we first clarify the definition of a solution by taking a correct energy and show the well-posedness of the problem. We next calculate canonical restriction of subdifferential of the energy. Although the general strategy is similar to those in [4, 5], it is nontrivial to implement the strategy.

Let us describe the content of this paper. We present here a general existence result for (1.1). For this purpose we use the nonlinear semigroup theory developed by Kōmura [39] and Brézis [15]. The main step in this direction is the identification of (1.1) as a gradient flow of an energy functional E . It turns out that the natural definition of $E : L^2(\Omega) \times L^2(\Gamma) \rightarrow \mathbb{R} \cup \{+\infty\}$

is as follows,

$$E(u, v) = \begin{cases} \int_{\Omega} |Du| + \int_{\Gamma} |\gamma u - v| d\mathcal{H}^{N-1} & \text{if } (u, v) \in BV(\Omega) \times L^2(\Gamma), \\ +\infty & \text{otherwise.} \end{cases}$$

In Section 3, we study the lower semi-continuity of E and related problems, because this is the precondition of the nonlinear semigroup theory. In Section 4, we state and prove the existence of solutions to (1.1). Here, our point of departure is the observation that (1.1) is a gradient flow of E . In fact, if $\tau = 1$, then (1.1) is the gradient flow of E with respect to the standard inner product in $H = L^2(\Omega) \times L^2(\Gamma)$. We notice that (1.1) may be viewed as a gradient flow of E with respect to a non-standard inner product in H , given by formula $((u_1, v_1), (u_2, v_2))_{\tau} = \int_{\Omega} u_1 u_2 dx + \tau \int_{\Gamma} v_1 v_2 d\mathcal{H}^{N-1}$. We comment on this in Section 4.

We also take advantage of the structure of E to notice the order preserving property of the flow and the comparison principle. This is also done in Section 4 and the analysis is based on the work by Brézis [14] and Kenmochi [37].

A very important part of the analysis, which on the one hand is technical, on the other hand is necessary for the study of facet evolution is the identification of the subdifferential, ∂E , and its canonical selection. This is performed in Section 5. This section closes with the remark on the relationship between the subdifferentials with respect to $(\cdot, \cdot)_{\tau}$ for different values of τ .

Section 6 prepares the tools for the analysis of facets. In particular, we adjust the notion of calibrability to the present setting, when we pay particular attention to the behavior of facets, touching the boundary of $\partial\Omega$ along Γ , where the dynamic boundary condition is set.

We also introduce the notion of coherency, which is useful, when we wish to address the phenomenon of the boundary layer detachment. We also state there sufficient and necessary condition for calibrability or coherency.

We study a number of explicit examples, which show different types of behavior. Section 7 offers an analysis of a one dimensional problem as a warm-up. In this case no boundary layer detachment occurs. The radially symmetric two-dimensional problems are treated in Section 8. We notice that a general Theorem 6.10 and its Corollary 6.1 imply that if $\Gamma = \partial\Omega$, where Ω is a ball, then radially symmetric facets touching Γ will be coherent, i.e. no boundary detachment occurs. The situation is different, when we consider an annulus with inner radius r_0 and $\Gamma = \partial B(0, r_0)$. In this case we pinpoint the situation of the boundary layer detachment.

2 Preliminaries

In this section, we begin with the basic notation used throughout this paper.

For an abstract Banach space X , we denote by $\|\cdot\|_X$ the norm of X , and when X is a Hilbert space, we denote by $(\cdot, \cdot)_X$ the inner product of X . In particular, in cases of Euclidean spaces, we uniformly denote by $|\cdot|$ the Euclidean norm, and we use “ $\cdot$ ” to denote the standard scalar product of two vectors. Additionally, for fixed dimensions $d, \ell \in \mathbb{N}$ and a bounded open set $U \subset \mathbb{R}^d$, we denote by $\|\cdot\|_{\infty}$ the supremum-norm in $L^{\infty}(U, \mathbb{R}^d)$, i.e. $\|w\|_{\infty} := \text{ess sup}_{x \in U} |w(x)|$, for $w \in L^{\infty}(U, \mathbb{R}^d)$.

For any proper functional $\Psi : X \rightarrow (-\infty, \infty]$ on a Hilbert space X , we denote by $D(\Psi)$ the effective domain of Ψ , i.e. $D(\Psi) := \left\{ w \in X \mid \Psi(w) < \infty \right\}$.

For any proper lower semi-continuous (l.s.c., in short) and convex function Φ defined on a Hilbert space X , we denote the subdifferential of Φ by $\partial\Phi$. The subdifferential $\partial\Phi$ corresponds to a weak differential of Φ , and in fact it is a maximal monotone graph in the product space $X \times X$. More precisely, for each $w_0 \in X$, the value $\partial\Phi(w_0)$ of the subdifferential at w_0 is defined as a set of all elements $\eta_0 \in X$ which satisfy the following variational inequality:

$$(\eta_0, w - w_0)_X \leq \Phi(w) - \Phi(w_0) \text{ for any } w \in D(\Phi).$$

The set $D(\partial\Phi) := \left\{ w \in X \mid \partial\Phi(w) \neq \emptyset \right\}$ is called the domain of $\partial\Phi$. We often use the notation “ $(w_0, \eta_0) \in \partial\Phi$ in $X \times X$ ”, to mean that “ $\eta_0 \in \partial\Phi(w_0)$ in X with $w_0 \in D(\partial\Phi)$ ”, by identifying the operator $\partial\Phi$ with its graph in $X \times X$.

Remark 2.1. An example of a subdifferential is the following set-valued sign function $\text{Sgn}^d : \mathbb{R}^d \rightarrow 2^{\mathbb{R}^d}$, given as:

$$\omega \in \mathbb{R}^d \mapsto \text{Sgn}^d(\omega) := \begin{cases} \frac{\omega}{|\omega|}, & \text{if } \omega \neq 0, \\ \{\tilde{\omega} \in \mathbb{R}^d \mid |\tilde{\omega}| \leq 1\}, & \text{if } \omega = 0. \end{cases}$$

It is easy to check that the set-valued function Sgn^d coincides with the subdifferential of the Euclidean norm $|\cdot| : \omega \in \mathbb{R}^d \mapsto |\omega| = \sqrt{\omega \cdot \omega} \in [0, \infty)$.

Notations in BV-theory. (cf. [3, 8, 24, 34]) For any $d \in \mathbb{N}$, we denote the d -dimensional Lebesgue measure by $\mathcal{L}^d$. The measure theoretical phrases, such as “a.e.”, “ dt ”, “ dx ”, etc are with respect to the Lebesgue measure in the corresponding dimension, unless specified otherwise.

Let $d \in \mathbb{N}$ be a fixed dimension and let $U \subset \mathbb{R}^d$ be a bounded open set. We denote by $\mathcal{M}(U)$ (resp. $\mathcal{M}_{\text{loc}}(U)$) the space of all finite Radon measures (resp. the space of all Radon measures) on U . In general, the space $\mathcal{M}(U)$ (resp. $\mathcal{M}_{\text{loc}}(U)$) is known as the dual of the Banach space $C_0(U)$ (resp. dual of the locally convex space $C_c(U)$).

A function $u \in L^1(U)$ is called a BV -function (resp. BV_{loc} -function) on U if and only if its distributional gradient Du is a finite Radon measure (resp. a Radon measure) on U , namely $Du \in \mathcal{M}(U, \mathbb{R}^d)$ (resp. $Du \in \mathcal{M}_{\text{loc}}(U, \mathbb{R}^d)$), and we denote by $BV(U)$ (resp. $BV_{\text{loc}}(U)$) the space of all BV -functions (resp. BV_{loc} -functions) on U . For any $u \in BV(U)$, the total variation measure $|Du| \in \mathcal{M}(U)$ of the gradient Du is called *the total variation measure of u* . Then, by [3, Proposition 3.6], we have,

$$|Du|(U) = \sup \left\{ \int_U u \operatorname{div} \varphi \, dx \mid \varphi \in C_c^1(U, \mathbb{R}^d) \text{ and } |\varphi| \leq 1 \text{ on } U \right\},$$

and we also write $\int_U |Du|$ for $|Du|(U)$.

As a function space, $BV(U)$ is a Banach space, endowed with the norm:

$$\|u\|_{BV(U)} := \|u\|_{L^1(U)} + |Du|(U), \text{ for any } u \in BV(U).$$

For any $u \in BV(U)$, we denote by Du^a (respectively, Du^s), the absolutely continuous part (respectively, the singular part of Du) with respect to $\mathcal{L}^d$. Consequently, one can observe that:

$$Du = Du^a + Du^s = \nabla u \mathcal{L}^d + \frac{Du^s}{|Du^s|} |Du^s| \quad \text{in } \mathcal{M}(U).$$

Here, $\frac{Du^s}{|Du^s|}$ denotes the Radon–Nikodým derivative of Du^s with respect to the total variation measure $|Du^s|$, and ∇u is the approximate differential of $u \in BV(U)$ (cf. [3, Definition 3.70]).

There exists a unique linear operator $\gamma_{\partial U} : BV(U) \rightarrow L^1(\partial U)$, called *trace operator* such that $\gamma_{\partial U} \varphi = \varphi|_{\partial U}$ on ∂U for any $u \in C^1(\bar{U})$.

Notations for the variational analysis (cf. [6]). Throughout this paper, let $N \in \mathbb{N}$ be a fixed dimension, let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $\Gamma \subset \mathbb{R}^N$ be a subset of the boundary $\partial\Omega$ which possibly coincides with the whole $\partial\Omega$. Also, we assume that the boundary $\partial\Omega$ has a smoothness of C^1 -class, and we simply denote by $\nu : \partial\Omega \rightarrow \mathbb{S}^{N-1}$ the unit outer normal on $\partial\Omega$, when $N > 1$ and $\gamma : BV(\Omega) \rightarrow L^1(\partial\Omega)$ is the trace onto $\partial\Omega$. On this basis, we define:

$$L_{\text{div}}^p(\Omega, \mathbb{R}^N) := \left\{ \omega \in L^p(\Omega, \mathbb{R}^N) \mid \text{div } \omega \in L^p(\Omega) \right\},$$

and $X_p(\Omega) := L_{\text{div}}^p(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$, for any $1 \leq p < \infty$.

Also, referring to the general theory for BV -functions, e.g. [6, Sections 1–2], we recall the following facts.

There exists a bounded linear operator $[(\cdot) \cdot \nu] : X_2(\Omega) \rightarrow L^\infty(\partial\Omega)$, such that

$$\begin{cases} \|[\omega \cdot \nu]\|_\infty \leq \|\omega\|_\infty \text{ for any } \omega \in X_2(\Omega), \\ [\tilde{\omega} \cdot \nu] = \tilde{\omega} \cdot \nu \text{ on } \partial\Omega, \text{ if } \tilde{\omega} \in C^1(\bar{\Omega}, \mathbb{R}^N). \end{cases} \quad (2.1)$$

Besides, for every $\omega \in X_2(\Omega)$ and $u \in BV(\Omega) \cap L^2(\Omega)$, there exists a finite Radon measure $(\omega, Du) \in \mathcal{M}(\Omega)$, such that (ω, Du) is absolutely continuous for $|Du|$,

$$\left| \frac{(\omega, Du)}{|Du|} \right| \leq \|\omega\|_\infty, \quad |Du|\text{-a.e. in } \Omega, \quad (2.2)$$

and

$$\int_\Omega (\omega, Du) = - \int_\Omega \text{div } \omega u \, dx + \int_{\partial\Omega} [\omega \cdot \nu] \gamma u \, d\mathcal{H}^{N-1}. \quad (2.3)$$

Moreover, for the absolutely continuous part $(\omega, Du)^a$ of (ω, Du) for $\mathcal{L}^N$ and the singular part $(\omega, Du)^s$, it follows that:

$$(\omega, Du) = (\omega, Du)^a + (\omega, Du)^s = \omega \cdot \nabla u \mathcal{L}^N + \frac{(\omega, Du)}{|Du|} |Du^s| \text{ in } \mathcal{M}(\Omega). \quad (2.4)$$

3 Energy and its lower semi-continuity

We want to write (1.1) as a gradient flow for a suitable energy functional E . We choose the following Hilbert space $H = L^2(\Omega) \times L^2(\Gamma)$ with the standard inner product,

$$((u_1, v_1), (u_2, v_2)) = (u_1, u_2)_\Omega + (v_1, v_2)_\Gamma,$$

where we write $(u_1, u_2)_\Omega = \int_\Omega u_1 u_2 dx$ and $(v_1, v_2)_\Gamma = \int_\Gamma v_1 v_2 d\mathcal{H}^{N-1}$. We define a functional $E : H \rightarrow [0, \infty]$, by setting:

$$E(u, v) = \begin{cases} \int_\Omega |Du| + \int_\Gamma |\gamma u - v| d\mathcal{H}^{N-1} & \text{if } (u, v) \in BV(\Omega) \times L^2(\Gamma), \\ +\infty & \text{otherwise.} \end{cases} \quad (3.1)$$

Remark 3.1. We could consider a more general, one-homogeneous function Φ in place of $|\cdot|$ above. However, this would create another layer of difficulty obscuring the main issue. On the other hand feasibility of such approach is suggested by Moll's paper [42].

The first step is to show that, E defined above, is lower semi-continuous in the L^2 topology.

Proposition 3.1. Let us suppose that $u \in BV(\Omega)$, $v \in L^2(\Gamma)$ and $\{(u_n, v_n)\}_{n=0}^\infty \subset H$ is any sequence converging to (u, v) in H . Then,

$$\liminf_{n \rightarrow \infty} E(u_n, v_n) \geq E(u, v).$$

Proof. If $\Gamma = \partial\Omega$, then this fact is well-known, see [28]. However, the definition of E includes integration over Γ , which may be essentially smaller than $\partial\Omega$, thus we prefer to include the proof. We use here the idea of Giaquinta-Modica-Souček, [28], to extend the functional $\int_\Omega |Du|$ to a bigger domain. We proceed by taking any region $\tilde{\Omega}$ with Lipschitz boundary and such that the following conditions hold:

- 1) $\Omega \subset \tilde{\Omega}$;
- 2) $\partial\tilde{\Omega} \cap \partial\Omega = \partial\Omega \setminus \Gamma$;
- 3) the region $\tilde{\Omega} \setminus \bar{\Omega}$ has a Lipschitz continuous boundary.

When $\phi \in L^1(\partial\Omega)$ is given, then we may find $\tilde{\phi} \in W^{1,1}(\tilde{\Omega} \setminus \bar{\Omega})$ such that $\gamma\tilde{\phi} = \phi$ on Γ , see [6, 19]. Then, we define the following space,

$$BV_{\Gamma, \phi}(\tilde{\Omega}) = \{u \in BV(\tilde{\Omega}) : u = \tilde{\phi} \text{ on } \tilde{\Omega} \setminus \bar{\Omega}\}.$$

It is a well-known fact that functional $BV(\tilde{\Omega}) \ni u \mapsto \int_{\tilde{\Omega}} |Du|$ is lower semi-continuous with respect to the L^2 . As a result, this functional is lower semi-continuous on $BV_{\Gamma, \phi}(\tilde{\Omega})$, a closed subspace of $BV(\tilde{\Omega})$. Once we realize that for $u \in BV_{\Gamma, \phi}(\tilde{\Omega})$, we have

$$Du = Du \llcorner \Omega + Du \llcorner \Gamma + \nabla u \llcorner \tilde{\Omega} \setminus \bar{\Omega},$$

where $Du \llcorner \Gamma = \nu(\phi - \gamma u) \mathcal{H}^{N-1} \llcorner \Gamma$, where γu is the trace of $u \in BV(\Omega)$, then

$$\int_{\tilde{\Omega}} |Du| = \int_\Omega |Du| + \int_\Gamma |\phi - \gamma u| d\mathcal{H}^{N-1} + \int_{\tilde{\Omega} \setminus \bar{\Omega}} |\nabla \tilde{\phi}| dx.$$

As a result, the functional $L^2(\Omega) \ni u \mapsto E(u, v) =: E_v(u)$ is lower semi-continuous. In order to complete the task, we have to consider $\liminf_{n \rightarrow \infty} E(u_n, v_n)$, when $(u_n, v_n) \rightarrow (u, v)$ in H . Since $|\gamma u_n - v_n| + |v_n - v| \geq |\gamma u_n - v|$, then we see,

$$\liminf_{n \rightarrow \infty} E(u_n, v_n) \geq \liminf_{n \rightarrow \infty} E_v(u_n) - \lim_{n \rightarrow \infty} \int_\Gamma |v_n - v| d\mathcal{H}^{N-1}.$$

Finally, our claim follows. □

Remark 3.2. We noticed in the course of the proof above that for a fixed $v \in L^2(\Gamma)$, functional $E_v(u)$ is lower semi-continuous. It is a relaxation, i.e. the lower semi-continuous envelope, of the following functional

$$L^2(\Omega) \ni u \mapsto E_v^\infty(u) = \begin{cases} \int_\Omega |Du| & \text{if } u \in BV(\Omega), \gamma|_\Gamma u = v, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, for any $a > 1$ functional $E_v^a : L^2(\Omega) \longrightarrow [0, \infty]$, given by

$$E_v^a(u) = \begin{cases} \int_\Omega |Du| + a \int_\Gamma |\gamma u - v| d\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

is not lower semi-continuous, because $E_v < E_v^a \leq E_v^\infty$ on $L^2(\Omega)$. The relaxation of E_v^a is E_v . On the other hand, it is easy to check that for any $a \in (0, 1)$ functional E_v^a is lower semi-continuous. Indeed, in this case $E_v^a = aE_v + (1-a) \int_\Omega |Du|$ and both ingredients are lower semi-continuous.

Remark 3.3. We recall that lower semi-continuity of E combined with its convexity implies sequential weak lower semi-continuity.

4 The evolution problem and the Comparison Principle

We recall two basic abstract facts from the theory of maximal monotone operators.

Theorem 4.1 (Well-posedness). Let $\mathcal{E} : X \longrightarrow [0, \infty]$ be a proper, lower semi-continuous, convex function on a Hilbert space X . Then, for any $w_0 \in D(\mathcal{E})$ there is a unique solution $w \in W_{loc}^{1,2}([0, \infty); X)$, such that

$$\frac{dw}{dt}(t) \in -\partial\mathcal{E}(w(t)), \quad \text{a.e. } t \in (0, \infty), \quad \text{with } w(0) = w_0, \quad \text{in } X.$$

Also, the function $[0, \infty) \ni t \mapsto \mathcal{E}(w(t)) \in [0, \infty)$ is absolutely continuous on any compact interval, and it satisfies that

$$\frac{d}{dt}\mathcal{E}(w(t)) = -\left\|\frac{dw}{dt}(t)\right\|_X^2 \quad \text{a.e. } t > 0.$$

In particular, if $w_0 \in D(\partial\mathcal{E})$, then $w \in W_{loc}^{1,\infty}([0, \infty); X)$, w is right-differentiable over $[0, \infty)$, and at every $t \geq 0$, the right derivative $\frac{d^+w}{dt}(t)$ satisfies

$$\frac{d^+w}{dt}(t) = -\partial^\circ\mathcal{E}(w(t)) \quad \text{in } X,$$

where $\partial^\circ\mathcal{E}$ denotes the minimal section of $\partial\mathcal{E}$.

This type of well-posedness of the gradient flow of a convex functional goes back to Kōmura. The above version can be found in Brézis, see [15] or Pazy, see [44]. Here, we note that $W_{loc}^{1,2}([0, \infty); X)$ and $W_{loc}^{1,\infty}([0, \infty); X)$ are contained in the class $W_{loc}^{1,1}(0, \infty; X)$ of all absolute continuous functions in (δ, T) for any $T > \delta > 0$ with values in the Hilbert space X .

In order to proceed, we recall the notion of Banach lattice. An ordered Banach space X with ordering $\geq$ is called a *vector lattice*, if the linear structure is compatible with the ordering, i.e.

$$\begin{aligned} f \geq g \text{ implies } f + h &\geq g + h \quad \text{for all } f, g, h \in X; \\ f \geq 0 \text{ implies } \lambda f &\geq 0 \quad \text{for all } f \in X, \lambda \geq 0. \end{aligned}$$

In addition, we require that any two elements $f, g \in X$ have a supremum, denoted by $f \vee g$ and infimum, denoted by $f \wedge g$. Besides, for $w \in X$, we denote by w_+ its positive part, i.e. $w_+ = w \vee 0 = \max(w, 0)$. We refer the interested reader for more details on the Banach lattice to [7].

Proposition 4.2 (Order preserving structure). Assume that a Hilbert space X is a vector lattice. Let us suppose that

$$\frac{d}{dt} \|\omega_+(t)\|_X^2 = 2(\omega_+(t), \omega'(t)) \quad \text{for any } \omega \in W_{loc}^{1,1}(0, \infty; X).$$

Let $\mathcal{E}$ in Theorem 4.1 fulfills

$$\mathcal{E}(w_1 \vee w_2) + \mathcal{E}(w_1 \wedge w_2) \leq \mathcal{E}(w_1) + \mathcal{E}(w_2) \quad \text{for all } w_1, w_2 \in D(\mathcal{E}).$$

If w_1 and w_2 are two solutions (in the sense of Theorem 4.1) of

$$\frac{dw}{dt}(t) \in -\partial\mathcal{E}(w(t)) \quad \text{in } X, \quad \text{a.e. } t > 0, \quad (4.1)$$

and if the initial data w_{10} and w_{20} satisfy

$$w_1(0) = w_{10} \leq w_{20} = w_2(0) \quad \text{in } X,$$

then

$$w_1(t) \leq w_2(t) \quad \text{in } X, \quad \text{for all } t > 0.$$

This type of argument is well-known. For example it is presented in the thesis of Brézis [14] and more generally in Kenmochi-Mizuta-Nagai, see [37]. We give here a proof since it is elementary.

Proof. By definition, we see that for a.e. $t > 0$

$$\begin{aligned} \mathcal{E}(\varphi) - \mathcal{E}(w_1(t)) &\geq (w'_1(t), w_1(t) - \varphi)_X \quad \text{for all } \varphi \in X, \\ \mathcal{E}(\hat{\varphi}) - \mathcal{E}(w_2(t)) &\geq (w'_2(t), w_2(t) - \hat{\varphi})_X \quad \text{for all } \hat{\varphi} \in X. \end{aligned}$$

In these inequalities, we take

$$\begin{cases} \varphi = w_1(t) + (w_2 - w_1)_+(t) = (w_1 \vee w_2)(t), \\ \hat{\varphi} = w_2(t) - (w_2 - w_1)_+(t) = (w_1 \wedge w_2)(t); \end{cases}$$

the last identities follow from the property of a vector lattice. Then one gets

$$\begin{aligned}\mathcal{E}((w_1 \vee w_2)(t)) - \mathcal{E}(w_1(t)) &\geq (w_1'(t), -(w_2 - w_1)_+(t))_X, \\ \mathcal{E}((w_1 \wedge w_2)(t)) - \mathcal{E}(w_2(t)) &\geq (w_2'(t), (w_2 - w_1)_+(t))_X\end{aligned}$$

for a.e. $t > 0$. Adding these two inequalities and invoking our assumption for $\mathcal{E}$ with respect to $\wedge$ and $\vee$, we see that

$$0 \geq ((w_2' - w_1')(t), (w_2 - w_1)_+(t))_X = \frac{1}{2} \frac{d}{dt} \|(w_2 - w_1)_+(t)\|_X^2 \quad \text{a.e. } t > 0.$$

We thus conclude that

$$\frac{d}{dt} \|(w_2 - w_1)_+(t)\|_X^2 \leq 0 \quad \text{a.e. } t > 0.$$

Thus, the order preserving property follows. $\square$

Remark 4.1. From the proof above, it is easy to claim a comparison principle saying that if w_1 is a subsolution and w_2 is a supersolution of (4.1), then $w_1 \leq w_2$ provided that $w_1(0) \leq w_2(0)$. Here, we say $w \in W_{loc}^{1,2}([0, \infty); X)$ is a *subsolution* if for a.e. $t > 0$ the inequality

$$\mathcal{E}(w(t) + h) - \mathcal{E}(w(t)) \geq (-w'(t), h)_X \quad \text{for all } h \in X \text{ and } h \geq 0$$

is fulfilled. A supersolution is defined in a symmetric way.

We now consider the gradient flow of E defined in (3.1) in a Hilbert space $H = L^2(\Omega) \times L^2(\Gamma)$ equipped with an inner product

$$((f_1, f_2), (g_1, g_2))_\tau = \int_\Omega f_1 g_1, dx + \tau \int_\Gamma f_2 g_2 d\mathcal{H}^{N-1}$$

for $f = (f_1, f_2), g = (g_1, g_2) \in H$. Here, $\tau > 0$ is a fixed parameter. The topology defined by the inner product $(\cdot, \cdot)_\tau$ is the same but its gradient flow is different. Formally, the gradient flow with respect to the $(\cdot, \cdot)_\tau$ inner product reads as eq. (1.1). Since it is clear that E in (3.1) is convex, lower semi-continuous with respect to the convergence in the standard inner product as well as with respect to $(\cdot, \cdot)_\tau$ and $H = \overline{D(E)}$, Proposition 3.1 enables us to apply Theorem 4.1 to get a well-posedness result.

Theorem 4.3. For any $U_0 := (u_0, v_0) \in D(E) = (BV(\Omega) \cap L^2(\Omega)) \times L^2(\Gamma)$ there is a unique solution $U := (u, v) \in W_{loc}^{1,2}([0, \infty); H)$ of the following problem

$$\begin{aligned}\frac{dU}{dt}(t) &:= \frac{d}{dt} (u(t), v(t)) \in -\partial_\tau E(U(t)) \equiv -\partial_\tau E(u(t), v(t)) \quad \text{in } H, \quad \text{a.e. } t > 0, \\ U(0) &\equiv (u(0), v(0)) = (u_0, v_0) \quad \text{in } H,\end{aligned} \quad (4.2)$$

where $\partial_\tau E$ denotes the subdifferential of E with respect to the inner product $(\cdot, \cdot)_\tau$. Also, the function $t \in [0, \infty) \rightarrow E(U(t)) \equiv E(u(t), v(t)) \in [0, \infty)$ is absolutely continuous on any compact interval, and it satisfies that

$$\begin{aligned}\frac{d}{dt} E(U(t)) &= - \left(\frac{dU}{dt}(t), \frac{dU}{dt}(t) \right)_\tau \\ &= - \left\| \frac{du}{dt}(t) \right\|_{L^2(\Omega)}^2 - \tau \left\| \frac{dv}{dt}(t) \right\|_{L^2(\Gamma)}^2 \quad \text{a.e. } t > 0.\end{aligned} \quad (4.3)$$

In particular, if $U_0 \in D(\partial_\tau E)$, then $U = (u, v) \in W_{loc}^{1,\infty}([0, \infty); H)$, U is right-differentiable over $[0, \infty)$, and at every $t > 0$, the right derivative $\frac{d^+ U}{dt}(t) := \frac{d^+}{dt}(u(t), v(t))$ satisfies

$$\frac{d^+}{dt}U(t) = -\partial_\tau^\circ E(U(t)) \quad \text{in } H, \quad \text{a.e. } t > 0. \quad \square$$

We notice that H has the desired lattice structure after we define

$$(f_1, f_2) \leq (g_1, g_2) \quad \text{for } f = (f_1, f_2), g = (g_1, g_2) \in H$$

if $f_1 \leq f_2$ a.e. in Ω and $g_1 \leq g_2$ $\mathcal{H}^{N-1}$ -a.e. on Γ . We also check that the functional E has the desired properties:

Proposition 4.4. If E is defined by formula (3.1), then

$$E(u_1 \wedge u_2, v_1 \wedge v_2) + E(u_1 \vee u_2, v_1 \vee v_2) \leq E(u_1, v_1) + E(u_2, v_2).$$

Proof. First, by referring to [48, Lemmas 2.2 and 3.1], we verify that

$$\int_\Omega |D(u_1 \vee u_2)| + \int_\Omega |D(u_1 \wedge u_2)| \leq \int_\Omega |Du_1| + \int_\Omega |Du_2|.$$

We also have to show that

$$\int_\Gamma (|u_1 \vee u_2 - v_1 \vee v_2| + |u_1 \wedge u_2 - v_1 \wedge v_2|) d\mathcal{H}^{N-1} = \int_\Gamma (|u_1 - v_1| + |u_2 - v_2|) d\mathcal{H}^{N-1}$$

where we identified $u_i, i = 1, 2$ with their traces on Γ .

Since the roles of u_1 and u_2 are interchangeable, we may assume that $u_1 \vee u_2 = u_1$ and $u_1 \wedge u_2 = u_2$. If $v_1 \geq v_2$, then there is nothing to prove, thus we may assume that $v_1 < v_2$. Finally, we have to check that

$$|u_1 - v_2| + |u_2 - v_1| = |u_1 - v_1| + |u_2 - v_2| \quad \text{for a.e. } x \in \Gamma.$$

However, this obviously holds for all u_2 . $\square$

The result we have just proved permits us to apply Proposition 4.2 to conclude the order preserving property.

Theorem 4.5 (Order preserving property). Let $U_i = (u_i, v_i), i = 1, 2$, be a solution in Theorem 4.3 starting from $U_{i_0} = (u_{i_0}, v_{i_0}) \in H$. If $U_{10} \leq U_{20}$, then $U_1(t) \leq U_2(t)$ for all $t > 0$, i.e., $u_1(t) \leq u_2(t), v_1(t) \leq v_2(t)$ for all $t > 0$.

5 The subdifferential and its canonical section

This section is devoted to the characterizations of the subdifferential of $E(u, v)$ given by (3.1) and its canonical section. Even though eq. (4.2) contains a parameter $\tau > 0$, we shall see that without the loss of generality, it is sufficient to calculate the subdifferential of $E(u, v)$ with respect to the standard inner product of H .

5.1 The representation of the subdifferential

The goal of this subsection is to prove the following proposition.

Theorem 5.1 (Representation of the subdifferential). Let $E(u, v)$ be given by (3.1), as a result it is a proper, lower semi-continuous and convex function on H . Then, for pairs of functions $(u, v) \in H$ and $(\xi, \zeta) \in H$, the following two statements are equivalent.

- (A) $(\xi, \zeta) \in \partial E(u, v)$ in H when $(u, v) \in D(\partial E)$.
- (B) $(u, v) \in BV(\Omega) \times L^2(\Gamma)$, and there exists a vector field $z \in X_2(\Omega)$, such that:
 - (b1) $\frac{(z, Du)}{|Du|} = 1$, $|Du|$ -a.e. in Ω , and moreover, $z \in \text{Sgn}^N(\nabla u)$ a.e. in Ω ;
 - (b2) $-[z \cdot \nu] \in \text{Sgn}(\gamma u - v)$ a.e. on Γ , and $[z \cdot \nu] = 0$ a.e. on $\partial\Omega \setminus \Gamma$;
 - (b3) $\xi = -\text{div } z$ in $L^2(\Omega)$, and $\zeta = [z \cdot \nu]$ in $L^2(\Gamma)$;

where Sgn is the abbreviation of the set-valued function $\text{Sgn}^1 : \mathbb{R} \rightarrow 2^{\mathbb{R}}$, defined in Remark 2.1, when $d = 1$.

For the proof of this proposition, we first prepare some additional notations with an auxiliary lemma.

Operator $\mathcal{A}$. We define a set-valued operator $\mathcal{A} \subset H \times H$ by letting:

$$(u, v) \in H \mapsto \mathcal{A}(u, v) := \left\{ (\xi, \zeta) \in H \left| \begin{array}{l} (\xi, \zeta) \text{ as in condition (b3), where } z \in \\ X_2(\Omega) \text{ satisfies conditions (b1)–(b2) in} \\ \text{Theorem 5.1} \end{array} \right. \right\}, \quad (5.1)$$

and we denote by $D(\mathcal{A})$ the domain of this operator, i.e.

$$D(\mathcal{A}) := \left\{ (\tilde{u}, \tilde{v}) \in H \mid \mathcal{A}(\tilde{u}, \tilde{v}) \neq \emptyset \right\}.$$

Relaxed convex function $E_\varepsilon(u, v)$. We define a sequence $\{E_\varepsilon\}_{\varepsilon>0}$ of lower semi-continuous, convex functionals $E_\varepsilon : H \rightarrow [0, \infty]$, by letting for any $\varepsilon > 0$,

$$E_\varepsilon(u, v) := \begin{cases} \int_{\Omega} \sqrt{|\nabla u|^2 + \varepsilon^2} dx + \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla u|^2 dx, \\ \quad \text{if } u \in H^1(\Omega) \text{ and } \gamma u = v \text{ a.e. on } \Gamma, \\ \infty, \text{ otherwise,} \end{cases} \quad (5.2)$$

Note that for every $\varepsilon > 0$, E_ε are proper on H . Also,

$$D(E_\varepsilon) = V := \left\{ (\tilde{u}, \tilde{v}) \in H^1(\Omega) \times H^1(\Gamma) \mid \gamma \tilde{u} = \tilde{v} \text{ a.e. on } \Gamma \right\}, \text{ for } \varepsilon > 0, \quad (5.3)$$

namely, the effective domains $D(E_\varepsilon)$, for $\varepsilon > 0$, are equal to a closed linear subspace V in $H^1(\Omega) \times H^1(\Gamma)$. The equality in (5.3) is essential to guarantee the lower semi-continuity of the convex functions E_ε , for $\varepsilon > 0$.

Lemma 5.1. Let us fix any constant $\varepsilon > 0$, and let us set:

$$D_\varepsilon := \left\{ (\tilde{u}, \tilde{v}) \in D(E_\varepsilon) \left| \begin{array}{l} \frac{\nabla \tilde{u}}{\sqrt{|\nabla \tilde{u}|^2 + \varepsilon^2}} + \varepsilon^2 \nabla \tilde{u} \in L^2_{\text{div}}(\Omega, \mathbb{R}^N), \\ [(\frac{\nabla \tilde{u}}{\sqrt{|\nabla \tilde{u}|^2 + \varepsilon^2}} + \varepsilon^2 \nabla \tilde{u}) \cdot \nu] \in L^2(\partial\Omega), \text{ and} \\ [(\frac{\nabla \tilde{u}}{\sqrt{|\nabla \tilde{u}|^2 + \varepsilon^2}} + \varepsilon^2 \nabla \tilde{u}) \cdot \nu] = 0 \text{ a.e. on } \partial\Omega \setminus \Gamma \end{array} \right. \right\}.$$

Then, the subdifferential $\partial E_\varepsilon \subset H \times H$ coincides with a single-valued operator $\mathcal{A}_\varepsilon \subset H \times H$, defined as follows:

$$(u, v) \in D_\varepsilon \subset H \mapsto \mathcal{A}_\varepsilon(u, v) := \begin{pmatrix} -\text{div} \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon^2}} + \varepsilon^2 \nabla u \right) \\ [(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon^2}} + \varepsilon^2 \nabla u) \cdot \nu] \end{pmatrix} \in H. \quad (5.4)$$

Proof. This lemma can be obtained as a straightforward consequence of standard variational methods (cf. [9, 22]). $\square$

Proof of Theorem 5.1. With the use of the operator $\mathcal{A}$ given by (5.1), the conclusion of the proposition can be rephrased as follows:

$$\mathcal{A} = \partial E \text{ in } H \times H. \quad (5.5)$$

We check this equality with the help of the following two Claims #1–#2.

Claim #1: $\mathcal{A} \subset \partial E$ in $H \times H$. Let us assume that:

$$(\xi, \zeta) \in \mathcal{A}(u, v) \text{ in } H \text{ with } (u, v) \in D(\mathcal{A}).$$

Then, in the light of (2.3) and (5.1), we can see that for any $(\varphi, \psi) \in D(E)$ we have:

$$\begin{aligned} & ((\xi, \zeta), (\varphi, \psi) - (u, v)) \\ &= \int_{\Omega} -\text{div } z (\varphi - u) dx + \int_{\Gamma} [z \cdot \nu] (\psi - v) d\mathcal{H}^{N-1} \\ &= \int_{\Omega} (z, D(\varphi - u)) - \int_{\partial\Omega} [z \cdot \nu] \gamma(\varphi - u) d\mathcal{H}^{N-1} + \int_{\Gamma} [z \cdot \nu] (\psi - v) d\mathcal{H}^{N-1} \\ &\leq \|z\|_{\infty} \int_{\Omega} |D\varphi| - \int_{\Omega} |Du| + \int_{\Gamma} (|\gamma\varphi - \psi| - |\gamma u - v|) d\mathcal{H}^{N-1} \\ &\leq E(\varphi, \psi) - E(u, v). \end{aligned}$$

Thus, $(\xi, \zeta) \in \partial E(u, v)$ in H .

Claim #2: $(\mathcal{A} + \mathcal{I}_H)H = H$. Since the inclusion $(\mathcal{A} + \mathcal{I}_H)H \subset H$ is trivial, our task can be reduced to show only the converse one.

Let us fix any $(f, g) \in H$. Then, applying Minty's theorem and Lemma 5.1, we can find a sequence of functions $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0} \subset V$ such that:

$$(f, g) - (u_\varepsilon, v_\varepsilon) \in \partial E_\varepsilon(u_\varepsilon, v_\varepsilon) \text{ in } H, \text{ for all } \varepsilon > 0. \quad (5.6)$$

Here, with (5.2) and Lemma 5.1 in mind, we multiply the both sides of (5.6) by $(u_\varepsilon, v_\varepsilon)$. Then, by using Young's inequality, one can immediately see that:

$$\begin{aligned} \frac{1}{2} \|(u_\varepsilon, v_\varepsilon)\|_H^2 + E_\varepsilon(u_\varepsilon, v_\varepsilon) &\leq \frac{1}{2} \|(f, g)\|_H^2 + E_\varepsilon(0, 0) \\ &\leq \frac{1}{2} \|(f, g)\|_H^2 + \varepsilon \mathcal{L}^N(\Omega), \text{ for all } \varepsilon > 0. \end{aligned} \quad (5.7)$$

Subsequently, invoking (3.1), (5.2), (5.7) and the compactness theorem of Rellich–Kondrashov type, we can find an approximating limit $(u, v) \in D(E)$ together with a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0, 1)$ and a sequence of functions $\{(u_n, v_n)\}_{n=1}^\infty := \{(u_{\varepsilon_n}, v_{\varepsilon_n})\}_{n=1}^\infty \subset V$, such that:

$$1 > \varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n \downarrow 0 \text{ as } n \rightarrow \infty, \quad (5.8)$$

$$\begin{cases} (u_n, v_n) \rightarrow (u, v) \text{ weakly in } H, \\ u_n \rightarrow u \text{ in } L^1(\Omega), \\ \varepsilon_n u_n \rightarrow 0 \text{ weakly in } H^1(\Omega), \end{cases} \quad \text{as } n \rightarrow \infty, \quad (5.9)$$

and for any $(\varphi, \psi) \in V$,

$$\begin{aligned} \int_{\Omega} \frac{\nabla u_n}{\sqrt{|\nabla u_n|^2 + \varepsilon_n^2}} \cdot \nabla \varphi \, dx + \int_{\Omega} \nabla(\varepsilon_n u_n) \cdot \nabla(\varepsilon_n \varphi) \, dx \\ = \int_{\Omega} (f - u_n) \varphi \, dx + \int_{\Gamma} (g - v_n) \psi \, d\mathcal{H}^{N-1}, \quad n \in \mathbb{N}. \end{aligned} \quad (5.10)$$

Additionally, since

$$\left| \frac{\nabla u_n}{\sqrt{|\nabla u_n|^2 + \varepsilon_n^2}} \right| \leq 1 \text{ a.e. in } \Omega, \text{ for } n = 1, 2, 3, \dots,$$

we may assume existence of a vector field $z \in L^\infty(\Omega, \mathbb{R}^N)$ such that

$$\begin{aligned} \frac{\nabla u_n}{\sqrt{|\nabla u_n|^2 + \varepsilon_n^2}} &\rightarrow z \text{ weakly-* in } L^\infty(\Omega, \mathbb{R}^N) \text{ as } n \rightarrow \infty, \\ \text{and } |z| &\leq 1 \text{ a.e. in } \Omega, \end{aligned} \quad (5.11)$$

by taking an additional subsequence, if necessary.

Now, applying the convergences (5.8)–(5.9) and (5.11) to the variational form (5.10), we can see that for any $(\varphi, \psi) \in V$, we have:

$$\int_{\Omega} z \cdot \nabla \varphi \, dx = \int_{\Omega} (f - u) \varphi \, dx + \int_{\Gamma} (g - v) \psi \, d\mathcal{H}^{N-1}. \quad (5.12)$$

In particular, taking any $\varphi_0 \in H_0^1(\Omega)$ and putting $(\varphi, \psi) = (\varphi_0, 0) \in V$ in (5.12), we have:

$$\int_{\Omega} z \cdot \nabla \varphi_0 \, dx = \int_{\Omega} (f - u) \varphi_0 \, dx, \text{ for any } \varphi_0 \in H_0^1(\Omega),$$

i.e.

$$-\operatorname{div} z = f - u \in L^2(\Omega) \text{ in } H^{-1}(\Omega). \quad (5.13)$$

Subsequently, for any $\tilde{\psi} \in H^1(\partial\Omega)$, we invoke [8, Proposition 5.6.3] with the C^1 -smoothness of $\partial\Omega$ to take an extension $\tilde{\psi}^{\text{ex}} \in H^1(\Omega)$ of $\tilde{\psi}$. Then, putting $(\varphi, \psi) = (\tilde{\psi}^{\text{ex}}, \tilde{\psi}) \in V$ in (5.12), we deduce from (5.13) that for any $\tilde{\psi} \in H^1(\partial\Omega)$, we have,

$$\int_{\partial\Omega} [z \cdot \nu] \tilde{\psi} \, d\mathcal{H}^{N-1} = \int_{\Gamma} (g - v) \tilde{\psi} \, d\mathcal{H}^{N-1} = \int_{\partial\Omega} [g - v]_0^{\text{ex}} \tilde{\psi} \, d\mathcal{H}^{N-1},$$

where $[g - v]_0^{\text{ex}} \in L^2(\partial\Omega)$ is the zero-extension of $g - v \in L^2(\Gamma)$. This implies that,

$$[z \cdot \nu](y) = \begin{cases} (g - v)(y), & \text{if } y \in \Gamma, \\ 0, & \text{if } y \in \partial\Omega \setminus \Gamma, \end{cases} \text{ for a.e. } y \in \partial\Omega. \quad (5.14)$$

Finally, we take any $(\tilde{\varphi}, \tilde{\psi}) \in V$, and put $(\varphi, \psi) = (u_n - \tilde{\varphi}, v_n - \tilde{\psi}) \in V$ in (5.10) to obtain that:

$$\begin{aligned} E_{\varepsilon_n}(u_n, v_n) &+ \int_{\Omega} (u_n - f)(u_n - \tilde{\varphi}) \, dx + \int_{\Gamma} (v_n - g)(v_n - \tilde{\psi}) \, d\mathcal{H}^{N-1} \\ &\leq \int_{\Omega} \frac{\nabla u_n}{\sqrt{|\nabla u_n|^2 + \varepsilon_n^2}} \cdot \nabla \tilde{\varphi} \, dx + \frac{\varepsilon_n^2}{2} \int_{\Omega} |\nabla \tilde{\varphi}|^2 \, dx + E_{\varepsilon_n}(0, 0) \\ &= \int_{\Omega} \frac{\nabla u_n}{\sqrt{|\nabla u_n|^2 + \varepsilon_n^2}} \cdot \nabla \tilde{\varphi} \, dx + \frac{\varepsilon_n^2}{2} \int_{\Omega} |\nabla \tilde{\varphi}|^2 \, dx + \varepsilon_n \mathcal{L}^n(\omega), \text{ for } n = 1, 2, 3, \dots \end{aligned} \quad (5.15)$$

Here, having in mind (3.1), (5.2), (5.8)–(5.9), (5.11) and the weak lower semi-continuity of E on H , let us take the limit-inf of both sides of (5.15). Then, we compute,

$$\begin{aligned} E(u, v) &+ \int_{\Omega} (u - f)(u - \tilde{\varphi}) \, dx + \int_{\Gamma} (v - g)(v - \tilde{\psi}) \, d\mathcal{H}^{N-1} \\ &\leq \liminf_{n \rightarrow \infty} E_{\varepsilon_n}(u_n, v_n) + \liminf_{n \rightarrow \infty} \int_{\Omega} (u_n - f)(u_n - \tilde{\varphi}) \, dx + \liminf_{n \rightarrow \infty} \int_{\Gamma} (v_n - g)(v_n - \tilde{\psi}) \, d\mathcal{H}^{N-1} \\ &\leq \liminf_{n \rightarrow \infty} \left(E_{\varepsilon_n}(u_n, v_n) - \int_{\Omega} (f - u_n)(u_n - \tilde{\varphi}) \, dx - \int_{\Gamma} (g - v_n)(v_n - \tilde{\psi}) \, d\mathcal{H}^{N-1} \right) \\ &\leq \int_{\Omega} z \cdot \nabla \tilde{\varphi} \, dx. \end{aligned}$$

Therefore, for any $(\tilde{\varphi}, \tilde{\psi}) \in V$, we have,

$$\begin{aligned} &\int_{\Omega} |Du| + \int_{\Gamma} |\gamma u - v| \, d\mathcal{H}^{N-1} \\ &\leq \int_{\Omega} z \cdot \nabla \tilde{\varphi} \, dx + \int_{\Omega} (f - u)(u - \tilde{\varphi}) \, dx + \int_{\Gamma} (g - v)(v - \tilde{\psi}) \, d\mathcal{H}^{N-1}. \end{aligned} \quad (5.16)$$

Additionally, by applying (2.3)–(2.4), (5.13)–(5.14) and (5.16), we can deduce that:

$$\begin{aligned}
& \int_{\Omega} |Du| + \int_{\Gamma} |\gamma u - v| d\mathcal{H}^{N-1} = \int_{\Omega} |\nabla u| dx + \int_{\Omega} |Du^s| + \int_{\Gamma} |\gamma u - v| d\mathcal{H}^{N-1} \\
& \leq - \int_{\Omega} \operatorname{div} z \tilde{\varphi} dx + \int_{\partial\Omega} [z \cdot \nu] \gamma \tilde{\varphi} d\mathcal{H}^{N-1} \\
& \quad + \int_{\Omega} (-\operatorname{div} z)(u - \tilde{\varphi}) dx + \int_{\Gamma} [z \cdot \nu](v - \tilde{\psi}) d\mathcal{H}^{N-1} \\
& = - \int_{\Omega} \operatorname{div} z u dx + \int_{\Gamma} [z \cdot \nu] v d\mathcal{H}^{N-1} \\
& = \int_{\Omega} (z, Du) - \int_{\partial\Omega} [z \cdot \nu] \gamma u d\mathcal{H}^{N-1} + \int_{\Gamma} [z \cdot \nu] v d\mathcal{H}^{N-1} \\
& = \int_{\Omega} \frac{(z, Du)}{|Du|} |Du| + \int_{\Gamma} -[z \cdot \nu](\gamma u - v) d\mathcal{H}^{N-1}. \\
& = \int_{\Omega} z \cdot \nabla u dx + \int_{\Omega} (z, Du)^s + \int_{\Gamma} -[z \cdot \nu](\gamma u - v) d\mathcal{H}^{N-1}. \tag{5.17}
\end{aligned}$$

In the meantime, from (2.1)–(2.2), (2.4) and (5.11), we can easily check that:

$$\left\{ \begin{array}{l} \left| \frac{(z, Du)}{|Du|} \right| \leq \|z\|_{\infty} \leq 1, \quad |Du| \text{-a.e. in } \Omega, \\ \text{with } \left| \int_{\Omega} z \cdot \nabla u dx \right| \leq \|z\|_{\infty} \int_{\Omega} |\nabla u| dx \leq \int_{\Omega} |\nabla u| dx, \\ \text{and } \left| \int_{\Omega} (z, Du)^s \right| \leq \int_{\Omega} \left| \frac{(z, Du)}{|Du|} \right| |Du^s| \leq \int_{\Omega} |Du^s|, \end{array} \right. \tag{5.18}$$

and

$$| -[z \cdot \nu] | \leq \|z\|_{\infty} \leq 1, \quad \text{a.e. on } \partial\Omega. \tag{5.19}$$

As a consequence from (5.17)–(5.19), it is inferred that:

$$\left\{ \begin{array}{l} \frac{(z, Du)}{|Du|} = 1, \quad |Du| \text{-a.e. in } \Omega, \text{ and in particular,} \\ \quad z \cdot \nabla u = |\nabla u|, \text{ and } z \in \operatorname{Sgn}^N(\nabla u), \text{ a.e. in } \Omega, \\ -[z \cdot \nu](\gamma u - v) = |\gamma u - v|, \\ \quad \text{i.e. } -[z \cdot \nu] \in \operatorname{Sgn}(\gamma u - v), \text{ a.e. on } \Gamma. \end{array} \right. \tag{5.20}$$

Taking into account (5.1), (5.13)–(5.14) and (5.20), we infer that:

$$\begin{aligned}
& (f - u, g - v) \in \mathcal{A}(u, v), \text{ in } H, \\
& \text{i.e. } (\mathcal{A} + \mathcal{I}_H)(u, v) \ni (f, g) \text{ in } H, \text{ with } (u, v) \in D(\mathcal{A}).
\end{aligned}$$

Indeed, Claim #2 follows.

Now, the rephrased conclusion (5.5) will be obtained by applying Minty's theorem to $\mathcal{A}$, and by using the maximality of the monotone graph $\mathcal{A} \subset \partial E$ in $H \times H$. $\square$

We have just characterized ∂E , the subdifferential of E with respect to the standard inner product of H . This corresponds to eq. (4.2) with $\tau = 1$. We would like to establish the relationship between ∂E and $\partial_\tau E$, i.e. the subdifferential of E with respect to the inner product $(\cdot, \cdot)_\tau$ in H . Thus, we could study (4.2) for any positive τ . Here is our observation.

Corollary 5.1. Let $\tau > 0$, and $U = (u, v) \in H$. Then, the domain $D(\partial E) = D(\mathcal{A})$ coincides with the domain $D(\partial_\tau E)$ of the subdifferential $\partial_\tau E$ of E with respect to the scalar product $(\cdot, \cdot)_\tau$ in H , and $(\xi, \zeta) \in \partial_\tau E(U)$ if and only if $(\xi, \tau\zeta) \in \partial E(U)$ in H .

Proof. We easily verify this lemma by using the following relationship

$$((\xi, \zeta), (h_1, h_2))_\tau = ((\xi, \tau\zeta), (h_1, h_2)) \quad \text{for all } (\xi, \zeta), (h_1, h_2) \in H. \quad \square$$

5.2 The canonical section

Once we described the subdifferential, we may set up the minimization necessary to select the canonical section of $\partial E(U)$. Here, we assume $\tau = 1$, but we shall see later that this does not lead to any loss of generality, see Lemma 6.1.

Proposition 5.2. If (ξ, ζ) is the canonical selection of $\partial E(U)$ with $U = (u, v) \in D(\partial E)$, then:
(a) $(\xi, \zeta) = (-\operatorname{div} z, [z \cdot \nu])$, where z is a minimizer of

$$\min \left\{ \mathcal{E}(z) \left| \begin{array}{l} z \in X_2(\Omega), z \in \operatorname{Sgn}^N(\nabla u), \text{ a.e.}, \\ \frac{(z, Du)}{|Du|} = 1, |Du| \text{-a.e.}, \text{ and} \\ -[z \cdot \nu] \in \operatorname{Sgn}(\gamma u - v), \mathcal{H}^{N-1} \text{-a.e.} \end{array} \right. \right\}, \quad (5.21)$$

where $X_2(\Omega)$ is defined in (2.1) and

$$\mathcal{E}(z) = \int_{\Omega} |\operatorname{div} z|^2 dx + \int_{\Gamma} |[z \cdot \nu]|^2 d\mathcal{H}^{N-1}.$$

Moreover, $\operatorname{div} z$ and $[z \cdot \nu]$ are determined uniquely.

(b) We assume that z is a minimizer of (5.21), $F_0 := \{x \in \Omega : |z(x)| < 1\}$ is open, we set $F := \bar{F}_0$. If the boundaries of F_0 and F are equal and they are Lipschitz continuous, then $\operatorname{div} z = \lambda = \operatorname{const}$ on F .

(c) If in addition to (b), we know that $|[z \cdot \nu]| < 1$ on $\Gamma_F := F \cap \Gamma$, then $\operatorname{div} z = -[z \cdot \nu] (= \lambda)$ on Γ_F .

Remark 5.1. In particular part (c) does not apply if $|[z \cdot \nu]| = 1$ on Γ_F . Parts (b) and (c) provide a set of necessary conditions for z to be a minimizer. Later, in Section 6, we will study this in greater detail as well as we will address the sufficient conditions, see Proposition 6.5.

Proof. Part (a) follows from Theorem 5.1 and the definition of the canonical section. Uniqueness of $\operatorname{div} z$ follows from strict convexity of the integrand.

In order to establish (b) and (c), we take any smooth vector field ϖ , having a support in the open set F_0 , such that $z + t\varpi \in \operatorname{Sgn}^N(\nabla u)$, i.e. $|z + t\varpi| \leq 1$, on Ω , for all $t \in \mathbb{R}$ with sufficiently small $|t|$. Since $-[(z + t\varpi) \cdot \nu] = -[z \cdot \nu]$ on $\partial\Omega$ for any $t \in \mathbb{R}$ and z is a minimizer

of the above problem, the function $t \in \mathbb{R} \mapsto E(z + t\varpi) \in [0, \infty)$ has a critical point at $t = 0$. On the other hand it is easy to compute $\frac{d}{dt}E(z + t\varpi)|_{t=0}$. Thus, we obtain,

$$\int_{\Omega} \operatorname{div} z \operatorname{div} \varpi \, dx + \int_{\partial\Omega} [z \cdot \nu][\varpi \cdot \nu] \, d\mathcal{H}^{N-1} = 0. \quad (5.22)$$

Now, we will complete (b). We notice that (5.22) simplifies if vector field v has a compact support contained in the interior of F . In this case, the boundary term drops out, so (5.22) takes the form,

$$\int_{F_0} \operatorname{div} z \operatorname{div} \varpi \, dx = 0.$$

The integration by parts yields $\nabla \operatorname{div} z = 0$ in F_0 . Additionally, since the boundary of F_0 is Lipschitz, we can say that $\operatorname{div} z = \text{const} =: \lambda$ on F ($= \bar{F}_0$).

In order to deduce (c), we take a vector field ϖ having the support in $F_0 \cup \{x \in \partial\Omega : |[z \cdot \nu]| < 1\}$, which is contained in $F_0 \cup \Gamma_F$. Then, (5.22) takes the form,

$$\begin{aligned} 0 &= \int_{\Omega} \operatorname{div} z \cdot \operatorname{div} \varpi \, dx + \int_{\partial\Omega} [z \cdot \nu][\varpi \cdot \nu] \, d\mathcal{H}^{N-1} \\ &= \int_F \lambda \operatorname{div} \varpi \, dx + \int_{\Gamma_F} [z \cdot \nu][\varpi \cdot \nu] \, d\mathcal{H}^{N-1} = \int_{\Gamma_F} (\lambda + [z \cdot \nu])[\varpi \cdot \nu] \, d\mathcal{H}^{N-1}. \end{aligned}$$

Since $[\varpi \cdot \nu]$ is arbitrary, we deduce that $\lambda + [z \cdot \nu] \equiv 0$ on Γ_F . Our claims follow. $\square$

Proposition 5.3. Let us suppose $\Omega \subset \mathbb{R}^2$ and Ω, Γ have radial symmetry, u_0 and v_0 depend only on the radius r . Then, for all $t > 0$, if $(-\operatorname{div} z, [z \cdot \nu])$ is the canonical section of $\partial E(U(t))$, then we can choose z of the form $z(x, t) = \varrho(r, t) \frac{x}{r}$, where $r = |x|$.

Proof. First, we notice that we can choose z depending only on r . The argument is based on the averaging with respect to the Haar measure on S^1 . The details are explained in [33].

Thus, $z(x) = \varrho(r) \mathbf{e}_r + \psi(r) \mathbf{e}_\varphi$, where $\mathbf{e}_r = \frac{x}{r}$ and $\mathbf{e}_\varphi = (-x_2, x_1)/r$. We notice that $1 \geq |z|^2 = \varrho^2 + \psi^2 \geq \varrho^2$. Moreover,

$$\operatorname{div} z = \operatorname{div} \varrho(r) \mathbf{e}_r = \frac{(\varrho(r)r)'}{r}.$$

In other words, the part of z , tangential to circles $\partial B(0, r)$, is divergence-free.

Thus, we may drop the tangential part of z , because it neither contributes to $\operatorname{div} z$, nor to the boundary trace. $\square$

Remark 5.2. Proposition 5.3 extends to radially symmetric domains in $\mathbb{R}^N$ and the data with the same symmetry. Here $\operatorname{div} z = (\varrho(r)r^{N-1})'/r^{N-1}$ for general N .

Finally, we can decide the form of the subdifferential in the one-dimensional case, but we restrict our attention to monotone initial condition u_0 . We set $\chi = 1$, if u_0 is increasing and $\chi = -1$, if u_0 is decreasing. We notice that the outer normals ν to Γ are in fact numbers, $\nu(0) = -1$ and $\nu(L) = 1$.

Proposition 5.4. Let us suppose that $U = (u, v) \in L^2(0, T; H)$ is a solution to (4.2), where $\Omega = (0, L)$, $\Gamma = \partial\Omega$, u_0 is monotone. We denote by (ξ, ζ) the canonical selection of the subdifferential of E at $U(t)$, $t > 0$, i.e. $\xi = -z_x$, $\zeta(i) = z(i) \cdot \nu(i)$, where $i \in \Gamma = \{0, L\}$. Let us consider $[a, b] \subset (0, L)$. We assume that $u_x(a^+)$, $u_x(b^-)$ exist and they are different from zero. Then,

- (a) $\text{Sgn}(u_x(a^+)) = \text{Sgn}(u_x(b^-)) = \text{Sgn}(u_x(x^\pm)) = \chi$ for all $x \in [a, b]$;
- (b) if $\gamma u = v$ at $x \in \Gamma$, then $z(i)\nu(i) \in [-1, 1]$, $i \in \Gamma$;
- (c) if $\gamma u \neq v$ at $x \in \Gamma$, then $|z(i)| = 1$, $i \in \Gamma$.

Proof. Part (a) follows from the fact that at any point x , where u_0 is differentiable and different from zero, we have $z(x) = \chi$. The set of such points in $[a, b]$ has a full measure. Since $z(x) = \chi = \text{Sgn}(\frac{d}{dx}u_0(x))$ and z must be continuous, we deduce that $z = \chi$ on $[a, b]$.

The proofs of the remaining parts is done by inspection of the conditions on the canonical section. $\square$

We know that the canonical selection is uniquely defined as the element of the subdifferential with the least norm. The structure of this minimization problem (5.21) is such that z has to be decided only where $Du = 0$. We would like to take advantage of this fact for the purpose of the localization of the problem. We explain it below.

Corollary 5.2. Let us suppose that $(-\text{div } z, [z \cdot \nu])$ is a canonical selection of ∂E , and $F_0 := \{x \in \Omega \mid |z(x)| < 1\} \subset \Omega$ is open with Lipschitz continuous boundary and $\partial F_0 = \partial \bar{F}_0$. We recall that F_0 is contained in the complement of the support of measure $|Du|$. Let F be the closure of a connected component of F_0 and let ν_F be the outer unit normal of ∂F . Additionally, let us suppose that $|[z \cdot \nu_F]| = 1$ for $\mathcal{H}^{N-1}$ -a.e. $x \in \partial F \cap \Omega$. Let F be a connected component of $\bar{F}_0$, and let ν_F be the outer unit normal of ∂F . Additionally, let us suppose that $|[z \cdot \nu_F]| = 1$ for $\mathcal{H}^{N-1}$ -a.e. $x \in \partial F \cap \Omega$.

Then, $(\text{div } z|_F, [z \cdot \nu]|_{\Gamma_F})$ minimizes the following functional

$$I(\zeta) = \int_F |\text{div } \zeta|^2 dx + \int_{\Gamma_F} |[\zeta \cdot \nu]|^2 d\mathcal{H}^{N-1}$$

in the set

$$\left\{ \zeta \in L^\infty(F, \mathbb{R}^N) \mid \begin{array}{l} \|\zeta\|_\infty \leq 1, [z \cdot \nu_F] = [\zeta \cdot \nu_F] \text{ on } \partial F \cap \Omega, \\ [\zeta \cdot \nu] = 0 \text{ on } \partial\Omega \setminus \Gamma_F \end{array} \right\}.$$

Proof. Indeed, if there is z_0 , such that $I(z_0) < I(z)$, then due to $[z \cdot \nu_F] = [z_0 \cdot \nu_F]$ on $\partial F \cap \Omega$, we see that for $\tilde{z} = z_0\chi_F + z\chi_{\Omega \setminus F}$, we have that

$$\text{div } \tilde{z} = \text{div } z_0\chi_F + \text{div } z\chi_{\Omega \setminus F}.$$

Hence, $\mathcal{E}(\tilde{z}) < \mathcal{E}(z)$, contrary to the fact that $(-\text{div } z, [z \cdot \nu])$ is a canonical selection of ∂E . $\square$

In next section, we will have a closer look at I .

5.3 Scaling out parameter τ

When we were calculating the subdifferential we used the standard inner product of H . This corresponds to parameter $\tau = 1$ in (4.2). We shall see here that in fact the parameter τ may be set to one by a proper dilating of the domain $\Omega \times (0, T)$. Indeed, we can show the following statement.

Let us suppose $U = (u, v) \in W_{loc}^{1,\infty}([0, \infty); H)$ is a solution to (4.2), with initial condition $U_0 = (u_0, v_0) \in D(\partial E)$. In other words, there is $z \in L^\infty(0, \infty; X_2)$ and $[z \cdot \nu] \in L^\infty((0, \infty) \times \Gamma)$ such that

$$\begin{aligned} u_t &= \operatorname{div} z & (x, t) &\in \Omega \times (0, T), \\ v_t &= -[z \cdot \nu] & (x, t) &\in \Gamma \times (0, T), \\ u(x, 0) &= u_0(x) & x &\in \Omega, \\ v(x, 0) &= v_0(x) & x &\in \Gamma. \end{aligned} \tag{5.23}$$

Here, $(-\operatorname{div} z, [z \cdot \nu])$ is the minimal section of $\partial E(U)$, i.e. it is a minimizer of (5.21).

For any $k \in \mathbb{N}$, any $A \subset \mathbb{R}^k$ and any $\tau > 0$, we set,

$$A_\tau = \{\tau x : x \in A\}. \tag{5.24}$$

Besides, we define $U^\tau = (u^\tau, v^\tau)(y, s)$, $z^\tau(y, s)$ and $\nu^\tau(y, s)$ by the formulas

$$U^\tau(y, s) = U(x, t), \quad z^\tau(y, s) = z(x, t), \quad \text{and} \quad \nu^\tau(y, s) = \nu(x, t), \tag{5.25}$$

where $y = \tau x \in \Omega_\tau$, $s = \tau t \in (0, \tau T)$. We immediately notice that

$$U_t = \tau U_s^\tau, \quad \operatorname{div}_x z = \tau \operatorname{div}_y z^\tau.$$

If we set $\mathcal{E}_\tau(\zeta)$ by formula

$$\mathcal{E}_\tau(\zeta) = \int_{\Omega_\tau} |\operatorname{div}_y \zeta|^2 dy + \frac{1}{\tau} \int_{\Gamma_\tau} |[\zeta \cdot \nu^\tau]|^2 d\mathcal{H}^{N-1},$$

for $\zeta \in X_2$ satisfying the conditions presented in (5.21), then we may check (this is the content of Lemma 6.1) that $\mathcal{E}_\tau(z^\tau) = \tau^{N-2} \mathcal{E}(z)$. Thus, z^τ is the minimal section of $\partial_\tau E$. Thus, we conclude that U^τ and z^τ form a solution to

$$\begin{aligned} u_t^\tau &= \operatorname{div} z^\tau & (y, s) &\in \Omega_\tau \times (0, \tau T), \\ \tau v_t^\tau &= -[z^\tau \cdot \nu^\tau] & (y, s) &\in \Gamma_\tau \times (0, \tau T), \\ u^\tau(y, 0) &= u_0^\tau(y) & y &\in \Omega_\tau, \\ v^\tau(y, 0) &= v_0^\tau(y) & y &\in \Gamma_\tau. \end{aligned} \tag{5.26}$$

In other words, we have shown:

Corollary 5.3. If U is a solution to (4.2) in $H = L^2(\Omega) \times L^2(\Gamma)$ with $\tau = 1$, then U^τ is a solution to (4.2) in $L^2(\Omega_\tau) \times L^2(\Gamma_\tau)$ with $\tau > 0$. $\square$

Of course, the converse statement is true. If $\tilde{U}$ is a solution to (5.26), then $\tilde{U}^{1/\tau}$ is a solution to (5.23). In order to see this, we apply the results we have shown to $\tilde{U}^{1/\tau}$ and we scale $\tilde{U}^{1/\tau}$ by τ^{-1} .

6 Calibrability and coherence

We shall introduce the notions of calibrability and coherency, when a facet touches the boundary of a given domain. In the following considerations, we assume that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with Lipschitz boundary. Let Γ be a relatively closed set in $\partial\Omega$ of positive $\mathcal{H}^{N-1}$ measure.

A compact set F in $\bar{\Omega}$ together with direction $\chi \in C^0(\Omega \setminus F; \{\pm 1\})$ is called a *facet* in Ω .

Remark 6.1. The definition above is as in [32], however, we can also define a facet as a flat part of the graph a solution u , see [30, §2.4]. In the present context, such a distinction does not matter, because we are talking about sets, where $\nabla u = 0$.

Let us consider a facet (F, χ) whose boundary ∂F is Lipschitz. Let ν_F be the outer unit normal field of ∂F . Let z be a vector field in F belonging to $X_2(F)$. We say that z is a *Cahn–Hoffman vector field* in F with (Ω, Γ) if

$$\|z\|_\infty \leq 1 \quad [z \cdot \nu_F] = \gamma\chi \quad \text{on} \quad \partial F \cap \Omega, \quad [z \cdot \nu_F] = 0 \quad \text{on} \quad (F \cap \partial\Omega) \setminus \Gamma \quad (6.1)$$

is fulfilled, where $\gamma\chi$ is the trace of χ taken from F^c , the complement of F . The totality of Cahn–Hoffman vector fields is denoted by $CH(F, \Omega, \Gamma)$, i.e.,

$$CH(F, \Omega, \Gamma) = \left\{ z \in X_2(F) \mid z \text{ fulfills (6.1)} \right\}.$$

We say that a facet (F, χ) with Lipschitz boundary is *admissible* if $CH(F, \Omega, \Gamma)$ is non empty.

We are interested in those Cahn–Hoffman vector fields, which minimize the localized problem of the canonical selection of ∂E . We argued in Corollary 5.2 that for $\tau = 1$ in eq. (4.2), the functional to minimize was I , defined there. We claim that for a given facet (F, χ) and a parameter $\tau > 0$ in eq. (4.2), we must consider the following functional in order to determine the minimal section of E ,

$$I_\tau(z) = \int_F |\operatorname{div} z|^2 dx + \frac{1}{\tau} \int_{\Gamma_F} |[z \cdot \nu]|^2 d\mathcal{H}^{N-1}, \quad z \in CH(F, \Omega, \Gamma),$$

where $\Gamma_F := \partial F \cap \Gamma$. We notice that $I = I_1$. Lemma 6.1 shows the relationship between I_1 and I_τ , proving that I_τ is indeed the localized problem of the canonical selection of ∂E .

In principle, we should check if I_τ attains its minimum. Functional I_τ is convex on a closed, convex set $CH(F, \Omega, \Gamma)$. We claim that it is lower semi-continuous with respect to L^2 -weak convergence of $\operatorname{div} z$. For this purpose, we have to check that the boundary integral is lower semi-continuous. We notice that if a test function $\varphi \in W^{1,2}(\Omega)$ and $\operatorname{div} z_n \rightharpoonup \operatorname{div} z$ in L^2 , then we may assume that $z_n \rightharpoonup z$ in L^2 , because $\|z_n\|_\infty \leq 1$. As a result,

$$\lim_{n \rightarrow \infty} \int_{\Gamma_F} [z_n \cdot \nu] \gamma \varphi d\mathcal{H}^{N-1} = \lim_{n \rightarrow \infty} \int_F \operatorname{div} (z_n \varphi) dx = \int_F \operatorname{div} (z \varphi) dx = \int_{\Gamma_F} [z \cdot \nu] \gamma \varphi d\mathcal{H}^{N-1}. \quad (6.2)$$

In order to claim that $[z_n \cdot \nu] \rightharpoonup [z \cdot \nu]$ in $L^2(\Gamma_F, \mathcal{H}^{N-1})$, we need to show

$$\lim_{n \rightarrow \infty} \int_{\Gamma_F} [z_n \cdot \nu] \psi d\mathcal{H}^{N-1} = \int_{\Gamma_F} [z \cdot \nu] \psi d\mathcal{H}^{N-1} \quad \text{for all } \psi \in L^2(\Gamma_F).$$

However, identity (6.2) combined with the standard mollification argument, yields the desired result. Hence, the boundary term is weakly lower semi-continuous, as we claimed.

Thus, there always exists a minimizer $z_0 \in CH(F, \Omega, \Gamma)$ of $I_\tau(z)$. Moreover, by strict convexity of I_τ with respect to $\operatorname{div} z_0$ and $[z_0 \cdot \nu]$ on Γ_F the values of $\operatorname{div} z_0$ and $[z_0 \cdot \nu]$ are uniquely determined although there are many minimizers z of $I_\tau(z)$, besides z_0 .

After these preparations, the following definition is justified.

Definition 6.1. An admissible facet (F, χ) is *calibrable* if there is a Cahn–Hoffman vector field z minimizing I_τ such that $\operatorname{div} z$ is constant in F and that $[z \cdot \nu]$ is constant on Γ .

Remark 6.2. The above notion of calibrability agrees with the conventional one when $\Omega = \mathbb{R}^N$.

Prompted by Definition 6.1, we introduce the notation.

$$SCH(F, \Omega, \Gamma) := \{z \in CH(F, \Omega, \Gamma) : \operatorname{div} z = \text{const on } F, [z \cdot \nu] = \text{const on } \Gamma\}.$$

Elements of $SCH(F, \Omega, \Gamma)$ will be called special Cahn–Hoffman vector fields.

We would like to establish the relationship between minimizers of I_τ and I_1 .

Lemma 6.1. A vector field $z \in CH(F_{\tau_0}, \Omega_{\tau_0}, \Gamma_{\tau_0})$ is a minimizer of I_{τ_0} if and only if $z^\tau \in CH(F_\tau, \Omega_\tau, \Gamma_\tau)$, where $z^\tau(x) = z(x \frac{\tau_0}{\tau})$ minimizes I_τ . Moreover, $I_\tau(z^\tau) = \tau^{N-2} I(z)$.

Proof. We may assume $\tau_0 = 1$. Obviously, $z \in CH(F, \Omega, \Gamma)$ if and only if $z^\tau \in CH(F_\tau, \Omega_\tau, \Gamma_\tau)$.

Then, we perform a change of variables in $I_\tau(z^\tau)$. Namely, after setting $y = x/\tau$, we obtain,

$$I_\tau(z^\tau) = \int_{F_\tau} |\operatorname{div}_x z^\tau|^2 dx + \frac{1}{\tau} \int_{(\Gamma_F)_\tau} |[z^\tau \cdot \nu]|^2 d\mathcal{H}^{N-1} = \tau^{N-2} I(z).$$

This multiplicative relationship $I_\tau(z^\tau) = \tau^{N-2} I(z)$ implies validity of our claim. $\square$

The relationship between values of $z \in CH(F, \Omega, \Gamma)$ on F and Γ_F is important for our considerations. Here is our basic observation.

Lemma 6.2. For z in $CH(F, \Omega, \Gamma)$, we denote the average of $\operatorname{div} z$ and $[z, \nu]$ by

$$\lambda_z = \frac{1}{|F|} \int_F \operatorname{div} z dx, \quad \mu_z = \frac{1}{\mathcal{H}^{N-1}(\Gamma_F)} \int_{\Gamma_F} [z \cdot \nu] d\mathcal{H}^{N-1}.$$

Then,

$$\lambda_z |F| = \mathcal{H}^{N-1}(\partial_+ F) - \mathcal{H}^{N-1}(\partial_- F) + \mu_z \mathcal{H}^{N-1}(\Gamma_F),$$

where $|F|$ denotes the Lebesgue measure of F . Here,

$$\partial_\pm F = \{x \in \partial F \cap \Omega \mid \gamma\chi = \pm 1\}.$$

Proof. Integration by parts yields

$$\begin{aligned} \lambda_z |F| &= \int_F \operatorname{div} z dx = \int_{\partial_+ F} [z \cdot \nu_F] d\mathcal{H}^{N-1} + \int_{\partial_- F} [z \cdot \nu_F] d\mathcal{H}^{N-1} + \int_{\Gamma_F} [z \cdot \nu] d\mathcal{H}^{N-1} \\ &= \mathcal{H}^{N-1}(\partial_+ F) - \mathcal{H}^{N-1}(\partial_- F) + \mu_z \mathcal{H}^{N-1}(\Gamma_F). \end{aligned} \quad \square$$

The above Lemma helps us to introduce a notion important in our analysis. Calibrability of a facet means that it moves as an entity. However, the bulk may move at a different velocity than the boundary layer. That is why we introduce the notion of coherency.

Definition 6.3. We shall say that facet (F, χ) is (τ, Γ) -coherent if there is a Cahn–Hoffman vector field z , minimizing I_τ such that

$$\tau \lambda_z + \mu_z = 0,$$

where λ_z and μ_z defined in Lemma 6.2.

Now, we are going to establish the relationship between these notions and establish the sufficient conditions for facet calibrability or (τ, Γ) -coherency. But first we state a simple fact about quadratic polynomials. Let a and b be positive constants. We consider,

$$f(\lambda, \mu) = a\lambda^2 + b\mu^2/\tau \quad (6.3)$$

under constraint

$$\lambda a = c + b\mu, \quad (c \in \mathbb{R}). \quad (6.4)$$

Proposition 6.4. Let $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}$ be the (unique) minimizer of (6.3) subject to (6.4), if and only if $\tau\lambda + \mu = 0$.

Proof. This is elementary. We set

$$g(\mu) = f((c + b\mu)/a, \mu)$$

and differentiate to get

$$g'(\mu) = 2b(c + b\mu)/a + 2b\mu/\tau.$$

Here, we know that the minimum point μ_0 coincides with the unique solution of $g'(\mu_0) = 0$. Additionally, due to the constraint $\lambda = (c + b\mu)/a$, the equation $g'(\mu_0) = 0$ is equivalent to $\lambda_0 + \mu_0/\tau = 0$, with $\lambda_0 = (c + b\mu_0)/a$. The proof of Proposition is now complete. $\square$

We will use this observation in the next Proposition, providing sufficient conditions for a minimizer.

Proposition 6.5. If $z_* \in CH(F, \Omega, \Gamma)$ is such that $\tau\lambda_* + \mu_* = 0$, where $\lambda_* = \lambda_{z_*} = \operatorname{div} z_*$ on F , $\mu_* = \mu_{z_*} = [z_* \cdot \nu]$ on Γ_F , then z_* is a minimizer of I_τ .

Proof. By the Schwarz inequality, for any vector field z , we have

$$\frac{1}{|F|} \left(\int_F \operatorname{div} z \, dx \right)^2 \leq \int_F (\operatorname{div} z)^2 \, dx, \quad \frac{1}{\mathcal{H}^{N-1}(\Gamma_F)} \left(\int_{\Gamma_F} [z \cdot \nu] \, d\mathcal{H}^{N-1} \right)^2 \leq \int_{\Gamma_F} [z \cdot \nu]^2 \, d\mathcal{H}^{N-1}.$$

Thus, we see that the definition of I_τ yields,

$$|F|\lambda_z^2 + \frac{\mathcal{H}^{N-1}(\Gamma_F)}{\tau} \mu_z^2 \leq \int_F (\operatorname{div} z)^2 \, dx + \frac{1}{\tau} \int_{\Gamma_F} [z \cdot \nu]^2 \, d\mathcal{H}^{N-1} = I_\tau(z). \quad (6.5)$$

We know by Lemma 6.2 that

$$|F|\lambda_z = c + \mu_z \mathcal{H}^{N-1}(\Gamma_F), \quad (6.6)$$

where c is a constant. By Proposition 6.4, the left-hand-side of (6.5) is minimized under the constraint (6.6) if and only if $\tau\lambda_{z_*} + \mu_{z_*} = 0$. Furthermore, our assumption on z_* yields us

$$I_\tau(z_*) = |F|\lambda_*^2 + \frac{\mathcal{H}^{N-1}(\Gamma_F)}{\tau} \mu_*^2 \leq I_\tau(z).$$

Thus, the proof is complete. $\square$

In other words, a special Cahn–Hoffman vector field z minimizes I_τ if we can ensure $\tau\lambda_z + \mu_z = 0$. Now, we may prove the converse statement.

Proposition 6.6. Let us suppose that facet (F, χ) is calibrable and (τ, Γ) -coherent. Then, there is $z_* \in SCH(F, \Omega, \Gamma)$ such that $\tau \operatorname{div} z_* + [z_* \cdot \nu] = 0$.

Proof. Since (F, χ) is calibrable, we can find $z_* \in SCH(F, \Omega, \Gamma)$ minimizing I_τ . In addition, we must have $\tau \operatorname{div} z_* + [z_* \cdot \nu] = 0$, due to the (τ, Γ) -coherency of (F, χ) and the uniqueness of minimizing $\operatorname{div} z_*$ and $[z_* \cdot \nu]$. $\square$

However, we must be prepared for the existence of calibrable facets, which are not (τ, Γ) -coherent.

Lemma 6.7. If for a facet (F, χ) there is z_* in $SCH(F, \Omega, \Gamma)$, with $\lambda := \operatorname{div} z_* \in \mathbb{R}$ and $\mu := [z_* \cdot \nu] \in \mathbb{R}$, such that $|\mu| = 1$ and $(\lambda + \mu/\tau) \operatorname{Sgn} \mu \leq 0$. Then, this facet is calibrable.

Proof. Let us suppose that $z_* \in SCH(F, \Omega, \Gamma)$ and $\operatorname{div} z_* = \lambda$, $[z_* \cdot \nu] = \mu$. We take a test vector field ζ such that $z_* + \zeta \in CH(F, \Omega, \Gamma)$. Thus, in particular, $[\zeta \cdot \nu_F] = 0$ on $\partial F \setminus \Gamma$, but $[\zeta \cdot \nu] \operatorname{Sgn} \mu \leq 0$ on Γ_F .

After having performed simple computations and the integration by parts, we arrive at

$$\begin{aligned} I_\tau(z_* + \zeta) - I_\tau(z_*) - I_\tau(\zeta) &= 2\lambda \int_F \operatorname{div} \zeta \, dx + \frac{2\mu}{\tau} \int_{\Gamma_F} [\zeta \cdot \nu] \, d\mathcal{H}^{N-1} \\ &= 2 \left(\lambda + \frac{\mu}{\tau} \right) \int_{\Gamma_F} [\zeta \cdot \nu] \, d\mathcal{H}^{N-1}. \end{aligned} \quad (6.7)$$

Thus, combining this information with $(\lambda + \mu/\tau) \operatorname{Sgn} \mu \leq 0$ and $[\zeta \cdot \nu] \operatorname{Sgn} \mu \leq 0$ on Γ_F yields that the right-hand-side in (6.7) is positive. Hence, our claim follows. $\square$

Theorem 6.8. Let (F, χ) be calibrable. Assume that (6.3) is minimized under (6.4) at some (λ_0, μ_0) with $|\mu_0| \leq 1$. Assume that there is $z_* \in SCH(F, \Omega, \Gamma)$ with $[z_* \cdot \nu] = \mu_0$. Then (F, χ) is (τ, Γ) -coherent.

Proof. This is just an application of Proposition 6.4. The assumption $|\mu_0| \leq 1$ is a necessary condition so that $[z_* \cdot \nu] = \mu_0$ since $\|z_*\|_\infty \leq 1$. $\square$

The property of coherency heavily depends on geometry of Γ . Here is a conjecture.

Conjecture 6.9. If Γ is strictly mean-convex near F , then an admissible facet (F, χ) is (τ, Γ) -coherent for all $\tau > 0$. Here, when we say that Γ is strictly mean-convex, we mean that there is a positive constant γ_0 such that $\kappa \geq \gamma_0$ on Γ , where κ is the inward mean curvature of Γ in $\partial\Omega$.

More generally, if $\inf_{x \in \Gamma \cap F} \kappa(x) > -1/\tau$, then we expect that (F, χ) is (τ, Γ) -coherent.

We shall show this conjecture with extra regularity assumptions on a minimizer of $I_\tau(z)$.

Theorem 6.10. Assume that $\partial\Omega$ is at least C^2 in a neighborhood of Γ and the mean curvature κ is estimated from below, $\kappa \geq \gamma_0$ where $\gamma_0 > 1/2 - 1/\tau$. Let $z_0 \in CH(F, \Omega, \Gamma)$ be a minimizer of I_τ . Assume that z_0 can be extended as a C^2 function in a neighborhood U of Γ_F in $\mathbb{R}^N$. Then, the following properties hold:

(i) $|[z_0 \cdot \nu]| < 1$ on Γ_F ,

(ii) $\tau \operatorname{div} z_0 + [z_0 \cdot \nu] = 0$ on Γ_F . In other words, (F, χ) is (τ, Γ) -coherent.

Proof. Let d_Σ be the distance function from a closed subset Σ of Γ_F . We recall a general formula

$$\operatorname{div} z = \operatorname{div}_T z + (m \cdot \nabla)(z \cdot m), \quad (6.8)$$

where div_T is the surface divergence on a hypersurface $\{d_\Sigma = c\}$ and $m = -\nabla d_\Sigma$, which is normal to $\{d_\Sigma = c\}$. Indeed,

$$\begin{aligned} \operatorname{div}_T z &= \operatorname{tr}(I - m \otimes m) \nabla z \\ \operatorname{tr}(m \otimes m \nabla z) &= \sum_{i,j} m_i m_j \partial_j z_i = (m \cdot \nabla)(z \cdot m) \end{aligned}$$

because $(m \cdot \nabla)m = 0$. This implies the desired decomposition (6.8). The formula (6.8) holds for a.e. c and for $\mathcal{H}^{N-1}$ -a.e. $x \in \{d_\Sigma = c\}$ if d_Σ is not C^1 , but Lipschitz continuous.

(i) We set $\Sigma = \{x \in \Gamma_F \mid [z_0 \cdot \nu] = 1\}$ and set

$$z_\varepsilon = \varphi_\varepsilon z_0 \quad \text{with} \quad \varphi_\varepsilon = \min(1, 1 - \varepsilon + d_\Sigma),$$

for $\varepsilon > 0$. We shall prove that $I_\tau(z_\varepsilon) < I_\tau(z_0)$ for sufficiently small $\varepsilon > 0$ assuming that Σ is non empty. We calculate

$$\begin{aligned} I_\tau(z_0) - I_\tau(z_\varepsilon) &= \left\{ \int_{\Sigma_\varepsilon} |\operatorname{div} z_0|^2 dx - \int_{\Sigma_\varepsilon} |\operatorname{div} z_\varepsilon|^2 dx \right\} \\ &\quad + \frac{1}{\tau} \left\{ \int_{\Gamma_F} ([z_0 \cdot \nu])^2 d\mathcal{H}^{N-1} - \int_{\Gamma_F} (z_\varepsilon \cdot \nu)^2 d\mathcal{H}^{N-1} \right\} \\ &= I + II, \end{aligned}$$

where $\Sigma_\varepsilon = \{x \in F \mid d_\Sigma(x) < \varepsilon\}$. To estimate I , we calculate

$$\operatorname{div} z_\varepsilon = \varphi_\varepsilon \operatorname{div} z_0 + \nabla \varphi_\varepsilon \cdot z_0$$

and observe that

$$I \geq -2 \int_{\Sigma_\varepsilon} (\varphi_\varepsilon \operatorname{div} z_0) \nabla \varphi_\varepsilon \cdot z_0 dx - \int_{\Sigma_\varepsilon} (\nabla \varphi_\varepsilon \cdot z_0)^2 dx = III + IV.$$

We use the decomposition formula (6.8) and the fact $z \in C^1(U)$ to get

$$\begin{aligned} III/2 &= - \int_{\Sigma_\varepsilon} (\varphi_\varepsilon \operatorname{div} z_0) \nabla \varphi_\varepsilon \cdot z_0 dx = \int_{\Sigma_\varepsilon} (\varphi_\varepsilon \operatorname{div}_T z_0) m \cdot z_0 dx \\ &\quad + \int_{\Sigma_\varepsilon} (\varphi_\varepsilon (m \cdot \nabla)(z_0 \cdot m)) (z_0 \cdot m) dx \\ &= \varepsilon \int_{\Sigma} (\operatorname{div}_T \nu) ([z_0 \cdot \nu]) d\mathcal{H}^{N-1} + O(\varepsilon^2) \\ &\quad + (1 - \varepsilon) \int_{\Sigma_\varepsilon} (m \cdot \nabla) \left(\frac{(z_0 \cdot m)^2}{2} \right) dx + O(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Since $[z_0 \cdot \nu] = 1$ on Σ , $|z_0| \leq 1$ in F and $z_0 \in C^1(U)$, then

$$\int_{\Sigma_\varepsilon} (m \cdot \nabla)(z_0 \cdot m)^2 dx \geq O(\varepsilon^2).$$

Finally, since $\kappa = \operatorname{div}_T \nu$, we observe that

$$\text{III} \geq 2\varepsilon \int_{\Sigma} \kappa d\mathcal{H}^{N-1} + O(\varepsilon^2).$$

It is easy to see that

$$IV = -\varepsilon \int_{\Sigma} (z_0 \cdot \nu)^2 d\mathcal{H}^{N-1} + O(\varepsilon^2) = -\varepsilon \int_{\Sigma} d\mathcal{H}^{N-1} + O(\varepsilon^2).$$

Thus, we observe that

$$I \geq \varepsilon \left(\int_{\Sigma} 2\kappa d\mathcal{H}^{N-1} - \int_{\Sigma} d\mathcal{H}^{N-1} \right) + O(\varepsilon^2).$$

It is easy to estimate

$$\begin{aligned} \tau \text{II} &= \int_{\Gamma_F} ((z_0 - z_\varepsilon) \cdot \nu) ((z_0 + z_\varepsilon) \cdot \nu) d\mathcal{H}^{N-1} \\ &= \varepsilon \int_{\Sigma} ([z_0 \cdot \nu]) \cdot (2[z_0 \cdot \nu]) d\mathcal{H}^{N-1} + O(\varepsilon^2) \\ &= 2\varepsilon \int_{\Sigma} d\mathcal{H}^{N-1} + O(\varepsilon^2). \end{aligned}$$

Thus,

$$I + \text{II} \geq \varepsilon \left(\int_{\Sigma} 2\kappa d\mathcal{H}^{N-1} + \left(\frac{2}{\tau} - 1 \right) \int_{\Sigma} d\mathcal{H}^{N-1} \right) + O(\varepsilon^2).$$

As a result, if $\kappa \geq \gamma_0$ with $\gamma_0 > 1/2 - 1/\tau$, then $I_\tau(z_0) - I_\tau(z_\varepsilon) > 0$ for small ε . We thus prove that $z_0 \cdot \nu < 1$ if z_0 is a minimizer. The inequality $z_0 \cdot \nu > -1$ can be proved similarly.

(ii) If $|[z_0 \cdot \nu]| < 1$ on Γ_F , then for any $h \in C^1(F)$ such that $|(z_0 + \varepsilon h) \cdot \nu| < 1$ on Γ_F and $|z + \varepsilon h| \leq 1$ in F . The first condition does not restrict $h_\nu = h \cdot \nu$. The second condition restricts the tangential component $h_T = h - (h \cdot \nu)\nu$ so that $|z_0 + \varepsilon h_\nu \cdot \nu + \varepsilon h_T| \leq 1$. Since z_0 is the minimizer, we obtain that

$$\begin{aligned} 0 &= \frac{1}{2} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} I_\tau(z_0 + \varepsilon h) = \int_F \operatorname{div} z_0 \operatorname{div} h dx + \frac{1}{\tau} \int_{\Gamma_F} ([z_0 \cdot \nu]) h_\nu d\mathcal{H}^{N-1} \\ &= - \int_F \nabla \operatorname{div} z_0 \cdot h dx + \int_{\partial F} \operatorname{div} z_0 (h \cdot \nu_F) d\mathcal{H}^{N-1} + \frac{1}{\tau} \int_{\Gamma_F} ([z_0 \cdot \nu]) h_\nu d\mathcal{H}^{N-1}. \end{aligned}$$

We take arbitrary C^1 function f near Γ_F and pick a test function $h \in C^1(F)$, satisfying $h_\nu = f$ on Γ_F . We require that the support of h is in an δ -neighborhood of Γ_F , and moreover, the

tangential component is controlled by f . The first term of above formula is $O(\delta)$ as $\delta \rightarrow 0$, as a result

$$\int_{\Gamma_F} (\tau \operatorname{div} z_0 + [z_0 \cdot \nu]) f d\mathcal{H}^{N-1} = 0.$$

This implies that

$$\tau \operatorname{div} z_0 + [z_0 \cdot \nu] = 0 \quad \text{on} \quad \Gamma_F.$$

□

Actually, we could relax the assumptions of this Theorem without weakening the claim.

Corollary 6.1. The conclusion of Theorem 6.10 holds if we assume that the mean curvature κ is estimated as follows,

$$\inf_{\Gamma_F} \kappa > -\frac{1}{\tau}.$$

Proof. We fix $\tau = \tau_0$, we consider Ω_{τ/τ_0} , (see formula (5.24)), a scaled domain. We notice that κ_{τ/τ_0} , the mean curvature of $(\Gamma_F)_{\tau/\tau_0}$, is equal to $\frac{\tau_0}{\tau} \kappa$. Our condition $\inf_{\Gamma_F} \kappa > \frac{1}{2} - \frac{1}{\tau_0}$ is equivalent to

$$\inf_{(\Gamma_F)_{\tau/\tau_0}} \frac{\tau}{\tau_0} \kappa_{\tau/\tau_0} > \frac{1}{2} - \frac{1}{\tau_0}.$$

In other words, $\kappa_{\tau/\tau_0} > \frac{\tau_0}{2\tau} - \frac{1}{\tau}$. However, we may take arbitrary small τ_0 . This means that κ may be as close to $-\frac{1}{\tau}$, as we wish. Thus, the condition

$$\inf_{\Gamma_F} \kappa > -\frac{1}{\tau}$$

is sufficient to guarantee existence of a Cahn–Hoffman vector field.

Suppose now that

$$\inf_{\Gamma_F} \kappa > -\frac{1}{\tau},$$

then there exists $\tau_0 > 0$ such that $\inf_{\Gamma_F} \kappa > \frac{\tau_0}{2\tau} - \frac{1}{\tau}$. Now, by scaling we consider the problem in $\Omega_{\tau_0/\tau}$, then the mean curvature condition is equivalent to

$$\inf_{\Gamma_F} \frac{\tau_0}{\tau} \kappa_{\tau_0/\tau} > \frac{\tau_0}{2\tau} - \frac{1}{\tau},$$

because $\kappa = \frac{\tau_0}{\tau} \kappa_{\tau_0/\tau}$. In other words,

$$\kappa_{\tau_0/\tau} > \frac{1}{2} - \frac{1}{\tau_0},$$

as desired. □

7 Instant facet formation in the one-dimensional problem

Before considering any two dimensional configuration, we would like to present a simple one-dimensional warm-up problem. We assume that our data contain exactly one facet touching the boundary, where we specify the dynamic boundary condition. Our goal is to capture the behavior of facets by constructing explicit solutions. This task involves monitoring the boundary behavior of solutions.

For the sake of simplicity, we consider only monotone initial condition u_0 . We shall write

$$\chi = 1 \quad \text{if } u_0 \text{ is increasing and } \chi = -1 \quad \text{if } u_0 \text{ is decreasing.} \quad (7.1)$$

Theorem 7.1. Let us suppose that $\Omega = (0, L)$, $\Gamma = \partial\Omega$ and $u_0 \in \text{Lip}(\Omega)$. We assume that u_0 is strictly monotone on $[0, b_0]$, $b_0 \in (0, L)$ and it has exactly one facet $([b_0, L], \chi)$ touching the boundary at $x = L$ and $v_0 = u_0|_{\Gamma}$. Then,

1) The vertical velocity of facet $([b(t), L], \chi)$ is

$$u_t = \frac{-\chi}{1 + L - b(t)}. \quad (7.2)$$

Facet $([b_0, L], \chi)$ calibrable and $(1, \Gamma)$ -coherent. Moreover, if we write $h_r(t) = u(\cdot, t)|_{[b(t), L]}$, then $\frac{dh_r}{dt} = u_t$ and $h_r(0) = u(L, 0)$. In addition, $b(t)$ is a solution to $h_r(t) = u_0(b(t))$, i.e. $b(t) = (u_0)^{-1}(h_r(t))$.

2) A new facet forms instantly at $x = 0$, its initial velocity is χ . If we denote by $a(t)$ the right endpoint of the new facet at $t > 0$, then its velocity is given by

$$u_t = \frac{\chi}{1 + a(t)}. \quad (7.3)$$

Facet $([0, a(t)], \chi)$ calibrable and $(1, \Gamma)$ -coherent. Moreover, if we write $h_l(t) = u(\cdot, t)|_{[0, a(t)]}$, then $\frac{dh_l}{dt} = u_t$ and $h_l(0) = u(0, 0)$. In addition, $a(t) = (u_0)^{-1}(h_l(t))$. In particular, the velocity of facet $[0, a(t)], \chi$ is continuous at $t = 0$.

3) The unique solution $U(t) = (u(t), v(t))$ is given by formula (7.10) below. In particular, for all $t \geq 0$, $\gamma u(t) = v(t)$, in other words, $\gamma u_t = v_t$.

Remark 7.1. Roughly speaking, the sign of the velocity of facet $([b(t), L], \chi)$ is opposite to the sign of the space derivative of solution u on $(a(t), b(t))$, while in case of facet $([0, a(t)], \chi)$ the signs of its velocity and u_x on $(a(t), b(t))$ agree.

We notice that a is a continuous function of time and so is the velocity of the facet at $x = 0$.

Proof of Theorem 7.1. We are going to construct semi-explicit solutions from the information about the subdifferential $\partial E(U)$. We will use the fact that the solution $U : [0, +\infty) \rightarrow H$ with the initial condition $U(0) = U_0 \in D(\partial E)$ is a locally Lipschitz continuous function. Moreover, for all $t \geq 0$, we have $U(t) \in D(\partial E)$ and

$$\frac{dU^+}{dt} = -\mathcal{A}^o(U), \quad \text{for all } t \geq 0, \quad (7.4)$$

where $\mathcal{A}^o(U)$ is the canonical section of $\partial E(U)$. Once we construct $\mathcal{A}^o(U_0)$, we will argue about the formula for the solution, which can be checked directly.

1) We learn from Proposition 5.4 that the subdifferential of $E(U_0)$ has the form $(\xi, \zeta) = (-z', z(L))$, where $z(x) \in \text{Sgn}(\frac{d}{dx}u_0(x))$ and $z(L) \in \text{Sgn}(u_0(L) - v_0) = \text{Sgn } 0$. Since u_0 is a.e. differentiable on $(0, b_0)$, then the continuous z must be equal to $\text{Sgn}(\frac{d}{dx}u_0(x))$ for a.e. $x \in (0, b_0)$. Hence, $z(x) = \chi$ for $x \in [0, b_0]$. Moreover, by Proposition 5.2, the optimal z has to be linear on facets. Hence, z takes the following form,

$$z(x) = \frac{\mu - \chi}{L - b_0}(x - b_0) + \chi, \quad x \in (b_0, L),$$

where $z(L) = \mu \in [-1, 1]$ has to be determined.

The variational problem set in (5.21) leads to the following simple question of minimization

$$\int_{b_0}^L \left(\frac{\mu - \chi}{L - b_0} \right)^2 dx + \mu^2$$

with respect to $\mu \in [-1, 1]$. The minimum is attained if and only if

$$\frac{\mu - \chi}{L - b_0} + \mu = 0, \quad (7.5)$$

i.e.

$$\mu = \frac{\chi}{1 + L - b_0}. \quad (7.6)$$

We notice that due to (7.5), after we identify $v_t(L)$ with $\frac{d^+}{dt}v(0)\Big|_{x=L}$, then we have,

$$v_t(L) = -[z \cdot \nu](L) = -\mu = \text{div } z. \quad (7.7)$$

In other words, facet $([b_0, L], \chi)$ and the boundary value move with the same velocity. According to the theory developed in Section 6 and formula (7.7), we conclude that facet $([b_0, L], \chi)$ is calibrable and $(1, \Gamma)$ -coherent.

Moreover, formula (7.7) shows that facet $([b(t), L], \chi)$ will expand, because if we set $h_r(t) = u(\cdot, t)|_{[b(t), L]}$, then $h_r(t)$ must satisfy the equation,

$$u_0(b(t)) = h_r(t), \quad t \geq 0$$

or

$$b(t) = u_0^{-1}(h_r(t))$$

which is well-defined for the expanding facet due to monotonicity of u_0 . We expect that (7.7) will continue to hold for later times, which combined with the above formula for b yields the following ODE for $h_r(t)$,

$$\frac{d}{dt}h_r = \frac{-\chi}{1 + L - u_0^{-1}(h_r)}, \quad h_r(0) = u_0(L). \quad (7.8)$$

The right-hand-side of this equation need not be Lipschitz continuous, but it is a decreasing function, thus there is a unique solution to (7.8).

2) Now, we turn our attention to $x = 0$. We note that at $t = 0$, we have

$$v_t(0) = -[z \cdot \nu] = \text{Sgn } u'_0(0) = \chi.$$

This condition means that a facet forms instantaneously and it moves with initial velocity χ . It is easy to derive a formula for the velocity of this facet. Namely, the argument which leads us to (7.2) yields (7.3) too. Moreover, if we set $h_l(t) = u(\cdot, t)|_{[0, a(t)]}$, then the expanding facet $([0, a(t)], \chi)$ must satisfy the equation $u_0(a(t)) = h_l(t)$. Thus, we come to the conclusion that h_l must satisfy the following ODE

$$\frac{d}{dt}h_l = \frac{-\chi}{1 + u_0^{-1}(h_l)}, \quad h_l(0) = u_0(0). \quad (7.9)$$

Moreover, facet $([0, a(t)], \chi)$ is calibrable and $(1, \Gamma)$ -coherent, (for $t > 0$). We may use the same argument, as in part 1), to establish this result.

3) In the above considerations, facets attached to Γ always move with the same velocity as the boundary layer. Moreover, we see that on $[a(t), b(t)]$, we have $u_t = 0$. Hence, we can summarize our computations in the following formula for u ,

$$u(x, t) = \begin{cases} h_l(t) & x \in [0, a(t)], \\ u_0(x) & x \in (a(t), b(t)), \\ h_r(t) & x \in [b(t), L], \end{cases} \quad (7.10)$$

where $t < T_{cr}$ and $a(T_{cr}) = b(T_{cr})$.

We can check by inspection that $U(\cdot, t) = (u(\cdot, t), \gamma u(\cdot, t))$ belongs to $D(\partial E)$ and it satisfies (7.4). Our claim follows. $\square$

8 A boundary layer behavior in the radial case in two dimensions

We would like to study properties of solution while taking advantage of the radial symmetry. We expect that the examples, we are going to present, look the same in all dimensions bigger than one. However, for the sake of definiteness, we restrict our attention to the planar case.

If Ω is a ball centered at the origin and $\Gamma = \partial\Omega$, then we will see that Corollary 6.1 implies that any radially symmetric facet (F, χ) touching Γ will be calibrable and $(1, \Gamma)$ -coherent. Nonetheless, we find it instructive to present the construction of minimizers of I_τ , (here $\tau = 1$), based on Proposition 6.5 and Lemma 6.7.

We also consider the case when Ω is an annulus and Γ is the boundary of the inner ball, then Theorem 6.8 in general is not applicable. Thus, we may expect to see the boundary layer detachment. In other words, we will see calibrable facets, which are not coherent. We will present detailed calculations in Subsection 8.2. Our argument depends on the form of the canonical selection of the subdifferential presented in Proposition 5.3.

We state our results for $\tau = 1$, which has the obvious advantage of notation simplicity. However, Corollary 5.3 and Lemma 6.1 tell us that once we have a result for $\tau = 1$, then we have the same result for any $\tau > 0$ on a scaled domain. We leave the details for the interested reader.

8.1 A ball

We want to take advantage of a possible simplification of the argument, when we consider a ball $B(0, R)$. We assume the radial symmetry of initial datum $u_0(x) = u_0(|x|)$. We restrict our attention to Lipschitz continuous u_0 and monotone $r \mapsto u_0(r)$ data. We use χ as defined in (7.1).

We assume that $\Gamma = \partial B(0, R)$ and we set $F := \bar{B}(0, R) \setminus B(0, \rho)$ with $\rho \in (0, R)$. We will consider facet (F, χ) , where both F and χ are defined above, which is attached to the boundary.

It is clear that a facet at the center of the ball must appear, see [29], however, in order to simplify the discussion, we assume that a facet $B(0, a_0)$, $a_0 > 0$ is already present, i.e. $U_0 \in D(\partial E)$. We will denote its evolving radius by $a \equiv a(t)$. We are not interested in any other interior facets.

We want to discover only short time dynamics before any possible facet collision occurs at $t = t_{cr}$. We argue, as in the proof of Theorem 7.1, that for this purpose, it is sufficient to study $\mathcal{A}^o(U_0)$, the canonical section of $\partial E(U_0)$, because $U \in C([0, +\infty); H)$, $U(t) \in D(\partial E)$ for all $t \geq 0$ and

$$\frac{d^+ U}{dt} + \mathcal{A}^o(U) = 0 \quad \text{for all } t \geq 0. \quad (8.1)$$

The explicit form of the minimal section of ∂E at U_0 is such that the formulas for the position of the facet depend continuously upon parameters, thus we can directly check that (8.1) holds until a facet collision occurs.

Now, we begin our analysis of the subdifferential. In fact, it is sufficient to consider the localized functional I_τ , in this case $\tau = 1$.

In principle, we have the following cases singled out in Proposition 5.2. If z is a minimizer of (5.21), i.e. I_1 , then we have either:

(1) Facet (F, χ) is calibrable and $(1, \Gamma)$ -coherent, i.e. there $z \in SCH(F, \Omega, \Gamma)$ minimizing I_1 and such that $\operatorname{div} z = \lambda$, $[z \cdot \nu] = \mu$ and $\mu = -\lambda$. This in turn implies that $|\lambda| = |\mu| \leq 1$. This happens when $\gamma u = v$ on Γ ;

or

(2) Facet (F, χ) is calibrable but not coherent, i.e. there is $z \in SCH(F, \Omega, \Gamma)$ minimizing I_1 and such that $\operatorname{div} z = \lambda$, $[z \cdot \nu] = \mu$ and $|\mu| = 1$, but $|\lambda| \neq 1$. We shall see that this cannot happen when $\gamma u = v$ on Γ .

Our task is to construct $z \in SCH(F, \Omega, \Gamma)$ for a given geometric configuration. We will prove that z minimizes I_1 by invoking Proposition 6.5 or Lemma 6.7.

We will first look for configurations corresponding to case (1), i.e. the $(1, \Gamma)$ -calibrability, the t -dependence is suppressed. Due to Proposition 5.3 the Cahn–Hoffman vector field z has the form $z(x) = w(|x|)\mathbf{e}_r$, where $\mathbf{e}_r = x/|x|$, for $x \neq 0$.

The monotonicity assumption on u_0 gives, $\nabla u_0 \neq 0$ for $|x| \in (a, \rho)$, hence $|z| = |\nabla u|/|\nabla u| = 1$. As a result, $z \cdot \mathbf{e}_r = w(r) = \chi$ for $|x| \in (a, \rho)$. In addition, continuity of z yields,

$$(z \cdot \nu_F)(\rho) = -\chi \quad \text{and} \quad [z \cdot \nu](R) = \mu,$$

where ν_F is the outer normal to the facet (F, χ) of the annulus $A(\rho, R) := B(0, R) \setminus \bar{B}(0, \rho)$. Indeed, the value of z is well-defined on $A(a, \rho)$, so the first equality holds. On the other hand,

since $\gamma u - v = 0$ on Γ and $-[z \cdot \nu](R) \in \text{Sgn}(\gamma u - v) = [-1, 1]$, we conclude that $[z \cdot \nu](R) = \mu$, where μ has to be determined. Summing up these conditions yields,

$$w(\rho) = \chi, \quad w(R) = \mu. \quad (8.2)$$

We seek w such that the divergence of z is constant on the facet and $|w(r)| \leq 1$, for all $r \in (\rho, R)$. Since $\text{div } z = w' + w/r$, then we want that w satisfy

$$\text{div } z = \frac{(rw)'}{r} = \lambda, \quad (8.3)$$

as well as boundary conditions (8.2), because due to Proposition 5.2, we need a minimal section of the subdifferential. Since we are considering case (1),

$$\text{div } z = -[z \cdot \nu] \quad \text{for } r = R. \quad (8.4)$$

Combining these conditions gives us $\lambda = -\mu$.

After simple calculations, we reach,

$$\mu = \frac{2\rho\chi}{R^2 - \rho^2 + 2R}. \quad (8.5)$$

We can also see that w has the form

$$w(r) = \frac{\rho\chi(r^2 - R^2 - 2R)}{(\rho^2 - R^2 - 2R)r},$$

and $|z(r)| = |w(r)| \leq 1$ for $r \in [\rho, R]$. The calculations above show that $z \in SCH(F, \Omega, \Gamma)$, see (8.3) and (8.4). Moreover, we invoke Proposition 6.5 to deduce that z is a minimizer of I_τ , with $\tau = 1$. Thus, we conclude that (F, χ) is calibrable and $(1, \Gamma)$ -coherent.

The bulk at $r = \rho$ moves with velocity

$$u_t = \text{div}(\chi \mathbf{e}_r) = \frac{\chi}{\rho}. \quad (8.6)$$

At the same time the sign of facet velocity is $-\chi$, so we conclude that they are going in the opposite directions. As a result, the facet expands.

We also have to follow the boundary behavior of the solution. Since

$$v_t = -[z \cdot \nu] = -\mu = \lambda, \quad (8.7)$$

we see that the velocities of the facet and the boundary value are equal. This implies that $\gamma u = v$ on Γ for $t \in [0, t_{cr}]$.

The missing piece of information is $\rho(t)$, the inner radius of facet $(\bar{B}(0, R) \setminus B(0, \rho(t)), \chi)$. Since the solution $U = (u, v)$ cannot have jumps, we deduce that

$$v(t) = u(\rho(t), t).$$

Taking into account the equations for $u(\rho, t)$, (8.6), and $v(t)$, (8.7), as well as (8.5), we deduce that

$$v_0 - \int_0^t \frac{2\chi\rho(s)}{R^2 - \rho^2(s) + 2R} ds = u_0(\rho) + \frac{\chi t}{\rho}. \quad (8.8)$$

Now, we consider the case $\rho = R$, i.e. a possibility of a facet formation at the boundary with the dynamic condition. Because for a given $t \geq 0$ the selection z is continuous, we notice that (while identifying v_t with $\frac{d^+}{dt}v$)

$$v_t = -[z \cdot \nu] = -\chi \mathbf{e}_r \cdot \nu = -\chi \quad \text{for } t = 0,$$

where ν is the outer normal to $\partial\Omega$. If we compare v_t above with u_t from eq. (8.6), then we see that the boundary value and the bulk move in the opposite directions. We notice that $v_t = -[z \cdot \nu] \in \text{Sgn}(\gamma u - v)$ required by (b2) of Theorem 5.1 (B) is possible if and only if $\gamma u = v$, thus a facet must be formed to preserving the continuity of solutions at Γ . At $t > 0$, we are back in the situation we have already studied.

We might say that we have a collision of the bulk and the boundary, which leads to the creation of a calibrable and $(1, \Gamma)$ -coherent facet at $\rho = R$.

We analyzed all possible inner radii ρ of the facet $F = \overline{A(\rho, R)}$, noticing that only case (1) occurred as long as $\gamma u = v$. Thus by the uniqueness of solution, we conclude that case (2) never happens.

The analysis of the ball is complete. We may collect our observations in a single statement.

Theorem 8.1. Let us assume that Ω is the ball $B(0, R)$ and the initial condition u_0 is radially symmetric, i.e. $u_0(x) = u_0(|x|)$, the function $r \mapsto u_0(r)$ is monotone, belongs to $C^2([0, R])$ and $u'_0(R) \neq 0$. Moreover, $\Gamma = \partial\Omega$ and v_0 is radially symmetric, i.e., v_0 is a constant and $v_0 = \gamma u_0$ and χ is defined in (7.1). Then,

(1) If a facet (F, χ) , where $F = \overline{A(\rho, R)}$, is present, then it is calibrable and $(1, \Gamma)$ -coherent. Moreover, it evolves with a velocity given by

$$u_t = \text{div } z \equiv \lambda := \frac{2\rho}{R^2 - \rho^2 + 2R} \chi, \quad (8.9)$$

where $\chi = 1$ for u_0 decreasing and $\chi = -1$ for u_0 increasing. Moreover, $|u_t| < 1$, as long as $\rho < R$ and $v(t) = \gamma u(t)$ for all $t \geq 0$.

(2) If $\rho = R$, then a facet touching the boundary is created, which moves according to (1) for $t > 0$. (3) The inner radius of the facet, $\rho(t)$, satisfies the following ODE,

$$\frac{d\rho}{dt} \left(u'_0(\rho) - \frac{\chi t}{\rho^2} \right) = -\frac{\chi}{\rho} - \frac{2\rho\chi}{R^2 - \rho^2 + 2R}, \quad \rho(0) = \rho_0. \quad (8.10)$$

Proof. Actually, we constructed explicitly $u(r, t)$, $v(t) \equiv u(R, t)$ and $z \in SCH(F, \Omega, \Gamma)$, as a section of the subdifferential ∂E . The computations we presented above indeed show that,

$$\frac{d^+ U}{dt} = \mathcal{A}^o(U), \quad U(0) = (u_0, v_0).$$

The position of the facet follows from the continuity of solutions. The differentiation of (8.8) with respect to time yields part (3). Here it is important that $u'_0(R) \neq 0$, otherwise the derivative of ρ at $t = 0$ may be infinite. We note that formulas (8.8) and (8.10) are meaningful also in the case $\rho = R$. $\square$

8.2 An annulus case

In order to set the notation, we recall that we write $A(r_{in}, R_{out}) = B(0, R_{out}) \setminus \bar{B}(0, r_{in})$ for an open annulus with inner radius r_{in} and outer radius R_{out} . We set $\Omega = A(r_0, R)$. We notice that the case of a facet touching $\partial B(0, R)$ is not different from the case of ball and it has been solved in the previous subsection. As previously, for the sake of definiteness, we assume that u_0 Lipschitz continuous, radially symmetric and monotone as a function of $|x|$.

8.2.1 Facet evolution

We assume that $\Gamma = \partial B(0, r_0)$. Moreover, we also assume the existence of the inner facet is (F, χ) , where $F = \bar{B}(0, \rho) \setminus B(0, r_0)$, $\rho > r_0$ and χ , defined in (7.1), indicates monotonicity of u_0 in case the initial condition depends only on the distance from the origin. With the help of Proposition 5.3, we deduce that the Cahn–Hoffman vector has the form of $z(x) = w(|x|)\mathbf{e}_r$. However, in order to proceed, we will assume that initially v_0 on Γ equals to the trace of u_0 on Γ . Contrary to the case of the ball, determining the behavior of $\rho(t)$ is more difficult and we will not do this since this is not the main point here.

Now, we begin our analysis of the subdifferential. We know that this is reduced to the study of minimizers of I_τ , here $\tau = 1$. In principle, as for the ball in Subsection 8.1, we have the following cases singled out in Proposition 5.2:

- (1) Facet (F, χ) is calibrable and coherent, provided that $\gamma u = v$ on Γ . In other words, there is $z \in SCH(F, \Omega, \Gamma)$ minimizing I_τ , such that $\operatorname{div} z = \lambda$ on F , $[z \cdot \nu] = \mu$ on Γ $\mu = -\lambda$ and $|\lambda| = |\mu| \leq 1$.
- (2) Facet (F, χ) is calibrable but not coherent, but then $\gamma u \neq v$ on Γ . That is, z , any minimizer of I_τ in $SCH(F, \Omega, \Gamma)$ is such that $\operatorname{div} z = \lambda$ on F , $[z \cdot \nu] = \mu$ on Γ , but $|\lambda| \neq 1$ and $|\mu| = 1$.

We will first look for configurations corresponding to case (1). Arguing as in the previous subsection, we see that $z \in CH(F, \Omega, \Gamma)$, of the form $z = w\mathbf{e}_r$ satisfies the following boundary conditions,

$$\mu = [z \cdot \nu](r_0) = -\mathbf{e}_r \cdot w(r_0)\mathbf{e}_r, \quad (z \cdot \nu_F)(\rho) = \mathbf{e}_r \cdot w(\rho)\mathbf{e}_r = \chi. \quad (8.11)$$

Moreover, we have

$$v_t = -[z \cdot \nu](r_0) = -\mu, \quad \operatorname{div} z = \lambda, \quad \lambda = -\mu.$$

This yields the following boundary value problem for an ODE,

$$\frac{(rw)'}{r} = \lambda, \quad w(r_0) = \lambda, \quad w(\rho) = \chi. \quad (8.12)$$

Solving this ODE yields

$$w(r) = \frac{\lambda r}{2} + \frac{c}{r},$$

where

$$\lambda = 2 \frac{\rho\chi}{\rho^2 - r_0^2 + 2r_0}, \quad c = \frac{\lambda}{2} r_0(2 - r_0). \quad (8.13)$$

Finally,

$$w(r) = \frac{\rho\chi(r^2 - r_0^2 + 2r_0)}{r(\rho^2 - r_0^2 + 2r_0)}, \quad c = \frac{\rho\chi r_0(2 - r_0)}{\rho^2 - r_0^2 + 2r_0}. \quad (8.14)$$

Remark 8.1. In case (1), we can derive the following ODE for $\rho = \rho(t)$:

$$\frac{d\rho}{dt} \left(u'_0(\rho) - \frac{\chi t}{\rho^2} \right) = -\frac{\chi}{\rho} - \frac{2\rho\chi}{\rho^2 - r_0^2 + 2r_0},$$

just as in Theorem 8.1 (3). Hence, we can say that $(u_t, v_t) = (\lambda, -\mu) = (\lambda, \lambda) \in C(F \times [0, t]) \times C(\Gamma_F \times [0, t])$, for any time t before the facet collision.

We have to check when $z \in SCH(F, \Omega, \Gamma)$, i.e.,

$$|\lambda| \leq 1 \quad \text{and} \quad |w(r)| \leq 1 \text{ for all } r \in (r_0, \rho).$$

In particular, we identify the case when the facet and the boundary layer move with the same velocity.

Proposition 8.2. Let us suppose that $\Omega = A(r_0, R)$, $\Gamma = \partial B(0, r_0)$, $u_0(x) = \tilde{u}_0(r)$, $\rho \geq r_0$, u_0 is of C^2 -class and $\tilde{u}_0$ is monotone. We also assume that (F, χ) is a facet, where $F = \overline{A(r_0, \rho)}$ and χ is given by (7.1). Moreover, $v_0 = \tilde{u}_0(r_0)$ and $z(x, t) = w(|x|, t) \frac{x}{|x|}$, where w is given by (8.14), λ is defined by (8.13).

If $r_0 > 2$, then $|\lambda| \leq 1$ and $|w(r)| \leq 1$ for all $r \in (r_0, \rho)$, as a result $z \in SCH(F, \Omega, \Gamma)$. Moreover, z is the minimal selection $\mathcal{A}^o(U)$, i.e. facet (F, χ) is calibrable and $(1, \Gamma)$ -coherent.

Proof. The formulas (8.13) and (8.14) for λ and w were derived on the premise that $\operatorname{div} z = -[z \cdot \nu]$ and $\operatorname{div} z$ is a constant.

It is easy to see that $|z(x)| = |w(|x|)| \leq 1$ if and only if $\rho + r_0 \geq 2$. Moreover, if λ is given by (8.13), then $|\lambda| \leq 1$ is equivalent to $0 \leq (\rho - r_0)(\rho + r_0 - 2)$, and the equality holds if and only if $\rho + r_0 = 2$. The last inequality is true if and only if $\rho + r_0 > 2$. Thus, $z(x, t) = w(|x|, t) \frac{x}{|x|}$ is in $SCH(F, \Omega, \Gamma)$, where $F = \overline{B(0, \rho)} \setminus B(0, r_0)$. We invoke Proposition 6.5 to deduce that z minimizes I_τ with $\tau = 1$. Hence, the facet (F, χ) is calibrable and coherent. Moreover, we see that $v_t = \gamma u_t$ on Γ . □

This proposition states that the facet and the boundary layer move at the same velocity.

The borderline case $r_0 = 2$ behaves like the previous one with some changes. We see from (8.13) that $\lambda = 2\chi/\rho$, thus $|\lambda| < 1$, because $\rho > r_0 = 2$. Moreover, formula (8.13) implies that $c = 0$, hence,

$$w(r) = \frac{\chi r}{\rho},$$

so $|w(r)| \leq 1$. We collect these observations below.

Proposition 8.3. Let us suppose that the hypotheses of Proposition 8.2 hold, but $r_0 = 2$. Then, $\lambda = 2\chi/\rho$, facet (F, χ) , where F is defined in the proposition above and is calibrable and coherent. Moreover, $\chi = v_t = \gamma u_t$ on Γ .

Proof. Clearly, $z(x, t) = w(|x|, t) \frac{x}{|x|}$, where w is given above, belongs to $SCH(F, \Omega, \Gamma)$. Moreover, $|\lambda| = 2/\rho < 1$, because $\rho > r_0 = 2$. Hence, Proposition 6.5 implies that z minimizes I_τ , with $\tau = 1$. As a result, we obtain the canonical section of ∂E . Thus, facet (F, χ) , where $F = \overline{B(0, \rho)} \setminus B(0, r_0)$, is calibrable and coherent. We also see that the velocities v_t and u_t on the facet are equal. □

More computations are required when $r_0 < 2$, they are presented in the course of proof of the proposition below.

Proposition 8.4. Let us suppose that the hypotheses of Proposition 8.2 hold, but $r_0 < 2$. Then, λ is given by (8.17) below and:

- 1) $|\lambda| > 1$ is equivalent to $\rho + r_0 < 2$. Then, facet (F, χ) is calibrable, but not $(1, \Gamma)$ -coherent and $|u_t| > |v_t|$. Since v_t and u_t have the same sign, as a result for $t > 0$ the boundary layer detaches.
- 2) $|\lambda| = 1$ is equivalent to $\rho + r_0 = 2$, and facet (F, χ) is calibrable and $(1, \Gamma)$ -coherent i.e. the boundary layer moves at the velocity of the bulk.
- 3) $|\lambda| < 1$ is equivalent to $\rho + r_0 > 2$. Then, facet (F, χ) is calibrable and $(1, \Gamma)$ -coherent.

Proof. We noticed in the course of proof of Proposition 8.2 that formulas (8.13) for λ yielded $|\lambda| \leq 1$ if and only if $r_0 + \rho \geq 2$. In other words, if $r_0 + \rho < 2$, then (8.13) and (8.14) are no longer correct, because they violate the condition $|w(r_0)| \leq 1$. Thus, we consider equation (8.12) for w , but with the boundary conditions specified below,

$$\frac{(rw)'}{r} = \lambda, \quad w(r_0) = \chi, \quad w(\rho) = \chi. \quad (8.15)$$

Obviously, we need to determine λ to be able to solve (8.15).

In order to justify the condition $w(r_0) = \chi$, we recall that if we had $|w(r_0)| = |[z \cdot \nu]| < 1$, then we would deduce from Proposition 5.2 that $\operatorname{div} z = -[z \cdot \nu]$, but this occurs when $r_0 + \rho \geq 2$.

For the purpose of determining λ we use Lemma 6.2. This leads to the following condition on λ , assuming u_t is the velocity of facet F ,

$$\int_F u_t = \int_F \operatorname{div} z = \int_{\partial F} [z \cdot \nu_F] d\mathcal{H}^1 = \chi \int_{\partial B(0, \rho)} d\mathcal{H}^1 - \chi \int_{\partial B(0, r_0)} d\mathcal{H}^1. \quad (8.16)$$

Since we look for z satisfying $\operatorname{div} z = \lambda$, then we obtain

$$\lambda \pi (\rho^2 - r_0^2) = 2\pi \chi (\rho - r_0).$$

As a result,

$$\lambda = \frac{2\chi}{\rho + r_0} \quad (8.17)$$

and its absolute value exceeds 1 if and only if $\rho + r_0 < 2$.

We easily see that the solution to (8.15) is

$$w(r) = \frac{\chi(r^2 + \rho r_0)}{r(\rho + r_0)}; \quad (8.18)$$

We must make sure that $|w(r)| \leq 1$. We notice that this condition is equivalent to

$$(\rho - r)(r_0 - r) \leq 0$$

which is always true for $r \in [r_0, \rho]$ with equalities at $r = r_0$ or $r = \rho$.

Formula (8.15) also shows that $\mu = -\chi$. Thus, we may apply Lemma 6.7 to deduce that $z \in SCH(F, \Omega, \Gamma)$, we constructed, minimizes I_1 . Thus, facet (F, χ) is calibrable, but not coherent because $\lambda + \mu \neq 0$.

At the same time due to the boundary conditions (8.15), we notice that $v_t = \chi$. In other words the boundary layer is slower than the bulk and it detaches, due to the lack of coherency.

If $\rho + r_0 = 2$, then the Cahn–Hoffman vector field, we constructed above, minimizes I_1 due to Lemma 6.7. Moreover, $\lambda + \mu = 0$, hence facet (F, χ) is calibrable and coherent. Finally, the velocities of the bulk at $r = r_0$ and the boundary layer are equal, so the solution is continuous.

The condition $\rho + r_0 > 2$ is equivalent to $|\lambda| = |u_t| < 1$. In this case we proceed as in the proof of Proposition 8.2 and solve equation (8.12). Then λ is given by (8.13) while (8.14) gives w . In the course of proof of Proposition 8.2, we learned that $|w(r)| \leq 1$ for $r \in (r_0, \rho)$ if and only if $r_0 + \rho > 2$. Now, Proposition 6.5 yields minimality of $z(x) = w(|x|)\mathbf{e}_r$, hence facet (F, χ) is calibrable. Moreover, since $\lambda + \mu = 0$, the facet is $(1, \Gamma)$ -coherent. $\square$

Remark 8.2. We notice that in part 1) the curvature of Γ may be in the interval $(-\infty, -1)$. In case 3), we have calibrable and coherent facet $\bar{A}(r_0, \rho)$ whose curvature is in the interval $(-1, 0)$, which is in accordance with Conjecture 6.9.

8.2.2 Boundary phenomena when $r_0 = \rho$

We want to analyze the situation occurring when the initial data contains no facet and $\Omega = A(r_0, R)$ while $\Gamma = \partial B(0, r_0)$ and $u_0|_\Gamma = v_0$. Exactly, as in the previous subsection, we do not address the question of the position of the inner radius of the facet.

We keep in mind that on the one hand

$$v_t = -[z \cdot \nu], \quad (8.19)$$

while on the other hand $[z \cdot \nu] \in \text{Sgn}(v - \gamma u)$. Meanwhile, since z is continuous and $z = \chi \mathbf{e}_r$ at $t = 0$, we deduce that (we write $\frac{d^+}{dt}v = v_t$),

$$v_t = \chi \quad \text{at } t = 0.$$

We may calculate the bulk velocity, (here $\frac{d^+}{dt}u = u_t$),

$$u_t = \text{div } z = \frac{\chi}{r} \quad \text{at } t = 0. \quad (8.20)$$

Thus, we are ready to state the final result:

Proposition 8.5. Let us suppose that $\Omega = A(r_0, R)$, $\Gamma = \partial B(0, r_0)$ and $u_0(x) = u_0(|x|)$, where function $r \mapsto u_0(r)$ is strictly monotone on $[r_0, R)$, $R > r_0$. We assume that no facet is contained in the data, moreover, $u_0(r_0) = v_0$. Then,

- 1) If $r_0 > 1$, then $0 < \frac{2R}{R^2 - r_0^2 + 2r_0} < |\gamma u_t| = |v_t| \leq 1$ on Γ , for t close to zero, so that a facet forms;
- 2) If $r_0 = 1$, then $\gamma u_t = v_t = \chi$ on Γ , for t close to zero, and no facet is created;
- 3) If $r_0 < 1$, then $|\gamma u_t| > |v_t| = 1$ on Γ , for t close to zero, so that the boundary layer detaches.

Proof. We have already calculated the bulk velocity in (8.20). We want to examine the speed of the bulk at $r = r_0$,

$$|\gamma u_t| = |\text{div } z| = \frac{1}{r_0} \quad \text{at } t = 0.$$

We also noticed that

$$v_t = \chi \quad \text{at } t = 0.$$

Thus, we see that the boundary layer moves faster than the bulk if and only if $r_0 > 1$, at the same time $\gamma u_t v_t \geq 0$. This means that a facet is formed and for $t > 0$. Furthermore, the results obtained earlier are applicable. Therefore, with (8.13) and $1 < r_0 \leq \rho < R$ in mind, we can see that

$$0 < \frac{2R}{R^2 - r_0^2 + 2r_0} < \frac{2\rho}{\rho^2 - r_0^2 + 2r_0} = |\gamma u_t| = |v_t| \leq 1,$$

until the facet collision occurs.

If $r_0 = 1$, then we see that $\gamma u_t = v_t = \chi$. As a result, the boundary layer moves with the bulk, and no new facet is created.

Finally, let us consider $r_0 < 1$, then

$$|\gamma u_t| = |\operatorname{div} z| = \frac{1}{r_0} > 1 \quad \text{at } t = 0.$$

On the other hand

$$v_t = -[z \cdot \nu] = \chi \quad \text{at } t = 0.$$

In other words, the bulk moves faster, so $v - \gamma u \neq 0$ and $\operatorname{Sgn}(v - \gamma u) = -\chi$. As a result, the boundary layer detaches and no facet forms, and $|\gamma u_t| > |v_t| = 1$, for $t > 0$ close to zero. $\square$

The observations we have made above imply the energy decay.

Proposition 8.6. Under the assumptions of Proposition 8.5, the total energy decays, i.e. $t \mapsto E(u(t), v(t))$ is a decreasing function, for $t > 0$ close to zero.

Proof. By Proposition 8.5, we can find a small positive constant δ_0 such that

$$|v_t| \geq \frac{2R}{R^2 - r_0^2 + 2r_0} > 0 \quad \text{on } [0, \delta_0].$$

Additionally, from (4.3) in Theorem 4.3, we can see that

$$\begin{aligned} \frac{d}{dt} E(u(t), v(t)) &= -|u_t(t)|_{L^2(\Omega)}^2 - |v_t(t)|_{L^2(\Gamma)}^2 \\ &\leq -2\pi r_0 |v_t(t)|^2 < -\frac{8\pi r_0 R^2}{(R^2 - r_0^2 + 2r_0)^2} < 0 \quad \text{a.e. } t \in (0, \delta_0). \end{aligned}$$

It implies that the function $t \mapsto E(u(t), v(t))$ is decreasing on the time-interval $[0, \delta_0]$. $\square$

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