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Brudno's theorem for $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts

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Abstract

We generalize Brudno's theorem of 1-dimensional shift dynamical system to $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts. That is to say, in $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshift, the Kolmogorov-Sinai entropy is equivalent to the Kolmogorov complexity density almost everywhere for an ergodic shift-invariant measure.

Keywords. Brudno's theorem, Kolmogorov-Sinai entropy, Kolmogorov complexity, Shannon-McMillan-Breiman theorem, Subshifts, $\mathbb{Z}^d$ -action, Universally typical sets

1 Introduction

In a topological dynamical system, A. A. Brudno defined a complexity of the trajectory of a point in the space by using the notion of Kolmogorov complexity, and showed the relationship between this quantity and the Kolmogorov-Sinai entropy [2]. As a preliminary step, Brudno considered the 1-dimensional shift dynamical system and showed that, for an ergodic shift-invariant measure, the Kolmogorov complexity density is equal to the Kolmogorov-Sinai entropy almost everywhere [2, Theorem 1.1].

A partial approach to generalize this theorem to a d -dimensional case is found in [10]. S. G. Simpson showed that, in $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts, there exists a point such that its Kolmogorov complexity density is coincident with the topological entropy [10]. Examining Simpson's proof, we see that what he showed substantively is that the Kolmogorov complexity density is equal to the Kolmogorov-Sinai entropy almost everywhere only for a measure of maximal entropy.

The purpose of this paper is to generalize the Brudno's theorem of the $\mathbb{Z}_+^1$ -action shift dynamical system to $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshifts. The main theorem is the following:

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Theorem 3.1 *If $\mu \in EM(S, \varsigma)$, then*

$$\mathcal{K}(\omega) = h_\varsigma(\mu), \quad \mu\text{-a.e. } \omega \in S.$$

Here S denotes $\mathbb{Z}^d$ (or $\mathbb{Z}_+^d$) subshift, ς denotes the shift action on S , $EM(S, \varsigma)$ denotes the set of all ergodic shift-invariant measures on the topological dynamical system (S, ς) , $\mathcal{K}(\omega)$ denotes the Kolmogorov complexity density of ω , and $h_\varsigma(\mu)$ denotes the Kolmogorov-Sinai entropy of the measure preserving dynamical system $(S, \mathfrak{B}(S), \mu, \varsigma)$. We give the rigorous definition of these terms in Section 2.

In Section 2, we introduce some basic mathematical notions in ergodic theory, Kolmogorov complexity and shift dynamical systems. We used [4, 7, 9, 11] as main references for this section. Using these basic notions, we define the Kolmogorov complexity density of each point of $\Sigma^{\mathbb{Z}^d}$ (or $\Sigma^{\mathbb{Z}_+^d}$) naturally. In Section 3, we prove the main theorem and give some examples. The proof essentially uses Shannon-McMillan-Breiman theorem and universally typical sets.

2 Some Mathematical Preliminaries

We first give quick reviews for some mathematical results related to the main theorem. We will not give proofs of theorems, see e.g. [4, 8]. We write $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. For an arbitrary fixed $d \in \mathbb{N}$, we set $G := \mathbb{Z}^d$ or $G := \mathbb{Z}_+^d$. For all $n \in \mathbb{N}$, let $\Lambda_n := \{g = (g_i)_{i=1}^d \in G : \forall i \in \{1, \dots, d\}, |g_i| < n\}$. Then we have

$$|\Lambda_n| = \begin{cases} (2n-1)^d & (G = \mathbb{Z}^d), \\ n^d & (G = \mathbb{Z}_+^d), \end{cases}$$

where we denote by $|A|$ the cardinality of a set A .

2.1 Ergodic theory

Let $(X, \mathfrak{B}, \mu, \mathcal{T})$ be a *measure preserving dynamical system* (m.p.d.s.), namely, $(X, \mathfrak{B}, \mu)$ be a probability space and $\mathcal{T} = (T^g)_{g \in G}$ be a measurable μ -invariant action of G on X . A set $A \in \mathfrak{B}$ is said to be $\mathcal{T}$ -invariant mod μ if and only if $\mu(T^{-g}A \triangle A) = 0$ holds for all $g \in G$, where $\triangle$ denotes the symmetric difference. We write $\mathcal{I}_\mu(\mathcal{T}) := \{A \in \mathfrak{B} : A \text{ is } \mathcal{T}\text{-invariant mod } \mu\}$. If $\mu(A) = 0$ or $\mu(A) = 1$ for all $A \in \mathcal{I}_\mu(\mathcal{T})$, then the m.p.d.s. $(X, \mathfrak{B}, \mu, \mathcal{T})$ is said to be *ergodic*. A family of measurable sets $\alpha = \{A_i\}_{i \in I}$ is called a μ -*partition* of X if $\mu(A_i \cap A_j) = 0$ ($i \neq j$), $\mu(X \setminus \bigcup_{i \in I} A_i) = 0$ and $\mu(A_i) > 0$ ($\forall i \in I$). Let α be a μ -partition of X . The *information* of α is the function I_α on X defined by $I_\alpha(x) := -\sum_{A \in \alpha} (\log_2 \mu(A)) \cdot 1_A(x)$ ($\forall x \in X$). The *entropy* of α is defined by the average information, i.e., $H_\mu(\alpha) := \int_X I_\alpha d\mu = \sum_{A \in \alpha} \varphi(\mu(A))$ where $\varphi(t) := -t \log_2 t$. From Kolmogorov complexity's point of view, we choose the binary logarithm $\log_2$ instead of $\log_e$. Let β be another μ -partition. The *common refinement* of α and β is $\alpha \vee \beta := \{A \cap B : A \in \alpha, B \in \beta, \mu(A \cap B) > 0\}$. We set $T^{-g}\alpha := \{T^{-g}A : A \in \alpha\}$ for each $g \in G$ and $\alpha^\Lambda :=$

$\bigvee_{g \in \Lambda} T^{-g} \alpha$ for a finite subset $\Lambda \subset G$. The *dynamical entropy* of the m.p.d.s. $(X, \mathfrak{B}, \mu, \mathcal{T})$ relative to the partition α is $h(\mu, \alpha, \mathcal{T}) := \inf_{n > 0} \frac{1}{|\Lambda_n|} H_\mu(\alpha^{\Lambda_n}) = \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} H_\mu(\alpha^{\Lambda_n})$.

Theorem 2.1 (Shannon-McMillan-Breiman) *Let $(X, \mathfrak{B}, \mu, \mathcal{T})$ be an ergodic m.p.d.s. and α be a μ -partition of X with $H_\mu(\alpha) < \infty$. Then*

$$h(\mu, \alpha, \mathcal{T}) = \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} I_{\alpha^{\Lambda_n}} \quad \text{in } L^1(X, \mu).$$

Moreover, if α is finite, then this convergence holds also for μ -a.s. $x \in X$.

The *Kolmogorov-Sinai entropy* of the m.p.d.s $(X, \mathfrak{B}, \mu, \mathcal{T})$ is defined by

$$h_{\mathcal{T}}(\mu) := \sup\{h(\mu, \alpha, \mathcal{T}) : \alpha \text{ is a } \mu\text{-partition with } H_\mu(\alpha) < \infty\}.$$

We denote by α^G the σ -algebra generated by all $T^{-g} \alpha$, $g \in G$. A μ -partition α is called a μ -generator if $\alpha^G = \mathfrak{B} \mod \mu$, where this equation means that $\forall A \in \mathfrak{B}, \exists B \in \alpha^G, \mu(A \triangle B) = 0$.

Theorem 2.2 (Kolmogorov-Sinai) *Let $(X, \mathfrak{B}, \mu, \mathcal{T})$ be a m.p.d.s. and α be a μ -generator such that $H(\alpha) < \infty$. Then $h_{\mathcal{T}}(\mu) = h(\mu, \alpha, \mathcal{T})$.*

Let $(X, \mathcal{T})$ be a *topological dynamical system* (t.d.s.), namely, X be a compact metrizable space and $\mathcal{T} = (T^g)_{g \in G}$ be a continuous action of G on X . In this setting we denote by $\mathfrak{B}(X)$ the Borel σ -algebra of X . We denote by $M(X)$ the set of all probability measures on the Borel measurable space $(X, \mathfrak{B}(X))$, by $M(X, \mathcal{T})$ the set of all $\mathcal{T}$ -invariant probability measures on $(X, \mathfrak{B}(X))$ and by $EM(X, \mathcal{T})$ the set of all ergodic members in $M(X, \mathcal{T})$, respectively.

2.2 Kolmogorov complexity

Let Σ be a finite set and $|\Sigma| \geq 2$. Without loss of generality, we set $\Sigma := \{0, 1, \dots, N\}$ where $N \in \mathbb{N}$. We define the set of all finite *strings* over Σ as

$$\Sigma^* := \bigcup_{n=0}^{\infty} \Sigma^n = \{\lambda, 0, 1, \dots, N, 00, 01, \dots, 0N, 10, \dots, 1N, \dots, NN, 000, \dots\},$$

where $\Sigma^0 = \{\lambda\}$ and λ denote the empty string. The *length* of $x \in \Sigma^*$ is denoted by $l(x)$. For all $x, y \in \Sigma^*$, we call x a *prefix* of y if there exists $z \in \Sigma^*$ such that $y = xz$. A set $A \subset \Sigma^*$ is said to be *prefix-free* if, for all $x \in A$, the elements of $A \setminus \{x\}$ are not prefixes of x . Let $\mathcal{D}$ be a subset of $\{0, 1\}^*$ and let f be a function from $\mathcal{D}$ to Σ^* . If $\mathcal{D} \subsetneq \{0, 1\}^*$, we call such a function f a *partial function* and write $f : \{0, 1\}^* \rightsquigarrow \Sigma^*$, and if $\mathcal{D} = \{0, 1\}^*$ then we call f a *total function*. A partial function $\phi : \{0, 1\}^* \rightsquigarrow \Sigma^*$ is said to be *partial recursive* if and only if there exists a Turing machine M such that ϕ is computed by M , i.e., for all $x \in \{0, 1\}^*$, M on input x halts if and only if $x \in \text{dom}(\phi)$, in that case, M

outputs $\phi(x)$. Moreover, if $\text{dom}(\phi)$ is prefix-free, then we call ϕ a *partial recursive prefix function*. Let $\phi : \{0, 1\}^* \rightsquigarrow \Sigma^*$ be a partial recursive prefix function. For all $x \in \Sigma^*$, the *complexity* of x with respect to ϕ is defined by

$$K_\phi(x) := \begin{cases} \min\{l(p) : p \in \phi^{-1}(x)\}, & (\phi^{-1}(x) \neq \emptyset), \\ \infty & (\phi^{-1}(x) = \emptyset). \end{cases}$$

A partial recursive prefix function $\phi : \{0, 1\}^* \rightsquigarrow \Sigma^*$ is said to be *additively optimal* if for all partial recursive prefix functions $\psi : \{0, 1\}^* \rightsquigarrow \Sigma^*$, there exists a constant $c_{\phi, \psi} \in \mathbb{R}$ such that for all $x \in \Sigma^*$, $K_\phi(x) \leq K_\psi(x) + c_{\phi, \psi}$. We fix such a function ϕ and define the *prefix Kolmogorov complexity* of $x \in \Sigma^*$ by $K(x) := K_\phi(x)$.

2.3 Shift dynamical system

Let $\Sigma := \{0, 1, \dots, N\}$ ($N \in \mathbb{N}$) and we set $\Omega := \Sigma^G$. By Tychonoff's theorem, Ω endowed with the product topology of the discrete topology on Σ is a compact topological space. For all $n \in \mathbb{N}$ and for all $s \in \Sigma^{\Lambda_n}$, we define the *cylinder set* of s by $\llbracket s \rrbracket := \{\omega \in \Omega : \omega \upharpoonright \Lambda_n = s\}$. We set

$$\Sigma^{\Lambda^*} := \bigcup_{n=0}^{\infty} \Sigma^{\Lambda_n}$$

where $\Sigma^{\Lambda_0} := \{\lambda\}$ and write $\llbracket V \rrbracket := \bigcup_{s \in V} \llbracket s \rrbracket$ for all $V \subset \Sigma^{\Lambda^*}$. Let $\sigma^g : \Omega \rightarrow \Omega$ denote the *shift* by $g \in G$, i.e., $(\sigma^g \omega)_i := \omega_{i+g}$ for all $\omega = (\omega_i)_{i \in G}$, and we write $\sigma := (\sigma^g)_{g \in G}$. Since σ is a continuous action of G on Ω , (Ω, σ) is a t.d.s.. Note that for all $\mu \in M(\Omega, \sigma)$, the partition $\{\llbracket s \rrbracket\}_{s \in \Sigma^{\Lambda_1}}$ is a μ -generator. A nonempty subset $S \subset \Omega$ is called a *subshift* if and only if S is shift-invariant and closed. If $S \subset \Omega$ is a subshift, then $(S, \sigma \upharpoonright S)$ is a t.d.s., where $\sigma \upharpoonright S := (\sigma^g \upharpoonright S)_{g \in G}$. We fix an arbitrary bijective computable function $f : \mathbb{Z}_+ \rightarrow G$ such that for all $n \in \mathbb{N}$, $f(\{0, 1, \dots, |\Lambda_n| - 1\}) = \Lambda_n$ and define $\mathcal{G} : \Sigma^{\Lambda^*} \rightarrow \Sigma^*$ as follows:

$$\mathcal{G}(s) := \begin{cases} s_{f(0)} \cdots s_{f(|\Lambda_n|-1)}, & s = (s_g)_{g \in \Lambda_n} \in \Sigma^{\Lambda_n} \ (n \in \mathbb{N}), \\ \lambda, & s = \lambda. \end{cases}$$

We define the prefix Kolmogorov complexity of $s \in \Sigma^{\Lambda^*}$ by $K(s) := K(\mathcal{G}(s))$.

Lemma 2.3 *For all $n, k \in \mathbb{N}$, $|\{s \in \Sigma^{\Lambda_n} : K(s) < k\}| < 2^k$.*

Proof. By [3, Theorem 7.2.4], we have for all $n, k \in \mathbb{N}$,

$$\begin{aligned} |\{s \in \Sigma^{\Lambda_n} : K(s) < k\}| &= |\{\mathcal{G}(s) \in \Sigma^{|\Lambda_n|} : K(\mathcal{G}(s)) < k\}| \\ &\leq |\{x \in \Sigma^* : K(x) < k\}| < 2^k. \end{aligned}$$

□

The *upper* and *lower Kolmogorov complexity density* of $\omega \in \Omega$ are defined by

$$\overline{\mathcal{K}}(\omega) := \limsup_{n \rightarrow \infty} \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|}, \quad \underline{\mathcal{K}}(\omega) := \liminf_{n \rightarrow \infty} \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|}.$$

If $\overline{\mathcal{K}}(\omega) = \underline{\mathcal{K}}(\omega)$, we simply denote them by $\mathcal{K}(\omega)$. The quantities $\overline{\mathcal{K}}(\omega)$ and $\underline{\mathcal{K}}(\omega)$ are independent of the choice of additively optimal partial recursive prefix function ϕ and $\mathcal{G}$, and uniquely defined.

Lemma 2.4 *The functions $\overline{\mathcal{K}}, \underline{\mathcal{K}} : \Omega \rightarrow \mathbb{R}$ are measurable.*

Proof. Let us show that $\overline{\mathcal{K}}$ is measurable. For all $x \in \mathbb{R}$, we have

$$\begin{aligned} \overline{\mathcal{K}}^{-1}((-\infty, x)) &= \left\{ \omega \in \Omega : \limsup_{n \rightarrow \infty} \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} < x \right\} \\ &= \bigcup_{k \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \left\{ \omega \in \Omega : \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} < x - \frac{1}{k} \right\}. \end{aligned}$$

Here

$$\left\{ \omega : \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} < x - \frac{1}{k} \right\} = \begin{cases} \bigcup_{l=0}^{\lceil |\Lambda_n|(x - \frac{1}{k}) - 1 \rceil} \{ \omega : K(\omega \upharpoonright \Lambda_n) = l \}, & x - \frac{1}{k} > 0, \\ \emptyset, & x - \frac{1}{k} \leq 0. \end{cases}$$

Since the set $\{ \omega \in \Omega : K(\omega \upharpoonright \Lambda_n) = l \}$ is cylinder, then the set $\overline{\mathcal{K}}^{-1}((-\infty, x))$ is measurable. Hence the function $\overline{\mathcal{K}}$ is measurable. The proof for $\underline{\mathcal{K}}$ is similar. $\square$

Remark 2.5 *Let C be a plain Kolmogorov complexity (that is not conditioned on prefix function). By [7, Example 3.1.4], we have for all $s \in \Sigma^{\Lambda^*}$*

$$C(\mathcal{G}(s)) \leq K(\mathcal{G}(s)) \leq C(\mathcal{G}(s)) + 2 \log C(\mathcal{G}(s)).$$

It means that C and K are asymptotically equal. Then we may use C to define $\overline{\mathcal{K}}, \underline{\mathcal{K}}$.

3 Relation between KS entropy and Kolmogorov complexity

Let $d \in \mathbb{N}$, $G = \mathbb{Z}^d$ or $G = \mathbb{Z}_+^d$, $\Sigma = \{0, 1, \dots, N\}$ ($N \in \mathbb{N}$) and $S \subset \Omega$ ($:= \Sigma^G$) be a subshift. Other notations are the same as before. We set $\varsigma := \sigma \upharpoonright S$. Note that (S, ς) is a t.d.s.. We now state the main result.

Theorem 3.1 *If $\mu \in EM(S, \varsigma)$, then*

$$\mathcal{K}(\omega) = h_\varsigma(\mu), \quad \mu\text{-a.e. } \omega \in S. \quad (3.1)$$

Remark 3.2 *Brudno's original result is on the case $G = \mathbb{Z}_+$ only [2]. In the case $G = \mathbb{Z}^d$ or $G = \mathbb{Z}_+^d$, Simpson showed that if μ is a measure of maximal entropy, then (3.1) holds [10]. Our theorem is a generalization of them.*

It is sufficient to prove the theorem for the case $S = \Omega$. Because, if $\mu \in EM(\Omega, \sigma)$ and $\mu(S) = 1$, then $\mu \upharpoonright S \in EM(S, \varsigma)$ and $h_\varsigma(\mu \upharpoonright S) = h_\sigma(\mu)$ hold where $\mu \upharpoonright S$ denotes the restriction of μ to S . So we prove the theorem about full shift (Ω, σ) .

Theorem 3.3 (μ -typical sets) *Let $\mu \in EM(\Omega, \sigma)$. For all $\epsilon > 0$ and $n \in \mathbb{N}$, we set*

$$\begin{aligned}\mathfrak{T}_\epsilon^{(n)} &:= \left\{ s \in \Sigma^{\Lambda_n} : 2^{-|\Lambda_n|(h_\sigma(\mu)+\epsilon)} < \mu([s]) < 2^{-|\Lambda_n|(h_\sigma(\mu)-\epsilon)} \right\}, \\ \mathfrak{T}_\epsilon &:= \liminf_{n \rightarrow \infty} [\mathfrak{T}_\epsilon^{(n)}].\end{aligned}$$

Then the following holds:

$$\mu(\mathfrak{T}_\epsilon) = \lim_{n \rightarrow \infty} \mu([\mathfrak{T}_\epsilon^{(n)}]) = 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\log_2 |\mathfrak{T}_\epsilon^{(n)}|}{|\Lambda_n|} \leq h_\sigma(\mu) + \epsilon.$$

Proof. It follows from Shannon-McMillan-Breiman theorem (Theorem 2.1). $\square$

Lemma 3.4 *If $\mu \in EM(\Omega, \sigma)$, then*

$$\underline{\mathcal{K}}(\omega) \geq h_\sigma(\mu), \quad \mu\text{-a.e. } \omega \in \Omega. \quad (3.2)$$

Proof. If $h_\sigma(\mu) = 0$, then (3.2) is obvious. Let $h_\sigma(\mu) > 0$ and fix an arbitrary $k \in \mathbb{N}$ such that $\frac{1}{k} < h_\sigma(\mu)$. For all $n \in \mathbb{N}$, we set $D_{n,k} := \left\{ s \in \Sigma^{\Lambda_n} : \frac{K(s)}{|\Lambda_n|} \leq h_\sigma(\mu) - \frac{1}{k} \right\}$. By Lemma 2.3, we have

$$|D_{n,k}| \leq 2^{|\Lambda_n|(h_\sigma(\mu) - \frac{1}{k}) + 1}. \quad (3.3)$$

We fix an arbitrary $\epsilon \in (0, \frac{1}{k})$ and set $\mathfrak{T}_{\frac{1}{k}-\epsilon}^{(n)}$ as in Theorem 3.3. Then, by Theorem 3.3, we have for μ -a.e. $\omega \in \Omega$,

$$\exists N_\omega \in \mathbb{N}, \forall n \geq N_\omega, \omega \in [\mathfrak{T}_{\frac{1}{k}-\epsilon}^{(n)}]. \quad (3.4)$$

On the other hand, by (3.3) and the definition of $\mathfrak{T}_{\frac{1}{k}-\epsilon}^{(n)}$, we have

$$\begin{aligned}\mu([D_{n,k}] \cap [\mathfrak{T}_{\frac{1}{k}-\epsilon}^{(n)}]) &= \mu \left(\bigcup_{s \in D_{n,k} \cap \mathfrak{T}_{\frac{1}{k}-\epsilon}^{(n)}} [s] \right) \leq \sum_{s \in D_{n,k} \cap \mathfrak{T}_{\frac{1}{k}-\epsilon}^{(n)}} \mu([s]) \\ &\leq 2^{|\Lambda_n|(h_\sigma(\mu) - \frac{1}{k}) + 1} \cdot 2^{-|\Lambda_n|(h_\sigma(\mu) - \frac{1}{k} + \epsilon)} = 2^{-|\Lambda_n|\epsilon + 1}.\end{aligned}$$

Hence $\sum_{n=1}^{\infty} \mu([D_{n,k}] \cap [\mathfrak{T}_{\frac{1}{k}-\epsilon}^{(n)}]) < \infty$ holds. Therefore, by the Borel-Cantelli lemma, for μ -a.e. $\omega \in \Omega$,

$$\exists N'_\omega \in \mathbb{N}, \forall n \geq N'_\omega, \omega \notin [D_{n,k}] \cap [\mathfrak{T}_{\frac{1}{k}-\epsilon}^{(n)}]. \quad (3.5)$$

By (3.4) and (3.5), for μ -a.e. $\omega \in \Omega$, we have

$$\exists N''_{\omega} \in \mathbb{N}, \forall n \geq N''_{\omega}, \omega \notin \llbracket D_{n,k} \rrbracket.$$

Since $\omega \notin \llbracket D_{n,k} \rrbracket$ means $\frac{K(\omega|\Lambda_n)}{|\Lambda_n|} > h_{\sigma}(\mu) - \frac{1}{k}$, we have for all $k \geq \lceil \frac{1}{h_{\sigma}(\mu)} \rceil + 1$,

$$\underline{\mathcal{K}}(\omega) \geq h_{\sigma}(\mu) - \frac{1}{k}, \quad \mu\text{-a.e. } \omega \in \Omega. \quad (3.6)$$

Hence

$$\begin{aligned} \mu(\{\omega : \underline{\mathcal{K}}(\omega) < h_{\sigma}(\mu)\}) &= \mu\left(\bigcup_{k=\lceil \frac{1}{h_{\sigma}(\mu)} \rceil + 1}^{\infty} \left\{\omega : \underline{\mathcal{K}}(\omega) < h_{\sigma}(\mu) - \frac{1}{k}\right\}\right) \\ &\leq \sum_{k=\lceil \frac{1}{h_{\sigma}(\mu)} \rceil + 1}^{\infty} \mu\left(\left\{\omega : \underline{\mathcal{K}}(\omega) < h_{\sigma}(\mu) - \frac{1}{k}\right\}\right) = 0. \end{aligned}$$

Therefore (3.2) holds. $\square$

The following theorem plays a key role to prove the inverse direction.

Theorem 3.5 (Universally typical sets) *For all rational number $h_0 \in (0, \log_2 |\Sigma|]$, there exists a sequence of subsets $\{\mathfrak{U}_{h_0}^{(n)} \subset \Sigma^{\Lambda_n}\}_n$ such that the following conditions hold:*

(1) *For all $\mu \in EM(\Omega, \sigma)$ with $h_{\sigma}(\mu) < h_0$,*

$$\mu(\mathfrak{U}_{h_0}) = \lim_{n \rightarrow \infty} \mu(\llbracket \mathfrak{U}_{h_0}^{(n)} \rrbracket) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log_2 |\mathfrak{U}_{h_0}^{(n)}|}{|\Lambda_n|} = h_0$$

hold where $\mathfrak{U}_{h_0} := \liminf_{n \rightarrow \infty} \llbracket \mathfrak{U}_{h_0}^{(n)} \rrbracket$.

(2) *The sequence of subsets $\{\mathfrak{U}_{h_0}^{(n)} \subset \Sigma^{\Lambda_n}\}_n$ is computable.*

Proof. See [5, Thoerem 3.1] and its proof. $\square$

Lemma 3.6 *If $\mu \in EM(\Omega, \sigma)$, then*

$$\overline{\mathcal{K}}(\omega) \leq h_{\sigma}(\mu), \quad \mu\text{-a.e. } \omega \in \Omega. \quad (3.7)$$

Proof. Let $n \in \mathbb{N}$ and $h_0 \in (h_{\sigma}(\mu), \log_2 |\Sigma|]$ be a rational number. We set $\mathfrak{U}_{h_0}^{(n)}$ and $\mathfrak{U}_{h_0}$ as in Theorem 3.5. By its definition, for all $\omega \in \mathfrak{U}_{h_0}$, we have

$$\exists N \in \mathbb{N}, \forall n > N, \omega \in \llbracket \mathfrak{U}_{h_0}^{(n)} \rrbracket. \quad (3.8)$$

Note that $\omega \in \llbracket \mathfrak{U}_{h_0}^{(n)} \rrbracket$ means $\omega \upharpoonright \Lambda_n \in \mathfrak{U}_{h_0}^{(n)}$, and here we can encode each $s \in \mathfrak{U}_{h_0}^{(n)}$ into $\lceil \log_2 |\mathfrak{U}_{h_0}^{(n)}| \rceil$ bits code. Then the following holds for all $\omega \in \mathfrak{U}_{h_0}$:

$$\exists N \in \mathbb{N}, \forall n > N, \frac{C(\mathcal{G}(\omega \upharpoonright \Lambda_n))}{|\Lambda_n|} \leq \frac{\log_2 |\mathfrak{U}_{h_0}^{(n)}| + \log_2 n + \text{const.}}{|\Lambda_n|}, \quad (3.9)$$

where C be a plain Kolmogorov complexity. As previously stated in Remark 2.5, C and K are asymptotically equal. Then by Theorem 3.5 and (3.9), we have $\overline{\mathcal{K}}(\omega) \leq h_0$ for all $\omega \in \mathfrak{U}_{h_0}$, namely, $\mathfrak{U}_{h_0} \subset \{\omega \in \Omega : \overline{\mathcal{K}}(\omega) \leq h_0\}$. Let $h_{0,k} \in (h_\sigma(\mu), \log_2 |\Sigma|]$ ($k \in \mathbb{N}$) be rational numbers with $\lim_{k \rightarrow \infty} h_{0,k} = h_\sigma(\mu)$. Then we have

$$\begin{aligned} \mu(\{\omega : \overline{\mathcal{K}}(\omega) > h_\sigma(\mu)\}) &= \mu\left(\bigcup_{k=1}^{\infty} \{\omega : \overline{\mathcal{K}}(\omega) > h_{0,k}\}\right) \\ &\leq \sum_{k=1}^{\infty} \mu(\{\omega : \overline{\mathcal{K}}(\omega) > h_{0,k}\}) \leq \sum_{k=1}^{\infty} \mu(\mathfrak{U}_{h_{0,k}}^c) = 0. \end{aligned}$$

Therefore (3.7) holds. $\square$

Theorem 3.1 follows from Lemma 3.4 and Lemma 3.6.

Remark 3.7 In Theorem 3.1, μ is not necessarily computable. Especially if μ is a computable measure, then Theorem 3.1 is easily seen by the following way: Let $\nu \in M(\Sigma^{\mathbb{Z}_+})$ be a computable measure such that for all $\omega \in \Omega$, $\mu(\llbracket \omega \upharpoonright \Lambda_n \rrbracket) = \nu(\llbracket \mathcal{G}(\omega \upharpoonright \Lambda_n) \rrbracket)$. By [6, THEOREM 5.1, LEMMA 5.2], if $\mathcal{G}(\omega)(:= \lim_{n \rightarrow \infty} \mathcal{G}(\omega \upharpoonright \Lambda_n), \omega \in \Omega)$ is Martin-Löff random with respect to ν , then there exist $c_1, c_2 > 0$ such that for all $n \in \mathbb{N}$

$$\begin{aligned} -\log_2 \nu(\llbracket \mathcal{G}(\omega \upharpoonright \Lambda_n) \rrbracket) - c_1 &< K(\mathcal{G}(\omega \upharpoonright \Lambda_n)) \\ &\leq -\log_2 \nu(\llbracket \mathcal{G}(\omega \upharpoonright \Lambda_n) \rrbracket) + 2 \log_2 l(\mathcal{G}(\omega \upharpoonright \Lambda_n)) + c_2. \end{aligned}$$

Then we have for μ -a.e. $\omega \in \Omega$

$$\mathcal{K}(\omega) = \lim_{n \rightarrow \infty} \frac{-\log_2 \mu(\llbracket \omega \upharpoonright \Lambda_n \rrbracket)}{|\Lambda_n|} = h_\sigma(\mu).$$

The last equality is derived from Shannon-McMillan-Breiman theorem.

Example 3.8 (d -dimensional Bernoulli shifts) Let (Ω, σ) be the $\mathbb{Z}^d$ or $\mathbb{Z}_+^d$ shift space as before. We fix a probability vector $q = (q_i : i \in \Sigma)$ on Σ and denote the corresponding Bernoulli measure on $\mathfrak{B}(\Omega)$ by $\mu := q^{\times G}$. Then, by Kolmogorov-Sinai theorem (Theorem 2.2), we can show that $h_\sigma(\mu) = \sum_{i \in \Sigma} \varphi(q_i)$. By Theorem 3.1, we have for μ -a.e. $\omega \in \Omega$

$$\mathcal{K}(\omega) = \sum_{i \in \Sigma} \varphi(q_i).$$

Corollary 3.9 *If $\mu \in M(\Omega, \sigma)$, then there exists $\mathcal{K}(\omega) = \lim_{n \rightarrow \infty} \frac{\mathcal{K}(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|}$ for μ -a.e. $\omega \in \Omega$ and the following holds:*

$$h_\sigma(\mu) = \mu(\mathcal{K}) = \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{s \in \Sigma^{\Lambda_n}} \mathcal{K}(s) \mu(\llbracket s \rrbracket).$$

Proof. Let $\mu = \int_{EM(\Omega, \sigma)} \nu d\rho(\nu)$ be the ergodic decomposition, where ρ be a probability measure on $EM(\Omega, \sigma)$ (see [4, 9, 11]). By Jacobs's theorem [11, Theorem 8.4] and Theorem 3.1, we have

$$\mu(\overline{\mathcal{K}}) = \int_{EM(\Omega, \sigma)} \left\{ \int_{\Omega} \overline{\mathcal{K}}(\omega) d\nu(\omega) \right\} d\rho(\nu) = \int_{EM(\Omega, \sigma)} h_\sigma(\nu) d\rho(\nu) = h_\sigma(\mu)$$

and $\mu(\underline{\mathcal{K}}) = h_\sigma(\mu)$ is also the same. Hence for μ -a.e. $\omega \in \Omega$ there exists $\mathcal{K}(\omega)$ and $\mu(\mathcal{K}) = h_\sigma(\mu)$ holds. On the other hand, by (3.9) and Lebesgue's convergence theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{s \in \Sigma^{\Lambda_n}} \mathcal{K}(s) \mu(\llbracket s \rrbracket) &= \lim_{n \rightarrow \infty} \int_{\Omega} \frac{\mathcal{K}(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} d\mu(\omega) \\ &= \int_{\Omega} \lim_{n \rightarrow \infty} \frac{\mathcal{K}(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} d\mu(\omega) = \mu(\mathcal{K}). \end{aligned}$$

□

Remark 3.10 *In the case $G = \mathbb{Z}_+$, Corollary 3.9 can be found in [1].*

Example 3.11 (d -dimensional Ising model) *Let $d \in \mathbb{N}$ and $\Sigma := \{+1, -1\}$. Here $+1$ and -1 represent “spin up” and “spin down” at the sites of a “lattice gas” on $G := \mathbb{Z}^d$, respectively. Let $\Omega := \Sigma^G$ be a configuration space and σ be a shift action of G on Ω . For d -dimensional Ising model, the local energy function $\psi : \Omega \rightarrow \mathbb{R}$ is defined by*

$$\psi(\omega) := -\beta \left(-\sum_{j=1}^d (\omega_{\mathbf{0}} \omega_{e_j} + \omega_{\mathbf{0}} \omega_{-e_j}) - B \omega_{\mathbf{0}} \right), \quad \omega \in \Omega,$$

where $\mathbf{0} := (0, \dots, 0)$, $e_j := (0, \dots, \overset{j\text{th}}{1}, \dots, 0) \in G$. Here $-\sum_{j=1}^d (\omega_{\mathbf{0}} \omega_{e_j} + \omega_{\mathbf{0}} \omega_{-e_j})$ represents the interaction between neighboring spins, $-B \omega_{\mathbf{0}}$ represents the effect of a magnetic field $B \in \mathbb{R}$ on the spin at site $\mathbf{0}$ and $\beta \geq 0$ denote the inverse temperature. Then the pressure of this model is given by

$$p(\psi) = \sup_{\mu \in M(\Omega, \sigma)} \mu(\mathcal{K} + \psi).$$

In mathematical point of view, this example is just a replacement of $h_\sigma(\mu)$ by $\mu(\mathcal{K})$, but it shows that the generalization of Brudno's theorem to $\mathbb{Z}^d$ -action (especially $d = 2$ or 3) has a physical background.

By using Brudno's theorem for multidimensional subshifts, we can construct a universally typical sets of multidimensional data as follows.

Theorem 3.12 (Universally typical sets using Brudno's theorem) *Let $h_0 > 0$ and $n \in \mathbb{N}$. We set*

$$\mathfrak{K}_{h_0}^{(n)} = \left\{ s \in \Sigma^{\Lambda_n} : \frac{K(s)}{|\Lambda_n|} < h_0 \right\} \quad \text{and} \quad \mathfrak{K}_{h_0} := \liminf_{n \rightarrow \infty} \llbracket \mathfrak{K}_{h_0}^{(n)} \rrbracket.$$

Then for all $\mu \in EM(\Omega, \sigma)$ with $h_\sigma(\mu) < h_0$ the following holds:

$$\mu(\mathfrak{K}_{h_0}) = \lim_{n \rightarrow \infty} \mu(\llbracket \mathfrak{K}_{h_0}^{(n)} \rrbracket) = 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{\log_2 |\mathfrak{K}_{h_0}^{(n)}|}{|\Lambda_n|} \leq h_0.$$

Proof. For all $\mu \in EM(\Omega, \sigma)$ with $h_\sigma(\mu) < h_0$, by Theorem 3.1, we have

$$\begin{aligned} 1 &= \mu \left(\left\{ \omega \in \Omega : \lim_{n \rightarrow \infty} \left| \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} - h_\sigma(\mu) \right| = 0 \right\} \right) \\ &= \mu \left(\bigcap_{\epsilon > 0} \bigcup_{N \in \mathbb{N}} \bigcap_{n > N} \left\{ \omega \in \Omega : h_\sigma(\mu) - \epsilon < \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} < h_\sigma(\mu) + \epsilon \right\} \right) \\ &\leq \mu \left(\bigcup_{N \in \mathbb{N}} \bigcap_{n > N} \left\{ \omega \in \Omega : \frac{K(\omega \upharpoonright \Lambda_n)}{|\Lambda_n|} < h_0 \right\} \right) = \mu(\mathfrak{K}_{h_0}) \leq \liminf_{n \rightarrow \infty} \mu(\llbracket \mathfrak{K}_{h_0}^{(n)} \rrbracket). \end{aligned}$$

Then $\mu(\mathfrak{K}_{h_0}) = \lim_{n \rightarrow \infty} \mu(\llbracket \mathfrak{K}_{h_0}^{(n)} \rrbracket) = 1$ holds. Since $\lim_{n \rightarrow \infty} \mu(\llbracket \mathfrak{K}_{h_0}^{(n)} \rrbracket) = 1$, $|\mathfrak{K}_{h_0}^{(n)}| \neq 0$ holds for sufficiently large $n \in \mathbb{N}$. Therefore, by Lemma 2.3, we have $0 \neq |\mathfrak{K}_{h_0}^{(n)}| < 2^{h_0|\Lambda_n|+1}$ ($n \gg 1$). Hence $\frac{\log_2 |\mathfrak{K}_{h_0}^{(n)}|}{|\Lambda_n|} < h_0 + \frac{1}{|\Lambda_n|}$ ($n \gg 1$). This completes the proof. $\square$

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