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# VANISHING RANGES FOR THE MOD $p$ COHOMOLOGY OF ALTERNATING SUBGROUPS OF COXETER GROUPS

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**ABSTRACT.** We obtain vanishing ranges for the mod  $p$  cohomology of alternating subgroups of finite  $p$ -free Coxeter groups. Here a Coxeter group  $W$  is  $p$ -free if the order of the product  $st$  is prime to  $p$  for every pair of Coxeter generators  $s, t$  of  $W$ . Our result generalizes those for alternating groups formerly proved by Kleshchev-Nakano and Burichenko. As a byproduct, we obtain vanishing ranges for the twisted cohomology of finite  $p$ -free Coxeter groups with coefficients in the sign representations. In addition, a weak version of the main result is proved for a certain class of infinite Coxeter groups.

## 1. INTRODUCTION

Let  $(W, S)$  be a Coxeter system, the pair of a Coxeter group  $W$  and the set  $S$  of Coxeter generators of  $W$ . The alternating subgroup  $A_W$  of  $W$  is the kernel of the sign homomorphism  $W \rightarrow \{\pm 1\}$  defined by  $s \mapsto -1$  ( $s \in S$ ). The symmetric group  $\Sigma_n$  on  $n$  letters is a finite Coxeter group (of type  $A_{n-1}$ ), and its alternating subgroup is nothing but the alternating group  $A_n$  on  $n$  letters. Cohomology of alternating groups has been studied by various authors. Among others, Kleshchev-Nakano [9, p. 354 Corollary] and Burichenko [7, Theorem 1.4] independently obtained vanishing ranges for the mod  $p$  cohomology of alternating groups:

**Theorem 1.1** (Kleshchev-Nakano, Burichenko). *Let  $p$  be an odd prime. The mod  $p$  cohomology  $H^k(A_n, \mathbb{F}_p)$  vanishes for  $0 < k < p - 2$ .*

The primary purpose of this paper is to generalize Theorem 1.1 to alternating subgroups of finite Coxeter groups, and thereby to give an alternative proof of Theorem 1.1. Our main result is the following:

**Theorem 1.2.** *Let  $p$  be an odd prime,  $W$  a finite  $p$ -free Coxeter group, and  $A_W$  the alternating subgroup of  $W$ . Then the mod  $p$  cohomology  $H^k(A_W, \mathbb{F}_p)$  vanishes for  $0 < k < p - 2$ .*

Here a Coxeter group is  $p$ -free if the order of the product  $st \in W$  is prime to  $p$  for every pair of Coxeter generators  $s, t \in S$ . Since symmetric groups  $\Sigma_n$  are  $p$ -free for  $p \geq 5$  as Coxeter groups, Theorem 1.1 is a special case of our theorem. Note that finiteness and  $p$ -freeness assumptions on  $W$  are necessary, and vanishing ranges for  $H^k(A_W, \mathbb{F}_p)$  are best possible. See §2.2, §6 and §7 below.

The key ingredients for the proof of Theorem 1.2 are

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- (1) the classification of finite Coxeter groups (see §3),
- (2) high connectivity of the Coxeter complex  $X_W$  (Proposition 5.1, 5.2),
- (3) high connectivity of the orbit space  $X_W/A_W$  (Proposition 5.3),

as well as some considerations of equivariant cohomology in §4. The proof is inspired by the arguments in [2], where the first author obtained vanishing ranges for the  $p$ -local homology of  $p$ -free Coxeter groups. As a byproduct of Theorem 1.2, we will obtain vanishing ranges for the twisted cohomology of finite  $p$ -free Coxeter groups with coefficients in the sign representations over  $\mathbb{F}_p$  (Theorem 6.1). As we remarked above, Theorem 1.2 no longer holds for infinite Coxeter groups. Instead, we will prove a weak version of Theorem 1.2 for a certain class of infinite Coxeter groups (Theorem 7.2).

## 2. PRELIMINARIES

**2.1. Coxeter groups.** In this subsection, we recall definitions and relevant facts concerning of Coxeter groups. References are [1, 4, 8]. Let  $S$  be a finite set. A Coxeter matrix is a symmetric matrix  $M = (m(s, t))_{s, t \in S}$  each of whose entries  $m(s, t)$  is a positive integer or  $\infty$  such that

- (1)  $m(s, s) = 1$  for all  $s \in S$ ,
- (2)  $2 \leq m(s, t) = m(t, s) \leq \infty$  for all distinct  $s, t \in S$ .

The *Coxeter system* associated to  $M$  is the pair  $(W, S)$  where  $W$  is the group generated by  $s \in S$  and the fundamental relations  $(st)^{m(s, t)} = 1$  ( $m(s, t) < \infty$ ):

$$W := \langle s \in S \mid (st)^{m(s, t)} = 1 (m(s, t) < \infty) \rangle.$$

The group  $W$  is called the *Coxeter group* associated to  $M$ , elements of  $S$  are called *Coxeter generators* of  $W$ , and the cardinality of  $S$  is called the *rank* of  $W$  and is denoted by  $|S|$  or  $\text{rank } W$ .

For a subset  $T \subseteq S$ , the subgroup  $W_T := \langle T \rangle$  of  $W$  generated by elements  $t \in T$  is called a (standard) *parabolic subgroup* (or a special subgroup in the literature). In particular,  $W_S = W$  and  $W_\emptyset = \{1\}$ . It is known that  $(W_T, T)$  is a Coxeter system associated to the restriction of the Coxeter matrix to  $T$ . Henceforth, we sometimes omit the reference to the Coxeter matrix  $M$  and the set of Coxeter generators  $S$  if there is no ambiguity.

Finally, given an odd prime number  $p$ , a Coxeter group  $W$  is called *p-free* in [2] if  $m(s, t)$  is prime to  $p$  for every pair of Coxeter generators  $s, t \in S$ . By convention,  $\infty$  is prime to all prime numbers. Since the order of the product  $st$  ( $s, t \in S$ ) is precisely  $m(s, t)$ , the definition agrees with the one given in the introduction. For every finite irreducible Coxeter group  $W$ , the range of odd prime numbers  $p$  such that  $W$  is  $p$ -free can be found in the Appendix (see §3 for the definition of irreducible Coxeter groups).

**2.2. Known results for cohomology of alternating subgroups.** As was defined in the introduction, the *alternating subgroup*  $A_W$  of a Coxeter group of  $W$  is the kernel of the sign homomorphism  $W \rightarrow \{\pm 1\}$  which assigns  $-1$  to  $s \in S$ . Alternating subgroups are also called rotation(al) subgroups in the literature. To the best of our knowledge, not much is known about cohomology of alternating

subgroups of Coxeter groups. Finite presentations of alternating subgroups of arbitrary Coxeter groups were given in Bourbaki [4, Chapitre IV §1, Exercise 9] as an exercise. The proof can be found in Brenti-Reiner-Roichman [5, Proposition 2.1.1]. From finite presentations, one can compute the first integral homology of alternating subgroups. Moreover, Maxwell [10] determined the Schur multiplier  $H^2(A_W, \mathbb{C}^\times) \cong H_2(A_W, \mathbb{Z})$  for alternating subgroups  $A_W$  of *finite* Coxeter groups  $W$ .

Before preceeding further, we remark on the  $p$ -freeness assumption in Theorem 1.2. Let  $p$  be an odd prime and  $D_{2m} := \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle$  the dihedral group of order  $2m$ , which is a Coxeter group of rank 2. It is  $p$ -free if and only if  $p$  does not divide  $m$ . The alternating subgroup of  $D_{2m}$  is the cyclic group of order  $m$  generated by  $st$ . Now suppose that  $p$  divides  $m$ . It is well-known that  $H^*(\mathbb{Z}/m, \mathbb{F}_p) \cong \mathbb{F}_p[u] \otimes E(v)$ , where  $\mathbb{F}_p[u]$  is the polynomial algebra generated by a two dimensional generator  $u$  and  $E(v)$  is the exterior algebra generated by a one dimensional generator  $v$ . In particular,  $H^1(\mathbb{Z}/m, \mathbb{F}_p) \cong \mathbb{F}_p$ , which shows the necessity of  $p$ -freeness assumption in Theorem 1.2.

### 3. THE CASE $|S| \leq p - 2$

In this section, we will prove Theorem 1.2 for  $|S| \leq p - 2$  by using the classification of finite Coxeter groups. Recall that the Coxeter matrix  $M = (m(s, t))_{s, t \in S}$  defining the Coxeter system  $(W, S)$  is represented by the *Coxeter graph*  $\Gamma$  whose vertex set is  $S$  and whose edges are the unordered pairs  $\{s, t\} \subset S$  with  $m(s, t) \geq 3$ . The edges  $\{s, t\}$  with  $m(s, t) \geq 4$  are labeled by those numbers. For convenience, we write  $W = W(\Gamma)$  and call it the Coxeter group of type  $\Gamma$ . A Coxeter group  $W(\Gamma)$  is called *irreducible* if  $\Gamma$  is connected, otherwise called *reducible*. For a reducible Coxeter group  $W(\Gamma)$ , if  $\Gamma$  consists of the connected components  $\Gamma_1, \Gamma_2, \dots, \Gamma_r$ , then  $W(\Gamma)$  is isomorphic to the internal direct product of parabolic subgroups  $W(\Gamma_i)$ 's each of which is irreducible:

$$W(\Gamma) = W(\Gamma_1) \times W(\Gamma_2) \times \cdots \times W(\Gamma_r).$$

The classification of Coxeter graphs for finite irreducible Coxeter groups is well-known. They consist of infinite families  $A_n$  ( $n \geq 1$ ),  $B_n$  ( $n \geq 2$ ),  $D_n$  ( $n \geq 4$ ),  $I_2(m)$  ( $m \geq 3$ ), and exceptional graphs  $H_3, H_4, F_4, E_6, E_7$  and  $E_8$ . The subscript stands for the rank of the corresponding Coxeter group. See the Appendix for the orders of finite irreducible Coxeter groups. Note that  $W(A_n)$  is isomorphic to the symmetric group of  $n + 1$  letters, and  $W(I_2(m))$  is isomorphic to the dihedral group of order  $2m$ .

**Proposition 3.1.** *Let  $p \geq 5$  be a prime and  $W$  a finite  $p$ -free Coxeter group with  $|S| \leq p - 2$ . Then  $W$  has no  $p$ -torsion.*

*Proof.* If  $W$  is a finite  $p$ -free Coxeter group, then  $W$  decomposes into the direct product of finite irreducible  $p$ -free Coxeter groups  $W \cong W_1 \times \cdots \times W_r$  with  $\sum_{i=1}^r \text{rank } W_i = \text{rank } W$ . So it suffices to prove the proposition when  $W$  is irreducible. Since  $|W(A_n)| = (n + 1)!$ ,  $|W(B_n)| = 2^n n!$  and  $|W(D_n)| = 2^{n-1} n!$ , they have no  $p$ -torsion when  $n \leq p - 2$ . As for  $W(I_2(m))$ , it is  $p$ -free if and only if it has

no  $p$ -torsion. These observations imply the proposition for  $p \geq 11$ , for all finite irreducible Coxeter groups of type other than  $A_n, B_n, D_n$  and  $I_2(m)$  have no  $p$ -torsion for  $p \geq 11$  (see the Appendix). Apart from Coxeter groups of type  $A_n, B_n, D_n$  and  $I_2(m)$ , finite irreducible Coxeter groups with rank at most  $p-2$  are,  $W(H_3)$  for  $p = 5$ , and  $W(F_4), W(H_3)$  and  $W(H_4)$  for  $p = 7$ . But  $W(H_3)$  is not 5-free, while  $W(F_4), W(H_3)$  and  $W(H_4)$  have no 7-torsion, proving the proposition.  $\square$

As an immediate consequence, we obtain the following corollary which implies Theorem 1.2 for  $|S| \leq p-2$ :

**Corollary 3.2.** *Let  $p \geq 5$  be a prime and  $W$  a finite  $p$ -free Coxeter group with  $|S| \leq p-2$ . Then  $H^k(W, \mathbb{F}_p) = H^k(A_W, \mathbb{F}_p) = 0$  for  $k > 0$ .*

#### 4. EQUIVARIANT COHOMOLOGY

Let  $G$  be a group. By a  $G$ -complex we mean a CW-complex  $X$  together with a continuous action of  $G$  on  $X$  which permutes the cells. A  $G$ -complex  $X$  is called admissible in [6] if, for each cell  $\sigma$  of  $X$ , the isotropy subgroup  $G_\sigma$  of  $\sigma$  fixes  $\sigma$  pointwise. Throughout this section,  $X$  is a finite dimensional, connected, admissible  $G$ -complex, and  $A$  is a trivial  $G$ -module (an abelian group equipped with the trivial  $G$ -action). We consider the equivariant cohomology  $H_G^*(X, A)$  (see [6, Chapter VII] for the definition and relevant facts).

**Lemma 4.1.** *If  $H^k(X, A) = 0$  for  $0 < k < d$  then  $H_G^k(X, A) \cong H^k(G, A)$  for  $0 \leq k < d$ .*

*Proof.* Consider the spectral sequence

$$E_2^{ij} = H^i(G, H^j(X, A)) \Rightarrow H_G^{i+j}(X, A)$$

(see [6, §VII.7]). We have  $E_2^{*,0} = H^*(G, A)$  and  $E_2^{*,j} = 0$  for  $0 < j < d$ . This proves  $H_G^k(X, A) \cong H^k(G, A)$  for  $0 \leq k < d$ .  $\square$

Next, consider the spectral sequence

$$(4.1) \quad E_1^{ij} = H^j(G, C^i(X, A)) \Rightarrow H_G^{i+j}(X, A).$$

By Shapiro's lemma,

$$(4.2) \quad E_1^{ij} \cong \prod_{\sigma \in \mathcal{E}_i} H^j(G_\sigma, A)$$

where  $\mathcal{E}_i$  is a set of representatives for  $G$ -orbits of  $i$ -cells of  $X$  (see [6, §VII.7]). Note that  $A$  in  $H^j(G_\sigma, A)$  is the trivial  $G_\sigma$ -module since  $X$  is admissible.

**Lemma 4.2.** *In the spectral sequence (4.1), we have  $E_2^{*,0} \cong H^*(X/G, A)$ .*

*Proof.* Recall that the differential  $d_1^{ij} : E_1^{ij} \rightarrow E_1^{i+1,j}$  of the spectral sequence is the map  $H^j(G, C^i(X, A)) \rightarrow H^j(G, C^{i+1}(X, A))$  induced by the coboundary operator of  $C^*(X, A)$ . Now there are isomorphisms

$$H^0(G, C^i(X, A)) \cong C^i(X, A)^G \cong C^i(X/G, A),$$

where the second isomorphism holds because  $X$  is admissible. These isomorphisms are compatible with differentials of the spectral sequence and coboundary operators of  $C^*(X, A)$  and  $C^*(X/G, A)$ , the lemma follows.  $\square$

**Lemma 4.3.** *If  $X$  satisfies the following conditions:*

- (1) *For each cell  $\sigma$  of  $X$ ,  $H^k(G_\sigma, A) = 0$  for  $0 < k < d$ ,*
- (2)  *$H^k(X/G, A) = 0$  for  $0 < k < d$ .*

*Then  $H_G^k(X, A) = 0$  for  $0 < k < d$ .*

*Proof.* Consider the spectral sequence (4.1). We have  $E_1^{*,j} = 0$  for  $0 < j < d$  by the isomorphism (4.2), and  $E_2^{i,0} \cong H^i(X/G, A) = 0$  for  $0 < i < d$  by Lemma 4.2, proving the lemma.  $\square$

Combining Lemma 4.1 and 4.3, we obtain the following proposition which will be used to prove Theorem 1.2:

**Proposition 4.4.** *Let  $G$  be a group and  $A$  a trivial  $G$ -module. If there exists a finite dimensional, admissible, connected  $G$ -complex  $X$  satisfying the following conditions:*

- (1) *For each cell  $\sigma$  of  $X$ ,  $H^k(G_\sigma, A) = 0$  for  $0 < k < d$ ,*
- (2)  *$H^k(X, A) = 0$  and  $H^k(X/G, A) = 0$  for  $0 < k < d$ .*

*Then  $H^k(G, A) = 0$  for  $0 < k < d$ .*

## 5. COXETER COMPLEXES AND THE PROOF OF THE MAIN THEOREM

Now we prove Theorem 1.2. To do so, first we recall the definition and properties of Coxeter complexes which are relevant to prove Theorem 1.2. A reference for Coxeter complexes is [1, Chapter 3]. Given a Coxeter group  $W$ , the *Coxeter complex*  $X_W$  of  $W$  is the poset of cosets  $wW_T$  ( $w \in W, T \subsetneq S$ ), ordered by reverse inclusion. It is known that  $X_W$  is indeed an  $(|S| - 1)$ -dimensional simplicial complex (see [1, Theorem 3.5]). The  $k$ -simplices of  $X_W$  are the cosets  $wW_T$  with  $k = |S| - |T| - 1$ . A coset  $wW_T$  is a face of  $w'W_{T'}$  if and only if  $wW_T \supseteq w'W_{T'}$ . In particular, the vertices are cosets of the form  $wW_{S \setminus \{s\}}$  ( $s \in S, w \in W$ ), while the maximal simplices are the singletons  $wW_\emptyset = \{w\}$  ( $w \in W$ ). The maximal simplex  $W_\emptyset = \{1\}$  is called the *fundamental chamber*. In what follows, we will not distinguish between  $X_W$  and its geometric realization. The following fact is well-known (see [1, Proposition 1.108]).

**Proposition 5.1.** *If  $W$  is a finite Coxeter group, then  $X_W$  is a triangulation of the  $(|S| - 1)$ -dimensional sphere  $\mathbb{S}^{|S|-1}$ .*

In case  $W$  is infinite, Serre proved the following result:

**Proposition 5.2** ([11, Lemma 4]). *If  $W$  is an infinite Coxeter group, then  $X_W$  is contractible.*

There is a simplicial action of  $W$  on  $X_W$  by left translation  $w' \cdot wW_T := w'wW_T$ . The isotropy subgroup of a simplex  $wW_T$  is precisely  $wW_T w^{-1}$  which fixes  $wW_T$  pointwise. Hence  $X_W$  is an admissible  $W$ -complex. Now let  $\Delta_W = \{W_T \mid T \subsetneq S\}$

be the subcomplex of  $X_W$ , which consists of the fundamental chamber  $W_\emptyset$  and its faces. Then  $\Delta_W$  is a set of representatives for the  $W$ -orbits of simplices, and hence  $\Delta_W$  is a strict fundamental domain for the action of  $W$  (see [1, Lemma 3.75]).

**Proposition 5.3.** *For any Coxeter group  $W$  not necessarily finite, the orbit space  $X_W/A_W$  is homeomorphic to the  $(|S| - 1)$ -dimensional sphere  $\mathbb{S}^{|S|-1}$ .*

*Proof.* Since  $X_W$  is an admissible  $A_W$ -complex, the orbit space  $X_W/A_W$  inherits a CW-structure whose cells correspond bijectively to  $A_W$ -orbits of simplices of  $X_W$ . Pick  $s_0 \in S$  arbitrary. As  $W = A_W \sqcup s_0 A_W$  and  $\Delta_W$  is a set of representatives for the  $W$ -orbits of simplices, each  $A_W$ -orbit of simplices is represented by either  $W_T$  or  $s_0 W_T$  ( $T \subsetneq S$ ). The isotropy subgroup of the fundamental chamber  $W_\emptyset$  is trivial, which implies  $W_\emptyset$  and  $s_0 W_\emptyset$  represent distinct  $A_W$ -orbits. On the other hand, for  $\emptyset \neq T \subsetneq S$  and  $t \in T$ , we have  $ts_0 \in A_W$  and  $ts_0 \cdot s_0 W_T = W_T$ , which implies  $W_T$  and  $s_0 W_T$  represent the same  $A_W$ -orbit. As a result,  $X_W/A_W$  can be identified with the cell complex obtained from  $\Delta_W \sqcup s_0 \Delta_W$  by identifying faces  $W_T$  and  $s_0 W_T$  ( $T \neq \emptyset$ ). We conclude that  $X_W/A_W$  is homeomorphic to  $\mathbb{S}^{|S|-1}$ .  $\square$

Now we prove Theorem 1.2 by induction on  $|S|$ . We may assume  $|S| > p - 2$  by Corollary 3.2. Consider the action of  $A_W$  on the Coxeter complex  $X_W$ . We have  $H^k(X_W, \mathbb{F}_p) = H^k(X_W/A_W, \mathbb{F}_p) = 0$  for  $0 < k < |S| - 1$  by Proposition 5.1 and 5.3, and hence we have

$$(5.1) \quad H^k(X_W, \mathbb{F}_p) = H^k(X_W/A_W, \mathbb{F}_p) = 0 \quad (0 < k < p - 2).$$

Moreover, for each simplex  $\sigma := wW_T$  of  $X_W$ , the isotropy subgroup  $(A_W)_\sigma$  of  $\sigma$  satisfies

$$(A_W)_\sigma = wW_T w^{-1} \cap A_W = w(W_T \cap A_W)w^{-1} = wA_{W_T} w^{-1} \cong A_{W_T}$$

because  $A_W$  is a normal subgroup of  $W$ . Since  $A_{W_T}$  is the alternating subgroup of  $W_T$  with  $|T| < |S|$ , we see that

$$(5.2) \quad H^k((A_W)_\sigma, \mathbb{F}_p) \cong H^k(A_{W_T}, \mathbb{F}_p) = 0 \quad (0 < k < p - 2)$$

by the induction assumption. Applying (5.1) and (5.2) to Proposition 4.4, Theorem 1.2 follows.

## 6. TWISTED COHOMOLOGY OF FINITE COXETER GROUPS

Given a Coxeter system  $(W, S)$ , let  $\mathbb{F}_p[-1]$  be the sign representation of  $W$  over  $\mathbb{F}_p$ . Namely,  $\mathbb{F}_p[-1] = \mathbb{F}_p$  as an abelian group, and each  $s \in S$  acts on  $\mathbb{F}_p[-1]$  as the multiplication by  $-1$ . As an application of Theorem 1.2, we will deduce the vanishing range for  $H^k(W, \mathbb{F}_p[-1])$  as follows:

**Theorem 6.1.** *Let  $W$  be a finite  $p$ -free Coxeter group. Then  $H^k(W, \mathbb{F}_p[-1]) = 0$  for  $k < p - 2$ .*

*Proof.* Observe that  $H^0(W, \mathbb{F}_p[-1]) \cong \mathbb{F}_p[-1]^W = 0$ . Let  $M := \text{Ind}_{A_W}^W(\mathbb{F}_p)$  be the induced module of the trivial  $A_W$ -module  $\mathbb{F}_p$ . Then  $H^k(W, M) \cong H^k(A_W, \mathbb{F}_p)$  by Shapiro's lemma. On the other hand, since  $p$  is odd,  $M$  decomposes into

$M \cong \mathbb{F}_p \oplus \mathbb{F}_p[-1]$ , which implies  $H^k(W, M) \cong H^k(W, \mathbb{F}_p) \oplus H^k(W, \mathbb{F}_p[-1])$  and hence

$$(6.1) \quad H^k(A_W, \mathbb{F}_p) \cong H^k(W, \mathbb{F}_p) \oplus H^k(W, \mathbb{F}_p[-1]).$$

Now Theorem 1.2 implies  $H^k(W, \mathbb{F}_p[-1]) = 0$  for  $0 < k < p - 2$ .  $\square$

The proof also implies  $H^k(W, \mathbb{F}_p) = 0$  holds for  $0 < k < p - 2$ , however, the first author proved a much stronger result in [2]. Now let  $\Sigma_p$  be the symmetric group on  $p$  letters and  $A_p$  the alternating group on  $p$  letters. It is known that  $H^{p-2}(\Sigma_p, \mathbb{F}_p[-1]) \cong \mathbb{F}_p$  (see [3, pp. 74–75]). So vanishing ranges in Theorem 6.1 are best possible. Moreover, the isomorphism (6.1) implies  $H^{p-2}(A_p, \mathbb{F}_p) \neq 0$ , which shows vanishing ranges in Theorem 1.2 are also best possible.

## 7. ALTERNATING SUBGROUPS OF INFINITE COXETER GROUPS

In general, Theorem 1.2 no longer holds for  $p$ -free Coxeter groups of infinite order. For example, let  $D_\infty = \langle s, t \mid s^2 = t^2 = 1 \rangle \cong \mathbb{Z}/2 * \mathbb{Z}/2$  be the infinite dihedral group. By definition, it is  $p$ -free for all  $p$  since  $m(s, t) = \infty$ . On the other hand, the alternating subgroup  $A_{D_\infty}$  of  $D_\infty$  is the infinite cyclic group generated by  $st \in D_\infty$  so that  $H^1(A_{D_\infty}, \mathbb{F}_p) \cong \mathbb{F}_p$  for all  $p$ . More generally, we have the following proposition:

**Proposition 7.1.** *If  $W$  is an infinite Coxeter group all of whose proper parabolic subgroups are of finite order, then  $H^{|S|-1}(A_W, \mathbb{F}_p) \neq 0$  for all  $p$ .*

*Proof.* Note first  $H^*(A_W, \mathbb{Q}) \cong H_{A_W}^*(X_W, \mathbb{Q})$  because  $X_W$  is contractible by Proposition 5.2. Consider the spectral sequence (4.1) and (4.2) with  $\mathbb{Q}$ -coefficients:

$$E_1^{ij} = \prod_{\sigma \in \mathcal{E}_i} H^j((A_W)_\sigma, \mathbb{Q}) \Rightarrow H_{A_W}^{i+j}(X_W, \mathbb{Q}).$$

Since the isotropy subgroup  $(A_W)_\sigma$  is finite for any  $\sigma$  by the assumption, we see that  $H^j((A_W)_\sigma, \mathbb{Q}) = 0$  for  $j > 0$  and hence  $E_1^{*,j} = 0$  for  $j > 0$ . In addition,  $E_2^{*,0} \cong H^*(\mathbb{S}^{|S|-1}, \mathbb{Q})$  by Proposition 4.2 and 5.3. As a result, we have

$$H^*(A_W, \mathbb{Q}) \cong H^*(\mathbb{S}^{|S|-1}, \mathbb{Q})$$

and hence  $H^{|S|-1}(A_W, \mathbb{F}_p) \neq 0$  for any prime  $p$  in virtue of the universal coefficient theorem.  $\square$

The assumption that all proper parabolic subgroups of  $W$  are finite is somewhat restrictive. Apart from finite Coxeter groups, it holds if and only if (a)  $W$  is an irreducible Euclidean reflection group or (b)  $W$  is a hyperbolic reflection group whose fundamental domain is a closed simplex contained entirely in the interior of the hyperbolic space. See [1, Remark 3.29] for precise. Now we prove the weak version of Theorem 1.2 for those Coxeter groups.

**Theorem 7.2.** *Let  $W$  be an infinite  $p$ -free Coxeter group all of whose proper parabolic subgroups are of finite order. Then  $H^k(A_W, \mathbb{F}_p) = 0$  for  $0 < k < \min\{p - 2, |S| - 1\}$ .*

*Proof.* Set  $d = \min\{p - 2, |S| - 1\}$  and consider the Coxeter complex  $X_W$ . For each  $k$ -simplex  $\sigma$  of  $X_W$ , the isotropy subgroup  $(A_W)_\sigma$  is isomorphic to  $A_{W_T}$  for some  $T \subsetneq S$  as in the proof of Theorem 1.2, and hence  $H^k((A_W)_\sigma, \mathbb{F}_p) = 0$  for  $0 < k < d$  by the assumption and Theorem 1.2. On the other hand, noting  $d \leq |S| - 1$ ,  $H^k(X_W, \mathbb{F}_p) = H^k(X_W/A_W, \mathbb{F}_p) = 0$  for  $0 < k < d$  by Proposition 5.2 and 5.3. Now the theorem follows from Proposition 4.4.  $\square$

The proof of the following corollary is similar to the one for Theorem 6.1:

**Corollary 7.3.** *Under the assumption of Theorem 7.2,  $H^k(W, \mathbb{F}_p[-1]) = 0$  holds for  $k < \min\{p - 2, |S| - 1\}$ .*

#### APPENDIX

The following is the table for the Coxeter graph  $\Gamma$ , the order  $|W(\Gamma)|$  of the corresponding finite irreducible Coxeter group  $W(\Gamma)$ , and the range of odd prime numbers  $p$  such that  $W(\Gamma)$  is  $p$ -free.

| $\Gamma$                | $ W(\Gamma) $                        | $p$ -freeness |
|-------------------------|--------------------------------------|---------------|
| $A_1$                   | 2                                    | $p \geq 3$    |
| $A_n$ ( $n \geq 2$ )    | $(n + 1)!$                           | $p \geq 5$    |
| $B_2$                   | 8                                    | $p \geq 3$    |
| $B_n$ ( $n \geq 3$ )    | $2^n n!$                             | $p \geq 5$    |
| $D_n$ ( $n \geq 4$ )    | $2^{n-1} n!$                         | $p \geq 5$    |
| $E_6$                   | $2^7 \cdot 3^4 \cdot 5$              | $p \geq 5$    |
| $E_7$                   | $2^{10} \cdot 3^4 \cdot 5 \cdot 7$   | $p \geq 5$    |
| $E_8$                   | $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ | $p \geq 5$    |
| $F_4$                   | $2^7 \cdot 3^2$                      | $p \geq 5$    |
| $H_3$                   | $2^3 \cdot 3 \cdot 5$                | $p \geq 7$    |
| $H_4$                   | $2^6 \cdot 3^2 \cdot 5^2$            | $p \geq 7$    |
| $I_2(m)$ ( $m \geq 3$ ) | $2m$                                 | $p \nmid m$   |

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