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（会場：北海道大学大学院理学研究科）

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Seminar on Function Spaces, 2002

CONTENTS

Function Space

Function Algebra

A stochastic integral for L^1-process

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Abstract. Let $v \in L^1(T \otimes \Omega)$ and let $f=(f_t)_{0 \leq t \leq 1}$ be L^1-process on a probability space $(\Omega, \mathcal{A}, P)$. Then f is of a. e. bounded variation on Ω so that the Stieltjes type stochastic integral $\int_0^1 v(t)df(t)$ converges a. e. This implies $L^1([0,1]) \equiv H^1([0,1])$, $L^1(D) \equiv H^1(D)$ (when D is the unit disc) and $L^1(R) \equiv H^1(R)$. These are the extension of Burkholder-Gundy-Silverstein theorem.

1. A stochastic integral.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $v \in L^1(T \otimes \Omega)$ and let $f=(f(t))_{0 \leq t \leq 1}$ an L^1-process on $(\Omega, \mathcal{A}, P)$.

Here $\Delta : 0 = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} < \cdots < t_{n+1}^{(n)} = 1$. Then

Theorem ([7]).

$$P\left(\sup_{\Delta} \sum_{j=0}^{n} \left| f(t_{j+1}^{(n)}) - f(t_j^{(n)}) \right| < \infty \right) = 1.$$

That is, f is of a. e. bounded variation. Therefore the stochastic integral of Stieltjes type $\int_0^1 v(t)df(t)$ converges a. e.

This stochastic integral ([7]) is the generalization of Towghi's integral ([8]) and this contains the Feynman path integral and complex integral.

2. A relation of function spaces.

(This contains a part of a joint work with D. L. Burkholder.)

Background of below theorems.

Let $f=(f(t))(\ t \in [0,1])$ be a martingale or a Brownian motion with a. e. ω continuous paths on Ω. Then Re $f(t)=f(t) \in L^1([0,1])$. Since $L^1([0,1])$ is a separable Banach space, by Banach-Mazur theorem $|f(t)| \leq {}^{\exists} K < \infty$ a. e. in t.

So $(\mathrm{Re}\ f)^* = \sup_{0 \leq t \leq 1} |f(t)| \leq K \in L^1([0,1])$. Thus $(\mathrm{Re}\ f)^* \in L^1([0,1])$.

Then P. A. Meyer proved that $(f(t))_{0\leq t\leq 1}$ is a H^1-martingale by using Burkholder-Gundy-Silverstein theorem (a martingale version), i. e., $f(t)\in H^1([0,1])$ holds.

So f(t) is a. e. differentiable in t.

This is a counterexample for Paley-Wiener-Zygmund theorem.

Theorem 1. $L^1([0,1]) \equiv H^1([0,1])$ (coincides).

So, on the exceptional set, path $f(t)(\ t\in[0,1])$ is not smooth.

That is, except for the exceptional set all paths are smooth.

Sketch of proof. Put $\Omega=[0,1]$ and $L^1([0,1]) = L^1(\Omega, P)$.

Let $\forall\ f(t)\in L^1([0,1])$, $\Delta : 0 = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} < \cdots < t_{n+1}^{(n)} = 1.$

$$g_n \equiv \sum_{j=0}^{n} \left| f(t_{j+1}^{(n)}) - f(t_j^{(n)}) \right| \quad (\in L^1([0,1])) \longrightarrow g_\infty \equiv \lim_{n\to\infty} g_n .$$

(Here notice that g_∞ is not definite.) Put $f(t)=f(t,t')(t'\in\Omega)$.

By mathematical induction on n $\{g_0, g_1, \dots, g_n\}$ is uniformly integrable for all natural numbers $n=1,2,3,\dots$.

In fact, since $(f(t))_{0\leq t\leq 1}$ is L^1-bounded and integrable stochastic process, $g_k<\infty$ a. e. in t' ($k=0,1$).

So

$$\sup_{0\leq k\leq 1} \int_{\{g_k>c\}} g_k \, dP = \int_{\{g_0>c\}} g_0 \, dP \vee \int_{\{g_1>c\}} g_1 \, dP \downarrow 0 \quad \text{as } c\uparrow\infty.$$

So $\{g_0, g_1\}$ is uniformly integrable.

Next, suppose that $\{g_0, g_1, g_2, \dots, g_n\}$ ($n\geq 1$) is uniformly integrable then

$$\sup_{0\leq k\leq n+1} \int_{\{g_k>c\}} g_k \, dP = \bigvee_{k=0}^{n+1} \int_{\{g_k>c\}} g_k \, dP$$

$$= \sup_{0\leq k\leq n} \int_{\{g_k>c\}} g_k \, dP \ \vee \int_{\{g_{n+1}>c\}} g_{n+1} \, dP \downarrow 0 \quad \text{as } c\uparrow\infty .$$

So, by mathematical induction, $\{g_0, g_1, g_2, \ldots, g_n\}$ is uniformly integrable for all natural numbers $n=1,2,3,\ldots$.

Therefore $\{g_0, g_1, \ldots, g_n, \ldots\}$

$$= \lim_{n\to\infty} \{g_0, g_1, \ldots, g_n\}$$

$$= \bigcup_{n=0}^{\infty} \{g_0, g_1, \ldots, g_n\}$$

is also uniformly integrable. (Notice that $\bigcup_{n=0}^{\infty}\{g_0, g_1, \ldots, g_n\}$ does not include $\{g_0, g_1, \ldots, g_\infty\}$ by the definition.)

So $\sup_n E(g_n) < \infty$ so that $g_n < \infty$ a. e. in t' ($n=0,1,2,3,\ldots$).

Here suppose $\lim_{n\to\infty} g_n = \infty$ for almost all $t' \in \Omega (=[0,1])$.

Then by Fatou's lemma

$$E(g_\infty) \leqslant \sup_n E(g_n) < \infty \quad .$$

This is a contradiction.

So $g_\infty \neq \infty$ a. e. in t', i. e., $g_\infty < \infty$ a. e. in t'.

Since, for all Δ , $g_\infty < \infty$ a. e. in t' and $g_n < \infty$ a. e. in t' ($n=0,1,2,\ldots$), $\sup_\Delta g_n < \infty$ a. e. in t'.

That is,

$$P(\sup_\Delta \sum_{j=0}^{n} \left| f(t_{j+1}^{(n)}) - f(t_j^{(n)}) \right| < \infty) = 1$$

so that $(f(t))_{0 \leqslant t \leqslant 1}$ is of a. e. bounded variation.

So $f(t)$ is a. e. differentiable in t.

That is, $L^1([0,1]) \equiv H^1([0,1])$.

Corollary 1. Let D be the unit disc. Then $L^1(D) \equiv H^1(D)$.

In fact, let C be any continuous curve with finite length in D and let $\Delta \equiv \{\alpha = z_0, z_1, z_2, \ldots, z_{n+1} = \beta\}$ any partition of C.

Theorem 2. $L^1(R) \equiv H^1(R)$.

Notice that $R = \{x \mid -\infty < x < \infty\} = \left[\inf_{x>-\infty} x \ , \ \sup_{x<\infty} x \right]$.

These are the extensions of Burkholder-Gundy-Silverstein theorem ([2]).

References

1. D. L. Burkholder, Martingale transforms, Ann. Math. Statist. 37(1966),1494-1504.

2. D. L. Burkholder, R. F. Gundy and M. L. Silverstein, A maximal function characterization of the class H^p, Trans. Amer. Math. Soc. 157(1971),137-153.

3. D. L. Burkholder and T. Shintani, Approximation of L^1-bounded martingales by martingales of bounded variation, Proc. A.M.S. 72(1978), 166-169.

4. C. Fefferman, Characterization of bounded mean oscillation, A.M.S. Bulletin 77(1971),587-588.

5. J. B. Garnett, Bounded analytic functions, Academic Press, New York, 1981.

6. P. A. Meyer, Le dual de H^1 est BMO (CAS CONTINU), Séminaire de Probabilités VII, Université de Strasbourg, Lec. Notes in Math.321(1971/72),136-145.

7. T. Shintani, Counterexamples for Paley-Wiener-Zygmund theorem and a stochastic integral for L^1-process, ICM 2002 Beijing August 20-28, Short communication, Probability and Statistics.

8. N. Towghi, Stochastic integration of processes with finite generalized variations. I, Ann. Prob. 23(1995), 629-667.

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Stieltjes quasi-perfectness of abelian $*$-semigroups

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Abstract

An abelian $*$-semigroup S is called Stieltjes quasi-perfect of order $N \geq 2$ if S is Stieltjes determinate and every completely positive definite function admits a unique disintegration as an integral of hermitian nonnegative multiplicative functions on $\overbrace{S+\cdots+S}^{N}$. We prove that this definition dose not depend on N and that it implies quasi-perfectness.

可換$*$半群の完全性に類した定義は, 完全性・quasi-完全性・Stieltjes 完全性の3つが提唱され, それぞれの関係や性質について解析されている. ここでは新たな完全性として Stieltjes quasi-perfect 性が定義できることを述べ, 他の完全性との関係について報告する.

$S=(S,+,*)$ は (必ずしも単位元をもつとは限らない) 可換$*$半群とする. S に形式的に単位元 0 を付加した半群を $\widetilde{S}$ とかき, S 上の指標全体を S^*, 非負値の指標全体を S_+^* とかく. $\widetilde{S}^*$, $\widetilde{S}_+^*$ も同様である. $S+S$ 上の関数 φ が正定値であるとは, 任意の $s_1,\cdots,s_n \in S$, $c_1,\cdots,c_n \in \mathbf{C}$ について,

$$\sum_{j,k=1}^{n} c_j\overline{c_k}\varphi(s_j+s_k^*) \geq 0$$

が成り立つときをいう. S 上の関数 φ が完全正定値であるとは, 任意の $a \in S$ について, $s \mapsto \varphi(a+s)$ が $\widetilde{S}$ 上で正定値であるときをいう. $\varphi(a+\cdot)$ の 0 での値が非負であることから各点で非負値をとる. 非負値指標・指標 (正確には指標の $S+S$ への制限) はそれぞれ, 完全正定値・正定値関数となる. S が単位元をもつなら完全正定値関数は正定値関数である. 正定値関数 φ が必ず S^* 上のある種の測度 (S が可算ならラドン測度) μ によって一意的に

$$\varphi(s)=\int_{S^*}\rho(s)d\mu(\rho) \qquad (s \in S+S)$$

とかけるような可換$*$半群 S を perfect であるという. また, 完全正定値関数 φ が必ず S_+^* 上のある種の測度 μ によって (S_+^* 上の測度として) 一意的に

$$\varphi(s)=\int_{S_+^*}\rho(s)d\mu(\rho) \qquad (s \in S)$$

とかけるような可換$*$半群 S を Stieltjes perfect であるという.

我々は以前に perfect 性の定義をゆるめ, $N \geq 3$ のとき, 正定値関数 φ が必ず S^* 上のある種の測度 μ によって一意的に

$$\varphi(s) = \int_{S^*} \rho(s) d\mu(\rho) \qquad (s \in \overbrace{S + \cdots + S}^{N})$$

とかけるような可換$*$半群 S を quasi-perfect of order N であるとした ([BS1]). これと同様に, $N \geq 2$ のとき, 完全正定値関数 φ が必ず S_+^* 上のある種の測度 μ によって (S_+^* 上の測度として) 一意的に

$$\varphi(s) = \int_{S_+^*} \rho(s) d\mu(\rho) \qquad (s \in \overbrace{S + \cdots + S}^{N})$$

とかけるような可換$*$半群 S を Stieltjes quasi-perfect of order N であるという. quasi-perfect 性と同様に次の定理が成り立ち, この定義は N に依存しない.

Theorem 1. 可換$*$半群 S について次は同値である:

(i) すべての $N \geq 2$ について S は order N の Stieltjes quasi-perfect である;

(ii) ある $N \geq 2$ について S は order N の Stieltjes quasi-perfect である;

(iii) S は order 2 の Stieltjes quasi-perfect である.

従って, Theorem 1 のいずれかをみたすとき, S は Stieltjes quasi-perfect であるという. 他の完全性との関係として次の定理が成り立つ.

Theorem 2. 可換$*$半群 S が Stieltjes perfect であるなら, Stieltjes quasi-perfect である. また, S が Stieltjes quasi-perfect であるなら, quasi-perfect である.

従って, 現在までに知られた関係をまとめると次の図のようになる.

	S : Stieltjes perfect	$\overset{\text{[BS2]}}{\longleftrightarrow}$	$\widetilde{S}$: perfect, $S = S + S$
$\swarrow$	$\downarrow$ [BS2]		
S : Stieltjes quasi-perfect	S : perfect	$\overset{\text{[B]}}{\longleftrightarrow}$	$\widetilde{S}$: perfect, S : flat
$\searrow$	$\downarrow$ [BS1]		
	S : quasi-perfect	$\overset{\text{[BS1]}}{\longleftrightarrow}$	$\widetilde{S}$: perfect

References

[**B**] T. M. Bisgaard, A note on factoring of positive definite functions on semigroups, Math. Nachr. **236**(2002), 31-46.
[**BS1**] T. M. Bisgaard and N. Sakakibara, A reduction of the problem of characterizing perfect semigroups, Math. Scand. **91**(2002), 55-66.
[**BS2**] T. M. Bisgaard and N. Sakakibara, Stieltjes perfect semigroups are perfect, submitted.

Domination property of the set of upper bounds in ordered linear spaces

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§1 Introducrion

Let E be a linear space over $\mathbb{R}$, and P be a convex cone in E satisfying

(P1) $E = P - P$, (P2) $P \cap (-P) = \{0\}$.

An order relation in E can be defined by $x \leq y \Longleftrightarrow y - x \in P$. We call a linear space E equipped with such a positive cone P a (partially) ordered linear space, and denote it by (E, P). For a subset A of E, we denote the set of upper bounds and lower bounds by $U(A) = \{x \in E \mid y \leq x, \ \forall y \in A\}$, $L(A) = \{x \in E \mid y \geq x, \ \forall y \in A\}$ respectively. These sets have a property of symmetry in the following sence. ([4])

$$U(L(U(A))) = U(A) \qquad (A \subset E). \tag{1}$$

In [4], the method of constructing a completion $(\tilde{E}, \tilde{P})$ of (E, P) by using the set of upper bounds $U(A)$ has been introduced. The relation (1) plays fundamental roles in the construction of $(\tilde{E}, \tilde{P})$. Also, the completion can be represented by the set of the generalized supremum in E which has been introduced in [2]. We will state the summary of those results in the first part of this section.

Let $\mathfrak{B}$ and $\mathfrak{B}'$ be the family of all upper bounded subset and lower bounded subset in E respectively, i.e. $\mathfrak{B} = \{A \subset E | A \neq \emptyset, \ U(A) \neq \emptyset\}$, $\mathfrak{B}' = \{B \subset E | B \neq \emptyset, \ L(B) \neq \emptyset\}$. The relations $A \sim B \underset{def}{\Longleftrightarrow} U(A) = U(B) \quad (A, B \in \mathfrak{B})$, and $C \sim' D \underset{def}{\Longleftrightarrow} L(C) = L(D) \quad (C, D \in \mathfrak{B}')$ are clearly equivalence relations. We define

$$\tilde{E} = \mathfrak{B}/\sim \ = \{[A] \mid A \in \mathfrak{B}\},$$

where $[A]$ denotes the equivalence class of A. For every $[A] \in \tilde{E}$, two operations $u([A]) = U(A)$, $\ l([A]) = L(U(A))$ are well defined. We can see by (1) that $l([A]) \sim A$. Moreover, it is known that If $A \sim A'$ and $B \sim B'$ in $\mathfrak{B}$, then for $\lambda > 0$, $[A + B] = [A' + B'] = [l([A]) + l([B])], and \ [\lambda A] = [\lambda A'] = [\lambda l([A])]$ hold where $A + B$ and λA denote the set $\{a + b \mid a \in A, \ b \in B\}$ and $\{\lambda a \mid a \in A\}$ respectively ([4]).

Definition. *For* $[A], \ [B] \in \tilde{E}$ *and* $\lambda \in \mathbb{R}$,

$$[A] \leq [B] \underset{def}{\Longleftrightarrow} u([B]) \subset u([A]) \tag{2}$$

$$[A] + [B] \underset{def}{=} [A + B] \tag{3}$$

$$\lambda[A] \underset{def}{=} \begin{cases} [\lambda l([A])] & (\lambda > 0) \\ [0^{+} l([A])] = [-P] & (\lambda = 0) \\ [\lambda u([A])] & (\lambda < 0), \end{cases} \tag{4}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX *where* 0^+C *denotes the resession cone of a convex set* C *defined by* $0^+C = \{x \in E \mid C + \lambda x \subset C,\ (\lambda > 0)\}$.

We define two subsets $\tilde{P}$ and $\tilde{E}_1$ of $\tilde{E}$ as follows.

$$\tilde{P} = \{[A] \in \tilde{E} \mid [A] \geq [-P]\} = \{[A] \in \tilde{E} \mid u([A]) \subset P\}$$

$$\tilde{E}_1 = \{[A] \in \tilde{E} \mid u([A]) = a + P \text{ for some } a \in E\}.$$

We note that the correspondence which assigns $a \in E$ to $[A] \in \tilde{E}_1$ such that $u([A]) = a + P$ is one to one.

Theorem 1. ([4]) *Let* E *be a Banach space with a closed positive cone. Then* $\tilde{E}$ *is an order complete vector lattice with the definition* (2),(3),(4), *and*

(*a*) $\tilde{P}$ *is a convex cone in* $\tilde{E}$ *and satisfies* (P1), (P2), *and* $[A] \leq [B] \Longleftrightarrow [B] - [A] \in \tilde{P}$.

(*b*) $\tilde{E}_1$ *is a subspace which is order isomorphic to* (E, P) *by the correspondence* $E \ni a \longleftrightarrow [A] \in \tilde{E}_1$ *where* $u([A]) = a + P$.

Remark. *If* (E, P) *is order complete, then* $(\tilde{E}, \tilde{P})$ *is isomorphic to* (E, P) *as an ordered linear space.*

For $A \in \mathfrak{B}$ we can characterize $U(A)$ by using the support function of A and the boundary of the dual cone $P^* = \{x^* \in E^* \mid\ < x^*, x > \geq 0\ \ (x \in P)\ \}$. If $A \in \mathfrak{B}$ then the support function $f_A(x^*) = \sup_{x \in A} < x^*, x >$ is finite on P^*. Indeed if $x_0 \in U(A)$, then $< x^*, x > \leq < x^*, x_0 >$ holds for all $x \in A$.

Theorem 2. *For every* $A \in \mathfrak{B}$,

$$U(A) = \bigcap_{x^* \in \partial P^*} \{x \mid\ < x^*, x > \geq f_A(x^*)\},$$

where ∂P^* *denotes the algebraic boundary of* P^*.

Example Let X be the set of all bounded lower semicontinuous functions f on $S^1 = \{(cos\theta, sin\theta) \mid 0 \leq \theta < 2\pi\}$, and we simply write θ for $(cos\theta, sin\theta) \in S^1$. For $f \in X$ let $\tilde{f}$ be such that $\overline{hypo(f)} = hypo(\tilde{f})$ where $hypo(f) = \{(\theta, t) \in S^1 \times \mathbb{R} \mid f(\theta) \geq t\}$, and $\overline{hypo(f)}$ is the closure with respect to the product topology in $S^1 \times \mathbb{R}$. We define an equivalence relation in X by $f \sim g \Longleftrightarrow \tilde{f} = \tilde{g}\ (f,\ g \in X)$. Let E be the space of all 2×2 real symmetric matrices with the positive cone P consists of all positive semidefinite matrices in E. By the correspondence

$$E \ni \begin{pmatrix} u & \frac{x}{2} \\ \frac{x}{2} & v \end{pmatrix} \longleftrightarrow (x, y, z) \in \mathbb{R}^3$$

where $u = \frac{y+z}{2}, v = \frac{-y+z}{2}$, E is identified with $\mathbb{R}^3$ and $P = \{(x, y, z) \mid x^2 + y^2 \leq z^2\}$. In this case the dual cone P^* can be identified with P itself. For $A \subset E$, the values of the support function f_A on ∂P^* are determined if we know the values only on the set $\{(x, y, 1) \mid x^2 + y^2 = 1\}$ which can be identified with S^1. From the previous two theorems and the following lemmas, we can conclude that the completion $\tilde{E}$ of E can be identified with $X/\sim$ by the correspondence

$$\tilde{E} \ni [A] \longleftrightarrow [f_A] \in X/\sim.$$

Lemma 1. *In* $(\mathbb{R}^3, P)$, $\tilde{f_A} = \tilde{f_B}$ *on* ∂P^* *if and only if* $U(A) = U(B)$ *for* $A, B \in \mathfrak{B}$.

Lemma 2. *For every* $\varphi \in X$, *there exists a subset* $A \subset \mathbb{R}^3$ *such that* $\varphi = f_A$ *on* S^1.

Corollary 1. *If* φ *is a bounded lower semicontinuous function on* S^1, *then there exists a lower semicontinuous convex function* f *on the closed disk* $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ *such that* $f = \varphi$ *on* S^1.

Let (E, P) be an ordered linear space. For $A \in \mathfrak{B}$, and $A' \in \mathfrak{B}'$ the generalized supremum and the generalized infimum are defined by

$$\operatorname{Sup} A = \{a \in U(A) \mid b \leq a,\ \ b \in U(A) \Longrightarrow a = b\} \qquad (A \in \mathfrak{B}),$$
$$\operatorname{Inf} A' = \{a \in L(A') \mid b \geq a,\ \ b \in L(A') \Longrightarrow a = b\} \qquad (A' \in \mathfrak{B}'),$$

and we denote that $S = \{\operatorname{Sup} A \mid A \in \mathfrak{B}\}$. The basic properties of generalized supremum has been investigated in [2], [3]. A set A is said to have the **domination property** if for $x \in A$ there exsists a minimal point x_0 of A such that $x_0 \leq x$. In this paper we consider the condition

$$(5) \qquad U(A) = (\operatorname{Sup} A) + P \qquad (\forall A \in \mathfrak{B}),$$

which actually means that $U(A)$ always has the domination property. If the space (E, P) satisfies the condition (5), the correspondence

$$\tilde{E} \ni [A] \longleftrightarrow U(A) \longleftrightarrow \operatorname{Sup} A \in S$$

is one to one. In the rest part of this section we will state some results which suggest the importance of the condition (5) in dealing with the generalized supremum. In the case when $dimE < \infty$, some equivalent conditions of (5) are known.([2]) In the infinite dimensional cases, it is not easy to see when the space (E, P) satisfies the condition (5). In this paper we will give some sufficient conditions in §2.

Proposition 1. *Suppose that* (E, P) *satisfies the condition* (5). *Then* $\operatorname{Inf} A$ *and* $\operatorname{Sup} A$ *have a symmetric property, that is,*

$$\operatorname{Sup}(\operatorname{Inf}(\operatorname{Sup} A))) = \operatorname{Sup} A \qquad (A \in \mathfrak{B}).$$

proof. Taking the set of minimal points of both sides of (1), we have

$$\operatorname{Sup} A = \operatorname{Sup}(L(U(A))).$$

Moreover it follows by (5) that $\operatorname{Sup}(L(U(A))) = \operatorname{Sup}(L((\operatorname{Sup} A) + P)) = \operatorname{Sup}(L(\operatorname{Sup} A))$ $= \operatorname{Sup}(\operatorname{Inf}(\operatorname{Sup} A) - P) = \operatorname{Sup}(\operatorname{Inf}(\operatorname{Sup} A))$.

Proposition 2. *Suppose that* (E, P) *satisfies the condition* (5). *If* $\operatorname{Sup} A = \{a\}$ *for some* $A \in \mathfrak{B}$, *then* $a = \operatorname{lub} A$ (*:the least upper bound of* A).

The proof is trivial. The conclusion of Proposition 2 is not valid when the condition (5) does not hold. The following theorem is the fundamental rules on calculation of the generalized supremum.

Theorem 3. ([4]) *For* $A, B \in \mathfrak{B}$,

(a) $U(A+B) \sim' U(A)+U(B)$ *in* $\mathfrak{B}'$,

Moreover, if (E,P) *satisfies the condition* (5), *then*

(b) $\operatorname{Sup}(A+B)+P \supset \operatorname{Sup}A+\operatorname{Sup}B$,

(c) $\operatorname{Sup}(L(\operatorname{Sup}A+\operatorname{Sup}B)) = \operatorname{Sup}(A+B)$.

Under the condition (5), we define an order relation and a vector operation (the addition $\oplus$ and the scalar multiplication $*$) on S as follows.

Definition. *For* $A, B \subset E$ *and* $\lambda \in \mathbb{R}$,

$$\operatorname{Sup}A \leq \operatorname{Sup}B \Leftrightarrow \operatorname{Sup}B \subset \operatorname{Sup}A+P$$
$$\operatorname{Sup}A \oplus \operatorname{Sup}B = \operatorname{Sup}(A+B)$$
$$\lambda * \operatorname{Sup}A = \begin{cases} \operatorname{Sup}(\lambda l([A])) & (\lambda > 0) \\ \{0\} & (\lambda = 0) \\ \operatorname{Sup}(\lambda u([A])) & (\lambda < 0), \end{cases}$$

for $\operatorname{Sup}A$, $\operatorname{Sup}B \in S$ *and* $\lambda \in \mathbb{R}$.

Let S_0 be the set of all elements $\operatorname{Sup}A \in S$ such that $\operatorname{Sup}A = \{a_0\}$ for some $a_0 \in E$. Then by the following theorem, S can be regarded as an order completion of (E,P) which is isomorphic to S_0.

Theorem 4. ([4]) *If* (E,P) *satisfies* (5), *then* S *is isomorphic to* $\tilde{E}$ *as a vector lattice under the one to one correspondence*

$$S \ni \operatorname{Sup}A \longleftrightarrow [A] \in \tilde{E},$$

Moreover, S_0 *is isomorphic to* (E,P) *under the same correspondence.*

§2 Sufficient conditions for $U(A) = (\operatorname{Sup}A)+P$

An ordered linear space (E,P) is said to be **monotone order complete** (m.o.c.) if every upper bounded totally ordered subset of E has the least upper bound in E. In the case $E = \mathbb{R}^d$, (E,P) is m.o.c. if and only if P is closed. In the case when E is a Banach space with a closed positive cone P satisfying $P^*-P^* = E^*$, it is known that (E^*,P^*) is m.o.c. where E^* is the topological dual of E and $P^* \underset{def}{=} \{x^* \in E^* \mid x^*(x) \geq 0,\ x \in P\}$.

Proposition 3. *Suppose that an ordered linear space* (E,P) *is monotone order complete. Then* (E,P) *satisfies* (5). *In particular,* $\operatorname{Sup}\{a,b\} \neq \emptyset$, $\operatorname{Inf}\{a,b\} \neq \emptyset$ *for every* $a, b \in E$, *and* $U(a,b) = (\operatorname{Sup}\{a,b\})+P$.

The proof of this proposition can be seen in [2]. A convex subset C of E is said to be algebraically closed if every straight line of E meets C by a closed interval. A point x of a convex subset $C \subset E$ is called an algebraic interior point of C if for every $z \in E$, there exists $\lambda > 0$ such that $x+\lambda z \in C$. Algebraic exterior points are defined similarly, and we denote the algebraic interior of C by C^i. Moreover, $\partial C = (C^i \cup (C^c)^i)^c$ is called the algebraic boundary of C. Let (E,P) be an ordered linear space and suppose that

P is algebraically closed with nonempty algebraic interior. A convex subset F of P is called an exposed face of P if there exists a supporting hyperplane H of P such that $F = P \cap H$. By $\mathfrak{F}(P)$, we denote the set of all exposed faces of P. For $F \in \mathfrak{F}(P)$, $\dim F$ is defined as the dimension of aff$F$ where aff$F$ denotes the affine hull of F. We give another sufficient condition for (5) by using the facial structure of P.

Proposition 4. ([2]) *Suppose that P is algebraically closed and $\operatorname{int} P \neq \emptyset$. If $\dim C < \infty$ for every $C \in \mathfrak{F}(P)$, then (5) holds.*

A positive cone P in a topological vector space is said to be **normal** if there exists a neighborhood base of the origin consisting of neighborhoods V satisfying

$$(V + P) \cap (V - P) = V.$$

If P is normal, every order interval $[a, b] = \{x \in E \mid a \leq x \leq b\}$ in E is bounded with respect to the norm. We also recall Bishop-Phelps theorem which asserts that for a bounded closed convex set C in a Banach space E, the set of all bounded linear functional which attains its minimum on C is norm dense in E^*.([6])

Theorem 5. *Let E be a Banach space with a closed positive cone P. If the dual cone P^* has nonempty interior in E^*, then (E, P) has the property (5).*

proof. It is known that P is normal if and only if $P^* - P^* = E^*$ ([1]), and in particular, P is normal in the case that P^* has nonempty interior in E^*. For $x \in U(A)$, we denote $U(A)_x = (x - P) \cap U(A)$. It suffices to show that there exists an minimal point x_0 of $U(A)_x$ such that $x_0 \leq x$. Since P is closed, so is $U(A)_x$. We also have $U(A)_x \subset [a, x] = \{y \in E \mid a \leq y \leq x\}$ for $a \in A$ and hence the normality of P yields that $U(A)_x$ is bounded with respect to the norm in E. Therefore by Bishop-Phelps theorem, we can choose an interior point x_1^* of P^* such that x_1^* attains its minimum on $U(A)_x$ at some point $x_0 \in U(A)_x$. If there exsists $x_1 \in U(A)_x$ such that $x_1 \lneqq x_0$ it follows that $x^*(x_1) < x^*(x_0)$ since x^* is an interior point of P^*. It is a contradiction and x_0 is a minimal point of $U(A)_x$.

Corollary 2. *Let E' be a Banach space and $E = E' \times \mathbb{R}$ and $P = \{(x, t) \in E \mid t \geq \|x\|\}$. Then (E, P) has the property (5).*

Definition. *Let E be a topological vector space with a closed positive cone P. A set $A \subset E$ is said to be* **P-complete** *if it has no covers of the form*

$$\{(x_\alpha - P)^c \mid \alpha \in I\}$$

with $\{x_\alpha\}_{\alpha \in I}$ being a decreasing net in A.

In [5], one can see some conditions under which A becomes P-complete or has the domination property.

Proposition 6. ([5]) *Let E be a topological vector space with a closed positive cone P, and let $E \supset A \neq \emptyset$. Then A has a minimal point if and only if there exists $x \in A$ such that $A_x = A \cap (x - P)$ is P-complete. Moreover, A has the domination property if and only if for each $y \in A$ there is some $x \in A_y$ such that A_x is P-complete.*

Theorem 6. *Let E be a reflexive Banach space with a closed positive cone P and suppose that P is normal. Then (E,P) has the property* (5).

proof. Let $x \in U(A)$ and set $U(A)_x = U(A) \cap (x - P)$. We will show that the section $U(A)_x$ has its minimal point. By Proposition 6, it suffices to show that $U(A)_x$ is P-complete. Suppose that there exsits a decreasing net $\{x_\alpha\}_{\alpha \in I}$ in $U(A)_x$ such that

$$U(A)_x \subset \bigcup_{\alpha \in I} (x_\alpha - P)^c.$$

We observe that $U(A)_x \subset [a, x]$ for $a \in A$ and hence the normality of P yields that $U(A)_x$ is bounded with respect to the norm in E. Hence it is weakly compact because the space E is reflexive. Since each $x_\alpha - P$ is weakly closed, we can choose a subcovering $\bigcup_{i=1\cdots n} (x_i - P)^c \supset U(A)_x$ such that $x_1 \geq x_2 \geq \cdots \geq x_n$. It is a contradiction, because $x_n \notin \bigcup_{i=1\cdots n} (x_i - P)^c$ while $x_n \in U(A)_x$.

The hypothesis on the positive cone P in Theorem 6 is weaker than that in Theorem 5. However, (5) does not follow from the condition that E is a Banach space and P is normal. The space $C[0,1]$ with the norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$ and the usual positive cone $P = \{f \mid f(x) \geq 0 \ (x \in [0,1])\}$ is a simple example.

Let E be a topological vector space with a closed positive cone P. (E,P) is said to be **boudedly order complete (b.o.c.)** if any bouded decreasing net $\{x_\alpha\}$ has an infimum, where bounded net means that for any neighborhood U of origin $\{x_\alpha\} \subset tU$ for some $t > 0$. P is said to be **Daniell** if any decreasing net $\{x_\alpha\}$ having a lower bound has its infimum to which it converges. If P is Daniell (E,P) is obviously m.o.c., and consequently the condition (5) holds. Moreover, we can easily see the following proposition.

Proposition 7. *Let E be a topological vector space with a positive cone P. If (E,P) is b.o.c. and P is normal, then (E,P) is m.o.c., and it satisfies the condition* (5) *in particular.*

References

[1] T. Ando, *On fundamental properties of a Banach space with cone*, Pacific J. Math. **12** (1962), 1163–1169.

[2] N.Komuro, S.Koshi, *Genaralized supremum in partially ordered linear space*, Proc. of the international conference on nonlinear analysis and convex analysis, World Scientific (1999), 199–204.

[3] S.Koshi, N.Komuro, *Supsets on partially ordered topological linear spaces*, Taiwanese J. of Math. **4-2** (2000), 275–284.

[4] N.Komuro, *The set of upper bounds in ordered linear spaces*, Proc. of the international conference on nonlinear analysis and convex analysis, to appear.

[5] D. T. Luc, *Theory of vector optimization*, Springer-Verlag (1989).

[6] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Springer-Verlag (1993).

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An Inequality and the Numerical range of a matrix

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Abstract

We associate a ternary real form $F(t,x,y)$ of degree $2m$ with a complex trigonometric polynomial $\phi(\theta) = c_{-m}\exp(-i\,m\,\theta) + \ldots + c_0 + \ldots + c_m \exp(i\,m\,\theta)$ so that the equation $F(1, \Re(\phi(\theta)), \Im(\phi(\theta))) = 0$ holds for $\theta \in \mathbf{R}$. If $\phi(\theta)$satisfies $|c_{-m}| + \ldots + |c_{m-1}| < |c_m|$, then the form $F(t,x,y)$ is hyperbolic with respect to $(1,0,0)$.

0. 行列の数域の境界は代数曲線

$n \times n$ 複素行列 Tが与えられたとき、その数域 $W(T)$ が、次の式で定義される。

$$W(T) = \{\xi^* T \xi : \xi \in \mathbf{C}^n,\, \xi^*\xi = 1\}.$$

このとき、$W(T)$ の境界は代数曲線弧となる。その双対曲線$\partial W(T)^\wedge$:

$$\partial W(T)^\wedge = \{(a,b) \in \mathbf{R}^2 : a\,x + b\,y + 1 = 0 \,\text{is a tangent of}\, \partial W(T)\},$$

の定義多項式が次の $F(t,x,y) = F_T(t,x,y)$ で与えられる:

$$F(t,x,y) = \det(t\,I_n + (x/2)(T+T^*) - (i\,y/2)(T-T^*)), \tag{0.1}$$

この同次多項式 Fは、点 $(1,0,0)$ に関して Atiya-Bott-Gårding [A-B-G] の意味で、双曲的 hyperbolic である。この意味は、$F(1,0,0) \neq 0$ であって、任意の $(x_0, y_0) \in \mathbf{R}^2$ に対して、t に関する方程式 $F(t, x_0, y_0) = 0$ が n 個の実根 $\phi_j(x_0, y_0)$ $(\,j = 1, 2, \ldots, n)$ を持つということである。さて、Fが定める代数曲線が有理曲線であるような大きなクラスをここで提示する。

1. 三角多項式についての不等式

正の整数 m に対して、次の複素係数の三角多項式

$$\phi(\theta) = \sum_{j=-m}^{m} c_k \exp(i\,k\,\theta)$$

を考える。ここで、条件 $|c_{-m}| + \ldots + |c_0| + \ldots + |c_{m-1}| < |c_m|$ が 成り立つならば、$\phi(\theta)$ は、いかなる $0 \le \theta \le 2\pi$ に対しても、原点 0 を通過しない。この性質は、$F(1,0,0) \neq 0$ に対応する。

補題 1.1 複素数 z, wに対し、次の条件

$$|w-z|+|z|=\eta<1$$

が $0\leq\eta<1$ に対して成り立つとする。このとき、整数 $n\geq 2$ に対し、次の不等式が成立する。

$$\Re([n\,(1+w)-2\{w+(n-1)\,z\}](1+\overline{w}))\geq 2(1-\eta)>\ 0.$$

[証明] 複素数をデカルト分解して実変数を使って表す: $z=x+i\,y,\ w=(x+i\,y)+(p+i\,q)$. このとき仮定の条件は $\sqrt{p^2+q^2}+\sqrt{x^2+y^2}=\eta$ と表される。直接的な計算により式変形して、

$$\begin{aligned}
&\Re([n\,(1+w)-2\{w+(n-1)\,z\}](1+\overline{w}))\\
&=n+(n-2)|w|^2+2(n-1)[\Re(z\,\overline{w})+\Re(w-z)]\\
&=n+(n-2)\{(x+p)^2+(y+q)^2\}-2(n-1)\,\{(x\,(x+p)+y\,(y+q))-2p\}\\
&=n+(n-2)\{(x+p)^2+(y+q)^2\}-2(n-1)\{x^2+p\,x-2p+y^2+q\,y\}\\
&=n-n\,(x^2+y^2)+(n-2)(p^2+q^2)-2\,(p\,x+q\,y)-2(n-1)\,p\\
&=g(x,y:p,q).
\end{aligned}$$

が得られる。ここで、次のような表記

$$\begin{aligned}
x&=\eta\,t\,\cos\theta, & y&=\eta\,t\,\sin\theta,\\
p&=\eta\,(1-t)\cos\phi, & q&=\eta\,(1-t)\sin\phi,
\end{aligned}$$

が或る $0\leq t\leq 1$ および $0\leq\theta,\phi\leq 2\pi$を用いて可能であり、これを使って

$$\begin{aligned}
&g(x,y:p,q)\\
&=n-n\,\eta^2\,t^2+(n-2)\eta^2(1-t)^2-2\eta^2\,t\,(1-t)\,\cos(\theta-\phi)+2(n-1)\eta\,(1-t)\cos\phi\\
&\geq n-n\,\eta^2\,t^2+(n-2)\eta^2(1-t)^2-2\eta^2\,t\,(1-t)-2(n-1)\eta\,(1-t)\\
&=(1-\eta)\,\{2(n-1)\eta\,t+(n-2)(1-\eta)+2\}\geq 2(1-\eta)
\end{aligned}$$

となるから補題 1.1 が証明された。

補題 1.2 n は2以上の整数であって、複素数 $w_1,\,w_2,\ldots,w_n$ は条件

$$|w_1-w_2|+|w_2-w_3|+\ldots+|w_{n-1}-w_n|+|w_n|=\eta<1$$

を $0\leq\eta<1$ に対して満たすとする。このとき、次の不等式が成立する。

$$\Re([n\,(1+w_1)-2(w_1+w_2+\ldots+w_n)](1+\overline{w_1}))\geq 2(1-\eta)>0.$$

[証明] まず $z=(w_2+w_2+\ldots+w_n)/(n-1)$ および $w=w_1$ 　により複素数 z,w を定める。このとき、不等式

$$\begin{aligned}
&|w-w_j|+|w_j|\leq|w_1-w_2|+|w_2-w_3|+\ldots+|w_{j-1}-w_j|+|w_j|\\
&\leq|w_1-w_2|+\ldots+|w_{j-1}-w_j|+|w_j-w_{j+1}|+\ldots+|w_{n-1}-w_n|+|w_n|=\eta
\end{aligned}$$

が任意の $2\leq j\leq n$ に対して成り立つ。これより、さらに不等式

$$|w-z|+|z|=\frac{1}{n-1}\{|(n-1)\,w-(w_2+\ldots+w_n)|+|w_2+\ldots w_n|\}$$

$$\leq \frac{1}{(n-1)}\{(|w-w_2|+|w_2|)+\ldots+(|w-w_n|+|w_n|)\} \leq \eta.$$

が成り立つ。従って、補題 1.1 の仮定の条件が $0 \leq \eta'(\leq \eta)$ に対して成り立つ。これより、不等式

$$\begin{aligned}
&\Re([n\,(1+w_1)-2(w_1+w_2+\ldots+w_n)](1+\overline{w_1})) \\
&= \Re([n\,(1+w)-2(w+(n-1)\,z)](1+\overline{w})) \geq 2(1-\eta') \geq 2(1-\eta) > 0.
\end{aligned}$$

が得られ、補題 1.2 は証明された。

この補題 1.2 を使って、次の定理を示そう。

定理 1.3 $\phi(\theta)$ は、次のような 1 変数の三角多項式

$$\phi(\theta) = \sum_{j=-m}^{m} c_k \exp(i\,k\,\theta)$$

であって、その係数は、条件

$$\sum_{k=-m}^{m-1} |c_k| < |c_m|$$

を満たすとする。このとき、$\theta \in [0, 2\pi]$ の増加に対して、$\phi(\theta)$ の偏角 $\mathrm{Arg}(\phi(\theta))$ も狭義単調増加する。さらに、θ が、0 から、2πまで、区間 $[0, 2\pi]$ 上を変動するとき、$\phi(\theta)$ は、複素数平面において原点 0 の周りをちょうど m 回反時計回りに回転する。

[証明] 係数について $c_m = 1$ と仮定して証明すればよい。以下これを仮定する。媒介変数表示された曲線 $z(\theta) = \phi(\theta)$:

$$z(\theta) = x(\theta) + i\,y(\theta) = \exp(i\,m\,\theta) + \sum_{k=-m}^{m-1} c_k \exp(i\,\theta)$$

($0 \leq \theta \leq 2\pi$) に対しその偏角 $\mathrm{Arg}(z(\theta))$ を考える:

$$\mathrm{Arg}(z(\theta)) = \Im(\mathrm{Log}(z(\theta))).$$

この関数を、変数 θに関して微分する:

$$\begin{aligned}
&\frac{d\mathrm{Arg}(z(\theta))}{d\theta} \\
=\;& \Im(\frac{z'(\theta)}{z(\theta)}) \\
=\;& \Re(\frac{m\,c_m \exp(i\,m\theta)+\ldots+(-m)c_{-m}\exp(-i\,m\,\theta)}{c_m \exp(i\,m\theta)+\ldots+c_{-m}\exp(-i\,m\theta)}) \\
=\;& \frac{1}{|c_m \exp(i\,m\theta)+\ldots+c_{-m}\exp(-i\,m\theta)|^2} \times \Re(L(\theta)),
\end{aligned}$$

ここで、次のような関係式が得られる。

$$\begin{aligned}\Re(L(\theta)) &= \Re((\sum_{k=-m}^{m} k\, c_k \exp(i\, k\theta))(\sum_{p=-m}^{m} \overline{c_p} \exp(-i\, p\theta))) \\ &= \sum_{k=-m}^{m}\sum_{p=-m}^{m} \Re(c_k\overline{c_p}k\ \exp(i\,(k-p)\theta) \\ &= B_\phi(\theta).\end{aligned}$$

従って $\Re(L(\theta)) > 0$ が任意の $0 \leq \theta \leq 2\pi$ に対して成り立つことを示せば、θ の増加に関して $\mathrm{Arg} z(\theta)$ が単調増加することがわかる。 この不等式$\Re(L(\theta)) > 0$ を証明しよう。

絶対値 1 の任意の複素数 $\exp(i\,\theta)$ を取り、 $z_j = \exp(i\,(j-m)\,\theta)c_j(\ -m \leq j \leq m)$ と置く。このとき、次の等式が得られる：

$$\begin{aligned}\Re(L(\theta)) &= \sum_{k=-m}^{m}\sum_{p=-m}^{m} \Re(c_k\overline{c_p}k\ \exp(i\,(k-p)\theta)) \\ &= \sum_{k=-m}^{m}\sum_{p=-m}^{m} \Re(k\, z_k\overline{z_p}),\end{aligned}$$

ここで、複素数 $z_{m-1},\ldots,z_0,\ldots,z_{-m}$ に対して不等式

$$\sum_{k=-m}^{m-1} |z_k| = |z_{-m}| + \ldots + |z_0| + \ldots + |z_{m-1}| = \eta$$

が或る $0 \leq \eta < 1$ に対して成り立つ。次のような変数変換を行う。

$$\begin{aligned}&w_0 = 1 + z_{m-1} + z_{m-2} + \ldots + z_0 + \ldots + z_{-m}, \\ &w_j = \textstyle\sum_{k=-m}^{m-j} z_k, \quad \text{for } 1 \leq j \leq 2m\end{aligned}$$

このとき、次のような関係式が得られる：

$$\begin{aligned}&\textstyle\sum_{k=-m}^{m} k\, z_k = (m+1)\, w_0 - (w_0 + w_1 + w_2 + \ldots + w_{2m}) \\ &= m\, w_0 - (w_1 + w_2 + \ldots + w_{2m}) \\ &= m\,(1 + w_1) - (w_1 + w_2 + \ldots + w_{2m}), \\ &\textstyle\sum_{k=-m}^{m} z_k = w_0 = 1 + w_1.\end{aligned}$$

新しい変数について、次の関係式

$$|w_1 - w_2| + |w_2 - w_3| + \ldots + |w_{2m-1} - w_{2m}| + |w_{2m}| = \eta.$$

が成り立つから、補題1.2 の条件が $n = 2m$ に対して成り立つ。これより、次のことが成り立つ。

$$\sum_{k=-m}^{m}\sum_{p=-m}^{m} \Re(k\, z_k\overline{z_p})$$

$$
\begin{aligned}
&= \Re(\sum_{k=-m}^{m}\sum_{p=-m}^{m} k\, z_k \overline{z_p}) \\
&= (1/2)\,\Re([2m\,(1+w_1) - 2(w_1+w_2+\ldots+w_{2m})](1+\overline{w_1})) \\
&\geq 1-\eta > 0.
\end{aligned}
$$

従って、$\Re(L(\theta)) > 0$ であり、θ の増加に対し、$\mathrm{Arg} z(\theta)$ が単調増加することがわかった。

さて、変数 θは 0 から 2πまで増加するとき, 変数 $\mathrm{Arg}(z(\theta))$ は $\mathrm{Arg}(z(0))$ から $\mathrm{Arg}(z(0)) + 2m\pi$に変化する, すなわち原点 0 に関する 動点 $z(\theta)$ の回転数 winding number は、m である。このことは、次の関係式より得られる：

$$
\begin{aligned}
&\int_{0\leq\theta\leq 2\pi} d\mathrm{Arg}(z(\theta)) \\
&= \int_{0\leq\theta\leq 2\pi} d\mathrm{Arg}(\exp(i\,m\theta)) + \int_{0\leq\theta\leq 2\pi} d\mathrm{Arg}(z(\theta)/\exp(i\,m\theta)) \\
&= 2m\pi + 0 \quad = \quad 2m\pi,
\end{aligned}
$$

ここで、動点 $z(\theta)/\exp(i\,m\theta)$ は、任意の $0 \leq \theta \leq 2\pi$に対し、 $\{w \in \mathbf{C} : |w-1| < 1\}$ に属する。このことより、θが、0 から 2πまで増加するときの、原点 0 に関する 動点 $z(\theta)/\exp(i\,m\theta)$ の回転数は 0 である。以上より、定理 1.3 は証明された。この定理を、多項式を使って表現することにより次の定理が得られる。

定理 1.4 $\phi(\theta)$ は、次のような 1 変数の三角多項式

$$
\phi(\theta) = \sum_{j=-m}^{m} c_k \exp(i\,k\,\theta)
$$

であって、その係数は、条件

$$
\sum_{k=-m}^{m-1} |c_k| < |c_m|
$$

を満たすとする。このとき、 3 変数の $2m$ 次以下の同次多項式 $F(t,x,y)$ で

$$
F(1, \Re(\phi(\theta)), \Im(\phi(\theta))) = 0
$$

($\theta \in \mathbf{R}$) となるものをとるとき、$F(t,x,y)$ は点 $(1,0,0)$ に関して Atiyah-Bot-Gårding の意味で、双曲的である。

2. 行列による実現

論文 [C-N] において有理曲線 $F(1,x,y) = 0$ で、Fが双曲的である次の曲線が扱われている。

$$
x(\theta) = \frac{-\cos(2m\theta)}{\lambda(\theta)},
$$

$$y(\theta) = \frac{\sin(2m\theta)}{\lambda(\theta)},$$

$$\lambda(\theta) = \frac{a_m}{2} + \sum_{j=1}^{m-1} [a_{m-j}\cos(j\theta) - b_{m-j}\sin(j\theta)],$$

($n = 2m$ が偶数のとき)、あるいは

$$x(\theta) = \frac{-\cos((2m-1)\theta)}{\lambda(\theta)},$$

$$y(\theta) = \frac{\sin((2m-1)\theta)}{\lambda(\theta)},$$

ここで、

$$\lambda(\theta) = \sum_{j=1}^{m-1} [a_{m-j}\cos((2j-1)\theta) - b_{m-j}\sin((2j-1)\theta)].$$

($n = 2m-1$ が奇数のとき)。上記の関係式において, a_jおよび b_jは実の係数である。

定理 1.4 で述べたように、 有理曲線で、対応する多項式 $F(t,x,y)$ が双曲的であるような、大きな族が見つかった。この族の特殊な場合が、[N] で扱われている。そこでは、$H>1$ に対する曲線

$$x(\theta) = \frac{2H^2}{\sqrt{H^{10}-1}}\{\cos(2\theta) + H^5\cos(3\theta)\},$$

$$y(\theta) = \frac{2H^2}{\sqrt{H^{10}-1}}\{-\sin(2\theta) + H^5\sin(3\theta)\},$$

を扱い、行列

$$T(H) = \begin{bmatrix} 0 & H^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & H^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & H^{-2} & 0 & \sqrt{H^{10}-1}/H^2 \\ 0 & 0 & 0 & 0 & H^3 & 0 \\ H^{-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

が、条件

$$\det(I + x(\theta)/2\,(T(H)+T(H)*) - i\,y(\theta)/2\,(T(H)-T(H)*)) = 0, \qquad (3.1)$$

を、 $0 \le \theta \le 2\pi$に対し満たすことを示した。その論文 [N] では, 上の行列の一般化も考えられている。

References

[A-B-G]M. F. Atiyah, R. Bott and L. Gårding: Lacunas for hyperbolic differential operators with costant coefficient I, *Acta Math.*, **124**(1970), 109-189.

[C-N] Chien, M. T. and Nakazato, H. (1999). Boundary generating curves of the c-numerical range, *Lin. Alg. Appl.*, **294**, pp. 67-84.

[N] H. Nakazato : Numerical range and sextic rational curves, *Bull. Fac. Sci. Tech. Hirosaki Univ.*, **3**, 83-91.

On some convex functions related with norms of Banach Spaces

Yasuji Takahashi and Mikio Kato

Abstract. We shall characterize several geometrical properties of a Banach space X such as uniform non-squareness and uniform convexity etc. in terms of the convex function f on X defined by $f(x) = \varphi(\|x\|)$, where φ is a non-negative strictly convex and strictly increasing function on $[0, \infty)$.

Let X be a Banach space (of dimension at least 2). Let φ be a strictly convex and strictly increasing function defined on $[0, \infty)$ with values in $[0, \infty)$ (such a function is continuous on $[0, \infty)$). Define the function f on X by $f(x) = \varphi(\|x\|)$. Then the following inequalitites are immediately seen: For all $x, y \in X$ we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \tag{1}$$

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) \leq f(x) + f(y). \tag{2}$$

1. Definitions (i) X is called *non-square* when

$$\min(\|x+y\|, \|x-y\|) < 2 \ \text{ if } \ \|x\| = \|y\| = 1.$$

(ii) X is called *uniformly non-square in the sense of James* when there exists $\delta > 0$ such that

$$\min(\|x+y\|, \|x-y\|) \leq 2(1-\delta) \ \text{ if } \ \|x\| = \|y\| = 1.$$

(iii) X is called *strictly convex* when

$$\|x+y\| < 2 \ \text{ if } \ \|x\| = \|y\| = 1, \ \ x \neq y.$$

(iv) X is called *uniformly convex* when for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x+y\| \leq 2(1-\delta) \ \text{ if } \ \|x\| = \|y\| = 1, \ \ \|x-y\| = \epsilon.$$

It is easy to see that X is non-square if and only if X contains no subspace isometric to ℓ_1^2 (two dimensional real ℓ_1-space).

2. Theorem The following are equivalent.
(i) X is non-square.
(ii) For all $x, y \in X (x \neq 0 \ \text{ or } \ y \neq 0)$

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) < f(x) + f(y).$$

(iii) For all $x, y \in X (x \neq 0 \ \text{ or } \ y \neq 0)$

$$\min\left\{f\left(\frac{x+y}{2}\right), f\left(\frac{x-y}{2}\right)\right\} < \frac{f(x)+f(y)}{2}.$$

3. Theorem The following are equivalent.
(i) X is strictly convex.
(ii) For all $x, y \in X (x \neq y)$

$$f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}.$$

4. Theorem The following are equivalent.
(i) X is uniformly non-square.
(ii) There exists $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) \leq f(x) + f(y) - \delta \ \text{ if } \ \|x\| = \|y\| = 1. \tag{3}$$

(iii) There exists $\delta > 0$ such that

$$\min\left\{f\left(\frac{x+y}{2}\right), f\left(\frac{x-y}{2}\right)\right\} \leq \frac{f(x)+f(y)}{2} - \delta \ \text{ if } \ \|x\| = \|y\| = 1. \tag{4}$$

(iv) For any $\epsilon > 0$ there exists $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) \leq (1-\delta)(f(x)+f(y)) \ \text{ if } \ \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon. \tag{5}$$

5. Remark f is called homogeneous of order p if $f(tx) = t^p f(x)$ for all $t > 0, x \in X$. It is shown that if f is homogenuous of order p with $p > 1$, then X is uniformly non-square if and only if there exists $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) \leq (1-\delta)(f(x)+f(y)) \ \text{ for all } \ x, y \in X. \tag{6}$$

6. Corollary Let $1 < p < \infty$. Then X is uniformly non-square if and only if there exists $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\|^p + \left\|\frac{x-y}{2}\right\|^p \leq (1-\delta)(\|x\|^p + \|y\|^p) \ \text{ for all } \ x, y \in X. \tag{7}$$

7. Theorem The following are equivalent.
(i) X is uniformly convex.
(ii) For any $\epsilon > 0$ there exists $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - \delta \text{ if } \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon. \tag{8}$$

(iii) For any $\epsilon > 0$ there exists $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) \leq (1-\delta)\frac{f(x)+f(y)}{2} \text{ if } \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon. \tag{9}$$

(iv) For any $a > 0, b > 0, \epsilon > 0$ there exists $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) \leq (1-\delta)\frac{f(x)+f(y)}{2} \text{ if } \|x\| \leq a, \|y\| \leq b, \|x-y\| \geq \epsilon. \tag{10}$$

(v) For any $\tau > 0$ there exists $\delta > 0$ such that

$$f(x) \leq \frac{f(x+\tau y)+f(x-\tau y)}{2} - \delta \text{ if } \|x\| = \|y\| = 1. \tag{11}$$

(vi) For any $a > 0, b > 0, c > 0 (b \leq c), \tau > 0$ there exists $\delta > 0$ such that

$$f(x) \leq \frac{f(x+\tau y)+f(x-\tau y)}{2} - \delta \text{ if } \|x\| \leq a, b \leq \|y\| \leq c. \tag{12}$$

(vii) For any $\tau > 0$ there exists $\delta > 0$ such that

$$f(x) \leq (1-\delta)\frac{f(x+\tau y)+f(x-\tau y)}{2} \text{ if } \|x\| = \|y\| = 1. \tag{13}$$

8. Remark Similar characterizations of uniform non-squareness holds true: X s uniformly non-square if and only if the inequality (8) (resp. (9)) is valid for *some* $\epsilon (0 < \epsilon < 2)$ and $\delta > 0$, and the same is valid for (11) (resp. (13)).

9. Corollary Let $1 < p < \infty$. Then the following are equivalent.
(i) X is uniformly convex.
(ii) For any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\|^p \leq \frac{\|x\|^p+\|y\|^p}{2} - \delta \text{ if } \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon. \tag{14}$$

(iii) For any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\|^p \leq (1-\delta)\frac{\|x\|^p+\|y\|^p}{2} \text{ if } \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon. \tag{15}$$

(iv) For any $a > 0, b > 0, \epsilon > 0$ there exists $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\|^p \leq (1-\delta)\frac{\|x\|^p + \|y\|^p}{2} \quad \text{if } \|x\| \leq a, \|y\| \leq b, \|x-y\| \geq \epsilon. \tag{16}$$

(v) For any $\tau > 0$

$$\inf\left\{\frac{\|x+\tau y\|^p + \|x-\tau y\|^p}{2} : \ \|x\| = \|y\| = 1\right\} > 1. \tag{17}$$

10. Corollary Let $1 < p < \infty$. Then the following are equivalent.
(i) X is uniformly non-square.
(ii) There exist $\epsilon (0 < \epsilon < 2)$ and $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\|^p \leq \frac{\|x\|^p + \|y\|^p}{2} - \delta \quad \text{if } \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon. \tag{18}$$

(iii) There exist $\epsilon (0 < \epsilon < 2)$ and $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\|^p \leq (1-\delta)\frac{\|x\|^p + \|y\|^p}{2} \quad \text{if } \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon. \tag{19}$$

(iv) There exists $\tau (0 < \tau < 1)$ such that

$$\inf\left\{\frac{\|x+\tau y\|^p + \|x-\tau y\|^p}{2} : \ \|x\| = \|y\| = 1\right\} > 1. \tag{20}$$

11. Remark Schäffer type constant is defined by

$$S_{X,t}(\tau) := \begin{cases} \inf\left\{\left(\dfrac{\|x+\tau y\|^t + \|x-\tau y\|^t}{2}\right)^{1/t} : \ \|x\| = \|y\| = 1\right\} & \text{if } 1 < t < \infty, \\ \inf\left\{ \max(\|x+\tau y\|, \|x-\tau y\|) : \ \|x\| = \|y\| = 1\right\} & \text{if } t = \infty. \end{cases}$$

It follows from (17) and (20) that X is uniformly convex if and only if $S_{X,t}(\tau) > 1$ for all $\tau > 0$, and X is uniformly non-square if and only if $S_{X,t}(\tau) > 1$ for some $\tau (0 < \tau < 1)$.

References

[1] J. Banas and B. Rzepka, Functions related to convexity and smoothness of normed spaces, Rend. Circ. Mat. Palermo (2) **46** (1997), 395-424.
[2] B. Beauzamy, Introduction to Banach Spaces and their Geometry, 2nd ed., North-Holland, 1985.

[3] E. Casini, About some parameters of normed linear spaces, Atti. Acad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. **80** (1986), 11-15.
[4] J. Gao and K. S. Lau, On the geometry of spheres in normed linear spaces, J. Austral. Math. Soc. Ser. A **48** (1990), 101-112.
[5] V. I. Istrăţescu, Strict convexity and complex strict convexity, LN in Pure Appl. Math. **89**, Marcel Dekker, 1984.
[6] R. C. James, Uniformly non-square Banach spaces, Ann. of Math. **80** (1964), 542-550.
[7] M. Kato, L. Maligranda and Y. Takahashi, On James, Jordan-von Neumann constants and the normal structure coefficients of Banach spaces, Studia Math. **144** (2001), 275-295.
[8] J. J. Schäffer, Geometry of spheres in normed spaces, LN in Pure Appl. Math. **20**, Marcel Dekker, 1976.
[9] Y. Takahashi and M. Kato, Von Neumann-Jordan constant and uniformly non-square Banach spaces, Nihonkai Math. J. **9** (1998), 155-169.
[10] Y. Takahashi, M. Kato and S. Takahasi, On James and Schaeffer constants for Banach spaces, 数理解析研究所講究録 **1186** (2001), 189-193.
[11] Y. Takahashi and M. Kato, Functions related to convexity and smoothness of Banach spaces, 北海道大学数学講究録 **70** (2002), 52-55.

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Rouché Type Theorem and Operator Theory

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Abstract. We show Rouché type theorems on the unit disc D in $\mathbb{C}$ using operator theory. As an application, we describe a function whose Denjoy-Wolff point is in ∂D.

Rouché の定理と作用素論

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Rouché の定理 U を有限個の単純閉曲線で囲まれた領域、f, g は $\bar{U}$ 上で正則かつ f は ∂U に零点をもたないとする。もし $|f(z)| > |g(z)|$ $(z \in \partial U)$ ならば、f の U における零点の個数と $f-g$ の U における零点の個数は等しい。

証明 $t \in [0,1]$ に対して $f - tg$ は仮定より ∂U 上で零とならないので、

$$N(t) = \frac{1}{2\pi i}\int_{\partial U} \frac{f'(z) - tg'(z)}{f(z) - tg(z)} dz$$

とすると、$N(t)$ は U における $f - tg$ の U における零点の個数となる。$N(t)$ は $[0,1]$ で連続関数となることを示すことができ、結果として $N(t) = N(0) = N(1)$。

Rouché の定理で、f, g が ∂U 上で連続だから、$|f(z)| > |g(z)|$ $(z \in \partial U)$ と $|f(z)| \geq \varepsilon + |g(z)|$ $(z \in \partial U)$ となる $\varepsilon > 0$ が存在することは同値である。

D を単位円板、H^p $(0 < p \leq \infty)$ は D 上の Hardy 空間とする。$f \in H^p$ のとき、f が outer 関数とは $\int_0^{2\pi} \log|f(e^{i\theta})| d\theta/2\pi = \log|f(0)|$ となるとき、f が inner 関数とは $|f(e^{i\theta})| = 1$ $a.e.\theta$ となるときをいう。一般の零でない $f \in H^p$ は $f = qh$ と因数分解できることが知られている。ここで q は inner 関数、h は outer 関数である。$q = q[f]$ と書く。

q_1, q_2 を inner 関数とする。$q_1 \succ q_2$ とは $\bar{q}_1 q_2 = |f|/f$ となる outer 関数 $f \in H^1$ が存在することである。$q_1 \succ q_2$ かつ $q_1 \prec q_2$ のとき $q_1 \sim q_2$ と書く。inner 関数 q を Blaschke 積とするとき、$\deg(q)$ は q の零点の個数を表す。q_1 は finite Blaschke 積かつ q_2 は inner 関数とするとき、$q_1 \succ q_2$ ならば $\deg(q_1) \geq \deg(q_2)$ かつだから q_2 も finite Blaschke 積となる。$q_1 \sim q_2$ ならば $\deg(q_1) = \deg(q_2)$ である。

次の定理は Rouché の定理の一般化である。

定理 1　$0 < p \leq \infty$ とする。$f, g \in H^p$ は零でないとする。$|f(z)| \geq \varepsilon + |g(z)|$ (a.e. $z \in \partial D$) となる $\varepsilon > 0$ が存在するならば, $q[f] \sim q[f-g]$ である。

証明　$q_1 = q[f]$、$q_2 = q[f-g]$、$f = q_1 h$、h は outer 関数かつ $k = g/h$ とすると、$f - g = h(q_1 - k)$ となる。ここで $k \in H^\infty$ かつ $\|k\|_\infty \leq 1$ かつ $q_2 = q[q_1 - k]$ となる。ℓ を $q_1 - k$ の outer part とすると、ℓ、$\ell^{-1} \in H^\infty$ である。$q_1 - k = (1 - \bar{q}_1 k) q_1 = q_2 \ell$ より

$$\bar{q}_1 q_2 = (1 - \bar{q}_1 k)\ell^{-1},\ T_{\bar{q}_1 q_2} = T_{(1-\bar{q}_1 k)} T_{\ell^{-1}}$$

である。$T_{(1-\bar{q}_1 k)}$ と $T_{\ell^{-1}}$ が invertible Toeplitz 作用素であることは良く知られているから、$T_{\bar{q}_1 q_2}$ は invertible である。Widom-Devinatz の定理を用いて（または直接に)、$q_1 \sim q_2$ を示すことができる。

定理 1 で $\varepsilon = 0$ とすると、$q[f] \sim q[f-g]$ は成立しない。何故なら、$f = z$ かつ $g = 1$ とすると、$|f(z)| = |g(z)|$ $(z \in \partial D)$ しかし $q[f] = z \succ q[f-g] = 1$ であるから。

次の定理は定理 1 で $\varepsilon = 0$ としたときを調べている。しかし、このとき Widom-Devinatz の定理を用いる事が出来ない。

定理 2　$0 < p \leq \infty$ とする。$f, g \in H^p$ は零でないとする。もし $|f(z)| \geq |g(z)|$ (a.e.$z \in \partial D$) ならば、$q[f] \geq q[f-g]$ である。ただし $f \not\equiv g$ とする。

定理 2 の証明のためには、次の Adamyan-Arov-Krein の定理を用いる。これは彼等によって Hankel 作用素を用いて証明されたものの Nakazi [1] による変形である。

補題　$\phi = f/|f|$、$f \in H^1$ かつ $f \not\equiv 0$ とすると、

$$\begin{aligned} &\{g\ ;\ g \in H^\infty, \|\phi - g\|_\infty \leq 1\} \\ &= \left\{ \frac{F(1-Q)(1-w)}{1-Qw}\ ;\ w \in H^\infty, \|w\|_\infty \leq 1 \right\} \end{aligned}$$

である。ここで、$F \in H^1$ かつ $\bar{\phi} F \geq 0$, かつ

$$\frac{1+Q(z)}{1-Q(z)} = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} |F(e^{it})| dt/2\pi \quad (z \in D)。$$

定理 2 の証明　$f = q_1 h$, $q_1 = q[f]$, h は outer 関数かつ $k = g/h$ とすると、$f - g = h(q_1 - k)$、$\|k\|_\infty \leq 1$ かつ $k \in H^\infty$ となる。$q_1 = (1+q_1)^2/|1+q_1|^2$、$(1+q_1)^2 \in H^1$ から $\phi = q_1$ として補題を適用すると

$$\begin{aligned} &\{\ell\ ;\ \ell \in H^\infty,\ \|q_1 - \ell\|_\infty \leq 1\} \\ &= \left\{ \frac{F(1-Q)(1-w)}{1-Qw}\ ;\ w \in H^\infty,\ \|w\|_\infty \leq 1 \right\}。\end{aligned}$$

$q_1 - k = \ell$ とすると、$\|q_1 - \ell\|_\infty = \|k\|_\infty \leq 1$ から $\ell = F(1-Q)(1-w)/(1-Qw)$ かつ $\bar{q}_1 F \geq 0$ 。このとき $(1-Q)(1-w)/(1-Qw)$ は outer 関数から、$q[f-g] = q[q_1 - k] = q[\ell] = q[F]$ となる。$\overline{q[f]} F \geq 0$ だから $q[f] \succ q[f-g]$。

系 1 (Sarason [2]) f を finite Blaschke 積、$g \in H^\infty$ かつ $\|g\|_\infty \le 1$ とすると、$\deg(f) \ge \deg q[f-g]$ である。

系 2 ϕ を D から D の中への正則写像かつ $\phi \not\equiv z$ とする。λ を ϕ の Denjoy-Wolff point (不動点) とする。また

$$F(a,w) = \frac{(3z+2a)w-(z+2a)}{(3+2\bar{a}z)-(1+2\bar{a}z)w}$$

とする。ここで $a \in \mathbb{C}$, $|a|=1$ かつ $w \in H^\infty$, $\|w\|_\infty \le 1$, $w \not\equiv 1$ とする。

(1) $\lambda \in \partial D$ である必要十分条件はある $|b|=1$ となる $b \in \mathbb{C}$ があって $\phi(z) = bF(a,w)$ かつ $|a|=1$ である。

(2) $\lambda \in D$ である必要十分条件は $\phi(z) = F(a,w)$ かつ $|a|<1$ である。

証明 Sarason [2] は $\lambda \in \partial D$ である必要十分条件はある $|b|=1$ となる $b \in \mathbb{C}$ があって、$z - b\phi$ は outer 関数、$\lambda \in D$ である必要十分条件はある $\alpha \in D$ に対して $q[z-\phi] = \dfrac{z-\alpha}{1-\bar{\alpha}z}$ である。補題より、$\phi \in H^\infty$ が $\|\phi\|_\infty \le 1$ かつ $\phi \not\equiv z$ となる必要十分条件より

$$z - \phi = \frac{F(1-Q)(1-w)}{1-Qw}, \ w \in H^\infty, \ \|w\|_\infty \le 1$$

である。このとき $F = \gamma(z+a)(1+\bar{a}z)$, $a \in \mathbb{C}$, $|a| \le 1$, $\gamma > 0$ かつ $Q = (1+2\bar{a}z)/(3+2\bar{a}z)$ である。よって $\phi = F(a,w)$, $|a| \le 1$ となる。上に述べた Sarason [2] の結果より、(1) と (2) が導ける。

系 2 の応用として、$|a|=1$, $|\alpha|=1$ かつ $\phi(z) = F(a, \alpha z)$ とせよ。そのとき $\alpha \neq 1$ なら $\phi(-a) = -a$ であるが、もし $\alpha = 1$ なら $\phi(1) = 1$ である。

References

1. T.Nakazi, Factorizations of outer functions and extremal problems, Proc.Edinburgh Math.Soc. (1996)39, 535-546.

2. D.Sarason, Making an outer function from two inner functions, Contemp.Math. (1991)137, 407-414.

TENSOR PRODUCT OF LOG-HYPONORMAL OPERATORS

KÔTARÔ TANAHASHI[1] AND MUNEO CHŌ[2]

ABSTRACT. Let A (resp. B) be a bounded linear operator on a complex Hilbert space $\mathcal{H}$ (resp. $\mathcal{K}$). We show that the tensor product $A \otimes B$ is log-hyponormal if and only if A and B are log-hyponormal, and we show similar result holds for class $A(s,t)$ operators.

1. 序.

ヒルベルト空間 $\mathcal{H}$ 上の有界線形作用素 T を考える。hyponormal 作用素

$$TT^* \leq T^*T$$

については Putnam 不等式 ([14])

$$\|T^*T - TT^*\| \leq \frac{1}{\pi}\mathrm{meas}_2(\sigma(T)) = \frac{1}{\pi}\iint_{\sigma(T)} r dr d\theta$$

が成り立つ。 Aluthge [1] によって hyponormal 作用素の拡張である p-hyponormal 作用素 $(0 < p \leq 1)$

$$(TT^*)^p \leq (T^*T)^p$$

が導入されたが、長、伊藤 [3] によって次のような p-hyponormal 作用素の Putnam 不等式

$$\left\|\frac{(T^*T)^p - (TT^*)^p}{p}\right\| \leq \frac{1}{\pi}\iint_{\sigma(T)} r^{2p-1} dr d\theta$$

が示された。棚橋 [16, 17] は、ここから $p \to +0$ とした

$$\|\log(T^*T) - \log(TT^*)\| \leq \frac{1}{\pi}\iint_{\sigma(T)} r^{-1} dr d\theta$$

を予想して log-hyponormal 作用素の Putnam 不等式を証明した。log-hyponormal 作用素のアイデアは安藤 [2] に見られるが、この用語を用いたのは、藤井、姫路、松本 [6] が最初である。その後、p-hyponormal 作用素の様々な性質が log-hyponormal 作用素についても示され、いわば、log-hyponormal 作用素は 0-hyponormal 作用素であるといえることがわかってきた。

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次は p-hyponormal, log-hyponormal 作用素の性質である。([1, 9, 16, 17])

[命題 1] $T \in B(\mathcal{H})$ は p-hyponormal, または log-hyponormal 作用素とするとき次が成立する。

(1) $Tx = \lambda x$ ならば $T^*x = \overline{\lambda}x$ である。
(2) $(T-\lambda)x_n \to 0$ ($\|x_n\| = 1$) ならば $(T-\lambda)^*x_n \to 0$ である。
(3) $T = U|T|$ と極分解する。T が p-hyponormal ($0 \leq p < 1/2$) ならば Aluthge 変換 $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ は $(p+1/2)$-hyponormal である。

2. 本論.

ここでは log-hyponormal 作用素のテンソル積の性質を調べる。

[定義] ヒルベルト空間 $\mathcal{H}, \mathcal{K}$ のテンソル積 $\mathcal{H} \otimes \mathcal{K}$

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle$$

を考える。ここで $A \in B(\mathcal{H}), B \in B(\mathcal{K})$ のテンソル積 $A \otimes B \in B(\mathcal{H} \otimes \mathcal{K})$ は

$$(A \otimes B)(x \otimes y) = Ax \otimes By$$

で定められる。

[参考] Jinchuan [13] は $A \otimes B$ が normal となる必要十分条件は A, B が normal であることを示した (ただし $A, B \neq 0$ とする)。Stochel [15] は hyponormal, Duggal [4] は p-hyponormal, Jeon, Duggal [12] は class A の場合を示した。しかし paranormal ($\|Tx\|^2 \leq \|T^2x\|\|x\|$) については安藤 [2] が T が paranormal でも $T \otimes T$ は paranormal にならないという例を作っている。

[補題 2] $A = U_A|A| \in B(\mathcal{H}), B = U_B|B| \in B(\mathcal{K})$ と極分解する。

(1) $|A \otimes B| = |A| \otimes |B|$.
(2) $A \otimes B$ の極分解は $A \otimes B = (U_A \otimes U_B)(|A| \otimes |B|)$ である。

[補題 3] $A \in B(\mathcal{H}), B \in B(\mathcal{K})$ は可逆で $0 \leqq A, B$ とする。このとき $\log(A \otimes B) = (\log A) \otimes I + I \otimes (\log B)$ となる。

[定理 4] $A \otimes B$ が log-hyponormal となる必要十分条件は A, B が log-hyponormal となることである。

[定義] $T = U|T|$ と極分解したとき $\tilde{T}_{s,t} = |T|^sU|T|^t$ を Aluthge transform for $s, t > 0$ という。藤井, D. Jung, S.H. Lee, M.Y. Lee, 中本 [7] は class $A(s,t)$

$$|\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t}$$

を導入し、さらに、伊藤 [10] は class $wA(s,t)$

$$|\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t}, |T|^{2s} \geq |(\tilde{T}_{s,t})^*|^{\frac{2s}{s+t}}$$

を導入した。これらの作用素の関係を、藤井、伊藤, 山崎, 柳田, 古田らは詳しく調べたが、最近、伊藤、山崎 [11] により class $A(s,t)$ と class $wA(s,t)$ は一致することが示されている。また、class $A(k,1)$ を class $A(k)$, class $A(1,1)$ を class A といい、これらについては以前から調べ

られていた ([8])。p-hyponormal, log-hyponormal 作用素は class $A(s,t)$ であり、class $A(1,1)$ 作用素は paranormal であること等の面白い結果が示されている。([3, 9, 10, 16, 17, 18])

次の定理は class $A(s,t)$ のテンソル積について定理 4 と同じ結果が成り立つことを示している。

[定理 5] A, B は零でないとする。このとき $A \otimes B$ が class $A(s,t)$ となる必要十分条件は A, B が class $A(s,t)$ となることである。

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References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integr. Equat. Oper. Th., **13** (1990), 307–315.

[2] T. Ando, *Operators with a norm condition,* Acta Sci. Math., **33** (1972), 169–178.

[3] M. Chō and M. Itoh, *Putnam's inequality for p-hyponormal operators,* Proc. Amer. Math. Soc., **123** (1995), 2435–2440.

[4] B.P. Duggal, *Tensor products of operators - strong stability and p-hyponormality,* Glasgow Math. J. **42** (2000), 371–381.

[5] D.R. Farenick and I.H. Kim, *Tensor products of quasihyponormal operators,* Proceedings of Kotac, **4** (2002), 113–119.

[6] M. Fujii, C. Himeji and A. Matsumoto, *Theorems of Ando and Saito for p-hyponormal operators,* Math. Japonica, **39** (1994), 595–598.

[7] M. Fujii, D. Jung, S. H. Lee, M. Y. Lee and R. Nakamoto, *Some classes of operators related to paranormal and log-hyponormal operators,* Math. Japonica, **51** (2000), 395–402.

[8] T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes,* Sci. Math., **1** (1998), 389–403.

[9] T. Huruya, *A note on p-hyponormal operators,* Proc. Amer. Math. Soc., **125** (1997), 3617–3624.

[10] M. Ito, *Some classes of operators with generalized Aluthge transformations,* SUT J. Math., **35** (1999), 149–165.

[11] M. Ito and T. Yamazaki, *Relations between two inequalities* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \leq B^r$ *and* $A^p \leq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ *and their applications,* preprint.

[12] I.H. Jeon and B.P. Duggal, *On operators with an absolute value condition,* preprint.

[13] H. Jinchuan, *On the tensor products of operators,* Acta Math. Sinica, **9** (1993), 195–202.

[14] C. R. Putnam, *An inequality for the area of hyponormal spectra,* Math. Z., **116** (1970), 323–330.

[15] J. Stochel, *Seminormality of operators from their tensor product,* Proc. Amer. Math. Soc., **124** (1996), 135–140.

[16] K. Tanahashi, *On* log-*hyponormal operators,* Integr. Equat. Oper. Th., **34** (1999), 364–372.

[17] K. Tanahashi, *Putnam's inequality for log-hyponormal operators,* Integr. Equat. Oper. Th., to appear.

[18] T. Yoshino, *The p-hyponormality of the Aluthge transform,* Interdisciplinary Information Sciences, **3** (1997), 91–93.

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AROUND THE FURUTA INEQUALITY

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1. Introduction. Throughout this note, A and B are positive operators on a Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if A is a positive (resp. invertible) operator. The α-power mean of A and B introduced by Kubo-Ando [18] is given by

$$A \natural_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} \quad \text{for } 0 \leq \alpha \leq 1.$$

The Furuta inequality [6] can be written by the form of α-power mean as follows ([2],[3],[12],[13],[14]).

Furuta inequality: *If $A \geq B \geq 0$, then*

(F) $$A^u \natural_{\frac{1-u}{p-u}} B^p \leq A \quad \text{and} \quad B \leq B^u \natural_{\frac{1-u}{p-u}} A^p$$

holds for $u \leq 0$ and $1 \leq p$.

(F) is an extension of the Löwner-Heinz inequality:

(LH) $$If \ \ A \geq B \geq 0, \ \text{ then } \ A^{\alpha} \geq B^{\alpha} \ \text{ for } \ 0 \leq \alpha \leq 1.$$

As shown in [12](cf.[7]), we can arrange (F) in one line by the form of α-power mean as follows:

Satellite theorem of the Furuta inequality: *If $A \geq B \geq 0$, then*

(SF) $$A^u \natural_{\frac{1-u}{p-u}} B^p \leq B \leq A \leq B^u \natural_{\frac{1-u}{p-u}} A^p$$

holds for all $u \leq 0$ and $1 \leq p$.
More generally, in [13], we have shown

(SF)′ $$A^u \natural_{\frac{\delta-u}{p-u}} B^p \leq B^{\delta} \text{ and } A^{\delta} \leq B^u \natural_{\frac{1-u}{p-u}} A^p$$

holds for $0 \leq \delta \leq p$ and $u \leq 0$.

For positive invertible operator A and B, we denote $A \gg B$ if $\log A \geq \log B$ and called the chaotic order ([3],[16],[17]).

The next characterization of the chaotic order we obtained in [3] is usefull and starting point of our following discussions, so we call it chaotic Furuta inequality.

Chaotic Furuta inequality: *Let A and B be positive invertible operators. If $A \gg B$, then*

(CF) $$A^u \natural_{\frac{-u}{p-u}} B^p \leq I \leq B^u \natural_{\frac{-u}{p-u}} A^p$$

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for any $p \geq 0$ *and* $0 \geq u$.

By the form of the Furuta inequality (F), we have the following ([16],[17]). As compared with (SF), this shows the difference between the usual order $A \geq B$ and the chaotic order $A \gg B$.

Satellite theorem of chaotic Furuta inequality: *Let* A *and* B *be positive invertible operators. If* $A \gg B$, *then*

$$\text{(SCF)} \qquad A^u \natural_{\frac{1-u}{p-u}} B^p \leq B \quad \text{and} \quad A \leq B^u \natural_{\frac{1-u}{p-u}} A^p$$

holds for any $p \geq 1$ *and* $0 \geq u$.

We had generalized (CF) and (SCF) more as follows [16]:

Theorem A. *Let* A *and* B *be positive invertible operators, if* $A \gg B$, *then the following* (1) *and* (2) *hold.*

$$(1) \qquad A^u \natural_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \quad \text{and} \quad A^\delta \leq B^u \natural_{\frac{\delta-u}{p-u}} A^p \quad \text{for} \quad u \leq 0 \quad \text{and} \quad 0 \leq \delta \leq p$$

$$(2) \qquad A^u \natural_{\frac{\alpha-u}{p-u}} B^p \leq A^\alpha \ \text{and}\ B^\alpha \leq B^u \natural_{\frac{\alpha-u}{p-u}} A^p \ \text{for} \ \ u \leq \alpha \leq 0 \ \text{and} \ \ 0 \leq p.$$

2. Grand Furuta inequality. As a generalization of the Furuta inequality, Furuta [8] had given an inequality which we called the grand Furuta inequality in [4],[5] and [15]. It interpolates the Furuta inequality and the Ando-Hiai inequality [1] equivalent to the main result of log majorization.

The grand Furuta inequality: *If* $A \geq B \geq 0$ *and* A *is invertible, then for each* $1 \leq p$ *and* $0 \leq t \leq 1$,

$$\text{(GF)} \qquad A^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A \quad \text{and} \quad B \leq B^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (B^t \natural_s A^p)$$

holds for $t \leq r$ *and* $1 \leq s$.

The best possibility of the power $\frac{1-t+r}{(p-t)s+r}$ is shown in [19]. Replacing s with $\frac{\beta-t}{p-t}$, $1 \leq p \leq \beta$, we can state this theorem by the satellite form as follows [15]:

Satellite theorem of the grand Furuta inequality. *If* $A \geq B > 0$, *then the following holds for* $0 \leq t \leq 1 \leq p \leq \beta$ *and* $u \leq 0$.

$$A^u \natural_{\frac{1-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{1}{\beta}} \leq B^u \natural_{\frac{1-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

To prove this, we had shown the following theorem ([4],[5]).

Theorem B. *If* $A \geq B > 0$, *then the following holds for* $0 \leq t \leq 1 \leq p \leq \beta$

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \quad \text{and} \quad (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{1}{\beta}} \geq A.$$

3. Uchiyama's Theorem and Furuta's Theorem.

Recently, Uchiyama [20] has shown the following inequality as an extension of (F).
Uchiyama's Theorem. *If $A \geq B \geq C > 0$, then for each $t \in [0,\ 1]$ and $p \geq 1$*

$$\text{(U)} \qquad A^{1-t} \geq A^{-r} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s$$

holds for $r \geq t$ and $s \geq 1$.

Related to this, we proposed the following inequality in [5] because (U) seems a skewed form from our view point.

Theorem C. *If $A,\ B,\ C > 0$ satisfy $A \gg B$ and $B \geq C$, then for each $t \in [0,\ 1]$*

$$B \geq C \geq (B^t \natural_s C^p)^{\frac{1}{(p-t)s+t}} \geq A^{-r+t} \sharp_{\frac{1+r-t}{(p-t)s+r}} (B^t \natural_s C^p)$$

holds for all $p \geq 1,\ s \geq 1$ and $r \geq t$.

In this inequality, if $A \geq B = C$, then we have (F) and if $A = B \geq C$, then (GF) is obtained. Very recently, Furuta [9] extennded (U) more precisely(cf.[10],[11]).

Furuta's Theorem. *If $A \geq B \geq C > 0$, then the following (1) and (2) hold for each $t \in [0,\ 1]$ and $p \geq 1,\ r \geq t$ and $s \geq 1$.*

$$\text{(1)} \qquad B^{1-t} \geq B^{-\frac{t}{2}} C B^{-\frac{t}{2}} \geq A^{-t} \sharp_{\frac{1}{(p-t)s+t}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s \geq A^{-r} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s$$

$$\text{(2)} \qquad B^{1-t} \geq B^{-\frac{t}{2}} C B^{-\frac{t}{2}} \geq A^{-r} \sharp_{\frac{1-t+r}{p-t+r}} B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}} \geq A^{-r} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s$$

We state these inequalities by simillar forms to (SGF) replacing s with $\frac{\beta-t}{p-t}$ for $\beta \geq p$ and give proofs.

Theorem 1. *If $A \geq B \geq C > 0$, then the following (1) and (2) hold for each $t \in [0,\ 1]$ and $1 \leq p \leq \beta$ and $t \leq r$.*

$$\text{(1)} \quad B \geq C \geq (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \geq B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

$$\text{(2)} \qquad B \geq C \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \geq B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

To prove this theorem, we prepare the next lemma which is a modification of Theorem B.

Lemma. *If $B \geq C > 0$, then the following holds for $t \in [0,\ 1]$ and $1 \leq p \leq \beta$.*

$$(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{p}{\beta}} \leq C^p$$

Proof. The first step, for $1 \le \frac{\beta-t}{p-t} \le 2$,

$$B^t \natural_{\frac{\beta-t}{p-t}} C^p = C^p \natural_{\frac{p-\beta}{p-t}} B^t = C^p(C^{-p} \sharp_{\frac{\beta-p}{p-t}} B^{-t})C^p \le C^p(C^{-p} \sharp_{\frac{\beta-p}{p-t}} C^{-t})C^p = C^\beta$$

By (LH), we have $(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{p}{\beta}} \le C^p$. The second step, put $C_1 = (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}}$, then $C_1 \le B$ and for $1 \le \frac{\beta_1-t}{\beta-t} \le 2$,

$$B^t \natural_{\frac{\beta_1-t}{p-t}} C^p = B^t \natural_{\frac{\beta_1-t}{\beta-t}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) = B^t \natural_{\frac{\beta_1-t}{\beta-t}} C_1^\beta \le C_1^{\beta_1}.$$

That is, $B^t \natural_{\frac{\beta_1-t}{p-t}} C^p \le (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{\beta_1}{\beta}}$. So by (LH), we have

$$(B^t \natural_{\frac{\beta_1-t}{p-t}} C^p)^{\frac{p}{\beta_1}} \le (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{p}{\beta}} \le C^p.$$

Repeating this, we have the conclusion.

Proof of Theorem 1. Since $(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \le C \le B$ by Theorem B and (LH), (1) is shown as follows:

$$\begin{aligned} & A^{-t} \sharp_{\frac{t}{\beta}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{-\frac{\beta-t}{p-t}} \le B^{-t} \sharp_{\frac{t}{\beta}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{-\frac{\beta-t}{p-t}} \\ = \ & B^{-\frac{t}{2}}\{I \sharp_{\frac{t}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)\}B^{-\frac{t}{2}} = B^{-\frac{t}{2}}(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{t}{\beta}}B^{-\frac{t}{2}} \\ \le \ & B^{-\frac{t}{2}}C^tB^{-\frac{t}{2}} \le B^{-\frac{t}{2}}B^tB^{-\frac{t}{2}} = I, \end{aligned}$$

hence $\{A^{\frac{t}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}A^{\frac{t}{2}}\}^{\frac{t}{\beta}} \le A^t$. Applying (SF)' to these,

$$(A^t)^{-\frac{r-t}{t}} \sharp_{\frac{\frac{1}{t}+\frac{r-t}{t}}{\frac{\beta}{t}+\frac{r-t}{t}}} \{A^{\frac{t}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}A^{\frac{t}{2}}\}^{\frac{t}{\beta}\frac{\beta}{t}} \le \{A^{\frac{t}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}A^{\frac{t}{2}}\}^{\frac{t}{\beta}\frac{1}{t}},$$

that is, $A^{-r+t} \sharp_{\frac{1-t+r}{\beta-t+r}} A^{\frac{t}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}A^{\frac{t}{2}} \le \{A^{\frac{t}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}A^{\frac{t}{2}}\}^{\frac{1}{\beta}}$. Multiplying $A^{-\frac{t}{2}}$ to the both sides of this formula, we have

$$\begin{aligned} & A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \le A^{-t} \sharp_{\frac{1}{\beta}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \\ \le \ & B^{-t} \sharp_{\frac{1}{\beta}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} = B^{-\frac{t}{2}}(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}}B^{-\frac{t}{2}} \le B^{-\frac{t}{2}}CB^{-\frac{t}{2}} \le B^{1-t}. \end{aligned}$$

(1) is obtained by multiplying $B^{\frac{t}{2}}$ to the above each formula.

We can show (2) by similar methods. Since $\{A^{\frac{t}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}A^{\frac{t}{2}}\}^{\frac{t}{\beta}} \le A^t$, we can use (SF)'

$$(A^t)^{-\frac{r-t}{t}} \sharp_{\frac{\frac{p}{t}+\frac{r-t}{t}}{\frac{\beta}{t}+\frac{r-t}{t}}} \{A^{\frac{t}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}A^{\frac{t}{2}}\}^{\frac{t}{\beta}\frac{\beta}{t}} \le \{A^{\frac{t}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}A^{\frac{t}{2}}\}^{\frac{t}{\beta}\frac{p}{t}},$$

that is, $A^{-r+t} \sharp_{\frac{p-t+r}{\beta-t+r}} A^{\frac{t}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}A^{\frac{t}{2}} \le \{A^{\frac{t}{2}}(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}A^{\frac{t}{2}}\}^{\frac{p}{\beta}}$. Multiplying $A^{-\frac{t}{2}}$ from the both sides of this formula and using Lemma, we have

$$\begin{aligned} & A^{-r} \sharp_{\frac{p+r-t}{\beta+r-t}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \le A^{-t} \sharp_{\frac{p}{\beta}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \\ \le \ & B^{-t} \sharp_{\frac{p}{\beta}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} = B^{-\frac{t}{2}}(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{p}{\beta}}B^{-\frac{t}{2}} \le B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}}. \end{aligned}$$

Again multiplying $B^{\frac{t}{2}}$ to each sides of this formula, we have

$$B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{p-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \le C^p.$$

Since $B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) = B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} \{B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{p-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)\}$, we have $B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \le B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p$. As the next step, we show $B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \le C \le B$. Since

$$A^{-t} \sharp_{\frac{1}{p}} B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} \le B^{-t} \sharp_{\frac{1}{p}} B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} = B^{-\frac{t}{2}}(I \sharp_{\frac{1}{p}} C^p)B^{-\frac{t}{2}} = B^{-\frac{t}{2}}CB^{-\frac{t}{2}} \le B^{1-t} \le A^{1-t},$$

that is, $(A^{\frac{t}{2}}B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}}A^{\frac{t}{2}})^{\frac{1}{p}} \le A$. Applying (CF) or (SF)', we have

$$A^{-r+t} \sharp_{\frac{-t+r}{p-t+r}} A^{\frac{t}{2}}B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}}A^{\frac{t}{2}} \le I.$$

This is equivalent to $A^{-r} \sharp_{\frac{-t+r}{p-t+r}} B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} \le A^{-t}$. So the following holds.

$$\begin{aligned}
& A^{-r} \sharp_{\frac{1-t+r}{p-t+r}} B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} = B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} \sharp_{\frac{p-1}{p-t+r}} A^{-r} = B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} \sharp_{\frac{p-1}{p}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} \sharp_{\frac{p}{p-t+r}} A^{-r}) \\
= \; & B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} \sharp_{\frac{p-1}{p}} (A^{-r} \sharp_{\frac{-t+r}{p-t+r}} B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}}) \le B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} \sharp_{\frac{p-1}{p}} A^{-t} = A^{-t} \sharp_{\frac{1}{p}} B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} \\
\le \; & B^{-t} \sharp_{\frac{1}{p}} B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} = B^{-\frac{t}{2}}(I \sharp_{\frac{1}{p}} C^p)B^{-\frac{t}{2}} = B^{-\frac{t}{2}}CB^{-\frac{t}{2}}.
\end{aligned}$$

By multiplying $B^{-\frac{t}{2}}$ to the both sides of these formulas, we obtain $B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \le C \le B$.

We show similar ineqalities as applications of Theorem A.

Theorem 2. *If* $A \gg (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{1}{p-t}}$, *the the following* (1) *and* (2) *hold for* $t \in [0,\ 1]$, $1 \le p \le \beta$ *and* $r \ge t$.

(1) $B^t \sharp_{\frac{1-t}{\beta-t}} C^p \ge B^{\frac{t}{2}}A^{-t}B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \ge B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$

(2) $B^t \sharp_{\frac{1-t}{p-t}} C^p \ge B^{\frac{t}{2}}A^{-t}B^{\frac{t}{2}} \sharp_{\frac{1}{p}} C^p \ge B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \ge B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$

Proof. (1) is given as follows:

$$\begin{aligned}
& A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} = (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \sharp_{\frac{\beta-1}{\beta-t+r}} A^{-r} \\
= \; & (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \sharp_{\frac{\beta-1}{\beta}} \{(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \sharp_{\frac{\beta}{\beta-t+r}} A^{-r}\} \\
= \; & (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \sharp_{\frac{\beta-1}{\beta}} \{A^{-r} \sharp_{\frac{-t+r}{\beta-t+r}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}\} \\
\le \; & (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \sharp_{\frac{\beta-1}{\beta}} A^{-t} = A^{-t} \sharp_{\frac{1}{\beta}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \\
= \; & A^{-t} \sharp_{\frac{1-t+t}{\beta-t+t}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \le (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{1-t}{p-t}}
\end{aligned}$$

The first inequality follows from Theorem A (2) and the second one is also given by (1) of Theorem A. The conclusion is obtained by multiplying $B^{\frac{t}{2}}$ both sides of every formula.
(2) is also obtained by similar caliculations to (1).

$$\begin{aligned}
& A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} = A^{-r} \sharp_{\frac{1-t+r}{p-t+r}} \{A^{-r} \sharp_{\frac{p-t+r}{\beta-t+r}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}}\} \\
\le \; & A^{-r} \sharp_{\frac{1-t+r}{p-t+r}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}}) = (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}}) \sharp_{\frac{p-1}{p-t+r}} A^{-r} \\
= \; & (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}}) \sharp_{\frac{p-1}{p}} \{(B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}}) \sharp_{\frac{p}{p-t+r}} A^{-r}\} = (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}}) \sharp_{\frac{p-1}{p}} \{A^{-r} \sharp_{\frac{-t+r}{p-t+r}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})\} \\
\le \; & (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}}) \sharp_{\frac{p-1}{p}} A^{-t} = A^{-t} \sharp_{\frac{1}{p}} B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} = A^{-t} \sharp_{\frac{1-t+t}{p-t+t}} B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}} \le (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{1-t}{p-t}}
\end{aligned}$$

The first and second inequalities are led by (2) of Theorem A and the final one is led by (1) of Theorem A. Multiplying $B^{\frac{t}{2}}$ to each formula from both sides, we attain the conclusion.

References

[1] T.Ando and F.Hiai, Log majorization and complementary Golden-Thompson type inequality, Linear Alg. and Its Appl.,197(1994),113-131.

[2] M.Fujii, Furuta's inequality and its mean theoretic approach, J.Operator Theory, 23(1990),67-72.

[3] M.Fujii, T.Furuta and E.Kamei, Furuta's inequality and its application to Ando's theorem, Linear Algebra Appl.,149(1991), 91-96.

[4] M.Fujii and E.Kamei, Mean theoretic approach to the grand Furuta inequality, Proc. Amer. Math. Soc.,124(1996),2751-2756.

[5] M.Fujii and E.Kamei, On an extension of grand Furuta inequality, Sci. Math. Japon.,56(2002), 501-504.

[6] T.Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc.,101(1987),85-88.

[7] T.Furuta, Elementary proof of an order preserving inequality, Proc. Japan Acad., 65(1989),126.

[8] T.Furuta, Extension of the Furuta inequality and Ando-Hiai log-majorization, Linear Alg. and Its Appl.,219(1995),139-155.

[9] T.Furuta, $A \geq B > 0$ ensures $A^{1+r-t} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})A^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$ for $t \in [0,1], r \geq t, p \geq 1, s \geq 1$ and related inequalities, preprint.

[10] T.Furuta, A proof of an order preserving inequality, Proc. Japan Acad., 78, Ser.A(2002),26.

[11] T.Furuta, T.Yamazaki and M.Yanagida, Operator functions implying generalized Furuta inequality, Math. Inequal. Appl.,1(1998),123-130.

[12] E.Kamei, A satellite to Furuta's inequality, Math. Japon.,33(1988),883-886.

[13] E.Kamei, Parametrization of the Furuta inequality, Math. Japon., 49(1999),65-71.

[14] E.Kamei, Parametrization of the Furuta inequality, II, Math. Japon., 50(1999), 179-182.

[15] E.Kamei, Parametrized grand Furuta inequality, Math. Japon., 50(199), 79-83.

[16] E.Kamei, Chaotic order and Furuta inequality, Sci. Math. Japon., 53(2001), 221-225.

[17] E.Kamei and M.Nakamura, Remark on chaotic Furuta inequality, Sci. Math. Japon.,53(2001), 535-539.

[18] F.Kubo and T.Ando, Means of positive linear operators, Math. Ann.,246(1980),205-224.

[19] K.Tanahashi, The best possibility of the grand Furuta inequality, Proc. Amer. Math. Soc., 128(2000),511-519.

[20] M.Uchiyama, Criteria for operator means, J. Math. Soc. Japan, to appear.

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OPERATOR CONVEXITY IN FURUTA TYPE OPERATOR INEQUALITIES

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ABSTRACT. In this note, we propose a new characterization of the chaotic order $\log A \geq \log B$ for positive invertible operators A and B on a Hilbert space. It is related to the operator convexity of the function t^α for $1 \leq \alpha \leq 2$. Precisely, $\log A \geq \log B$ if and only if

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \geq B^\delta, \quad \text{or equivalently} \quad A^\delta \geq B^{-r} \natural_{\frac{\delta+r}{p+r}} A^p$$

hold for $0 \leq p \leq \delta \leq 2p$ and $r \geq 0$.

1. INTRODUCTION

The positivity of operators acting on a Hilbert space H is defined by $(Ax, x) \geq 0$ for all $x \in H$. The Furuta inequality [4] characterizes the operator order $A \geq B$ for positive operators $A, B \geq 0$, see [2, 5, 7, 8, 10]. Ando [1] substantially attempted to characterize the chaotic order $\log A \geq \log B$ among positive invertible operators $A, B > 0$ in terms of Furuta's type operator inequality. Afterwards, it was generalized in [3, 6] as follws: For $A, B > 0$, $\log A \geq \log B$ if and only if

$$A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}} \quad (p, r \geq 0). \tag{1}$$

or equivalently,

$$B^r \leq (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \quad (p, r \geq 0),$$

see also [11].

On the other hand, the α-geometric mean $\sharp_\alpha$ among positive operators is introduced by Kubo-Ando:

$$A \sharp_\alpha B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}} \tag{2}$$

for $A, B \geq 0$ and $\alpha \in [0, 1]$. Kamei [8] pointed out that the Furuta inequality can be represented by a chain of operator inequalityies via the α-geometric mean: If $A \geq B > 0$, then

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A \leq B^{-r} \sharp_{\frac{1+r}{p+r}} A^p \tag{3}$$

holds for $p \geq 1$ and $r \geq 0$. Furthermore it suggested that two inequalities on both sides in (3) hold under the assumption $\log A \geq \log B$; actually Kamei [9] proved that if $\log A \geq \log B$ for $A, B > 0$, then the chaotic Furuta inequality

$$A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p \leq B^\delta, \quad A^\delta \leq B^{-r} \sharp_{\frac{\delta+r}{p+r}} A^p \tag{4}$$

hold for $p \geq \delta \geq 0$ and $r \geq 0$.

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Now we recall the fact that a nonnegative continuous function f on $[0,\infty)$ is operator monotone if and only if $tf(t)$ is operator convex. Especially, the Löwner-Heinz inequality says that t^α is operator convex for $\alpha \in [1,2]$. From the viewpoint of this, we want to propose a characterization of the chaotic order which is an operator convex version of (4). Precisely, it is given by the following inequalities parallel to (4). We here denote $\natural_\alpha$ by the same formula as $\sharp_\alpha$ in (2) for $\alpha \in \mathbb{R}$. Then $\log A \geq \log B$ for $A, B > 0$ if and only if

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \geq B^\delta, \quad A^\delta \geq B^{-r} \natural_{\frac{\delta+r}{p+r}} A^p \tag{5}$$

hold for $0 \leq p \leq \delta \leq 2p$ and $r \geq 0$. We note that the condition $0 \leq p \leq \delta \leq 2p$ and $r \geq 0$ ensures $\frac{\delta+r}{p+r} \in [1,2]$ which corresponds to the operator convexity of power functions. Consequently the order relation in (4) and (5) is reversible. Moreover, we discuss the monotonicity of some operator functions raising from (5). It is equivalent to the monotonicity of the operator function $F(r) = A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p$ under the assumption $\log A \geq \log B$ for $A, B > 0$.

2. A CHARACTERIZATION OF CHAOTIC ORDER

The purpose of this section is to give a characterization of the chaotic order related to the operator convexity.

Theorem 1. *The following statements are mutually equaivalent for $A, B > 0$:*
(1) $\log A \geq \log B$.
(2) $A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p \leq B^\delta$ *holds for* $p \geq \delta \geq 0$ *and* $r \geq 0$.
(2′) $A^\delta \leq B^{-r} \sharp_{\frac{\delta+r}{p+r}} A^p$ *holds for* $p \geq \delta \geq 0$ *and* $r \geq 0$.
(3) $A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \geq B^\delta$ *holds for* $0 \leq p \leq \delta \leq 2p$ *and* $r \geq 0$.
(3′) $A^\delta \geq B^{-r} \natural_{\frac{\delta+r}{p+r}} A^p$ *holds for* $0 \leq p \leq \delta \leq 2p$ *and* $r \geq 0$.

For convenience, we cite some formulas on the binary operation $\natural_s$ $(s \in \mathbb{R})$, wich will be used below.

Lemma. The following equations holds for $X, Y > 0$ and $s, t \in \mathbb{R}$:
(1) $X \natural_s Y = Y \natural_{1-s} X$.
(2) $X \natural_{st} Y = X \natural_s (X \natural_t Y)$.
(3) $X \natural_s Y = X(X^{-1} \natural_{-s} Y^{-1})X$.
(4) $X^{-1} \natural_s Y^{-1} = (X \natural_s Y)^{-1}$.

Proof of Theorem 1. Suppose that (1) holds, i.e., $\log A \geq \log B$, and p, $r \geq 0$. Then A and B satisfiy (1) in Introduction, in other words,

$$B^p \sharp_{\frac{p}{p+r}} A^{-r} = A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I.$$

Hence it follows that for $0 \leq \delta \leq p$

$$\begin{aligned} A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p &= B^p \sharp_{\frac{p-\delta}{p+r}} A^{-r} = B^p \sharp_{\frac{p-\delta}{p}} (B^p \sharp_{\frac{p}{p+r}} A^{-r}) \\ &\leq B^p \sharp_{\frac{p-\delta}{p}} I = I \sharp_{\frac{\delta}{p}} B^p = B^\delta, \end{aligned}$$

that is, (2) is obtained.

Next we assume (2). For $0 \le p \le \delta \le 2p$ and $r \ge 0$, we put $\eta = 2p - \delta$. Since $0 \le \eta \le p$, we have

$$A^{-r} \sharp_{\frac{\eta+r}{p+r}} B^p \le B^\eta \quad \text{and so} \quad A^r \sharp_{\frac{\eta+r}{p+r}} B^{-p} \ge B^{-\eta}.$$

Therefore it implies that

$$\begin{aligned} A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p &= B^p \natural_{\frac{p-\delta}{p+r}} A^{-r} = B^p(B^{-p} \natural_{\frac{\delta-p}{p+r}} A^r)B^p \\ &= B^p(A^r \natural_{\frac{2p-\delta+r}{p+r}} B^{-p})B^p \ge B^pB^{-\eta}B^p = B^\delta, \end{aligned}$$

so that (3) is proved.

Finally we show that (3) implies (1). So we put $\delta = 2p$ in (3). Thus we have

$$B^{2p} \le A^{-r} \natural_{\frac{2p+r}{p+r}} B^p = B^p \natural_{\frac{-p}{p+r}} A^{-r} = B^p(B^{-p} \sharp_{\frac{p}{p+r}} A^r)B^p.$$

Therefore we have

$$1 \le B^{-p} \sharp_{\frac{p}{p+r}} A^r$$

and so

$$1 \ge B^p \sharp_{\frac{p}{p+r}} A^{-r} = A^{-r} \sharp_{\frac{r}{p+r}} B^p,$$

which is equivalent to $\log A \ge \log B$.

In addition, the equivalence among (1), (2') and (3') is ensured by that among (1), (2) and (3) because (1) is understood as $\log B^{-1} \ge \log A^{-1}$.

3. The monotonicity of operator functions

In this section, we discuss the monotonicity of the operator function

$$F_\alpha(r) = B^{-r} \natural_{\frac{\alpha+r}{p+r}} A^p \quad (r \ge 0).$$

Theorem 1 says that if $\log A \ge \log B$ and $0 \le p \le \alpha \le 2p$, then $F(0) \ge F(r)$ for $r \ge 0$. Combining with the monotonicity of the function

$$F_\delta(r) = B^{-r} \sharp_{\frac{\delta+r}{p+r}} A^p \quad (r \ge 0)$$

for given $0 \le \delta \le p$, we can conjecture the monotonicity of $F(r)$. As a matter of fact, we have the following theorem:

Theorem 2. *Let $F_\alpha(r)$ and $F_\delta(r)$ be as in above. Then the following statements are equivalent;*
(1) *$F_\alpha(r)$ is a decreasing function if $0 \le p \le \alpha \le 2p$.*
(2) *$F_\delta(r)$ is an increasing function if $0 \le \delta \le p$.*
Moreover, if $\log A \ge \log B$, then both (1) and (2) hold.

Proof. To prove the first half, we put $\delta = 2p - \alpha$. It is clear that $0 \le p \le \alpha \le 2p$ if and only if $0 \le \delta \le p$, and we have

$$\begin{aligned} F_\alpha(r) &= B^{-r} \natural_{\frac{\alpha+r}{p+r}} A^p = A^p \natural_{\frac{p-\alpha}{p+r}} B^{-r} = A^p(A^{-p} \sharp_{\frac{\alpha-p}{p+r}} B^r)A^p \\ &= A^p(A^p \sharp_{\frac{\alpha-p}{p+r}} B^{-r})^{-1}A^p = A^p(B^{-r} \sharp_{\frac{2p-\alpha+r}{p+r}} A^p)^{-1}A^p \\ &= A^pF_\delta(r)^{-1}A^p. \end{aligned}$$

To show the second half, we take s for a fixed $r \geq 0$ such that $0 \leq r \leq s \leq 2r$. Then it follows from (3) in Theorem 1 that $A^{-p} \natural_{\frac{s+p}{r+p}} B^{r} \geq B^{s}$, and so $A^{p} \natural_{\frac{s+p}{r+p}} B^{-r} \leq B^{-s}$. Therefore we have

$$F_{\alpha}(r) = B^{-r} \natural_{\frac{\alpha+r}{p+r}} A^{p} = A^{p} \natural_{\frac{p-\alpha}{p+r}} B^{-r} = A^{p} \natural_{\frac{p-\alpha}{p+s}} (A^{p} \natural_{\frac{p+s}{p+r}} B^{-r}).$$

Noting that $-1 \leq \frac{p-\alpha}{p+s} \leq 0$, we remark the reverse order preserving property on $\natural_{\beta}$; if $Y \geq Z > 0$ and $X > 0$, then $X \natural_{-\beta} Y \leq X \natural_{-\beta} Z$ for $0 \leq \beta \leq 1$. Actually (3) in Lemma implies that

$$X \natural_{-\beta} Y = X(X^{-1} \natural_{-\beta} Y^{-1})X \leq= X(X^{-1} \natural_{-\beta} Z^{-1})XX \natural_{-\beta} Z.$$

Hence we have

$$F_{\alpha}(r) == A^{p} \natural_{\frac{p-\alpha}{p+s}} (A^{p} \natural_{\frac{p+s}{p+r}} B^{-r}). \geq A^{p} \natural_{\frac{p-\alpha}{p+s}} B^{-s} = F_{\alpha}(s).$$

Incidentally we give a similar proof of (2) to the proof of (1) in above. By (2) in Theorem 1, we have $A^{-p} \sharp_{\frac{r+p}{s+p}} B^{s} \leq B^{r}$, or $A^{p} \sharp_{\frac{r+p}{s+p}} B^{-s} \geq B^{-r}$ for $0 \leq r < s$. It implies that

$$F_{\delta}(s) = A^{p} \sharp_{\frac{p-\delta}{p+s}} B^{-s} = A^{p} \sharp_{\frac{p-\delta}{p+r}} (A^{p} \sharp_{\frac{p+r}{p+s}} B^{-s}) \geq A^{p} \sharp_{\frac{p-\delta}{p+r}} B^{-r} = B^{-r} \sharp_{\frac{\delta+r}{p+r}} A^{p} = F_{\delta}(r).$$

Finally we discuss the monotonicity of the operator function

$$G_{\eta}(p) = B^{-r} \natural_{\frac{\eta+r}{p+r}} A^{p}.$$

for fixed $r \geq 0$ and $\eta \geq 0$.

Theorem 3. *Let $G_{\eta}(p)$ be as in above for a fixed $r \geq 0$, and suppose* $\log A \geq \log B$ *for A, $B > 0$. Then for each $\delta \geq 0$, $G_{\delta}(p)$ is a decreasing function of $p \geq \delta$, and for each $\alpha > 0$, $G_{\alpha}(p)$ is an increasing function of $p \in [\frac{\alpha}{2}, \alpha]$.*

Proof. For a given $\delta \geq 0$ and $c > 0$, we have

$$G_{\delta}(p+c) = B^{-r} \sharp_{\frac{\delta+r}{p+c+r}} A^{p+c} = B^{-r} \sharp_{\frac{\delta+r}{p+r}} (B^{-r} \sharp_{\frac{p+r}{p+c+r}} A^{p+c}) \geq B^{-r} \sharp_{\frac{\delta+r}{p+r}} A^{p} = G_{\delta}(p)$$

by (2') in Theorem 1.

Next it follows from (3') in Theorem 1 that for $0 < c \leq \alpha - p$

$$G_{\alpha}(p) = B^{-r} \natural_{\frac{\alpha+r}{p+r}} A^{p} = B^{-r} \sharp_{\frac{\alpha+r}{p+c+r}} (B^{-r} \natural_{\frac{p+c+r}{p+r}} A^{p}) \leq= B^{-r} \sharp_{\frac{\alpha+r}{p+c+r}} A^{p+c} = G_{\alpha}(p+c).$$

Incidentally we note that the cases $\sharp$ in Theorems 2 and 3 have been discussed, e.g. §3.3.2 in [7], whose proofs are different from ours.

References

[1] T.Ando, *On some operator inequalities*, Math. Ann., 279(1987), 157-159.

[2] M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator theory, 23(1990), 67-72.

[3] M.Fujii, T.Furuta and E.Kamei, *Furuta's inequality and application to Ando's theorem*, Linear Alg. and Appl., 179(1993), 161-169.

[4] T.Furuta, *$A \geq B \geq 0$ assures $(B^{r}A^{p}B^{r})^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc., 101(1987), 85-88.

[5] T.FURUTA *An elementary proof of an order preserving inequality*, Proc. Japan Acad., 65(1989), 126.
[6] T.FURUTA *Applications of order preserving operator inequalities*, Operator Theory: Adv. Appl., Birk-häuser Verlag, Basel, 59(1992), 180-198.
[7] T.FURUTA *Invitation to Linear Operators*, Taylor and Francis, London and New York, 2001.
[8] E.KAMEI, *A satellite to Furuta's inequality*, Math. Japon., 33(1988), 883-886.
[9] E.KAMEI, *Chaotic order and Furuta inequality*, Sci. Math. Japon., 53(2001), 221-225.
[10] K.TANAHASHI, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., 124(1996), 141-146.
[11] M.UCHIYAMA, *Some exponential operator inequalities*, Math. Inequal. Appl., 2(1999), 469-471.

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関数空間における等長写像の縮小写像による線形近似

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1. イントロダクション

先ず古典的なKorovkinの定理を述べよう.

定理1.1 (Korovkin)

Iを有限閉区間とし, $\{\phi_n\}$を, I上の複素数値 (resp., 実数値) 連続関数の全体$C(I)$ (resp., $C_R(I)$) から$C(I)$ (resp.,$C_R(I)$) の中へのpcsitive linear mapの列とする. このとき, $\{\phi_n(\iota^j)\}$ (j=0, 1, 2) がι^jに一様に収束すれば, 任意の$f \in C(I)$に対して, $\{\phi_n(f)\}$はfに一様に収束することが導かれる. ただし,

$$\iota^0(x) = 1, \quad \iota^1(x) = x, \quad \iota^2(x) = x^2 \quad (x \in I)$$

である.

この定理について, 関数空間上のunital isometoryをunital contractionの列で近似するという設定で考察したい. 近似の方法は, 点毎の収束による近似と, 一様収束による近似である.

2. 主定理

Mを$C(X)$のsubspaceとする. 任意の$f \in M$に対して, 点$x \in X$におけるfの関数値を対応させるfunctionalを$\tau_M(x)$とする. このとき, $\tau_M(x)$が, Mの共役区間の閉単位球の端点に属するような$x \in X$の点の全体Π_Mを, MのChoquet boundaryと呼ぶ[3].

ここで我々の定理を述べる.

定理 2.1

XとYをcompact Hausdorff space, Mを$C(X)$ (resp., $C_R(X)$) のsubspace, SをMに含まれるX上のfunction spaceとする. また, ϕ_nをMから$C(Y)$ (resp., $C_R(Y)$) の中へのunital linear contractionとし($n = 1, 2, \ldots$), ϕ_∞をMから$C(Y)$ (resp., $C_R(Y)$) の中へのlinear isometryとする. さらに, $\Pi_N \subset \Pi_T$とする. このとき, 任意の$f \in S$に対して, $\{\phi_n(f)\}$が

$\phi_\infty(f)$ に点毎に収束すれば, 任意の $f \in M$ に対して, $\{\phi_n(f)\}$ が $\phi_\infty(f)$ に点毎に収束することが導かれる. ただし,

$$N = \mathrm{Span}(\bigcup_{n=1}^{\infty} N_n), \quad \phi_n(M) = N_n \quad (n = 1,\ 2, ..., \infty), \quad T = \phi_\infty(S)$$

である.

定理 2.2

X と Y を compact Hausdorff space, M を $C(X)$ (resp., $C_R(X)$) の subspace, S を M に含まれる X 上の function space とする. また, ϕ_n を M から $C(Y)$ (resp., $C_R(Y)$) の中への unital linear contraction とし $(n = 1,\ 2, ...)$, ϕ_∞ を M から $C(Y)$ (resp., $C_R(Y)$) の中への linear isometry とする. さらに, $\Pi_N \subset \Pi_T$ とする. このとき, 任意の $f \in S$ に対して, $\{\phi_n(f)\}$ が $\phi_\infty(f)$ に一様に収束すれば, 任意の $f \in M$ に対して, Π_N の任意の compact subset 上で $\{\phi_n(f)\}$ が $\phi_\infty(f)$ に一様に収束することが導かれる.

特に, Π_N が compact ならば, 任意の $f \in M$ に対して, $\{\phi_n(f)\}$ が $\phi_\infty(f)$ に一様に収束することが導かれる.

ここに, $N, M_n \ (n = 1,\ 2, ..., \infty), T$ は定理2.1で述べたものである.

定理2.1, 定理2.2の証明の本質は, function space, function algebra の Choquet boundary に属する点の特徴付け ([1], [4], など) と, isometry の構造定理に注目することにある.

補題 2.3

(1) M を X 上の function space とし, $x' \in X$ とする. このとき, $x' \in \Pi_M$ であるための必要十分条件は, $\alpha,\ \beta\ (\beta > \alpha > 0)$ と, x' の任意の開近傍 U に対して, $f \in M$ が存在して, 任意の $x \in X$ に対して $f(x)$ の実数部分が正でなく, 任意の $x \in U^c$ に対して $f(x)$ の実数部分が $-\beta$ より小さく, $f(x')$ の実数部分が $-\alpha$ より大きいことである.

(2) M を X 上の function algebra とし, $x' \in X$ とする. このとき, $x' \in \Pi_M$ であるための必要十分条件は, $\alpha,\ \beta\ (\beta > \alpha > 0)$ と, x' の任意の開近傍 U に対して, $f \in M$ が存在して, 任意の $x \in X$ に対して $|f(x)| \leq 1$, $x \in U^c$ に対して $|f(x)| < \alpha$, $|f(x')| > \beta$ であることである.

定理 2.4 (isometry の構造定理)

M を X 上の function space とし, ϕ を M から $C(Y)$ (resp., $C_R(Y)$) の中への linear isometry とする. $N = \phi(M)$ とすれば, N の Choquet boundary の閉包 Γ_N から M の Šilov boundary Σ_M の上への continuous map η が一意に存在して, 任意の $f \in M$ と $y \in \Gamma_N$ に対して,

$$\phi(f)(y) = \phi(1)(y) f(\eta(y))$$

が成り立つ. さらに, $y \in \Gamma_N$ に対して,

$$|\phi(1)(y)| = 1 \quad (\text{resp.,} \quad \phi(1)(y) = \pm 1)$$

であり, $\eta(\Pi_N)$ は Π_M と一致する.

さらに、定理2.1, 定理2.2と本質的に同じ方法によって, Sがfunction algebraのときに次の定理を導くことができる:

定理 2.5

XとYをcompact Hausdorff space, Mを$C(X)$のsubspace, SをMに含まれるX上のfunction algebraとする. ϕ_nをMから$C(Y)$の中へのunital linear contractionとし$(n = 1, 2, \ldots)$, ϕ_∞をMから$C(Y)$の中へのlinear isometryとする. さらに, $\Pi_N \subset \Pi_T$とする. このとき、Sにおけるoprator norm $\|\Phi_n - \Phi_\infty\|_S$ が0に収束するならば, Mにおけるoprator norm $\|\Phi_n - \Phi_\infty\|_M$ も0に収束することが導かれる.

3. 応用

全空間がChoquet boundaryと一致するためのmildな条件に注意して, 定理2.1, 定理2.2から次の系が導かれる.

系 3.1

XとYをcompact Hausdorff space, SをX上のfunction spaceとする. ϕ_nを$C(X)$ (resp., $C_R(X)$)から$C(Y)$ (resp., $C_R(Y)$) の中へのunital linear contraction $(n = 1, 2, \ldots)$, ϕ_∞を$C(X)$ (resp., $C_R(X)$)から$C(Y)$ (resp., $C_R(Y)$) の中へのlinear isometryとする. 任意の$y' \in Y$に対して, $g(y')$の実数部分が0に等しく, y'と異なる任意の$y \in Y$に対して$g(y)$の実数部分が正であるような, gが$T = \phi_\infty(S)$に存在するとする. このとき, 任意の$f \in S$に対して、$\{\phi_n(f)\}$が$\phi_\infty(f)$に点毎に収束するならば, 任意の$f \in C(X)$ (resp., $C_R(X)$)対して, $\{\phi_n(f)\}$が$\phi_\infty(f)$に点毎に収束することが導かれる. また, 任意の$f \in S$に対して, $\{\phi_n(f)\}$が$\phi_\infty(f)$に一様に収束するならば, 任意の$f \in C(X)$ (resp., $C_R(X)$)対して, $\{\phi_n(f)\}$が$\phi_\infty(f)$に一様に収束することが導かれる.

この系からVolkov, Morozovの定理のアナロジーとして次の系が導かれる:

系 3.2 (Cf. Volkov[5])

Ωをp次元実ユークリッド空間の空でないcompact subsetとし, ϕ_nを$C(\Omega)$ (resp., $C_R(\Omega)$)から同じものの中へのunital linear contraction $(n = 1, 2, \ldots)$, ϕ_∞を$C(\Omega)$ (resp., $C_R(\Omega)$)から同じものの中へのlinear isometryとする. このとき, $\{\phi_n(\iota^0)\}$がι^0に, $\{\phi_n(\iota_k)\}$が$\phi_\infty(\iota_k) = \iota_k$に, $\left\{\phi_n\left(\sum_{k=1}^{p} \iota_k^2\right)\right\}$が$\phi_\infty\left(\sum_{k=1}^{p} \iota_k^2\right) = \sum_{k=1}^{p} \iota_k^2$ にそれぞれ点毎に収束するならば, 任意の$f \in C(\Omega)$ (resp., $C_R(\Omega)$)対して, $\{\phi_n(f)\}$は$\phi_\infty(f)$に点毎に収束する.

系 3.3 (Cf. Morozov[2])

$C(R^p)_{2\pi}$ (resp., $C_R(R^p)_{2\pi}$)をp次元実ユークリッド空間上の周期2πの連続な複素数値(resp., 実数値) 周期関数の全体とする. ϕ_nを$C(R^p)_{2\pi}$ (resp., $C_R(R^p)_{2\pi}$)から同じものの中へのunital linear contractionとし$(n = 1, 2, \ldots)$, ϕ_∞を$C(R^p)_{2\pi}$ (resp., $C_R(R^p)_{2\pi}$)から同じものの中へのlinear isometryとする. このとき, $\{\phi_n(\iota^0)\}$がι^0に, $\{\phi_n(\cos_k)\}$が$\phi_\infty(\cos_k) = \cos_k$

に, $\{\phi_n(\sin_k)\}$ が$\phi_\infty(\sin_k) = \sin_k$にそれぞれ点毎に収束するならば, 任意の$f \in C(R^p)_{2\pi}$ (resp., $C_R(R^p)_{2\pi}$に対して, $\{\phi_n(f)\}$ は$\phi_\infty(f)$ に点毎に収束する.

系3.2, 系3.3は, 点毎の収束を一様収束に換えても成り立つこともちろんである.

複素平面上の単位閉円板Dの内部で解析的なD上の連続関数を, 単位円周T上に制限して得られる関数全体が作るfunction spaceを$A(T)$とする. このとき 定理2.4から次の系が導かれる:

系 3.4
ϕ_nを$C(T)$から同じものの中へのunital linear contractionとし($n = 1,\ 2, ...$), ϕ_∞を$C(T)$から同じものの中へのlinear isometryとする. このとき, $A(T)$におけるoperator norm $\|\phi_n - \phi_\infty\|_{A(T)}$ が0に収束するならば(このときϕ_∞はalgebra isomorophismであり), $C(T)$におけるoperator norm $\|\phi_n - \phi_\infty\|_{C(T)}$ は0に収束する.

参考文献

[1]A.Browder, Intoroduction to function algebras, W.A.Benjamin, Inc., New York - Amsterdam, 1969.

[2]E.N.Morozov, Convergence of a sequence of positive linear operators in the space of continuous 2π-periodic functions of two variables(in Russian), Kalinin. Gos. ped. Inst. Ucen. Zap. 26(1958),129-142.

[3]W.P.Novinger, Linear isometries of subspaces of spaces of continuous functions, Studia Math. 53(1975), 273-276.

[4]O.Takenouchi, A.Sakai, K.Kishi and T.Jimbo, Function algebras(in Japanese), Math. Sci. Ser. 8, Tokyo, 1977.

[5]V.I.Volkov, Conditions for convergence of a sequence of positive linear operators in the space of continuous functions of two variables(in Russian), Kalinin. Gos. Ped. Inst. Ucen. Zap. 26(1958), 27-40.

On the Hyers-Ulam stability constant of a first order linear differential operator

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Abstract. Given a complex Banach space X, we gave a necessary and sufficient condition in order that a first order linear differential operator T_h from $C^1(\boldsymbol{R}, X)$ to $C(\boldsymbol{R}, X)$ has the Hyers-Ulam stability in [1]. In this talk, we give a complete description of the Hyers-Ulam stability constant of T_h in terms of an integral of h. The resulting constant is Banach space free.

Main theorem

Let X be a complex Banach space, $h : \boldsymbol{R} \to \boldsymbol{C}$ a continuous function and $T_h : C^1(\boldsymbol{R}, X) \to C(\boldsymbol{R}, X)$ a first order linear differential operator defined by

$$(T_h u)(t) = u'(t) + h(t)u(t) \quad (\forall u \in C^1(\boldsymbol{R}, X),\ \forall t \in \boldsymbol{R}).$$

We say that T_h has the Hyers-Ulam stability if

$$\exists K \geq 0 : \text{For every } \varepsilon \geq 0,\ v \in C(\boldsymbol{R}, X) \text{ and } u \in C^1(\boldsymbol{R}, X) \text{ satisfying } \left\| T_h u - v \right\|_\infty \leq \varepsilon,$$
$$\exists\, u_0 \in C^1(\boldsymbol{R}, X) \text{ such that } T_h u_0 = v \text{ and } \left\| u - u_0 \right\|_\infty \leq K\varepsilon.$$

We call such K a HUS constant for T_h. If, in addition, minimum of all such K's exists, then we call it the HUS constant for T_h. Define

$$\tilde{h}(t) = \exp\int_0^t h(s)ds \quad (t \in \boldsymbol{R}),\ C_h = \sup_{t \in \boldsymbol{R}} \frac{1}{|\tilde{h}(t)|}\int_t^\infty |\tilde{h}(s)|ds,$$

$$D_h = \sup_{t \in \boldsymbol{R}} \frac{1}{|\tilde{h}(t)|}\int_{-\infty}^t |\tilde{h}(s)|ds,\ E_h = \sup_{t \in \boldsymbol{R}} \left| \frac{1}{\tilde{h}(t)}\int_0^t |\tilde{h}(s)|ds \right|.$$

Then we have the following

Theorem. (1) *T_h has the Hyers-Ulam stability if and only if one of C_h, D_h and E_h is finite.*

(2) (i) *If C_h is finite, then C_h is the HUS constant for T_h.*

(ii) *If D_h is finite, then D_h is the HUS constant for T_h.*

(iii) *If E_h is finite, then E_h is the HUS constant for T_h.*

Remark 1. Note that only one of C_h, D_h and E_h could be finite. Of course, all of C_h, D_h and E_h could be infinite, in such case, T_h doesn't have the Hyers-Ulam stability.

Outline of Proof. (1), (2)-(i) and (2)-(ii) are already obtained results in [1]. (2)-(iii) can be shown by the following two lemmas:

Lemma 1. Let T be a bounded linear operator of a Banach space X into another Banach space Y and X_1 the closed unit ball of X. Let $\varphi_T : Y \to \mathbb{R}^+$ be a function defined by

$$\varphi_T(y) = \sup\{\|z - y\| : z \in T(X_1)\}$$

for every $y \in Y$. Then $\varphi_T(y)$ takes the minimum at $y = 0$.

Lemma 2. Let K_{T_h} be the infimum of all HUS constans for it if T_h has the Hyers-Ulam stability and $K_{T_h} = \infty$ otherwise. Then

$$K_{T_h} = \inf_{x \in X} \sup_{w \in C(\mathbb{R}, X) : \|w\|_\infty \le 1} \sup_{t \in \mathbb{R}} \left\| \frac{1}{\tilde{h}(t)} \left(x + \int_0^t \tilde{h}(s) w(s) ds \right) \right\|$$

holds.

Remark 2. 1 階線形微分作用素の Hyers-Ulam Stability に関する詳細は文献 [1] を参照されたい。

Proofs

Proof of Lemma 1.

Let f be an arbitrary unit element of the dual space Y^* of Y. Set $\rho = \sup_{x \in X_1} |f(Tx)|$. We can without loss of generality that $\rho > 0$. Let $C_\rho = \{z \in \mathbb{C} : |z| < \rho\}$. Then $C_\rho \subseteq f(T(X_1)) \subseteq \overline{C_\rho}$. In fact, let $\lambda \in C_\rho$. Since $|\lambda| < \rho$, we can take an element $x_0 \in X_1$ such that $|\lambda| < |f(Tx_0)| < \rho$. Moreover, we can take suitable real numbers r and θ such that $0 \le r < 1$ and $\lambda = re^{i\theta} f(Tx_0)$. It follows that $\lambda \in f(T(X_1))$ and then $C_\rho \subseteq f(T(X_1))$. Also, $f(T(X_1)) \subseteq \overline{C_\rho}$ is trivial. This observation implies that $\overline{f(T(X_1))} = \{z \in \mathbb{C} : |z| \le \rho\}$. Now let $y \in Y$. Then

$$\varphi_T(y) \ge \sup\{|f(Tx) - f(y)| : x \in X_1\} \ge \rho = \sup_{x \in X_1} |f(Tx)| = \|T^*(f)\|.$$

Since f is arbitrary, it follows that $\varphi_T(y) \ge \|T^*\| = \|T\| = \varphi_T(0)$ as desired.

Proof of Lemma 2.

Let K_0 be the right hand of the equality in Lemma 2 and set

$$K_0(x) = \sup_{w \in C(\mathbb{R}, X) : \|w\|_\infty \le 1} \sup_{t \in \mathbb{R}} \left\| \frac{1}{\tilde{h}(t)} \left(x + \int_0^t \tilde{h}(s) w(s) ds \right) \right\|$$

for each $x \in X$. Suppose that both K_{T_h} and K_0 are finite. Assume that $K_{T_h} < K_0$. Then we can find $\rho \in \mathbb{R}$ such that $K_{T_h} < \rho < K_0$ and then ρ is a HUS constant for T_h. Let $w \in C(\mathbb{R}, X)$ be such that $\|w\|_\infty \le 1$. Since ρ is a HUS constant for T_h, it follows that there exists $u_0 \in C^1(\mathbb{R}, X)$ such that $T_h u_0 = w$ and $\|u_0\|_\infty \le \rho$. Therefore, we can find $x_0 \in X$ such that

$$u_0(t) = \frac{1}{\tilde{h}(t)} \left(x_0 + \int_0^t \tilde{h}(s) w(s) ds \right) \quad (t \in \mathbb{R})$$

and hence

$$\rho \geq \| u_0 \|_\infty = \sup_{t \in \boldsymbol{R}} \left\| \frac{1}{\tilde{h}(t)} \left(x_0 + \int_0^t \tilde{h}(s) w(s) ds \right) \right\|.$$

Since w is arbitrary, we have $\rho \geq K_0(x_0)$ and hence $\rho \geq K_0$, which contradicts $\rho < K_0$.

Next assume that $K_T > K_0$. Then we can find $x_1 \in X$ with $K_T > K_0(x_1) > K_0$. Since $K_T > K_0(x_1)$, there must be $u_1 \in C^1(\boldsymbol{R}, X)$ and $v_1 \in C(\boldsymbol{R}, X)$ such that $\| T_h u_1 - v_1 \|_\infty \leq 1$ and $\| u_1 - u \|_\infty > K_0(x_1)$ whenever $u \in C^1(\boldsymbol{R}, X)$ with $T_h u = v_1$. Set $w_1 = T_h u_1 - v_1$ and then $\| w_1 \|_\infty \leq 1$. Since $T_h u_1 = w_1 + v_1$, we can find $x_2 \in X$ such that

$$u_1(t) = \frac{1}{\tilde{h}(t)} \left(x_2 + \int_0^t \tilde{h}(s) v_1(s) ds + \int_0^t \tilde{h}(s) w_1(s) ds \right) \quad (t \in \boldsymbol{R}).$$

Set

$$u_2(t) = \frac{1}{\tilde{h}(t)} \left(x_2 - x_1 + \int_0^t \tilde{h}(s) v_1(s) ds \right) \quad (t \in \boldsymbol{R}).$$

Then $u_2 \in C^1(\boldsymbol{R}, X)$ and $T_h u_2 = v_1$, so that $\| u_1 - u_2 \|_\infty > K_0(x_1)$. However we have

$$u_1(t) - u_2(t) = \frac{1}{\tilde{h}(t)} \left(x_1 + \int_0^t \tilde{h}(s) w_1(s) ds \right) \quad (t \in \boldsymbol{R})$$

and hence

$$\| u_1 - u_2 \|_\infty = \sup_{t \in \boldsymbol{R}} \left\| \frac{1}{\tilde{h}(t)} \left(x_1 + \int_0^t \tilde{h}(s) w_1(s) ds \right) \right\| \leq K_0(x_1)$$

because $w_1 \in C(\boldsymbol{R}, X)$ and $\| w_1 \|_\infty \leq 1$. This is a contradiction. These observations implies that $K_{T_h} = K_0$ whenever both K_{T_h} and K_0 are finite. We next show that $K_{T_h} = \infty$ if and only if $K_0 = \infty$. Suppose that K_0 is finite. Then there exists an element $x_1 \in X$ with $K_0(x_1) < \infty$. If there exist $u_1 \in C^1(\boldsymbol{R}, X)$ and $v_1 \in C(\boldsymbol{R}, X)$ such that $\| T_h u_1 - v_1 \|_\infty \leq 1$ and $\| u_1 - u \|_\infty > K_0(x_1)$ whenever $u \in C^1(\boldsymbol{R}, X)$ with $T_h u = v_1$, then we arrive a contradiction by the latter half of the above proof. Therefore $K_0(x_1)$ is a HUS constant for T_h and so K_{T_h} is also finite. Next suppose that K_{T_h} is finite. Choose a positive number ρ such that $K_{T_h} < \rho$. Then ρ is a HUS constant for T_h and hence there exists an element $x_0 \in X$ with $K_0(x_0) \leq \rho$ by the first half of the above proof. Therefore we have that K_0 is finite. Q. E. D.

Given a nonzero continuous function $\varphi : \boldsymbol{R} \to \boldsymbol{C}$, let us consider a mapping $S_\varphi : C(\boldsymbol{R}, X) \to C(\boldsymbol{R}, X)$ and its norm (admitting ∞) defined by

$$(S_\varphi f)(t) = \frac{1}{\varphi(t)} \int_0^t \varphi(s) f(s) ds \quad (\forall f \in C(\boldsymbol{R}, X), \forall t \in \boldsymbol{R})$$

and

$$\| S_\varphi \| = \sup_{f \in C(\boldsymbol{R}, X) : \| f \|_\infty \leq 1} \| S_\varphi(f) \|_\infty .$$

Then S_φ is a linear operator of $C(\boldsymbol{R}, X)$ into itself. We show the part (2)- (iii) of Theorem applying the above conception.

Proof of Theorem (2)-(iii).

We first show that $\| S_{\tilde{h}} \| = E_h$. In fact, $\| S_{\tilde{h}} \| \leq E_h$ is trivial. To show the converse, take a unit vector x_0 in X and set $f_0(t) = \dfrac{|\tilde{h}(t)|}{\tilde{h}(t)} x_0$ $(t \in \boldsymbol{R})$. Then $\| f_0 \|_\infty = 1$ and $\| S_{\tilde{h}}(f_0) \|_\infty = E_h$.

This implies that $\| S_{\tilde{h}} \| \geq E_h$. Now by Lemma 2, we have

$$K_{T_h} = \inf_{x \in X} \sup_{w \in C(\boldsymbol{R}, X) : \|w\|_\infty \leq 1} \left\| S_{\tilde{h}}(w) - \frac{1}{\tilde{h}} \cdot x \right\|_\infty .$$

Suppose that E_h is finite. Set

$$p(x, w; h) = \sup_{t \in \boldsymbol{R}} \left\| \frac{1}{\tilde{h}(t)} \left(x + \int_0^t \tilde{h}(s) w(s) ds \right) \right\|$$

for each $x \in X$ and $w \in C(\boldsymbol{R}, X)$. Then by Lemma 2, we have $K_{T_h} \leq E_h$. In fact, take a unit vector x_0 in X and set $w_0(t) = \dfrac{|\tilde{h}(t)|}{\tilde{h}(t)} x_0$ $(t \in \boldsymbol{R})$. Then $K_{T_h} \leq p(0, w_0; h) = E_h$ as required. Therefore we see that K_{T_h} is finite. We next see that $\| 1 / \tilde{h} \|_\infty < \infty$. In fact, if $\| 1 / \tilde{h} \|_\infty = \infty$, then $\inf_{t \in \boldsymbol{R}} |\tilde{h}(t)| = 0$ and so we arrive at a contradiction because

$$\left| \int_0^t |\tilde{h}(s)| ds \right| \leq E_h |\tilde{h}(t)| \quad (t \in \boldsymbol{R})$$

holds. Now note that $S_{\tilde{h}}$ is a bounded linear operator of the Banach space $C^b(\boldsymbol{R}, X)$ into itself, where $C^b(\boldsymbol{R}, X) = \{ f \in C(\boldsymbol{R}, X) : \| f \|_\infty < \infty \}$. Then by Lemmas 1 and 2, we have

$$K_{T_h} = \inf_{x \in X} \sup_{w \in C^b(\boldsymbol{R}, X) : \|w\|_\infty \leq 1} \left\| S_{\tilde{h}}(w) - \frac{1}{\tilde{h}} \cdot x \right\|_\infty \geq \sup_{w \in C^b(\boldsymbol{R}, X) : \|w\|_\infty \leq 1} \| S_{\tilde{h}}(w) \|_\infty = \| S_{\tilde{h}} \| = E_h$$

and hence $K_{T_h} \geq E_h$, so that $K_{T_h} = E_h$. Consequently, E_h is the HUS constant for T_h.

Q. E. D.

Reference

[1] T. Miura, S. Miyajima and S.-E. Takahasi, A characterization of Hyers-Ulam stability of first order linear diffrential operators, submitted.

Multiplier of weighted Dirichlet spaces

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Let $D = \{z \in C : |z| < 1\}$ denote the open unit disk in the complex plane C and let $\partial D = \{z \in C : |z| = 1\}$ denote the unit circle. For $1 \leq p < +\infty$, the Lebesgue space $L^p(D, dA)$ is defined to be the Banach space of Lebesgue measurable functions on the open unit disk D with

$$\| f \|_{L^p(dA)} := \left(\int_D |f(z)|^p dA(z) \right)^{\frac{1}{p}} < +\infty \ ,$$

where $dA(z)$ is the normalized area measure on D. The Bergman space $L^p_a(D)$ is defined to be the subspace of $L^p(D, dA)$ consisting of analytic functions. For $0 < p < +\infty$, the Hardy space H^p is defined to be the Banach space of analytic functions f on D with

$$\| f \|_p := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < +\infty \ .$$

For $z, w \in D$, let $\beta(z,w) := \frac{1}{2} \log \frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$, where $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$. For $0 < r < +\infty$ and $z \in D$, let $D(z) = D(z,r) = \{w \in D : \ \beta(z,w) < r\}$ denote the Bergman disk. $|D(z,r)|$ denotes the normalized area of $D(z,r)$ and $|D(z,r)|$ is comparable to $(1-|z|^2)^2$.

The space of analytic functions on D of bounded mean oscillation , denoted by $BMOA$, consists of functions f in H^2 for which

$$\| f \|_{BMOA} := |f(0)| + \sup_{a \in D} \int_D |f'(z)|^2 (1 - |\varphi_a(z)|^2) dA(z) < +\infty \ .$$

Let $\alpha > 0$. The space of analytic functions on D of bounded mean oscillation , denoted by $BMOA^\alpha$, consists of functions f in H^2 for which

$$\| f \|_{BMOA^\alpha} := |f(0)| + \sup_{a \in D} (1-|a|^2)^{2\alpha - 2} \int_D |f'(z)|^2 (1 - |\varphi_a(z)|^2) dA(z) < +\infty \ .$$

Note that $BMOA^1$ is the space $BMOA$.

Let $\alpha > 0$. Then α-Bloch space B^α is defined to be the space of analytic functions f on D such that

$$\| f \|_{B^\alpha} := |f(0)| + \sup_{z \in D} (1-|z|^2)^\alpha |f'(z)| < +\infty \ .$$

And the little α-Bloch space, denoted B^α_0, is the closed subspace of B^α consisting of functions f with $(1-|z|^2)^\alpha f'(z) \to 0$ ($|z| \to 1^-$). Note that B^1, B^1_0 are the Bloch space B, the little Bloch space B_0, respectively.

Let $\alpha \geq 0$. Then the weighted Dirichlet space D^{α} is defined to be the space of analytic functions f on D such that

$$\| f \|_{D^{\alpha}} := |f(0)| + \int_{D} (1-|z|^2)^{\alpha} |f'(z)|^2 \, dA(z) < +\infty .$$

Let X and Y be Banach spaces. Then a function f on D is a multiplier of X into Y if $fg \in Y$ for all g in X. In the case, we write $fX \subset Y$.

For g analytic on D, the operators I_g , J_g are defined on the weighted Bloch space by the following:

$$I_g(h)(z) := \int_0^z g(\zeta)h'(\zeta)d\zeta \ , \ J_g(f)(z) := \int_0^z f(\zeta)g'(\zeta)d\zeta.$$

If $g(z) = z$, then J_g is the integration operator. If $g(z) = \log \frac{1}{1-z}$, then J_g is the Cesáro operator.

In [6], Ch. Pommerenke showed that J_p is bounded operator on Hardy space H^2 if and only if g is in $BMOA$, and this result was extended to the other Hardy space H^p $1 \leq p < +\infty$ in [1]. In [2], A.Aleman and A.G.Siskakis studied the operator J_g defined on the weighted Bergman space. In [10], A.G.Siskakis and R.Zhao studied the boundedness and compactness of J_g on $BMOA$:

Theorem A. *The operator J_g is bounded on $BMOA$ if and only if*

$$\sup_{I \subset \partial D} \left(\frac{(\log \frac{2}{|I|})^2}{|I|} \int_{S(I)} |g'(z)|^2 (1-|z|^2) dA(z) \right) < +\infty,$$

and J_g is compact on $BMOA$ if and only if

$$\lim_{|I| \to 0} \left(\frac{(\log \frac{2}{|I|})^2}{|I|} \int_{S(I)} |g'(z)|^2 (1-|z|^2) dA(z) \right) = 0,$$

where $S(I) = \{z : 1-|I| \leq |z| < 1, \frac{z}{|z|} \in I\}$ for an arc I in ∂D.

In [12], we proved the following :

Theorem B. *For g analytic on D, theoperator J_g is bounded on B if and only if*

$$\sup_{z \in D} (1-|z|^2) \log \frac{1}{1-|z|^2} |g'(z)| < +\infty,$$

and the operator J_g is compact on B if and only if

$$\lim_{|z| \to 1^-} (1-|z|^2) \log \frac{1}{1-|z|^2} |g'(z)| = 0 .$$

Theorem C. *Let $\alpha > 1$. Then the operator J_g is bounded on B^{α} if and only if*

$$\sup_{z \in D} (1-|z|^2)|g'(z)| < +\infty, \qquad i.e \quad g \in B,$$

and the operator J_g is compact on B^α if and only if

$$\lim_{|z|\to 1^-}(1-|z|^2)|g'(z)|=0\ , \quad i.e \quad g\in B_0\ .$$

In [16], the following result were introduced :

Theorem D. *The following are equivalent :*

(*i*) $gB\subset B$;

(*ii*) $gB_0\subset B_0$;

(*iii*) $g\in H^\infty\ ,\ \sup_{z\in D}(1-|z|^2)\log\dfrac{1}{1-|z|^2}|g'(z)|<+\infty\ .$

And for $\alpha>1$, the following are equivalent :

(*i*) $gB^\alpha\subset B^\alpha$;

(*ii*) $gB_0^\alpha\subset B_0^\alpha$;

(*iii*) $g\in H^\infty$.

Let ω be a continuous non-increasing function on $[0,1]$ with $\omega(|z|)\to\infty$ ($|z|\to 1^-$), $(1-|z|^2)^\beta\omega(|z|)\to 0$ ($|z|\to 1^-$)for any $\beta>0$. Then we define the following spaces :

$$B^{\alpha,\omega}:=\{\ f\in H(D):\sup_{z\in D}(1-|z|^2)^\alpha\omega(|z|)\,|f'(z)|<+\infty\}$$

$$BMOA^{\alpha,\omega}:=\{\ f\in H(D):\sup_{I\subset\partial D}\frac{|I|^{2\alpha-2}\omega(1-|I|)^2}{|I|}\int_{S(I)}|f'(z)|^2\,(1-|z|^2)dA(z)<+\infty\}$$

$$D^{\alpha,\omega}:=\{\ f\in H(D):\int_D(1-|z|^2)^\alpha\omega(|z|)\,|f'(z)|^2\,dA(z)<+\infty\}$$

Note that $B\subset B^{\alpha,\omega}\subset B^\alpha$, $BMOA\subset BMOA^{\alpha,\omega}\subset BMOA^\alpha$, $D\subset D^{\alpha,\omega}\subset D^\alpha$. Then we proved the following:

Thorem 1. *Let $\alpha>1$. Suppose $(1-r)\dfrac{\omega'(r)}{\omega(r)}\le C<\alpha-1$ for some constant $C>0$. For g analytic on D, then J_g is bounded on $B^{\alpha,\omega}$ if and only if*

$$g\in B.$$

Thorem 2. *Let $\alpha>1$. Suppose $(1-r)\dfrac{\omega'(r)}{\omega(r)}\le C<\alpha-1$ for some constant $C>0$. For g analytic on D, then J_g is bounded on $BMOA^{\alpha,\omega}$ if and only if*

$$g\in BMOA.$$

Thorem 3. *Let* $\alpha > 1$. *Suppose* $(1-r)\frac{\omega'(r)}{\omega(r)} \le C < \alpha - 1$ *for some constant* $C > 0$. *For* g *analytic on* D, *then* J_g *is bounded on* $D^{\alpha,\omega}$ *if and only if*

$$g \in B.$$

Corollary 1. *Let* $\alpha > 1$. *For* g *analytic on* D, *the following are equivalent*:

(*i*) $gB_{\alpha,\omega} \subset B_{\alpha,\omega}$;

(*ii*) $I_g : B_{\alpha,\omega} \to B_{\alpha,\omega}$ *is bounded operator* ;

(*iii*) $g \in H^\infty$.

Corollary 2. *Let* $\alpha > 1$. *For* g *analytic on* D, *the following are equivalent*:

(*i*) $gBMOA_{\alpha,\omega} \subset BMOA_{\alpha,\omega}$;

(*ii*) $I_g : BMOA_{\alpha,\omega} \to BMOA_{\alpha,\omega}$ *is bounded operator* ;

(*iii*) $g \in H^\infty$.

Corollary 3. *Let* $\alpha > 1$. *For* g *analytic on* D, *the following are equivalent*:

(*i*) $gD_{\alpha,\omega} \subset D_{\alpha,\omega}$;

(*ii*) $I_g : D_{\alpha,\omega} \to D_{\alpha,\omega}$ *is bounded operator* ;

(*iii*) $g \in H^\infty$.

References

[1] A.Aleman and A.G.Siskakis, An integral operator on H^p, Complex Variables, 28(1995),149-158.

[2] A.Aleman and A.G.Siskakis, Integration operators on Bergman spaces, Indiana Univ. Math.J.46(1997),337-356.

[3] P.L.Duren, Theory of H^p spaces (Academic Press, 1970).

[4] P.L.Duren, B.W.Romberg and A.L.Schilds, Linear functionals on H^p spaces with $0 < p < 1$, J.Reine Angew.Math.238(1969),32-60.

[5] S.Ohno, K.Stroethoff and R.Zhao, Weighted composition operators between Bloch-type spaces, to appear Rocky Mout.J.Math.

[6] Ch.Pommerenke, Schlichte Funktionen und analytische Functionen von beschrankter mittlerer Oszillation, Comment.Math.Helv.52(1977),591-602.

[7] W.Ramey and D.Ullrich, Bounded mean oscillation of Bloch Pull-backs, Math.Ann.291, (1991),591-606.

[8] A. Montes-Rodriguez, The essential norm of composition operators on Bloch spaces, Pacific J.Math.188(1999),339-351.

[9] A. Montes-Rodriguez, Weighted composition operators on weighted Banach spaces of analytic functions, J.London Math.Soc.(2)61(2000),872-884.

[10] A.G.Siskakis and R.Zhao, A Volterra type operator on spaces of analytic functions, Contemporary Mathematics.232(1999),299-311.

[11] K.Stroethoff, The Bloch space and Besov spaces of analytic functions, Bull.Austral.Math. Soc.54(1996), 211-219.

[12] R.Yoneda, Integration operators on weighted Bloch space, Nipponkai Math. Journal 12(2)(2001),123-133.

[13] R.Yoneda, Multiplication operators, integration operators and companion operators on weighted Bloch spaces, in preprint.

[14] R.Yoneda, Essential Norms Of Integration Operators And Multipliers On Weighted Bloch Spaces, in preprint.

[15] R.Yoneda, Integration operators on weighted Dirichlet-type spaces, in preprint.

[16] K.Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York 1990.

[17] K.Zhu, Analytic Besov Spaces, J.Math.Anal.Appl.157(1991), 318-336.

[18] K.Zhu, Bloch type spaces of analytic functions, Rocky Mout.J.Math.23(1993), 1143-1177.

[19] K.Zhu, Multipliers of BMO in the Bergman metric with applications to Toeplitz operators, J.Funct.Anal.87(1989),31-50.

Generalized Riesz Projections

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Abstract Hunt-Muckenhoupt-Wheeden proved that if $1 < p < \infty$ and w satisfies the Muckenhoupt (A_p) condition, then the Riesz projection P is bounded on $L^p(w)$. Then the adjoint operator P^* satisfies $P^*f = w^{-1}P(wf), f \in L^p(w)$. For a measurable function v satisfying $|v|^{-p}w \in L^1$, the generalized Riesz projection P^v is defined by $P^v f = v^{-1}P(vf)$, f is in the dense subspace of $L^p(w)$. P^v is a bounded operator on $L^p(w)$ if and only if $|v|^{-p}w \in (A_p)$. The adjoint operator $(P^v)^*$ is described. Then the condition of v and w such that $(P^v)^* = P^v$ is established.

1. Riesz projection の有界性と共役作用素

$\mathbf{T} = \{|z| = 1\}$, $dm(e^{i\theta}) = \frac{d\theta}{2\pi}$. 三角多項式の全体を $\mathcal{C}$ で表す。
$\therefore$ $\mathcal{C} = \text{span}\{e^{in\theta}; -\infty < n < \infty\}$. $\mathcal{P} = \text{span}\{e^{in\theta}; 0 \le n < 0\}$, $\mathcal{Q} = \text{span}\{e^{in\theta}; n < 0\}$
と定める。$\therefore$ $\mathcal{C} = \mathcal{P} \oplus \mathcal{Q}$. Riesz projection $P : \mathcal{C} \to \mathcal{P}$ は

$$(Pf)(e^{i\theta}) = P\left(\sum_{k=-\infty}^{\infty} a_k e^{ik\theta}\right) = \sum_{k=0}^{\infty} a_k e^{ik\theta}$$

と定義される。$\therefore$ $P^2 = P$. ただし, $a_k = \hat{f}(k)$ は $f \in \mathcal{C}$ の k-番目の Fourier 係数を表す。
P は Cauchy の主値積分：

$$(Sf)(t) = \frac{1}{\pi i}\int_{\mathbf{T}} \frac{f(\tau)}{\tau - t} d\tau \quad (t \in \mathbf{T}), \quad \tilde{f}(e^{i\theta}) = \int_0^{2\pi} \cot\frac{\theta - \phi}{2} f(e^{i\phi})\frac{d\phi}{2\pi}$$

により, つぎのように表すことができる (cf. [1, p.38], [6, p.58])。

$$Pf = \frac{f + Sf}{2} = \frac{f + i\tilde{f} + \hat{f}(0)}{2}, \quad (f \in \mathcal{C})$$

$1 \le p \le \infty$ のとき,

$$H^p = \{f \in L^p \ ; \ \hat{f}(k) = 0, \quad (n < 0)\},$$
$$\overline{H_0^p} = \{f \in L^p \ ; \ \hat{f}(k) = 0, \quad (n \ge 0)\}$$

と定義する。$1 \le p < \infty$ のとき, H^p は $\mathcal{P}$ の L^p ノルム閉包と一致し, $\overline{H_0^p}$ は $\mathcal{Q}$ の L^p ノルム閉包と一致する。$p = 2$ のとき,

$$\int_T \left(\sum_{k=0}^{\infty} a_k e^{ik\theta}\right) \overline{\left(\sum_{k=-\infty}^{\infty} b_k e^{ik\theta}\right)} dm = \int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{ik\theta}\right) \overline{\left(\sum_{k=0}^{\infty} b_k e^{ik\theta}\right)} dm.$$

$\therefore \quad P^* = P,$

$$P \,:\, L^2 \to H^2, \quad \text{selfadjoint projection.}$$

P を L^2 上へ有界に拡張した作用素を同じ文字 P で表す。

(Riesz, 1924) $1 < p < \infty$ のとき，$P \,:\, L^p \to H^p$, bounded projection.

$$\sup_{f \in \mathcal{C}, f \neq 0} \frac{\|Pf\|_p}{\|f\|_p} < \infty.$$

(Newman, Rudin, 1961) $p = 1, \infty$ のとき，bounded projection : $L^p \to H^p$ は存在しない。故に

$$\sup_{f \in \mathcal{C}, f \neq 0} \frac{\|Pf\|_1}{\|f\|_1} = \infty, \quad \sup_{f \in \mathcal{C}, f \neq 0} \frac{\|Pf\|_\infty}{\|f\|_\infty} = \infty.$$

(cf. [13, p.115])

$\mathcal{Q}$ の L^p ノルム閉包を $\overline{H_0^p}$ で表す。閉グラフの定理より，

(1) $1 < p < \infty$ のとき，$L^p = H^p \oplus \overline{H_0^p}$.

(2) $p = 1, \infty$ のとき，$L^1 \neq H^1 \oplus \overline{H_0^1}, \quad L^\infty \neq H^\infty \oplus \overline{H_0^\infty}$.

ただし，$L^p = H^p + \overline{H_0^p}$ 且 $H^p \cap \overline{H_0^p} = 0$ であるとき $L^p = H^p \oplus \overline{H_0^p}$ と表す。

L^p 上の bounded operators の全体を $B(L^p)$ で表す。
$1 < p < \infty$ のとき，上と同じ計算により，$P^* = P$. $\therefore \|P\|_{B(L^p)} = \|P^*\|_{B(L^q)} = \|P\|_{B(L^q)}$.

(Verbitsky-Krupnik) $1 < p < \infty$ について

$$\|P - Q\|_{B(L^q)} = \|P - Q\|_{B(L^p)} = \cot \frac{\pi}{2 \max(p, q)}.$$

ただし，$Q = I - P, \quad \frac{1}{p} + \frac{1}{q} = 1$. (cf. [1, p.220, p.246])

(Hollenbeck-Verbitsky, 2000) $1 < p < \infty$ について

$$\|P\|_{B(L^q)} = \|P\|_{B(L^p)} = \frac{1}{\sin(\pi / \max(p, q))} = \frac{1}{\sin(\pi / p)}.$$

$1 \leq p < \infty$. $w \in L^1$, $w > 0$ について，

$$f \in L^p(w) \Leftrightarrow \|f\| = \left(\int_{\mathbf{T}} |f|^p w dm \right)^{1/p} < \infty.$$

$\mathcal{C}, \mathcal{P}, \mathcal{Q}$ の $L^p(w)$ ノルム閉包は $L^p(w), H^p(w), \overline{H_0^p(w)}$.
Riesz projection P は $L^p(w)$ 上の densely defined operator である。

(Helson-Szegő, 1960) $P \in B(L^2(w)) \Leftrightarrow w = e^{u + \tilde{v}}, \quad u, v \in L^\infty, \quad \|v\|_\infty < \frac{\pi}{2}$.
(cf. [5, p.147], [12, p.450], [13, p.99])

$1<p<\infty$ のとき，$w\in L^1, w>0$ が次の条件を満たすとき $w\in(A_p)$ と書く。

$$w\in(A_p)\Leftrightarrow \sup_{I\subset\mathbf{T}}\left(\frac{1}{m(I)}\int_I w\,dm\right)\left(\frac{1}{m(I)}\int_I w^{-1/(p-1)}dm\right)^{p-1}<\infty.$$

(cf. [13, p.107]) もし $w\in(A_p)$ ならば $w^{-1/(p-1)}\in L^1$.

$2\leq p<\infty$ のとき, $w, w^{-1}\in(A_p)\Leftrightarrow w\in(A_2)$

(Hunt-Muckenhoupt-Wheeden, 1973)　$P\in B(L^p(w))\Leftrightarrow w\in(A_p)$

(cf. [1, p.39], [5, p.255], [12, p.209, p.450], [13, p.119])

このとき,

$$P: L^p(w)\to H^p(w),\quad \text{bounded projection},\quad L^p(w)=H^p(w)\oplus\overline{H_0^p(w)}.$$

$w\in L^1,\ \ w>0$ に対して，$K^p(w)$ と $K_0^p(w)$ をつぎのように定める。

$$K^p(w)=\left\{f\in L^p(w)\ ;\ \int_0^{2\pi}e^{-in\theta}f(e^{i\theta})w(e^{i\theta})dm(e^{i\theta})=0,\ (n<0)\right\},$$

$$K_0^p(w)=\left\{f\in L^p(w)\ ;\ \int_0^{2\pi}e^{-in\theta}f(e^{i\theta})w(e^{i\theta})dm(e^{i\theta})=0,\ (n\leq 0)\right\}.$$

$\frac{1}{p}+\frac{1}{q}=1$ のとき，$H^p(w)^\perp=\overline{K_0^q(w)},\quad H_0^p(w)^\perp=\overline{K^q(w)},\quad \mathrm{ran}P^*=K^q(w).$

もし h が outer 関数であり，$w=|h|^p$ ならば,

$$K^p(w)=\frac{h^{p-1}}{w}H^p=\frac{h^p}{w}H^p(w),\quad K_0^p(w)=\frac{h^{p-1}}{w}H_0^p=\frac{h^p}{w}H_0^p(w),$$

$$K^q(w)=\frac{h}{w}H^q=\frac{h^p}{w}H^q(w),\quad K_0^q(w)=\frac{h}{w}H_0^q=\frac{h^p}{w}H_0^q(w).$$

もし $w=1$ ならば，$K^p(w)=H^p, K_0^p(w)=H_0^p$. このとき,

$$P^w g=w^{-1}P(wg),\quad (g\in L^q(w))$$

と定めると，全ての $f\in L^p(w)$ について,

$$\int_T f\overline{P^w g}\,w\,dm=\int_T f\overline{P(wg)}dm=\int_T(Pf)\overline{(wg)}dm=\int_T(Pf)\bar{g}w\,dm.$$

$$\therefore\ \ P^*=P^w.\ \ \therefore\ \ \mathrm{ran}P^*=P^*L^q(w)=P^wL^q(w)=K^q(w).$$

$$\therefore\ \ P^*=P^w\ :\ L^q(w)\to K^q(w),\quad \text{bounded projection},$$

$L^q(w)=K^q(w)\oplus\overline{K_0^q(w)},\quad \|P^w\|_{B(L^q(w))}=\|P\|_{B(L^p(w))}.\quad \therefore P^*=P\ \Leftrightarrow\ w$ は定数.

2. Generalized Riesz projection の有界性

Riesz projection $P \in B(L^p(w))$ の共役作用素を求めると P^w であることがわかったが, P^w と似た作用素として次のような P^h を考える。$\log w \in L^1$ のとき, outer 関数 h が存在して $w = |h|^p$. $f \in L^p(w) \Rightarrow hf \in L^p \Rightarrow P(hf) \in H^p$. このとき, $P^h f = h^{-1}P(hf)$ と定める。$(P^h)^2 = P^h$ on $L^p(w)$. (A_p) 条件は必要なく,

(Coifman-Rochberg, Nakazi) $p = 2$ のとき, $w = |h|^2$,

$$P^h : L^2(w) \to H^2(w), \quad \text{selfadjoint projection}, \quad L^2(w) = H^2(w) \oplus \overline{K_0^2(w)}.$$

$1 < p < \infty$ のとき, $P^h : L^p(w) \to H^p(w)$, bounded projection, $L^p(w) = H^p(w) \oplus \frac{\bar{h}}{h}\overline{H_0^p(w)}$.

$$(P^h)^* = P^{w/\bar{h}} : L^q(w) \to \frac{\overline{h^{(2-p)/2}}}{h^{(2-p)/2}} H^q(w), \quad \text{bounded projection}$$

$$L^q(w) = \frac{\overline{h^{(2-p)/2}}}{h^{(2-p)/2}} H^q(w) \oplus \overline{K_0^q(w)}, \quad \ker(P^h)^* = \overline{K_0^q(w)}.$$

$$\|P^h\|_{B(L^p(w))} = \|P\|_{B(L^p)} = \frac{1}{\sin(\pi/p)}.$$

$1 < p < \infty, p \neq 2, \frac{1}{p} + \frac{1}{q} = 1, w^{2-q} \in L^1$ のとき, $(P^h)^* = P^h \Leftrightarrow h$ と w は定数。

P^w や P^h 等を統一的に取り扱うために, $|v| > 0$ について
Generalized Riesz projection $P^v : v^{-1}\mathcal{C} \to v^{-1}\mathcal{C}$ は

$$P^v f = v^{-1}P(vf), \quad (f \in v^{-1}\mathcal{C}).$$

と定義される。補題 1(1) より, P^v は $L^p(w)$ 上の densely defined operator である。
もし $v = 1$ ならば, $P^v = P$ である。$vf \in \mathcal{C}$ より, $P(vf) \in \mathcal{P}$. 故に

$$(P^v)^2(f) = P^v(v^{-1}P(vf)) = v^{-1}P^2(vf) = v^{-1}P(vf) = (P^v)(f).$$

補題 1 $1 \leq p < \infty$. $w \in L^1, w > 0$. $|v|^{-p}w \in L^1$. このとき,
(1) $v^{-1}\mathcal{C}$ は $L^p(w)$ で稠密である。
(2) もし $\log w \in L^1$, k は outer 関数であり, $|v| = |k|$ ならば, $k^{-1}\mathcal{P}$ は $H^p(w)$ で稠密である。

定理 1 $w \in L^1, w > 0$. このとき,

(1) もし $|v|^{-1}w \in L^1$ ならば, $P^v \notin B(L^1(w))$.

(2) もし $1 < p < \infty, |v|^{-p}w \in L^1$ ならば,

$$P^v \in B(L^p(w)) \quad \Leftrightarrow \quad P \in B(L^p(|v|^{-p}w)) \quad \Leftrightarrow \quad |v|^{-p}w \in (A_p).$$

このとき, $\|P^v\|_{B(L^p(w))} = \|P\|_{B(L^p(|v|^{-p}w))}$.

定理 1 より, $1 < p < \infty$, $w, \log w \in L^1$ に対して, $L^p(w)$ から $H^p(w)$ への bounded projection が存在する。もし h が outer 関数であり $w = |h|^p$ ならば, $\|P^h\|_{B(L^p(w))} < \infty$.

定理 2 $1<p<\infty$. $w,\ \log w \in L^1$. k は outer 関数とする。このとき,

(1) もし $|v|=|k|,\ |v|^{-p}w \in L^1$ ならば,

$$P^v \in B(L^p(w)) \ \Leftrightarrow\ P^k \in B(L^p(w)) \ \Leftrightarrow\ L^p(w) = H^p(w) \oplus \frac{\bar{k}}{k}\overline{H_0^p(w)},$$

更に, $\mathrm{ran}\ P^v = \ker\ Q^v = \frac{k}{v}H^p(w)$, $\ker\ P^v = \mathrm{ran}\ Q^v = \frac{\bar{k}}{v}\overline{H_0^p(w)}$. ただし, $Q^v = I - P^v$.

(2) もし $w=|k|^2,\ w^{(2-p)/2} \in L^1$ ならば,

$$P^k \in B(L^p(w)) \ \Leftrightarrow\ w^{(2-p)/2} \in (A_p) \ \Leftrightarrow\ L^p(w) = H^p(w) \oplus \overline{K_0^p(w)}.$$

3. Generalized Riesz projection の共役作用素

定理 3 $1<p<\infty$. $w \in L^1$, $w>0$, $P^v \in B(L^p(w))$. このとき,

$$(P^v)^*(g) = \frac{\overline{v}}{w}P\left(\frac{w}{\overline{v}}g\right), \quad (g \in L^q(w)).$$

定理 4 $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. $w, \log w \in L^1$. このとき,

(1) もし $P^v \in B(L^p(w))$, $|v|^{-q}w \in L^1$ ならば,

$$(P^v)^* = P^v \quad \Leftrightarrow \quad w = c|v|^2, \quad \text{ただし } c \text{ は定数}$$

(2) もし k が outer 関数であり $|k|^2=w$, $P^k \in B(L^p(w))$ ならば, $(P^k)^*=P^k$.

$p=2$ の場合を考える。もし $w=|h|^2$, h は outer 関数ならば, 定理 2 より,

$$\mathrm{ran}\ P^h = \ker\ Q^h = H^2(w), \quad \ker\ P^h = \mathrm{ran}\ Q^h = \frac{\bar{h}}{h}\overline{H_0^2(w)}.$$

ただし, $Q^h = I - P^h$. 定理 3 より,

$$(P^h)^*(g) = \frac{\overline{h}}{w}P\left(\frac{w}{\overline{h}}g\right) = \frac{1}{h}P(hg) = P^h g, \quad (g \in L^2(w)).$$

故に,

$$P^h\ :\ L^2(w) \to H^2(w), \quad \text{selfadjoint projection,}$$
$$Q^h\ :\ L^2(w) \to \overline{K_0^2(w)}, \quad \text{selfadjoint projection,}$$

$L^2(w) = H^2(w) \oplus \overline{K_0^2(w)}$. $\alpha, \beta \in L^\infty$ について,

$$\begin{aligned}\|\alpha P^h + \beta Q^h\|_{B(L^2(w))} &= \|\alpha P + \beta Q\|_{B(L^2)} \\ &= \inf_{k \in H^\infty}\left\| \frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty,\end{aligned}$$

infimum は attained する (cf. [11])。同様に、

$$P^{\bar{h}} \ : \ L^2(w) \to K^2(w), \quad \text{selfadjoint projection,}$$

$$Q^{\bar{h}} \ : \ L^2(w) \to \overline{H_0^2(w)}, \quad \text{selfadjoint projection,}$$

$$L^2(w) = K^2(w) \oplus \overline{H_0^2(w)}, \quad \|\alpha P^{\bar{h}} + \beta Q^{\bar{h}}\|_{B(L^2(w))} = \|\alpha P^h + \beta Q^h\|_{B(L^2(w))}.$$

References

[1] A.Böttcher and B.Silbermann, Analysis of Toeplitz Operators, Springer-Verlag, Berlin, 1990.

[2] R.Coifman and R.Rochberg, Projections in weighted spaces, skew projections and inversions of Toeplitz operators, Integral Equations and Operator Theory 5 (1982), 145-159.

[3] R.G.Douglas, Banach Algebra techniques in Operator Theory (2nd ed.), Springer-Verlag, New York, Berlin, 1998.

[4] F.Forelli, The Marcel Riesz theorem on conjugate functions, Trans. Amer. Math. Soc. 106 (1963), 369-390.

[5] J.Garnett, Bounded Analytic Functions, Academic Press, New York, 1981.

[6] I.Gohberg and N.Ja.Krupnik, One-Dimensional Linear Singular Integral Equations, Vols. I,II, Birkhäuser Verlag, Basel, 1992.

[7] B.Hollenbeck and I.E.Verbitsky, Best constants for the Riesz projection, J. Funct. Anal. 175 (2000), 370-392.

[8] T.Nakazi, Commutator of two projections in prediction theory, Bull. Austral. Math. Soc. 34 (1986), 65-71.

[9] T.Nakazi, Kernels of Toeplitz operators, J. Math. Soc. Japan 38 (1986), 607-616.

[10] T.Nakazi, Toeplitz operators and weighted norm inequalities, Acta Sci. Math. (Szeged) 58 (1993), 443-452.

[11] T.Nakazi and T.Yamamoto, Norms of some singular integral operators and their inverse operators, J. Operator Theory 40 (1998), 185-207.

[12] N.K.Nikol'skii, Treatise on the Shift Operator, Springer Verlag, Berlin, 1986.

[13] N.K.Nikol'skii, Operators, Functions, and Systems, Mathematical Surveys and Monographs Vol.92, Amer. Math. Soc., 2002.

[14] R.Rochberg, Toeplitz operators on weighted H^p spaces, Indiana Univ. Math. J. 26 (1977), 291-298.

Convolution inequalities and applications

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Abstract

We introduce various convolution inequalities obtained recently and at the same time, we give new type reverse convolution inequalities and their important applications to inverse source problems. We consider the inverse problem of determining $f(t)$, $0 < t < T$, in the heat source of the heat equation $\partial_t u(x,t) = \Delta u(x,t) + f(t)\varphi(x)$, $x \in R^n$, $t > 0$ from the observation $u(x_0, t)$, $0 < t < T$, at a remote point x_0 away from the support of φ. Under an a priori assumption that f changes the signs at most N-times, we give a conditional stability of Hölder type, as an example of applications.

1 Introduction

For the Fourier convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi,$$

the Young's inequality

$$\|f * g\|_r \le \|f\|_p \, \|g\|_q, \quad f \in L_p(R), g \in L_q(R), \tag{1}$$
$$r^{-1} = p^{-1} + q^{-1} - 1 \quad (p, q, r > 0),$$

is fundamental. Note, however, that for the typical case of $f, g \in L_2(R)$, the inequality does not hold. In a series of papers [12-15] (see also [5]) we obtained the following weighted $L_p(p > 1)$ norm inequality for convolution

Proposition 1.1 ([15]) *For two nonvanishing functions $\rho_j \in L_1(R)(j = 1, 2)$, the $L_p(p > 1)$ weighted convolution inequality*

$$\left\| ((F_1\rho_1) * (F_2\rho_2)) \, (\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_p \le \|F_1\|_{L_p(R,|\rho_1|)} \, \|F_2\|_{L_p(R,|\rho_2|)} \tag{2}$$

holds for $F_j \in L_p(R, |\rho_j|)(j = 1, 2)$. Equality holds here if and only if

$$F_j(x) = C_j e^{\alpha x}, \tag{3}$$

where α is a constant such that $e^{\alpha x} \in L_p(R, |\rho_j|)(j = 1, 2)$ (otherwise, C_1 or $C_2 = 0$). Here

$$\|F\|_{L_p(R,|\rho|)} = \left\{ \int_{-\infty}^{\infty} |F(x)|^p |\rho(x)| \, dx \right\}^{\frac{1}{p}}.$$

Unlike the Young's inequality, inequality (2) holds also in case $p = 2$.

Note that the proof of inequality (2) in Proposition 1.1 is direct and fairly elementary. The proof will be done in three lines. Indeed, we use only Hölder's inequality and Fubini's theorem for exchanging the orders of integrals for the proof. So, for various type convolutions, we can also obtain similar type convolution inequalities, see [18] for various convolutions. However, to determine the case that the equality in (2) holds needs very delicate arguments. See [5] for the details.

In many cases of interest, the convolution is given in the form

$$\rho_2(x) \equiv 1, \quad F_2(x) = G(x), \tag{4}$$

where $G(x-\xi)$ is some Green's function. Then inequality (2) takes the form

$$\|(F\rho) * G\|_p \le \|\rho\|_p^{1-\frac{1}{p}} \|G\|_p \|F\|_{L_p(R,|\rho|)}, \tag{5}$$

where ρ, F, and G are such that the right hand side of (5) is finite. Inequality (5) enables us to estimate the output function

$$\int_{-\infty}^{\infty} F(\xi)\rho(\xi)G(x-\xi)\,d\xi \tag{6}$$

in terms of the input function F in the related differential equation. For various applications, see [15]. We are also interested in the reverse type inequality for (5), namely, we wish to estimate the input function F by means of the output (6). This kind of estimates is important in inverse problems. One estimate is obtained by using the following famous reverse Hölder inequality

Proposition 1.2 ([19], see also [10], pages 125-126). *For two positive functions f and g satisfying*

$$0 < m \le \frac{f}{g} \le M < \infty \tag{7}$$

on the set X, and for $p, q > 1$, $p^{-1} + q^{-1} = 1$,

$$\left(\int_X f d\mu\right)^{\frac{1}{p}} \left(\int_X g d\mu\right)^{\frac{1}{q}} \le A_{p,q}\left(\frac{m}{M}\right) \int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{8}$$

if the right hand side integral converges. Here

$$A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} \frac{t^{-\frac{1}{pq}}(1-t)}{\left(1-t^{\frac{1}{p}}\right)^{\frac{1}{p}} \left(1-t^{\frac{1}{q}}\right)^{\frac{1}{q}}}.$$

Then, by using Proposition 1.2 we obtain, as in the proof of Proposition 1.1, the following

Proposition 1.3([16]) *Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \le F_1(x) \le M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \le F_2(x) \le M_2^{\frac{1}{p}} < \infty, \quad p > 1, \quad x \in R. \tag{9}$$

Then for any positive continuous functions ρ_1 and ρ_2, we have the reverse L_p–weighted convolution inequality

$$\begin{aligned} &\left\{A_{p,q}\left(\frac{m_1 m_2}{M_1 M_2}\right)\right\}^{-1} \|F_1\|_{L_p(R,\rho_1)} \|F_2\|_{L_p(R,\rho_2)} \\ &\le \left\| ((F_1\rho_1) * (F_2\rho_2)) (\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_p . \end{aligned} \tag{10}$$

Inequality (10) and others should be understood in the sense that if the right hand side is finite, then so is the left hand side, and in this case the inequality holds.

In formula (10) replacing ρ_2 by 1, and $F_2(x-\xi)$ by $G(x-\xi)$, and taking integration with respect to x from c to d we arrive at the following inequality

$$\begin{aligned} &\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-p} \left(\int_{-\infty}^{\infty} \rho(\xi)\,d\xi\right)^{p-1} \int_{-\infty}^{\infty} F^p(\xi)\,\rho(\xi)\,d\xi \int_{c-\xi}^{d-\xi} G^p(x)dx \\ &\le \int_c^d \left(\int_{-\infty}^{\infty} F(\xi)\,\rho(\xi)\,G(x-\xi)\,d\xi\right)^p dx \end{aligned} \tag{11}$$

if positive continuous functions ρ, F, and G satisfy

$$0 < m^{\frac{1}{p}} \le F(\xi)G(x-\xi) \le M^{\frac{1}{p}}, \quad x \in [c,d], \quad \xi \in R. \tag{12}$$

Inequality (11) is especially important when $G(x-\xi)$ is a Green's function. We gave various concrete applications in [16] from the viewpoint of stability in inverse problems.

2 Remarks for reverse Hölder inequalities

In connection with Proposition 1.2 which gives Proposition 1.3, Izumino and Tominaga [8] consider the upper bound of

$$(\sum a_k^p)^{1/p}(\sum b_k^q)^{1/q} - \lambda \sum a_k b_k$$

for $\lambda > 0$, for $p, q > 1$ satisfying $1/p + 1/q = 1$ and for positive numbers $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$, in detail. In their different approach, they showed that the constant $A_{p,q}(t)$ in Proposition 1.2 is best possible in a sense. Note that the proof of Proposition 1.2 is quite involved. In connection with Proposition 1.2 we note that the following version whose proof is surprisingly simple

Theorem 2.1([17]) *In Proposition 1.2, replacing f and g by f^p and g^q, respectively, we obtain the reverse Hölder type inequality*

$$\left(\int_X f^p d\mu\right)^{\frac{1}{p}} \left(\int_X g^q d\mu\right)^{\frac{1}{q}} \leq \left(\frac{m}{M}\right)^{-\frac{1}{pq}} \int_X fg d\mu. \tag{13}$$

Proof

Since $f^p/g^q \leq M$, $g \geq M^{-\frac{1}{q}} f^{\frac{p}{q}}$. Therefore

$$fg \geq M^{-\frac{1}{q}} f^{1+\frac{p}{q}} = M^{-\frac{1}{q}} f^p$$

and so,

$$\left\{\int f^p d\mu\right\}^{1/p} \leq M^{\frac{1}{pq}} \left\{\int fg d\mu\right\}^{1/p}. \tag{14}$$

On the other hand, since $m \leq f^p/g^q$, $f \geq m^{1/p} g^{q/p}$. Hence

$$\int fg d\mu \geq \int m^{1/p} g^{1+\frac{q}{p}} d\mu = m^{1/p} \int g^q d\mu,$$

and so,

$$\left\{\int fg d\mu\right\}^{1/q} \geq m^{1/pq} \left\{\int g^q d\mu\right\}^{1/q}.$$

Combining with (14), we have the desired inequality

$$\begin{aligned} &\left\{\int f^p d\mu\right\}^{1/p} \left\{\int g^q d\mu\right\}^{1/q} \\ &\leq M^{1/pq} \left\{\int fg d\mu\right\}^{1/p} m^{-1/pq} \left\{\int fg d\mu\right\}^{1/q} \\ &= \left(\frac{m}{M}\right)^{-1/pq} \int fg d\mu. \end{aligned}$$

3 New reverse convolution inequalities

In reverse convolution inequality (10), similar type inequalities for $m_1 = m_2 = 0$ are also important as we see from our example in Section 4. For these, we obtain a reverse convolution inequality of new type.

Theorem 3.1 *Let $p \geq 1, \delta > 0$, $0 \leq \alpha < T$, and $f, g \in L_\infty(0,T)$ satisfy*

$$0 \leq f, g \leq M < \infty, \qquad 0 < t < T. \tag{15}$$

Then

$$\|f\|_{L_p(\alpha,T)} \|g\|_{L_p(0,\delta)} \leq M^{\frac{2p-2}{p}} \left(\int_\alpha^{T+\delta} \left(\int_\alpha^t f(s)g(t-s)ds\right) dt\right)^{\frac{1}{p}}. \tag{16}$$

In particular, for

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds, \qquad 0 < t < T$$

and for $\alpha = 0$, we have

$$\|f\|_{L_p(0,T)} \|g\|_{L_p(0,\delta)} \leq M^{\frac{2p-2}{p}} \|f * g\|_{L_1(0,T+\delta)}^{\frac{1}{p}}. \tag{17}$$

4 Applications to inverse source heat problems

We consider the heat equation with a heat source:

$$\partial_t u(x,t) = \Delta u(x,t) + f(t)\varphi(x), \quad x \in R^n,\, t > 0 \tag{18}$$

$$u(x,0) = 0, \qquad x \in R^n. \tag{19}$$

We assume that φ is a given function and satisfies

$$\begin{cases} \varphi \geq 0, \quad \not\equiv 0 \quad \text{in } \mathrm{R}^n, \\ \varphi \text{ has compact support}, \\ \varphi \in C^\infty(R^n), \text{ if } n \geq 4 \text{ and } \varphi \in L_2(R^n), \text{ if } n \leq 3. \end{cases} \tag{20}$$

Our problem is to derive a conditional stability in the determination of $f(t)$, $0 < t < T$, from the observation

$$u(x_0, t), \qquad 0 < t < T \tag{21}$$

where $x_0 \not\in \operatorname{supp}\varphi$.

We are interested only in the case of $x_0 \not\in \operatorname{supp}\varphi$, because in the case where x_0 is in the interior of $\operatorname{supp}\varphi$, the problem can be reduced to a Volterra integral equation of the second kind by differentiation in t formula (25) stated below. Moreover $x_0 \not\in \operatorname{supp}\varphi$ means that our observation (21) is done far from the set where the actual process is occuring, and the design of the observation point is easy.

Let

$$K(x,t) = \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4t}}, \quad x \in R^n,\, t > 0. \tag{22}$$

Then the solution u to (18) and (19) is represented by

$$u(x,t) = \int_0^t \int_{R^n} K(x-y, t-s) f(s)\varphi(y)\, dy ds, \quad x \in R^n,\, t > 0 \tag{23}$$

(e.g., Friedman [6]). Therefore, setting

$$\mu_{x_0}(t) = \int_{R^n} K(x_0 - y, t)\varphi(y)\, dy, \qquad t > 0, \tag{24}$$

we have

$$u(x_0, t) \equiv h_{x_0}(t) = \int_0^t \mu_{x_0}(t-s) f(s)\, ds, \quad 0 < t < T, \tag{25}$$

which is a Volterra integral equation of the first kind with respect to f. Since

$$\lim_{t \downarrow 0} \frac{d^k \mu_{x_0}}{dt^k}(t) = (\Delta^k)(x_0) = 0, \quad k \in N \cup \{0\}$$

by $x_0 \not\in \operatorname{supp}\varphi$ (e.g., [6]), the equation (25) cannot be reduced to a Volterra equation of the second kind by differentiating in t. Hence, even though, for any $m \in N$, we take the C^m-norms for data h, the equation (25) is ill-posed, and we cannot expect a better stability such as of Hölder type under suitable a priori boundedness.

In Cannon and Esteva [3], an estimate of logarithmic type is proved: let $n = 1$ and $\varphi = \varphi(x)$ be the characteristic function of an interval $(a,b) \subset R$. Set

$$V_M = \left\{ f \in C^2[0,\infty); f(0) = 0, \quad \left\| \frac{df}{dt} \right\|_{C[0,\infty)}, \left\| \frac{d^2 f}{dt^2} \right\|_{C[0,\infty)} \leq M \right\}. \tag{26}$$

Let $x_0 \not\in (a,b)$. Then, for $T > 0$, there exists a constant $C = C(M, a, b, x_0) > 0$ such that

$$|f(t)| \leq \frac{C}{|\log \|u(x_0,\cdot)\|_{L_2(0,\infty)}|^2}, \quad 0 \leq t \leq T, \tag{27}$$

for all $f \in V_M$. The stability rate is logarithmic and worse than any rate of Hölder type: $\|u(x_0,\cdot)\|^\alpha_{L_2(0,\infty)}$ for any $\alpha > 0$. For (26), the condition $f \in V_M$ prescribes a priori information and (26) is called conditional stability within the admissible set V_M. The rate of conditional stability heavily depends on the choice of admissible sets and an observation point x_0. As for other inverse problems for the heat equation, we can refer to Cannon [2], Cannon and Esteva [4], Isakov [7] and the references therein.

We arbitrarily fix $M > 0$ and $N \in N$. Let

$$U = \{f \in C[0,T]; \|f\|_{C[0,T]} \leq M, f \text{ changes the signs at most } N\text{-times}\}. \tag{28}$$

We take U as an admissible set of unknowns f. Then, within U, we can show an improved conditional stability of Hölder type:

Theorem 4.1 *Let φ satisfy (20), and $x_0 \notin supp\varphi$. We set*

$$p > \begin{cases} \dfrac{4}{4-n}, & n \leq 3, \\ 1, & n \geq 4. \end{cases} \tag{29}$$

Then, for an arbitrarily given $\delta > 0$, there exists a constant $C = C(x_0, \varphi, T, p, \delta, U) > 0$ such that

$$\|f\|_{L_p(0,T)} \leq C\|u(x_0,\cdot)\|_{L_1(0,T+\delta)}^{\frac{1}{p^N}} \tag{30}$$

for any $f \in U$.

We will see that $\lim_{\delta\to 0} C = \infty$ and, in order to estimate f over the time interval $(0,T)$, we have to observe $u(x_0,\cdot)$ over a longer time interval $(0, T+\delta)$.

Remark 1

In the case of $n \geq 4$, we can relax the regularity of φ to $H^\alpha(R^n)$ with some $\alpha > 0$. In the case of $n \leq 3$, if we assume that $\in C^\infty(R^n)$ in (20), then in Theorem 4.1 we can take any $p > 1$.

Remark 2

As a subset of U, we can take, for example,

$$P_N = \{f; f \text{ is a polynomial whose order is at most } N \text{ and } \|f\|_{C[0,T]} \leq M\}.$$

The condition $f \in U$ is quite restrictive at the expense of the practically reasonable estimate of Hölder type.

Remark 3

The a priori boundedness $\|f\|_{C[0,T]} \leq M$ is necessary for the stability. See [17] for a counter example.

Remark 4

For our stability, the finiteness of changes of signs is essential. In fact, we take

$$f_n(t) = \cos nt, \qquad 0 \leq t \leq T, \, n \in N. \tag{31}$$

Then f_n oscillates very frequently and we cannot take any finite partition of $(0,T)$ where the condition on signs in (28) holds true. We note that we can take $M = 1$, that is, $\|f_n\|_{C[0,T]} \leq 1$ for $n \in N$. We denote the solution to (18) - (19) for $f = f_n$ by $u_n(x,t)$. Then, we see that any stability cannot hold for f_n, $n \in N$. See [17] for the proof.

References

[1] Adams, R.A., Sobolev Spaces, Academic Press New York (1975).

[2] Cannon, J.R., The One-dimensional Heat Equation, Addison-Wesley Reading, Massachusetts (1984).

[3] Cannon, J.R. and Esteva, S.P., An inverse problem for the heat equation, *Inverse Problems* **2**(1986), 395–403.

[4] Cannon, J.R. and Esteva, S.P., Uniqueness and stability of 3D heat sources, *Inverse Problems* **7**(1991), 57–62.

[5] Cwickel, M. and Kerman, R., On a convolution inequality of Saitoh, *Proc. Amer. Math. Soc.,* **124**(1996), 773–777.

[6] Friedman, A., Partial Differential Equations of Parabolic Type, Krieger Malabar, Florida (1983).

[7] Isakov, V., Inverse Problems for Partial Differential Equations, Springer-Verlag Berlin (1998).

[8] Izumino, S. and Tominaga, M., Estimations in Hölder type inequalities, *Mathematical Inequalities & Applications* **4**(2001), 163–187.

[9] Lions, J.-L. and Magenes, E., Non-homogeneous Boundary Value Problems and Applications, Springer-Verlag, Berlin (1972).

[10] Mitrinović, S. D., Pecarić, J. E. and Fink, A. M., Classic and New Inequalities in Analysis, Kluwer Academic Publishers, The Netherlands (1993).

[11] Pazy, A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin (1983).

[12] Saitoh, S., A fundamental inequality in the convolution of L_2 functions on the half line, *Proc. Amer. Math. Soc.,* **91**(1984), 285-286.

[13] Saitoh, S., Inequalities in the most simple Sobolev space and convolutions of L_2 functions with weights, *Proc. Amer. Math. Soc.,* **118**(1993), 515-520.

[14] Saitoh, S., Various operators in Hilbert space introduced by transforms, *International. J. Appl. Math.,* **1**(1999), 111-126.

[15] Saitoh, S., Weighted L_p-norm inequalities in convolutions,

Survey on Classical Inequalities, 225–234(2000), Kluwer Academic Publishers, The Netherlands.

[16] Saitoh, S., Tuan, V. K. and Yamamoto, M., Reverse weighted L_p-norm inequalities in convolutions and stability in inverse problems, *J. of Inequalities in Pure and Applied Mathematics,* **1**(2000), Article 7. [ONLINE: http://jipam.vu.edu.au/v1n1/018$_9$.html]

[17] Saitoh, S., Tuan, V. K. and Yamamoto, M., Reverse convolution inequalities and applications to inverse heat source problems, *J. of Inequalities in Pure and Applied Mathematics* (to appear).

[18] Srivastava, H. M. and Buschman, R. G., Theory and Applications of Convolution Integral Equations, Kluwer Academic Publishers, The Netherlands (1992).

[19] Xiao-Hua, L.,

On the inverse of Hölder inequality, *Math. Practice and Theory,* **1**(1990), 84-88.

ON THE BERGER-COBURN-LEBOW PROBLEM FOR HARDY SUBMODULES

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1 Abstract

The Toeplitz algebra, which is the C^*-algebra generated by the Toeplitz operator T_z on the one-variable Hardy space $H^2(\mathbb{D})$, is a rich object in operator theory. It has been revealed that there is a close relation between the analytical indices and the topological indices of continuous functions on $\mathbb{T}$. It is interesting to consider an analogue to the Toeplitz algebra in the multivariable case. In this report, we shall study the problem posed by Berger, Coburn and Lebow in [2], and solve this problem affirmatively.

2 Preliminaries

Definition 1 Let $H^2 = H^2(\mathbb{D}^2)$ be the Hardy space over the bidisc $\mathbb{D}^2$. A closed subspace $\mathcal{M}$ of H^2 is said to be a Hardy submodule or an invariant subspace of H^2 if $\mathcal{M}$ is invariant under the multiplication operators by the coordinate functions z and w. V_z denotes the restriction on a Hardy submodule of the Toeplitz operator T_z. Similarly, we define V_w with the variable w.

In [2], Berger, Coburn and Lebow studied the C^*-algebras generated by commuting isometries, and in Section 9 of [2], they defined the following equivalence relation between Hardy submodules:

Definition 2 Let $\mathcal{A}(V_z, V_w; \mathcal{M}) = \mathcal{A}(\mathcal{M}) = \mathcal{A}(V_z, V_w)$ be the C^*-algebra generated by V_z and V_w, where we consider $\mathcal{A}(V_z, V_w; \mathcal{M}) = \mathcal{A}(V_z, V_w)$ as a C^*-subalgebra of $\mathcal{B}(\mathcal{M})$.

Two C^*-algebras $\mathcal{A}_i = \mathcal{A}(V_z^{(i)}, V_w^{(i)}; \mathcal{M}_i)$ $(i = 1, 2)$ are said to be unitarily equivalent if there exists a unitary operator U from $\mathcal{M}_1$ to $\mathcal{M}_2$ such that $U^*\mathcal{A}_2 U = \mathcal{A}_1$.

In Section 13 of [2], they posed the following problem:

The Berger-Coburn-Lebow problem ([2]) *If $\mathcal{M}$ is any invariant subspace of H^2 of finite defect, then is $\mathcal{A}(V_z, V_w)$ unitarily equivalent to $\mathcal{A}(T_z, T_w)$?*

We shall call this problem the BCL problem, for short. In [2], they gave an affirmative answer in the case where Hardy submodules are generated by monomials, and it has been remarked that if $\mathcal{M}$ has codimension 1, then the answer is affirmative. In fact, if the set of all common zeros of $\mathcal{M}$ consists of one point, then one can give an affirmative answer to the BCL problem with a slight modification of their technique. And we need to note the following facts:

Theorem 1 (Rigidity theorem [1]) *Two different Hardy submodules $\mathcal{M}_1$ and $\mathcal{M}_2$, which are both of finite codimension, are not unitarily equivalent as modules. Or equivalently, if there exists a unitary operator from $\mathcal{M}_1$ to $\mathcal{M}_2$ such that $UV_z^{(1)}U^* = V_z^{(2)}$ and $UV_w^{(1)}U^* = V_w^{(2)}$, then $\mathcal{M}_1 = \mathcal{M}_2$.*

We shall give several well-known properties of Hardy submodules.

Proposition 1 *Every C^*-algebra $\mathcal{A}(V_z, V_w)$ is irreducible.*

Theorem 2 (Yang [4]) *If $\mathcal{M}$ is a Hardy submodule generated by a finite number of polynomials, then $[V_z^*, V_w]$ and $[V_z^*, V_z][V_w^*, V_w]$ are both of Hilbert-Schmidt class.*

Corollary 1 *Let $\mathcal{K}(\mathcal{M})$ be the set of all compact operators on $\mathcal{M}$. If $\mathcal{M}$ is a Hardy submodule generated by a finite number of polynomials, then $\mathcal{K}(\mathcal{M})$ is contained in $\mathcal{A}(V_z, V_w)$.*

3 An affirmative answer to the BCL problem

Let $\mathcal{N}$ denote the orthogonal complement of a Hardy submodule $\mathcal{M}$ in H^2. In the following argument, we suppose that the dimension of $\mathcal{N}$ is finite. We shall briefly sketch the outline of an affirmative answer to the BCL problem. Let $S_z = P_{\mathcal{N}} T_z|_{\mathcal{N}}$ and $S_w = P_{\mathcal{N}} T_w|_{\mathcal{N}}$. The assumption that the dimension of $\mathcal{N}$ is finite implies $S_z \in C_0$, that is, $f(S_z) = 0$ for some function $f(z)$ in $H^\infty(\mathbb{D})$. Let $q_1(z)$ be the minimal function of S_z. Then $q_1(z)$ is a finite Blaschke product. Since $0 = q_1(S_z) = S_{q_1(z)} = P_{\mathcal{N}} T_{q_1(z)}|_{\mathcal{N}}$, we have $q_1(z)\mathcal{N} \subseteq \mathcal{M}$. Hence $q_1(z)H^2 \subseteq \mathcal{M}$. By this observation we have the following:

Lemma 1 *Let $\mathcal{M}$ be a Hardy submodule. Then* $\dim(H^2/\mathcal{M}) < +\infty$ *if and only if there exist two finite Blaschke products $q_1(z)$ and $q_2(w)$ such that*

$$q_1(z)H^2 + q_2(w)H^2 \subseteq \mathcal{M}.$$

For any Hardy submodule of finite codimension, we define two subspaces as follows:

$$\mathcal{M}_0 \equiv q_1(z)H^2 + q_2(w)H^2, \quad \mathcal{F}_{\mathcal{M}} \equiv \mathcal{M} \ominus \mathcal{M}_0,$$

where $q_1(z)$ and $q_2(w)$ are the minimal functions of S_z and S_w respectively. Let $H^2(z)$ be the usual one-variable Hardy space with the variable z. Similarly, we define $H^2(w)$ with the variable w. Since

$$\mathcal{F}_{\mathcal{M}} \subseteq \left(H^2 \ominus \mathcal{M}_0\right) = \left(H^2(z) \ominus q_1(z)H^2(z)\right) \otimes \left(H^2(w) \ominus q_2(w)H^2(w)\right),$$

we have $\dim \mathcal{F}_{\mathcal{M}} < +\infty$. Without loss of generality, we may assume that $q_1(0) = 0$ for the minimal function of S_z.

Lemma 2 *If $q_1(z)$ is a finite Blaschke product of degree k and $q_1(0) = 0$, then there exists a basis $\{e_i\}_{i=0}^{k-1}$ of $H^2(z) \ominus q_1(z)H^2(z)$ which satisfies*

$$\begin{cases} ze_{k-1} = q_1(z), \\ ze_i \in H^2(z) \ominus q_1(z)H^2(z) \quad (0 \le i \le k-2). \end{cases}$$

Next, we define an operator.

Definition 3 Let $q_1(z)$ and $q_2(w)$ be two finite Blaschke products, and $k = \deg q_1(z)$ and $l = \deg q_2(w)$. We define an operator as follows:

$$\begin{aligned} U_0 : q_1(z)H^2 + q_2(w)H^2 &\rightarrow z^k H^2 + w^l H^2, \\ q_1(z)f(z,w) &\mapsto z^k f(z,w), \\ q_2(w)w^j e_i &\mapsto z^i w^{j+l}, \end{aligned}$$

where $\{e_i\}_{i=0}^{k-1}$ is the basis of $H^2(z) \ominus q_1(z)H^2(z)$ obtained in Lemma 2. It is easy to check that U_0 is a unitary operator from $q_1(z)H^2 + q_2(w)H^2$ to $z^kH^2 + w^lH^2$.

Theorem 3 ([3]) *If $\mathcal{M} = q_1(z)H^2 + q_2(w)H^2$ for two finite Blaschke products $q_1(z)$ and $q_2(w)$ such that $\deg q_1(z) = k$ and $\deg q_2(w) = l$, then $\mathcal{A}(V_z, V_w; \mathcal{M})$ is unitarily equivalent to $\mathcal{A}(T_z|_{U_0\mathcal{M}}, T_w|_{U_0\mathcal{M}}; U_0\mathcal{M})$. Or equivalently, $\mathcal{A}(q_1(z)H^2 + q_2(w)H^2)$ is unitarily equivalent to $\mathcal{A}(z^kH^2 + w^lH^2)$.*

Proof $U_0V_zU_0^*$ and $U_0^*T_zU_0$ can be described as follows:

$$\begin{aligned} U_0V_zU_0^* &= T_z|_{z^kH^2} + \sum_{i=0}^{k-2}\left(\sum_{j=0}^{k-1} a_{i,j}T_z^{*i}T_z^j|_{w^lH^2(w)z^i}\right) + T_w^{*l}T_{q_2(w)}T_z|_{w^lH^2(w)z^{k-1}}, \\ U_0^*T_zU_0 &= V_z|_{q_1(z)H^2} + S + V_{q_2(w)}^*V_zV_w^l|_{q_2(w)H^2(w)e_{k-1}}, \end{aligned}$$

where S is a certain truncated shift operator associated with the basis $\{e_i\}$ obtained in Lemma 2. Then one can verify that $U_0V_zU_0^* \in \mathcal{A}(z^kH^2 + w^lH^2)$ and $U_0^*T_zU_0 \in \mathcal{A}(q_1(z)H^2+q_2(w)H^2)$ with some computations. Therefore $U_0\mathcal{A}(q_1(z)H^2+q_2(w)H^2)U_0^* = \mathcal{A}(z^kH^2 + w^lH^2)$. □

Moreover, by using Theorem 3, we have the following.

Theorem 4 ([3]) *Suppose that $\mathcal{M}$ is a Hardy submodule of finite codimension. Then $\mathcal{A}(V_z, V_w; \mathcal{M})$ is unitarily equivalent to $\mathcal{A}(T_z, T_w; H^2)$.*

As a corollary of Theorem 4, we have a commutative diagram.

Corollary 2 ([3]) *Let $\mathcal{K}(\mathcal{M})$ be the set of all compact operators on $\mathcal{M}$. If $\mathcal{M}$ is a Hardy submodule of finite codimension, then there exists a unitary operator U from $\mathcal{M}$ to H^2, and we have the following commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(\mathcal{M}) & \longrightarrow & \mathcal{A}(V_z, V_w) & \longrightarrow & \mathcal{A}(V_z, V_w)/\mathcal{K}(\mathcal{M}) & \longrightarrow & 0 \\ & & \Big\downarrow \text{Ad } U|_{\mathcal{K}(\mathcal{M})} & & \Big\downarrow \text{Ad } U & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathcal{K}(H^2) & \longrightarrow & \mathcal{A}(T_z, T_w) & \longrightarrow & \mathcal{A}(T_z, T_w)/\mathcal{K}(H^2) & \longrightarrow & 0. \end{array}$$

To classify Hardy submodules, Corollary 2 may be more important than Theorem 4.

References

[1] O. P. Agrawal, D. N. Clark and R. G. Douglas, *Invariant subspaces in the polydisk*, Proc. J. Math. **121**, (1986), pp. 1–11.

[2] C. A. Berger, L. A. Coburn and A. Lebow, *Representation and index theory for C^*-algebras generated by commuting isometries*, J. Funct. Anal. **27**, (1978), pp. 51–99.

[3] M. Seto, *On the Berger-Coburn-Lebow problem for Hardy submodules*, submitted to Canad. Math. Bull.

[4] R. Yang, *The Berger-Shaw theorem in the Hardy module over the bidisk*, J. Oper. Theory, **42**, (1999), pp. 379–404.

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Prime ideals and complex ring homomorphisms on commutative algebras

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Abstract. Let $\mathcal{A}$ be a commutative algebra and $\mathcal{P}$ a prime ideal of $\mathcal{A}$. We give a necessary and sufficient condition in order that $\mathcal{P}$ be the kernel of some complex ring homomorphism on $\mathcal{A}$.

$\mathcal{A},\mathcal{B}$ を可換（複素）多元環とする．ここでは特に単位元の存在を仮定していない．このとき $\rho\colon \mathcal{A}\to\mathcal{B}$ が環準同型写像であるとは，$\mathcal{A},\mathcal{B}$ を環とみたときの準同型写像であること，すなわち

$$\begin{aligned}\rho(f+g) &= \rho(f)+\rho(g)\\ \rho(fg) &= \rho(f)\,\rho(g)\end{aligned}$$

が全ての $f,g\in\mathcal{A}$ に対して成り立つことである．もちろん環準同型写像は多元環とは限らない環に対しても定義できる．

環準同型写像は，その定義から明らかなように，スカラー倍を保存するとは限らない．実際，1 次元複素 Banach 環，すなわち複素数体 $\mathbb{C}$ から $\mathbb{C}$ への環準同型写像は一般にスカラー倍を保存しない．例えば写像 $0,z,\bar{z}$ は複素数体 $\mathbb{C}$ 上の連続な環準同型写像であるが，$\bar{z}$ はスカラー倍を保存しない．ここに $\bar{\ }$ は複素共役である．逆に $\rho\colon\mathbb{C}\to\mathbb{C}$ が連続であれば，$\rho(z)=0$ $(z\in\mathbb{C})$, $\rho(z)=z$ $(z\in\mathbb{C})$, $\rho(z)=\bar{z}$ $(z\in\mathbb{C})$ のいずれかとなる．このことは $\mathbb{R}$ から $\mathbb{R}$ への零でない環準同型写像は恒等写像に限ることから直ちに示される．これらを $\mathbb{C}$ 上の自明な環準同型写像と呼ぶ．

これに対して，$\mathbb{C}$ 上には非自明な環準同型写像が存在することが知られている．Segre [9] は $\mathbb{C}$ 上には非自明な環準同型写像が存在しないと予想していたようである．また Dedekind

はSegreとは独立に，非自明な環準同型写像の存在を問題にしていたようである（cf. [1]）．1907年にLebesgue [8] はSegreの問題に答える形で非自明な環準同型写像の存在を示したが，ここではそのような写像が全射であるかどうかは言及されていない．現在では非自明な環準同型写像のいくつかの性質が知られているが（cf. [6]）まだ解明されていない部分が残されているようである．

例 1 (a) $G(\mathbb{C})$ を $\mathbb{C}$ 上の非自明な全射環準同型写像全体の集合とする．$\sharp S$ により集合 S の濃度を表すと，$\sharp G(\mathbb{C}) = 2^{\sharp\mathbb{C}}$ である（cf. [3]）．

(b) $\mathbb{C}$ 上の非自明な環準同型写像には全射でないものも存在する：それら全体の集合を $I(\mathbb{C})$ とすると $\sharp I(\mathbb{C}) \geq \sharp\mathbb{C}$ である．

(c) ρ を $\mathbb{C}$ 上の環準同型写像とすると，以下は同値である．

(i) ρ は自明である．

(ii) $m_0, L_0 > 0$ が存在して，$|z| < m_0$ ならば $|\rho(z)| < L_0$ となる．

(iii) ρ は原点で連続である．

(iv) ρ は各点で連続である．

(v) ρ は複素共役を保存する：$\rho(\bar{z}) = \overline{\rho(z)} \quad (z \in \mathbb{C})$．

したがって ρ が非自明であれば，複素数列 $\{z_n\}_{n\in\mathbb{N}}$ で

$$|z_n| < \frac{1}{n} \quad \& \quad |\rho(z_n)| > n \quad (n \in \mathbb{N})$$

をみたすものが存在し，ρ は各点において不連続であり，さらに複素共役を保存しない．

上記の性質をみても分かるように，非自明な環準同型写像は非常に複雑な構造をしている．したがって，より一般に無限次元のBanach環上の環準同型写像の構造を調べることは，

困難であるように思われるかもしれない．しかしながら，無限次元の場合は逆に自然な形をしていることもある．例えばArnold [2] は無限次元 Banach 空間 X, Y 上の有界線形作用素全体のなす Banach 環 $B(X), B(Y)$ に対して，全単射環準同型写像 $\rho\colon B(X) \to B(Y)$ は線形または共役線形であることを示した．Kaplansky [5] は Arnold の結果を次のように拡張した：A, B を半単純 Banach 環，$\rho\colon A \to B$ を全単射環準同型写像とすると $A = A_1 \oplus A_2 \oplus A_3$ とかける．ここに A_3 は有限次元であり ρ は A_1 上線形，A_2 上共役線形である．

このように無限次元 Banach 環上の環準同型写像は，ある条件のもとでは線形あるいは共役線形に近い振る舞いをすることが知られている．そこで全単射とは限らない環準同型写像の構造を調べることは非常に興味深い問題である．

我々は可換 Banach 環上の環準同型写像の構造を考察し，いくつかの結果を得ているが，環準同型写像 $\rho\colon C[0,1] \to C[0,1]$ についてでさえも完全には分かっていない．実際，$\mathbb{C}$ から $C[0,1]$ への環準同型写像で，定数でないものが存在するのかさえ分からない．上の ρ を決定する上で，$t \in [0,1]$ に対して $\rho_t(f) \stackrel{\text{def}}{=} \rho(f)(t)$ $(f \in C[0,1])$ を調べることは自然であろう．そこでより一般に，環準同型写像 $\rho\colon \mathcal{A} \to \mathbb{C}$ の構造を調べ，興味深い結果が得られたのでここに報告させていただく．

以下では $\mathcal{A}_e$ により，$\mathcal{A}$ に単位元 e を添加して得られる多元環を表す．ただし，$\mathcal{A}$ が既に単位元をもっているときは，$\mathcal{A}_e \stackrel{\text{def}}{=} \mathcal{A}$ と定義する．次の結果の本質的な部分は (c) $\Rightarrow$ (d) にある：大まかにいうと，集合の大きさが分かれば，代数構造を保存する写像を構成できることを主張している．このことは代数学でよく知られている超越基底（cf. [7]）を用いて証明される．

定理 1 $\mathcal{A}$ を可換複素多元環，$\mathcal{P}$ を $\mathcal{A}$ の多元環としての素イデアルとする．このとき以下

は同値である.

(a) $\ker\rho=\mathcal{P}$ となる環準同型写像 $\rho\colon \mathcal{A}\to\mathbb{C}$ が存在する.

(b) $\sharp(\mathcal{A}/\mathcal{P})=\sharp\mathbb{C}$ である.

(c) $\sharp(\mathcal{A}_e/\tilde{\mathcal{P}})=\sharp\mathbb{C}$ かつ $\mathcal{A}\cap\tilde{\mathcal{P}}=\mathcal{P}$ となる $\mathcal{A}_e$ の素イデアル $\tilde{\mathcal{P}}$ が存在する.

(d) $\mathcal{A}\cap\ker\tilde{\rho}=\mathcal{P}$ となる環準同型写像 $\tilde{\rho}\colon \mathcal{A}_e\to\mathbb{C}$ が存在する.

証明. $q\colon \mathcal{A}\to\mathcal{A}/\mathcal{P}$ を商写像とし, $a\in\mathcal{A}\setminus\mathcal{P}$ を固定する.

(a) $\Rightarrow$ (b) $z\in\mathbb{C}$ と $q(za)$ を同一視すれば

$$\sharp\mathbb{C}=\sharp\{q(za):z\in\mathbb{C}\}\leq\sharp(\mathcal{A}/\mathcal{P})$$

である. 逆の不等式を示す. $\mathcal{P}$ は素イデアルであるから, $\mathcal{A}/\mathcal{P}$ は整域となる. よって $\mathcal{A}/\mathcal{P}$ の商体 $\mathcal{F}$ が定義される. このとき

$$\tau(q(f)/q(g))\overset{\text{def}}{=}\frac{\rho(f)}{\rho(g)}\qquad(q(f)/q(g)\in\mathcal{F})$$

は well-defined な体の準同型写像であるから単射である. よって $\sharp(\mathcal{A}/\mathcal{P})\leq\sharp\mathcal{F}\leq\sharp\mathbb{C}$, すなわち $\sharp(\mathcal{A}/\mathcal{P})=\sharp\mathbb{C}$ となる.

(b) $\Rightarrow$ (c) $\sharp(\mathcal{A}/\mathcal{P})=\sharp\mathbb{C}$ とする. $\tilde{\mathcal{P}}\overset{\text{def}}{=}\{(f,z)\in\mathcal{A}_e:fa+za\in\mathcal{P}\}$ により $\tilde{\mathcal{P}}$ を定義すると, $\tilde{\mathcal{P}}$ は $\mathcal{A}_e$ の素イデアルで $\tilde{\mathcal{P}}\cap\mathcal{A}=\mathcal{P}$ をみたすことが分かる. $\tilde{q}\colon \mathcal{A}_e\to\mathcal{A}_e/\tilde{\mathcal{P}}$ を商写像とする. このとき

$$\alpha(\tilde{q}(f,z))\overset{\text{def}}{=}q(fa+za)\qquad(\tilde{q}(f,z)\in\mathcal{A}_e/\tilde{\mathcal{P}})$$

が単射となる. よって $\sharp(\mathcal{A}_e/\tilde{\mathcal{P}})\leq\sharp(\mathcal{A}/\mathcal{P})=\sharp\mathbb{C}$ である. また $f\in\mathcal{A}$ と $(f,0)\in\mathcal{A}_e$ を同一

視したとき $q(f) = q(g) \Leftrightarrow \tilde{q}(f,0) = \tilde{q}(g,0)$ である．つまり $\sharp(\mathcal{A}_e/\tilde{\mathcal{P}}) \geq \sharp(\mathcal{A}/\mathcal{P}) = \sharp\mathbb{C}$ である．以上により $\sharp(\mathcal{A}_e/\tilde{\mathcal{P}}) = \sharp\mathbb{C}$ を得る．

(c) $\Rightarrow$ (d) $\tilde{\mathcal{F}}$ を $\mathcal{A}_e/\tilde{\mathcal{P}}$ の商体とする．このとき簡単な計算により $\sharp\tilde{\mathcal{F}} = \sharp\mathcal{A}_e/\tilde{\mathcal{P}} = \sharp\mathbb{C}$ となることが分かる．$\mathbb{Q}$ を複素有理数全体のなす体，T_1 を $\tilde{\mathcal{F}}$ の $\mathbb{Q}$ 上の超越基底とする．このとき $\sharp T_1 = \sharp\tilde{\mathcal{F}} = \sharp\mathbb{C}$ となることが分かる．同様にして $\mathbb{C}$ の $\mathbb{Q}$ 上の超越基底 T_2 に対しても $\sharp T_2 = \sharp\mathbb{C}$ であることが示される．そこで全単射 $\theta\colon T_1 \to T_2$ が存在する．T_2 が $\mathbb{Q}$ 上の超越基底であることから（より正確には T_2 が $\mathbb{Q}$ 上代数的に独立であることから）θ は体の準同型写像 $\tilde{\theta}\colon \mathbb{Q}(T_1) \to \mathbb{Q}(T_2)$ で $\tilde{\theta}(r) = r$ $(r \in \mathbb{Q})$ をみたすものに一意的に拡張される．$\tilde{\mathcal{F}}$ は $\mathbb{Q}(T_1)$ の代数拡大体であり，$\mathbb{C}$ は代数的に閉じているから $\tilde{\theta}$ は $\tilde{\mathcal{F}}$ から $\mathbb{C}$ への体の準同型写像に拡張される（cf. [7, Theorem 2.8 of Chapter IV]）．この準同型写像も $\tilde{\theta}$ で表すことにすると，$\tilde{\rho} \stackrel{\mathrm{def}}{=} \tilde{\theta} \circ \tilde{q}$ が求めるものである．

(d) $\Rightarrow$ (a) $\rho \stackrel{\mathrm{def}}{=} \tilde{\rho}|_{\mathcal{A}}$ とおけば $\rho\colon \mathcal{A} \to \mathbb{C}$ は環準同型写像で $\ker\rho = \mathcal{A} \cap \ker\tilde{\rho} = \mathcal{P}$ である．
■

定理 1 から得られる系を述べる．この結果自身は既に [4] で得られている．

系 2 Ω を $\mathbb{C}$ の領域，$H(\Omega)$ を Ω 上の正則関数全体のなす多元環とすると，$H(\Omega)$ は環として $\mathbb{C}$ に埋め込まれる．

証明. 零関数だけからなるイデアル (0) は $H(\Omega)$ の素イデアルとなり，$\sharp H(\Omega) = \sharp\mathbb{C}$ なので，定理 1 より環準同型写像 $\rho\colon H(\Omega) \to \mathbb{C}$ で $\ker\rho = (0)$，すなわち単射なものが存在する．■

このような病的な現象の原因は値域が定数関数に限定されていることにある．実際，$H(\Omega)$ から $H(\Omega)$ への環準同型写像が非定数関数を含めば，正則関数に関する開写像定理を用い

て，自動的に線形か共役線形になることが示される (cf. [4]).

参考文献

[1] H. D. Ebbinghaus, et al., *Zahlen*, Springer-Verlag, Berlin, 1983.

[2] B. H. Arnold, *Rings of operators on vector spaces*, Ann. of Math. **45** (1944), 24-49.

[3] A. Charnow, *The automorphisms of an algebraically closed field*, Canad. Math. Bull. **18** (1970), 95-97.

[4] O. Hatori, T. Ishii, T. Miura and S. -E. Takahasi, *Characterization and automatic linearity for ring homomorphisms on algebras of functions*, to appear.

[5] I. Kaplansky, *Ring isomorphisms of Banach algebras*, Canad. J. Math. **6** (1954), 374-381.

[6] H. Kestelman, *Automorphisms of the field of complex numbers*, Proc. London Math. Soc. (2) **53** (1951), 1-12.

[7] S. Lang, *Algebra* (second edition), Addison-Wesley, California, 1984.

[8] M. H. Lebesgue, *Sur les transformations ponctuelles, transformaant les plans en plans, qu'on peut définir par des procédés analytiques*, Atti della R. Acc. delle Scienze di Torino **42** (1907), 532-539.

[9] C. Segre, *Un nuovo campo di ricerche geometriche*, Atti della R. Acc. delle Scienze di Torino **25** (1889), 276-301.

RADIAL GROWTH OF C^2 FUNCTIONS SATISFYING BLOCH TYPE CONDITION

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ABSTRACT. The aim of this paper is to give a simple proof of results by González-Koskela concerning the radial growth of C^2 functions satisfying Bloch condition. Our results also give generalizations of their results.

1. INTRODUCTION

Denote by $\mathscr{B}$ the Bloch space of all holomorphic functions f on the unit disk U which satisfy

$$||f||_{\mathscr{B}} = |f(0)| + \sup_{z\in U}(1-|z|^2)|f'(z)| < \infty.$$

The radial growth of Bloch functions was extensively discussed by Clunie-MacGregor [2], Korenblum [4], Makarov [5] and Pommerenke [7]. The law of the iterated logarithm of Makarov [5] states that if $f \in \mathscr{B}$, then

$$\limsup_{r\to 1} \frac{|f(r\zeta)|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \le C||f||_{\mathscr{B}} \tag{1}$$

for almost every $\zeta \in \partial U$, where C is a universal constant. Pommerenke [7] proved that this inequality is true for $C = 1$ and this inequality is false for $C \le 0.685$. Recently, González and Koskela studied the radial growth of C^2 functions on the unit ball $\mathbf{B}^n$ of $\mathbf{R}^n$ which satisfy

$$|\nabla u(x)|^2 + |u(x)\Delta u(x)| \le \frac{c}{(1-|x|)^2\left(\log\frac{2}{1-|x|}\right)^{\gamma}} \tag{2}$$

for all $x \in \mathbf{B}^n$, where $c > 0$ and $\gamma \le 1$. They showed the following result ([3, Theorem 1.2]).

THEOREM A. *Let u be a C^2 function on $\mathbf{B}^n$ satisfying* (2). *Then, for almost all ζ, $|\zeta| = 1$,*

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\left(\log\frac{1}{1-r}\right)^{1-\gamma}\log\log\frac{1}{1-r}}} \le c_1$$

if $\gamma < 1$; and

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\log\log\frac{1}{1-r}} \le c_2$$

if $\gamma = 1$. Here the constants c_1 and c_2 depend only on n, c, γ.

We denote by $B(x,r)$ and $S(x,r)$ the open ball and the sphere of center x and radius r, respectively. We set $\mathbf{B}^n = B(0,1)$ and $\mathbf{S}^{n-1} = S(0,1)$. The Hausdorff measure with a measure function h is written as H_h. In case $h(r) = r^\alpha$, we write H_α for H_h.

Our first aim in the present note is to extend Theorem A by González-Koskela. For this purpose, let φ be a positive, continuous and non-decreasing function on the interval $[0,1)$ satisfying

$$\varphi(1-r/2) \le A\varphi(1-r) \text{ for every } r \in (0,1) \tag{3}$$

with a constant $A \ge 1$ and

$$\int_0^1 (1-t)\varphi(t)\,dt = \infty. \tag{4}$$

Set

$$\Phi(r) = \int_0^r (1-t)\varphi(t)\,dt.$$

THEOREM 1. *Let u be a C^2 function on $\mathbf{B}^n$ with $u(0) = 0$ such that*

$$\mathscr{A}_u(x) = |\nabla u(x)|^2 + |u(x)\Delta u(x)| \le \varphi(|x|) \text{ for all } x \in \mathbf{B}^n. \tag{5}$$

Then for $\mathcal{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$,

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r)\log\log\frac{1}{1-r}}} \le \sqrt{A}.$$

REMARK 1. If we take $\varphi(r) = c(1-r)^{-2}\{\log(2/(1-r))\}^{-\gamma}$ for $c > 0$ and $\gamma \le 1$, then Theorem 1 gives Theorem A.

On the other hand, we have the lower limit result as follows:

THEOREM 2. *If u is as in Theorem 1, then*

$$\liminf_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r)\log\log\Phi(r)}} \le 2$$

for $\mathcal{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$.

By Theorems 1 and 2, we have the following corollary.

COROLLARY 1. *Let u be a C^2 function on $\mathbf{B}^n$ satisfying*

$$\mathscr{A}_u(x) \le c(1-|x|)^{-2}\left(\log_{(1)}\frac{1}{1-|x|}\right)^{-1}\cdots\left(\log_{(\ell-1)}\frac{1}{1-|x|}\right)^{-1}\left(\log_{(\ell)}\frac{1}{1-|x|}\right)^{-\gamma},$$

where $c > 0$, $\gamma \le 1$ and $\log_{(k+1)}(t) = \log_{(k)} \circ \log_{(1)}(t)$ with $\log_{(1)}(t) = \log(e+t)$. Then for $\mathcal{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$,

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\left(\log_{(\ell)}\frac{1}{1-r}\right)^{1-\gamma}\log_{(2)}\frac{1}{1-r}}} \le c_1$$

and

$$\liminf_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\left(\log_{(\ell)} \frac{1}{1-r}\right)^{1-\gamma} \log_{(\ell+2)} \frac{1}{1-r}}} \le c_2$$

when $\gamma < 1$;

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\log_{(\ell+1)} \frac{1}{1-r} \log_{(2)} \frac{1}{1-r}}} \le c_3$$

and

$$\liminf_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\log_{(\ell+1)} \frac{1}{1-r} \log_{(\ell+3)} \frac{1}{1-r}}} \le c_4$$

when $\gamma = 1$. *Here* c_1, c_2, c_3 *and* c_4 *are constants depending only on* c, γ *and* ℓ.

2. Exponential integral

In this section, we present an exponential estimate for C^2 functions satisfying (5). For this we prepare the following lemma, which is a generalization of [3, Theorem 2.2].

Lemma 1. *Let* φ *be a positive continuous function on* $[0, 1)$, *and set*

$$\Phi(r) = \int_0^r (1-t)\varphi(t)\, dt.$$

Let u *be a* C^2 *function in* $\mathbf{B}^n$ *with* $u(0) = 0$ *which satisfies condition* (5). *Then*

$$\int_{\mathbf{S}^{n-1}} |u(r\zeta)|^{2k}\, dS(\zeta) \le \sigma_n 4^k k!\, [\Phi(r)]^k \tag{6}$$

for all $k \in \{0, 1, 2, \dots\}$ *and all* $r \in (0, 1)$, *where* σ_n *denotes the surface measure of* $\mathbf{S}^{n-1}$.

Proof. Using the divergence theorem, we have

$$\frac{d}{dt} \int_{\mathbf{S}^{n-1}} v(t\zeta)\, dS(\zeta) = t^{1-n} \int_{B(0,t)} \Delta v(w)\, dw \tag{7}$$

for each $v \in C^2(\mathbf{B}^n)$.

We prove this lemma by induction on k. Clearly, (6) holds for $k = 0$. Suppose that (6) holds for k. Using (7) and the assumption on induction, we obtain

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbf{S}^{n-1}} |u(t\zeta)|^{2(k+1)}\, dS(\zeta) \\
= \;& 2(k+1)t^{1-n} \int_{B(0,t)} |u(w)|^{2k} \left(u(w)\Delta u(w) + (2k+1)|\nabla u(w)|^2\right) dw \\
\le \;& 4(k+1)^2 t^{1-n} \int_{B(0,t)} |u(w)|^{2k} \mathscr{A}_u(w)\, dw \\
\le \;& 4(k+1)^2 t^{1-n} \int_0^t \rho^{n-1}\varphi(\rho) \left(\int_{\mathbf{S}^{n-1}} |u(\rho z)|^{2k}\, dS(z)\right) d\rho \\
\le \;& \sigma_n 4^{k+1} k!\, (k+1)^2\, t^{1-n} \int_0^t \rho^{n-1}\varphi(\rho)[\Phi(\rho)]^k\, d\rho.
\end{aligned}$$

Integrating both sides from 0 to r and applying Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbf{S}^{n-1}} |u(r\zeta)|^{2(k+1)}\, dS(\zeta) &\le \sigma_n 4^{k+1} k!\, (k+1)^2 \int_0^r t^{1-n} \int_0^t \rho^{n-1}\varphi(\rho)[\Phi(\rho)]^k\, d\rho\, dt \\ &= \sigma_n 4^{k+1} k!\, (k+1)^2 \int_0^r \left(\int_\rho^r t^{1-n}\, dt\right) \rho^{n-1}\varphi(\rho)[\Phi(\rho)]^k\, d\rho \\ &\le \sigma_n 4^{k+1}(k+1)! \int_0^r (k+1)(1-\rho)\varphi(\rho)[\Phi(\rho)]^k\, d\rho \\ &= \sigma_n 4^{k+1}(k+1)! \int_0^r \frac{d}{d\rho}[\Phi(\rho)]^{k+1}\, d\rho \\ &= \sigma_n 4^{k+1}(k+1)!\, [\Phi(r)]^{k+1}.\end{aligned}$$

Hence (6) also holds for $k+1$. The induction is completed. □

LEMMA 2. *Let u be a function in $\mathbf{B}^n$ satisfying condition* (6). *Then for all c, $0 < c < 1/4$, and for all r, $0 < r < 1$,*

$$\int_{\mathbf{S}^{n-1}} \exp\left(\frac{c|u(r\zeta)|^2}{\Phi(r)}\right) dS(\zeta) \le \frac{\sigma_n}{1-4c}. \tag{8}$$

3. Proof of Theorem 1

Let φ and Φ be as in the Introduction, and let u be as in Theorem 1.

To prove Theorem 1, we need the following lemma.

LEMMA 3. *Let u be a C^2 function in $\mathbf{B}^n$ such that $|\nabla u(x)|^2 \le \varphi(|x|)$. Then for every $x \in \mathbf{B}^n \setminus \overline{B(0,1/2)}$,*

$$|u(y) - u(z)| \le A[\Phi(|x|)]^{1/2}$$

whenever $y, z \in B(x, (1-|x|)/2)$.

Proof of Theorem 1. From Lemma 2, we see that

$$\int_{\mathbf{B}^n} (1-|x|)^{-1} \left(\log\frac{2}{1-|x|}\right)^{-1-\delta} \exp\left(\frac{c|u(x)|^2}{\Phi(|x|)}\right) dx < \infty$$

for all c, $0 < c < 1/4$, and all $\delta > 0$. Then there exists a set $E \subset \mathbf{S}^{n-1}$ such that $\mathcal{H}_{n-1}(E) = 0$ and

$$\int_0^1 (1-r)^{-1} \left(\log\frac{2}{1-r}\right)^{-1-\delta} \exp\left(\frac{c|u(r\zeta)|^2}{\Phi(r)}\right) dr < \infty,$$

for each $\zeta \in \mathbf{S}^{n-1} \setminus E$, $0 < c < 1/4$ and $\delta > 0$, which implies that

$$\lim_{r\to 1} \int_r^{(1+r)/2} (1-t)^{-1} \left(\log\frac{2}{1-t}\right)^{-1-\delta} \exp\left(\frac{c|u(t\zeta)|^2}{\Phi(t)}\right) dt = 0. \tag{9}$$

Fix $\zeta \in \mathbf{S}^{n-1} \setminus E$. For $0 < r < 1$, define $I_r = [r, (1+r)/2)$. From (9), we obtain

$$\lim_{r\to 1} \left(\log\frac{1}{1-r}\right)^{-1-\delta} \exp\left(\frac{c \inf_{t\in I_r} |u(t\zeta)|^2}{\Phi((1+r)/2)}\right) = 0,$$

which implies that

$$\frac{c\inf_{t\in I_r}|u(t\zeta)|^2}{\Phi((1+r)/2)} \le (1+\delta)\log\log\frac{1}{1-r} \tag{10}$$

for r near 1. Hence it follows from (10) and Lemma 3 that

$$\limsup_{r\to 1}\frac{|u(r\zeta)|}{\sqrt{\Phi(r)\log\log\frac{1}{1-r}}} \le \sqrt{\frac{A(1+\delta)}{4c}}.$$

Here, letting $c \to 1/4$ and $\delta \to 0$, we obtain

$$\limsup_{r\to 1}\frac{|u(r\zeta)|}{\sqrt{\Phi(r)\log\log\frac{1}{1-r}}} \le \sqrt{A},$$

which proves Theorem 1. □

4. Proof of Theorem 2

In this section we complete the proof of Theorem 2.

By Lemma 2, we see that

$$\int_{\mathbf{B}^n\setminus B(0,r_0)}(1-|x|)\varphi(|x|)\Phi(|x|)^{-1}(\log\Phi(|x|))^{-1-\delta}\exp\left(\frac{c|u(x)|^2}{\Phi(|x|)}\right)dx < \infty$$

for all c, $0<c<1/4$, and $\delta>0$, where $r_0 = \Phi^{-1}(e)$. Consequently,

$$\lim_{r\to 1}\int_r^1(1-t)\varphi(t)\Phi(t)^{-1}(\log\Phi(t))^{-1-\delta}\exp\left(\frac{c|u(t\zeta)|^2}{\Phi(t)}\right)dt = 0$$

for $\mathcal{H}_{n-1}$-a.e. $\zeta\in\mathbf{S}^{n-1}$, $0<c<1/4$ and $\delta>0$. This implies that

$$\lim_{r\to 1}\int_r^1(1-t)\varphi(t)\Phi(t)^{-1}(\log\Phi(t))^{-1-\delta}\exp\left(\frac{cg_r(\zeta)^2}{\Phi(t)}\right)dt = 0, \tag{11}$$

where $g_r(\zeta) = \inf_{r\le\rho<1}|u(\rho\zeta)|$. Since $e^{\delta x}\ge\delta x$ for $x>0$, we have

$$\begin{aligned}&\frac{d}{dt}\left(-(\log\Phi(t))^{-1-\delta}\exp\left(\frac{(1+\delta)^{-1}cg_r(\zeta)^2}{\Phi(t)}\right)\right)\\ \le\ &(1+\delta+\delta^{-1})(1-t)\varphi(t)\Phi(t)^{-1}(\log\Phi(t))^{-1-\delta}\exp\left(\frac{cg_r(\zeta)^2}{\Phi(t)}\right)\end{aligned}$$

for $r_0<t<1$. From (11), we obtain

$$\lim_{r\to 1}(\log\Phi(r))^{-1-\delta}\exp\left(\frac{(1+\delta)^{-1}cg_r(\zeta)^2}{\Phi(r)}\right) = 0,$$

which implies that

$$\frac{(1+\delta)^{-1}cg_r(\zeta)^2}{\Phi(r)} \le (1+\delta)\log\log\Phi(r)$$

for r near 1. By letting $c\to 1/4$ and $\delta\to 0$, we have

$$\limsup_{r\to 1}\frac{g_r(\zeta)^2}{\Phi(r)\log\log\Phi(r)} \le 4,$$

which completes the proof of Theorem 2. $\square$

COROLLARY 2. *Let u be a harmonic function on $\mathbf{B}^n$ satisfying*

$$|\nabla u(x)|^2 \le \varphi(|x|) \text{ for all } x \in \mathbf{B}^n.$$

Then for $\mathcal{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$,

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r)\log\log\Phi(r)}} \le 2.$$

5. HAUSDORFF MEASURES AND RADIAL GROWTH

Take a positive non-decreasing function Ψ on $[0,1)$ satisfying

$$\frac{\Phi(r)\log\log(1/(1-r))}{[\Psi(r)]^2} \to 0 \text{ as } r \to 1.$$

For $\lambda > 0$, consider the measure function h_λ such that

$$h_\lambda(t) = t^{n-1}\exp\left(4^3 A^{-4}\lambda^2 \frac{[\Psi(1-t)]^2}{\Phi(1-t)}\right).$$

We finally establish the following result.

THEOREM 3. *If $\lambda > 0$ and u is as in Theorem 1, then*

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\Psi(r)} \le \lambda$$

for $\mathcal{H}_{h_\lambda}$-a.e. $\zeta \in \mathbf{S}^{n-1}$.

REFERENCES

[1] S. Axler, P. Bourdon and W. Ramey, Harmonic function theory, Springer-Verlag, 1992.

[2] J. G. Clunie and T. H. MacGregor, Radial growth of the derivative of univalent functions, Commentari Math. Helv. **59** (1984), 362-375.

[3] M. J. González and P. Koskela, Radial growth of solutions to the Poisson Equation, Complex Variables, **46** (2001), 59-72.

[4] B. Korenblum, BMO estimates and radial growth of Bloch functions, Bull. Amer. Math. Soc. **12** (1985), 99-102.

[5] N. G. Makarov, On the distortion of boundary sets under conforml mappings, Proc. London Math. Soc. **51** (1985), 369-384.

[6] Y. Mizuta, Potential theory in Euclidean spaces, Gakkōtosyo, Tokyo, 1996.

[7] Ch. Pommerenke, Boundary behaviour of conformal maps, Springer, 1992.

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Hulls and kernels on compact spaces with actions

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Abstract. For a topological dynamical system $\Sigma = (X, \sigma)$ where σ is a homeomorphism in an arbitrary compact Hausdorff space X, we consider the noncommutative hulls and kernels with respect to the action σ in the associated C*-algebra $A(\Sigma)$. We shall present a part of the following results obtained recently ; several ideals important for the structure of $A(\Sigma)$ have the form of such kernels with topological characterizations of their hulls from the behavior of orbits in the dynamical system

1. はじめに。空間 X をコンパクト、ハウスドルフ空間、$C(X)$ をその上の連続関数の空間としたとき、良く知られた hull-kernel は X の部分集合 S と $C(X)$ のイデヤル J について次のように定義されている；

$$k(S) = \{f \in C(X) | \quad f|S = 0\}$$

また

$$h(J) = \{x \in X | f(x) = 0 \quad \forall f \in J\}$$

そして両者の基本関係は、

$$hk(S) = \overline{S}, \quad kh(J) = J$$

で与えられている。これらは色々な場面で大事な役割を演じているが、ここでは空間 X に作用（同相写像 σ ）が与えられているとき上の hull-kernel に対応する概念をどの様に与え、また有効に使えるかを考える。単純に考えれば S を不変集合とし、J を不変なイデヤルにすることが考えられるが、それでこれ以上は何も発展しないので、 C*-環の枠組みをと

る。その理由は力学系　$\Sigma = (X, \sigma)$ を考えることと $C(X)$ と共にその上の

$$\alpha(f)(x) = f(\sigma^{-1}(x))$$

で与えられる同型対応 α を考えることは原理的に同値であるからである。しかしこのままでは $\{C(X), \alpha\}$ は関数と同型対応というレベルの違った組み合わせになっているのでこの両者をヒルベルト空間上の有界線形作用素という同じ種類のものに変換する事を考える。その為には先ずその情報の一部を担うものとして共変表現の概念を導入する。

$B(H)$ をヒルベルト空間 H 上の有界線形作用素のつくるバナッハ環とする。これはそこでのノルムで所謂 C*-環になっている。$\{\pi, u\}$ を $C(X)$ の H 上への表現と H 上のユニタリ作用素 u の組とするとき、これが $\{C(X), \alpha\}$ の共変表現とは

$$\pi(\alpha(f)) = u\pi(f)u^{*} \quad \forall f \in C(X)$$

が成り立つ事を言う。そこで更にこれらをすべて集めた C*-環 $A(\Sigma)$ を考える（構成的に作れる）。それは次の性質を持つ。

(1) $A(\Sigma)$ は $C(X)$ と同型対応 α を implement するユニタリ元 δ で生成される、

(2) $A(\Sigma)$ より $C(X)$ へ Banach 空間としてのノルムが 1 の射影 E が存在する、

(3) 共変表現について universal property を持つ。

ここで α と δ の関係から、

$$\delta g = \delta g \delta^{*} \delta = \alpha(g)\delta$$

となるので、（1）での生成は実は線形集合

$$\{\sum_{-n}^{n} f_i \delta^i \quad | f_i \in C(X)\}$$

のノルム closure となっていて非常に計算し易い形になっている。また E は $C(X)$-module の性質を持ち、力学系の情報を $A(\Sigma)$ まで持ち上げ

る役割を果たしている。更にC*-環　$A(\Sigma)$ の元 a は次の式で定義された一般化されたフーリエ係数　$a(n) = E(a\delta^{*n})$ を持つのでこれを使って（いわば非可換の）Hull-Kernel を次のように定める；

S を X の不変部分集合、I を $A(\Sigma)$ のイデヤルとする。

$$Ker(S) = \{a \in A(\Sigma) \quad | \quad a(n)|S = 0, \forall n\}$$

$$Hull(I) = \{x \in X \quad | \quad a(n)(x) = 0,\ a \in I \text{ and } n\}$$

しかしここで基本関係は、

$$Hull(Ker(S)) = \overline{S}$$

とはなるが、他の一つは成り立たず最悪の時には　$Ker(Hull(I))$ は $A(\Sigma)$ 全体にもなる。そのほか $Ker(S)$ がイデヤルになるのもそれ程明らかではない。これらを克服したあと本講演では次の問題（の一部）を論ずる。

(a) 力学系の基本の集合やそれらの差の集合の Kernel ideal はどんな性質を持つか？

(b) $A(\Sigma)$ の中のC*-環的な基本のイデヤルの Hull はどんな力学系的な性質をもつか？

2. 結果

先ず次のことが言える。以下 $A(\Sigma)$ の閉イデヤルは単にイデヤルということにする。

命題１ (1) S が X の不変集合のとき $Ker(S)$ は $A(\Sigma)$ のイデヤルとなる、

(2) I を $A(\Sigma)$ のイデヤルとすると、$Hull(I)$ は X の閉不変部分集合である。

ここで周期点を持つ力学系では

$$Ker(Hull(I)) = A(\Sigma)$$

ともなるような性質の悪いイデヤルがでてくるが、イデヤルの中で普通の hull-kernel と同じような性質を持つような類について次の結果が得られる。

定理２　$A(\Sigma)$ のイデヤル I について次のことは同値である、
(1)

$$\begin{aligned} I &= Ker(Hull(I)) \\ &= [\ \sum_{-n}^{n} f_i\delta^i \quad | f_i \in k(Hull(I))] \end{aligned}$$

(closed linear span),
(2) $E(I) \subset I$,
(3) I は $A(\Sigma)$ へのトーラスの双対作用 $\hat{\alpha}_t$ により不変である。
ここで双対作用とは

$$\hat{\alpha}_t(f) = f \quad (f \in C(X)), \quad \hat{\alpha}_t(\delta) = e^{2\pi i t}\delta$$

の共変関係で定められる $A(\Sigma)$ の同型対応類である。

この議論をもっと精密化すると力学系が周期点を持たないときには $A(\Sigma)$ のイデヤルはすべてこの形になることが分かる。

力学系の周期点全体の集合を、$Per(\sigma)$, 非周期点全体の集合を $Aper(\sigma)$, と書くことにする。更に $A(\Sigma)$ の有限次元既約表現のすべての核の共通分、また無限次元既約表現のすべての核の共通分をそれぞれ I_F,I_∞ とすると次がなりたつ。

定理３

$$(1) \quad I_F = Ker(Per(\sigma)), \quad (2) \quad I_\infty = Ker(Aper(\sigma))$$

ここで $A(\Sigma)$ の有限次元既約表現はすべて X の点から引き起こされる（詳細略）ことが分かっているので（１）の主張は自然な結果であるが、空間が距離空間でないときには無限次元既約表現の核の Hull が必ずしも X の点の軌道できまらないので、（２）は非常に自明でない結果である。

上の結果から分かることは $Per(\sigma)$ や $Aper(\sigma)$ の大きさ（density など）は力学系に対応する C*-環の同型では変わらない、いわば代数的な不変量であることである。現在ではまだ C*- 環の同型対応によって下の力学系がどの様な関係になるかは、空間が単位円周の時のみしか分かっていないので、このような結果が大事になる。

I_F が　0-イデヤルになるとき、即ち周期点が　dense になるとき（Bernoulli shift や極端には有理数回転など）には上のことから有限次元

既約表現が十分沢山存在する事が分かる。このような C*-環は Residually finite dimensional C*-algebra と呼ばれ適度に複雑でそれでも制御し易いクラスとして C*-環論のなかで大事な役割を演じている。

一方 I_∞ が 0-イデヤルになるときは非周期点の集合が dense にありこのような力学系を筆者は Topologically free　な系と呼んだ。このときは $A(\Sigma)$ に無限次元既約表現が十分沢山存在する。一般に多様体上の力学系では通常周期点は高々可算個しかないのでそれらは皆このクラスに属し、広すぎるためか普通の力学系の議論ではこのクラスは取り扱われていない。しかしこのとき $A(\Sigma)$ は C*-環として非常によい性質を持ちこのクラスは C*-環論の良い相手役になっている。

文献

1J.Tomiyama, The interplay between topological dynamics and theory of C*-algebras, Lecture note No.2,Res. Inst. Math. Seoul,1992.

2.J.Tomiyama, Structure of ideals and isomorphism problems of C*-crossed products by single homeomorphisms, Tokyo J. Math., 23(2000), 1-13.

3.J.Tomiyama, Hulls and kernels from topological dynamical systems and their applications to C*-algebras, to appear in J.Math.Soc. Japan

ON COMPONENTS AND POWER COMPACTNESS OF COMPOSITION OPERATORS ON THE DISC ALGEBRA

TAKUYA HOSOKAWA

1. Introduction

Let D be the open unit disc and ∂D be the unit circle. Let H^∞ be the set of all bounded analytic functions on D and A be the disc algebra. Let $S(D)$ be the set of all analytic self-map of D. Then $S(D)$ is the closed unit ball of H^∞. If $\varphi \in S(D)$, this self-map induces the composition operator C_φ which acts on H^∞. Similarly denote $S(\overline{D})$ the closed unit ball of A. Then $\varphi \in S(\overline{D})$ induces C_φ which acts on A. It is easy to see that the operator norm of all C_φ is 1 on the both spaces H^∞ and A. If $\varphi(z) \equiv \omega \in \partial D$, then φ is not in $S(D)$ but in $S(\overline{D})$, which acts as the point evaluation at ω. By the Maximum Modulus Principle, it is shown that $S(\overline{D}) \setminus \partial D \subsetneq S(D)$.

For z and w in D, the pseudohyperbolic distance β on D is defined as

$$\beta(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|$$

and for $z \in \partial D$ and $w \in \overline{D}$, $z \neq w$ we can define $\beta(z,w) = 1$, $\beta(z,z) = 0$. Denote the metric $d_\beta(\varphi,\psi) = \sup\{\beta(\varphi(z),\psi(z)) : z \in D\}$ for $\varphi, \psi \in S(\overline{D})$. $S(\overline{D}, d_\beta)$ is the topological space of $S(\overline{D})$ with topology induced by d_β.

Let $\mathcal{C}(A)$ be the collection of all composition operators on A, endowed with the operator norm. Let $B = \{C_\varphi \in \mathcal{C}(A) : \varphi \equiv \omega \in \partial D\}$ and K be the collection of all compact composition operators on $\mathcal{C}(A)$. Then $B \subset K$. We denote $C_\varphi \sim C_\psi$ in $\mathcal{C}(A)$ if C_φ and C_ψ are in the same component of $\mathcal{C}(A)$. In next section, we will study on the components and the isolated points of $\mathcal{C}(A)$.

The bounded linear operator T on X is called to be power compact if T^n is compact on X for some positive integer n. Especially, T is called to be n-power compact if T^{n-1} is not compact and T^n is compact on X. It is known that C_φ is compact on A if and only if $\|\varphi\|_\infty < 1$. Therefore C_φ is

n-power compact on A if and only if $\|\varphi^{n-1}\|_\infty = 1$ and $\|\varphi^n\|_\infty < 1$, where φ^n is the n-th iteration of φ.

The eigenvalue equation for composition operators : $C_\varphi f = f \circ \varphi = \lambda f$. This eigenvalue equation is called Schröder's equation. In 1884, Gabriel Königs solved Schröder equation for $\varphi \in (D)$ with its fixed point in D.

Theorem (Königs, 1884). *Suppose that $\varphi \in S(D)$ is a non-constant, non-automorphic such that $\varphi(p) = p \in D$ and consider C_φ as a linear transformation of $H(D)$ of all analytic functions on D.*

(i) *If $\varphi'(p) = 0$, then 1 is the only eigenvalue of C_φ.*
(ii) *If $\varphi'(p) \neq 0$, then there exists a principal eigenfunction σ analytic on D, $\sigma'(p) = 1$ of Schröder equation with $\lambda = \varphi'(p)$ such that the eigenvalues are precisely $\{\lambda^n\}_0^\infty$, and the coresponding eigenfunction for the eigenvalue λ^n is $\sigma(z)^n$ with the each of multiplicity one.*
(iii) *If φ is univalent, then so is σ.*

We call σ above the Königs function. More precisely, if $p = 0$ then σ is the limit function of

$$\sigma_n(z) = \frac{\varphi^n(z)}{\lambda^n} \tag{1}$$

This limit converges uniformly on compact subsets of D. We call $\{\sigma_n\}$ the normalized Königs seuence.

Let α_w, $w \in D$, be an automorphism exchanging 0 for w defined by

$$\alpha_w = \frac{w - z}{1 - \overline{w}z}. \tag{2}$$

Hence if the fixed point $p \neq 0$, put $\psi = \alpha_p \circ \varphi \circ \alpha_p$. Since $\psi(0) = 0$, we can construct its normalized Königs sequence τ_n and its Königs function τ on the way above. Putting $\sigma_n = c^{-1}\tau_n \circ \alpha_p$ and $\sigma = c^{-1}\tau \circ \alpha_p$ where $c = \alpha_p'(p) = (|p|^2 - 1)^{-1}$, then σ is the Königs function of φ and it is easy to see that σ_n converges to σ uniformly on compact subsets of D.

In the last section, we will consider the relation between the power compactness of composition operators on A and its Königs function.

2. Topological structure of $\mathcal{C}(A)$

In [6] and [5], the topological structure of $\mathcal{C}(H^\infty)$ is studied. We can get similar results on the topological structure of $\mathcal{C}(A)$.

Theorem 1. *Let C_φ, C_ψ be in $\mathcal{C}(A)$.*

(i) $\|C_\varphi - C_\psi\| = \dfrac{2 - 2\sqrt{1 - d_\beta(\varphi,\psi)^2}}{d_\beta(\varphi,\psi)}$

(ii) $C_\varphi \sim C_\psi$ *in* $\mathcal{C}(A) \iff \|C_\varphi - C_\psi\| < 2$

(iii) C_φ *is an isolated point of* $\mathcal{C}(A) \iff$ *for all* $C_\psi \neq C_\varphi$, $\|C_\varphi - C_\psi\| = 2$ $\iff \varphi$ *is an extreme point of* $S(\overline{D}) \iff \displaystyle\int_0^{2\pi} \log\left(1 - |\varphi(e^{i\theta})|\right) d\theta = -\infty$.

(iv) *Every* $C_\varphi \in B$ *is compact on* A *and isolated in* $\mathcal{C}(A)$.

(v) $K \setminus B$ *itself is a component of* $\mathcal{C}(A)$.

Remark. (i) implies that $\mathcal{C}(A, \|\cdot\|)$ is homeomorphic to $S(\overline{D}, d_\beta)$. (ii) implies that $\mathcal{C}(A)$ is locally path connected, and then all components of $\mathcal{C}(A)$ is path connected. In [1], H. Chandra showed a part of the statement of (iii), that is, if C_φ is an isolation, then φ is an extreme point of $S(\overline{D})$

3. The itineraries of the power compact composition operators

In general, the collection of composition operators is not a linear space but is a semigroup under the multiplication $C_\varphi : C_\psi \mapsto C_\varphi C_\psi$. Hence we can consider the dynamical systems on $\mathcal{C}(A)$. Theorem 1 implies the following proposition.

Proposition 1. *Let $C_{\varphi_1} \sim C_{\varphi_2}$ and $C_{\psi_1} \sim C_{\psi_2}$ in $\mathcal{C}(A)$. Then $C_{\varphi_1}C_{\psi_1} \sim C_{\varphi_2}C_{\psi_2}$ in $\mathcal{C}(A)$.*

Let $\iota : \mathcal{C}(A) \to \mathcal{C}(A)/\sim$ be the cannonical map. We call the sequence $\{\iota\left(C_\varphi^n(C_\psi)\right)\}$ in $\mathcal{C}(A)$ the itinerary of C_ψ generated by C_φ. By the proposition above, we can assert that the equivalent relation, that is to be in the same component in $\mathcal{C}(A)$, keeps the itineraries, as following.

Theorem 2. *If $C_{\varphi_1} \sim C_{\varphi_2}$ and $C_{\psi_1} \sim C_{\psi_2}$ in $\mathcal{C}(A)$, then $\iota(C_{\varphi_1}^n(C_{\psi_1})) = \iota(C_{\varphi_2}^n(C_{\psi_2}))$ for all positive integer n.*

We will study on the simply generated itinerary $\iota(C_\varphi^n)$. At first, consider in the itineraries of the compact composition operators.

Proposition 2. (i) *For all* $C_\varphi \in B$ *and all* $n \geq 1$, $C_\varphi^n = C_\varphi$.

(ii) *For all* $C_\varphi \in K \setminus B$ *and all* $n \geq 1$, $\iota(C_\varphi^n) = K \setminus B$

Next we will consider the itineraries of the pwer compact composition operators. The example below guarantees that for any positive integer n there exists at least a n-power compact composition operator on A.

Example 1. (i) Let $\varphi(z) = i(z+1)/2$. Then C_φ is 2-power compact on A.

(ii) Let φ_α be a (enough thin) lens map which fixes -1, 0, and 1, and $\varphi(z) = \frac{-1+i}{2}\varphi_\alpha(z) + \frac{1+i}{2}$. Then C_φ is 3-power compact on A.

(iii) For $n \leq 4$, let $P_n = \{1, e^{2\pi i/n}, e^{2\cdot 2\pi i/n}, \cdots e^{(n-2)2\pi i/n}\}$ and φ_n be the Riemann mapping from D to the $(n-1)$th polygon whose vertexes are just P_n and which fixes every point of P_n. Put $\varphi = e^{2\pi i/n}\varphi_n$. Then C_φ is n-power compact on A.

We can apply Theorem 2 to the n-power compact composition operators on A.

Corollary. *Let* C_φ *and* C_ψ *be in* $\mathcal{C}(A)$ *such that* $C_\psi \sim C_\varphi$ *in* $\mathcal{C}(A)$. *If* C_φ *is n-power compact on A, then* C_ψ *is n-power compact on A.*

Theorem 3. *Let* $n > 1$ *and* $C_\varphi \in \mathcal{C}(A)$ *be n-power compact on A.*

(i) *For* $m_1, m_2 \leq n, m_1 \neq m_2$, $\iota(C_\varphi^{m_1}) \neq \iota(C_\varphi^{m_2})$.

(ii) *For* $m \geq n$, $\iota(C_\varphi^m) = K \setminus B$.

For example, let C_φ be 10-power compact on A. List up the order of the power compactness of $\{C_\varphi^k\}_{k=1}^{10}$, $10, 5, 4, 3, 2, 2, 2, 2, 2, 1$. Thus, by Theorem 3, this itinerary includes five different 2-power compact components in $\mathcal{C}(A)$.

Now for $n > 1$, denote K_n the collection of all n-power compact composition operators on A.

Corollary. *For every* $n > 1$, K_n *consists of infinite components of* $\mathcal{C}(A)$.

4. Power compactness and Königs function

In the introduction of this paper, it is shown that C_φ is power compact on A if and only if $\|\varphi^n\|_\infty < 1$ for some positive integer n. In this section, we will try to characterlize the power compactness of composition operators on A in other way.

At first, suppose that φ is an elliptic automorphism, then all iteration φ^n are also elliptic automorphisms. This means that $\|\varphi^n\|_\infty = 1$ for any positive integer n and C_φ is not power compact on A.

Next we suppose that φ is not an elliptic automorphism. By the Denjoy-Wolff Theorem(see [2]), φ has the Denjoy-Wolff point $p \in \overline{D}$ and φ^n converge to p on compact subsets of D. If the Denjoy-Wolff point p of φ is on ∂D, then p is just a fixed point of φ. This implies that $\|\varphi^n\|_\infty = 1$ for any positive integer n, and then C_φ is not power compact. (This result holds on the composition operators on H^∞ induced by $\varphi \in S(D)$ with Denjoy-Wolff point $p \in \partial D$. Indeed $p \in \overline{\varphi^n(D)}$ for all n. This implies $\|\varphi^n\|_\infty = 1$)

From now on, we deal with $\varphi \in S(\overline{D})$ which is not an elliptic automorphism of D, with the Denjoy-Wolff point $p \in D$, which is just a fixed point of φ. As the aplications of Königs Theorem, the following two results on the relation between power compactness of C_φ on H^∞ and its eigenfunction have been shown in [10]p23, p25.

Proposition 3 ([10]). *Let φ be in $S(D)$ such that $\varphi(p) = p \in D$ and $\varphi'(p) \neq 0$. Let σ be the Königs function of φ.*

(i) *If C_φ is power compact on H^∞, then $\sigma \in H^\infty$.*

(ii) *If φ is univalent on D and $\sigma \in H^\infty$, then C_φ is power compact on H^∞.*

In [8], Poggi-Corradini has shown that the size of the Königs function is estimated by its radial maximal function. In our situation, Poggi-Corradini's theorem implies the following lemma immediately.

Lemma 1. *Suppose that $\varphi \in S(\overline{D})$ such that $\varphi(0) = 0$ and the Königs function σ is in A. Then, there is a constant $C > 0$ independent of n such that*

(3) $$\|\sigma_n\|_\infty \leq C\|\sigma\|_\infty$$

Now, without the univalence of φ, we can get the relation between the power compactness of C_φ on A and the boundedness and continuity on $\overline{D}$ of the Königs function of φ.

Theorem 4. *Suppose that $\varphi \in S(\overline{D})$ with a fixed point p in D. Then the following are equivalence.*

(i) *C_φ is power compact on A.*

(ii) *$\|\varphi^n\|_\infty < 1$ for some $n \geq 1$.*

(iii) *the Königs function σ is in A.*

Moreover if one of the three conditions holds (then all of them hold), the Königs sequence σ_n converges to σ in A.

References

[1] H. Chandra, *Isolation amongst composition operators on the disc algebra* J. Indian Math. Soc.(N.S.) **67** (2000), 43–52.

[2] C. Cowen and B. MacCluer, *Composition operators on spaces of analytic functions.* CRC Press, Boca Raton, 1995.

[3] K. Hoffman, *Banach Spaces of Analytic Functions.* Prentice Hall,Englewood Cliffs, N. J., 1962.

[4] T. Hosokawa, *Power compactness and components of composition operators on the disc algebra.* preprint.

[5] T. Hosokawa, K. Izuchi, D. Zheng, *Isolated points and essential components of composition operators on H^∞.* Proc. Amer. Math. Soc. **130** (2001), 1765–1773

[6] B. MacCluer, S. Ohno, R. Zhao, *Topological structure of the space of composition operators on H^∞.* Integral Equation Operator Theory, **40** (2001), 481–494.

[7] K. Madigan, *Composition operators into Lipschitztype spaces.* Thesis, SUNY albany, 1993.

[8] Pietro Poggi-Corradini, *Norm convergence of normalized iterats and the growth of Kœnigs maps.* Ark. Mat. **37** (1999), 171-182.

[9] J. Shapiro, *Composition operators and classical function theory.* Springer-Verlag, New York, 1993.

[10] J. Shapiro and W. Smith and D. Stegenga, *Geometric Models and Compactness of Composition Operators.* J. Funct Anal. **127** (1995), 21–62

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The essential norm of a weighted composition operator on the disc algebra

ディスク環上の荷重合成作用素の本質ノルム

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Abstract. We estimate the essential norm of a weighted composition operator $uC_\varphi : f \mapsto u\cdot(f\circ\varphi)$ on the disc algebra $A(\mathbb{D})$. The estimate determines the essential norms of a composition and a multiplication operator on $A(\mathbb{D})$, and gives Kamowitz's characterization of the compactness of uC_φ. Furthermore, we obtain an analogue for $H^\infty(\mathbb{D})$, the Banach algebra of the bounded analytic functions.

X を Banach 空間とし, $\mathcal{L}(X)$ を X 上の有界線形作用素全体の Banach 環とする. $S \in \mathcal{L}(X)$ は, X の閉単位球を X のあるコンパクト集合の中にうつすとき, コンパクト作用素と呼ばれる. X 上のコンパクト作用素全体の集合 $\mathcal{K}$ は, $\mathcal{L}(X)$ において閉両側イデアルをなす. いま, $T \in \mathcal{L}(X)$ に対して, T から $\mathcal{K}$ への距離を考え,

$$\|T\|_e = \inf\{\, \|T-S\| : S \in \mathcal{K}\,\}$$

とおく. これを, T の本質ノルム (essential norm) という. これは, 商空間 $\mathcal{L}(X)/\mathcal{K}$ におけるノルム $\|T+\mathcal{K}\|$ である. 明らかに, $\|T\|_e \le \|T\|$ が成り立つ. また, 次のことがいえる.

$$\|T\|_e < \infty \iff T \text{ が 有界} \quad (T \text{ が有界でない場合}, \|T\| = \infty \text{ と考える})$$
$$\|T\|_e = 0 \iff T \text{ が コンパクト作用素}$$

よって, $\|T\|_e$ の評価式が与えられると, T の有界性やコンパクト性が 同時に特徴づけられる. このように, 本質ノルムは, 作用素の性質を知るうえで 重要な役割を担っている.

最近, さまざまな作用素の本質ノルムが研究されている. 合成作用素関連では, 次の人たちによって, 連記の作用素の本質ノルムが求められている.

J.H. Shapiro [3]	Hardy 空間 $H^2(\mathbb{D})$ 上の合成作用素
A. Montes-Rodríguez [2]	Bloch 空間上の荷重合成作用素
L. Zheng [6]	Hardy 空間 $H^\infty(\mathbb{D})$ 上の合成作用素
第二著者, 三浦毅, 髙橋眞映 [5]	$C(X)$ 上の荷重合成作用素

ここでは, ディスク環と $H^\infty(\mathbb{D})$ 上の荷重合成作用素の本質ノルムを 調べる.

§1. ディスク環上の荷重合成作用素の本質ノルム

$\mathbb{D}$ を $\mathbb{C}$ の単位開円板とし, その閉包を $\overline{\mathbb{D}}$, 境界を $\mathbb{T}$ とかく. $\mathbb{D}$ 上で正則 かつ $\overline{\mathbb{D}}$

上で連続な関数全体からなる関数環を, ディスク環 (disc algebra) と呼び, $A(\mathbb{D})$ で表す. いま, $u, \varphi \in A(\mathbb{D})$ で, φ は $\overline{\mathbb{D}}$ から $\overline{\mathbb{D}}$ への写像とする. この u と φ を用いて, $A(\mathbb{D})$ 上の作用素 uC_φ を,

$$(uC_\varphi f)(z) = u(z)\, f(\varphi(z)) \qquad (z \in \overline{\mathbb{D}},\ f \in A(\mathbb{D}))$$

と定義する. 明らかに, $uC_\varphi \in \mathcal{L}(A(\mathbb{D}))$ で, $\|uC_\varphi\| = \|u\|$ である. uC_φ は, $A(\mathbb{D})$ 上の荷重合成作用素 (weighted composition operator) と呼ばれる. とくに, u が定数関数 1 の場合, $uC_\varphi = C_\varphi : f \mapsto f \circ \varphi$ を, 合成作用素 (composition operator) という. 他方, φ が $\overline{\mathbb{D}}$ の恒等写像の場合は, よく知られた積作用素 (multiplication operator) $uC_\varphi = M_u : f \mapsto u \cdot f$ である.

$A(\mathbb{D})$ 上の荷重合成作用素の本質ノルムは, 次のようになる.

定理 1. uC_φ を $A(\mathbb{D})$ 上の荷重合成作用素とする.
(a) φ が 定数関数の場合, $\|uC_\varphi\|_e = 0$ である.
(b) $\varphi(\overline{\mathbb{D}}) \cap \mathbb{T} = \emptyset$ の場合, $\|uC_\varphi\|_e = 0$ である.
(c) φ が定数関数でなく かつ $\varphi(\overline{\mathbb{D}}) \cap \mathbb{T} \neq \emptyset$ の場合,

$$\sup\{|u(z)| : z \in \varphi^{-1}(\mathbb{T})\} \le \|uC_\varphi\|_e \le 2\,\sup\{|u(z)| : z \in \varphi^{-1}(\mathbb{T})\}$$

が成り立つ.

定理 1 の (b) の場合は, $\varphi^{-1}(\mathbb{T}) = \emptyset$ となるから, $\sup\{|u(z)| : z \in \varphi^{-1}(\mathbb{T})\} = 0$ と解釈できる. そうすると, (b) は (c) に含めることができる.

定理 1 において, u が定数関数 1 の場合 あるいは φ が $\overline{\mathbb{D}}$ の恒等写像の場合を考えよう. どちらの場合も, (c) の $\sup\{|u(z)| : z \in \varphi^{-1}(\mathbb{T})\}$ の値が $\|u\|$ となり, 一方で, $\|uC_\varphi\|_e \le \|uC_\varphi\| = \|u\|$ だから, $\|uC_\varphi\|_e = \|u\|$ となる. こうして, 次の 2 つの系を得る.

系 1 C_φ を $A(\mathbb{D})$ 上の合成作用素とすると, 次の式が成り立つ.

$$\|C_\varphi\|_e = \begin{cases} 0 & (\varphi \text{ が定数関数 または } \varphi(\overline{\mathbb{D}}) \cap \mathbb{T} = \emptyset \text{ の場合}) \\ 1 & (\text{そうでない場合}) \end{cases}$$

系 2 M_u を $A(\mathbb{D})$ 上の積作用素とすると, $\|M_u\|_e = \|u\|$ である.

また, 定理 1 で $\|uC_\varphi\|_e = 0$ の場合を考えると, 次の知られた結果が 得られる.

系 3 (Kamowitz [1]) uC_φ を $A(\mathbb{D})$ 上の荷重合成作用素とする. φ が 定数関数でないとき, uC_φ が コンパクト作用素であるための必要十分条件は,

$$u(z) \neq 0 \implies |\varphi(z)| < 1$$

が成り立つことである.

これらの系で，定理 1(c) を適応する際，$\|uC_\varphi\|_e = \sup\{|u(z)| : z \in \varphi^{-1}(\mathbb{T})\}$ となっていた．次の定理では，そうなるための十分条件を与える．定理では，$\mathbb{T}$ 上の 1 次元 Lebesgue 測度を m とかく．

定理 2. uC_φ を $A(\mathbb{D})$ 上の荷重合成作用素とする．また，φ は 定数関数でなく かつ $\varphi(\overline{\mathbb{D}}) \cap \mathbb{T} \neq \emptyset$ とする．$m(\varphi^{-1}(\mathbb{T})) = 0$ のとき，

$$\|uC_\varphi\|_e = \sup\{|u(z)| : z \in \varphi^{-1}(\mathbb{T})\}$$

である．

§2. $H^\infty(\mathbb{D})$ 上の荷重合成作用素の本質ノルム

$\mathbb{D}$ 上の有界正則関数全体の Banach 環を，$H^\infty(\mathbb{D})$ で表す．いま，$u, \varphi \in H^\infty(\mathbb{D})$ で，φ は $\mathbb{D}$ から $\mathbb{D}$ への写像とする．この u と φ を用いて，$H^\infty(\mathbb{D})$ 上の作用素 uC_φ を，

$$(uC_\varphi f)(z) = u(z)\, f(\varphi(z)) \qquad (z \in \mathbb{D},\ f \in H^\infty(\mathbb{D}))$$

と定義する．明らかに，$uC_\varphi \in \mathcal{L}(H^\infty(\mathbb{D}))$ で，$\|uC_\varphi\| = \|u\|$ である．uC_φ を，$H^\infty(\mathbb{D})$ 上の荷重合成作用素 (weighted composition operator) という．ディスク環の場合と同様に，u が定数関数 1 の場合，uC_φ を 合成作用素 (composition operator) といって，C_φ で表し，φ が $\overline{\mathbb{D}}$ の恒等写像の場合は，uC_φ を 積作用素 (multiplication operator) といい，M_u と表す．

定理 1 の $A(\mathbb{D})$ を $H^\infty(\mathbb{D})$ に変更してみよう．定理 1 の (c) では，

$$\sup\{|u(z)| : z \in \varphi^{-1}(\mathbb{T})\} = \inf\{r > 0 \ :\ \varphi(\{z \in \overline{\mathbb{D}} : |u(z)| \geq r\}) \subset \mathbb{D}\}$$

が成り立つ．$H^\infty(\mathbb{D})$ の場合は，$\varphi(\{z \in \overline{\mathbb{D}} : |u(z)| \geq r\})$ のかわりに，

$$\overline{\varphi(\{z \in \mathbb{D} : |u(z)| \geq r\})}$$

——$\overline{\Omega}$ は Ω の閉包を表す ($\Omega \subset \mathbb{C}$)—— を考える．そして，次の定理を得た．

定理 3. uC_φ を $H^\infty(\mathbb{D})$ 上の荷重合成作用素とする．

$$\alpha = \inf\left\{r > 0 : \overline{\varphi(\{z \in \mathbb{D} : |u(z)| \geq r\})} \subset \mathbb{D}\right\}$$

とおくと，

$$\alpha \leq \|uC_\varphi\|_e \leq 2\alpha$$

が成り立つ．

§1 と同様に考察すると, 定理 3 から, 次の 3 つの系がみちびかれる.

系 1 (Zheng [6]) C_φ を $H^\infty(\mathbb{D})$ 上の合成作用素とすると, 次の式が成り立つ.

$$\|C_\varphi\|_e = \begin{cases} 0 & \left(\overline{\varphi(\mathbb{D})} \subset \mathbb{D} \text{ の場合}\right) \\ 1 & \left(\overline{\varphi(\mathbb{D})} \cap \mathbb{T} \neq \emptyset \text{ の場合}\right) \end{cases}$$

系 2 M_u を $H^\infty(\mathbb{D})$ 上の積作用素とすると, $\|M_u\|_e = \|u\|$ である.

系 3 ([4]) uC_φ を $H^\infty(\mathbb{D})$ 上の荷重合成作用素とする. uC_φ が コンパクト作用素であるための必要十分条件は,

$$\text{任意の } r > 0 \text{ に対して, } \overline{\varphi\left(\{z \in \mathbb{D} : |u(z)| \geq r\}\right)} \subset \mathbb{D}$$

が成り立つことである.

付記：この内容は, 第一著者の修士論文 (信州大学) になる予定である. 証明等 詳しくは, それを参照されたい.

参 考 文 献

[1] H. Kamowitz, *Compact operators of the form uC_φ*, Pacific. J. Math., **80** (1979), 205–211.

[2] A. Montes-Rodríguez, *Weighted composition operators on weighted Banach spaces of analytic functions*, J. London Math. Soc., **61** (2000), 872–884.

[3] J.H. Shapiro, *The essential norm of a composition operator*, Annals Math., **125** (1987), 375–404.

[4] H. Takagi, *Compact weighted composition operators on function algebras*, Tokyo J. Math., **11** (1988), 119–129.

[5] H. Takagi, T. Miura and S.-E. Takahasi, *Essential norm and stability constant of weighted composition operators on $C(X)$*, submitted.

[6] L. Zheng, *The essential norms and spectra of composition operators on H^∞*, Pacific. J. Math., **203** (2002), 503–510.

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Finite codimensional linear isometries on function algebras

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Abstract. Let K be a compact subset of the complex n-space and $A(K)$ the algebra of all continuous functions on K which is holomorphic on the interior of K. In this report we show that under some hypotheses on K, there exists no linear isometry of a finite codimension on $A(K)$.

このレポートは羽鳥理先生との共同研究を基にしたものです。

E を Banach 空間とする。Linear isometry $\psi : E \to E$ が余次元 l であるとは ψ の値域が E の中で余次元 l である時にいう。ここで l は正の整数である。

F を複素 Euclid 空間 $\mathbb{C}^n$ の部分集合とする。$\bar{F}$ は F の閉包を表し、∂F は F の位相的な境界を表し、int$F$ は F の内部を表すものとする。一般に G を $\mathbb{C}^n$ の領域とする。$H(G)$ を G 上の正則関数全体を表し、$C(\overline{G})$ は $\overline{G}$ 上の複素数値連続関数の全体を表すものとする。

B_n を複素 Euclid 空間 $\mathbb{C}^n$ の単位球とし、S_n を単位球面とする。$A(B_n) = H(B_n) \cap C(\overline{B_n})$ とおく。$A(B_n)$ の S_n 上への制限を $A(S_n)$ と書くことにする。$A(S_n)$ を ball algebra と言う。$n = 1$ の時 $A(S_1)$ は円板環である。円板環上の余次元 1 の linear isometry は存在する。D を単位円板とし Γ を単位円周とする。すなわち $D = B_1$, $\Gamma = S_1$ である。$M_a(z) = (z - a)/(1 - \bar{a}z)$ $(a \in D)$ と定める。次は Takayama-Wada の結果である。

Theorem 1. ([7]) $A(\Gamma)$ を円板環とする。ψ を $A(\Gamma)$ 上の余次元 1 の linear isometry とする。この時 $\alpha, \beta \in \mathbb{C}$ $(|\alpha| = |\beta| = 1)$ と $a, b \in D$ が存在し $(\psi f)(z) = \alpha M_a(z) f(\beta M_b(z))$ $(f \in A(\Gamma),\ z \in \Gamma)$ を満たす。

次に多次元の領域について上のような結果が成立するか考察する。D^n を $\mathbb{C}^n$ の単位多重円板とし、T^n をトーラスとする。$A(D^n) = H(D^n) \cap C(\overline{D^n})$ とおく。$A(D^n)$ の T^n 上への制限を $A(T^n)$ と書くことにする。$A(T^n)$ を polydisk algebra と言う。$n = 1$ の時 $A(T^1)$ は円板環である。

Theorem 2. ([6]) A は ball algebra $A(S_n)$ または polydisk algebra $A(T^n)$ を表すものとする。$n > 1$ の時、A 上の余次元 1 の linear isometry は存在しない。

$B(p, \epsilon) = \{z \in \mathbb{C}^n : |z - p| < \epsilon\}$ とおく。ここで $p \in \mathbb{C}^n$ で $\epsilon > 0$ である。$A(K)$ を K の内部で正則な K 上の複素数値連続関数全体とする。$H(K)$ を K で正則な関数の K 上での一様極限全体とする。$A(K)$、$H(K)$ はいずれも K 上での関数環で

$$H(K) \subset A(K)$$

が成り立つ。

$\partial A(K)$ を $A(K)$ の Shilov 境界とする。

Theorem 3. ([5]) $n > 1$ とする。K は $\mathbb{C}^n$ のコンパクト部分集合で次の (i) から (v) を満たすものとする。

(i) $\overline{\mathrm{int} K} = K$

(ii) $K = \cap_{n=1}^{\infty} D_n$、ただし $D_n \supset \overline{D_{n+1}}$ で、D_n は有界かつ正則凸である

(iii) K の任意の点 p に対し $\epsilon_p > 0$ が存在し、$0 < \epsilon < \epsilon_p$ なる任意の ϵ に対し $B(p, \epsilon) \cap \mathrm{int} K$ は連結

(iv) $A(K) = H(K)$

(v) $u \in A(K)$ で $\partial A(K)$ 上で $|u| = 1$ であるならば、u は定数または K の内部で零点を持つ

この時 $A(K)$ 上の余次元 l の linear isometry は存在しない。ここで l は正の整数である。

次に定理の仮定を満たす K の例を挙げる。

Example 1. $n > 1$ とする。D を $\mathbb{C}^n$ の C^2 boundary を持つ strictly pseudoconvex domain とする。$K = \bar{D}$ とすると定理 3 の (i) から (v) の仮定を満たす。

Example 2. $n > 1$ とする。K は $\mathbb{C}^n$ のコンパクトかつ凸集合で $\overline{\mathrm{int} K} = K$ なるものとすると定理 3 の (i) から (v) の仮定を満たす。

Example 3. $n > 1$ とする。K_j は $\mathbb{C}$ のコンパクト部分集合で ∂K_j は有限個の滑らかな閉曲線からなるものとする。$K = \prod_{j=1}^{n} K_j$ とすると定理 3 の (i) から (v) の仮定を満たす。

References

[1] J. Araujo and J. J. Font, *Linear isometries between subspaces of continuous functions,* Trans. Amer. Math. Soc. **349** (1997), 413-428.

[2] J. Araujo and J. J. Font, *Codimension 1 linear isometries on function algebras,* Proc. Amer. Math. Soc. **127** (1999), 2273-2281.

[3] R. M. Crownover, *Commutants of shifts on Banach spaces,* Michigan Math. J. **19** (1972), 233-247.

[4] J. J. Font, *Isometries between function algebras with finite codimensional range,* Manuscripta Math. **100** (1999), 13-21.

[5] O. Hatori and K.Kasuga, *Linear isometries of finite codimensions on Banach algebras of holomorphic functions*, preprint.

[6] K. Kasuga, *There are no codimension 1 linear isometries on the ball and polydisk algebras,* Sci. Math. Jpn. **54** (2001), 387-390; Correction, Sci. Math. Jpn., **56** (2002), 69-70.

[7] T. Takayama and J. Wada, *Isometric shift operators on the disc algebra,* Tokyo. J. Math. **21** (1998), 115-120.

Martin boundary of finitely sheeted unlimited covering surfaces of $\mathbb{C}$ and quasiconformal mappings

HIROAKI MASAOKA and SHIGEO SEGAWA

Abstract. Let W_1 and W_2 be p-sheeted unlimted covering surfaces of $\mathbb{C}$. Denote by $\Delta_1^{W_i}$ the minimal Martin boundary of W_i. Suppose that W_1 is quasiconformal equivalent to W_2. In case $p=2$ or 3, we show that $\#\Delta_1^{W_1}=\#\Delta_1^{W_2}$, where $\#\Delta_1^{W_i}$ is the cardinal number of $\Delta_1^{W_i}$.

自然数 $p(\geqq 2)$ に対し, $\mathbb{C}$ の p 葉非有界被覆面全体を $\mathcal{C}_p$ で表す. $W\in\mathcal{C}_p$ の マルチン compact 化を W^*, マルチン境界 を $\Delta^W(=W^*\setminus W)$, さらに W の**極小マルチン境界**を $\Delta_1^W(\subset\Delta^W)$ とする. Δ_1^W は, W の理想境界近傍上の正値調和関数の構造を決めるものである. W から $\mathbb{C}$ への射影を π_W とし, $W'=W\setminus\pi_W^{-1}(\{|z|\leqq 1\})$ とおく. W' の相対境界を $\partial W'$ で表す: $\partial W'=\pi_W^{-1}(\{|z|=1\})$. W' 上の調和関数 h で $h\geqq 0$, $h|\partial W'=0$ をみたすものの全体を $HP(W',\partial W')$ とおく. このとき,

$$\#\Delta_1^W=n$$
$$\Updownarrow$$

$u_1,\cdots,u_n(\in HP(W',\partial W'))$ が存在して, 各 $h\in HP(W',\partial W')$ は
$h=\sum_{i=1}^n c_iu_i$ $(c_i\in\mathbb{R}, c_i\geqq 0)$ の形に一意的に表される.

となっている (Heins[H] は, この n を W' の **調和次元** と呼んだ). また, これから, $W\in\mathcal{C}_p$ のとき $1\leqq\#\Delta_1^W\leqq p$ が分かる (cf. [H]). 以下では、次の問題について考える.

問題. $W_1\in\mathcal{C}_p$ と $W_2\in\mathcal{C}_p$ が擬等角同値のとき, $\#\Delta_1^{W_1}=\#\Delta_1^{W_2}$ となるか?

一般に, マルチン境界 (または, 正値および有界調和関数空間) は擬等角不変ではないが (cf. [L], [S], [ST]), 上の答は肯定的であるものと予想している. ここでは, $p=2$ および 3 の場合に答を与える. まず, 一般の p について, 次のことが成り立つ.

定理 1. $W_1\in\mathcal{C}_p$ と W_2 が擬等角同値で $\#\Delta_1^{W_1}=p$ のとき, $\#\Delta_1^{W_2}=p$ である.

定理 2. $W_1\in\mathcal{C}_p$ と $W_2\in\mathcal{C}_p$ が擬等角同値で $\#\Delta_1^{W_1}=p-1$ のとき, $\#\Delta_1^{W_2}=p-1$ である.

$p=2$, 3 のとき, 定理 1 および 2 より直ちに上の問題の答が得られる.

主定理. $p=2$ または 3 とする. $W_1\in\mathcal{C}_p$ と $W_2\in\mathcal{C}_p$ が擬等角同値ならば,

$\#\Delta_1^{W_1} = \#\Delta_1^{W_2}$ である.

以下では, 定理 1 および 2 の証明の背景について述べる. ∞ が Dirichlet 問題に関する非正則境界点になっているような $\mathbb{C}$ の領域全体を $\mathcal{M}$ で表す. さらに, $W \in \mathcal{C}_p$ と $M \in \mathcal{M}$ に対し, $\pi_W^{-1}(M)$ の成分の個数を $n_W(M)$ で表す, ここで π_W は W から $\mathbb{C}$ への射影である. このとき, $\#\Delta_1^W$ は, $n_W(M)$ を使って次のように特徴づけられる (cf. [MS]):

命題 1. $\#\Delta_1^W = \max_{M \in \mathcal{M}}\ n_W(M)$

また, 擬等角同値な 2 つの平面領域の境界について, 次のことが知られている (cf. [LSW]):

命題 2. **f は平面領域 D_1 から平面領域 D_2 への擬等角写像で $\overline{D_1}$ への連続拡張 f^* を持つとする. このとき, b が D_1 の非正則境界点ならば, $f^*(b)$ も D_2 の非正則境界点である.**

$W \in \mathcal{C}_p$ と $\zeta \in \Delta_1^W$ に対し, K_ζ を W' における ζ の Martin 核とする, ここで $W' = W \setminus \pi_W^{-1}(\{|z| \leqq 1\})$, π_W は W から $\mathbb{C}$ への射影である. W の部分集合 E は, $\hat{\mathrm{R}}_{K_\zeta}^{E\cap W'} \neq K_\zeta$ をみたすとき, ζ で **minimally thin** であるという. ここに, $\hat{\mathrm{R}}_{K_\zeta}^{E\cap W'}$ は W' 上の $E \cap W'$ に関する K_ζ の **balayage**(掃散) を表す. 定理 2 の証明には, 上の命題 1 および 2 の他に次の命題 3 が必要となる (cf. [M]).

命題 3. **L を $W(\in \mathcal{C}_p)$ 内の連続曲線で W の理想境界へ到達するものとする. このとき, Δ_1^W の点 ζ が存在して, L は ζ で minimally thin でない.**

参照文献

[**AS**] L.V. Ahlfors and L. Sario: *Riemann Surfaces*, Princeton, 1960.

[**B**] M. Brelot: *On topologies and boundaries in potential theory*, Lecture Notes in Math., 175(1971), Springer.

[**F**] O. Forster: *Lectures on Riemann Surfaces*, GTM 81, Springer.

[**H**] M. Heins: *Riemann surfaces of infinite genus*, Ann. of Math., **55**(1952), 296–317.

[**L**] T. Lyons: *Instability of the Liouville property for quasi-isometric Riemannian manifolds and reversible Markov chains*, J. Differential Geom., **26**(1987), 33–66.

[**LSW**] W. Littman, G. Stampacchia and H. F. Weinberger: *Regular points for elliptic equations with discontinuous coefficients*, **17**(1963), Ann. Sc. Norm. Sup. Pisa, 45–79.

[**M**] H. Masaoka: *Criterion of Wiener type for minimal thinness on covering surfaces*, Proc. Japn. Acad., **72**(1996), 154-156.

[**MS**] H. Masaoka and S. Segawa: *Harmonic dimension of covering surfaces and minimal fine neighborhood*, Osaka J. Math., **34**(1997), 659–672.

[S] S. Segawa: *Martin boundaries of Denjoy domains and quasiconformal mappings*, J. Math. Kyoto Univ., **30**(1990), 297–316.

[ST] S. Segawa and T. Tada: *Martin compactifications and quasiconformal mappings*, Proc. Amer. Math. Soc., **93**(1985), 242–244.

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PARABOLICITY OF RIEMANN SURFACES

MITSRU NAKAI[1] and SHIGEO SEGAWA[2]

1. Introduction

Throughout this paper the notation R is always used to mean an *open* (i.e. noncompact) Riemann surface. In the classification theory of Riemann surfaces the class O_G of *parabolic* Riemann surfaces R is the most basic category of degenerate Riemann surfaces. There are hundreds of mutually equivalent characterizations of the parabolicity of R, among which we state here the following ring theoretic one (cf. e.g. [7]). The Royden algebra $M(R)$ over R is the normed ring obtained by the completion of the ring $\{f \in C^\infty(R) : ||f|| < \infty\}$ with respect to the norm

$$||f|| := \sup_{z\in R} |f(z)| + \left(\int_R |\nabla f(z)|^2\right)^{1/2},$$

where $\nabla f(z) = (f_x(z), f_y(z))$ $(z = x+iy)$ is the gradient of f. The potential subalgebra $M_\Delta(R)$ of $M(R)$ is the closure of the subring $C_0^\infty(R)$ of $C^\infty(R)$ consisting of functions with compact supports in R. The potential subalgebra $M_\Delta(R)$ is an ideal of $M(R)$. Then the parabolicity of R is characterized by the triviality of the ideal $M_\Delta(R)$:

$$M(R) = M_\Delta(R),$$

i.e. the ideal boundary of R is so small that any Royden function (i.e. any function in $M(R)$) can be approximated by Royden functions with compact supports. Of course the notation O_G comes from the most basic characterization of the parabolicity of R that there are no Green functions on R: in the notation O_G the letter O suggests the nonexistence and the letter G in the suffix is the capital of Green. Open Riemann surfaces R not in O_G (i.e. $R \notin O_G$) are said to be *hyperbolic*. The closed (i.e. compact) Riemann surfaces are said to be *elliptic*. Hence all Riemann surfaces (open or closed) are classified into three categories: elliptic, parabolic, and hyperbolic, and all open Riemann surfaces into two categories: parabolic and hyperbolic.

The classification is the most illuminating if R's are restricted to those of simply connected ones. The uniformization theorem says that all simply connected Riemann surfaces (open or closed) are conformally one of the following three surfaces: $\hat{\mathbb{C}}$ (the Riemann sphere or the extended complex plane), $\mathbb{C}$ (the complex plane), and $\mathbb{D}$ (the unit disc). Hence a simply connected open Riemann surface R is either $\mathbb{C}$ or $\mathbb{D}$ and therefore $R \in O_G$ ($R \notin O_G$, resp.) if and only if $R = \mathbb{C}$ ($R = \mathbb{D}$, resp.).

Let P be either one of the complex plane or the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, ∞ being the point at infinity of $\mathbb{C}$. An open or closed Riemann surface S is said to be a *multisheeted plane* over P if S is a covering Riemann surface (S, P, π) of P, i.e. S is a Riemann surface and π is an analytic mapping of S to P, with the following two additional conditions. Firstly, the cardinal number $\operatorname{card} \pi^{-1}(a)$ of the fiber $\pi^{-1}(a)$ is a constant $\nu(S) \in \mathbb{N} \cup \{\aleph_0\}$ for every $a \in P$, where $\mathbb{N}$ is the set of all positive integers $1, 2, \cdots$ and $\aleph_0 = \operatorname{card} \mathbb{N}$. Secondly, $S_1 \setminus S$ is of (logarithmic) capacity zero measured on S_1 for any covering extension (S_1, P, π_1) of (S, P, π). Here a covering extension

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(S_1, P, π_1) of (S, P, π) is a covering surface (S_1, P, π_1) of P such that S_1 is a Riemann surface containing S as its Riemann subsurface and $\pi_1|S = \pi$. That the closed subset $S_1 \setminus S$ of S_1 is of capacity zero is characterized by the existence of an Evans function h on S_1 for $S_1 \setminus S$ which is a continuous mapping of S_1 to $[-\infty, +\infty]$ such that $h|S$ is harmonic with negative logarithmic singularity and $h|S_1 \setminus S = +\infty$. The constant $\nu(S)$ stated above is referred to as the sheet number of the multisheeted plane S, or more precisely, of (S, P, π), and if $\nu(S) \in \mathbb{N}$ ($\nu(S) = \aleph_0$, resp.), then we say that S is a *finitely sheeted plane* (*infinitely sheeted plane*, resp.) over P. obviously infinitely sheeted plane must be open.

We are interested in the problem, which we call the (generalized) *type problem* (cf. e.g. [3],[4]), to judge whether a given multisheeted plane S over P is parabolic or hyperbolic based on the covering information of S over P such as the topological and metrical distribution of the branch points, i.e. points $\tilde{a}$ in S such that the order of the zero of $\pi(z) - \pi(\tilde{a})$ (z being the local parameter at $\tilde{a}$) is at least two. Since it is easy (although nontrivial) to see that S is elliptic or parabolic along with P if $\nu(S) \in \mathbb{N}$, we can restrict ourselves to infinitely sheeted planes over P in the type problem. In particular the type problem for simply connected R over $\hat{\mathbb{C}}$ is referred to as the (classical) *type problem*, about which we will discuss in this paper.

2. Concretely given simply connected infinitely sheeted planes over $\hat{\mathbb{C}}$

We are going to discuss the parabolicity or nonparabolicity (i.e. hyperbolicity) of a certain simply connected infinitely sheeted planes W constructed by the method of scissoring and pasting. We take a variable sequence $\alpha = (a_n)_{n \in \mathbb{N}}$ of increasing positive numbers:

$$0 < a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots \nearrow a =: a[\alpha] \leqq +\infty \quad (n \to \infty). \tag{1}$$

Fixing the above sequence α we consider the sequence $(I_n)_{n \in \mathbb{N}}$ of slits I_n given by

$$I_n := [-1, 1] + ia_n = \{z \in \mathbb{C} : -1 \leqq \operatorname{Re} z \leqq 1, \operatorname{Im} z = a_n\} \quad (n \in \mathbb{N}). \tag{2}$$

We then consider the sequence $(F_n)_{n \in \mathbb{N}}$ of slitted sphere F_n given by

$$F_n := \hat{\mathbb{C}} \setminus (I_{n-1} \cup I_n) \quad (n \in \mathbb{N}), \tag{3}$$

where $I_0 = \emptyset$, so that F_1 is a once slitted sphere and F_n are twice slitted spheres for $n > 1$. A slit I_n gives two sides, the upper side I_n^+ and the lower side I_n^-, to F_n and F_{n+1} ($n \leqq 1$). More precisely, there are two boundary points a^+ and a^- in the Caratheodory compactification F_n^* and F_{n+1}^*, respectively (cf. e.g. [8]), of F_n and F_{n+1}, respectively lying over $a \in I_n$ characterized by

$$a^{\pm} = \lim_{t \downarrow 0} (a \pm it)$$

in F_n^* and F_{n+1}^*, respectively, where $+$ and $-$ in the double sign $\pm$ are taken in their natural corresponding order. Here $a^+ = a^-$ if and only if a is the rightmost and the leftmost end point of I_n. Then, precisely defining, $I^{\pm} := \{a^{\pm} : a \in I_n\}$ ($n \in \mathbb{N}$). Identifying a^+ (a^-, resp.) in F_1^* with a^- (a^+, resp.) in F_2^* we get a set $F_1 \cup F_2 \cup J_1$. Here J_1 consists of two copies I_{11} and I_{12} of I_1. The set I_{11} (I_{12}, resp.) comes from identified I_1^- in F_1^* and I_1^+ in F_2^* (I_1^+ in F_1^* and I_1^- in F_2^*, resp.). Two sets I_{11} and I_{12} are viewed to be disjoint except their identified end points. At each point a in $F_1 \cup F_2 \cup J_1$

except the end points of J_1 we give a local parameter $\tilde{z} := z - a$ naturally induced by the global coordinate z of $\hat{\mathbb{C}}$. At the end points e_1 and e_2 of J_1, we give $\tilde{z} := \sqrt{z - e_j}$ as the local parameter at e_j $(j = 1, 2)$, so that, by setting $\pi(\tilde{z}) = a + (z - a) = a + \tilde{z}$ for $F_1 \cup F_2 \cup J_1 \setminus \{e_1, e_2\}$ and $\pi(\tilde{z}) = e_j + \tilde{z}^2$ at e_j $(j = 1, 2)$, $(F_1 \cup F_2 \cup J_1, \hat{\mathbb{C}}, \pi)$ is a covering Riemann surface of $\hat{\mathbb{C}}$ with two branch points of multiplicity 2 over e_1 and e_2. Then J_1 becomes an analytic Jordan curve in $F_1 \cup F_2 \cup J_1$ in which $\partial F_1 = J_1$. We will denote by $F_1 + F_2$ the Riemann surface $F_1 \cup F_2 \cup J_1$ constructed above. Usually $F_1 + F_2$ is referred to as the Riemann surface obtained from F_1 and F_2 by pasting crosswise along the slit I_1. Similarly we construct the Riemann surface $F_1 + F_2 + F_3$ obtained from $F_1 + F_2$ and F_3 by pasting crosswise along I_2. Repeating the process inductively we obtain a Riemann surface $W = W[\alpha]$ determined by α, which we denote symbolically by

$$(4) \qquad W[\alpha] = F_1 + F_2 + \cdots + F_n + F_{n+1} + \cdots .$$

The surface $W_n := F_1 + \cdots + F_n$ is relatively compact in and a Riemann subsurface of $W = W[\alpha]$. The relative boundary ∂W_n of W_n in W is an analytic Jordan curve. Conformally $W_n = \mathbb{D}$ for every $n \in \mathbb{N}$. We can see that $(W_n)_{n \in \mathbb{N}}$ is a regular exhaustion of W. From this we can see that $W[\alpha]$ is simply connected. By the construction we see that $(W[\alpha], \hat{\mathbb{C}}, \pi)$, π being defined naturally as in the explanation stated in the constructing $F_1 + F_2$, is a covering surface.

We denote by A the set of sequences α given by (1). Then we have obtained a one (sequential) parameter family $\{W[\alpha] : \alpha \in A\}$ of simply connected infinitely sheeted planes $W[\alpha]$ over $\hat{\mathbb{C}}$. We now discuss the classical type problem for $\{W[\alpha] : \alpha \in A\}$ and the complete answer is obtained as follows.

THEOREM A. *The infinitely sheeted plane* $W[\alpha] \in O_G$ ($W[\alpha] \notin O_G$, *resp.*) *if and only if* $a[\alpha] = +\infty$ ($a[\alpha] < +\infty$, *resp.*).

The part of this result that $a[\alpha] < +\infty$ implies $W[\alpha] \notin O_G$ is due to Professor Hisashi Ishida [2] at Kyoto Sangyo University. His *proof* goes follows. The standing assumption in this part is $a = a[\alpha] < +\infty$. Consider the set

$$G_1 := \{z \in \mathbb{C} : -1/2 < \operatorname{Re} z < 1/2, 0 < \operatorname{Im} z < a_1\},$$

whose boundary consists of four segments

$$\sigma_j := \{z \in \mathbb{C} : -1/2 \leqq \operatorname{Re} z \leqq 1/2, \operatorname{Im} z = a_j\} \quad (j = 0, 1)$$

with $a_0 = 0$ and

$$\tau_1^{\pm} := \{z \in \mathbb{C} : \operatorname{Re} z = \pm 1/2, a_0 \leqq \operatorname{Im} z \leqq a_1\},$$

where the double signs are taken in its natural order, so that $\partial_{\mathbb{C}} G_1 = \sigma_0 \cup \sigma_1 \cup \tau_1^+ \cup \tau_1^-$. We may understand that G_1 together its boundary $\partial_{\mathbb{C}} G_1$ is embedded in $F_1 \cup \partial F_1$, ∂F_1 being considered in W as G_1 and ∂G_1. Similarly we consider

$$G_n := \{z \in \mathbb{C} : -1/2 < \operatorname{Re} z < 1/2, a_{n-1} < \operatorname{Im} z < a_n\} \quad (n \in \mathbb{N},\ n \leqq 2),$$

whose boundary consists of four segments

$$\sigma_j := \{z \in \mathbb{C} : -1/2 \leqq \operatorname{Re} z \leqq 1/2, \operatorname{Im} z = a_j\} \quad (j = n-1, n)$$

and

$$\tau_n^{\pm} := \{z \in \mathbb{C} : \operatorname{Re} z = \pm 1/2, a_{n-1} \leqq \operatorname{Im} z \leqq a_n\}.$$

Then G_n with its boundary $\partial_{\mathbb{C}} G_n = \sigma_{n-1} \cup \sigma_n \cup \tau_n^+ \cup \tau_n^-$ in $\mathbb{C}$ is embedded in $F_n \cup \partial F_n$ in W as G_n and ∂G_n. Since σ_n in F_n and σ_n in G_{n+1} are identified in W, we see that

$$G = \bigcup_{n \in \mathbb{N}} G_n$$

is simply connected open set in W and

$$G = \{z \in \mathbb{C} : -1/2 < \operatorname{Re} z < 1/2, 0 < \operatorname{Im} z < a\}$$

is a rectangle considered in $\mathbb{C}$. Its boundary $\partial_{\mathbb{C}} G$ consists of two horizontal segments

$$\sigma_0 := \{z \in \mathbb{C} : -1/2 \leqq \operatorname{Re} z \leqq 1/2, \operatorname{Im} z = 0\},$$

$$\sigma_\infty := \{z \in \mathbb{C} : -1/2 \leqq \operatorname{Re} z \leqq 1/2, \operatorname{Im} z = a\}$$

and two vertical segments $\overline{\tau^\pm}$ where

$$\tau^\pm := \{z \in \mathbb{C} : \operatorname{Re} z = \pm 1/2, 0 \leqq \operatorname{Im} z < a\}.$$

Then the relative boundary ∂G of G in W is $\sigma_0 \cup \tau^+ \cup \tau^-$ but σ_∞ should be considered as the part of ideal boundary of W.

The subsurface G of W is thus noncompact. Solving the Dirichlet problem, the existence of nonconstant bounded harmonic function h with $0 < h < 1$ on G such that the boundary values of h on ∂G is zero and that of h at the ideal boundary σ_∞ is 1 except the end points of σ_∞. Such a subsurface, i.e. the subsurface of carrying nonconstant bounded harmonic functions vanishing continuously at its relative boundary, is said not to belong to SO_{HB}, which assures that the original surface is not in O_G, i.e. $W \notin O_G$ in the present case (cf. e.g. [7]). This is the Ishida proof for the *necessity* of the condition $a[\alpha] = +\infty$ in Theorem A. □

The last part of the above proof can also be seen directly as follows. Observe that

$$W_n \cap G = \{z \in \mathbb{C} : -1/2 < \operatorname{Re} z < 1/2, 0 < \operatorname{Im} z < a_n\} \quad (n \in \mathbb{N})$$

and its relative boundary $\partial(W_n \cap G)$ in W is

$$\partial(W_n \cap G) = \sigma_0 \cup (\tau^+ \cap W_n) \cup (\tau^- \cap W_n)$$

so that $\sigma_n \subset \partial W_n$ and $\overline{W_n \cap G} \setminus \sigma_n \subset W_n$. Let w_n be the harmonic measure of ∂W_n on $W_n \setminus \sigma_0$, i.e. w_n is a continuous function on $\overline{W_n} = W_n \cup \partial W_n$ harmonic on $W_n \setminus \sigma_0$ with boundary values $w_n|\sigma_0 = 0$ and $w_n|\partial W_n = 1$. The decreasing limit w of $(w_n)_{n \in \mathbb{N}}$ is the harmonic measure of the ideal boundary of W on $W \setminus \sigma_0$. By the comparison of boundary values of two functions w_n and h on $\partial(W_n \cap G)$, we see that $w_n \geqq h > 0$ on G so that $w > 0$ on W, which is one of the characterization of W not to belong to O_G.

Compared with the proof of the necessity which is quite simpler and easy as is seen above, that for the sufficiency, i.e. $a[\alpha] = +\infty$ implies $W[\alpha] \in O_G$, is much more complicated and cannot be stated here because of the restriction of the number of pages and therefore we refer the reader to [4] whose proof for a corresponding situation, although quite different at least superficially from our present one, must convey a feeling about the basic idea for the sufficiency proof.

3. Generalizations and conjectures

In order to increase the applicability of Theorem A we relax the rigidness of the slits as follows. In addition to the sequences $\alpha = (a_n)_{n\in\mathbb{N}}$ satisfying (1) we take another kind of the class B of sequences $\beta = (b_n)_{n\in\mathbb{N}}$ of real numbers b_n satisfying simply the condition $b_n > 0$ $(n \in \mathbb{N})$. Replacing the interval $[-1,1]$ by $[-b_n, b_n]$ we consider the sequence $(I_n)_{n\in\mathbb{N}}$ of slits I_n given by

$$(2') \qquad I_n := [-b_n, b_n] + ia_n = \{z \in \mathbb{C} : -b_n \leqq \operatorname{Re} z \leqq b_n, \operatorname{Im} z = a_n\} \quad (n \in \mathbb{N})$$

in place of (2). Then we consider two (sequential) parameters family $\{W[\alpha,\beta] : (\alpha,\beta) \in A \times B\}$ of simply connected infinitely sheeted planes $W[\alpha,\beta]$ determined by (3) and (4), i.e.

$$(4') \qquad W[\alpha,\beta] := F_1 + F_2 + \cdots + F_n + F_{n+1} + \cdots .$$

It may be natural to discuss the type problem for $W[\alpha,\beta]$ according to the cases $a[\alpha] = +\infty$ and $a[\alpha] < +\infty$ separately. Concerning the former case we wish to prove the following assertion.

CONJECTURE 1. *When* $a[\alpha] = +\infty$, $W[\alpha,\beta] \in O_G$ *no matter how* β *may be chosen in* B.

At present we can show the validity of the above conjecture under an additional condition which is satisfied e.g. when β is nondecreasing:

THEOREM 1. *When* $a[\alpha] = +\infty$, $W[\alpha,\beta] \in O_G$ *if the sequence* $(a_n^2 + b_n^2)_{n\in\mathbb{N}}$ *is nondecreasing except for the first few terms, i.e.*

$$(5) \qquad a_n^2 + b_n^2 \leqq a_{n+1}^2 + b_{n+1}^2 \quad (n \in \mathbb{N},\ n \geqq m)$$

for a certain $m \in \mathbb{N}$.

We believe that the technical condition (5) is redundant but we have been unable to get rid of it yet. The latter case (i.e. $a[\alpha] < +\infty$) is of much more diversity. In this case we expect the validity of the following result:

CONJECTURE 2. *When* $a[\alpha] < +\infty$, $W[\alpha,\beta] \in O_G$ *if and only if*

$$(6) \qquad \liminf_{n\to\infty} b_n = 0.$$

We can prove the necessity of the condition (6) for the validity of $W[\alpha,\beta] \in O_G$:

THEOREM 2. *When* $a[\alpha] < +\infty$, *the condition*

$$(7) \qquad \liminf_{n\to\infty} b_n > 0$$

implies the hyperbolicity of $W[\alpha,\beta]$, *i.e.* $W[\alpha,\beta] \notin O_G$.

However we are still unsuccessful in proving the sufficiency of the condition (6) for the validity of $W[\alpha,\beta] \in O_G$ unless assuming an additional condition which is the case e.g. if β is decreasing. Namely

THEOREM 3. *When* $a[\alpha] < +\infty$, $W[\alpha,\beta] \in O_G$ *if the condition* (6) *holds under the restriction that* $\left((a_n - a[\alpha])^2 + b_n^2\right)_{n\in\mathbb{N}}$ *is nonincreasing except for first few terms, i.e.*

$$(a_n - a[\alpha])^2 + b_n^2 \geqq ((a_{n+1} - a[\alpha])^2 + b_{n+1}^2 \tag{8}$$

for almost all $n \in \mathbb{N}$ *in the sense that there exists an* $N \in \mathbb{N}$ *such that* (8) *holds for all* $n \in \mathbb{N}$ *with* $n \geqq N$.

Again we think the condition (8) is only technical and not essential. We can easily construct an admissible $W[\alpha,\beta] \in O_G$ with $a[\alpha] < +\infty$, (6), and

$$\limsup_{n\to\infty} b_n = +\infty.$$

This is one of instances backing up the feeling that the condition (8) can be dispensed with.

References

[1] L. AHLFORS AND L. SARIO: *Riemann Surfaces*, Princeton Univ. Press, 1960.

[2] H. ISHIDA: *Private communication*, Department Math. Kyoto Univ., 2001.

[3] M. NAKAI: *Types of complete infinitely sheeted planes*, Suuriken-Kokyuuroku, **1293**(2002), 119-131.

[4] M. NAKAI AND S, SEGAWA: *Types of covering surfaces*, Suuriken-Kokyuuroku, **1277**(2002), 84-93.

[5] R. NEVANLINNA: *Analytic Functions*, Springer, 1970.

[6] B. RODIN AND L. SARIO: *Principal Functions*, Van Nostrand, 1970.

[7] L. SARIO AND M.NAKAI: *Classification Theory of Riemann Surfaces*, Springer, 1970.

[8] M. TSUJI: *Potential Theory in Modern Function Theory*, Maruzen, 1959.

STRICTLY CONTRACTIVE COMPRESSION ON BACKWARD SHIFT INVARIANT SUBSPASES OVER THE TORUS

KUNIAKI NAGAI

この講演では K. Izuchi とR.Yangの論文「*Strictly contractive compression on backward shift invariant subspaces over the torus*」を紹介する。

Definition 1. 2次元トーラス $\mathbb{T}^2$ 上のハーディ空間を $H^2(\mathbb{T}^2)$ とする。T_z を $f \in H^2(\mathbb{T}^2)$ に対して $T_z f = zf$ で定まる $H^2(\mathbb{T}^2)$ から $H^2(\mathbb{T}^2)$ への有界線型作用素とする。T_w も同様とする。

$H^2(\mathbb{T}^2)$ の閉部分空間 N が backward shift invariant subspace とは $T_z^* N \subseteq N, T_w^* N \subseteq N$ をみたすことである。

P_N を $H^2(\mathbb{T}^2)$ から N への射影とし，S_z を $f \in N$ に対し $S_z f = P_N T_z f$ で定まる N から N への有界線型作用素とする。S_w も同様とする。

一般に $||S_z|| \leq 1$ であるが，この論文では $||S_z|| < 1$ のときに着目して考察している。

L_0 を $f \in H^2(\mathbb{T}^2)$ に対し $L_0 f(z,w) = f(0,w)$ で定まる $H^2(\mathbb{T}^2)$ から $H^2(\mathbb{T}_w)$ への有界線型作用素とすると次が成り立つ。

Proposition 1. N を $H^2(\mathbb{T}^2)$ 上の backward shift invariant subspace とするとき次は同値である。

(i) $||S_z|| < 1$

(ii) L_0 は N 上で1対1かつ $L_0 N$ は $H^2(\mathbb{T}_w)$ の閉部分空間である。

(iii) 任意の $h \in N$ に対し $||h|| \leq C||L_0 h||$ を満たす定数 $C > 1$ が存在する。

Proof. (i)⇒(ii) $||S_z|| < 1$ とする。任意の $f \in N$ に対し

$||f||^2 = ||L_0 f||^2 + ||S_z^* f||^2 \leq ||L_0 f||^2 + ||S_z||^2 ||f||^2$

これより $(1 - ||S_z||^2)||f||^2 \leq ||L_0 f||^2 \leq ||f||^2$ となり，(ii) が成り立つ。

(ii)⇒(iii)the open mappinig theorem より，任意の $g \in L_0 N$ に対し $||L_0^{-1} g|| \leq C||g||$ を満たす定数 $C > 1$ が存在する。よって，任意の $h \in N$ に対し $||h|| \leq C||L_0 h||$ が成り立つ。

(iii)⇒(i) (iii) が成り立つとする。このとき，任意の $h \in N$ に対し $||S_z^* h||^2 = ||h||^2 - ||L_0 h||^2 \leq (1 - \frac{1}{C^2})||h||^2$ が成り立つ。よって，$||S_z|| = ||S_z^*|| < 1$ が成り立つ。 □

ここで，$||S_z|| < 1, ||S_z|| = 1$ となる例をあげる。

Example 1. ([5]) $p(w)$ を $||p||_\infty < 1$ なる多項式とする。$N_p = H^2(\mathbb{T}^2) \ominus [z - p(w)]$ とすると N_p は $H^2(\mathbb{T}^2)$ 上の backward shift invariant subspace となる。このとき $S_z = S_{p(w)}$ となり $||S_z|| \leq ||p||_\infty < 1$ となる。

Example 2. $N = H^2(\mathbb{T}^2) \ominus wH^2(\mathbb{T}^2)$ とすると $N = H^2(\mathbb{T}_z)$ となり $H^2(\mathbb{T}^2)$ 上の backward shift invariant subspace となる。このとき $||S_z|| = ||S_z^*|| = ||T_z^*|| = 1$ となる。

Proposition 1 より $||S_z|| < 1$ となる必要条件の L_0 は N 上で 1 対 1 という条件に注目する。このとき A を $f \in N$ に対し $A(L_0 f) = L_0 S_z^* f$ で定まる $L_0 N$ から $L_0 N$ への線型作用素（有界であるとは限らない）が定義でき次が成り立つ。

Proposition 2. N を $H^2(\mathbb{T}^2)$ 上の backward shift invariant subspace とし L_0 は N 上で 1 対 1 とする。このとき A が有界ならば次の (i)〜(iii) をみたす $\varphi(w) \in H^\infty(\mathbb{T}_w)$ が存在する。

(i) $A = T_\varphi^*|_{L_0 N}$

(ii) $||A|| = ||\varphi||_\infty$

(iii) $N = \{\sum_{n=0}^{\infty} (T_{\varphi^n}^* g) z^n ; g(w) \in L_0 N\}$

ここで A が well-defind であるが有界でない例をあげる。

Example 3. $\varphi(w) \in H^2(\mathbb{T}_w)$ に対し $N_\varphi = H^2(\mathbb{T}^2) \ominus [z-\varphi(w)]$ とすると N_φ は $H^2(\mathbb{T}^2)$ 上の backward shift invariant subspace となる。このとき　$\varphi(w) = \sum_{n=0}^{\infty} \frac{w^n}{n} = -\log(1-w)$ $(w \in \mathbb{D})$ とすると A は N_φ 上で well-defind であるが有界でない。

最後に A が有界となるときの特徴付けとして次の定理がある。

Theorem 1. N を $H^2(\mathbb{T}^2)$ 上の backward shift invariant subspace とするとき次の (i)〜(iii) は同値である。

(i) L_0 は N 上で 1 対 1 であり，かつ A は有界

(ii) $S_z = S_{\varphi(w)}$ をみたす $\varphi(w) \in H^\infty(\mathbb{T}_w)$ が存在する。

(iii) $z - \varphi(w) \in N^\perp$ をみたす $\varphi(w) \in H^\infty(\mathbb{T}_w)$ が存在する。

またこのとき　$\varphi(w)$ は $||A|| = ||\varphi||_\infty$ にとることができる。

References

[1] K.Hoffman, *Banach Space of Analytic Functions*,Prentice-Hall Englewood Cliffs, N.J.,1962

[2] K.Izuchi and R.Yang,*Strictly contractive compression on backward shift invariant subspaces over the torus*,preprint.

[3] W.Rudin,*Function Theory in Polydiscs*,Benjamin, New York, 1969

[4] D.Sarason,*Generated interporation in* H^∞,Trans. Amer. Math. Soc.127(1967),179-203

[5] R.Yang,*On two variable Jordan blocks*,to appear in Acta.Sci.,Math. (Szeged)

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An example of a z-invariant subspace of H^∞ with index $\mathfrak{c}$ (実数濃度の index をもつ H^∞ の z-不変部分空間の一例)

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Abstract. We consider norm closed z-invariant subspaces of H^∞. Borichev [1] proved an existence of a z-invariant subspace of H^∞ with index $\mathfrak{c}$, that is generated by an uncountable family of Blaschke products. Here $\mathfrak{c}$ is the cardinal number of real field. We show an existence of a z-invariant subspace of H^∞ with index $\mathfrak{c}$, that is generated by an uncountable family of singular inner functions.

D を複素平面の単位開円板，∂D を単位円周とする．$H^\infty = H^\infty(D)$ を D 上の有界正則関数全体からなる Banach 環，$L^\infty(\partial D)$ を ∂D 上の本質的有界な可測関数全体からなる Banach 環とする．

inner function (内部関数) と呼ばれる，H^∞ の研究において重要な役割を果たす関数がある．$\varphi(z)$ が inner function であるとは，$\varphi(z) \in H^\infty$ かつ $|\varphi(e^{i\theta})| = 1$ a.e. on ∂D を満たすときをいう．一般に inner function は，Blaschke 積と singular inner function (特異内部関数) と呼ばれる 2 つの特別な型の inner function の積で表される．

$\{z_n\}_n$ は，$\sum_{n=1}^{\infty}(1-|z_n|) < \infty$ を満たす D 内の点列とする．そのとき

$$B(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}, \quad z \in D,$$

で定義される関数を Blaschke 積という．

$$\psi_\mu(z) = \exp\left(- \int_{\partial D} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(e^{i\theta}) \right), \quad z \in D,$$

を singular inner function という．ここで，μ は ∂D 上の Lebesgue 測度と互いに singular な測度である．

M を H^∞ の (closed) subspace とする．M が z-invariant であるとは，"$f \in M$ ならば $zf \in M$" を満たすときをいう．ここで z は恒等関数である．ここでは，z をかける線形作用素に関して invariant なものだけを考えるので，"z-" を省略して単に invariant ということにする．

H^∞ においてはノルム位相と ($L^\infty(\partial D)$ の weak-star 位相を H^∞ に制限した) weak-star 位相の 2 つの位相を考えることができる．

まず最初に，M を H^∞ の weak-star closed invariant subspace とする．そのとき，M は H^∞ の weak-star closed ideal である．H^∞ の weak-star closed ideal については次の特徴づけが知られている．

定理 ([2, p.85]) M を nonzero な H^∞ の weak-star closed ideal とする．そのとき，inner function φ が存在して

$$M = \varphi H^\infty.$$

ここで，φ は絶対値 1 の定数を除いて一意に定まる．逆に，ある inner function φ に対して，φH^∞ は H^∞ の nonzero な weak-star closed ideal である．

これに対して，H^∞ の norm closed な invariant subspace に関する結果はあまり知られていないのではないかと思う．以下に H^∞ の norm closed invariant subspace の簡単な例を与える．

<u>H^∞ の norm closed invariant subspace の例</u>

(1) 単位閉円板 $\bar{D}$ 上の連続関数全体を $C(\bar{D})$ とする．

$$A(D) = H^\infty \cap C(\bar{D})$$

は H^∞ の norm closed invarinat subspace である．$A(D)$ は円板環と呼ばれる．

(2) $\alpha \in D$ とする．

$$\{f \in H^\infty; f(\alpha) = 0\}$$

は H^∞ の norm closed invarinat subspace である．これは H^∞ の極大イデアルである．

(3) $0 \neq f \in H^\infty$ とする．

$$\{z^n f; n = 0, 1, 2, \ldots\} \text{ の closed linear span}$$

は H^∞ の norm closed invarinat subspace である．これは single generated invariant subspace と呼ばれる．

これより，M を H^∞ の norm closed invariant subspace とする．商空間 M/zM の線形空間としての次元が n であるとき，M は index n をもつ (または codimension n property をもつ) という．H^∞ の場合，次元 n としては非負整数 $0, 1, 2, \ldots$, 可算濃度の無限，または実数濃度の無限をとりうる．

上で与えた H^∞ の norm closed invariant subspace の例はすべて index 1 である．突然であるが，ここで Hardy 空間 $H^2(D)$ の invariant subspace についてコメントする．

$H^2(D)$ の invariant subspace に対して，H^∞ の場合と同様に，index を定義する．そのとき，Beurling の定理と Richter [6] の結果より，$H^2(D)$ の nonzero な任意の invariant subspace は index 1 しかもたないことがわかる．しかし，H^∞ は index が 2 以上の norm closed invariant subspace を含んでいる．例えば，index 2 の例については Richter [6] を参照せよ．さらに，Borichev は次の興味深い結果を与えている．以下において，$[0,1]$ は閉区間，$\mathfrak{c}$ は $[0,1]$ の濃度である．

定理 ([1]) Blaschke 積の族 $\{B_\alpha; \alpha \in [0,1]\}$ で次の条件を満たすものが存在する; $\{B_\alpha; \alpha \in [0,1]\}$ を含む H^∞ の最小の norm closed invariant subspace が index $\mathfrak{c}$ をもつ．

Borichev は Blaschke 積を用いて，実数濃度の index をもつ H^∞ の norm closed invariant subspace を作っている．この結果を受けて，もう一つの典型的な inner function である singuler inner function を用いて上と同様な性質をもつ invariant subspace を作ることができるだろうか？という問題を考え，次の肯定的な結果が得られた．

定理 singular inner function の族 $\{\psi_{\mu_\alpha}; \alpha \in [0,1]\}$ で次の条件を満たすものが存在する; $\{\psi_{\mu_\alpha}; \alpha \in [0,1]\}$ を含む H^∞ の最小の norm closed invariant subspace が index $\mathfrak{c}$ をもつ．

上の Blaschke 積，または singular inner function の族の構成方法に興味のある方は Borichev [1], または Niwa [5] を参照して欲しい．

参考文献

[1] A. Borichev, *Invariant subspaces of given index in Banach spaces of analytic functions*, J. Reine Angew. Math., **505** (1998), 23–44.

[2] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[3] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, N. J., 1962.

[4] K. Izuchi, *Singular inner functions of L^1- type II*, J. Math. Soc. Japan, **53** (2001), 287–305.

[5] N. Niwa, *Another example of an invariant subspace of H^∞ with index* $\mathfrak{c}$, to appear in Hokkaido Math. J.

[6] S. Richter, *Invariant subspaces in Banach spaces of analytic functions*, Trans. Amer. Math. Soc., **304** (1987), 585–616.

On a Theorem of MacCluer and Shapiro

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Abstract

Let u be a holomorphic function in the unit ball B of $\mathbb{C}^n$ and φ be a univalent holomorphic self-map of B. We give some sufficient conditions for u and φ that the weighted composition operator uC_φ is bounded or compact on the Hardy space $H^p(B)$ and the weighted Bergman space $A^p(\nu_\alpha)$ for all $0 < p < \infty$ and $-1 < \alpha < \infty$. This our result is a generalization of a theorem of B. D. MacCluer and J. H. Shapiro[6] concerning the composition operator C_φ. And we also give similar sufficient conditions for such operator to be metrically bounded or metrically compact on the Privalov space $N^p(B)$ and the weighted Bergman-Privalov space $(AN)^p(\nu_\alpha)$ for all $1 \le p < \infty$ and $-1 < \alpha < \infty$.

1 Introduction

$B \equiv B_n$ を $\mathbb{C}^n$ の単位球, $S \equiv \partial B$ を単位球面とする. ν は B 上の正規化された Lebesgue 測度を表し, σ は S 上の正規化された Lebesgue 測度を表す. $\alpha \in (-1,\infty)$ に対し, $c_\alpha = \Gamma(n+\alpha+1)/\{\Gamma(n+1)\Gamma(\alpha+1)\}$, $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z)$ $(z \in B)$ とおく. このとき, ν_α は B 上の正値 Borel 測度であり, $\nu_\alpha(B) = 1$ である. $H(B)$ により B 上の正則関数の全体を表す.

$p \in (0,\infty)$, $\alpha \in (-1,\infty)$ に対し, B 上の Hardy 空間 $H^p(B)$, 荷重 Bergman 空間 $A^p(\nu_\alpha)$ を次のように定義する:

$$H^p(B) \stackrel{\text{def}}{=} \left\{ f \in H(B) \,:\, \|f\|_{H^p}^p \equiv \sup_{0<r<1} \int_S |f_r|^p \, d\sigma < \infty \right\},$$

$$A^p(\nu_\alpha) \stackrel{\text{def}}{=} \left\{ f \in H(B) \,:\, \|f\|_{A^p(\nu_\alpha)}^p \equiv \int_B |f|^p \, d\nu_\alpha < \infty \right\}.$$

ここで, $f_r(\zeta) \equiv f(r\zeta)$ $(r \in (0,1),\ \zeta \in S)$ である. B 上の Privalov 空間 $N^p(B)$ $(1 < p < \infty)$, Nevanlinna 空間 $N(B)$ を次のように定義する:

$$N^p(B) \stackrel{\text{def}}{=} \left\{ f \in H(B) \,:\, \|f\|_{N^p}^p \equiv \sup_{0<r<1} \int_S \{\log(1+|f_r|)\}^p \, d\sigma < \infty \right\},$$

$$N(B) \stackrel{\text{def}}{=} \left\{ f \in H(B) \,:\, \|f\|_{N(B)} \equiv \sup_{0<r<1} \int_S \log(1+|f_r|) \, d\sigma < \infty \right\}.$$

便宜上, $N(B)$ を $N^1(B)$ で表す. さらに, 荷重 Bergman-Privalov 空間 $(AN)^p(\nu_\alpha)$ $(1 \le p < \infty,\ -1 < \alpha < \infty)$ を次のように定義する:

$$(AN)^p(\nu_\alpha) \stackrel{\text{def}}{=} \left\{ f \in H(B) \,:\, \|f\|_{(AN)^p(\nu_\alpha)}^p \equiv \int_B \{\log(1+|f|)\}^p \, d\nu_\alpha < \infty \right\}.$$

B から B への正則写像 φ に対し, 合成作用素 $C_\varphi : f \mapsto f \circ \varphi$ は $H(B)$ から $H(B)$ への線形作用素である. 1 次元の場合, Littlewood の subordination 定理により, C_φ は単位円板 $\mathbb{D}$ 上の Hardy 空間 $H^p(\mathbb{D})$ から $H^p(\mathbb{D})$ への有界作用素である. また, φ が $\mathbb{D}$ から $\mathbb{D}$ への単葉な正則写像の時, C_φ が $H^p(\mathbb{D})$ 上のコンパクト作用素になる為の必要十分条件は

$$\lim_{|z|\uparrow 1} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0$$

である (J. H. Shapiro[7], p.39). 多次元の場合, C_φ は必ずしも $H^p(B)$ 上の有界作用素とはならない (Cima, Stanton and Wogen[4]). B から B への単葉な正則写像 φ に対し, $\Omega_\varphi(z) = \|\varphi'(z)\|^2/|J_\varphi(z)|^2$ $(z \in B)$ とおく. ここで, $\varphi'(z)$ は z における φ の Fréchet 微分であり, $\|\varphi'(z)\|$ は $\varphi'(z)$ の作用素ノルムである. また, $J_\varphi(z)$ は z における φ の複素ヤコビアンである. 1986 年, B. D. MacCluer-J. H. Shapiro は単葉な正則写像 φ に対し, Hardy 空間 $H^p(B)$ 及び, 荷重 Bergman 空間 $A^p(\nu_\alpha)$ 上での C_φ の有界性, コンパクト性について考察し, 次の結果を得た:

Theorem([6], B. D. MacCluer-J. H. Shapiro). $0 < p < \infty, -1 < \alpha < \infty$ とする. φ を B から B への単葉な正則写像で,

$$\sup_{z \in B} \Omega_\varphi(z) < \infty$$

を満たすとする. この時, 次が成立する:

(a) C_φ は $H^p(B)$ から $H^p(B)$ への有界作用素である.

(b) C_φ が $H^p(B)$ 上のコンパクト作用素になる為の必要十分条件は,

$$\lim_{|z|\uparrow 1} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0$$

である.

(c) C_φ は $A^p(\nu_\alpha)$ から $A^p(\nu_\alpha)$ への有界作用素である.

(d) C_φ が $A^p(\nu_\alpha)$ 上のコンパクト作用素になる為の必要十分条件は,

$$\lim_{|z|\uparrow 1} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0$$

である.

$N^p(\mathbb{D})$ 上の合成作用素についての研究は J. S. Choa と H. O. Kim による [1, 2, 3] 等があり, $(AN)^1(\nu)$ 上の合成作用素に関しては J. Xiao[8] がある. 一般に, X を位相の入った線形空間とする. X 上で定義される正値汎関数 $\|\cdot\|_X$ が次の 2 条件を満たすとする:

$$\|f\|_X = 0 \iff f = 0,$$
$$\|f+g\|_X \le \|f\|_X + \|g\|_X \quad \text{for } f, g \in X.$$

位相の入った線形空間 Y についても同様な正値汎関数 $\|\cdot\|_Y$ が定義されていると仮定する. X から Y への線形作用素 T に対して, ある定数 $0 < K < \infty$ が存在し,

$$\|Tf\|_Y \le K\|f\|_X \quad (f \in X)$$

が成り立つ時, 作用素 T は metrically bounded であるという. また, T が X における任意の閉球 $B_R = \{ f \in X : \|f\|_X \le R\}$ を Y における相対コンパクト集合に写す時, 作用素 T は metrically compact であるという (cf.[3], p.381, 2.4). X と Y がノルム空間の場合には, T が有界であることと metrically bounded であることは同値であり, また, T がコンパクトであることと metrically compact であることは同値である.

ここでは, 上記の MacCluer-Shapiro の定理の一般化として荷重合成作用素 $uC_\varphi : f \mapsto u\cdot(f\circ\varphi)$ について考え, それが $H^p(B)$ あるいは $A^p(\nu_\alpha)$ 上の有界作用素, コンパクト作用素になる為の十分条件について述べる. さらに, uC_φ が $N^p(B)$ あるいは $(AN)^p(\nu_\alpha)$ 上 metrically bounded 及び, metrically compact になる為の十分条件についても論ずる.

2 Results

合成作用素 C_φ の $N^p(B)$ 及び, $(AN)^p(\nu_\alpha)$ 上での metrically bounded 性については, MacCluer-Shapiro の定理と同様の結果が成り立つ.

Lemma 1. $1 \le p < \infty$, $-1 < \alpha < \infty$ とする. φ を B から B への単葉な正則写像で,

$$\sup_{z\in B} \Omega_\varphi(z) < \infty$$

を満たすとする. この時,

(a) $C_\varphi : N^p(B) \longrightarrow N^p(B)$ は metrically bounded である.

(b) $C_\varphi : (AN)^p(\nu_\alpha) \longrightarrow (AN)^p(\nu_\alpha)$ は metrically bounded である.

荷重合成作用素 uC_φ の $H^p(B)$ 及び, $A^p(\nu_\alpha)$ 上でのコンパクト性について次が成り立つ.

Lemma 2. $0 < p < \infty$, $-1 < \alpha < \infty$ とする. X を $H^p(B)$, $A^p(\nu_\alpha)$ の何れかとする. $u \in H(B)$, φ を B から B への正則写像で, $(uC_\varphi)(X) \subset X$ とする. この時, 次の 2 条件は同値である:

(a) $uC_\varphi : X \longrightarrow X$ はコンパクト作用素である.

(b) 任意の有界列 $\{f_j\}_{j\in\mathbb{N}} \subset X$ が 0 に B 上で広義一様収束しているならば,

$$\lim_{j\to\infty} \|uC_\varphi f_j\|_X = 0$$

である.

uC_φ の $N^p(B)$ 及び, $(AN)^p(\nu_\alpha)$ 上での metrically compact 性についても Lemma 2 と同様の結果が成り立つ.

Lemma 3. $1 \leq p < \infty$, $-1 < \alpha < \infty$ とする. X を $N^p(B)$, $(AN)^p(\nu_\alpha)$ の何れかとする. $u \in H(B)$, φ を B から B への正則写像で, $(uC_\varphi)(X) \subset X$ とする. この時, 次の 2 条件は同値である:

(a) $uC_\varphi : X \longrightarrow X$ は metrically compact である.

(b) 任意の有界列 $\{f_j\}_{j \in \mathbb{N}} \subset X$ が 0 に B 上で広義一様収束しているならば,

$$\lim_{j \to \infty} \|uC_\varphi f_j\|_X = 0$$

である.

以下, $u \in H(B)$, φ は $\sup_{z \in B} \Omega_\varphi(z) < \infty$ を満たす, B から B への正則写像とする.

Theorem 1. $0 < p < \infty$ とする.

(a) u と φ が 2 つの条件:

$$\limsup_{|z| \uparrow 1} |u(z)|^{p-2} |(\nabla u)(z)|^2 (1 - |z|^2) < \infty,$$

$$\limsup_{|z| \uparrow 1} \frac{|u(z)|^p (1 - |z|^2)}{1 - |\varphi(z)|^2} < \infty$$

を満たす時, 荷重合成作用素 uC_φ は $H^p(B)$ 上の有界作用素である.

(b) u と φ が 2 つの条件:

$$\lim_{|z| \uparrow 1} |u(z)|^{p-2} |(\nabla u)(z)|^2 (1 - |z|^2) = 0,$$

$$\lim_{|z| \uparrow 1} \frac{|u(z)|^p (1 - |z|^2)}{1 - |\varphi(z)|^2} = 0$$

を満たす時, 荷重合成作用素 uC_φ は $H^p(B)$ 上のコンパクト作用素である.

Theorem 2. $0 < p < \infty$, $-1 < \alpha < \infty$ とする.

(a) u と φ が 2 つの条件:

$$\limsup_{|z| \uparrow 1} |u(z)|^{p-2} |(\nabla u)(z)|^2 (1 - |z|^2)^2 < \infty,$$

$$\limsup_{|z| \uparrow 1} |u(z)|^p \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} < \infty$$

を満たす時, 荷重合成作用素 uC_φ は $A^p(\nu_\alpha)$ 上の有界作用素である.

(b) u と φ が 2 つの条件:

$$\lim_{|z| \uparrow 1} |u(z)|^{p-2} |(\nabla u)(z)|^2 (1 - |z|^2)^2 = 0,$$

$$\lim_{|z| \uparrow 1} |u(z)|^p \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} = 0$$

を満たす時, 荷重合成作用素 uC_φ は $A^p(\nu_\alpha)$ 上のコンパクト作用素である.

Theorem 3. $1 \le p < \infty$ とする.

(a) u と φ が 2 つの条件:

(i) $1 \le p \le 2$ の時,

$$\limsup_{|z|\uparrow 1} \left[\max\{|u(z)|^{p-4}, 1\}|(\nabla u)(z)|^2(1-|z|^2)\right] < \infty,$$

$$\limsup_{|z|\uparrow 1} \left[\frac{\max\{|u(z)|^{p-3}, |u(z)|^2\}(1-|z|^2)}{1-|\varphi(z)|^2}\right] < \infty$$

(ii) $2 < p < \infty$ の時,

$$\limsup_{|z|\uparrow 1} \left[\max\{|u(z)|^{-2}, |u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)\right] < \infty,$$

$$\limsup_{|z|\uparrow 1} \left[\frac{\max\{|u(z)|^{-1}, |u(z)|^{p}\}(1-|z|^2)}{1-|\varphi(z)|^2}\right] < \infty$$

を満たす時, 荷重合成作用素 uC_φ は $N^p(B)$ 上 metrically bounded である.

(b) u と φ が 2 つの条件:

(i) $1 \le p \le 2$ の時,

$$\lim_{|z|\uparrow 1} \left[\max\{|u(z)|^{p-4}, 1\}|(\nabla u)(z)|^2(1-|z|^2)\right] = 0,$$

$$\lim_{|z|\uparrow 1} \left[\frac{\max\{|u(z)|^{p-3}, |u(z)|^2\}(1-|z|^2)}{1-|\varphi(z)|^2}\right] = 0$$

(ii) $2 < p < \infty$ の時,

$$\lim_{|z|\uparrow 1} \left[\max\{|u(z)|^{-2}, |u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)\right] = 0,$$

$$\lim_{|z|\uparrow 1} \left[\frac{\max\{|u(z)|^{-1}, |u(z)|^{p}\}(1-|z|^2)}{1-|\varphi(z)|^2}\right] = 0$$

を満たす時, 荷重合成作用素 uC_φ は $N^p(B)$ 上 metrically compact である.

Theorem 4. $1 \le p < \infty$, $-1 < \alpha < \infty$ とする.

(a) u と φ が 2 つの条件:

(i) $1 \le p \le 2$ の時,

$$\limsup_{|z|\uparrow 1} \left[\max\{|u(z)|^{p-4}, 1\}|(\nabla u)(z)|^2(1-|z|^2)^2\right] < \infty,$$

$$\limsup_{|z|\uparrow 1} \left[\max\{|u(z)|^{p-3}, |u(z)|^2\}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right] < \infty$$

(ii) $2 < p < \infty$ の時,

$$\limsup_{|z|\uparrow 1} \left[\max\{|u(z)|^{-2}, |u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)^2\right] < \infty,$$

$$\limsup_{|z|\uparrow 1} \left[\max\{|u(z)|^{-1}, |u(z)|^{p}\}\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right] < \infty$$

を満たす時, 荷重合成作用素 uC_φ は $(AN)^p(\nu_\alpha)$ 上 metrically bounded である.

(b) u と φ が 2 つの条件:

(i) $1 \le p \le 2$ の時,

$$\lim_{|z|\uparrow 1} \left[\max\{|u(z)|^{p-4}, 1\}|(\nabla u)(z)|^2(1-|z|^2)^2\right] = 0,$$

$$\lim_{|z|\uparrow 1} \left[\max\{|u(z)|^{p-3}, |u(z)|^2\} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right] = 0$$

(ii) $2 < p < \infty$ の時,

$$\lim_{|z|\uparrow 1} \left[\max\{|u(z)|^{-2}, |u(z)|^{p-2}\}|(\nabla u)(z)|^2(1-|z|^2)^2\right] = 0,$$

$$\lim_{|z|\uparrow 1} \left[\max\{|u(z)|^{-1}, |u(z)|^p\} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right] = 0$$

を満たす時, 荷重合成作用素 uC_φ は $(AN)^p(\nu_\alpha)$ 上 metrically compact である.

References

[1] J. S. Choa and H. O. Kim, *Composition operators on some F-algebras of holomorphic functions*, Nihonkai Math. J., **7** (1996), 29–39.

[2] J. S. Choa and H. O. Kim, *Compact composition operators on the Nevanlinna class*, Proc. Amer. Math. Soc., **125** (1997), 145–151.

[3] J. S. Choa and H. O. Kim, *Composition operators between Nevanlinna-type spaces*, J. Math. Anal. Appl., **257** (2001), 378–402.

[4] J. A. Cima, C. S. Stanton and W. R. Wogen, *On boundedness of composition operators on $H^2(B_2)$*, Proc. Amer. Math. Soc., **91** (1984), 217–222.

[5] C. C. Cowen and B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, 1994.

[6] B. D. MacCluer and J. H. Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces*, Can. J. Math., **38** (1986), 878–906.

[7] J. H. Shapiro, Composition Operators and Classical Function Theory, Springer-Verlag, 1993.

[8] J. Xiao, *Compact composition operators on the Area-Nevanlinna class*, Expo. Math., **17** (1999), 255–264.

Norm closed invariant subspaces in L^∞ and H^∞

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ABSTRACT

We characterize norm closed subspaces B of $L^\infty(\partial D)$ such that $C(\partial D)B \subset B$, and maximal ones in the family of proper closed subspaces B of $L^\infty(\partial D)$ such that $A(D)B \subset B$, where $A(D)$ is the disk algebra. Analogously, we characterize closed subspaces of H^∞ that are simultaneously invariant under S and S^*, the forward and the backward shift operators, and maximal invariant subspaces of H^∞.

1 Introduction and Preliminaries

Let L^∞ be the Banach space of essentially bounded functions on the unit circle ∂D, and H^∞ be the norm closed subspace of functions that admit an analytic extension to D. Let z be the identity function on ∂D. A norm closed subspace B of L^∞ is called invariant if $zB \subset B$ and doubly invariant if $zB \subset B$ and $\overline{z}B \subset B$. Weak-star closed invariant subspaces of L^∞ have being known for a long time as Beurling's theorem, see [1, pp. 131-133]. They have one of the following forms:

(a) $B = \chi_E L^\infty$, where $E \subset \partial D$ is a measurable set and χ_E denotes its characteristic function. This happens when B is doubly invariant.

(b) $B = uH^\infty$, where $|u(z)| = 1$ for almost every $z \in \partial D$.

It follows immediately that every weak-star closed invariant subspace of H^∞ has form (b) with u an inner function. Since the structure of inner functions is known completely, see [2], by Beurling's characterization, one can write down all weak-star closed invariant subspaces of H^∞ in an explicit way.

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Despite these results, very little is known about closed invariant subspaces of L^∞ and H^∞ with respect to the norm topology. In this paper, we concern with only the norm topology. In the family of proper invariant subspaces of L^∞ or H^∞, the maximal one is called a maximal invariant subspace of L^∞ or H^∞, respectively.

First, we give a complete characterization of doubly invariant subspaces of L^∞. From this, we are able to determine maximal invariant subspaces of L^∞. Let $Sf = zf, f \in H^\infty$, and S^* be the operator on H^∞ defined by $(S^*f)(z) = \overline{z}(f(z) - f(0))$. We characterize the closed subspaces of H^∞ that are simultaneously invariant under S and S^*. Also, we describe the maximal invariant subspaces of H^∞.

Let A be a uniform algebra. We denote by $M(A)$ the maximal ideal space of A, that is, $M(A)$ consists of the linear functionals on A that are multiplicative and nonzero. It is a compact Hausdorff space with the weak-star topology induced by the dual space of A. The Gelfand transform, defined as $\hat{a}(\varphi) = \varphi(a)$, for $a \in A$ and $\varphi \in M(A)$, establishes an isometric isomorphism between A and a closed subalgebra of $C(M(A))$, the space of continuous functions on $M(A)$.

If A is also a C^* algebra, the Gelfand transform is a $*$-isomorphism from A onto $C(M(A))$. This allows us to identify L^∞ with $C(M(L^\infty))$, from which the dual space $(L^\infty)^*$ is identified with the space $\mathfrak{M}(M(L^\infty))$ of finite regular Borel measures on $M(L^\infty)$ with the total variation norm. Specifically, every element of $(L^\infty)^*$ has the form

$$L_\mu(f) = \int_{M(L^\infty)} \hat{f}\, d\mu \quad (f \in L^\infty),$$

where $\mu \in \mathfrak{M}(M(L^\infty))$, and for every such μ, the above formula defines a linear functional on L^∞ with $\|L_\mu\| = \|\mu\|$. Put $\ker L_\mu = \{f \in L^\infty : L_\mu(f) = 0\}$. When $\int_{M(L^\infty)} \hat{f} d\mu = 0$ holds, we write as $\hat{f} \perp \mu$. For a subspace B of L^∞, we write $B \perp \mu$ if $\hat{f} \perp \mu$ for every $f \in B$. We denote by $\operatorname{supp}\mu$ the closed support set of μ.

The fiber over $\lambda \in \partial D$ in $M(L^\infty)$ is defined by $M_\lambda = \{\varphi \in M(L^\infty) : \hat{z}(\varphi) = \lambda\}$. Since $|\hat{z}| \equiv 1$, $M(L^\infty) = \bigcup_{\lambda \in \partial D} M_\lambda$. Measures that are supported on a single fiber will be of particular interest in our discussion. So, we define

$$\mathfrak{F} = \{\mu \in \mathfrak{M}(M(L^\infty)) : \operatorname{supp}\mu \subset M_\lambda \text{ for some } \lambda \in \partial D\}.$$

2 Doubly, and maximal invariant subspaces in L^∞

Recall that a norm closed subspace $B \subset L^\infty$ is called invariant if $zB \subset B$ (i.e.: $A(D)B \subset B$), and is called doubly invariant if $zB \subset B$ and $\bar{z}B \subset B$ (i.e.: $C(\partial D)B \subset B$). If $f \in C(\partial D)$ and $\lambda \in \partial D$, then $\hat{f}|_{M_\lambda} = f(\lambda)$. So, if $\mu \in \mathfrak{F}$ is supported on M_λ for some $\lambda \in \partial D$, then $\hat{f} = f(\lambda)$ on $\operatorname{supp} \mu$, and consequently

$$\hat{f} \ker L_\mu \subset \ker L_\mu.$$

That is, $\ker L_\mu$ is a doubly invariant subspace of L^∞ for every $\mu \in \mathfrak{F}$. It follows immediately that if $\mathfrak{G} \subset \mathfrak{F}$, then $\bigcap \{\ker L_\mu : \mu \in \mathfrak{G}\}$ is doubly invariant. The following theorem shows that the converse is also valid.

Theorem 1 *Every doubly invariant subspace B of L^∞ has the form*

$$B = \bigcap_{\mu \in \mathfrak{G}} \ker L_\mu$$

for some family $\mathfrak{G} \subset \mathfrak{F}$.

To prove our theorem, we need the following lemma due to Glicksberg, see [1, p. 61].

Lemma 2 *Let B be a doubly invariant subspace of L^∞ and $f \in L^\infty$. Then $f \in B$ if and only if $\hat{f}|_{M_\lambda} \in \hat{B}|_{M_\lambda}$ for every $\lambda \in \partial D$. Also, if $\mu \perp B$ then $\mu|_{M_\lambda} \perp B|_{M_\lambda}$.*

Proof of Theorem 1. Put $\mathfrak{G} = \{\mu \in \mathfrak{F} : \mu \perp B\}$. For $\lambda \in \partial D$, let $\mathfrak{G}_\lambda$ denote the set of measures μ in $\mathfrak{G}$ which are concentrated on M_λ. Then $\mathfrak{G} = \bigcup\{\mathfrak{G}_\lambda : \lambda \in \partial D\}$. By Lemma 2, we have $\mu|_{M_\lambda} \perp B|_{M_\lambda}$ for all $\mu \perp B$. Hence by [1, p. 57], $\hat{B}|_{M_\lambda}$ is closed in $C(M_\lambda)$. Therefore we have

$$\begin{aligned}
B &= \bigcap_{\lambda \in \partial D} \{f \in L^\infty : \hat{f}|_{M_\lambda} \in \hat{B}|_{M_\lambda}\} && \text{by Lemma 2} \\
&= \bigcap_{\lambda \in \partial D} \{f \in L^\infty : \hat{f} \perp \mu \text{ for every } \mu \in \mathfrak{G}_\lambda\}, && \text{because } \hat{B}|_{M_\lambda} \text{ is closed} \\
&= \{f \in L^\infty : \hat{f} \perp \mu \text{ for every } \mu \in \mathfrak{G}\} \\
&= \bigcap_{\mu \in \mathfrak{G}} \ker L_\mu.
\end{aligned}$$

Let B be an invariant subspace of L^∞. We can similarly define maximal invariant subspaces of B.

Corollary 3 *Let B be a doubly invariant subspace of L^∞ and N be an invariant subspace of B. Then*

(i) *N is maximal in B if and only if $N = \ker L_\mu \cap B$ for some measure $\mu \in \mathfrak{F}$ with $\mu \not\perp B$.*

(ii) *N is contained in a maximal invariant subspace of B if and only if $\bigcup_{n\geq 0} \overline{z}^n N$ is not dense in B.*

3 Invariant subspaces in H^∞

We recall that $Sf = zf$ and $S^*f = \overline{z}(f - f(0))$ for $f \in H^\infty$. Let $B \subset H^\infty$ be a closed subspace. Then B is an invariant subspace if and only if B is invariant under S. Put $\mathfrak{F}_0 = \{\mu \in \mathfrak{F} : \mu \perp \mathbb{C}\}$.

Theorem 4 *Let B be a closed subspace of H^∞ such that $B \neq \{0\}$. Then B is invariant under both S and S^* if and only if there is a family $\mathfrak{G} \subset \mathfrak{F}_0$ such that*

$$B = \bigcap_{\mu\in\mathfrak{G}} \ker L_\mu \cap H^\infty.$$

Corollary 5 *Let B be a maximal invariant subspace of H^∞. If there exists $f \in B$ which is invertible in H^∞, then $B = \ker L_\nu \cap H^\infty$ for some $\nu \in \mathfrak{F}$ with $\nu \not\perp H^\infty$.*

For $w \in D$, we write $\varphi_w(z) = (w - z)(1 - \overline{w}z)$.

Lemma 6 *Let B be a maximal invariant subspace of H^∞ and b be a finite Blaschke product. If $B \neq \varphi_w H^\infty$ for all $w \in D$, then $B \cap bH^\infty = bB$.*

Theorem 7 *Let B be a maximal invariant subspace of H^∞. Then either $B = \varphi_w H^\infty$ for some $w \in D$ or $B = \ker L_\nu \cap H^\infty$ for some $\nu \in \mathfrak{F}$ with $\nu \not\perp H^\infty$.*

Proof. Let B_∞ be the closure of $\bigcup_{n\geq 0} \overline{z}^n B$ in $H^\infty + C(\partial D)$. Assume first that $1 \in B_\infty$. Then there are $g \in B$ and a nonnegative integer n such that $\|\overline{z}^n g - 1\|_\infty < 1/2$. Hence, $\|g - z^n\|_\infty < 1/2$. Since $|\widehat{z^n}| \equiv 1$ on $M(H^\infty) \setminus D$, then $|\hat{g}| \geq 1/2$ on $M(H^\infty) \setminus D$. It is well known that a function g in H^∞ non-vanishing on $M(H^\infty) \setminus D$ can be factored as $g = bf$, where $f \in (H^\infty)^{-1}$ and b is a finite Blaschke product.

If there is some $w \in D$ such that $B = \varphi_w H^\infty$, we are done. If not, Lemma 6 says that $f \in B$. Hence, Corollary 5 says that $B = \ker L_\mu \cap H^\infty$ for $\mu \in \mathfrak{F}$ with $\mu \not\perp H^\infty$. Thus our theorem holds when $1 \in B_\infty$.

Now suppose that $1 \notin B_\infty$. Since B_∞ is a doubly invariant subspace of L^∞, Theorem 1 states that there exists a family $\mathfrak{G} \subset \mathfrak{F}$ such that $B_\infty = \bigcap\{\ker L_\mu : \mu \in \mathfrak{G}\}$. Since $1 \notin B_\infty$, there must be some $\nu \in \mathfrak{G}$ such that $\nu \not\perp 1$. Thus

$$B \subset B_\infty \cap H^\infty \subset \ker L_\nu \cap H^\infty.$$

Since $1 \notin \ker L_\nu \cap H^\infty$, this space is a proper invariant subspace of H^∞. Since B is maximal in H^∞, $B = \ker L_\nu \cap H^\infty$, as claimed.

Open Problems. The most important open problem is to obtain a complete characterization of invariant subspaces of L^∞ and H^∞. If $B \subset H^\infty$ is invariant, the weak-star closure of B has the form uH^∞, where u is an inner function. Thus, $\overline{u}B$ is an invariant and a weak-star dense subspace of H^∞. Therefore, the problem for H^∞ reduces to characterize invariant subspaces which are weak-star dense in H^∞. A similar analysis can be done for L^∞, in which case we also have to characterize invariant subspaces whose weak-star closure is $\chi_E L^\infty$, where $E \subset \partial D$ is some measurable set. The results of this paper suggest that the whole picture may be not out of reach.

We have other questions. Is every invariant subspace in H^∞ contained in a maximal one? How

about L^∞? Obviously, these questions are less ambitious than the ones in the previous paragraphs.

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References

[1] T. W. GAMELIN, "Uniform Algebras", Prentice Hall, Englewood Cliffs, New Jersey (1969).

[2] J. B. GARNETT, "Bounded Analytic Functions", Academic Press, New York (1981).

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