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Spectral analysis of an abstract pair interaction model (抽象的な対相互作用モデルのスペクトル解析)

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Abstract

We consider an abstract pair-interaction model in quantum field theory with a coupling constant $\lambda \in \mathbb{R}$ and analyze the Hamiltonian $H(\lambda)$ of the model. In the massive case, there exist constants $\lambda_c < 0$ and $\lambda_{c,0} < \lambda_c$ such that, for each $\lambda \in (\lambda_{c,0}, \lambda_c) \cup (\lambda_c, \infty)$, $H(\lambda)$ is diagonalized by a proper Bogoliubov transformation, so that the spectrum of $H(\lambda)$ is explicitly identified, where the spectrum of $H(\lambda)$ for $\lambda > \lambda_c$ is different from that for $\lambda \in (\lambda_{c,0}, \lambda_c)$. As for the case $\lambda < \lambda_{c,0}$, we show that $H(\lambda)$ is unbounded from above and below. In the massless case, λ_c coincides with $\lambda_{c,0}$.

Contents

1	Introduction	2
2	Preliminaries	4
2.1	The abstract boson Fock Space	4
2.2	Bogoliubov Transformation	5
2.3	Hamiltonians	6
3	The Main Theorem	9
3.1	Assumptions	9
3.2	The Main Theorem	11
4	Definitions and properties of some functions and operators	12
4.1	Functions D and $D_{\pm}$	12
4.2	Operators $R_{\pm}$	16
4.3	Operators $\Omega_{\pm}$	20
4.4	Operators U and V	24
5	Commutation relations	26
5.1	Relations between U and V	28
5.2	Hilbert-Schmidtness of V	29
6	Analysis in the case $\lambda > \lambda_c$	31
6.1	Time evolution	31
6.2	Proof of Theorem 3.6 (1)	32

7	Analysis in the case $\lambda_{c,0} < \lambda < \lambda_c$	34
7.1	Properties of X, Y and ξ	34
7.2	Commutation relations	37
7.3	Proof of Theorem 3.6 (2)	38
8	Analysis in the case $\lambda < \lambda_{c,0}$	39
9	Generalization of the ϕ^2-model	40
10	Appendix	42

1 Introduction

This thesis is based on the joint work [6]. We consider an abstract pair-interaction model in quantum field theory. The Hamiltonian of the model is of the form

$$H(\lambda) := d\Gamma_b(T) + \frac{\lambda}{2}\Phi_s(g)^2$$

acting in the boson Fock space $\mathcal{F}_b(\mathcal{H})$ over a Hilbert space $\mathcal{H}$ (see Subsection 2.1), where T is a self-adjoint operator on $\mathcal{H}$, $d\Gamma_b(T)$ is the second quantization operator of T , $\Phi_s(g)$ is the Segal field operator with test vector g in $\mathcal{H}$ (see Subsection 2.1) and $\lambda \in \mathbb{R}$ is a coupling constant. A model of this type is called a ϕ^2 -model.

There have been many studies on massive or massless ϕ^2 -models in concrete forms or abstract forms (see, e.g., [4, 8, 9, 11, 12, 16]). In [11] and [16], the (essential) self-adjointness of the Hamiltonian of a ϕ^2 -model is proved in the case where $\lambda > 0$ or $|\lambda|$ is sufficiently small. In [11], the existence of a ground state of a ϕ^2 -model also is shown in the case where the quantum field under consideration is massive and $\lambda > 0$.

It is a well known that Hamiltonians with linear and/or quadratic interactions in quantum fields may be analyzed by the method of Bogoliubov transformations (see, e.g., [1, 2, 3, 4, 7, 8, 10, 12]). A typical Bogoliubov transformation is constructed from bounded linear operators U, V and a conjugation operator J on $\mathcal{H}$ satisfying the following equations:

$$\begin{cases} U^*U - V^*V &= I, \\ U_J^*V - V_J^*U &= 0, \\ UU^* - V_JV_J^* &= I, \\ UV^* - V_JU_J^* &= 0, \end{cases} \quad (1.1)$$

where $A_J := JAJ$ and A^* is the adjoint of a densely defined linear operator A . It is well known that there is a unitary operator $\mathbb{U}$ on $\mathcal{F}_b(\mathcal{H})$ which implements the Bogoliubov

transformation in question if and only if V is Hilbert-Schmidt [7, 13, 14, 15]. Moreover, it is shown that, under the condition that V is Hilbert-Schmidt and suitable additional conditions, the Hamiltonian under consideration is unitarily equivalent via $\mathbb{U}$ to a second quantization operator up to a constant addition. For example, the Pauli-Fierz model with dipole approximation, which can be regarded as a kind of ϕ^2 -model, is analyzed by this method in [10].

Recently, a general quadratic form Hamiltonian with a coupling constant $\lambda \in \mathbb{R}$ has been analyzed in [12] and it is shown that, in the case of a massive quantum field, under suitable conditions, the Hamiltonian is diagonalized by a Bogoliubov transformation. In [8], the sufficient condition formulated in [12] to obtain the result just mentioned has been extended. The spectrum of the standard pair-interaction model in physics, which is a concrete realization of the abstract pair-interaction model, is formally known [9] in the case where $\lambda > \lambda_{c,0}$ and $\lambda \neq \lambda_c$ for the constants λ_c and $\lambda_{c,0}$ which satisfy $\lambda_{c,0} < \lambda_c$. The paper [4] gives a rigorous proof for that in the framework of the boson Fock space theory over $\mathcal{H} = L^2(\mathbb{R}^d)$ for any $d \in \mathbb{N}$ and $\lambda > \lambda_c$.

One of the motivations for the present work is to extend the theory developed in [4] with $\mathcal{H} = L^2(\mathbb{R}^d)$ to the theory with $\mathcal{H}$ being an abstract Hilbert space including the case where $\lambda < \lambda_c$. It is a well known fact (see [9]) that the spectral properties of the standard pair-interaction model may depend on whether $\lambda > \lambda_c$ or $\lambda < \lambda_c$. Hence it is important to clarify this aspect mathematically. Therefore we analyze our model also for the region $\lambda < \lambda_c$. We show that, in the massive case with $\lambda \in (\lambda_{c,0}, \lambda_c)$ also, the method of Bogoliubov transformations can be applied to prove that the Hamiltonian $H(\lambda)$ is unitarily equivalent to a second quantization operator up to a constant addition. Then we see that the spectrum of $H(\lambda)$ for $\lambda \in (\lambda_{c,0}, \lambda_c)$ is different from that for $\lambda > \lambda_c$. In the massless case, $\lambda_{c,0}$ coincides with λ_0 .

The main results of the present paper include the following (1)–(3) (see Theorem 2.8 for more details): (1) Identification of the spectra of $H(\lambda)$ for $\lambda > \lambda_c$. (2) Identification of the spectra of $H(\lambda)$ for $\lambda_{c,0} < \lambda < \lambda_c$ it is only in the massive case; in the massless case, $\lambda_{c,0} = \lambda_c$. In this case, bound states different from the ground state appear. (3) Unboundedness of $H(\lambda)$ from above and below for $\lambda < \lambda_{c,0}$.

The outline of this paper is as follows. In Section 2, we define our model and recall a fundamental fact in a general theory of Bogoliubov transformations. We prove the (essential) self-adjointness of $H(\lambda)$ (Theorem 2.3). Then we state the main theorem of this paper (Theorem 3.6). In Section 3, we construct the operators U and V which are used to define the Bogoliubov transformation we need. In Section 4, we show that U and V satisfy (1.1) and V is Hilbert-Schmidt. In Section 5, we prove Theorem 2.8 (1) and calculate the ground

state energy of $H(\lambda)$ in the case $\lambda > \lambda_c$. In Section 6, we prove Theorem 2.8 (2). In Section 7, we prove Theorem 2.8 (3). In Section 8, we consider a slightly generalized Hamiltonian which is of the form $H(\eta, \lambda) := H(\lambda) + \eta \Phi_S(f)$ for $\eta \in \mathbb{R}$ and $f \in \mathcal{H}$. Applying the methods and results in the preceding sections, we analyze $H(\eta, \lambda)$ and identify the spectra of it. In Appendix, we state some basic facts in the theory of boson Fock space.

2 Preliminaries

2.1 The abstract boson Fock Space

Let $\mathcal{H}$ be a Hilbert space over the complex field $\mathbb{C}$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The inner product is linear in the second variable and anti-linear in the first one. The symbol $\| \cdot \|_{\mathcal{H}}$ denotes the norm associated with it. We omit $\mathcal{H}$ in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$, respectively if there is no danger of confusion. For each non-negative integer $n = 0, 1, 2, \dots$, $\otimes_s^n \mathcal{H}$ denotes the n -fold symmetric tensor product Hilbert space of $\mathcal{H}$ with convention $\otimes_s^0 \mathcal{H} := \mathbb{C}$. Then

$$\mathcal{F}_b(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_s^n \mathcal{H}$$

is called the boson Fock space over $\mathcal{H}$. For a dense subspace $\mathcal{D}$ in $\mathcal{H}$, $\hat{\otimes}_s^n \mathcal{D}$ denotes the algebraic n -fold symmetric tensor product of $\mathcal{D}$ with $\hat{\otimes}_s^0 \mathcal{H} := \mathbb{C}$. Then

$$\mathcal{F}_{b, \text{fin}}(\mathcal{D}) := \hat{\oplus}_{n=0}^{\infty} \hat{\otimes}_s^n \mathcal{D}$$

is a dense subspace of $\mathcal{F}_b(\mathcal{H})$, where $\hat{\oplus}_{n=0}^{\infty} \mathcal{D}_n$ stands for the algebraic direct sum of subspace $\mathcal{D}_n \subset \otimes_s^n \mathcal{H}$, $n = 0, 1, 2, \dots$. The finite particle vector subspace

$$\mathcal{F}_{b,0}(\mathcal{H}) := \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_b(\mathcal{H}) \left| \begin{array}{l} \psi^{(n)} \in \otimes_s^n \mathcal{H}, \ n \geq 0, \text{ there is an integer } n_0 \in \mathbb{N} \\ \text{such that } \psi^{(n)} = 0, \text{ for all } n \geq n_0 \end{array} \right. \right\}$$

satisfies $\mathcal{F}_{b, \text{fin}}(\mathcal{D}) \subset \mathcal{F}_{b,0}(\mathcal{H}) \subset \mathcal{F}_b(\mathcal{H})$, in particular, it is dense in $\mathcal{F}_b(\mathcal{H})$. For a linear operator T on a Hilbert space, the domain of T will be denoted by $D(T)$.

For a densely defined closable operator T on $\mathcal{H}$, let $T_b^{(n)}$ be the densely defined closed operator on $\otimes_s^n \mathcal{H}$ defined by

$$T_b^{(n)} := \begin{cases} \overline{\sum_{j=1}^n I \otimes \cdots \otimes I \otimes \overbrace{T}^{j\text{-th}} \otimes I \otimes \cdots \otimes I \upharpoonright \hat{\otimes}_s^n D(T)}, & n \geq 1, \\ 0, & n = 0, \end{cases}$$

where I denotes the identity operator on $\mathcal{H}$, $\overline{A}$ denotes the closure of a closable operator A and $A \upharpoonright \mathcal{M}$ denotes the restriction of a linear operator A on a subspace $\mathcal{M}$. The operator

$$\mathrm{d}\Gamma_{\mathrm{b}}(T) := \oplus_{n=0}^{\infty} T_{\mathrm{b}}^{(n)}$$

is called the second quantization operator of T . If T is self-adjoint or non-negative, then so is $\mathrm{d}\Gamma_{\mathrm{b}}(T)$. For each $f \in \mathcal{H}$, there exists a unique densely defined closed operator $A(f)$ on $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$ such that its adjoint $A(f)^*$ is given as follows:

$$D(A(f)^*) := \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{b}}(\mathcal{H}) \left| \sum_{n=1}^{\infty} n \|S_n(f \otimes \psi^{(n-1)})\|^2 < \infty \right. \right\},$$

$$(A(f)^*\psi)^{(n)} = \sqrt{n} S_n(f \otimes \psi^{(n-1)}), \quad n \in \mathbb{N}, \quad (A(f)^*\psi)^{(0)} = 0 \text{ for } \psi \in D(A(f)^*),$$

where S_n is the symmetrization operator on the n -fold tensor product $\otimes^n \mathcal{H}$ of $\mathcal{H}$. The operator $A(f)$ (resp. $A(f)^*$) is called the annihilation (resp. creation) operator with test vector f . We have

$$\mathcal{F}_{\mathrm{b},0}(\mathcal{H}) \subset D(A(f)) \cap D(A(f)^*)$$

for all $f \in \mathcal{H}$ and $A(f)$ and $A(f)^*$ leave $\mathcal{F}_{\mathrm{b},0}(\mathcal{H})$ invariant. Moreover, they satisfy the following commutation relations:

$$[A(f), A(g)^*] = \langle f, g \rangle, \quad [A(f), A(g)] = 0, \quad [A(f)^*, A(g)^*] = 0, \quad \text{for all } f, g \in \mathcal{H} \quad (2.1)$$

on $\mathcal{F}_{\mathrm{b},0}(\mathcal{H})$, where $[A, B] := AB - BA$ is the commutator of linear operators A and B . The relation (2.1) is called the canonical commutation relations (CCR) over $\mathcal{H}$. The symmetric operator

$$\Phi_{\mathrm{s}}(f) := \frac{1}{\sqrt{2}}(A(f) + A(f)^*), \quad f \in \mathcal{H}$$

is called the Segal field operator with test vector f . We write its closure by the same symbol.

2.2 Bogoliubov Transformation

In this subsection, we define a Bogoliubov transformation and recall an important theorem about it. For a conjugation J on $\mathcal{H}$ (i.e., J is an anti-linear operator on $\mathcal{H}$ satisfying $\|Jf\| = \|f\|$ for all $f \in \mathcal{H}$ and $J^2 = I$) and a linear operator A on $\mathcal{H}$, we define

$$A_J := JAJ.$$

Definition 2.1. Let U and V be bounded linear operators on $\mathcal{H}$ and J be a conjugation on $\mathcal{H}$. For each $f \in \mathcal{H}$, let a linear operator $B(f)$ on $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$ be given by

$$B(f) := A(Uf) + A(JVf)^*.$$

Then the correspondence $(A(\cdot), A(\cdot)^*) \mapsto (B(\cdot), B(\cdot)^*)$ is called a Bogoliubov transformation.

By $\mathcal{F}_{b,0}(\mathcal{H}) \subset D(B(f))$, the adjoint $B(f)^*$ exists and the equation $B(f)^* = A(Uf)^* + A(JVf)$ holds on $\mathcal{F}_{b,0}(\mathcal{H})$ for each $f \in \mathcal{H}$. If the equations

$$U^*U - V^*V = I, \quad U_J^*V - V_J^*U = 0$$

hold, then the Bogoliubov transformation preserves CCR, i.e., it holds that

$$[B(f), B(g)^*] = \langle f, g \rangle, [B(f), B(g)] = 0, [B(f)^*, B(g)^*] = 0, \text{ for all } f, g \in \mathcal{H},$$

on $\mathcal{F}_{b,0}(\mathcal{H})$. The following theorem is well known (see [14, 15]):

Theorem 2.2. *Let $\mathcal{H}$ be separable and the operators U and V satisfy (1.1). Then there exists a unitary operator $\mathbb{U}$ on $\mathcal{F}_b(\mathcal{H})$ such that*

$$\mathbb{U}\overline{B(f)}\mathbb{U}^{-1} = A(f), \quad f \in \mathcal{H}$$

if and only if V is Hilbert-Schmidt.

2.3 Hamiltonians

For a self-adjoint operator T on $\mathcal{H}$, constants $\lambda, \eta \in \mathbb{R}$ which are called coupling constants, and vectors $f, g \in \mathcal{H}$, we define Hamiltonians $H(\lambda)$ and $H(\eta, \lambda)$ by

$$H(\lambda) := d\Gamma_b(T) + \frac{\lambda}{2}\Phi_s(g)^2, \quad H(\eta, \lambda) := H(\lambda) + \eta\Phi_s(f).$$

If $g = 0$, then $H(\lambda)$ and $H(\eta, \lambda)$ are well-known operators. Thus, we always assume that $g \neq 0$ in the present paper. If $g \in D(T^{-1/2})$, let the constant be defined by

$$\lambda_{c,0} := -\|T^{-1/2}g\|^{-2}.$$

Theorem 2.3. *Suppose that T is an injective, non-negative, self-adjoint operator on $\mathcal{H}$. Let $f \in D(T^{-1/2})$ and $g \in D(T^{-1/2}) \cap D(T)$. Then the following (1)-(3) hold:*

(1) *Let*

$$\lambda_T(g) := \|T^{-1/2}g\|^{-1}(\|T^{-1/2}g\| + \|T^{1/2}g\|)^{-1} \quad (2.2)$$

and $|\lambda| < \lambda_T(g)$. Then $H(\eta, \lambda)$ is self-adjoint with $D(H(\eta, \lambda)) = D(d\Gamma_b(T))$ and essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. Moreover, $H(\eta, \lambda)$ is bounded from below.

(2) *Let $|\lambda| \geq \lambda_T(g)$ and $f \in D(T^{1/2})$. Then $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. Moreover, if $\lambda \geq \lambda_T(g)$, then $H(\eta, \lambda)$ is self-adjoint.*

- (3) Let $f \in D(T^{1/2})$. Then $\overline{H(\lambda_{c,0})}$ is bounded from below. Moreover, if $\lambda > \lambda_{c,0}$, then $\overline{H(\eta, \lambda)}$ is also bounded from below for all $\eta \in \mathbb{R}$ and $D(d\Gamma_b(T)^{1/2}) = D(\overline{H(\eta, \lambda)} + M)^{1/2}$ for all constant $M \geq 0$ satisfying $\overline{H(\eta, \lambda)} + M \geq 0$.

Proof. (1) For any $\lambda \in \mathbb{R}$, by using (2.1), (10.1), (10.2) and [5, Theorem 5.18.], there are constants $a, b \geq 0$ such that for all $\psi \in D(d\Gamma_b(T))$,

$$\left\| \frac{\lambda}{2} \Phi_s(g)^2 \psi \right\| \leq \frac{|\lambda|}{4} (a \|d\Gamma_b(T)\psi\| + b \|\psi\|).$$

In particular, we can choose a and b which satisfy $a|\lambda|/4 < 1$ if $|\lambda| < \lambda_T(g)$. We remark that, to obtain the factor $\lambda_T(g)$, we need to deform terms $\|A(g)^{*2}\psi\|^2$, $\|A(g)^*A(g)\psi\|^2$ and $\|A(g)^2\psi\|^2$ coming from $\|\Phi_s(g)^2\psi\|^2$ ($\psi \in \mathcal{F}_{b,0}(\mathcal{H})$) to $\|A(g)A(g)^*\psi\|^2$ + a marginal term respectively. Thus, for $|\lambda| < \lambda_T(g)$, by the Kato-Rellich theorem, $H(\lambda)$ is self-adjoint. It is well known that $\Phi_s(f)$ is infinitesimally small with respect to $d\Gamma_b(T)$. Hence, by the Kato-Rellich theorem, for $|\lambda| < \lambda_T(g)$, $H(\eta, \lambda)$ is self-adjoint.

- (2) Firstly, we show that, for any $f \in D(T^{1/2})$ and $\eta, \lambda \in \mathbb{R}$, $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_b(T)$. By (10.1), (10.2) and [5, Theorem 5.18.], we can see that there exists $a > 0$ such that $\|H(\eta, \lambda)\psi\| \leq a\|(d\Gamma_b(T) + I)\psi\|$ for all $\psi \in D(d\Gamma_b(T))$. Let $f \in D(T)$. Then by (2.1) and (10.3), for any $\psi \in \mathcal{F}_{b,\text{fin}}(D(T))$, we have

$$\begin{aligned} & \langle H(\eta, \lambda)\psi, (d\Gamma_b(T) + I)\psi \rangle - \langle (d\Gamma_b(T) + I)\psi, H(\eta, \lambda)\psi \rangle \\ &= \frac{\lambda}{\sqrt{2}} (\langle \Phi_s(g)\psi, A(Tg)\psi \rangle - \langle A(Tg)\psi, \Phi_s(g)\psi \rangle) + \frac{\eta}{\sqrt{2}} (\langle \psi, A(Tf)\psi \rangle - \langle A(Tf)\psi, \psi \rangle). \end{aligned}$$

Thus, by (10.1) and (10.2), we obtain

$$|\langle H(\eta, \lambda)\psi, (d\Gamma_b(T) + I)\psi \rangle - \langle (d\Gamma_b(T) + I)\psi, H(\eta, \lambda)\psi \rangle| \leq C\|(d\Gamma_b(T) + I)^{1/2}\psi\|^2, \quad (2.3)$$

where $C := \{|\lambda|\|T^{1/2}g\|(\|g\| + 2\|T^{-1/2}g\|) + \sqrt{2}|\eta|\|T^{1/2}f\|\}$. By a limiting argument, using the fact that $\mathcal{F}_{b,\text{fin}}(D(T))$ is a core of $d\Gamma_b(T)$ and $d\Gamma_b(T)$ -boundedness of $\Phi_s(g)^2$, we can show that for $f \in D(T^{1/2})$ and $\psi \in D(d\Gamma_b(T))$, (2.3) holds. Thus, by the Nelson commutator theorem, for all $\eta, \lambda \in \mathbb{R}$, $H(\eta, \lambda)$ is essentially self-adjoint and $\overline{H(\eta, \lambda)}$ is essentially self-adjoint on any core of $d\Gamma_b(T)$. The equation $\overline{H(\eta, \lambda)} \upharpoonright \mathcal{D} = \overline{H(\eta, \lambda)} \upharpoonright \mathcal{D}$ holds for any core $\mathcal{D}$ of $d\Gamma_b(T)$. Hence $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta, \lambda \in \mathbb{R}$. Next we show that, if $\lambda > \|T^{-1/2}g\|^{-1}(\|T^{-1/2}g\| + \|T^{1/2}g\|)^{-1}$, then $H(\eta, \lambda)$ is self-adjoint. We can show that, for $\lambda > 0$ and any $0 < \varepsilon < 1$, there is a constant $c_\varepsilon > 0$ such that

$$(1 - \varepsilon)\|d\Gamma_b(T)\psi\|^2 + \left\| \frac{\lambda}{2} \Phi_s(g)^2 \psi \right\|^2 \leq \|H(\eta, \lambda)\psi\|^2 + c_\varepsilon \|\psi\|^2, \quad \psi \in D(d\Gamma_b(T)).$$

Hence $H(\eta, \lambda)$ is closed. In particular, it is self-adjoint.

- (3) From the fact that $\Phi_s(f)$ is infinitesimally small with respect to $d\Gamma_b(T)$, for any $\varepsilon > 0$, $\varepsilon d\Gamma_b(T) + \eta \Phi_s(f)$ is bounded from below. By (10.1), for any $\varepsilon > 0$ and $\psi \in D(d\Gamma_b(T)^{1/2})$,

$$|\langle \psi, A(f)\psi \rangle| \leq \|T^{-1/2}f\| \left(\varepsilon \|d\Gamma_b(T)^{1/2}\psi\|^2 + \frac{1}{4\varepsilon} \|\psi\|^2 \right).$$

Hence if the assertion follows for $\eta = 0$, then so is for all η . Thus we show that the assertion follows for $\eta = 0$. If $\lambda > 0$, then clearly $H(\lambda) \geq 0$. Let $\lambda < 0$. By (10.1) and (10.2), for any $\psi \in D(d\Gamma_b(T)^{1/2})$, it follows that

$$\|\Phi_s(g)\psi\|^2 \leq 2\|T^{-1/2}g\|^2 \|d\Gamma_b(T)^{1/2}\psi\|^2 + \|g\|^2 \|\psi\|^2.$$

Thus for any $\psi \in D(d\Gamma_b(T))$,

$$\begin{aligned} \langle \psi, H(\lambda)\psi \rangle &= \|d\Gamma_b(T)^{1/2}\psi\|^2 + \frac{\lambda}{2} \|\Phi_s(g)\psi\|^2 \\ &\geq (1 + \lambda\|T^{-1/2}g\|^2) \|d\Gamma_b(T)^{1/2}\psi\|^2 + \frac{\lambda}{2} \|g\|^2 \|\psi\|^2. \end{aligned} \quad (2.4)$$

Hence $H(\lambda)$ is bounded from below if $\lambda \geq \lambda_{c,0}$.

Let $\lambda \geq \lambda_{c,0}$ and $M \geq 0$ be a constant satisfying $H(\lambda) + M \geq 0$. Then for any $\psi \in D(d\Gamma_b(T)) = D(H(\lambda))$,

$$\|(\overline{H(\lambda)} + M)^{1/2}\psi\|^2 \leq (1 + |\lambda|\|T^{-1/2}g\|^2) \|d\Gamma_b(T)^{1/2}\psi\|^2 + \left(\frac{|\lambda|}{2} \|g\|^2 + M \right) \|\psi\|^2. \quad (2.5)$$

By the fact that $D(d\Gamma_b(T))$ is a core of $d\Gamma_b(T)^{1/2}$, we have $D(d\Gamma_b(T)^{1/2}) \subset D((\overline{H(\lambda)} + M)^{1/2})$ and (2.5) holds on $D(d\Gamma_b(T)^{1/2})$.

In the case $\lambda > 0$, the fact that $\Phi_s(g)^2$ is non-negative implies that $\|H(\lambda)^{1/2}\psi\| \geq \|d\Gamma_b(T)^{1/2}\psi\|$ holds for any $\psi \in D(d\Gamma_b(T))$. In the case $0 > \lambda > \lambda_{c,0}$,

$$\|d\Gamma_b(T)^{1/2}\psi\|^2 \leq \frac{1}{1 + \lambda\|T^{-1/2}g\|^2} \left\{ \|(\overline{H(\lambda)} + M)^{1/2}\psi\|^2 - \left(\frac{\lambda}{2} \|g\|^2 + M \right) \|\psi\|^2 \right\}$$

holds for any $\psi \in D(d\Gamma_b(T))$ by (2.4). Hence for $\lambda > \lambda_{c,0}$ there is a constant $a, b \geq 0$ such that for any $\psi \in D(d\Gamma_b(T))$,

$$\|d\Gamma_b(T)^{1/2}\psi\| \leq a\|(\overline{H(\lambda)} + M)^{1/2}\psi\| + b\|\psi\|. \quad (2.6)$$

By a functional calculus, $D(d\Gamma_b(T))$ is a core of $(\overline{H(\lambda)} + M)^{1/2}$. This fact and (2.6) imply that $D((\overline{H(\lambda)} + M)^{1/2}) \subset D(d\Gamma_b(T)^{1/2})$ and (2.6) holds on $D((\overline{H(\lambda)} + M)^{1/2})$. $\square$

Remark 2.4. By [3, Lemma 13-15], if $\mathcal{H}$ is separable, then Theorem 2.3 takes the following forms:

Let $\mathcal{H}$ be separable, T be a non-negative, injective self-adjoint operator, $f \in D(T^{-1/2})$ and $g \in D(T^{-1/2}) \cap D(T^{1/2})$. Then the following (1)-(3) hold:

- (1) Let $\lambda > \lambda_{c,0}$. Then $H(\eta, \lambda)$ is self-adjoint with $D(H(\eta, \lambda)) = D(d\Gamma_b(T))$ and essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. Moreover, $H(\eta, \lambda)$ is bounded from below.
- (2) Let $\lambda \leq \lambda_{c,0}$ and $f \in D(T^{1/2})$. Then $H(\eta, \lambda)$ is essentially self-adjoint on any core of $d\Gamma_b(T)$ for all $\eta \in \mathbb{R}$. In particular, if $\eta = 0$ and $\lambda = \lambda_{c,0}$, then $H(\lambda_{c,0}) = H(0, \lambda_{c,0})$ is bounded from below.
- (3) Let $\lambda > \lambda_{c,0}$. Then $D(d\Gamma_b(T)^{1/2}) = D((H(\eta, \lambda) + M)^{1/2})$ for all constant $M \geq 0$ satisfying $H(\eta, \lambda) + M \geq 0$.

3 The Main Theorem

3.1 Assumptions

To prove our main theorem stated later (Theorem 3.6), we need some assumptions. For a closed operator A , $\sigma(A)$ denotes the spectrum of A . If A is self-adjoint, then $\sigma_{ac}(A)$ (resp. $\sigma_p(A)$, $\sigma_{sc}(A)$) denotes the absolutely continuous (resp. point, singular continuous) spectrum of A . For a self-adjoint operator A which is bounded from below,

$$E_0(A) := \inf \sigma(A)$$

is called the lowest energy of A . In particular, it is called the ground state energy of A if $E_0(A) \in \sigma_p(A)$. In this case, an eigenvector of A with eigenvalue $E_0(A)$ is called a ground state of A . The ground state is said to be unique if $\dim \text{Ker}(A - E_0(A)) = 1$. For linear operators A and B , the symbol $A \subset B$ means that $D(A) \subset D(B)$ and $Af = Bf$ for all $f \in D(A)$, i.e., B is an extension of A .

Definition 3.1. Let T be a self-adjoint operator on $\mathcal{H}$ and $\{E(B) \mid B \in \mathbf{B}^1\}$ be the spectral measure associated with T on the Borel field $\mathbf{B}^1$ on $\mathbb{R}$. The operator T is called purely absolutely continuous if, for each $f \in \mathcal{H}$, the measure $\|E(\cdot)f\|^2$ on $\mathbf{B}^1$ is absolutely continuous with respect to the one-dimensional Lebesgue measure.

Definition 3.2. For a purely absolutely continuous self-adjoint operator T and vectors $f, g \in \mathcal{H}$, $\psi_{g,f}$ denotes the Radon-Nikodym derivative of the finite complex Borel measure $\langle g, E(\cdot)f \rangle$ on $\mathbf{B}^1$. In particular, we set $\psi_g := \psi_{g,g}$.

Assumption 3.3. (1) The operator T is a non-negative, purely absolutely continuous self-adjoint operator.

(2) The fixed vector $g \in \mathcal{H}$ satisfies $g \in D(\hat{T}^{-1/2}) \cap D(T^{1/2})$ and $Jg = g$, where $\hat{T} := T - E_0$, $E_0 := E_0(T)$ and J is a conjugation on $\mathcal{H}$ satisfying $JD(T) \subset D(T)$ and $JT\psi = TJ\psi$ for any $\psi \in D(T)$ (i.e., $JT \subset TJ$),

(3) $\sup_{E_0 < x} x^{\pm 1} \psi_g(x) < \infty$ and $\psi_g(x) > 0$ for all $x \in (E_0, \infty)$,

(4) $\psi_g \in C([E_0, \infty)) \cap C^1((E_0, \infty))$ and $\lim_{x \downarrow E_0} x^{-1} \psi'_g(x) = 0 = \lim_{x \rightarrow \infty} x^{-1} \psi'_g(x)$.

Remark 3.4. The operator T is injective since it is a purely absolutely continuous self-adjoint operator. Since T has no eigenvector, the inverse of $\hat{T}$ exists. Assumption 3.3 (2) implies that $T_J = T$. In general, for a self-adjoint operator A and a conjugation J , we can choose a vector $f \in D(A)$ satisfying $Jf = f$ if $A_J = A$. Thus the vector g in Assumption 3.3 (2) exists. By Assumption 3.3 (3), one can easily show that $\sup_{x \in \sigma(T)} \psi_g(x) < \infty$ and, for each $f \in \mathcal{H}$, the functions $\psi_{g,f}$ and $\psi_{T^{\pm 1/2}g,f}$ are in $L^2(\mathbb{R})$ and the maps $: f \mapsto \psi_{g,f}, \psi_{T^{\pm 1/2}g,f}$ are bounded. Actually, for any $h \in \mathcal{H}$ and $B \in \mathbf{B}^1$, the following inequality holds

$$|\langle E(B)h, f \rangle|^2 \leq \|E(B)h\|^2 \|E(B)f\|^2$$

by Schwarz's inequality. Thus we obtain $|\psi_{h,f}(\mu)|^2 \leq \psi_h(\mu) \psi_f(\mu)$ for almost all $\mu \in \mathbb{R}$ with respect to the Lebesgue measure. Hence, by Assumption 3.3 (3), we have the boundedness of the mappings. Moreover, we see that for any $F \in L^2(\mathbb{R})$, $g \in D(F(T))$, where $F(T)$ denotes the operator defined by $F(T) := \int_{\mathbb{R}} F(\mu) dE(\mu)$. In particular, g is in $D(\psi_{g,f}(T))$ for any $f \in \mathcal{H}$.

Lemma 3.5. Let T be a self-adjoint operator such that $JT \subset TJ$. Then

(1) $E(B)_J = E(B)$, for all $B \in \mathbf{B}^1$.

(2) Let F be a Borel measurable function on $\mathbb{R}$. Then $F(T)_J = F^*(T)$, where F^* is complex conjugation of F .

Proof. These are proved by using the spectral theorem. □

3.2 The Main Theorem

In this subsection, we state the main theorem of the present paper. Let λ_c be a constant defined by

$$\lambda_c := - \left(\int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2} d\|E(\mu)g\|^2 \right)^{-1} < 0.$$

Then, by a functional calculus, we obtain $\lambda_{c,0} \leq \lambda_c$, and $\lambda_{c,0} = \lambda_c$ if and only if $E_0 = 0$.

Theorem 3.6. *Let $\mathcal{H}$ be separable. Then the following (1)-(3) hold:*

- (1) *Let T and g satisfy Assumption 3.3. If $\lambda > \lambda_c$, then there are a unitary operator $\mathbb{U}$ on $\mathcal{F}_b(\mathcal{H})$ and a constant $E_g \in \mathbb{R}$ such that*

$$\mathbb{U}H(\lambda)\mathbb{U}^{-1} = d\Gamma_b(T) + E_g. \quad (3.1)$$

In particular, $\mathbb{U}^{-1}\Omega_0$ is the unique ground state of $H(\lambda)$, where $\Omega_0 := (1, 0, 0, \dots) \in \mathcal{F}_b(\mathcal{H})$ is the Fock vacuum, and

$$\sigma(H(\lambda)) = \{E_g\} \cup [E_0 + E_g, \infty), \quad (3.2)$$

$$\sigma_{ac}(H(\lambda)) = [E_0 + E_g, \infty), \quad \sigma_p(H(\lambda)) = \{E_g\}, \quad \sigma_{sc}(H(\lambda)) = \emptyset. \quad (3.3)$$

- (2) *Let T and g satisfy Assumption 3.3 and $E_0 > 0$. If $\lambda_{c,0} < \lambda < \lambda_c$, then there exist a unitary operator $\mathbb{V}$ on $\mathcal{F}_b(\mathcal{H})$, an injective non-negative self-adjoint operator ξ on $\mathcal{H}$ and a constant $E_b \geq 0$ such that ξ has a ground state and*

$$\mathbb{V}H(\lambda)\mathbb{V}^{-1} = d\Gamma_b(\xi) + E_g - E_b.$$

In particular, $\mathbb{V}^{-1}\Omega_0$ is the unique ground state of $H(\lambda)$, and

$$\sigma(H(\lambda)) = \cup_{n=0}^{\infty} \{n\beta + E_g - E_b\} \cup [E_0 + E_g - E_b, \infty),$$

$$\sigma_{ac}(H(\lambda)) = [E_0 + E_g - E_b, \infty),$$

$$\sigma_p(H(\lambda)) = \cup_{n=0}^{\infty} \{n\beta + E_g - E_b\}, \quad \sigma_{sc}(H(\lambda)) = \emptyset,$$

where $\beta > 0$ is the discrete ground state energy of ξ .

- (3) *Let T be a non-negative, injective self-adjoint operator. If $g \in D(T^{-1/2})$ and $\lambda < \lambda_{c,0}$, then $H(\lambda)$ is unbounded from above and below.*

Example 3.7. A concrete realization of the abstract model is given as follows (see [9, Chapter 12]):

$$\mathcal{H} \leftrightarrow L^2(\mathbb{R}^d), \quad T \leftrightarrow \omega, \quad g \leftrightarrow \frac{\hat{\rho}}{\sqrt{\omega}},$$

where ω is the multiplication operator associated with the function $\omega(k) := \sqrt{|k|^2 + m^2}$, $k \in \mathbb{R}^d$ for a fixed $m \geq 0$ and $\hat{\rho}$ is the Fourier transform of a function $\rho \in L^2(\mathbb{R}^d)$ satisfying $\hat{\rho}/\sqrt{\omega} \in L^2(\mathbb{R}^d)$. Assume that $\hat{\rho}$ is rotation invariant, i.e., there exists a function v on $[0, \infty)$ such that $\hat{\rho}(k) = v(|k|)$ for all $k \in \mathbb{R}^d$. Then we have $\psi_g(s) = |S^{d-1}| \omega_1^{-1}(s)^{d-2} |v(\omega_1^{-1}(s))|^2$ for $s > m$, where $|S^{d-1}|$ is the surface area of the $(d-1)$ -dimensional unite sphere with convention $|S^0| = 2\pi$ and $\omega_1(r) = \sqrt{r^2 + m^2}$, $r \geq 0$. Set $\psi_g(m) := 0$. Hence, with J being the complex conjugation, the following conditions (2)'-(4)' imply that the present model satisfies Assumption 3.3:

(2)' $\hat{\rho}(k)^* = \hat{\rho}(k)$ and

$$\hat{\rho} \in L^2(\mathbb{R}^d), \int_{\mathbb{R}^d} \frac{|\hat{\rho}(k)|^2}{|k|^2} dk < \infty.$$

(3)' $\hat{\rho}$ is rotation invariant. $\sup_{k \in \mathbb{R}^d} \omega(k)^{\pm 1/2} |k|^{(d-2)/2} |\hat{\rho}(k)| < \infty$. $\hat{\rho}(k) > 0$, for all $k \neq 0$.

(4)' $v \in C^1([0, \infty))$ and

$$\begin{aligned} \lim_{|k| \rightarrow 0} |k|^{d-4} \hat{\rho}(k) \{(d-2)\hat{\rho}(k) + 2|k|v'(|k|)\} &= 0, \\ \lim_{|k| \rightarrow \infty} |k|^{d-4} \hat{\rho}(k) \{(d-2)\hat{\rho}(k) + 2|k|v'(|k|)\} &= 0. \end{aligned}$$

We can show that ψ_g is right continuous at m by $\int_{\mathbb{R}^d} |\hat{\rho}(k)|^2 |k|^{-2} dk < \infty$ and $v \in C^1([0, \infty))$. Thus, $\psi_g \in C([m, \infty))$. For example, one can easily check that the function

$$\hat{\rho}(k) := \exp\left(-\frac{1}{|k|^2} - |k|^2\right), \quad k \in \mathbb{R}^d \setminus \{0\}, \quad \hat{\rho}(0) := 0$$

satisfies the above conditions (2)'-(4)'.

4 Definitions and properties of some functions and operators

In this section, we introduce some functions and operators. We assume that $\mathcal{H}$ is separable and Assumption 3.3 from this section to Section 6.

4.1 Functions D and $D_{\pm}$

Lemma 4.1. *Let $D : \mathbb{C} \setminus (0, \infty) \rightarrow \mathbb{C}$ be the function*

$$D(z) := 1 + \lambda \int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2 - z} d\|E(\mu)g\|^2, \quad z \in \mathbb{C} \setminus (0, \infty).$$

Then D is well-defined and analytic in $\mathbb{C} \setminus [0, \infty)$. Moreover, the following hold :

- (1) For all $\lambda > \lambda_c$, $D(z)$ has no zeros in $\mathbb{C} \setminus [0, \infty)$.
- (2) For all $\lambda < \lambda_c$, $D(z)$ has a unique simple zero in the negative real axis $(-\infty, 0)$.

Proof. If $\text{Im} z \neq 0$ (resp. $\text{Re} z < 0$), then for any $n \in \mathbb{N}$,

$$\int_{[E_0, \infty)} \left| \frac{\mu}{(\mu^2 - E_0^2 - z)^n} \right| d\|E(\mu)g\|^2 \leq c^{-n} \|T^{1/2}g\|^2 < \infty,$$

where c is $|\text{Im} z|$ (resp. $|\text{Re} z|$). If $z = 0$, then

$$\int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2} d\|E(\mu)g\|^2 \leq \|\hat{T}^{-1/2}g\|^2 < \infty.$$

Thus, by using the Lebesgue dominated convergence theorem, D is well-defined and analytic in $\mathbb{C} \setminus [0, \infty)$.

- (1) If $\lambda = 0$, then $D(z) = 1$ for all $z \in \mathbb{C} \setminus (0, \infty)$, so it has no zeros. Let $\lambda \neq 0$ and $z = x + iy \in \mathbb{C} \setminus (0, \infty)$. Then we see that

$$\text{Im } D(z) = y\lambda \int_{[E_0, \infty)} \frac{\mu}{(\mu^2 - E_0^2 - x)^2 + y^2} d\|E(\mu)g\|^2.$$

Thus $\text{Im } D(z) = 0$ is equivalent to $y = 0$. Therefore $D(z) = 0$ if and only if $D(x) = 0$. Let $y = 0$. In the case $\lambda > 0$, one has $D(x) > 0$ for all $x \in (-\infty, 0]$. Thus D has no zeros. Next, we consider the case $\lambda < 0$. We have for $x < 0$,

$$D'(x) = \lambda \int_{[E_0, \infty)} \frac{\mu}{(\mu^2 - E_0^2 - x)^2} d\|E(\mu)g\|^2 < 0.$$

Thus D is monotone decreasing in $(-\infty, 0)$. If $\lambda > \lambda_c$, then $D(0) > 0$. Hence D has no zeros.

- (2) Let $\lambda < \lambda_c$. We can see that

$$D(0) = 1 + \lambda \int_{[E_0, \infty)} \frac{\mu}{\mu^2 - E_0^2} d\|E(\mu)g\|^2 = 1 - \frac{\lambda}{\lambda_c} < 0.$$

By the Lebesgue dominated convergence theorem, $D(x) \rightarrow 1$ as $x \rightarrow -\infty$. Since D is monotone decreasing in $(-\infty, 0)$, D has a unique simple zero in $(-\infty, 0)$.

□

Let

$$\phi_g(x) := \psi_g(\sqrt{x})\chi_{[E_0^2, \infty)}(x), \quad x \in \mathbb{R},$$

where χ_B is the characteristic function of $B \in \mathbf{B}^1$.

Lemma 4.2. *The following hold :*

(1) *The function ϕ_g satisfies $\phi_g \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\sup_{x \in \mathbb{R}} |\phi'_g(x)| < \infty$.*

(2) *Let*

$$A_\varepsilon^{(1)}(x) := \frac{x}{\pi(x^2 + \varepsilon^2)}, \quad A_\varepsilon^{(2)}(x) := \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}, \quad x \in \mathbb{R}, \quad \varepsilon > 0$$

*be the conjugate poisson kernel and the poisson kernel respectively and $f * h$ denote the convolution of functions f and h . Let*

$$(H_\varepsilon f)(s) := \frac{1}{\pi} \int_{|x-s| \geq \varepsilon} \frac{f(x)}{s-x} dx, \quad (Hf)(s) := \lim_{\varepsilon \downarrow 0} (H_\varepsilon f)(s), \quad s \in \mathbb{R}, \quad \varepsilon > 0,$$

where Hf is called the Hilbert transform of f . Then for all $x \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} (A_\varepsilon^{(1)} * \phi_g)(x) = (H\phi_g)(x), \quad \lim_{\varepsilon \downarrow 0} (A_\varepsilon^{(2)} * \phi_g)(x) = \phi_g(x),$$

hold uniformly in x .

Proof. For any $h > 0$, by Assumption 3.3 (1), (4) and the mean value theorem, there exists $\theta \in (E_0 + h/2, E_0 + h)$ such that

$$\int_{E_0+h/2}^{E_0+h} \frac{\psi_g(\mu)}{\mu - E_0} d\mu = \frac{h}{2} \frac{\psi_g(\theta)}{\theta - E_0}.$$

This fact and $\theta < E_0 + h$ imply that

$$\|E([E_0, E_0 + h])\hat{T}^{-1/2}g\|^2 = \int_{[E_0, E_0+h]} \frac{\psi_g(\mu)}{\mu - E_0} d\mu > \frac{\psi_g(\theta)}{2}. \quad (4.1)$$

By taking the limit $h \downarrow 0$ and Assumption 3.3 (1), the left hand side of (4.1) tends to zero. Thus we obtain $\lim_{h \rightarrow E_0+0} \psi_g(h) = 0$. This fact and $\psi_g \in C([E_0, \infty))$ imply that $\psi_g(E_0) = 0$. Since ψ_g is the Radon-Nikodym derivative of $\|E(\cdot)g\|^2$ and $E_0 \leq T$, we have $\psi_g(x) = 0$ for $x < E_0$. Thus $\phi_g \in C(\mathbb{R})$. By the differentiability of ψ_g , we obtain $\phi'_g(x) = \psi'_g(\sqrt{x})/(2\sqrt{x})$ for $x > E_0^2$ and $\phi'_g(x) = 0$ for $x < E_0^2$. Thus, ϕ'_g is continuous on $(-\infty, E_0^2) \cup (E_0^2, \infty)$. Since $\phi'_g(x) = 0$ for $x < E_0^2$ and $\lim_{h \rightarrow 0+0} (E_0 + h)^{-1} \psi_g(E_0 + h) = 0$, we have $\lim_{h \rightarrow 0} \phi'_g(E_0^2 + h) = 0$. By this fact and the l'Hôpital theorem, we obtain $\lim_{h \rightarrow 0+0} (\phi_g(E_0^2 + h) - \phi_g(E_0^2))/h = 0$. We

have $\lim_{h \rightarrow 0-0} (\phi_g(E_0^2 + h) - \phi_g(E_0^2))/h = 0$ since $\phi_g(x) = 0$ for $x < E_0^2$. Thus ϕ_g is continuous at E_0^2 . Hence $\phi_g \in C^1(\mathbb{R})$. By the fact that $\psi'_g(x) = 0$ for $x < E_0$ and Assumption 3.3 (4) imply that $\phi_g \in C^1(\mathbb{R})$ and $\phi'_g(E_0^2) = 0$. By Assumption 3.3 (2) and a change of variable, we have $\phi_g \in L^1(\mathbb{R})$. We obtain $\phi_g \in L^2(\mathbb{R})$ by Assumption 3.3 (3) and a change of variable. The inequality $\sup_{x \in \mathbb{R}} |\phi'_g(x)| < \infty$ is given by Assumption 3.3 (4). The assertion (1) holds. Next we consider the assertion (2). By (1), in particular, ϕ_g is bounded and uniformly continuous. Thus it is easy to see that $A_\varepsilon^{(2)} * \phi_g$ converges uniformly to ϕ_g . Moreover, by (1), Hölder's inequality, the mean value theorem and a similar estimate to the proof of [17, Theorem 92.], we can show that $(A_\varepsilon^{(1)} * \phi_g)(x) - (H_\varepsilon \phi_g)(x)$ tends to 0 uniformly in x as $\varepsilon \downarrow 0$. Hence the assertion (2) holds. $\square$

Detailed studies of the Hilbert transform are given in [17].

Lemma 4.3. *For all $s \geq 0$, $D_\pm(s) := \lim_{\varepsilon \downarrow 0} D(s \pm i\varepsilon)$ are uniformly convergent and continuous in $s \geq 0$ with*

$$D_\pm(s) = 1 - \frac{\lambda\pi}{2} (H\phi_g)(E_0^2 + s) \pm i \frac{\lambda\pi}{2} \psi_g \left(\sqrt{E_0^2 + s} \right), \quad s \geq 0. \quad (4.2)$$

Proof. For any $s \geq 0$ and $\varepsilon > 0$, we have by a change of variable

$$D(s \pm i\varepsilon) = 1 - \frac{\lambda\pi}{2} (A_\varepsilon^{(1)} * \phi_g)(E_0^2 + s) \pm i \frac{\lambda\pi}{2} (A_\varepsilon^{(2)} * \phi_g)(E_0^2 + s).$$

Thus, by Lemma 4.2, $D_\pm$ converge uniformly in $s \geq 0$ and (4.2) holds. The continuity of $D_\pm$ is due to the uniform convergence. $\square$

Remark 4.4. *For all $\mu \in [E_0, \infty)$, we have*

$$i\pi\lambda\psi_g(\mu) = D_+(\mu^2 - E_0^2) - D_-(\mu^2 - E_0^2). \quad (4.3)$$

Lemma 4.5. *Let $\lambda \neq \lambda_c$, then $\delta := \inf_{s \geq 0} |D_\pm(s)| > 0$.*

Proof. If $\lambda = 0$, then clearly $D_\pm(s) = 1 > 0$ for all $s \in [0, \infty)$. Let $\lambda \neq 0, \lambda_c$. Then $D_\pm(0) = D(0) \neq 0$. Hence, by the continuity of $D_\pm$, $D_\pm$ has no zeros near $s = 0$. For any $\varepsilon > 0$ and $s > E_0^2 + 1$, we have

$$\begin{aligned} (H_\varepsilon \phi_g)(s) &= I_1^{(\varepsilon)}(s) + \sum_{j=2}^4 I_j(s), \\ I_1^{(\varepsilon)}(s) &= \int_\varepsilon^1 \frac{\phi_g(s-x) - \phi_g(s+x)}{x} dx, \quad I_2(s) = \int_{E_0^2}^{s-1} \frac{\phi_g(x)}{s-x} dx, \\ I_3(s) &= \int_{s+1}^{2s} \frac{\phi_g(x)}{s-x} dx, \quad I_4(s) = \int_{2s}^\infty \frac{\phi_g(x)}{s-x} dx. \end{aligned}$$

Then, by the Lebesgue dominated convergence theorem, each $I_j(s)$, $j = 2, 3, 4$ tends to zero as $s \rightarrow \infty$. By the mean value theorem and the property that $\phi'_g(x) \rightarrow 0$ as $x \rightarrow \infty$, we have $\lim_{s \rightarrow \infty} \lim_{\varepsilon \downarrow 0} I_1^{(\varepsilon)}(s) = 0$. Hence we can see that $(H\phi_g)(s) \rightarrow 0$ as $s \rightarrow \infty$. This fact implies that $\inf_{s_0 \leq s} \operatorname{Re} D_{\pm}(s) > 0$ for a sufficiently large number $s_0 > 0$. In addition, $\operatorname{Im} D_{\pm}(s)$ are positive for any closed interval included in $(0, \infty)$ by Assumption 3.3 (3) and the continuity of ψ_g . Hence we can see that $\inf_{s \geq 0} |D_{\pm}(s)| > 0$. $\square$

Remark 4.6. By Lemmas 4.3 and 4.5, we can see that there are constants $c, d, \varepsilon_0 > 0$ with $0 < c < d$ such that

$$c \leq \left| \frac{D(s \pm i\varepsilon)}{D_{\pm}(s)} \right| \leq d \quad (4.4)$$

for all $s \geq 0, 0 < \varepsilon < \varepsilon_0$.

4.2 Operators $R_{\pm}$

Through this subsection, we assume $\lambda \neq \lambda_c$.

Lemma 4.7. One can define bounded operators $R_{\pm}$ on $\mathcal{H}$ as follows:

$$R_{\pm}f := -\lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \frac{R_{\mu'^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_{\pm}(\mu'^2 - E_0^2)} d \langle T^{1/2}g, E(\mu')f \rangle, \quad f \in \mathcal{H},$$

where $R_z(A) := (A - z)^{-1}$ is the resolvent of a linear operator A at $z \in \rho(A)$ (the resolvent set of a linear operator A).

Proof. For a fixed $\varepsilon > 0$ and any $f \in \mathcal{H}$,

$$\int_{[E_0, \infty)} \left\| \frac{R_{\mu'^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_{\pm}(\mu'^2 - E_0^2)} \right\| d \|E(\mu')f\|^2 \leq \frac{\|f\|^2 \|T^{1/2}g\|}{\delta \varepsilon} < \infty$$

by Lemma 4.5 and a property of a resolvent. Thus we can define linear operators $R_{\pm}^{(\varepsilon)}$ on $\mathcal{H}$ by

$$R_{\pm}^{(\varepsilon)}f := -\lambda \int_{[E_0, \infty)} \frac{R_{\mu'^2 \pm i\varepsilon}(T^2)T^{1/2}g}{D_{\pm}(\mu'^2 - E_0^2)} d \langle T^{1/2}g, E(\mu')f \rangle$$

in the sense of Bochner integral with the polarization identity. For any $h, f \in \mathcal{H}$,

$$\begin{aligned} & \langle h, R_{\pm}^{(\varepsilon)}f \rangle \\ &= -\lambda \int_{[E_0, \infty)} \frac{\langle h, R_{\mu'^2 \pm i\varepsilon}(T^2)T^{1/2}g \rangle}{D_{\pm}(\mu'^2 - E_0^2)} d \langle T^{1/2}g, E(\mu')f \rangle \\ &= -\lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu^{1/2}}{(\mu^2 - \mu'^2 \mp i\varepsilon)D_{\pm}(\mu'^2 - E_0^2)} d \langle h, E(\mu)g \rangle d \langle T^{1/2}g, E(\mu')f \rangle, \end{aligned}$$

where we have used a functional calculus. By change of variables in the Lebesgue-Stieltjes integration, a functional calculus and Fubini's theorem, we have

$$\langle h, R_{\pm}^{(\varepsilon)} f \rangle = \frac{\lambda\pi}{2} \int_{[E_0, \infty)} - (A_{\varepsilon}^{(1)} * \phi_{g,f}^{\pm})(\mu^2) \mu^{1/2} \mp i (A_{\varepsilon}^{(2)} * \phi_{g,f}^{\pm})(\mu^2) \mu^{1/2} d \langle h, E(\mu) g \rangle,$$

where $\phi_{g,f}^{\pm}(x) = \psi_{g,f}(\sqrt{x}) x^{-1/4} D_{\pm}(x - E_0^2)^{-1} \chi_{[E_0^2, \infty)}(x)$, $x \in \mathbb{R}$. We have $\phi_{g,f}^{\pm} \in L^2(\mathbb{R})$ by Remark 3.4, and the function $(A_{\varepsilon}^{(j)} * \phi_{g,f}^{\pm})(\mu^2) \mu^{1/2}$ ($\mu \in \mathbb{R}$) is in $L^2(\mathbb{R})$ for each $j = 1, 2$. Thus, by a change of variable, we have

$$\begin{aligned} & \left\| R_{\pm}^{(\varepsilon)} f - \left(-\frac{\pi\lambda}{2} (H\phi_{g,f}^{\pm})(T^2) T^{1/2} g \mp \frac{1}{2} A_{\pm} f \right) \right\|^2 \\ & \leq \left(\frac{\lambda\pi}{2} \right)^2 c_g \int_{[E_0^2, \infty)} |(A_{\varepsilon}^{(1)} * \phi_{g,f}^{\pm})(x) - (H\phi_{g,f}^{\pm})(x)|^2 dx \\ & \quad + \left(\frac{\lambda\pi}{2} \right)^2 c_g \int_{[E_0^2, \infty)} |(A_{\varepsilon}^{(2)} * \phi_{g,f}^{\pm})(x) - \phi_{g,f}^{\pm}(x)|^2 dx, \end{aligned}$$

where $c_g := \sup_{x \in [E_0, \infty)} \psi_g(x)$ and the linear operators

$$A_{\pm} f := i\pi\lambda\psi_{g,f}(T) D_{\pm}(T^2 - E_0^2)^{-1} g, \quad f \in \mathcal{H}$$

are well-defined (see Remark 3.4 and Lemma 4.5). Hence, by $\phi_{g,f}^{\pm} \in L^2(\mathbb{R})$, we have

$$R_{\pm}^{(\varepsilon)} f \rightarrow -(\pi\lambda/2)(H\phi_{g,f}^{\pm})(T^2) T^{1/2} g \mp (1/2) A_{\pm} f \text{ as } \varepsilon \downarrow 0.$$

Moreover, by change of variables, the isometricity of Hilbert transform and Remark 3.4, we can show that the inequalities

$$\|(H\phi_{g,f}^{\pm})(T^2) T^{1/2} g\| \leq \frac{c_g}{\delta} \|f\|, \quad \|A_{\pm} f\| \leq \frac{\pi|\lambda|c_g}{\delta} \|f\|$$

hold for all $f \in \mathcal{H}$. Hence $R_{\pm}$ are bounded. $\square$

By the definition of the adjoint operator, $R_{\pm}^* := (R_{\pm})^*$ are given as follows: for $f \in \mathcal{H}$,

$$\begin{aligned} R_{\pm}^{(\varepsilon)*} f &= \lambda \int_{[E_0, \infty)} R_{\mu'^2 \pm i\varepsilon}(T^2) D_{\mp}(T^2 - E_0^2)^{-1} T^{1/2} g \, d \langle T^{1/2} g, E(\mu') f \rangle, \\ R_{\pm}^* f &= \lim_{\varepsilon \downarrow 0} R_{\pm}^{(\varepsilon)*} f. \end{aligned} \tag{4.5}$$

For a densely defined linear operator A on a Hilbert space, $A^{\sharp}$ denotes A or A^* .

Lemma 4.8. *The ranges of $R_{\pm}^{\sharp}$ are included in $D(T^{-1}) \cap D(T)$ and $R_{\pm}^{\sharp}$ map $D(T)$ into $D(T^2)$.*

Proof. For any $f, h \in \mathcal{H}$, we have

$$\langle h, R_{\pm} f \rangle = \frac{\lambda\pi}{2} \int_{[E_0, \infty)} - (H\phi_{g,f}^{\pm})(\mu^2) \mu^{1/2} \mp i \frac{\psi_{g,f}(\mu)}{D_{\pm}(\mu^2 - E_0^2)} d \langle h, E(\mu)g \rangle. \quad (4.6)$$

By a change of variable, we have

$$(H\phi_{g,f}^{\pm})(\mu^2) = \left(H\psi_{T^{-1/2}g,f}^{\pm} \right)(\mu) + \left(H\psi_{T^{-1/2}g,f}^{\pm} \right)(-\mu), \quad \mu \in \mathbb{R}, \quad (4.7)$$

where $\psi_{h,f}^{\pm}(x) := \psi_{h,f}(x)D_{\pm}(x^2 - E_0)^{-1}\chi_{[E_0, \infty)}(x)$, $x \in \mathbb{R}$ for $h, f \in \mathcal{H}$. Thus we see by Assumption 3.3 (3) and a functional calculus that $\text{Ran}(R_{\pm}) \subset D(T^{-1})$. The equation

$$\mu (H\phi_{g,f}^{\pm})(\mu^2) = \left(H\psi_{T^{1/2}g,f}^{\pm} \right)(\mu) - \left(H\psi_{T^{1/2}g,f}^{\pm} \right)(-\mu), \quad \mu \in \mathbb{R}, \quad (4.8)$$

(4.6), Assumption 3.3 (3) and operational calculus imply that $\text{Ran}(R_{\pm}) \subset D(T)$. For any $f \in D(T)$ and $\mu \in \mathbb{R}$,

$$\mu^2 (H\phi_{g,f}^{\pm})(\mu^2) = \left(H\psi_{T^{1/2}g,Tf}^{\pm} \right)(\mu) + \left(H\psi_{T^{1/2}g,Tf}^{\pm} \right)(-\mu) + \frac{2}{\pi} \int_{[E_0, \infty)} \psi_{T^{1/2}g,f}^{\pm}(x) dx.$$

Hence $R_{\pm}f \in D(T^2)$ and the following equation holds for any $h \in \mathcal{H}$,

$$\begin{aligned} \langle h, T^2 R_{\pm} f \rangle &= \frac{\lambda\pi}{2} \int_{[E_0, \infty)} - \left\{ \left(H\psi_{T^{1/2}g,Tf}^{\pm} \right)(\mu) + \left(H\psi_{T^{1/2}g,Tf}^{\pm} \right)(-\mu) + \frac{2c}{\pi} \right\} \mu^{1/2} d \langle h, E(\mu)g \rangle \\ &\quad \mp i \frac{\lambda\pi}{2} \int_{[E_0, \infty)} \psi_{T^{1/2}g,Tf}^{\pm}(\mu) \mu^{1/2} d \langle h, E(\mu)g \rangle, \end{aligned}$$

where $c := \int_{\mathbb{R}} \psi_{T^{1/2}g,f}^{\pm}(x) dx$. In quite the same manner as in the case of $R_{\pm}$, we can prove the statement for $R_{\pm}^*$. $\square$

Lemma 4.9. *The operator equations $(R_{\pm})_J = R_{\mp}$ hold.*

Proof. This follows from Assumption 3.3 (1) and Lemma 3.5. $\square$

Lemma 4.10. *The operator equation $R_- = R_+ \gamma + A_-$ holds, where*

$$\gamma := D_+(T^2 - E_0^2)D_-(T^2 - E_0^2)^{-1}$$

is a bounded operator.

Proof. The first resolvent formula gives that, for any $\mu', \mu'' \in \mathbb{R}, \varepsilon > 0$,

$$R_{\mu'^2 - i\varepsilon}(T^2) - R_{\mu'^2 + i\varepsilon}(T^2) = -2i\varepsilon R_{\mu'^2 - i\varepsilon}(T^2) R_{\mu'^2 + i\varepsilon}(T^2).$$

Then, for any $f \in \mathcal{H}$,

$$\begin{aligned} R_-^{(\varepsilon)} f &= -\lambda \int_{[E_0, \infty)} \frac{R_{\mu'^2+i\varepsilon}(T^2)T^{1/2}g}{D_-(\mu'^2-E_0^2)} d\langle T^{1/2}g, E(\mu')f \rangle \\ &\quad + 2i\lambda\varepsilon \int_{[E_0, \infty)} \frac{R_{\mu'^2+i\varepsilon}(T^2)R_{\mu'^2-i\varepsilon}(T^2)T^{1/2}g}{D_-(\mu'^2-E_0^2)} d\langle T^{1/2}g, E(\mu')f \rangle. \end{aligned}$$

Thus, by a change of variable, we have for any $h \in \mathcal{H}$

$$\begin{aligned} \langle h, R_-^{(\varepsilon)} f \rangle &= \langle h, R_+^{(\varepsilon)} \gamma f \rangle + 2i\lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\langle h, E(\mu)g \rangle d\langle T^{1/2}g, E(\mu')f \rangle \\ &\quad \times \frac{\mu^{1/2}\varepsilon}{\{(\mu^2 - \mu'^2)^2 + \varepsilon^2\}D_-(\mu'^2 - E_0^2)} \\ &= \langle h, R_+^{(\varepsilon)} \gamma f \rangle + i\pi\lambda \int_{[E_0, \infty)} (A_\varepsilon^{(2)} * \phi_{g,f}^-)(\mu^2)\mu^{1/2} d\langle h, E(\mu)g \rangle. \end{aligned}$$

By a property of the Poisson kernel, the function $(A_\varepsilon^{(2)} * \phi_{g,f}^-)(\mu^2)\mu^{1/2}$ ($\mu \in \mathbb{R}$) converges to $\psi_{g,f}(\mu)/D_-(\mu^2 - E_0^2)$ as $\varepsilon \rightarrow +0$ in the sense of $L^2(\mathbb{R})$. Hence the continuity of the inner product with $L^2(\mathbb{R})$ implies that

$$\begin{aligned} \langle h, R_- f \rangle &= \langle h, R_+ \gamma f \rangle + i\pi\lambda \int_{[E_0, \infty)} \frac{\psi_{g,f}(\mu)}{D_-(\mu^2 - E_0^2)} d\langle h, E(\mu)g \rangle \\ &= \langle h, R_+ \gamma f \rangle + \langle h, A_- f \rangle. \end{aligned}$$

Since f and h are arbitrary, one obtains the conclusion. $\square$

By the definitions of $A_\pm$, we have

$$(A_-)^* = -A_+.$$

Lemma 4.11. *For any Borel measurable function F on $\mathbb{R}$, $A_\pm F(T) \subset F(T)A_\pm$.*

Proof. For any $f \in D(F(T))$, an operational calculus implies that $\psi_{g,F(T)f} = F\psi_{g,f} \in L^2(\mathbb{R})$. This fact imply that $\psi_{g,f}(T)g \in D(F(T))$ and $F(T)\psi_{g,f}(T)g = \psi_{g,F(T)f}(T)g$. Hence $A_\pm f \in D(F(T))$ and $F(T)A_\pm f = A_\pm F(T)f$ by Lemma 4.5. $\square$

Lemma 4.12. *The following operator equations hold:*

$$A_- R_\pm^* = (\gamma - I)R_\pm^*, \quad A_-(A_-)^* = -A_- - (A_-)^*.$$

Proof. By applying Lemma 4.11 to the case $F = \chi_B$, one can easily see that $A_{\pm}E(B) = E(B)A_{\pm}$ hold for any $B \in \mathbf{B}^1$. For any $f, h \in \mathcal{H}$, we have

$$\begin{aligned} & \left\langle (A_-)^* h, R_{\pm}^{(\varepsilon)*} f \right\rangle \\ &= \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{i\pi\lambda^2 \mu^{1/2} \psi_g(\mu)}{(\mu^2 - \mu'^2 \mp i\varepsilon) D_{\mp}(\mu^2 - E_0^2) D_{-}(\mu^2 - E_0^2)} d\langle h, E(\mu)g \rangle d\langle T^{1/2}g, E(\mu')f \rangle. \end{aligned}$$

Then, since γ and $E(B)$ commute on $\mathcal{H}$ for any $B \in \mathbf{B}^1$, (4.3) gives

$$\begin{aligned} & \left\langle (A_-)^* h, R_{\pm}^{(\varepsilon)*} f \right\rangle \\ &= \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu^{1/2}}{(\mu^2 - \mu'^2 \mp i\varepsilon) D_{\mp}(\mu^2 - E_0^2)} d\langle h, E(\mu)(\gamma - 1)g \rangle d\langle T^{1/2}g, E(\mu')f \rangle \\ &= \left\langle h, (\gamma - 1)R_{\pm}^{(\varepsilon)*} f \right\rangle. \end{aligned}$$

Thus, by a limiting argument, we obtain $A_- R_{\pm}^* = (\gamma - 1)R_{\pm}^*$. Moreover, (4.3) and the equation $(A_-)^* = -A_+$ imply that

$$\begin{aligned} \langle h, A_- (A_-)^* f \rangle &= -(i\pi\lambda)^2 \int_{[E_0, \infty)} \frac{\psi_{g,f}(\mu) \psi_g(\mu)}{D_+(\mu^2 - E_0^2) D_-(\mu^2 - E_0^2)} d\langle h, E(\mu)g \rangle \\ &= -i\pi\lambda \int_{[E_0, \infty)} \frac{(D_+(\mu^2 - E_0^2) - D_-(\mu^2 - E_0^2)) \psi_{g,f}(\mu)}{D_+(\mu^2 - E_0^2) D_-(\mu^2 - E_0^2)} d\langle h, E(\mu)g \rangle \\ &= -\langle h, (A_-)^* f + A_- f \rangle. \end{aligned}$$

Hence the equation $A_- (A_-)^* = -A_- - (A_-)^*$ holds. $\square$

4.3 Operators $\Omega_{\pm}$

In this subsection we consider the bounded operators

$$\Omega_{\pm} := I + R_{\pm}.$$

Let $x_0 < 0$ be the zero of $D(z)$ given in Lemma 4.1 (2) and

$$U_b := \sqrt{\frac{\lambda}{D'(x_0)}} R_{E_0^2 + x_0}(T^2) T^{1/2} g, \quad P := \langle U_b, \cdot \rangle U_b.$$

Then, by functional calculus, we see that $\|U_b\| = 1$, $U_b \in D(T^{-1}) \cap D(T^2)$ and

$$TU_b = \sqrt{\lambda/D'(x_0)} T^{-1/2} g + (E_0^2 + x_0) T^{-1} U_b.$$

Hence P is a projection operator.

Lemma 4.13. *Let $\lambda \neq \lambda_c$. Then the following equations hold:*

$$\Omega_{\pm}^* \Omega_{\pm} = I, \quad (4.9)$$

$$\Omega_{\pm} \Omega_{\pm}^* = I - \theta(\lambda_c - \lambda)P, \quad (4.10)$$

where θ is the Heaviside function:

$$\theta(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Remark 4.14. *Lemma 4.13 implies that $\Omega_{\pm}$ are unitary operators if $\lambda > \lambda_c$ and partial isometries with their final subspace $\text{Ran}(I - P)$ if $\lambda < \lambda_c$.*

Proof. (1) We first prove (4.9).

It is sufficient to prove that $R_{\pm}^* R_{\pm} = -(R_{\pm} + R_{\pm}^*)$ hold. For any $f, h \in \mathcal{H}$ and $\varepsilon > 0$,

$$\begin{aligned} \langle R_{\pm}^{(\varepsilon)} h, R_{\pm}^{(\varepsilon)} f \rangle &= \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} d \langle h, E(\mu') T^{1/2} g \rangle d \langle T^{1/2} g, E(\mu'') f \rangle \\ &\quad \times \left\langle \frac{R_{\mu'^2 \pm i\varepsilon}(T^2) T^{1/2} g}{D_{\pm}(\mu'^2 - E_0^2)}, \frac{R_{\mu''^2 \pm i\varepsilon}(T^2) T^{1/2} g}{D_{\pm}(\mu''^2 - E_0^2)} \right\rangle. \end{aligned}$$

By the definition of the function D , we have

$$\lambda \langle T^{1/2} g, R_z(T^2) T^{1/2} g \rangle = D(z - E_0^2) - 1, \quad z \in \mathbb{C} \setminus (E_0^2, \infty).$$

By this formula and the resolvent identity, we obtain

$$\begin{aligned} \langle R_{\pm}^{(\varepsilon)} h, R_{\pm}^{(\varepsilon)} f \rangle &= \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} d \langle h, E(\mu') T^{1/2} g \rangle d \langle T^{1/2} g, E(\mu'') f \rangle \\ &\quad \times \frac{D(\mu'^2 - E_0^2 \mp i\varepsilon) - D(\mu''^2 - E_0^2 \pm i\varepsilon)}{(\mu'^2 - \mu''^2 \mp 2i\varepsilon) D_{\mp}(\mu'^2 - E_0^2) D_{\pm}(\mu''^2 - E_0^2)}. \\ &= - \langle E_{\pm}^{(\varepsilon)} h, R_{\pm}^{(2\varepsilon)} f \rangle - \langle R_{\pm}^{(2\varepsilon)} h, E_{\pm}^{(\varepsilon)} f \rangle, \end{aligned}$$

where the operators $E_{\pm}^{(\varepsilon)}$ on $\mathcal{H}$ are given as follows:

$$E_{\pm}^{(\varepsilon)} := D(T^2 - E_0^2 \pm i\varepsilon) D_{\pm}(T^2 - E_0^2)^{-1}.$$

The inequality (4.4) implies that $E_{\pm}^{(\varepsilon)}$ are bounded for all $0 < \varepsilon < \varepsilon_0$. Thus, by the Lebesgue dominated convergence theorem, we have $\text{s-lim}_{\varepsilon \downarrow 0} E_{\pm}^{(\varepsilon)} = I$. Hence we obtain $R_{\pm}^* R_{\pm} = -(R_{\pm} + R_{\pm}^*)$.

(2) We next prove (4.10) for $\lambda \neq \lambda_c$.

It is sufficient to prove that $R_{\pm}R_{\pm}^* = -(R_{\pm} + R_{\pm}^*) - \theta(\lambda_c - \lambda)P$ holds. For any $f, h \in \mathcal{H}$ and a fixed $\varepsilon > 0$, (4.5) implies

$$\begin{aligned} & \left\langle R_{\pm}^{(\varepsilon)*} h, R_{\pm}^{(\varepsilon)*} f \right\rangle \\ &= \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} d \langle h, E(\mu) T^{1/2} g \rangle d \langle T^{1/2} g, E(\mu') f \rangle \\ & \quad \times \langle R_{\mu^2 \pm i\varepsilon}(T^2) D_{\mp}(T^2 - E_0^2)^{-1} T^{1/2} g, R_{\mu'^2 \pm i\varepsilon}(T^2) D_{\mp}(T^2 - E_0^2)^{-1} T^{1/2} g \rangle. \end{aligned}$$

Then, by operational calculus, we see that

$$\begin{aligned} & \left\langle R_{\pm}^{(\varepsilon)*} h, R_{\pm}^{(\varepsilon)*} f \right\rangle \\ &= \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d \langle h, E(\mu) T^{1/2} g \rangle d \langle T^{1/2} g, E(\mu') f \rangle d \|E(\mu'') g\|^2 \\ & \quad \times \frac{\mu''}{(\mu''^2 - \mu^2 \pm i\varepsilon)(\mu''^2 - \mu'^2 \mp i\varepsilon) D_{\pm}(\mu''^2 - E_0^2) D_{\mp}(\mu''^2 - E_0^2)} \\ &= \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu^2 - \mu'^2 \mp 2i\varepsilon} J_{\varepsilon}^{\pm}(\mu, \mu') d \langle h, E(\mu) T^{1/2} g \rangle d \langle T^{1/2} g, E(\mu') f \rangle, \quad (4.11) \end{aligned}$$

where, for any $\mu, \mu' \in [E_0, \infty)$,

$$\begin{aligned} & J_{\varepsilon}^{\pm}(\mu, \mu') \\ &= \int_{[E_0, \infty)} \frac{\lambda \mu''}{D_{\pm}(\mu''^2 - E_0^2) D_{\mp}(\mu''^2 - E_0^2)} \left(\frac{1}{\mu''^2 - \mu^2 \pm i\varepsilon} - \frac{1}{\mu''^2 - \mu'^2 \mp i\varepsilon} \right) d \|E(\mu'') g\|^2. \end{aligned}$$

Then, by a change of variable and (4.3), one can show that

$$J_{\varepsilon}^{\pm}(\mu, \mu') = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} I_{\varepsilon, R}^{\pm}(\mu, \mu'),$$

where, for $R > 0$,

$$I_{\varepsilon, R}^{\pm}(\mu, \mu') := \int_0^R \left(\frac{1}{D_{+}(s)} - \frac{1}{D_{-}(s)} \right) G_{\mu, \mu'}^{\varepsilon, \pm}(s) ds$$

and

$$G_{\mu, \mu'}^{\varepsilon, \pm}(z) := \frac{1}{z - \mu'^2 + E_0^2 \mp i\varepsilon} - \frac{1}{z - \mu^2 + E_0^2 \pm i\varepsilon}, \quad z \in \mathbb{C}.$$

For $0 < \eta < \varepsilon$ and $R > 0$, let C_i ($i = 1, 2, 3$) be the curve given as follows:

$$\begin{aligned} C_1 : & \theta_1(t) = R - t - i\eta, \quad t : 0 \rightarrow R, \\ C_2 : & \theta_2(t) = \eta e^{-it}, \quad t : \pi/2 \rightarrow (3\pi)/2, \\ C_3 : & \theta_3(t) = t + i\eta, \quad t : 0 \rightarrow R. \end{aligned}$$

Then, for $C = C_1 + C_2 + C_3$, we have by the Lebesgue dominated convergence theorem,

$$I_{\varepsilon,R}^{\pm}(\mu, \mu') = \lim_{\eta \downarrow 0} \int_C \frac{1}{D(z)} G_{\mu,\mu'}^{\varepsilon,\pm}(z) dz.$$

We take R such that $R > \max\{\mu^2 - E_0^2, \mu'^2 - E_0^2\}$ and define a curve $C_4 : \theta_4(t) = \sqrt{\eta^2 + R^2} e^{-it}, t : t_s \rightarrow t_f$, for $t_s := \arctan(\eta/R)$ and $t_f = 2\pi - t_s$. We consider two cases separately.

- (i) The case $\lambda > \lambda_c$. In this case, the function $G_{\mu,\mu'}^{\varepsilon,\pm}(z)/D(z), z \in \mathbb{C} \setminus (0, \infty)$ has two simple poles at $z = \mu^2 - E_0^2 \mp i\varepsilon$ and $z = \mu'^2 - E_0^2 \pm i\varepsilon$. Then, by the residue theorem, we have

$$\begin{aligned} \int_C \frac{1}{D(z)} G_{\mu,\mu'}^{\varepsilon,\pm}(z) dz &= 2\pi i \left(\frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} \right) \\ &\quad - \int_{C_4} \frac{1}{D(z)} G_{\mu,\mu'}^{\varepsilon,\pm}(z) dz. \end{aligned}$$

Thus, as η tends to 0, we have

$$\begin{aligned} I_{\varepsilon,R}^{\pm}(\mu, \mu') &= 2\pi i \left(\frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} \right) \\ &\quad - \lim_{\eta \downarrow 0} \int_{C_4} \frac{1}{D(z)} G_{\mu,\mu'}^{\varepsilon,\pm}(z) dz. \end{aligned}$$

The definition of line integral implies

$$\int_{C_4} \frac{1}{D(z)} G_{\mu,\mu'}^{\varepsilon,\pm}(z) dz = -i \int_{t_s}^{2\pi - t_s} \frac{G_{\mu,\mu'}^{\varepsilon,\pm}(\sqrt{\eta^2 + R^2} e^{-it}) \sqrt{\eta^2 + R^2} e^{-it}}{D(\sqrt{\eta^2 + R^2} e^{-it})} dt.$$

By the triangle inequality, for any $t \in [t_s, t_f]$,

$$|G_{\mu,\mu'}^{\varepsilon,\pm}(\sqrt{\eta^2 + R^2} e^{-it})| \leq \frac{|\mu^2 - \mu'^2 \pm 2i\varepsilon|}{(R - |\mu^2 - E_0^2 \pm i\varepsilon|)(R - |\mu'^2 - E_0^2 \mp i\varepsilon|)}.$$

On the other hand, by Lemma 4.5, (4.4) and the Lebesgue dominated convergence theorem, there are constants $\tilde{R} > 0$ and $c_0 > 0$ such that $|D(z)| \geq c_0$ for all $|z| \geq \tilde{R}$. Thus we have

$$I_{\varepsilon,R}^{\pm}(\mu, \mu') = 2\pi i \left(\frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} \right) + O(R^{-1}) \quad (R \rightarrow \infty),$$

where $O(\cdot)$ stands for the well known Landau symbol. Therefore we have

$$J_{\varepsilon}^{\pm}(\mu, \mu') = \frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)}$$

for each $\mu, \mu' \in [E_0, \infty)$. Thus, by (4.11), we have

$$\left\langle R_{\pm}^{(\varepsilon)*} h, R_{\pm}^{(\varepsilon)*} f \right\rangle = - \left\langle \left(R_{\pm}^{(2\varepsilon)} \right)^* h, \left(E_{\pm}^{(\varepsilon)} \right)^{-1} f \right\rangle - \left\langle \left(E_{\pm}^{(\varepsilon)} \right)^{-1} h, \left(R_{\pm}^{(2\varepsilon)} \right)^* f \right\rangle.$$

As in the proof in (1), we obtain $\text{s-lim}_{\varepsilon \downarrow 0} \left(E_{\pm}^{(\varepsilon)} \right)^{-1} = I$. Therefore we obtain

$$\lim_{\varepsilon \downarrow 0} \left\langle R_{\pm}^{(\varepsilon)*} h, R_{\pm}^{(\varepsilon)*} f \right\rangle = - \langle R_{\pm}^* h, f \rangle - \langle h, R_{\pm}^* f \rangle.$$

Thus we obtain the desired result.

- (ii) The case $\lambda < \lambda_c$. In this case, $G_{\mu, \mu'}^{\varepsilon, \pm}(z)/D(z)$ has a simple pole at $z = x_0$ in addition to $z = \mu^2 - E_0^2 \mp i\varepsilon$ and $z = \mu'^2 - E_0^2 \pm i\varepsilon$. The residue R_0 of $G_{\mu, \mu'}^{\varepsilon, \pm}(z)/D(z)$ at $z = x_0$ is give by

$$R_0 = \frac{1}{D'(x_0)} \frac{\mu'^2 - \mu^2 \pm 2i\varepsilon}{(x_0 - \mu'^2 + E_0^2 \mp i\varepsilon)(x_0 - \mu^2 + E_0^2 \pm i\varepsilon)}.$$

Thus we have

$$J_{\varepsilon}^{\pm}(\mu, \mu') = \frac{1}{D(\mu'^2 - E_0^2 \pm i\varepsilon)} - \frac{1}{D(\mu^2 - E_0^2 \mp i\varepsilon)} + R_0$$

and also

$$\frac{\lambda}{\mu^2 - \mu'^2 \mp 2i\varepsilon} R_0 = - \frac{\lambda}{D'(x_0)} \frac{1}{(\mu'^2 - E_0^2 - x_0 \pm i\varepsilon)(\mu^2 - E_0^2 - x_0 \mp i\varepsilon)}.$$

This implies that

$$\begin{aligned} & \lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu^2 - \mu'^2 \mp 2i\varepsilon} R_0 d \langle h, E(\mu) T^{1/2} g \rangle d \langle T^{1/2} g, E(\mu') f \rangle \\ &= - \langle h, U_b \rangle \langle U_b, f \rangle = - \langle h, P f \rangle. \end{aligned}$$

Thus we obtain the desired result. □

4.4 Operators U and V

In this subsection, we investigate the operators U and V defined as follows:

$$U := \frac{1}{2}(T^{-1/2}\Omega_+T^{1/2} + T^{1/2}\Omega_+T^{-1/2}), V := \frac{1}{2}(T^{-1/2}\Omega_+T^{1/2} - T^{1/2}\Omega_+T^{-1/2}),$$

which are used to construct a Bogoliubov transformation. Then, by Lemma 4.8, we can see that $D(U) = D(V) = D(T^{-1/2}) \cap D(T^{1/2})$.

Lemma 4.15. *The operators U and V are bounded.*

Proof. By (4.6) and Lemma 4.8 we have

$$\begin{aligned}\langle h, T^{-1/2} R_{\pm} T^{1/2} f \rangle &= \frac{\lambda\pi}{2} \int_{[E_0, \infty)} - \left(H \phi_{T^{1/2}g, f}^{\pm} \right) (\mu^2) \mp i \frac{\psi_{g, f}(\mu)}{D_{\pm}(\mu^2 - E_0^2)} d \langle h, E(\mu) g \rangle, \quad (4.12) \\ \langle h, T^{1/2} R_{\pm} T^{-1/2} f \rangle &= \frac{\lambda\pi}{2} \int_{[E_0, \infty)} - \left(H \phi_{T^{-1/2}g, f}^{\pm} \right) (\mu^2) \mu \mp i \frac{\psi_{g, f}(\mu)}{D_{\pm}(\mu^2 - E_0^2)} d \langle h, E(\mu) g \rangle.\end{aligned}\quad (4.13)$$

By Assumption 3.3 (3), (4.7), (4.8) and a property of Hilbert transform, we can show that

$$\|T^{-1/2} R_{\pm} T^{1/2} f\|, \|T^{1/2} R_{\pm} T^{-1/2} f\| \leq \frac{|\lambda|\pi(C_g + c_g)}{2\delta} \|f\|,$$

where $C_g := (\sup_{E_0 < x} x^{-1} \psi_g(x))^{1/2} (\sup_{E_0 < x} x \psi_g(x))^{1/2}$. Hence the operators $T^{-1/2} R_{\pm} T^{1/2}$ and $T^{1/2} R_{\pm} T^{-1/2}$ are bounded. $\square$

In the same way as in the proof of Lemma 4.15, we see that $T^{-1/2} R_{\pm}^* T^{1/2}$ and $T^{1/2} R_{\pm}^* T^{-1/2}$ are bounded on each domain $D(T^{1/2})$ and $D(T^{-1/2})$. In what follows, we write the closed extensions of U and V by the same symbol respectively. Then

$$U^* = \frac{1}{2} (\overline{T^{-1/2} \Omega_+^* T^{1/2}} + \overline{T^{1/2} \Omega_+^* T^{-1/2}}).$$

Lemma 4.16. *The operators $U^{\sharp}$ and $V^{\sharp}$ leave $D(T^{-1/2})$ (resp. $D(T^{1/2}), D(T)$) invariant.*

Proof. By applying Lemma 4.8 and using the equation

$$U^{\sharp} = I + \frac{1}{2} \left(\overline{T^{-1/2} R_+^{\sharp} T^{1/2}} + \overline{T^{1/2} R_+^{\sharp} T^{-1/2}} \right),$$

one can easily see that the assertion for $U^{\sharp}$ is true. The proof for $V^{\sharp}$ is similar. $\square$

Lemma 4.17. *Let $F(x) = x^{\pm 1/2}, x^{\pm 1}$, a.e. $x \in (0, \infty)$. Then*

$$\Omega_+ F(T) \Omega_+^* = (\Omega_+)_J F(T) (\Omega_+^*)_J \quad \text{on } D(F(T)). \quad (4.14)$$

Proof. By Lemma 4.8, the domain of each side of (4.14) includes $D(F(T))$. By Lemmas 4.11 and 4.12, we have

$$\begin{aligned}(\Omega_+)_J F(T) (\Omega_+^*)_J &= R_+ F(T) R_+^* + R_+ \{ (A_-)^* + I \} F(T) \gamma + F(T) \gamma^* (A_- + I) R_+^* \\ &\quad + F(T) \{ A_- (A_-)^* + A_- + (A_-)^* + I \} \\ &= R_+ F(T) R_+^* + R_+ F(T) + F(T) R_+^* + F(T) \\ &= \Omega_+ F(T) \Omega_+^*.\end{aligned}$$

$\square$

5 Commutation relations

In this section, we prove that the pair (U, V) satisfies the condition (1.1), V is Hilbert-Schmidt and

$$B(f) := A(Uf) + A(JVf)^*, \quad f \in \mathcal{H}$$

satisfies some commutation relations with $H(\lambda)$. We denote the closure of $B(f)$ by the same symbol. By Lemma 4.16, we have $D(d\Gamma_b(T)^{1/2}) \subset D(B(f)) \cap D(B(f)^*)$ for all $f \in D(T^{-1/2})$.

Theorem 5.1. *The following commutation relations hold:*

(1) For any $f \in D(T)$ and $\psi \in \mathcal{F}_{b, \text{fin}}(D(T))$,

$$[H(\lambda), B(f)]\psi = -B(Tf)\psi. \quad (5.1)$$

(2) For any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(d\Gamma_b(T))$,

$$\langle H(\lambda)\phi, B(f)\psi \rangle - \langle B(f)^*\phi, H(\lambda)\psi \rangle = -\langle \phi, B(Tf)\psi \rangle. \quad (5.2)$$

(3) For any $f \in D(T^{-1/2}) \cap D(T)$, $B(f)$ maps $D(d\Gamma_b(T)^{3/2})$ into $D(d\Gamma_b(T))$ and for any $\psi \in D(d\Gamma_b(T)^{3/2})$,

$$[H(\lambda), B(f)]\psi = -B(Tf)\psi. \quad (5.3)$$

The both sides of (5.1), (5.2) and (5.3) have meaning by Lemma 4.16. To prove this theorem, we prove the following lemma:

Lemma 5.2. *For any $f \in D(T)$, the following equations hold:*

$$[U, T]f = (VT + TV)f = \frac{\lambda}{2} \langle D_-(T^2 - E_0^2)^{-1}g, f \rangle g, \quad (5.4)$$

$$(V^*J - U^*)g = -D_-(T^2 - E_0^2)^{-1}g. \quad (5.5)$$

Proof. For any $f, h \in D(T^{-1/2}) \cap D(T^{3/2})$, we obtain

$$\langle h, [U, T]f \rangle = \frac{1}{2} (\langle T^{1/2}R_+^*T^{-1/2}h, Tf \rangle - \langle Th, T^{1/2}R_+T^{-1/2}f \rangle).$$

Then, for each $\varepsilon > 0$, we have

$$\begin{aligned} & \left\langle T^{1/2}R_{\pm}^{(\varepsilon)*}T^{-1/2}h, Tf \right\rangle - \left\langle Th, T^{1/2}R_{\pm}^{(\varepsilon)}T^{-1/2}f \right\rangle \\ &= \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu'^2 - \mu^2}{(\mu'^2 - \mu^2 \pm i\varepsilon)D_{\pm}(\mu'^2 - E_0^2)} d\langle h, E(\mu)g \rangle d\langle g, E(\mu')f \rangle \\ &= \lambda \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{D_{\pm}(\mu'^2 - E_0^2)} d\langle h, E(\mu)g \rangle d\langle E(\mu')g, f \rangle \mp i\varepsilon \left\langle T^{-1/2}h, R_{\pm}^{(\varepsilon)}T^{-1/2}f \right\rangle. \end{aligned}$$

Taking the limit $\varepsilon \downarrow 0$, we have

$$\langle T^{1/2} R_{\pm}^* T^{-1/2} h, T f \rangle - \langle T h, T^{1/2} R_{\pm} T^{-1/2} f \rangle = \langle h, \lambda \langle D_{\mp}(T^2 - E_0^2)^{-1} g, f \rangle g \rangle.$$

Thus we have

$$\langle h, [U, T] f \rangle = \frac{\lambda}{2} \langle h, \langle D_{-}(T^2 - E_0^2)^{-1} g, f \rangle g \rangle.$$

Since $D(T^{-1/2}) \cap D(T^{3/2})$ is a core of T , the equation (5.4) holds for $f \in D(T)$. To prove (5.5), we note that

$$\begin{aligned} (V^* J - U^*) g &= \frac{1}{2} (T^{1/2} \Omega_+^* T^{-1/2} J - T^{-1/2} \Omega_+^* T^{1/2} J - T^{1/2} \Omega_+^* T^{-1/2} - T^{-1/2} \Omega_+^* T^{1/2}) g \\ &= -T^{-1/2} \Omega_+^* T^{1/2} g, \end{aligned}$$

where we have used $Jg = g$. Thus, for any $f \in \mathcal{H}$, we obtain

$$\begin{aligned} &\langle f, (V^* J - U^*) g \rangle \\ &= -\langle f, g \rangle - \lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \langle f, R_{\mu'^2 + i\varepsilon}(T^2) D_{-}(T^2 - E_0^2)^{-1} g \rangle d\|E(\mu') T^{1/2} g\|^2 \\ &= -\langle f, g \rangle + \lambda \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu'}{\mu'^2 - \mu^2 + i\varepsilon} d\|E(\mu') g\|^2 \frac{1}{D_{-}(\mu^2 - E_0^2)} \langle f, E(\mu) g \rangle \\ &= -\langle f, g \rangle + \int_{[E_0, \infty)} \frac{D_{-}(\mu^2 - E_0^2) - 1}{D_{-}(\mu^2 - E_0^2)} d\langle f, E(\mu) g \rangle \\ &= -\langle f, D_{-}(T^2 - E_0^2)^{-1} g \rangle. \end{aligned}$$

Hence (5.5) holds. $\square$

Proof of Theorem 5.1.

- (1) By Lemma 4.16, for any $f \in D(T)$, $B(f)$ leaves $\mathcal{F}_{b, \text{fin}}(D(T))$ invariant and $H(\lambda)$ maps $\mathcal{F}_{b, \text{fin}}(D(T))$ into $\mathcal{F}_{b, \text{fin}}(\mathcal{H}) \subset D(B(f))$. Thus, by using (2.1) and (10.3), we have for any $\psi \in \mathcal{F}_{b, \text{fin}}(D(T))$,

$$[H(\lambda), B(f)]\psi = \left\{ -A(TUf) + A(TJVf)^* - \frac{\lambda}{\sqrt{2}} \langle f, (V^* J - U^*) g \rangle \Phi_s(g) \right\} \psi.$$

Hence by Lemma 5.2, (5.1) holds.

- (2) By Lemma 4.16 and fundamental properties of the annihilation operators and creation operators, we can see that, for any $f \in D(T^{-1/2})$, $D(d\Gamma_b(T)^{1/2}) \subset D(B(f))$. For any $\psi, \phi \in D(d\Gamma_b(T))$, there are sequences $\psi_n, \phi_n \in \mathcal{F}_{b, \text{fin}}(D(T))$, $n \in \mathbb{N}$ such that

$\psi_n \rightarrow \psi, \phi_n \rightarrow \phi, d\Gamma_b(T)\psi_n \rightarrow d\Gamma_b(T)\psi, d\Gamma_b(T)\phi_n \rightarrow d\Gamma_b(T)\phi$ as $n \rightarrow \infty$, since $\mathcal{F}_{b,\text{fin}}(D(T))$ is a core of $d\Gamma_b(T)$. By (1), we have

$$\langle H(\lambda)\phi_n, B(f)\psi_k \rangle - \langle B(f)^*\phi_n, H(\lambda)\psi_k \rangle = -\langle \phi_n, B(Tf)\psi_k \rangle$$

for all $n, k \in \mathbb{N}$ and $f \in D(T^{-1/2}) \cap D(T)$. By the inequalities (10.1) and (10.2) and the $d\Gamma_b(T)$ -boundedness of $\Phi_s(g)^2$, we obtain that $\{B(f)\psi_n\}_{n=1}^\infty$, $\{B(f)\phi_n\}_{n=1}^\infty$, $\{\Phi_s(g)^2\psi_n\}_{n=1}^\infty$, $\{\Phi_s(g)^2\phi_n\}_{n=1}^\infty$ and $\{B(Tf)\psi_n\}_{n=1}^\infty$ converge. Hence we obtain (5.2).

- (3) By Lemma 4.16 and fundamental properties of the annihilation operators and creation operators, we see that, for any $f \in D(T^{-1/2}) \cap D(T)$, $B(f)$ maps $D(d\Gamma_b(T)^{3/2})$ into $D(d\Gamma_b(T))$. Therefore, by (5.2) and the density of $D(d\Gamma_b(T))$, we have (5.3). $\square$

5.1 Relations between U and V

Lemma 5.3. *Let $\lambda \neq \lambda_c$. Then the following equations hold:*

$$\begin{cases} U^*U - V^*V = I, \\ U_J^*V - V_J^*U = 0, \\ UU^* - V_JV_J^* = I - \theta(\lambda_c - \lambda)Q_+, \\ UV^* - V_JU_J^* = \theta(\lambda_c - \lambda)Q_-, \end{cases} \quad (5.6)$$

where

$$Q_\pm := \frac{1}{2} (\langle T^{1/2}U_b, \cdot \rangle T^{-1/2}U_b \pm \langle T^{-1/2}U_b, \cdot \rangle T^{1/2}U_b)$$

are bounded operators on $\mathcal{H}$.

Proof. It is sufficient to prove (5.6) on $D(T^{-1/2}) \cap D(T^{1/2})$. Using (4.9), one can show that the first equation in (4.9) holds. We have

$$U_J^*V - V_J^*U = \frac{1}{2}(-T^{1/2}(\Omega_+^*)_J\Omega_+T^{-1/2} + T^{-1/2}(\Omega_+^*)_J\Omega_+T^{1/2}).$$

Multiplying the equation by $(\Omega_+)_J$ from the left, and using Lemma 4.17, we obtain

$$\begin{aligned} (\Omega_+)_J(U_J^*V - V_J^*U) &= (\Omega_+)_J(-T^{1/2}(\Omega_+^*)_J\Omega_+T^{-1/2} + T^{-1/2}(\Omega_+^*)_J\Omega_+T^{1/2}) \\ &= \Omega_+(-T^{1/2}\Omega_+^*\Omega_+T^{-1/2} + T^{-1/2}\Omega_+^*\Omega_+T^{1/2}) = 0. \end{aligned}$$

By (4.9), this implies that $U_J^*V - V_J^*U = 0$. By Lemma 3.5 and Lemma 4.17, we have

$$\begin{aligned}
V_J V_J^* &= \frac{1}{4} \{ T^{-1/2}(\Omega_+ T \Omega_+^*)_J T^{-1/2} - T^{-1/2}(\Omega_+ \Omega_+^*)_J T^{1/2} \\
&\quad - T^{1/2}(\Omega_+ \Omega_+^*)_J T^{-1/2} + T^{1/2}(\Omega_+ T^{-1} \Omega_+^*)_J T^{1/2} \} \\
&= \frac{1}{4} (T^{-1/2} \Omega_+ T \Omega_+^* T^{-1/2} - T^{-1/2} \Omega_+ \Omega_+^* T^{1/2} \\
&\quad - T^{1/2} \Omega_+ \Omega_+^* T^{-1/2} + T^{1/2} \Omega_+ T^{-1} \Omega_+^* T^{1/2}) \\
&= V V^*.
\end{aligned}$$

Hence, by direct calculations and (4.10), one obtains $U U^* - V_J V_J^* = I - \theta(\lambda_c - \lambda) Q_+$. Similarly one can prove the last equation in (5.6) (note that $P_J = P$). $\square$

5.2 Hilbert-Schmidtness of V

In this subsection, we show that V is Hilbert-Schmidt. Then we can use Theorem 2.2 in the case of $\lambda > \lambda_c$.

Lemma 5.4. *The operator V is Hilbert-Schmidt.*

Proof. On $D(T^{-1/2}) \cap D(T^{1/2})$, V^*V is calculated as follows:

$$\begin{aligned}
V^*V &= \frac{1}{4} (T^{-1/2} R_+ T^{1/2} + T^{1/2} R_+^* T^{-1/2} + T^{1/2} [R_+^*, T^{-1}] R_+ T^{1/2} \\
&\quad + T^{1/2} R_+ T^{-1/2} + T^{-1/2} R_+^* T^{1/2} + T^{-1/2} [R_+^*, T] R_+ T^{-1/2} \\
&\quad + T^{1/2} R_+^* R_+ T^{-1/2} + T^{-1/2} R_+^* R_+ T^{1/2}) \\
&= \frac{1}{4} (T^{1/2} [R_+^*, T^{-1}] R_+ T^{1/2} + T^{-1/2} [R_+^*, T] R_+ T^{-1/2}),
\end{aligned}$$

where we have used the formula $R_+^* R_+ = -(R_+ + R_+^*)$ in the proof of Lemma 4.13 and Lemma 4.8. Thus, for any $f \in D(T^{-1/2}) \cap D(T^{1/2})$ and $\varepsilon > 0$, we have

$$\begin{aligned}
&\left\langle f, (T^{1/2} [R_+^{(\varepsilon)*}, T^{-1}] R_+^{(\varepsilon)} T^{1/2} + T^{-1/2} [R_+^{(\varepsilon)*}, T] R_+^{(\varepsilon)} T^{-1/2}) f \right\rangle \\
&= \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\lambda \mu'}{(\mu'^2 - \mu^2 + i\varepsilon) D_+(\mu'^2 - E_0^2)} d \left\langle [T^{-1}, R_+^{(\varepsilon)}] T^{1/2} f, E(\mu) T^{1/2} g \right\rangle d \langle E(\mu') g, f \rangle \\
&\quad + \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\lambda \mu}{(\mu'^2 - \mu^2 + i\varepsilon) D_+(\mu'^2 - E_0^2)} d \left\langle [T, R_+^{(\varepsilon)}] T^{-1/2} f, E(\mu) T^{-1/2} g \right\rangle d \langle E(\mu') g, f \rangle.
\end{aligned}$$

Then, for any $B \in \mathbf{B}^1$, we can see

$$\begin{aligned}
&\left\langle [T^{-1}, R_+^{(\varepsilon)}] T^{1/2} f, E(B) T^{1/2} g \right\rangle \\
&= \lambda \int_B \int_{[E_0, \infty)} \frac{\mu'' - \mu}{(\mu''^2 - \mu^2 - i\varepsilon) D_-(\mu''^2 - E_0^2)} d \langle f, E(\mu'') g \rangle d \|E(\mu) g\|^2. \tag{5.7}
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned} & \left\langle [T, R_+^{(\varepsilon)}] T^{-1/2} f, E(B) T^{-1/2} g \right\rangle \\ &= \lambda \int_B \int_{[E_0, \infty)} \frac{\mu - \mu''}{(\mu''^2 - \mu^2 - i\varepsilon) D_-(\mu''^2 - E_0^2)} d\langle f, E(\mu'') g \rangle d\|E(\mu) g\|^2. \end{aligned}$$

Thus, by the formula of a change of variable in Lebesgue-Stieltjes integration and Fubini's theorem, we have

$$\begin{aligned} & \left\langle f, (T^{1/2} [R_+^{(\varepsilon)*}, T^{-1}] R_+^{(\varepsilon)} T^{1/2} + T^{-1/2} [R_+^{(\varepsilon)*}, T] R_+^{(\varepsilon)} T^{-1/2}) f \right\rangle \\ &= \lambda^2 \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\|E(\mu) g\|^2 d\langle f, E(\mu'') g \rangle d\langle E(\mu') g, f \rangle \\ & \quad \times \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'^2 - \mu^2 + i\varepsilon)(\mu''^2 - \mu^2 - i\varepsilon) D_+(\mu'^2 - E_0^2) D_-(\mu''^2 - E_0^2)}. \end{aligned}$$

Then it is easy to see that for any $\mu, \mu', \mu'' \in [E_0, \infty)$,

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'^2 - \mu^2 + i\varepsilon)(\mu''^2 - \mu^2 - i\varepsilon) D_+(\mu'^2 - E_0^2) D_-(\mu''^2 - E_0^2)} \\ &= \frac{1}{(\mu' + \mu)(\mu'' + \mu) D_+(\mu'^2 - E_0^2) D_-(\mu''^2 - E_0^2)}. \end{aligned}$$

For any $\varepsilon > 0$ and $\mu, \mu', \mu'' \in [E_0, \infty)$, we have, by Lemma 4.5 and the arithmetic-geometric mean inequality,

$$\left| \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'^2 - \mu^2 + i\varepsilon)(\mu''^2 - \mu^2 - i\varepsilon) D_+(\mu'^2 - E_0^2) D_-(\mu''^2 - E_0^2)} \right| \leq \frac{1}{4\delta^2 \mu \sqrt{\mu' \mu''}}.$$

On the other side, for any $\alpha, \beta \in \mathbb{C}$, we see

$$\begin{aligned} & \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu \sqrt{\mu' \mu''}} d\|E(\mu) g\|^2 d\|E(\mu'')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2 \\ &= \|T^{-1/2} g\|^2 \|T^{-1/4}(f + \alpha g)\|^2 \|T^{-1/4}(f + \beta g)\|^2 < \infty. \end{aligned}$$

Thus, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\|E(\mu) g\|^2 d\|E(\mu'')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2 \\ & \quad \times \frac{(\mu - \mu')(\mu - \mu'')}{(\mu'^2 - \mu^2 + i\varepsilon)(\mu''^2 - \mu^2 - i\varepsilon) D_+(\mu'^2 - E_0^2) D_-(\mu''^2 - E_0^2)} \\ &= \int_{[E_0, \infty)} \int_{[E_0, \infty)} \int_{[E_0, \infty)} d\|E(\mu) g\|^2 d\|E(\mu'')(f + \alpha g)\|^2 d\|E(\mu')(f + \beta g)\|^2 \\ & \quad \times \frac{1}{(\mu' + \mu)(\mu'' + \mu) D_+(\mu'^2 - E_0^2) D_-(\mu''^2 - E_0^2)}. \end{aligned}$$

In particular, for each $\alpha, \beta = \pm 1, \pm i$, the polarization identity and Fubini's theorem give

$$\langle f, V^* V f \rangle = \frac{\lambda^2}{4} \int_{[E_0, \infty)} |\langle f, R_{-\mu}(T) D_{-}(T^2 - E_0^2)^{-1} g \rangle|^2 d\|E(\mu)g\|^2.$$

Let $\{e_n\}_{n=1}^{\infty} \subset D(T^{-1/2}) \cap D(T^{1/2})$ be a CONS of $\mathcal{H}$. The termwise integration implies that

$$\begin{aligned} \sum_{n=1}^{\infty} \langle e_n, V^* V e_n \rangle &= \frac{\lambda^2}{4} \int_{[E_0, \infty)} \|R_{-\mu}(T) D_{-}(T^2 - E_0^2)^{-1} g\|^2 d\|E(\mu)g\|^2 \\ &= \frac{\lambda^2}{4} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{(\mu' + \mu)^2 |D_{-}(\mu'^2 - E_0^2)|^2} d\|E(\mu')g\|^2 d\|E(\mu)g\|^2 \quad (5.8) \\ &\leq \frac{\lambda^2}{16\delta^2} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{1}{\mu' \mu} d\|E(\mu')g\|^2 d\|E(\mu)g\|^2 < \infty, \end{aligned}$$

where we have used the arithmetic-geometric mean inequality and Lemma 4.5. Hence V is Hilbert-Schmidt. $\square$

Lemma 5.5. *If $\lambda > \lambda_c$, then there is a unitary operator $\mathbb{U}$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $f \in \mathcal{H}$,*

$$\mathbb{U}B(f)\mathbb{U}^{-1} = A(f).$$

Proof. By Lemma 5.3 and Lemma 5.4, we can apply Theorem 2.2. $\square$

6 Analysis in the case $\lambda > \lambda_c$

In this section we prove Theorem 3.6 (1). Before starting the proof, we need to know a property of the Hamiltonian $H(\lambda)$.

6.1 Time evolution

Theorem 6.1 (Time evolution). *If $\lambda > \lambda_{c,0}$, then for all $f \in D(T^{-1/2})$, $\psi \in D(d\Gamma_b(T)^{1/2})$ and $t \in \mathbb{R}$,*

$$e^{itH(\lambda)} B(f) e^{-itH(\lambda)} \psi = B(e^{itT} f) \psi, \quad (6.1)$$

$$e^{itH(\lambda)} B(f)^* e^{-itH(\lambda)} \psi = B(e^{itT} f)^* \psi. \quad (6.2)$$

Proof. It is sufficient to prove (6.1), because (6.2) follows from taking the adjoint of (6.1). We define a function $v : \mathbb{R} \rightarrow \mathbb{C}$ by $v(t) := \langle \phi, e^{itH(\lambda)} B(e^{-itT} f) e^{-itH(\lambda)} \psi \rangle$, $t \in \mathbb{R}$ for any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(d\Gamma_b(T))$. Then v is well-defined by an operational

calculus and Theorem 2.3. The function v is differentiable and, by Theorem 5.1 (2), we have for any $t \in \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt}v(t) &= i \langle H(\lambda)e^{-itH(\lambda)}\phi, B(e^{-itT}f)e^{-itH(\lambda)}\psi \rangle - i \langle B(e^{-itT}f)^*e^{-itH(\lambda)}\phi, H(\lambda)e^{-itH(\lambda)}\psi \rangle \\ &\quad + i \langle e^{-itH(\lambda)}\phi, B(Te^{-itT}f)e^{-itH(\lambda)}\psi \rangle \\ &= 0. \end{aligned}$$

Hence $v(t) = v(0)$ for all $t \in \mathbb{R}$. Hence the equation

$$\langle \phi, e^{itH(\lambda)}B(e^{-itT}f)e^{-itH(\lambda)}\psi \rangle = \langle \phi, B(f)\psi \rangle$$

holds for all $t \in \mathbb{R}$. By replacing f with $e^{itT}f$, one has for all $\psi \in D(d\Gamma_b(T))$,

$$e^{itH(\lambda)}B(f)e^{-itH(\lambda)}\psi = B(e^{itT}f)\psi.$$

Since $D(d\Gamma_b(T))$ is a core of $(H(\lambda) + M)^{1/2}$ and $D(H(\lambda) + M)^{1/2} = D(d\Gamma_b(T)^{1/2})$ by Theorem 2.3 (3), we obtain (6.1) for $f \in D(T^{-1/2}) \cap D(T)$ and $\psi \in D(d\Gamma_b(T)^{1/2})$. Finally we extend (6.1) for all $f \in D(T^{-1/2})$. Let $f \in D(T^{-1/2})$ and $\psi \in D(d\Gamma_b(T)^{1/2})$. Then we set $f_n := E((-\infty, n])f$ for each $n \in \mathbb{N}$. Then $f_n \in D(T^{-1/2}) \cap D(T)$ for all $n \in \mathbb{N}$ and one can easily show that $f_n \rightarrow f$, $T^{-1/2}f_n \rightarrow T^{-1/2}f$ as $n \rightarrow \infty$ by using a functional calculus and the Lebesgue dominated convergence theorem. Thus we have $Uf_n \rightarrow Uf$, $JVf_n \rightarrow JVf$ as $n \rightarrow \infty$ by the boundedness of U and V . By using the linearity of the Hilbert transform and that of the map $f \mapsto \psi_{g,f}$, (4.12), (4.13) and (4.7), we can show that $T^{-1/2}Uf_n \rightarrow T^{-1/2}Uf$, $T^{-1/2}JVf_n \rightarrow T^{-1/2}JVf$ as $n \rightarrow \infty$. Therefore we obtain $B(f_n)\phi \rightarrow B(f)\phi$ and $B(e^{itT}f_n)\phi \rightarrow B(e^{itT}f)\phi$ as $n \rightarrow \infty$ for any $\phi \in D(d\Gamma_b(T)^{1/2})$ by [3, Lemma 4-28]. By the preceding result, we have for any $n \in \mathbb{N}$,

$$B(f_n)e^{-itH(\lambda)}\psi = e^{-itH(\lambda)}B(e^{itT}f_n)\psi.$$

The equation $D(d\Gamma_b(T)^{1/2}) = D((H(\lambda) + M)^{1/2})$ in Theorem 2.3 (3) implies that

$$e^{-itH(\lambda)}D(d\Gamma_b(T)^{1/2}) = D(d\Gamma_b(T)^{1/2}).$$

Hence, by taking the limit $n \rightarrow \infty$, we obtain (6.1) for $f \in D(T^{-1/2})$, $\psi \in D(d\Gamma_b(T)^{1/2})$. $\square$

6.2 Proof of Theorem 3.6 (1)

In this subsection, we assume that $\lambda > \lambda_c$.

Lemma 6.2. *Let $\Omega := \mathbb{U}^{-1}\Omega_0$, where $\mathbb{U}$ is the unitary operator in Lemma 5.5. Then there is an eigenvalue E_g of $H(\lambda)$ and Ω is the corresponding eigenvector: $H(\lambda)\Omega = E_g\Omega$.*

Proof. In general, by [3, Proposition 4-4] for a dense subspace $\mathcal{D} \subset \mathcal{H}$, if $\psi \in \cap_{f \in \mathcal{D}} D(A(f))$ satisfies $A(f)\psi = 0$ for all $f \in \mathcal{D}$, then there is a constant $\alpha \in \mathbb{C}$ such that $\psi = \alpha\Omega_0$. Thus, by Lemma 5.5, if $B(f)\phi = 0$ for all $f \in D(T^{-1/2})$, there is a constant $\alpha \in \mathbb{C}$ such that $\phi = \alpha\Omega$. For any $f \in D(T^{-1/2})$ and $t \in \mathbb{R}$,

$$B(f)e^{-itH(\lambda)}\Omega = e^{-itH(\lambda)}B(e^{itT}f)\Omega = 0$$

by Lemma 5.5 and Theorem 6.1. Thus, for each $t \in \mathbb{R}$, there is a constant $\alpha(t) \in \mathbb{C}$ such that $e^{-itH(\lambda)}\Omega = \alpha(t)\Omega$. Then we have $|\alpha(t)| = 1, \alpha(t+s) = \alpha(t)\alpha(s)$ for all $t, s \in \mathbb{R}$, since $\{e^{-itH(\lambda)}\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group. Thus there exists a constant $E_g \in \mathbb{R}$ such that $\alpha(t) = e^{-itE_g}, t \in \mathbb{R}$. The differentiation of the equation $e^{-itH(\lambda)}\Omega = e^{-itE_g}\Omega$ in t implies that $\Omega \in D(H(\lambda))$ and $\Omega \in \text{Ker}(H(\lambda) - E_g)$. $\square$

Proof of Theorem 3.6 (1).

The subspace $\mathcal{U} := \mathcal{L}(\{B(f_1)^* \cdots B(f_n)^*\Omega, \Omega \mid f_j \in D(T^{-1/2}), j = 1, \dots, n, n \in \mathbb{N}\})$ is dense in $\mathcal{F}_b(\mathcal{H})$ by the fact that $\mathcal{U} = \mathbb{U}^{-1}\mathcal{F}_{b, \text{fin}}(D(T^{-1/2}))$, where $\mathcal{L}(\mathcal{D})$ denotes the subspace algebraically spanned by the vectors in a subset $\mathcal{D}$ of a Hilbert space. By Lemma 6.1 and Lemma 10.3, for any $t \in \mathbb{R}$ and $f_j \in D(T^{-1/2}), j = 1, \dots, n$, we have

$$\begin{aligned} e^{itH(\lambda)}B(f_1)^* \cdots B(f_n)^*\Omega &= B(e^{itT}f_1)^* \cdots B(e^{itT}f_n)^*e^{itH(\lambda)}\Omega \\ &= B(e^{itT}f_1)^* \cdots B(e^{itT}f_n)^*e^{itE_g}\Omega \\ &= e^{itE_g}\mathbb{U}^{-1}e^{itd\Gamma_b(T)}A(f_1)^* \cdots A(f_n)^*\Omega_0 \\ &= \mathbb{U}^{-1}e^{it(d\Gamma_b(T)+E_g)}\mathbb{U}B(f_1)^* \cdots B(f_n)^*\Omega. \end{aligned}$$

By this equation and a limiting argument, we obtain $\mathbb{U}e^{itH(\lambda)}\mathbb{U}^{-1} = e^{it(d\Gamma_b(T)+E_g)}$. By the unitary covariance of functional calculus, we have

$$\mathbb{U}e^{itH(\lambda)}\mathbb{U}^{-1} = e^{it\mathbb{U}H(\lambda)\mathbb{U}^{-1}}, \quad t \in \mathbb{R}.$$

Hence (3.1) holds. The equation (3.1) and the well-known spectral properties of $d\Gamma_b(T)$ imply that E_g is the ground state energy of $H(\lambda)$ and Ω is the unique ground state of $H(\lambda)$. $\square$

Lemma 6.3. *The ground state energy E_g is given as follows:*

$$E_g = \frac{\lambda}{4}\|g\|^2 - \text{Tr}(T^{1/2}V^*VT^{1/2}), \quad (6.3)$$

$$\text{Tr}(T^{1/2}V^*VT^{1/2}) = \frac{\lambda^2}{4} \int_{[E_0, \infty)} \int_{[E_0, \infty)} \frac{\mu}{(\mu + \mu')^2 |D_-(\mu^2 - E_0^2)|^2} d\|E(\mu)g\|^2 d\|E(\mu')g\|^2. \quad (6.4)$$

Proof. The operator $\mathbb{U}$ leaves $D(d\Gamma_b(T))$ invariant by Theorem 3.6 (1). In particular, $\mathbb{U}\Omega_0 \in D(d\Gamma_b(T)^{1/2})$. Thus, by Lemma 10.4, the isometricity of $\mathbb{U}$ and the definition of $B(\cdot)$, we have $\langle \Omega_0, (H(\lambda) - E_g)\Omega_0 \rangle = \text{Tr}(T^{1/2}V^*VT^{1/2})$. By the definition of $H(\lambda)$ and (2.1), we have $\langle \Omega_0, H(\lambda)\Omega_0 \rangle = \lambda\|g\|^2/4$. Hence (6.3) holds. The formula (6.4) can be proved in the same way as (5.8). $\square$

7 Analysis in the case $\lambda_{c,0} < \lambda < \lambda_c$

In Section 5, we proved Theorem 3.6 (1). But the proof is valid only for the case $\lambda > \lambda_c$. Therefore it is necessary to find another pair of operators U and V if one wants to use a Bogoliubov transformation for the spectral analysis of $H(\lambda)$ in the case $\lambda \leq \lambda_c$. In this section we assume that T and g satisfy Assumption 3.3, $E_0 > 0$ and $\lambda_{c,0} < \lambda < \lambda_c$. Under these conditions, we can define the operators ξ, X, Y and $T_\pm$ as follows:

$$\begin{aligned}\xi &:= \Omega_+ T \Omega_+^* + \beta P, \\ X &:= U \Omega_+^* + T_+ P, \quad Y := V \Omega_+^* + T_- P, \\ T_\pm &:= \frac{1}{2}(\beta^{1/2} T^{-1/2} \pm \beta^{-1/2} T^{1/2}),\end{aligned}$$

where $\beta := (E_0^2 + x_0)^{1/2}$.

Remark 7.1. *The definition of x_0 implies that*

$$E_0^2 + x_0 \begin{cases} > 0, & \text{if } \lambda_{c,0} < \lambda < \lambda_c, \\ = 0, & \text{if } \lambda = \lambda_{c,0}, \\ < 0, & \text{if } \lambda < \lambda_{c,0}. \end{cases}$$

Thus, in the case $\lambda_{c,0} < \lambda < \lambda_c$, we see that the inequality $0 < \beta < E_0$ holds. Let

$$C(f) := A(Xf) + A(JYf)^*, f \in \mathcal{H}.$$

Then $C(f)$ is a densely defined closable operator. We denotes its closure by the same symbol.

7.1 Properties of X, Y and ξ

In this subsection, we study the operators X, Y and ξ . Firstly, we consider ξ . Let

$$\tilde{T} := \Omega_+ T \Omega_+^*.$$

Lemma 7.2. *The operator $\tilde{T}$ is a self-adjoint operator with $D(\tilde{T}) = D(T)$.*

Proof. By Lemma 4.8 we see that $D(\tilde{T}) = D(T)$. Hence $\tilde{T}$ is symmetric. For any $\phi \in D((\tilde{T})^*)$ and $\psi \in D(T) = D(\tilde{T})$, we have $\langle \Omega_+^*(\tilde{T})^*\phi, \psi \rangle = \langle \Omega_+^*\phi, T\psi \rangle$. This implies that $\Omega_+^*\phi \in D(T)$. Hence $\tilde{T}$ is self-adjoint. $\square$

Lemma 7.3. *The spectra of $\tilde{T}$ are as follows:*

$$\sigma(\tilde{T}) = \{0\} \cup \sigma(T), \sigma_{\text{ac}}(\tilde{T}) = \sigma(T), \sigma_{\text{p}}(\tilde{T}) = \{0\}, \sigma_{\text{sc}}(\tilde{T}) = \emptyset.$$

Proof. We define a family of projection operators $\{E_P(B) \mid B \in \mathbf{B}^1\}$ on $\mathcal{H}$ as follows: $E_P(B) = 0$ if $0 \notin B$ and $E_P(B) = P$ if $0 \in B$ for each $B \in \mathbf{B}^1$. By the definition of the spectral measure, we can see that $\{E_{\tilde{T}}(B) := \Omega_+ E(B) \Omega_+^* + E_P(B) \mid B \in \mathbf{B}^1\}$ is a spectral measure. Using a functional calculus, we see that $E_{\tilde{T}}(\cdot)$ is the spectral measure of $\tilde{T}$. The absolutely continuous part (resp. singular part) of $\tilde{T}$ is $\tilde{T} \upharpoonright \text{Ran}(I - P)$ (resp. $\tilde{T} \upharpoonright \text{Ran}(P)$) since T is absolutely continuous and $\Omega_{\pm}$ are partial isometries. Thus we see $\sigma(\tilde{T}) = \{0\} \cup \sigma_{\text{ac}}(\tilde{T}), \sigma_{\text{p}}(\tilde{T}) = \{0\}, \sigma_{\text{sc}}(\tilde{T}) = \emptyset$.

We next show that $\sigma_{\text{ac}}(\tilde{T}) = \sigma(T)$. For any $\mu \in \sigma(T)$, there is a sequence $\psi_n \in D(T), n \in \mathbb{N}$ such that $\|\psi_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \|(T - \mu)\psi_n\| = 0$. For each $n \in \mathbb{N}$, there is a $\phi_n \in \text{Ran}(I - P)$ such that $\psi_n = \Omega_+^*\phi_n$. Then $\|\phi_n\| = \|\Omega_+\psi_n\| = \|\psi_n\| = 1$ and $\|(\tilde{T} - \mu)\phi_n\| = \|(T - \mu)\psi_n\| \rightarrow 0$ as $n \rightarrow \infty$. Thus we have $\mu \in \sigma(\tilde{T} \upharpoonright \text{Ran}(I - P)) = \sigma_{\text{ac}}(\tilde{T})$. For any $\mu \in \sigma_{\text{ac}}(\tilde{T})$, there is a sequence $\eta_n \in D(\tilde{T}) \cap \text{Ran}(I - P)$ such that $\|\eta_n\| = 1$ and $\lim_{n \rightarrow \infty} \|(\tilde{T} - \mu)\eta_n\| = 0$. Then we easily see that $\Omega_+^*\eta_n \in D(T)$ for all $n \in \mathbb{N}$. The equation $\Omega_+\Omega_+^*\eta_n = \eta_n$ implies that $\|\Omega_+^*\eta_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\|(T - \mu)\Omega_+^*\eta_n\| = \|(\tilde{T} - \mu)\eta_n\| \rightarrow 0, \quad n \rightarrow \infty.$$

Thus $\mu \in \sigma(T)$. Hence $\sigma_{\text{ac}}(\tilde{T}) = \sigma(T)$. $\square$

Lemma 7.4. *The operator ξ is an injective, non-negative self-adjoint operator with $D(\xi) = D(T)$ and we have the following equations:*

$$\sigma(\xi) = \{\beta\} \cup \sigma(T), \sigma_{\text{ac}}(\xi) = \sigma(T), \sigma_{\text{p}}(\xi) = \{\beta\}, \sigma_{\text{sc}}(\xi) = \emptyset. \quad (7.1)$$

In particular, β is the ground state energy of ξ , which is an isolated eigenvalue of ξ , and U_{b} is the unique ground state of ξ .

Proof. By Lemma 7.3 and the spectral property of direct sum of self-adjoint operators, we have the equation (7.1). Thus β is an isolated ground state energy by Remark 7.1. By $\Omega_+^*U_{\text{b}} = 0$, U_{b} is a ground state of ξ . Assume that $f \in \text{Ker}(\xi - \beta)$ satisfies $(I - P)f \neq 0$. Then $\Omega_+^*f \neq 0$ by Lemma 4.13. This implies that $T\Omega_+^*f = \beta\Omega_+^*f$, but this contradicts Assumption 3.3 (1). Hence $(I - P)f = 0$ and this implies that the ground state of ξ is unique. $\square$

Lemma 7.5. *The operators $\xi^{\pm 1/2}$ are given by*

$$\xi^{1/2} = \Omega_+ T^{1/2} \Omega_+^* + \beta^{1/2} P, \quad (7.2)$$

$$\xi^{-1/2} = \Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P \quad (7.3)$$

with $D(\xi^{\pm 1/2}) = D(T^{\pm 1/2})$.

Proof. We can show in the same way as in the proof of Lemma 7.4 that the right hand side of (7.2) is non-negative, self-adjoint operator with its domain $D(T^{1/2})$. We have $\xi \subset (\Omega_+ T^{1/2} \Omega_+^* + \beta^{1/2} P)^2$. Since a self-adjoint operator has no non-trivial symmetric extension, (7.2) holds. In the same way as in the proof of (7.2), we can show that the right hand side of (7.3) is a self-adjoint operator. We have $D(\Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P) \subset \text{Ran}(\xi^{1/2})$ and $\xi^{1/2}(\Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P) = I$ on $D(\Omega_+ T^{-1/2} \Omega_+^*)$. Hence $\Omega_+ T^{-1/2} \Omega_+^* + \beta^{-1/2} P \subset \xi^{-1/2}$. Thus the equation (7.3) holds. $\square$

Next, we study X and Y .

Lemma 7.6. *The operators $X^\sharp$ and $Y^\sharp$ leave $D(T^{-1/2})$ (resp. $D(T^{1/2}), D(T)$) invariant.*

Proof. The assertion follows from Lemma 4.8, Lemma 4.16, Lemma 7.5 and the definitions of X and Y . $\square$

Lemma 7.7. *The following equations hold:*

$$\begin{cases} X^* X - Y^* Y &= I, \\ X_J^* Y - Y_J^* X &= 0, \\ X X^* - Y_J Y_J^* &= I, \\ X Y^* - Y_J X_J^* &= 0. \end{cases} \quad (7.4)$$

Proof. The operator P (resp. $T_\pm$) satisfies $P_J = P$ (resp. $(T_\pm)_J = T_\pm$). By (4.10), we have $\Omega_+^* U_b = 0$. Hence we obtain $(U^* \pm V^*) T^{\pm 1/2} U_b = 0$ and $(U^* T_\pm - V^* T_\mp) U_b = 0$. The equations $T_+ T_+ - T_- T_- = I$ and $T_+ T_- - T_- T_+ = 0$ hold on $D(T^{-1}) \cap D(T)$. By (5.6) and direct calculations, we have $X^* X - Y^* Y = I$ and $X_J^* Y - Y_J^* X = 0$. By similar calculations, we have $X X^* - Y_J Y_J^* = I$ and $X Y^* - Y_J X_J^* = 0$ on $D(T^{-1/2}) \cap D(T^{1/2})$. Then, by a limiting argument, we obtain (7.4). $\square$

Lemma 7.8. *The operator Y is Hilbert-Schmidt.*

Proof. We can easily show that the assertion follows from Lemma 5.4, Lemma 7.6 and the choice a CONS $\{e_n\}_{n=0}^\infty \subset D(T^{-1/2}) \cap D(T^{1/2})$ with $e_0 = U_b$. $\square$

Lemma 7.9. *There is a unitary operator $\mathbb{V}$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $f \in \mathcal{H}$,*

$$\mathbb{V} C(f) \mathbb{V}^{-1} = A(f).$$

Proof. By Theorem 2.2, (7.4) and Lemma 7.8, we can prove this assertion. $\square$

7.2 Commutation relations

Theorem 7.10. *The following commutation relations hold:*

(1) *For any $f \in D(T)$ and $\psi \in \mathcal{F}_{\text{b,fin}}(D(T))$,*

$$[H(\lambda), C(f)]\psi = -C(\xi f)\psi.$$

(2) *For any $f \in D(T^{-1/2}) \cap D(T)$ and $\psi, \phi \in D(d\Gamma_{\text{b}}(T))$,*

$$\langle H(\lambda)\phi, C(f)\psi \rangle - \langle C(f)^*\phi, H(\lambda)\psi \rangle = -\langle \phi, C(\xi f)\psi \rangle.$$

(3) *For any $f \in D(T^{-1/2}) \cap D(T)$, $C(f)$ maps $D(d\Gamma_{\text{b}}(T)^{3/2})$ into $D(d\Gamma_{\text{b}}(T))$ and for any $\psi \in D(d\Gamma_{\text{b}}(T)^{3/2})$,*

$$[H(\lambda), C(f)]\psi = -C(\xi f)\psi.$$

Theorem 7.10 follows, in the same manner as in the proof of Theorem 5.1, from Lemma 4.16, Lemma 7.5 and the next lemma:

Lemma 7.11. *For any $f \in D(T)$ the following equations hold:*

$$-TXf + \frac{\lambda}{2} \langle (Y^*J - X^*)g, f \rangle g = -X\xi f, \quad (7.5)$$

$$TJYf + \frac{\lambda}{2} \langle f, (Y^*J - X^*)g \rangle g = -JY\xi f. \quad (7.6)$$

Remark 7.12. *By Lemma 4.16 and the definition of ξ , the both sides of (7.5) and (7.6) have meaning.*

Proof. Let $a := \sqrt{\lambda/D'(x_0)}$. Then we can see by the definition of x_0 and (5.5),

$$(Y^*J - X^*)g = -\Omega_+ D_-(T^2 - E_0^2)^{-1}g + \frac{\beta^{-1/2}a}{\lambda} U_{\text{b}}.$$

We have

$$\begin{aligned} TT_{\pm}U_{\text{b}} &= \frac{1}{2}(\beta^{1/2}T^{1/2}U_{\text{b}} \pm \beta^{-1/2}T^{3/2}U_{\text{b}}) \\ &= \frac{1}{2}(\beta^{1/2}T^{1/2}U_{\text{b}} \pm \beta^{3/2}T^{-1/2}U_{\text{b}} \pm \beta^{-1/2}ag). \end{aligned} \quad (7.7)$$

Thus, for any $f \in D(T)$, we have

$$\begin{aligned} &-TXf + \frac{\lambda}{2} \langle (Y^*J - X^*)g, f \rangle g \\ &= -TU\Omega_+^*f - \frac{\lambda}{2} \langle D_-(T^2 - E_0^2)^{-1}g, \Omega_+^*f \rangle g - TT_+Pf + \frac{\beta^{-1/2}a}{2} \langle U_{\text{b}}, f \rangle g. \end{aligned}$$

Then, by (5.4) and (7.7), we have

$$\begin{aligned} -TXf + \frac{\lambda}{2} \langle (Y^*J - X^*)g, f \rangle g &= -UT\Omega_+^*f - \beta \langle U_b, f \rangle T_+U_b \\ &= -X(\Omega_+T\Omega_+^* + \beta P)f. \end{aligned}$$

Thus we obtain (7.5). Similarly one can prove (7.6). $\square$

7.3 Proof of Theorem 3.6 (2)

Theorem 7.13. *For all $f \in D(T^{-1/2})$, $\psi \in D(d\Gamma_b(T)^{1/2})$ and $t \in \mathbb{R}$,*

$$\begin{aligned} e^{itH(\lambda)}C(f)e^{-itH(\lambda)}\psi &= C(e^{it\xi}f)\psi, \\ e^{itH(\lambda)}C(f)^*e^{-itH(\lambda)}\psi &= C(e^{it\xi}f)^*\psi. \end{aligned}$$

Proof. These are proved in the same way as in the proof of Theorem 6.1 by Theorem 7.10. $\square$

Lemma 7.14. *Let $\Omega := \mathbb{V}^{-1}\Omega_0$ where $\mathbb{V}$ is the unitary operator in Lemma 7.9. Then:*

- (1) *There is an eigenvalue $\tilde{E}_g$ of $H(\lambda)$ and Ω is an eigenvector of $H(\lambda)$ with the eigenvalue $\tilde{E}_g$.*
- (2) *The following equation holds:*

$$\mathbb{V}H(\lambda)\mathbb{V}^{-1} = d\Gamma_b(\xi) + \tilde{E}_g.$$

- (3) *The constant $\tilde{E}_g$ is given as follows:*

$$\tilde{E}_g = E_g - \beta \|T_-U_b\|^2. \quad (7.8)$$

Proof. The assertions (1) and (2) can be proved in the same way as in the proof of Theorem 3.6 (1).

- (3) We have

$$\tilde{E}_g = \frac{\lambda}{4} \|g\|^2 - \text{Tr}(\xi^{1/2}Y^*Y\xi^{1/2})$$

in the same way as in the proof of Lemma 6.2. Then, by Lemma 7.5, we have

$$\xi^{1/2}Y^*Y\xi^{1/2} = \Omega_+T^{1/2}V^*VT^{1/2}\Omega_+^* + \Omega_+T^{1/2}V^*\beta^{1/2}T_-P + \beta^{1/2}PT_-VT^{1/2}\Omega_+^* + \beta PT_-T_-P.$$

We choose a CONS $\{e_n\}_{n=0}^\infty \subset D(T)$ satisfying $e_0 = U_b$. Then it is easy to see that $\{\Omega_+^*e_n\}_{n=1}^\infty$ is a CONS of $\mathcal{H}$ by Lemma 4.13. Hence we have

$$\begin{aligned} \text{Tr}(\xi^{1/2}Y^*Y\xi^{1/2}) &= \sum_{n=1}^\infty \langle e_n, \Omega_+T^{1/2}V^*VT^{1/2}\Omega_+^*e_n \rangle + \beta \|T_-U_b\|^2 \\ &= \text{Tr}(T^{1/2}V^*VT^{1/2}) + \beta \|T_-U_b\|^2. \end{aligned}$$

Thus we obtain (7.8). $\square$

In particular, $H(\lambda)$ have eigenvectors as follows:

$$\phi_n := \mathbb{V}^{-1}A(U_b)^{*n}\Omega_0, \quad H(\lambda)\phi_n = (n\beta + \tilde{E}_g)\phi_n, \quad n \in \mathbb{N} \cup \{0\}.$$

Hence the spectral properties of $H(\lambda)$ as stated in Theorem 3.6 (2) follow.

8 Analysis in the case $\lambda < \lambda_{c,0}$

In this section, we show that $H(\lambda)$ is unbounded from above and below.

Theorem 8.1. *Let $g \in D(T^{-1/2})$. Then $H(\lambda)$ is unbounded above for any $\lambda \in \mathbb{R}$. If $\lambda < \lambda_{c,0}$, then $H(\lambda)$ is unbounded below.*

Proof. For any $f \in D(T) \setminus \{0\}$, we set $\psi_n := a_n A(f)^{*n} \Omega_0$, $a_n \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N} \cup \{0\}$. Then we have the following equations:

$$\begin{aligned} d\Gamma_b(T)\psi_n &= n \frac{a_n}{a_{n-1}} A(Tf)^* \psi_{n-1}, \quad A(g)\psi_n = n \langle g, f \rangle \frac{a_n}{a_{n-1}} \psi_{n-1}, \\ \|\psi_n\|^2 &= |a_n|^2 n! \|f\|^{2n}, \quad \|A(g)^* \psi_n\|^2 = \|g\|^2 \|\psi_n\|^2 + \|A(g)\psi_n\|^2, \end{aligned}$$

where $\psi_{-1} := 0$. Then we have

$$\langle \psi_n, H(\lambda)\psi_n \rangle = \|\psi_n\|^2 \left(\frac{\lambda}{4} \|g\|^2 + n \frac{2\|T^{1/2}f\|^2 + \lambda |\langle g, f \rangle|^2}{2\|f\|^2} \right).$$

We take f such that $\langle g, f \rangle = 0$. Then we have $\langle \psi_n, H(\lambda)\psi_n \rangle / \|\psi_n\|^2 \rightarrow \infty$ as $n \rightarrow \infty$ for any $\lambda \in \mathbb{R}$. Thus $H(\lambda)$ is unbounded above for any $\lambda \in \mathbb{R}$.

Let $\phi_N := \sum_{n=0}^N \psi_n$, $N = 0, 1, 2, \dots$. Then we have $\|\phi_N\|^2 = \sum_{n=0}^N \|\psi_n\|^2$ and

$$\begin{aligned} \langle \phi_N, H(\lambda)\phi_N \rangle &= \sum_{n=2}^N \|\psi_n\|^2 \left(\frac{\lambda \|g\|^2}{4} + n \frac{2\|T^{1/2}f\|^2 + \lambda |\langle g, f \rangle|^2}{2\|f\|^2} + \frac{\lambda}{2} \operatorname{Re} \frac{a_{n-2}^*}{a_n^*} \frac{\langle g, f \rangle^2}{\|f\|^4} \right) \\ &\quad + \|\psi_1\|^2 \left(\frac{\lambda \|g\|^2}{4} + \frac{\|T^{1/2}f\|^2}{\|f\|^2} + \frac{\lambda |\langle g, f \rangle|^2}{2\|f\|^2} \right) + \frac{\lambda \|\psi_0\|^2 \|g\|^2}{4}. \end{aligned}$$

Let $a_0 := 1$, $a_n := n^{-3/4} n!^{-1/2}$, $n \in \mathbb{N}$ and, for any $0 < \delta$, $0 < \varepsilon < 1$,

$$\begin{aligned} f &= f_\delta := \frac{T^{-1}E((\delta, \infty))g}{\|T^{-1}E((\delta, \infty))g\|}, \\ c_\lambda(\varepsilon, \delta) &:= \|T^{1/2}f_\delta\|^2 \left\{ 1 + \frac{\lambda}{2} (2 - \varepsilon) \|T^{-1/2}E((\delta, \infty))g\|^2 \right\}. \end{aligned}$$

Then $\sum_{n=0}^{\infty} \|\psi_n\|^2$ converges and, for any $N \in \mathbb{N}$,

$$\langle \phi_N, H(\lambda) \phi_N \rangle = \sum_{n=2}^N \|\psi_n\|^2 n c_{\lambda}(\varepsilon, \delta) + \frac{\lambda}{2} \sum_{n=2}^N \|\psi_n\|^2 \left(\frac{a_{n-2}}{a_n} - n(1 - \varepsilon) \right) \langle g, f_{\delta} \rangle^2 + C_N, \quad (8.1)$$

where

$$C_N := \frac{\lambda \|g\|^2}{4} \sum_{n=0}^N \|\psi_n\|^2 + \|\psi_1\|^2 \left(\|T^{1/2} f_{\delta}\|^2 + \frac{\lambda}{2} \langle g, f_{\delta} \rangle^2 \right).$$

For all $0 < \delta, 0 < \varepsilon < 1$, we have

$$-\frac{2}{\|T^{-1/2} E((\delta, \infty)) g\|^2 (2 - \varepsilon)} < \lambda_{c,0}. \quad (8.2)$$

The left hand side of (8.2) tends to $\lambda_{c,0}$ as $\varepsilon, \delta \downarrow 0$. Since $\lambda < \lambda_{c,0}$, we can take a pair (ε, δ) satisfying $c_{\lambda}(\varepsilon, \delta) < 0$. We fix such a pair. There is a $n_0 \in \mathbb{N}$ such that $a_{n-2}/a_n - n(1 - \varepsilon) > 0$ for all $n \geq n_0$. Hence we can see that $\langle \phi_N, H(\lambda) \phi_N \rangle / \|\phi_N\|^2$ tends to $-\infty$ as $N \rightarrow \infty$, because the first term of the right hand side of (8.1) tends to $-\infty$ as $N \rightarrow \infty$. $\square$

9 Generalization of the ϕ^2 -model

In this section we consider $H(\eta, \lambda)$ defined in Subsection 2.3.

Assumption 9.1. *We need the following assumptions:*

- (1) $f \in D(T^{1/2})$ and $g \in D(T^{-1/2}) \cap D(T^{1/2})$,
- (2) $f \in D(T^{-1})$ and $\operatorname{Re} \langle T^{-1} f, g \rangle = 0$,
- (3) $f, g \in D(T^{-1})$ and $\operatorname{Re} \langle T^{-1} f, g \rangle \neq 0$.

We can prove a slight generalization of Theorem 3.6.

Theorem 9.2. *Let $\mathcal{H}$ be separable. Then the following (1)-(5) hold:*

- (1) *Suppose that Assumption 3.3 and, Assumption 9.1 (2) or (3) hold. Let $\lambda > \lambda_c$. Then there is a unitary operator $\mathbb{U}$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $\eta \in \mathbb{R}$,*

$$\mathbb{U} H(\eta, \lambda) \mathbb{U}^{-1} = d\Gamma_b(T) + E_g + E_{f,g},$$

where the constant $E_{f,g} \in \mathbb{R}$ is defined by

$$E_{f,g} = -\frac{\eta^2}{2} \|T^{-1/2} f\|^2 + \frac{(\operatorname{Re} \langle T^{-1} f, g \rangle)^2 \eta^2 \lambda}{2(1 + \lambda \|T^{-1/2} g\|^2)}.$$

- (2) Suppose that Assumption 3.3 and, Assumption 9.1 (2) or (3) hold. Let $E_0 > 0$ and $\lambda_{c,0} < \lambda < \lambda_c$. Then there are a unitary operator $\mathbb{V}$ on $\mathcal{F}_b(\mathcal{H})$ and a non-negative, injective self-adjoint operator ξ on $\mathcal{H}$ such that, for all $\eta \in \mathbb{R}$,

$$\mathbb{V}H(\eta, \lambda)\mathbb{V}^{-1} = d\Gamma_b(\xi) + E_g - E_b + E_{f,g}.$$

- (3) Let T be a non-negative, injective self-adjoint operator and suppose that f and g satisfy Assumption 9.1 (1) and (2). Then there is a unitary operator $\mathbb{W}$ on $\mathcal{F}_b(\mathcal{H})$ such that, for all $\eta \in \mathbb{R}$,

$$\mathbb{W}\overline{H(\eta, \lambda_{c,0})}\mathbb{W}^{-1} = \overline{H(\lambda_{c,0})} - \frac{\eta^2}{2}\|T^{-1/2}f\|^2.$$

- (4) Let T be a non-negative, injective self-adjoint operator and suppose that f and g satisfy Assumption 9.1 (1) and (3). Then, for all $\eta \in \mathbb{R} \setminus \{0\}$,

$$\sigma(\overline{H(\eta, \lambda_{c,0})}) = \mathbb{R}, \quad \sigma_p(\overline{H(\eta, \lambda_{c,0})}) = \emptyset.$$

- (5) Let T be a non-negative, injective self-adjoint operator and suppose that f and g satisfy Assumption 9.1 (1). Moreover, suppose that Assumption 9.1 (2) or (3) holds. Let $\lambda < \lambda_{c,0}$. Then, for all $\eta \in \mathbb{R}$, $\overline{H(\eta, \lambda)}$ is unbounded from above and below.

Theorem 9.2 is immediately proved by the following lemma and Theorem 3.6.

Lemma 9.3. *Let T be a non-negative, injective self-adjoint operator, $f \in D(T^{-1})$ and $g \in D(T^{-1/2}) \cap D(T)$.*

- (1) *Let $\operatorname{Re} \langle T^{-1}f, g \rangle = 0$. Then there is a unitary operator $\mathbb{U}_1$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $\eta, \lambda \in \mathbb{R}$,*

$$\mathbb{U}_1\overline{H(\eta, \lambda)}\mathbb{U}_1^{-1} = \overline{H(\lambda)} - \frac{\eta^2}{2}\|T^{-1/2}f\|^2. \quad (9.1)$$

- (2) *Let $\operatorname{Re} \langle T^{-1}f, g \rangle \neq 0$ and $g \in D(T^{-1})$.*

- (i) *If $\lambda \neq \lambda_{c,0}$, then there is a unitary operator $\mathbb{U}_2$ on $\mathcal{F}_b(\mathcal{H})$ such that for all $\eta \in \mathbb{R}$,*

$$\mathbb{U}_2\overline{H(\eta, \lambda)}\mathbb{U}_2^{-1} = \overline{H(\lambda)} + E_{f,g}.$$

- (ii) *If $\lambda = \lambda_{c,0}$, then for all $\eta \in \mathbb{R} \setminus \{0\}$,*

$$\sigma(\overline{H(\eta, \lambda_{c,0})}) = \mathbb{R}, \quad \sigma_p(\overline{H(\eta, \lambda_{c,0})}) = \emptyset. \quad (9.2)$$

Proof. Let $\mathbb{U}_1 := e^{-i\Phi_s(i\eta T^{-1}f)}$ for any $\eta \in \mathbb{R}$. Then, by direct calculations, we obtain

$$\mathbb{U}_1 H(\eta, \lambda) \mathbb{U}_1^{-1} = H(\lambda) - \frac{\eta^2}{2} \|T^{-1/2}f\|^2 - \lambda\eta\kappa\Phi_s(g) + \frac{\lambda}{2}\eta^2\kappa^2 \quad (9.3)$$

on $\mathcal{F}_{b,\text{fin}}(D(T))$ for all $\eta, \lambda \in \mathbb{R}$, where $\kappa := \text{Re}\langle T^{-1}f, g \rangle$. In the case of (1), we have (9.1) by $\kappa = 0$ and a limiting argument. Next, we prove (2). We assume that $g \in D(T^{-1})$ and $\text{Re}\langle T^{-1}f, g \rangle \neq 0$. Let $\mathbb{V}_1 := e^{i\Phi_s(i\alpha T^{-1}g)}$ for any $\alpha \in \mathbb{R}$ and define a unitary operator $\mathbb{U}_2 := \mathbb{V}_1 \mathbb{U}_1$. Then it follows that

$$\begin{aligned} \mathbb{U}_2 H(\eta, \lambda) \mathbb{U}_2^{-1} &= H(\lambda) + \left(\alpha + \lambda\alpha \|T^{-1/2}g\|^2 - \lambda\eta\kappa \right) \Phi_s(g) \\ &\quad - \frac{\eta^2}{2} \|T^{-1/2}f\|^2 + \frac{\lambda}{2}\eta^2\kappa^2 + \frac{\alpha}{2} \|T^{-1/2}g\|^2 \left(\alpha + \lambda\alpha \|T^{-1/2}g\|^2 - 2\lambda\eta\kappa \right) \end{aligned}$$

on $\mathcal{F}_{b,\text{fin}}(D(T))$ in the same way as (9.3). For $\lambda \neq \lambda_{c,0}$, let $\alpha = \lambda\eta\kappa(1 + \lambda\|T^{-1/2}g\|^2)^{-1}$. Then we obtain

$$\mathbb{U}_2 \overline{H(\eta, \lambda)} \mathbb{U}_2^{-1} = \overline{H(\lambda)} - \frac{\eta^2}{2} \|T^{-1/2}f\|^2 + \frac{\lambda\eta^2\kappa^2}{2(1 + \lambda\|T^{-1/2}g\|^2)} \quad (9.4)$$

by a limiting argument. If $\lambda = \lambda_{c,0}$, then, for all $\eta, \alpha \in \mathbb{R}$, we have

$$\mathbb{U}_2 \overline{H(\eta, \lambda_{c,0})} \mathbb{U}_2^{-1} = \overline{H_g(-\kappa\eta\lambda_{c,0}, \lambda_{c,0})} - \frac{\eta^2}{2} \|T^{-1/2}f\|^2 + \frac{\lambda_{c,0}\eta^2\kappa^2}{2} + \kappa\eta\alpha$$

in the same way as (9.4), where $H_g(\nu, \lambda_{c,0}) := H(\lambda_{c,0}) + \nu\Phi_s(g)$ for all $\nu \in \mathbb{R}$. We can see that $\sigma(\overline{H_g(\nu, \lambda_{c,0})}) = \mathbb{R}$ and $\sigma_p(\overline{H_g(\nu, \lambda_{c,0})}) = \emptyset$ for all $\nu \in \mathbb{R} \setminus \{0\}$, because $\mathbb{V}_1 \overline{H_g(\nu, \lambda_{c,0})} \mathbb{V}_1^{-1} = \overline{H_g(\nu, \lambda_{c,0})} + \nu\alpha\|T^{-1/2}g\|^2$ and $\alpha \in \mathbb{R}$ is arbitrary. Hence we have (9.2). $\square$

Remark 9.4. If $\mathcal{H}$ is separable, then the condition $g \in D(T^{-1/2}) \cap D(T)$ in the above lemma is weakened to the condition $g \in D(T^{-1/2}) \cap D(T^{1/2})$.

10 Appendix

In this section, we recall some known facts in the Fock space theory. Let T be a non-negative, injective self-adjoint operator on $\mathcal{H}$.

Lemma 10.1. [5, Theorem 5.16.]

Let $f \in D(T^{-1/2})$ and $\psi \in D(d\Gamma_b(T)^{1/2})$. Then $\psi \in D(A(f)) \cap D(A(f)^*)$ and the following inequalities hold:

$$\|A(f)\psi\| \leq \|T^{-1/2}f\| \|d\Gamma_b(T)^{1/2}\psi\|, \quad (10.1)$$

$$\|A(f)^*\psi\|^2 \leq \|T^{-1/2}f\|^2 \|d\Gamma_b(T)^{1/2}\psi\|^2 + \|f\|^2 \|\psi\|^2. \quad (10.2)$$

Lemma 10.2. [5, Proposition 5.10.] *For any $f \in D(T)$, the following commutation relations hold on $\mathcal{F}_{b, \text{fin}}(D(T))$:*

$$[d\Gamma_b(T), A(f)] = -A(Tf), \quad [d\Gamma_b(T), A(f)^*] = A(Tf)^*. \quad (10.3)$$

Lemma 10.3. [5, Lemma 5.21.] *For any $t \in \mathbb{R}$ and $f \in \mathcal{H}$, the following equations hold:*

$$e^{itd\Gamma_b(T)} A(f)^\sharp e^{-itd\Gamma_b(T)} = A(e^{itT} f)^\sharp.$$

Lemma 10.4. [5, Theorem 5.21.] *Assume that $\mathcal{H}$ be separable. Let $\{e_n\}_{n=1}^\infty \subset D(T^{1/2})$ be a CONS of $\mathcal{H}$. Then, for any $\psi \in D(d\Gamma_b(T)^{1/2})$, $\sum_{n=1}^\infty \|A(T^{1/2}e_n)\psi\|^2$ converges and the following equation holds:*

$$\sum_{n=1}^\infty \|A(T^{1/2}e_n)\psi\|^2 = \|d\Gamma_b(T)^{1/2}\psi\|^2.$$

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