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Bertrand and Mannheim curves of framed curves in the 3-dimensional Euclidean space

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Abstract

A Bertrand curve is a special class of space curves that the principal normal line of the curve and the principal normal line of another curve are the same. On the other hand, a Mannheim curve is also a special class of space curves that the principal normal line of the curve and the bi-normal line of another curve are the same. By definitions, the other curves are parallel curves to the direction of the principal normal vector. Even if regular cases, the existence conditions of the Bertrand and Mannheim curves seem to be wrong. Moreover, parallel curves may have singular points. As smooth curves with singular points, we consider framed curves in the Euclidean space. Then we define Bertrand and Mannheim curves of framed curves. Moreover, we clarify the Bertrand and Mannheim curves are depend of the moving frame.

1 Introduction

Bertrand and Mannheim curves are classical objects in differential geometry ([1, 2, 3, 4, 7, 17, 18, 19, 22]). A Bertrand curve is a special class of space curves that the principal normal line of the curve and the principal normal line of another curve are the same. On the other hand, a Mannheim curve is also a special class of space curves that the principal normal line of the curve and the bi-normal line of another curve are the same. Bertrand curves have been applied in computer-aided geometric design (cf. [21]). Moreover, there are a lot of investigations for other space form (for instance, [8, 16, 20]). By definitions, the other curves are parallel curves to the direction of the principal normal vector. Even if regular cases, the existence conditions of the Bertrand and Mannheim curves seem to be wrong. In order to define principal normal vector, the non-degenerate condition is needed. The non-degenerate condition is equivalent to the condition that the curvature does not vanish. Parallel curves does not satisfy the non-degenerate condition in general. We clarify existence conditions of Bertrand and Mannheim curves of regular space curves in §2. Moreover, parallel curves may have singular points. The locus of the singular points of parallel curves is the evolute of the original curves, see [6, 9, 10, 13]. We consider smooth curves with singular points. As smooth curves with singular points, we introduced framed curves in the Euclidean space in [12]. Then we define

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Bertrand and Mannheim curves of framed curves in §3 and §4, respectively. We give existence conditions of Bertrand and Mannheim curves of framed curves, respectively (Theorems 3.3 and 4.3). Moreover, we clarify Bertrand and Mannheim curves are depend of the moving frame (Remarks 3.7 and 4.8). We also give a difference between non-degenerate regular space curves and framed curves (Theorem 4.5).

All maps and manifolds considered in this paper are differentiable of class C^∞.

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2 Preliminaries

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$, where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ and the vector product is given by

$$\boldsymbol{a} \times \boldsymbol{b} = \det \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^3$. Let S^2 be the unit sphere in $\mathbb{R}^3$, that is, $S^2 = \{\boldsymbol{a} \in \mathbb{R}^3 || \boldsymbol{a}| = 1\}$. We denote the 3-dimensional smooth manifold $\{(\boldsymbol{a}, \boldsymbol{b}) \in S^2 \times S^2 | \boldsymbol{a} \cdot \boldsymbol{b} = 0\}$ by Δ.

We quickly review the theories of regular cases of Bertrand curves, Mannheim curves, and framed curves.

Since almost classical books does not treat singular points, they imply regularity and non-degenerate conditions. Therefore, even if regular cases, the existence conditions of the Bertrand and Mannheim curves seem to be wrong in classical (and recent) books [1, 2, 3, 7, 18, 22]. We clarify existence conditions of Bertrand and Mannheim curves.

2.1 Regular space curves

Let I be an interval of $\mathbb{R}$ and let $\gamma : I \to \mathbb{R}^3$ be a regular space curve, that is, $\dot{\gamma}(t) \neq 0$ for all $t \in I$, where $\dot{\gamma}(t) = (d\gamma/dt)(t)$. We assume $\dot{\gamma}(t) \times \ddot{\gamma}(t) \neq 0$ for all $t \in I$ and say that γ is *non-degenerate* (or, a *non-degenerate condition*). The non-degenerate condition is equivalent to the condition that the curvature of γ is non-zero.

If we take the arc-length parameter s, that is, $|\gamma'(s)| = 1$ for all s, then the tangent vector, the principal normal vector and the bi-normal vector are given by

$$\boldsymbol{t}(s) = \gamma'(s),\ \boldsymbol{n}(s) = \frac{\gamma''(s)}{|\gamma''(s)|},\ \boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s),$$

where $\gamma'(s) = (d\gamma/ds)(s)$. Then $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ is a moving frame of $\gamma(s)$ and we have the Frenet-Serret formula:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \\ \boldsymbol{b}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{pmatrix},$$

where

$$\kappa(s) = |\gamma''(s)|,\ \tau(s) = \frac{\det(\gamma'(s), \gamma''(s), \gamma'''(s))}{\kappa^2(s)}.$$

If we take a general parameter t, then the tangent vector, the principal normal vector and the bi-normal vector are given by

$$\boldsymbol{t}(t) = \frac{\dot\gamma(t)}{|\dot\gamma(t)|},\ \boldsymbol{n}(s) = \boldsymbol{b}(t) \times \boldsymbol{t}(t),\ \boldsymbol{b}(t) = \frac{\dot\gamma(t) \times \ddot\gamma(t)}{|\dot\gamma(t) \times \ddot\gamma(t)|}.$$

Then $\{\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)\}$ is a moving frame of $\gamma(t)$ and we have the Frenet-Serret formula:

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \\ \dot{\boldsymbol{b}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot\gamma(t)|\kappa(t) & 0 \\ -|\dot\gamma(t)|\kappa(t) & 0 & |\dot\gamma(t)|\tau(t) \\ 0 & -|\dot\gamma(t)|\tau(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \\ \boldsymbol{b}(t) \end{pmatrix},$$

where

$$\kappa(t) = \frac{|\dot\gamma(t) \times \ddot\gamma(t)|}{|\dot\gamma(t)|^3},\ \tau(t) = \frac{\det(\dot\gamma(t), \ddot\gamma(t), \dddot\gamma(t))}{|\dot\gamma(t) \times \ddot\gamma(t)|^2}.$$

Note that in order to define $\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t), \kappa(t)$ and $\tau(t)$, we assume that γ is not only regular, but also non-degenerate.

2.2 Bertrand curves of regular space curves

Definition 2.1 Let γ and $\overline{\gamma} : I \to \mathbb{R}^3$ be different non-degenerate curves. We say that γ and $\overline{\gamma}$ are *Bertrand mates* if there exists a smooth function $\lambda : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\boldsymbol{n}(t)$ and $\boldsymbol{n}(t) = \pm\overline{\boldsymbol{n}}(t)$ for all $t \in I$.

We also say that $\gamma : I \to \mathbb{R}^3$ is a *Bertrand curve* if there exists another non-degenerate curve $\overline{\gamma} : I \to \mathbb{R}^3$ such that γ and $\overline{\gamma}$ are Bertrand mates.

If γ and $\overline{\gamma}$ are Bertrand mates, then the principal normal line of γ and the principal normal line of $\overline{\gamma}$ are the same for each points. Note that if we take $-\lambda$ instead of λ, then we may assume that $\boldsymbol{n}(t) = \overline{\boldsymbol{n}}(t)$.

By a parameter change, we may assume that s is the arc-length parameter of γ.

Lemma 2.2 *Let $\gamma : I \to \mathbb{R}^3$ be non-degenerate with the arc-length parameter. Under the notation in Definition 2.1, if γ and $\overline{\gamma}$ are Bertrand mates, then λ is a non-zero constant.*

Proof. By differentiating $\overline{\gamma}(s) = \gamma(s) + \lambda(s)\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (1 - \lambda(s)\kappa(s))\boldsymbol{t}(s) + \lambda'(s)\boldsymbol{n}(s) + \lambda(s)\tau(s)\boldsymbol{b}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{n}}(s)$, we have $\lambda'(s) = 0$ for all $s \in I$. Therefore λ is a constant. If $\lambda = 0$, then $\overline{\gamma}(t) = \gamma(t)$, so λ is a non-zero constant. □

Theorem 2.3 *Let $\gamma : I \to \mathbb{R}^3$ be non-degenerate with the arc-length parameter. Suppose that $\tau(s) \neq 0$ for all $s \in I$ and A is a non-zero constant. Then γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ if and only if there exists a constant B such that $A\kappa(s) + B\tau(s) = 1$ and $B\kappa(s) - A\tau(s) \neq 0$ for all $s \in I$.*

Proof. Suppose that $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ and $\boldsymbol{n}(s) = \overline{\boldsymbol{n}}(s)$ for all $s \in I$. Note that s is not the arc-length parameter of $\overline{\gamma}$. By differentiating $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (1 - A\kappa(s))\boldsymbol{t}(s) + A\tau(s)\boldsymbol{b}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{n}}(s)$, there exists a smooth function $\theta : I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\boldsymbol{b}}(s) \\ \overline{\boldsymbol{t}}(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \boldsymbol{b}(s) \\ \boldsymbol{t}(s) \end{pmatrix}.$$

Then $|\dot{\overline{\gamma}}(s)| \sin\theta(s) = A\tau(s)$ and $|\dot{\overline{\gamma}}(s)| \cos\theta(s) = 1 - A\kappa(s)$. It follows that

$$-A\cos\theta(s)\tau(s) + (1 - A\kappa(s))\sin\theta(s) = 0. \tag{1}$$

By differentiating $\overline{\boldsymbol{t}}(s) = \sin\theta(s)\boldsymbol{b}(s) + \cos\theta(s)\boldsymbol{t}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\overline{\boldsymbol{n}}(s) = \theta'(s)\cos\theta(s)\boldsymbol{b}(s) - \theta'(s)\sin\theta(s)\boldsymbol{t}(s) + (-\sin\theta(s)\tau(s) + \cos\theta(s)\kappa(s))\boldsymbol{n}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{n}}(s)$, $\theta'(s) = 0$ for all $s \in I$. Therefore θ is a constant. By $\tau(s) \neq 0$ and $|\dot{\overline{\gamma}}(s)| \sin\theta = A\tau(s)$, we have $\sin\theta \neq 0$. By the equation (1), we have $A\kappa(s) + A(\cos\theta/\sin\theta)\tau(s) = 1$. Hence, if we put $B = A\cos\theta/\sin\theta$, then $A\kappa(s) + B\tau(s) = 1$ for all $s \in I$. Moreover,

$$|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s) = -\sin\theta\tau(s) + \cos\theta\kappa(s) = \frac{\sin\theta}{A}(-A\tau(s) + B\kappa(s)).$$

Since $\overline{\kappa}(s) \neq 0$, we have $B\kappa(s) - A\tau(s) \neq 0$ for all $s \in I$.

Conversely, suppose that there exists a constant B such that $A\kappa(s) + B\tau(s) = 1$, $B\kappa(s) - A\tau(s) \neq 0$ and $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ for all $s \in I$. It follows that

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (1 - A\kappa(s))\boldsymbol{t}(s) + A\tau(s)\boldsymbol{b}(s) = \tau(s)(B\boldsymbol{t}(s) + A\boldsymbol{b}(s)).$$

Since $|\dot{\overline{\gamma}}(s)| = \sqrt{A^2 + B^2}|\tau(s)|$, we have $\overline{\boldsymbol{t}}(s) = \mathrm{sgn}(\tau(s))(1/\sqrt{A^2 + B^2})(B\boldsymbol{t}(s) + A\boldsymbol{b}(s))$, where $\mathrm{sgn}(\tau(s)) = 1$ if $\tau(s) > 0$ and $\mathrm{sgn}(\tau(s)) = -1$ if $\tau(s) < 0$. By differentiating $\overline{\boldsymbol{t}}(s)$, we have $|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\overline{\boldsymbol{n}}(s) = \mathrm{sgn}(\tau(s))(1/\sqrt{A^2 + B^2})(B\kappa(s) - A\tau(s))\boldsymbol{n}(s)$. By the condition, we have $\boldsymbol{n}(s) = \pm\overline{\boldsymbol{n}}(s)$ for all $s \in I$. □

By a direct calculation and the proof of Theorem 2.3, we have the curvature and the torsion of $\overline{\gamma}$

Proposition 2.4 *Let γ and $\overline{\gamma} : I \to \mathbb{R}^3$ be different non-degenerate curves. Under the same assumptions in Theorem 2.3, suppose that γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ and $A\kappa(s) + B\tau(s) = 1$ for all $s \in I$, where B is a constant. Then the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ of $\overline{\gamma}$ are given by*

$$\overline{\kappa}(s) = \frac{|B\kappa(s) - A\tau(s)|}{(A^2 + B^2)|\tau(s)|}, \quad \overline{\tau}(s) = \frac{1}{(A^2 + B^2)\tau(s)}.$$

Proof. Since $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$, we have

$$\dot{\overline{\gamma}}(s) = (1 - A\kappa(s))\boldsymbol{t}(s) + A\tau(s)\boldsymbol{b}(s) = \tau(s)(B\boldsymbol{t}(s) + A\boldsymbol{b}(s)).$$

Therefore,

$$\begin{aligned}
\ddot{\overline{\gamma}}(s) &= \tau'(s)(B\boldsymbol{t}(s)+A\boldsymbol{b}(s))+\tau(s)(B\kappa(s)-A\tau(s))\boldsymbol{n}(s),\\
\dddot{\overline{\gamma}}(s) &= \tau''(s)(B\boldsymbol{t}(s)+A\boldsymbol{b}(s))+2\tau'(s)(B\kappa(s)-A\tau(s))\boldsymbol{n}(s)\\
&\quad +\tau(s)(B\kappa'(s)-A\tau'(s))\boldsymbol{n}(s)+\tau(s)(B\kappa(s)-A\tau(s))(-\kappa(s)\boldsymbol{t}(s)+\tau(s)\boldsymbol{b}(s)).
\end{aligned}$$

Since

$$\begin{aligned}
|\dot{\overline{\gamma}}(s)| &= |\tau(s)|(A^2+B^2)^{\frac{1}{2}},\\
|\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)| &= \tau(s)^2|B\kappa(s)-A\tau(s)|(A^2+B^2)^{\frac{1}{2}},\\
\det(\dot{\overline{\gamma}}(s),\ddot{\overline{\gamma}}(s),\dddot{\overline{\gamma}}(s)) &= \tau(s)^3(B\kappa(s)-A\tau(s))^2,
\end{aligned}$$

we have the curvature and the torsion as

$$\begin{aligned}
\overline{\kappa}(s) &= \frac{|\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)|}{|\dot{\overline{\gamma}}(s)|^3}=\frac{|B\kappa(s)-A\tau(s)|}{(A^2+B^2)|\tau(s)|},\\
\overline{\tau}(s) &= \frac{\det(\dot{\overline{\gamma}}(s),\ddot{\overline{\gamma}}(s),\dddot{\overline{\gamma}}(s))}{|\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)|^2}=\frac{1}{(A^2+B^2)\tau(s)}.
\end{aligned}$$

□

As a corollary of Proposition 2.4, we have a well-known result that $\tau(s)\overline{\tau}(s)$ is a positive constant.

On the other hand, $A\kappa(s)+B\tau(s)=1$ and $B\kappa(s)-A\tau(s)=0$ for all $s\in I$ if and only if $\kappa(s)=A/(A^2+B^2)$ and $\tau(s)=B/(A^2+B^2)$. It follows that γ is a helix up to congruence, that is, $\gamma(s)$ is given by

$$\gamma(s)=\left(A\cos\frac{s}{\sqrt{A^2+B^2}},A\sin\frac{s}{\sqrt{A^2+B^2}},\frac{Bs}{\sqrt{A^2+B^2}}\right).$$

By a direct calculation, we have $\boldsymbol{n}(s)=(-\cos(s/\sqrt{A^2+B^2}),-\sin(s/\sqrt{A^2+B^2}),0)$. Hence

$$\overline{\gamma}(s)=\gamma(s)+\lambda\boldsymbol{n}(s)=\left((A-\lambda)\cos\frac{s}{\sqrt{A^2+B^2}},(A-\lambda)\sin\frac{s}{\sqrt{A^2+B^2}},\frac{Bs}{\sqrt{A^2+B^2}}\right),$$

where λ is a constant. If $\lambda=A$, then $\overline{\gamma}(s)=(0,0,Bs/\sqrt{A^2+B^2})$. Then $\overline{\gamma}$ is degenerate, that is, $\overline{\kappa}(s)=0$ for all $s\in I$. In this case, if $\lambda\neq A$, then $\overline{\gamma}$ is non-degenerate and γ and $\overline{\gamma}$ are Bertrand mates, since

$$\overline{\boldsymbol{n}}(s)=\mathrm{sgn}(A-\lambda)\left(-\cos\frac{s}{\sqrt{A^2+B^2}},-\sin\frac{s}{\sqrt{A^2+B^2}},0\right),$$

where $\mathrm{sgn}(A-\lambda)=1$ if $A>\lambda$ and $\mathrm{sgn}(A-\lambda)=-1$ if $A<\lambda$.

2.3 Mannheim curves of regular space curves

Definition 2.5 Let γ and $\overline{\gamma}:I\to\mathbb{R}^3$ be different non-degenerate curves. We say that γ and $\overline{\gamma}$ are *Mannheim mates* if there exists a smooth function $\lambda:I\to\mathbb{R}$ such that $\overline{\gamma}(t)=\gamma(t)+\lambda(t)\boldsymbol{n}(t)$ and $\boldsymbol{n}(t)=\pm\overline{\boldsymbol{b}}(t)$ for all $t\in I$.

We also say that $\gamma : I \to \mathbb{R}^3$ is a *Mannheim curve* if there exists another non-degenerate curve $\overline{\gamma} : I \to \mathbb{R}^3$ such that γ and $\overline{\gamma}$ are Mannheim mates.

If γ and $\overline{\gamma}$ are Mannheim mates, then the principal normal line of γ and the bi-normal line of $\overline{\gamma}$ are the same for each points. Note that if we take $-\lambda$ instead of λ, then we may assume that $\boldsymbol{n}(t) = \overline{\boldsymbol{b}}(t)$.

By a parameter change, we may assume that s is the arc-length parameter of γ.

Lemma 2.6 *Let $\gamma : I \to \mathbb{R}^3$ be non-degenerate with the arc-length parameter. Under the notation in Definition 2.5, if γ and $\overline{\gamma}$ are Mannheim mates, then λ is a non-zero constant.*

Proof. By differentiating $\overline{\gamma}(s) = \gamma(s) + \lambda(s)\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (1 - \lambda(s)\kappa(s))\boldsymbol{t}(s) + \lambda'(s)\boldsymbol{n}(s) + \lambda(s)\tau(s)\boldsymbol{b}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{b}}(s)$, we have $\lambda'(s) = 0$ for all $s \in I$. Therefore λ is a constant. If $\lambda = 0$, then $\overline{\gamma}(t) = \gamma(t)$, so λ is a non-zero constant. □

Theorem 2.7 *Let $\gamma : I \to \mathbb{R}^3$ be non-degenerate with the arc-length parameter. Suppose that $\tau(s) \neq 0$ for all $s \in I$ and A is a non-zero constant. Then γ and $\overline{\gamma}$ are Mannheim mates with $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ if and only if $A(\kappa^2(s) + \tau^2(s)) = \kappa(s)$ and $\kappa(s)\tau'(s) - \kappa'(s)\tau(s) > 0$ for all $s \in I$.*

Proof. Suppose that $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$ and $\boldsymbol{n}(s) = \overline{\boldsymbol{b}}(s)$ for all $s \in I$. Note that s is not the arc-length parameter of $\overline{\gamma}$. By differentiating $\overline{\gamma}(s) = \gamma(s) + A\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (1 - A\kappa(s))\boldsymbol{t}(s) + A\tau(s)\boldsymbol{b}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{b}}(s)$, there exists a smooth function $\theta : I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\boldsymbol{t}}(s) \\ \overline{\boldsymbol{n}}(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \boldsymbol{b}(s) \\ \boldsymbol{t}(s) \end{pmatrix}.$$

Then $|\dot{\overline{\gamma}}(s)|\cos\theta(s) = A\tau(s)$ and $-|\dot{\overline{\gamma}}(s)|\sin\theta(s) = 1 - A\kappa(s)$. It follows that

$$A\tau(s)\sin\theta(s) + (1 - A\kappa(s))\cos\theta(s) = 0. \tag{2}$$

By differentiating $\overline{\boldsymbol{t}}(s) = \cos\theta(s)\boldsymbol{b}(s) - \sin\theta(s)\boldsymbol{t}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\overline{\boldsymbol{n}}(s) = -\theta'(s)\sin\theta(s)\boldsymbol{b}(s) - \theta'(s)\cos\theta(s)\boldsymbol{t}(s) - (\cos\theta(s)\tau(s) + \sin\theta(s)\kappa(s))\boldsymbol{n}(s).$$

Since $\boldsymbol{n}(s) = \overline{\boldsymbol{b}}(s)$,

$$\cos\theta(s)\tau(s) + \sin\theta(s)\kappa(s) = 0 \tag{3}$$

for all $s \in I$. By $\tau(s) \neq 0$ and $|\dot{\overline{\gamma}}(s)|\cos\theta(s) = A\tau(s)$, we have $\cos\theta(s) \neq 0$. Hence $\sin\theta(s) \neq 0$. Since $\overline{\boldsymbol{n}}(s) = \sin\theta(s)\boldsymbol{b}(s) + \cos\theta(s)\boldsymbol{t}(s)$, we have $|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s) = -\theta'(s)$. By the equations (2) and (3), we have

$$A(\kappa^2(s) + \tau^2(s)) = \kappa(s). \tag{4}$$

By differentiating (3), we have

$$-\theta'(s)\sin\theta(s)\tau(s)+\cos\theta(s)\tau'(s)+\theta'(s)\cos\theta(s)\kappa(s)+\sin\theta(s)\kappa'(s)=0.$$

Hence $\theta'(s)=(-\kappa(s)\tau'(s)+\kappa'(s)\tau(s))/(\kappa^2(s)+\tau^2(s))$. Since $|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)>0$, we have $\kappa(s)\tau'(s)-\kappa'(s)\tau(s)>0$ for all $s\in I$.

Conversely, suppose that $A(\kappa^2(s)+\tau^2(s))=\kappa(s)$, $\kappa(s)\tau'(s)-\kappa'(s)\tau(s)>0$ and $\overline{\gamma}(s)=\gamma(s)+A\boldsymbol{n}(s)$ for all $s\in I$. By differentiating $\overline{\gamma}(s)=\gamma(s)+A\boldsymbol{n}(s)$, we have

$$\begin{aligned}
\dot{\overline{\gamma}}(s) &= |\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(t)=(1-A\kappa(s))\boldsymbol{t}(s)+A\tau(s)\boldsymbol{b}(s)=A\frac{\tau(s)}{\kappa(s)}(\tau(s)\boldsymbol{t}(s)+\kappa(s)\boldsymbol{b}(s)),\\
\ddot{\overline{\gamma}}(s) &= \frac{d}{ds}(|\dot{\overline{\gamma}}(s)|)\overline{\boldsymbol{t}}(s)+|\dot{\overline{\gamma}}(s)|^2\overline{\kappa}(s)\overline{\boldsymbol{n}}(s)\\
&= A\left(\frac{\tau(s)}{\kappa(s)}\right)'(\tau(s)\boldsymbol{t}(s)+\kappa(s)\boldsymbol{b}(s))+A\frac{\tau(s)}{\kappa(s)}(\tau'(s)\boldsymbol{t}(s)+\kappa'(s)\boldsymbol{b}(s)).
\end{aligned}$$

Therefore, $|\dot{\overline{\gamma}}(s)|^3\overline{\kappa}(s)\overline{\boldsymbol{b}}(s)=A^2(\tau(s)/\kappa(s))^2(\kappa(s)\tau'(s)-\kappa'(s)\tau(s))\boldsymbol{n}(s)$. By the condition, we have $\boldsymbol{n}(s)=\overline{\boldsymbol{b}}(s)$. It follows that γ and $\overline{\gamma}$ are Mannheim mates. □

By a direct calculation and the proof of Theorem 2.7, we have the curvature and the torsion of $\overline{\gamma}$

Proposition 2.8 *Let γ and $\overline{\gamma}: I\to\mathbb{R}^3$ be different non-degenerate curves. Under the same assumptions in Theorem 2.7, suppose that γ and $\overline{\gamma}$ are Mannheim mates with $\overline{\gamma}(s)=\gamma(s)+A\boldsymbol{n}(s)$ and $A(\kappa^2(s)+\tau^2(s))=\kappa(s)$ for all $s\in I$. Then the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ of $\overline{\gamma}$ are given by*

$$\overline{\kappa}(s)=\frac{\kappa(s)(\kappa(s)\tau'(s)-\kappa'(s)\tau(s))}{|A\tau(s)|(\kappa^2(s)+\tau^2(s))^{\frac{3}{2}}},\quad \overline{\tau}(s)=\frac{\kappa(s)}{A\tau(s)}=\frac{\kappa^2(s)+\tau^2(s)}{\tau(s)}.$$

Proof. Since $\overline{\gamma}(s)=\gamma(s)+A\boldsymbol{n}(s)$, we have

$$\dot{\overline{\gamma}}(s)=(1-A\kappa(s))\boldsymbol{t}(s)+A\tau(s)\boldsymbol{b}(s)=A\frac{\tau(s)}{\kappa(s)}(\tau(s)\boldsymbol{t}(s)+\kappa(s)\boldsymbol{b}(s)).$$

Therefore,

$$\begin{aligned}
\ddot{\overline{\gamma}}(s) &= A\left(\frac{\tau(s)}{\kappa(s)}\right)'(\tau(s)\boldsymbol{t}(s)+\kappa(s)\boldsymbol{b}(s))+A\frac{\tau(s)}{\kappa(s)}(\tau'(s)\boldsymbol{t}(s)+\kappa'(s)\boldsymbol{b}(s)),\\
\dddot{\overline{\gamma}}(s) &= A\left(\frac{\tau(s)}{\kappa(s)}\right)''(\tau(s)\boldsymbol{t}(s)+\kappa(s)\boldsymbol{b}(s))+2A\left(\frac{\tau(s)}{\kappa(s)}\right)'(\tau'(s)\boldsymbol{t}(s)+\kappa'(s)\boldsymbol{b}(s))\\
&\quad +A\frac{\tau(s)}{\kappa(s)}(\tau'(s)\kappa(s)-\kappa'(s)\tau(s))\boldsymbol{n}(s).
\end{aligned}$$

By the proof of Theorem 2.7, we have $\kappa(s)\tau'(s)-\kappa'(s)\tau(s)>0$ for all $s\in I$. Since

$$\begin{aligned}
|\dot{\overline{\gamma}}(s)| &= \frac{|A\tau(s)|}{\kappa(s)}(\kappa^2(s)+\tau^2(s))^{\frac{1}{2}},\\
|\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)| &= A^2\left(\frac{\tau(s)}{\kappa(s)}\right)^2(\kappa(s)\tau'(s)-\kappa'(s)\tau(s)),\\
\det(\dot{\overline{\gamma}}(s),\ddot{\overline{\gamma}}(s),\dddot{\overline{\gamma}}(s)) &= A^3\left(\frac{\tau(s)}{\kappa(s)}\right)^3(\kappa(s)\tau'(s)-\kappa'(s)\tau(s))^2,
\end{aligned}$$

we have the curvature and the torsion as

$$\begin{aligned}\overline{\kappa}(s) &= \frac{|\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)|}{|\dot{\overline{\gamma}}(s)|^3} = \frac{\kappa(s)(\kappa(s)\tau'(s)-\kappa'(s)\tau(s))}{|A\tau(s)|(\kappa^2(s)+\tau^2(s))^{\frac{3}{2}}},\\ \overline{\tau}(s) &= \frac{\det(\dot{\overline{\gamma}}(s),\ddot{\overline{\gamma}}(s),\dddot{\overline{\gamma}}(s))}{|\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)|^2} = \frac{\kappa(s)}{A\tau(s)} = \frac{\kappa^2(s)+\tau^2(s)}{\tau(s)}.\end{aligned}$$

□

Note that $A(\kappa^2(s)+\tau^2(s))=\kappa(s)$ and $\kappa(s)\tau'(s)-\kappa'(s)\tau(s)=0$ for all $s\in I$ if and only if there exists a constant B such that $\kappa(s)=1/(A(1+B^2))$ and $\tau(s)=B/(A(1+B^2))$. It follows that γ is a helix up to congruence, that is, $\gamma(s)$ is given by

$$\gamma(s)=\left(A\cos\frac{s}{A\sqrt{1+B^2}},A\sin\frac{s}{A\sqrt{1+B^2}},\frac{Bs}{\sqrt{1+B^2}}\right).$$

By a direct calculation, we have $\boldsymbol{n}(s)=(-\cos(s/A\sqrt{1+B^2}),-\sin(s/A\sqrt{1+B^2}),0)$. Hence

$$\overline{\gamma}(s)=\gamma(s)+\lambda\boldsymbol{n}(s)=\left((A-\lambda)\cos\frac{s}{A\sqrt{1+B^2}},(A-\lambda)\sin\frac{s}{A\sqrt{1+B^2}},\frac{Bs}{\sqrt{1+B^2}}\right),$$

where λ is a constant. If $\lambda=A$, then $\overline{\gamma}(s)=(0,0,Bs/\sqrt{1+B^2})$. Then $\overline{\gamma}$ is degenerate, that is, $\overline{\kappa}(s)=0$ for all $s\in I$. If $\lambda\neq A$, then $\overline{\gamma}$ is non-degenerate. However, in this case, γ and $\overline{\gamma}$ are not Mannheim mates, since

$$\overline{\boldsymbol{b}}(s)=\frac{1}{\sqrt{B^2+(\frac{A-\lambda}{A})^2}}\left(B\sin\frac{s}{A\sqrt{1+B^2}},B\cos\frac{s}{A\sqrt{1+B^2}},\frac{A-\lambda}{A}\right).$$

Remark 2.9 If there exist non-zero constants A,C and a constant B such that $A\kappa(s)+B\tau(s)=1$ and $C(\kappa^2(s)+\tau^2(s))=\kappa(s)$, then $\kappa(s)=(1-B\tau(s))/A$ and

$$(A^2+B^2)C\tau^2(s)+(-2BC+AB)\tau(s)+C-A=0.$$

If there exists a solution, then $\tau(s)$ and $\kappa(s)$ are constants. Hence, $\kappa(s)\tau'(s)-\kappa'(s)\tau(s)=0$ for all $s\in I$. It follows that there is no Bertrand and Mannheim curves of regular space curves (cf. Theorem 4.5 and Remark 4.8).

2.4 Framed curves in the 3-dimensional Euclidean space

A framed curve in the 3-dimensional Euclidean space is a smooth space curve with a moving frame, in detail see [12].

Definition 2.10 We say that $(\gamma,\nu_1,\nu_2):I\to\mathbb{R}^3\times\Delta$ is a *framed curve* if $\dot{\gamma}(t)\cdot\nu_1(t)=0$ and $\dot{\gamma}(t)\cdot\nu_2(t)=0$ for all $t\in I$. We say that $\gamma:I\to\mathbb{R}^3$ is a *framed base curve* if there exists $(\nu_1,\nu_2):I\to\Delta$ such that (γ,ν_1,ν_2) is a framed curve.

We denote $\boldsymbol{\mu}(t)=\nu_1(t)\times\nu_2(t)$. Then $\{\nu_1(t),\nu_2(t),\boldsymbol{\mu}(t)\}$ is a moving frame along the framed base curve $\gamma(t)$ in $\mathbb{R}^3$ and we have the Frenet type formula,

$$\begin{pmatrix}\dot{\nu_1}(t)\\ \dot{\nu_2}(t)\\ \dot{\boldsymbol{\mu}}(t)\end{pmatrix}=\begin{pmatrix}0&\ell(t)&m(t)\\ -\ell(t)&0&n(t)\\ -m(t)&-n(t)&0\end{pmatrix}\begin{pmatrix}\nu_1(t)\\ \nu_2(t)\\ \boldsymbol{\mu}(t)\end{pmatrix},\ \dot{\gamma}(t)=\alpha(t)\boldsymbol{\mu}(t),$$

where $\ell(t) = \dot{\nu}_1(t) \cdot \nu_2(t)$, $m(t) = \dot{\nu}_1(t) \cdot \boldsymbol{\mu}(t), n(t) = \dot{\nu}_2(t) \cdot \boldsymbol{\mu}(t)$ and $\alpha(t) = \dot{\gamma}(t) \cdot \boldsymbol{\mu}(t)$. We call the mapping (ℓ, m, n, α) *the curvature of the framed curve* (γ, ν_1, ν_2). Note that t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$.

Definition 2.11 Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. We say that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are *congruent as framed curves* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that $\widetilde{\gamma}(t) = A(\gamma(t)) + \boldsymbol{a}$, $\widetilde{\nu}_1(t) = A(\nu_1(t))$ and $\widetilde{\nu}_2(t) = A(\nu_2(t))$ for all $t \in I$.

We have the existence and uniqueness theorems for framed curves in terms of the curvatures in [12], also see [11].

Theorem 2.12 (Existence Theorem for framed curves) *Let $(\ell, m, n, \alpha) : I \to \mathbb{R}^4$ be a smooth mapping. Then, there exists a framed curve $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ whose curvature is given by (ℓ, m, n, α).*

Theorem 2.13 (Uniqueness Theorem for framed curves) *Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves with curvatures (ℓ, m, n, α) and $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$, respectively. Then (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are congruent as framed curves if and only if the curvatures (ℓ, m, n, α) and $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$ coincide.*

Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature of the framed curve (ℓ, m, n, α). For the normal plane of $\gamma(t)$, spanned by $\nu_1(t)$ and $\nu_2(t)$, there is some ambient of framed curves similarly to the case of the Bishop frame of a regular space curve (cf. [5]). We define $(\widetilde{\nu}_1(t), \widetilde{\nu}_2(t)) \in \Delta_2$ by

$$\begin{pmatrix} \widetilde{\nu}_1(t) \\ \widetilde{\nu}_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix},$$

where $\theta(t)$ is a smooth function. Then $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ is also a framed curve and $\widetilde{\boldsymbol{\mu}}(t) = \boldsymbol{\mu}(t)$. By a direct calculation, we have

$$\begin{aligned} \dot{\widetilde{\nu}}_1(t) &= (\ell(t) - \dot{\theta}(t))\sin\theta(t)\nu_1(t) + (\ell(t) - \dot{\theta}(t))\cos\theta(t)\nu_2(t) \\ &\quad + (m(t)\cos\theta(t) - n(t)\sin\theta(t))\boldsymbol{\mu}(t), \\ \dot{\widetilde{\nu}}_2(t) &= -(\ell(t) - \dot{\theta}(t))\cos\theta(t)\nu_1(t) + (\ell(t) - \dot{\theta}(t))\sin\theta(t)\nu_2(t) \\ &\quad + (m(t)\sin\theta(t) + n(t)\cos\theta(t))\boldsymbol{\mu}(t). \end{aligned}$$

If we take a smooth function $\theta : I \to \mathbb{R}$ which satisfies $\dot{\theta}(t) = \ell(t)$, then we call the frame $\{\widetilde{\nu}_1(t), \widetilde{\nu}_2(t), \boldsymbol{\mu}(t)\}$ an *adapted frame* along the framed base curve $\gamma(t)$. It follows that the Frenet-Serret type formula is given by

$$\begin{pmatrix} \dot{\widetilde{\nu}}_1(t) \\ \dot{\widetilde{\nu}}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \widetilde{m}(t) \\ 0 & 0 & \widetilde{n}(t) \\ -\widetilde{m}(t) & -\widetilde{n}(t) & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\nu}_1(t) \\ \widetilde{\nu}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{5}$$

where $\widetilde{m}(t)$ and $\widetilde{n}(t)$ are given by

$$\begin{pmatrix} \widetilde{m}(t) \\ \widetilde{n}(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} m(t) \\ n(t) \end{pmatrix}.$$

We also consider a special moving frame along a framed base curve under a condition. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with $m^2(t) + n^2(t) \neq 0$. Then we define $(\boldsymbol{n}_1(t), \boldsymbol{n}_2(t)) \in \Delta$ by

$$\boldsymbol{n}_1(t) = \frac{m(t)\nu_1(t) + n(t)\nu_2(t)}{\sqrt{m^2(t) + n^2(t)}},\ \boldsymbol{n}_2(t) = \frac{-n(t)\nu_1(t) + m(t)\nu_2(t)}{\sqrt{m^2(t) + n^2(t)}}.$$

By a direct calculation, $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed immersion and $\boldsymbol{n}_1(t) \times \boldsymbol{n}_2(t) = \boldsymbol{\mu}(t)$. We call the moving frame $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ a *Frenet type frame* along $\gamma(t)$. Then the Frenet-Serret type formula is given by

$$\begin{pmatrix} \dot{\boldsymbol{n}}_1(t) \\ \dot{\boldsymbol{n}}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & L(t) & M(t) \\ -L(t) & 0 & 0 \\ -M(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1(t) \\ \boldsymbol{n}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},\ \dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t), \tag{6}$$

where

$$L(t) = \frac{m(t)\dot{n}(t) - \dot{m}(t)n(t) + \ell(t)(m^2(t) + n^2(t))}{m^2(t) + n^2(t)},\ M(t) = \sqrt{m^2(t) + n^2(t)}.$$

Therefore, the curvature of the framed immersion $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is given by $(L, M, 0, \alpha)$.

Since the original frame $\{\nu_1(t), \nu_2(t), \boldsymbol{\mu}(t)\}$ and the Frenet type frame $\{\boldsymbol{n}_1(t),\ \boldsymbol{n}_2(t),\ \boldsymbol{\mu}(t)\}$ have the common unit vector $\boldsymbol{\mu}(t)$ and the same orientation, the Frenet type frame is one of a rotated frame along $\gamma(t)$.

Let $\gamma : I \to \mathbb{R}^3$ be non-degenerate. If we take $\nu_1(t) = \boldsymbol{n}(t)$ and $\nu_2(t) = \boldsymbol{b}(t)$, then $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed curve and we have $\boldsymbol{n}_1(t) = -\boldsymbol{n}(t), \boldsymbol{n}_2(t) = -\boldsymbol{b}(t), \boldsymbol{\mu}(t) = \boldsymbol{t}(t)$. This is the reason why we call $\{\boldsymbol{n}_1(t), \boldsymbol{n}_2(t), \boldsymbol{\mu}(t)\}$ the Frenet type frame along $\gamma(t)$.

As a special case of a framed curve, let us consider a spherical Legendre curve, in detail see [23]. We say that $(\gamma, \nu) : I \to \Delta \subset S^2 \times S^2$ is a spherical Legendre curve if $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. We call γ a frontal and ν a dual of γ.

We define $\boldsymbol{\mu}(t) = \gamma(t) \times \nu(t)$. Then $\boldsymbol{\mu}(t) \in S^2$, $\gamma(t) \cdot \boldsymbol{\mu}(t) = 0$ and $\nu(t) \cdot \boldsymbol{\mu}(t) = 0$ for all $t \in I$. It follows that $\{\gamma(t), \nu(t), \boldsymbol{\mu}(t)\}$ is a moving frame along the frontal $\gamma(t)$.

Let $(\gamma, \nu) : I \to \Delta$ be a spherical Legendre curve. Then we have

$$\begin{pmatrix} \dot{\gamma}(t) \\ \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m(t) \\ 0 & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{7}$$

where $m(t) = \dot{\gamma}(t) \cdot \boldsymbol{\mu}(t)$ and $n(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$.

We say that the pair of functions (m, n) is the curvature of the spherical Legendre curve $(\gamma, \nu) : I \to \Delta$.

3 Bertrand curves of framed curves

Let (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves with the curvature (ℓ, m, n, α) and $(\overline{\ell}, \overline{m}, \overline{n}, \overline{\alpha})$, respectively. Suppose that γ and $\overline{\gamma}$ are different curves, that is, $\gamma \not\equiv \overline{\gamma}$.

Definition 3.1 We say that framed curves (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are *Bertrand mates* (or, $(\nu_1, \overline{\nu}_1)$-mates) if there exists a smooth function $\lambda : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\nu_1(t)$ and $\nu_1(t) = \overline{\nu}_1(t)$ for all $t \in I$.

We also say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a *Bertrand curve* if there exists another framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ such that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates.

Lemma 3.2 *Under the notation in Definition 3.1, if (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates, then λ is a non-zero constant.*

Proof. By differentiating $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\nu_1(t)$, we have

$$\overline{\alpha}(t)\overline{\boldsymbol{\mu}}(t) = (\alpha(t) + \lambda(t)m(t))\boldsymbol{\mu}(t) + \lambda(t)\ell(t)\nu_2(t) + \dot{\lambda}(t)\nu_1(t)$$

for all $t \in I$. Since $\overline{\nu}_1(t) = \nu_1(t)$, we have $\dot{\lambda}(t) = 0$ for all $t \in I$. Therefore λ is a constant. If $\lambda = 0$, then $\gamma(t) = \overline{\gamma}(t)$ for all $t \in I$. Hence λ is a non-zero constant. □

We give a necessary and sufficient condition of a Bertrand curve for a framed curve.

Theorem 3.3 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). Then (γ, ν_1, ν_2) is a Bertrand curve if and only if there exist a non-zero constant λ and a smooth function $\theta : I \to \mathbb{R}$ such that*

$$\lambda\ell(t)\cos\theta(t) - (\alpha(t) + \lambda m(t))\sin\theta(t) = 0 \tag{8}$$

for all $t \in I$.

Proof. Suppose that (γ, ν_1, ν_2) is a Bertrand curve. By Lemma 3.2, there exist another framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ and a non-zero constant $\lambda \in \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t)$ and $\nu_1(t) = \overline{\nu}_1(t)$ for all $t \in I$. By differentiating $\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t)$, we have

$$\overline{\alpha}(t)\overline{\boldsymbol{\mu}}(t) = (\alpha(t) + \lambda m(t))\boldsymbol{\mu}(t) + \lambda\ell(t)\nu_2(t).$$

Since $\nu_1(t) = \overline{\nu}_1(t)$, there exists a function $\theta : I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\nu}_2(t) \\ \overline{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \nu_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}. \tag{9}$$

Then we have

$$\overline{\alpha}(t)\sin\theta(t) = \lambda\ell(t), \quad \overline{\alpha}(t)\cos\theta(t) = \alpha(t) + \lambda m(t). \tag{10}$$

It follows that $\lambda\ell(t)\cos\theta(t) - (\alpha(t) + \lambda m(t))\sin\theta(t) = 0$ for all $t \in I$.

Conversely, suppose that $\lambda\ell(t)\cos\theta(t) - (\alpha(t) + \lambda m(t))\sin\theta(t) = 0$ for all $t \in I$. We define a mapping $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ by

$$\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t),\ \overline{\nu}_1(t) = \nu_1(t),\ \overline{\nu}_2(t) = \cos\theta(t)\nu_2(t) - \sin\theta(t)\boldsymbol{\mu}(t).$$

Then $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ is a framed curve. Therefore, (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates. □

Proposition 3.4 *Suppose that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates, where $\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t)$ and $\lambda\ell(t)\cos\theta(t) - (\alpha(t) + \lambda m(t))\sin\theta(t) = 0$ for all $t \in I$. Then the curvature $(\overline{\ell}, \overline{m}, \overline{n}, \overline{\alpha})$ of $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ is given by*

$$\begin{aligned} \overline{\ell}(t) &= \ell(t)\cos\theta(t) - m(t)\sin\theta(t), \\ \overline{m}(t) &= \ell(t)\sin\theta(t) + m(t)\cos\theta(t), \\ \overline{n}(t) &= n(t) - \dot{\theta}(t), \\ \overline{\alpha}(t) &= \lambda\ell(t)\sin\theta(t) + (\alpha(t) + \lambda m(t))\cos\theta(t). \end{aligned} \tag{11}$$

Proof. By the equation (9), we have $\overline{\nu}_2(t) = \cos\theta(t)\nu_2(t) - \sin\theta(t)\boldsymbol{\mu}(t)$. By differentiating, we have

$$\begin{aligned} -\overline{\ell}(t)\overline{\nu}_1(t) + \overline{n}(t)\overline{\boldsymbol{\mu}}(t) &= (-\ell(t)\cos\theta(t) + m(t)\sin\theta(t))\nu_1(t) \\ &\quad +(-\dot{\theta}(t) + n(t))\sin\theta(t)\nu_2(t) + (n(t) - \dot{\theta}(t))\cos\theta(t)\boldsymbol{\mu}(t). \end{aligned}$$

Since $\nu_1(t) = \overline{\nu}_1(t)$, we have $\overline{\ell}(t) = \ell(t)\cos\theta(t) - m(t)\sin\theta(t)$. Again by (9), $\overline{n}(t) = n(t) - \dot{\theta}(t)$. Moreover, by differentiating $\overline{\boldsymbol{\mu}}(t) = \sin\theta(t)\nu_2(t) + \cos\theta(t)\boldsymbol{\mu}(t)$, we have

$$\begin{aligned} -\overline{m}(t)\overline{\nu}_1(t) - \overline{n}(t)\overline{\nu}_2(t) &= (-\ell(t)\sin\theta(t) - m(t)\cos\theta(t))\nu_1(t) \\ &\quad +(\dot{\theta}(t) - n(t))\cos\theta(t)\nu_2(t) + (n(t) - \dot{\theta}(t))\sin\theta(t)\boldsymbol{\mu}(t). \end{aligned}$$

Since $\nu_1(t) = \overline{\nu}_1(t)$, we have $\overline{m}(t) = \ell(t)\sin\theta(t) + m(t)\cos\theta(t)$. By the equation (10), we also have $\overline{\alpha}(t) = \lambda\ell(t)\sin\theta(t) + (\alpha(t) + \lambda m(t))\cos\theta(t)$. □

By Theorem 3.3, we have the following result.

Corollary 3.5 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α).*

(1) If $\ell(t) = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is a Bertrand curve.

(2) If there exists a non-zero constant λ such that $\alpha(t) + \lambda m(t) = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is a Bertrand curve.

Proof. (1) If we take $\theta(t) = 0$, then the equation (8) is satisfied. (2) If we take $\theta(t) = \pi/2$, then the equation (8) is satisfied. □

Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with curvature (ℓ, m, n, α). If we take an adapted frame $\{\widetilde{\nu}_1, \widetilde{\nu}_2, \boldsymbol{\mu}\}$, then the curvature is given by $(0, \widetilde{m}, \widetilde{n}, \alpha)$, see (5). By Theorem 3.3 or Corollary 3.5, we have the following.

Corollary 3.6 *For an adapted frame, $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is always a Bertrand curve.*

Remark 3.7 By Corollary 3.6, we found that the notion of Bertrand curves depend of the moving frame. Even if regular space curves, the same phenomenon occurs when we consider the other moving frames (for instance, Bishop frame [5]).

Next we consider the principal normal direction like as regular cases. Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). Since $\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t)$, $\boldsymbol{\mu}(t)$ is the unit tangent direction and

$$\begin{aligned} \ddot{\gamma}(t) &= \dot{\alpha}(t)\boldsymbol{\mu}(t) - \alpha(t)m(t)\nu_1(t) - \alpha(t)n(t)\nu_2(t), \\ \dot{\gamma}(t) \times \ddot{\gamma}(t) &= \alpha^2(t)n(t)\nu_1(t) - \alpha^2(t)m(t)\nu_2(t), \\ (\dot{\gamma}(t) \times \ddot{\gamma}(t)) \times \dot{\gamma}(t) &= -\alpha^3(t)m(t)\nu_1(t) - \alpha^3(t)n(t)\nu_2(t), \end{aligned}$$

the principal normal direction is given by $\pm\boldsymbol{n}_1(t)$ away from singular points of γ (that is, $\alpha(t) \neq 0$). Hence we consider Frenet type frame $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{\mu}\}$, see (6) in §2.4.

Corollary 3.8 *Let (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. Then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\overline{\gamma}, \overline{\boldsymbol{n}}_1, \overline{\boldsymbol{n}}_2)$ are Bertrand mates if and only if there exists a constant λ such that the equations (8) and (11) are satisfied, where $\theta(t)$ is a constant.*

Proof. Since the curvature $n(t) = \overline{n}(t) = 0$ and Proposition 3.4, $\dot{\theta}(t) = 0$ for all $t \in I$. It follows that θ is a constant. By Theorem 3.3, we have the result. □

Remark 3.9 When $(m(t), n(t)) = (0, 0)$ at some points, if there exist a non-negative smooth function $r : I \to \mathbb{R}$ and a smooth function $\phi : I \to \mathbb{R}$ such that $m(t) = r(t)\cos\phi(t), n(t) = r(t)\sin\phi(t)$, then we can consider the same result for Corollary 3.8.

We give a construction of Bertrand curves of framed curves by using spherical Legendre curves, see (7) in §2.4. For regular cases see [17].

Theorem 3.10 *Let $(\gamma, \nu) : I \to \Delta$ be a spherical Legendre curve with the curvature (m, n). Suppose that λ, φ are non-zero constants with $\sin\varphi \neq 0$, $\boldsymbol{c} \in \mathbb{R}^3$ is a constant vector. Set*

$$\begin{aligned}
\widetilde{\gamma}(t) &= \lambda \int m(t)\gamma(t)dt + \lambda\cot\varphi \int m(t)\nu(t)dt + \boldsymbol{c},\\
\widetilde{\nu}_1(t) &= \boldsymbol{\mu}(t) = \gamma(t) \times \nu(t),\\
\widetilde{\nu}_2(t) &= \cos\varphi\gamma(t) + \sin\varphi\nu(t).
\end{aligned}$$

Then $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed curve with the curvature $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$,

$$\widetilde{\ell}(t) = -\cos\varphi m(t) + \sin\varphi n(t), \widetilde{m}(t) = -\sin\varphi m(t) - \cos\varphi n(t), \widetilde{n}(t) = 0, \widetilde{\alpha}(t) = \frac{\lambda m(t)}{\sin\varphi}.$$

Moreover, $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is a Bertrand curve.

Proof. By definition, $|\widetilde{\nu}_1(t)| = |\widetilde{\nu}_2(t)| = 1$ and $\widetilde{\nu}_1(t) \cdot \widetilde{\nu}_2(t) = 0$ for all $t \in I$. By a direct calculation, we have

$$\widetilde{\boldsymbol{\mu}}(t) = \widetilde{\nu}_1(t) \times \widetilde{\nu}_2(t) = \boldsymbol{\mu}(t) \times (\cos\varphi\gamma(t) - \sin\varphi\nu(t)) = \sin\varphi\gamma(t) + \cos\varphi\nu(t).$$

Since $\dot{\widetilde{\gamma}}(t) = \lambda m(t)\gamma(t) + \lambda\cot\varphi m(t)\nu(t) = (\lambda m(t)/\sin\varphi)\widetilde{\boldsymbol{\mu}}(t)$, $\dot{\widetilde{\gamma}}(t) \cdot \widetilde{\nu}_1(t) = \dot{\widetilde{\gamma}}(t) \cdot \widetilde{\nu}_2(t) = 0$ for all $t \in I$. Therefore, $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is a framed curve. By $\dot{\widetilde{\nu}}_1(t) = -m(t)\gamma(t) - n(t)\nu(t)$ and $\dot{\widetilde{\nu}}_2(t) = (\cos\varphi m(t) - \sin\varphi n(t))\boldsymbol{\mu}(t)$, we have the curvature of $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ by

$$\begin{aligned}
\widetilde{\ell}(t) &= \dot{\widetilde{\nu}}_1(t) \cdot \widetilde{\nu}_2(t) = -\cos\varphi m(t) + \sin\varphi n(t),\\
\widetilde{m}(t) &= \dot{\widetilde{\nu}}_1(t) \cdot \widetilde{\boldsymbol{\mu}}(t) = -\sin\varphi m(t) - \cos\varphi n(t),\\
\widetilde{n}(t) &= \dot{\widetilde{\nu}}_2(t) \cdot \widetilde{\boldsymbol{\mu}}(t) = 0,\\
\widetilde{\alpha}(t) &= \dot{\widetilde{\gamma}}(t) \cdot \widetilde{\boldsymbol{\mu}}(t) = \frac{\lambda m(t)}{\sin\varphi}.
\end{aligned}$$

If we take $\theta(t) = -\varphi$, then we have

$$\begin{aligned}
&\lambda\widetilde{\ell}(t)\cos\theta(t) - (\widetilde{\alpha}(t) + \lambda\widetilde{m}(t))\sin\theta(t)\\
&= \lambda(-\cos\varphi m(t) + \sin\varphi n(t))\cos\varphi + \left(\frac{\lambda m(t)}{\sin\varphi} + \lambda(-\sin\varphi m(t) - \cos\varphi n(t))\right)\sin\varphi\\
&= 0,
\end{aligned}$$

for all $t \in I$. By Theorem 3.3, $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is a Bertrand curve. □

4 Mannheim curves of framed curves

Let (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves with the curvature (ℓ, m, n, α) and $(\overline{\ell}, \overline{m}, \overline{n}, \overline{\alpha})$, respectively. Suppose that γ and $\overline{\gamma}$ are different curves, that is, $\gamma \not\equiv \overline{\gamma}$.

Definition 4.1 We say that framed curves (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are *Mannheim mates* (or, $(\nu_1, \overline{\nu}_2)$-mates) if there exists a smooth function $\lambda : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\nu_1(t)$ and $\nu_1(t) = \overline{\nu}_2(t)$ for all $t \in I$.

We also say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a *Mannheim curve* if there exists another framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ such that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates.

Lemma 4.2 *Under the notation in Definition 4.1, if (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates, then λ is a non-zero constant.*

Proof. By differentiating $\overline{\gamma}(t) = \gamma(t) + \lambda(t)\nu_1(t)$, we have

$$\overline{\alpha}(t)\overline{\boldsymbol{\mu}}(t) = (\alpha(t) + \lambda(t)m(t))\boldsymbol{\mu}(t) + \lambda(t)\ell(t)\nu_2(t) + \dot{\lambda}(t)\nu_1(t)$$

for all $t \in I$. Since $\overline{\nu}_2(t) = \nu_1(t)$, we have $\dot{\lambda}(t) = 0$ for all $t \in I$. Therefore λ is a constant. If $\lambda = 0$, then $\gamma(t) = \overline{\gamma}(t)$ for all $t \in I$. Hence λ is a non-zero constant. □

We give a necessary and sufficient condition of a Mannheim curve for a framed curve.

Theorem 4.3 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). Then (γ, ν_1, ν_2) is a Mannheim curve if and only if there exist a non-zero constant λ and a smooth function $\phi : I \to \mathbb{R}$ such that*

$$\lambda\ell(t)\sin\phi(t) + (\alpha(t) + \lambda m(t))\cos\phi(t) = 0 \tag{12}$$

for all $t \in I$.

Proof. Suppose that (γ, ν_1, ν_2) is a Mannheim curve. By Lemma 4.2, there exist another framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ and a non-zero constant $\lambda \in \mathbb{R}$ such that $\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t)$ and $\nu_1(t) = \overline{\nu}_2(t)$ for all $t \in I$. Then we have $\overline{\alpha}(t)\overline{\boldsymbol{\mu}}(t) = (\alpha(t) + \lambda m(t))\boldsymbol{\mu}(t) + \lambda\ell(t)\nu_2(t)$. Since $\nu_1(t) = \overline{\nu}_2(t)$, there exists a function $\phi : I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\boldsymbol{\mu}}(t) \\ \overline{\nu}_1(t) \end{pmatrix} = \begin{pmatrix} \cos\phi(t) & -\sin\phi(t) \\ \sin\phi(t) & \cos\phi(t) \end{pmatrix} \begin{pmatrix} \nu_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}. \tag{13}$$

Then we have

$$\overline{\alpha}(t)\cos\phi(t) = \lambda\ell(t), \quad -\overline{\alpha}(t)\sin\phi(t) = \alpha(t) + \lambda m(t). \tag{14}$$

It follows that $\lambda\ell(t)\sin\phi(t) + (\alpha(t) + \lambda m(t))\cos\phi(t) = 0$ for all $t \in I$.

Conversely, suppose that $\lambda\ell(t)\sin\phi(t) + (\alpha(t) + \lambda m(t))\cos\phi(t) = 0$ for all $t \in I$. We define a mapping $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ by

$$\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t),\ \overline{\nu}_1(t) = \sin\phi(t)\nu_2(t) + \cos\phi(t)\boldsymbol{\mu}(t),\ \overline{\nu}_2(t) = \nu_1(t).$$

Then $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ is a framed curve. Therefore, (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates. □

Proposition 4.4 *Suppose that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates, where $\overline{\gamma}(t) = \gamma(t) + \lambda\nu_1(t)$ and $\lambda\ell(t)\sin\phi(t) + (\alpha(t) + \lambda m(t))\cos\phi(t) = 0$ for all $t \in I$. Then the curvature $(\overline{\ell}, \overline{m}, \overline{n}, \overline{\alpha})$ of $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ is given by*

$$\begin{aligned} \overline{\ell}(t) &= -\ell(t)\sin\phi(t) - m(t)\cos\phi(t), \\ \overline{m}(t) &= \dot{\phi}(t) - n(t), \\ \overline{n}(t) &= \ell(t)\cos\phi(t) - m(t)\sin\phi(t), \\ \overline{\alpha}(t) &= \lambda\ell(t)\cos\phi(t) - (\alpha(t) + \lambda m(t))\sin\phi(t). \end{aligned} \tag{15}$$

Proof. By the equation (13), we have $\overline{\boldsymbol{\mu}}(t) = \cos\theta(t)\nu_2(t) - \sin\theta(t)\boldsymbol{\mu}(t)$. By differentiating, we have

$$\begin{aligned} -\overline{m}(t)\overline{\nu}_1(t) + \overline{n}(t)\overline{\nu}_2(t) &= (-\ell(t)\cos\phi(t) + m(t)\sin\phi(t))\nu_1(t) \\ &\quad +(-\dot{\phi}(t) + n(t))\sin\phi(t)\nu_2(t) + (n(t) - \dot{\phi}(t))\cos\phi(t)\boldsymbol{\mu}(t). \end{aligned}$$

Since $\nu_1(t) = \overline{\nu}_2(t)$, we have $\overline{n}(t) = \ell(t)\cos\phi(t) - m(t)\sin\phi(t)$. Again by (13), $\overline{m}(t) = \dot{\phi}(t) - n(t)$. Moreover, by differentiating $\overline{\nu}_1(t) = \sin\phi(t)\nu_2(t) + \cos\phi(t)\boldsymbol{\mu}(t)$, we have

$$\begin{aligned} \overline{\ell}(t)\overline{\nu}_2(t) + \overline{m}(t)\overline{\boldsymbol{\mu}}(t) &= (-\ell(t)\sin\phi(t) - m(t)\cos\phi(t))\nu_1(t) \\ &\quad +(\dot{\phi}(t) - n(t))\cos\phi(t)\nu_2(t) + (n(t) - \dot{\phi}(t))\sin\phi(t)\boldsymbol{\mu}(t). \end{aligned}$$

Since $\nu_1(t) = \overline{\nu}_2(t)$, we have $\overline{\ell}(t) = -\ell(t)\sin\phi(t) - m(t)\cos\phi(t)$. By the equation (14), we have $\overline{\alpha}(t) = \lambda\ell(t)\cos\phi(t) + (\alpha(t) + \lambda m(t))\sin\phi(t)$. □

As a difference between non-degenerate regular space curves and framed curves, we have a relation between Bertrand and Mannheim curves of framed curves.

Theorem 4.5 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α). Then (γ, ν_1, ν_2) is a Bertrand curve if and only if (γ, ν_1, ν_2) is a Mannheim curve.*

Proof. Suppose that (γ, ν_1, ν_2) is a Bertrand curve. By Theorem 3.3, there exist a non-zero constant λ and a smooth function $\theta : I \to \mathbb{R}$ such that $\lambda\ell(t)\cos\theta(t) - (\alpha(t) + \lambda m(t))\sin\theta(t) = 0$ for all $t \in I$. If $\phi(t) = \theta(t) + \pi/2$, then we have $\lambda\ell(t)\sin\phi(t) + (\alpha(t) + \lambda m(t))\cos\phi(t) = 0$ for all $t \in I$. By Theorem 4.3, (γ, ν_1, ν_2) is a Mannheim curve.

Conversely, suppose that (γ, ν_1, ν_2) is a Mannheim curve. By Theorem 4.3, there exist a non-zero constant λ and a smooth function $\phi : I \to \mathbb{R}$ such that $\lambda\ell(t)\sin\phi(t) + (\alpha(t) + \lambda m(t))\cos\phi(t) = 0$ for all $t \in I$. If $\theta(t) = \phi(t) - \pi/2$, then we have $\lambda\ell(t)\cos\theta(t) - (\alpha(t) + \lambda m(t))\sin\theta(t) = 0$ for all $t \in I$. By Theorem 3.3, (γ, ν_1, ν_2) is a Bertrand curve. □

If (γ, ν_1, ν_2) is a framed curve, then (γ, ν_2, ν_1) is also a framed curve. Since we can choose a moving frame, the above result immediately also prove by the definitions of Bertrand and Mannheim curves of framed curves. On the other hand, for regular cases, the moving frame is fixed. Therefore, the above result does not hold (cf. Remark 2.9).

By Theorem 4.3, we have the following result.

Corollary 4.6 *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α).*

(1) If $\ell(t) = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is a Mannheim curve.

(2) If there exists a non-zero constant λ such that $\alpha(t) + \lambda m(t) = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is a Mannheim curve.

Proof. (1) If we take $\phi(t) = \pi/2$, then the equation (12) is satisfied. (2) If we take $\theta(t) = 0$, then the equation (12) is satisfied. □

If we take an adapted frame $\{\widetilde{\nu}_1, \widetilde{\nu}_2, \boldsymbol{\mu}\}$, then the curvature is given by $(0, \widetilde{m}, \widetilde{n}, \alpha)$, see (5). By Theorem 4.3 or Corollary 4.6, we have the following.

Corollary 4.7 *For an adapted frame, $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is always a Mannheim curve.*

Remark 4.8 By Corollary 4.7, we also found that the notion of Mannheim curves depend of the frame. By Corollaries 3.6 and 4.7, $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is Bertrand and Mannheim curves (cf. Remark 2.9).

Next we consider the principal normal direction like as regular cases (see (6) and Corollary 3.8).

Corollary 4.9 *Let (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. Then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\overline{\gamma}, \overline{\boldsymbol{n}}_1, \overline{\boldsymbol{n}}_2)$ are Mannheim mates if and only if there exist a constant λ and a smooth function $\phi : I \to \mathbb{R}$ such that*

$$\begin{aligned}
&\lambda L(t)\sin\phi(t) + (\alpha(t) + \lambda M(t))\cos\phi(t) = 0,\\
&L(t)\cos\phi(t) - M(t)\sin\phi(t) = 0,\\
&\overline{L}(t) = -L(t)\sin\phi(t) - M(t)\cos\phi(t),\\
&\overline{M}(t) = \dot{\phi}(t),\\
&\overline{\alpha}(t) = \lambda L(t)\cos\phi(t) - (\alpha(t) + \lambda M(t))\sin\phi(t)
\end{aligned}$$

for all $t \in I$.

Proof. Since the curvature $n(t) = \overline{n}(t) = 0$ and Proposition 4.4, $\overline{M}(t) = \dot{\phi}(t)$ and $L(t)\cos\phi(t) - M(t)\sin\phi(t) = 0$ for all $t \in I$. By Theorem 4.3, we have the result. □

However, if we consider $(\overline{\gamma}, \overline{\boldsymbol{n}}_2, \overline{\boldsymbol{n}}_1)$, we have the following result.

Corollary 4.10 *Let (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. Then $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ and $(\overline{\gamma}, \overline{\boldsymbol{n}}_2, \overline{\boldsymbol{n}}_1)$ are Mannheim mates if and only if there exists a constant λ such that the equations* (12) *and* (15) *are satisfied, where $\phi(t)$ is a constant.*

Proof. Since the curvature $n(t) = \overline{m}(t) = 0$ and Proposition 4.4, $\dot{\phi}(t) = 0$. It follows that ϕ is a constant. By Theorem 4.3, we have the result. □

By Theorem 3.10 and Corollary 4.10, we can construct Mannheim curves of framed curves by using spherical Legendre curves.

As related topics, we investigate singularities of parallel curves (circler evolutes), see [14]. Also, the definition and properties of evolutes (spherical evolutes) of framed immersions are given in [13]. Furthermore, we consider higher dimensional cases of Bertrand and Mannheim curves of framed curves in [15].

References

[1] Y. Aminov, *Differential geometry and the topology of curves. Translated from the Russian by V. Gorkavy.* Gordon and Breach Science Publishers, Amsterdam, 2000.

[2] T. Banchoff, S. Lovett, *Differential geometry of curves and surfaces.* A K Peters, Ltd., Natick, MA, 2010.

[3] M. Berger, B. Gostiaux, *Differential geometry: manifolds, curves, and surfaces. Translated from the French by Silvio Levy.* Graduate Texts in Mathematics, 115. Springer-Verlag, New York, 1988.

[4] J. Bertrand, Mémoire sur la théorie des courbes à double courbure. *J. de methématiques pures et appliquées.* **15** (1850), 332–350.

[5] R. L. Bishop, There is more than one way to frame a curve. *Amer. Math. Monthly.* **82** (1975), 246–251.

[6] J. W. Bruce, P. J. Giblin, *Curves and Singularities. A geometrical introduction to singularity theory. Second edition.* Cambridge University Press, Cambridge, 1992.

[7] M. P. do Carmo, *Differential geometry of curves and surfaces. Translated from the Portuguese.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976.

[8] J. H. Choi, T. H. Kang, Y. H. Kim, Bertrand curves in 3-dimensional space forms. *Appl. Math. Comput.* **219** (2012), 1040–1046.

[9] T. Fukunaga, M. Takahashi, Evolutes of fronts in the Euclidean plane. *J. Singul.* **10** (2014), 92–107.

[10] T. Fukunaga, M. Takahashi, Evolutes and involutes of frontals in the Euclidean plane. *Demonstr. Math.* **48** (2015), 147–166.

[11] T. Fukunaga, M. Takahashi, Existence conditions of framed curves for smooth curves. J. Geom. **108** (2017), 763–774.

[12] S. Honda, M. Takahashi, Framed curves in the Euclidean space. *Adv. Geom.* **16** (2016), 265–276.

[13] S. Honda, M. Takahashi, Evolutes and focal surfaces of framed immersions in the Euclidean space. *Proc. Royal Soc. of Edinburgh Section A: Mathematics* (2019). DOI: 10.1017/prm.2018.84

[14] S. Honda, M. Takahashi, Circler evolutes and involutes of framed curves in the Euclidean space. *In preparation.* (2019).

[15] S. Honda, M. Takahashi, Bertrand and Mannheim curves of framed curves in higher dimensional Euclidean space. *In preparation.* (2019).

[16] J. Huang, L. Chen, S. Izumiya, D. Pei, Geometry of special curves and surfaces in 3-space form. *J. Geom. Phys.* **136** (2019), 31–38.

[17] S. Izumiya, N. Takeuchi, Generic properties of helices and Bertrand curves. *J. Geom.* **74** (2002), 97–109.

[18] W. Kühnel, *Differential geometry. Curves-surfaces-manifolds. Translated from the 1999 German original by Bruce Hunt.* Student Mathematical Library, 16. American Mathematical Society, Providence, RI, 2002.

[19] H. Liu, F. Wang, Mannheim partner curves in 3-space. *J. Geom.* **88** (2008), 120–126.

[20] P. Lucas, J. A. Ortega-Yagües, A variational characterization and geometric integration for Bertrand curves. *J. Math. Phys.* **54** (2013), 043508, 12 pp.

[21] S. G. Papaioannou, D. Kiritsis, An application of Bertrand curves and surfaces to CADCAM. *Comput.-Aided Des.* **17** (1985), 348–352.

[22] Dirk J. Struik, *Lectures on classical differential geometry. Reprint of the second edition.* Dover Publications, Inc., New York, 1988.

[23] M. Takahashi, Legendre curves in the unit spherical bundle over the unit sphere and evolutes. *Contemp. Math.* **675** (2016), 337–355.

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