Equivariant Čech-de Rham theory and its applications

Ko Fujisawa

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0 Introduction

In this thesis, we discuss equivariant cohomology, equivariant characteristic classes and their localization. Of our major interest is *Čech-de Rham cohomology.* As well known, the Čech-de Rham cohomology of a smooth manifold is a hypercohomology joining the Čech complex and the de Rham complex, which has been introduced for proving the equivalence between these two cohomology theories (cf. Bott-Tu [4]). Afterwards, Tatsuo Suwa has successfully established the Čech-de Rham theory as a tool for computing and describing explicit formulae at the level of cocycles :

$$\text{de Rham theory} \rightsquigarrow \text{Čech-de Rham theory.}$$

Indeed, it yields several applications such as localization formulae of characteristic classes and index theorems of vector fields on possibly singular varieties (Suwa [17, 19], Brasselet-Seade-Suwa [6]) and also index theorems for fixed points of holomorphic self-maps (Abate-Bracci-Tovena [1], Bracci-Suwa [5]).

The main aim of this thesis is to establish two different types of equivariant versions of the Čech-de Rham theory.

First, we recall equivariant de Rham cohomology constructed by the Cartan model. Let M be a smooth manifold with a compact Lie group action G, then naturally the Lie algebra $\mathfrak{g}$ and de Rham complex $\Omega^*(M)$ admit the adjoint action Ad_g and the action via pullback g^*. An *equivariant differential form* α is defined by a polynomial map $\alpha : \mathfrak{g} \to \Omega^*(M)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\alpha"] \arrow[d, "Ad_g"'] & \Omega^*(M) \arrow[d, "g^*"] \\
\mathfrak{g} \arrow[r, "\alpha"] & \Omega^*(M)
\end{tikzcd}$$

The Cartan complex $\Omega^*_G(M)$ is the commutative differential graded algebra of equivariant differential forms, and $H^*(\Omega^*_G(M))$ is the equivariant de Rham cohomology. Our first attempt is to build its Čech version. Given a G-invariant open covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of M, the associated Čech Cartan complex $\Omega^*_G(\mathcal{U})$ is defined, and its cohomology, called the *equivariant Čech-de Rham cohomology*, satisfies

$$H^*(\Omega^*_G(\mathcal{U})) \simeq H^*(\Omega^*_G(M)).$$

Thanks to the Chern-Weil theory, for a complex vector bundle with $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ an open covering, Suwa has developed his method for dealing with characteristic classes very explicitly at the level of Čech-de Rham cocycles by collecting arbitrary connections ∇_α on U_α. We follow his arguments in the setting of the Catran model. Of our particular interest is to describe the equivariant characteristic classes and their localization at the level of cocycles in an explicit and constructible way. Let M be a G-manifold and $\pi : E \to M$ a smooth G-equivariant complex vector bundle of rank l with the zero section $\Sigma \simeq M$. Put $\mathcal{W} = \{W_0, W_1\}$ with $W_0 = E \setminus \Sigma$ and $W_1 = E$. The *equivariant*

Thom form is explicitly given as an element of the relative equivariant Čech-de Rham complex

$$(0, \pi^*\varepsilon_{eq}, -\psi_{eq}) \in \Omega^l_G(\mathcal{W}, W_0),$$

where ε_{eq} is the equivariant Euler form and ψ_{eq} is the equivariant angular form such that $d_{eq}\psi_{eq} = -\pi^*\varepsilon_{eq}$ (Theorem2.24). A main result is an explicit expression of the *universal equivariant Thom form* for the trivial $U(l)$-equivariant bundle $\mathbb{C}^l \to \{0\}$, that involves an $\mathfrak{u}(l)^*$-valued differential form whose constant term is just the classical Bochner-Martinelli kernel (Theorem4.7). The above equivariant Thom form of E is now obtained from this universal form via the equivariant Chern-Weil map. In our approach, it may also be obtained via the localization of equivariant characteristic classes; the equivariant Thom class ψ^E_{eq} is equal to the localized equivariant top Chern class with respect to the diagonal section s_Δ:

$$\Psi^E_{eq} = c^l_\Sigma(\pi^*E, s_\Delta)_{eq}$$

(Theorem 4.11). Finally, we establish some applications. The first one is an essential version of equivariant Riemann-Roch theorem:

$$ch^*_\Sigma(\lambda_{\pi^*E^*}, s_\Delta)_{eq} = \Psi^E_{eq} \cdot td^{-1}(\pi^*E)_{eq}$$

(Theorem 4.14). The second is to present the Atiyah-Bott localization theorem in our setting (Theorem 4.20).

The most emphasized point is as follows. In the theory of Mathai-Quillen [15], the equivariant Thom form is introduced through the fermionic integral and supersymmetry arguments, and in this context, Paradan-Vergne [16] described equivariant Thom forms for oriented real vector bundles in several variants of de Rham complex. In contrast, our approach is elementary and simply minded – by means of our Čech-de Rham setting, we use only definite integrals for computations without employing the Mathai-Quillen framework. Our method presents the basis for further researches; for instance, it is promising to study $\bar{\partial}$-Thom forms and Atiyah classes in equivariant Čech-Dolbeault theory in complex holomorphic context.

The first construction of equivariant Čech-de Rham cohomology using the Cartan complex with some major results on the equivariant Thom form are collectively included in my research paper which has been published this year [8].

The second definition of equivariant cohomology uses a topological method, called the Borel construction: the universal principle bundle $EG \to BG$ induces

$$M_G = EG \times_G M = (EG \times M)/G$$

and its singular cohomology is usually called the equivariant cohomology of M: $H^*_G(M) := H^*(M_G)$. The Borel construction can also be formulated in a categorical approach, called the *simplicial method* (Segal, Dupont [7]) so that M_G corresponds to a contravariant functor $(M_G)_\bullet : [n] \mapsto G^n \times M$ from the simplicial category Δ to the category of manifolds. The associated de Rham complex $\Omega(G_\bullet \times M)$ is called *simplicial de Rham* *complex*. Our second aim is to present its Čech version. Given a G-invariant covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ we define a simplicial space $\mathcal{U}M_G$ and establish the following expected identity:

$$H^*(\Omega(\mathcal{U}M_G)) \simeq H^*(\Omega(M_G))$$

(Theorem 5.11). This approach must have potential applications to some new de-Rham theories for differentialble/topological stacks and to studies on secondary characteristic classes, and so on.

As a summary, cohomologies of the following four complexs are isomorphic to $H^*_G(M)$;

$$\begin{array}{cccccc}
\text{(Equi. de Rham theory)} & \Omega_G(M) & \rightsquigarrow & \Omega_G(\mathcal{U}) & \text{(Equi. Čech-de Rham theory)} \\
 & \updownarrow & & \updownarrow & \\
\text{(Simplicial de Rham theory)} & \Omega(M_G) & \rightsquigarrow & \Omega(\mathcal{U}M_G) & \text{(Simplicial Čech-de Rham theory)}
\end{array}$$

The present thesis is organized as follows. The first chapter is devoted to basic materials about Lie group actions on manifolds. In §2, we introduce the well known de Rham theory under the setting of G-action. First, recall the definition of equivariant differential forms and the Cartan complex, and then we describe its Čech version by following Suwa's construction [19, 17]. In particular, the fiber integral in our setting is described in detail. The highlight in this section is to express the equivariant Thom form in an explicit way. In §3, we go into the Chern-Weil theory in our equivariant Čech-de Rham setting. Equivariant characteristic classes, especially equivariant Chern classes, are discussed. In §4, we then take up the localization of the equivariant equivariant characteristic classes. In particular, we show that our localized equivariant top Chern form provides an explicit formula of the universal $U(l)$-equivariant Thom form. As applications, we show the equivariant version of the Riemann-Roch theorem for the zero locus of a section of a complex vector bundle and the Atiyah-Bott localization theorem in our setting. In the final section, §5, we switch to the Borel construction of equivariant cohomology – established is the *simplicial Čech-de Rham theory*.

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1 Preliminary

1.1 Actions of Lie groups and Lie algebras

Let M be a smooth manifold and G a Lie group.

Definition 1.1. An *smooth (left) action* of G on M is a group homomorphism $\tau : G \to \mathrm{Diff}(M)$, where $\mathrm{Diff}(M)$ is the group of diffeomorphisms on M, such that the the evaluation map

$$\mathrm{ev} : G \times M \to M, \quad (g, m) \mapsto \tau(g)(m)$$

is smooth. We will denote $\tau(g)(m)$ by $g \cdot m$ briefly. A smooth manifold together with G-action is called a *G-manifold*

Remark. In the above definition, if $\tau : G \to \mathrm{Diff}(M)$ is an anti-homomorphism(i.e. $\tau_{gh} = \tau_h \circ \tau_g$), we call τ a G-right action.

Hereafter, an action always means a smooth action.

Example 1.2 (one-parameter group of differomorphisms)**.** If X is a complete vector field on M, then

$$\phi : \mathbb{R} \to \mathrm{Diff}(M), \quad t \mapsto \phi_t = \exp(tX)$$

is a smooth action of $\mathbb{R}$ on M.

Example 1.3 (adjoint action, coadjoint action)**.** Let $\mathfrak{g}$ be the Lie algebra of G. Then the *adjoint action* of G on $\mathfrak{g}$ is

$$\mathrm{Ad} : G \to GL(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g$$

In particular, the adjoint action of $GL(n, \mathbb{R})$ on $\mathfrak{gl}(n, \mathbb{R})$ is

$$\mathrm{Ad}_g(X) = gXg^{-1}$$

Similarly the *coadjoint action* of G on $\mathfrak{g}^*$ is

$$\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*), \quad g \mapsto \mathrm{Ad}^*_g$$

where Ad^* is defined by $\langle Ad^*_g \xi, X\rangle = \langle \xi, Ad_{g^{-1}} X\rangle$ for all $\xi \in \mathfrak{g}^*, X \in \mathfrak{g}$

Definition 1.4. Let M,N be G-manifolds. A smooth map $f : M \to N$ is called *G-equivariant*, if it commutes with the group action, i.e. $f(g \cdot m) = g \cdot f(m)$ for all $g \in G$ and $m \in M$

Next we consider the *infinitesimal* action of a Lie algebra. Let $\mathfrak{X}(M)$ denote the Lie algebra of vector fields on M with bracket $[X, Y] = X \circ Y - Y \circ X$ for $X, Y \in \mathfrak{X}(M)$.

Definition 1.5. Let $\tau : G \to \mathrm{Diff}(M)$ be a smooth action. The *infinitesimal action* of $\mathfrak{g}$ on M

$$d\tau : \mathfrak{g} \to \mathfrak{X}(M), \quad X \mapsto X_M$$

is defined by

$$X_M(m) = \frac{d}{dt}\bigg|_{t=0} \exp(-tX) \cdot m$$

for $m \in M$

The infinitesimal action has the following property (it is easy to check):

Proposition 1.6. The infinitesimal action $d\tau : \mathfrak{g} \to \mathfrak{X}(M)$ is a Lie algebra homomorphism, i.e.

$$[X, Y]_M = [X_M, Y_M]$$

for all $X, Y \in \mathfrak{g}$

Proof. Let us first note that if M_1, M_2 are G-manifolds, and $f : M_1 \to M_2$ is G-equivariant map, then X_{M_1}, X_{M_2} are f-related. Since Lie brakets of pairs of f-related vector fields are again f-related, so it follows that if f is onto, and $X \to X_{M_1}$ is a Lie algebra homomorphism, then so is $X \to X_{M_2}$. Consider the following map:

$$F : G \times M \to M, \quad (a, m) \mapsto a^{-1} \cdot m$$

Letting G act on $G \times M$ by $g \cdot (a, m) := (ag^{-1}, m)$, F is a G-equivariant map. $X_{G\times M}$ may be identified with the left-invariant vector field generated by X. Since F is onto, and $X \to X_{G\times M}$ is a Lie algebra homomorphism, so is $X \to X_M$. $\square$

1.2 Orbits and Stabilizers

Let $\tau : G \to \mathrm{Diff}(M)$ be a smooth action and H a subgroup of G.

Definition 1.7.

(1) The *orbit* of G through $m \in M$ is $G \cdot m = \{g \cdot m \mid g \in G\}$

(2) The *stabilizer* of $m \in M$ is the subgroup $G_m = \{g \in G \mid g \cdot m = m\}$

(3) The space $M/G = \{G \cdot m \mid m \in M\}$ is called the orbit space for given action. We will equip M/G with the natural quotient topology.

(4) The set $M^H = \{m \in M \mid H \subset G_m\}$ is called the *fixed point set* of H.

Proposition 1.8. The orbit $G \cdot m$ is an immersed submanifold of M

Proof. Consider the evaluation map $\mathrm{ev}_m : G \to M, g \mapsto g \cdot m$. It is equivariant with respect to the left G action on G and the the G action on G : $\mathrm{ev}_m \circ L_g = \tau_g \circ \mathrm{ev}_m$. Since dL_g and $d\tau_g$ are bijective at each tangent space, it follows from the above equation that the map ev_m is of constant rank. By constant rank theorem, $\mathrm{ev}_m(G) = G \cdot m$ is an immersed submanifold of M. $\square$

By Cartan's closed subgroup theorem, we see that the stabilizer G_m is a Lie subgroup of G, since $G_m = \mathrm{ev}_m^{-1}(m)$. Then we have:

Proposition 1.9. The Lie algebra of the stabilizer G_m is expressed in the following:

$$\mathfrak{g}_m = \{X \in \mathfrak{g} \mid X_M(m) = 0\}$$

Proof. The Lie algebra of G_m is $\mathfrak{g}_m = \{X \in \mathfrak{g} \mid \exp(tX) \in G_m, \text{for any } t \in \mathbb{R}\}$ It follows that $\exp(-tX) \cdot m = m$ for $X \in \mathfrak{g}$. Taking derivative at $t = 0$, we get

$$\mathfrak{g}_m \subset \{X \in \mathfrak{g} \mid X_M(m) = 0\}$$

Conversely, if $X_M(m) = 0$, then the constant path $\gamma(t) \equiv m$ is the maximal integral curve of the vector field X_M through m. By uniqueness of the maximal integral curve through m,

$$\exp(tX) \cdot m = \gamma(t) = m.$$

Therefore $\exp(tX) \in G_m$ for all $t \in \mathbb{R}$. So $X \in \mathfrak{g}_m$. □

1.3 Proper Actions

In general the topology of the orbit space M/G might be very bad, e.g. non-Hausdorff. We will put conditions on the action to make the quotient space Hausdorff.

Definition 1.10. A smooth G-action on M is *proper* if the map

$$F : G \times M \to M \times M, \quad (g, m) \to (g \cdot m, m)$$

is proper(the pre-image of any compact set is compact).

Proposition 1.11. If G is compact, any smooth G-action is proper.

Proof. If $K \subset M \times M$ is a compact set, then $L = \mathrm{pr}_1(K) \cup \mathrm{pr}_2(K) \subset M$ is a compact set where $\mathrm{pr}_1, \mathrm{pr}_2$ is the projection to the first and second factors respectively. It follows that

$$F^{-1}(K) \subset F^{-1}(L \times L) \subset G \times L$$

is a closed set in a compact set $G \times L$, and thus $F^{-1}(K)$ is compact. □

Proposition 1.12. If G acts on M properly, the orbit space M/G is Hausdorff

Proof. Suppose the orbits of p and q can't be separated in M/G. Let U_n and V_n be balls in M of radius $\frac{1}{n}$ around p and q respectively with some metric on M. Then $G \cdot U_n \cap G \cdot V_n \neq \phi$. Namely, there exists $p_n \in U_n, q_n \in V_n$ and $g_n \in G$ such that $p_n = g_n \cdot q_n$. The sequence $(p_n, q_n) = (g_n \cdot q_n, q_n)$ is contained in the image of action map F which is proper. It follows that $(p_n, q_n) \to (p, q)$ such that $p = g \cdot q$ for some $g \in G$. Hence the orbits p and q coincide. Contradiction. □

We claimed that an orbit $G \cdot m$ is an immersed submanifold in Proposition 1.8. Under a proper action, moreover, we have the following theorem:

Theorem 1.13. Suppose G acts on M properly. Then each orbits $G{\cdot}m$ is an embedded, closed submanifold of M, with

$$T_m(G \cdot m) = \{X_M(m) \mid X \in \mathfrak{g}\}$$

Proof. Since the evaluation map ev_m is proper, its image $G \cdot m$ is closed in M. We have already seen that $G \cdot m$ is an immersed submanifold of M, so there is a neighborhood U of e in G such that $\mathrm{ev}_m(U)$ is an embedded submanifold of M near m. Since $\mathrm{ev}_m(U) = \mathrm{ev}_m(UG_m)$, we may replace U by UG_m so that $U \supset G_m$. Since $G \cdot m = G \cdot m'$ for any $m' \in G \cdot m$, to show $G \cdot m = \mathrm{ev}_m(G)$ is an embedded submanifold of M, it suffices to show that there exists a small neighborhood W of m in M such that $W \cap \mathrm{ev}_m(G) = W \cap \mathrm{ev}_m(U)$. We proceed by contradiction. Suppose there is no small neighborhood W of m in M such that $W \cap \mathrm{ev}_m(G) = W \cap \mathrm{ev}_m(U)$, then there exists a sequence of points $g_k \cdot m$ converging to m with $g_k \notin U$. The set $\{(m, g_k \cdot m)\} \cup \{(m, m)\}$ is compact. So by properness, the sequence $\{(g_k, m)\}$ is contained in a compact set. By passing to a subsequence we may assume $g_k \to g_\infty$. Since $g_k \cdot m \to m$, we see $g_\infty \cdot m = m$, i.e. $g_\infty \in G_m \subset U$. Since U is open, we conclude that $g_k \in U$ for large k, contradiction. Finally since $G \cdot m$ is a submanifold, its the tangent space should be the image of the tangent map, i.e. $T_m(G \cdot m) = (d\mathrm{ev}_m)_e(\mathfrak{g})$. However, for any $X \in \mathfrak{g}$ and any $f \in C^\infty(M)$,

$$\begin{aligned}(d\mathrm{ev}_m)_e(-X)(f) = -X_e(f \circ \mathrm{ev}_m) &= \left.\frac{d}{dt}\right|_{t=0} (f \circ \mathrm{ev}_m)(\exp(-tX)) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\exp(-tX) \cdot m) = X_M(m)(f),\end{aligned}$$

so the image of the tangent map $(d\mathrm{ev}_m)_e$ is $\{X_M(m) \mid X \in \mathfrak{g}\}$. ☐

We now consider the structure of a fixed point set of a proper Lie group action.

Lemma 1.14. Let M be a G-manifold, with G compact, and $m \in M^G$. Then there exists a G-invariant open neighborhood U of m and G-equivariant diffeomorphism $T_mM \cong U$ taking 0 to m.

Proof. Choosing a G-invariant Riemannian metric on M, the exponential map $\exp_0 : T_mM \to M$ for this metric is G-equivariant, and its differential at the identity is invertible. Hence it defines a G-equivariant diffeomorphism $B_\varepsilon(0) \to U \subset M$ for $\varepsilon > 0$ sufficiently small. But $B_\varepsilon(0)$ is diffeomorphic to all of T_mM , choosing a diffeomorphism preserving radial directions. ☐

Proposition 1.15. Suppose G acts on M properly. Then, for any subgroup H of G, the each component of the set M^H of H-fixed points is a closed embedded submanifold of M.

Proof. We first observe that $M^H = M^{\overline{H}}$, where $\overline{H}$ is the closure of H. Indeed, if m is in M^H, and $\{h_i\} \subset H$ a sequence converging to $h \in \overline{H}$, then $h \cdot m = \lim_i h_i \cdot m = m$ since G-action is smooth.This implies that $M^H \subset M^{\overline{H}}$ (the converse is obvious). Since G_m is compact, $\overline{H}$ is also compact. Thus we assume that H is a compact embedded Lie subgroup of G. By the Lemma 1.14(with $G = H$), there exists a H-invariant open neighborhood U of m and H-equivariant diffeomorphism $T_mM \cong U$ taking 0 to m. Thus, U^H corresponds to $(T_mM)^H$, which is a linear subspace of T_mM □

1.4 Proper Free Actions

If G acts on M properly, then M/G is Hausdorff. But it is still possible that the orbit space M/G is not a manifold. Then we will consider a proper free action to make M/G a manifold.

Definition 1.16. The G-action on M is *free* if $G_m = \{e\}$ for any $m \in M$

Theorem 1.17. If $\tau : G \to \mathrm{Diff}(M)$ is a proper free action, then the orbit space M/G is a manifold and the quotient map $\pi : M \to M/G$ is a submersion.

Proof. Since the action is free, the map ev_m is bijective. Moreover, $(d\mathrm{ev}_m)_e$ is injective, since

$$\mathrm{Ker}(d\mathrm{ev}_m)_e = \{X \in \mathfrak{g} \mid X_M(m) = 0\} = \mathrm{Lie}(G_m) = \{0\}.$$

Since ev_m is a constant rank map, according to Sard ' s theorem $(d\mathrm{ev}_m)_g$ has to be surjective for all g. It follows $(d\mathrm{ev}_m)_e$ is invertible, and in particular, $T_m(G \cdot m) \simeq \mathfrak{g}$. Choose a submanifold $S \subset M$ with $m \in S$ such that

$$T_m(G \cdot m) \oplus T_mS = T_mM$$

.

We claim that the evaluation map $\mathrm{ev} : G \times M \to M$ restricts to a smooth map $\mathrm{ev} : G \times S \to M$ whose tangent map at the point (e, m) is

$$(d\mathrm{ev})_{e,m}(X, Y) = (d\mathrm{ev}_m)_e X_e + Y_m$$

which is invertible. In fact, to check the above formula, we take any $f \in C^\infty(M)$,

$$(d\mathrm{ev})_{e,m}(X, Y)(f) = (X, Y)_{e,m}(f \circ \mathrm{ev}) = X(f \circ \mathrm{ev}_m) + Y(f \circ \tau_e) = (d\mathrm{ev}_m)_e X(f) + Y(f).$$

The invertibility follows from the uniqueness of the direct sum decomposition above together with the invertibility of the map $(d\mathrm{ev}_m)_e : \mathfrak{g} \to T_m(G \cdot m)$. By continuity, the tangent map $d\mathrm{ev}$ is still invertible at (e, m') for $m' \in S$ close to m. By choosing S small we may assume this is the case for all points in S. It follows from the commutative diagram

$$\begin{tikzcd} G\times S \arrow[r, "\mathrm{ev}"] \arrow[d, "L_{g_1}\times id"'] & M \arrow[d, "\tau_{g_1}"] \\ G\times S \arrow[r, "\mathrm{ev}"] & M \end{tikzcd}$$

that $\mathrm{ev} : G \times S \to M$ has invertible tangent map everywhere, and thus is a local diffeomorphism onto its image. We claim that this is in fact a diffeomorphism onto its image if we choose S small enough. If not, we can choose two sequences (g_1^k, m_1^k) and (g_2^k, m_2^k) such that $m_1^k \to m, m_2^k \to m$ and $g_1^k \cdot m_1^k = g_2^k \cdot m_2^k$. Note that if we denote $g^k = (g_1^k)^{-1} g_2^k$, then $g^k \cdot m_2^k = m_1^k$. Since ev is bijective near (e, m), there is a neighborhood U of e in G so that $g^k \notin U$ for all k. Now the set $\{(g^k \cdot m_2^k, m_2^k)\} \cup \{(m,m)\}$ is compact in $M \times M$. By properness, the sequence $\{(g^k, m_2^k)\}$ is contained in a compact set, thus has a converging subsequence. In other words, the sequence $\{g^k\}$ has a convergent subsequence. However, the only limit point of the set $\{g^k\}$ has to be e since the limit point must take m to m, and the action is free. This contradicts with the fact that no g^k lies in U. Now denote by V the image of $G\times S$ under ev in M. Then the diffeomorphism $\mathrm{ev} : G \times S \to V \subset M$ identifies V/G with S, hence gives a manifold structure on M/G near the orbit $G \cdot m$. Obviously the quotient map $\pi : M \to M/G$ is a submersion. $\square$

In particular, if H is a closed Lie subgroup of G, then one can easily that the H-action on G defined by

$$\tau_h : G \to G, \quad g \mapsto gh^{-1}$$

is proper and free. So we have:

Corollary 1.18. If H is a closed Lie subgroup of G, then the quotient G/H is a manifold and the quotient map $G \to G/H$ is a smooth submersion.

Since the stabilizer G_m is a closed group of G, G/G_m is a manifold. Moreover we have the following:

Proposition 1.19. Suppose G acts on M smoothly. Then a map

$$\phi : G/G_m \to G \cdot m, \quad G_m \cdot g \mapsto g \cdot m$$

is a diffeomorphism.

Proof. ϕ is obviously surjective. It is injective since if $G_m\cdot g_1 \neq G_m\cdot g_2$, then $g_1 g_2^{-1} \notin G_m$, so $g_1 \cdot m \neq g_2 \cdot m$. So it is a bijective map from G/G_m to its image, $G \cdot m$. Note that the tangent space of G/G_m at $G_m \cdot e$ is $\mathfrak{g}/\mathfrak{g}_m$. According to $\mathrm{ev}_m = \phi \circ \pi$, we have $(d\mathrm{ev}_m)_e = (d\phi)_{G_m\cdot e} \circ (d\pi)_e$. It follows $(d\phi)_{G_m\cdot e}(X + g_m) = X_M(m)$. In other words, $d\phi$ is bijective at $G_m \cdot e$. By equivariance,

$$\begin{tikzcd} G/G_m \arrow[r, "L_g"] \arrow[d, "\phi"'] & G/G_m \arrow[d, "\phi"] \\ G\cdot m \arrow[r, "\tau_g"] & G\cdot m \end{tikzcd}$$

$d\phi$ is bijective everywhere. If follows that ϕ is locally diffeomorphism everywhere. Since it is bijective, it is globally diffeomorphism. $\square$

1.5 The Slice Theorem

First of all, to formulate the slice theorem, we consider *a principal bundle* and an *equivariant vector bundle.*

Definition 1.20. A *principal G-bundle* over a manifold M is a manifold P together with a right G-action on P and a smooth map $\pi : P \to M$, having the following *local triviality* property; For each $m \in M$, there exists an open neighborhood U of m and G-equivariant diffeomorphism

$$\pi^{-1}(U) \to U \times G$$

where the G-action on the right hand side is $(m, g_1) \cdot g = (m, g_1 g)$

Definition 1.21. Suppose $\pi : P \to M$ is a G-principal bundle and G acts on a topology space F. Then G acts on the product space $P \times F$ by

$$g \cdot (p, y) = (p \cdot g^{-1}, g \cdot y)$$

Since this action is obviously proper and free, we get a smooth vector bundle

$$P \times_G F := (P \times F)/G \to M, \quad [p, y] \mapsto \pi(p),$$

where $[p, y]$ is the equivalence class of $(p, y) \in P \times F$ in $P \times_G F$. This bundle is called a *associated bundle.* If F is a vector space on which G acts linearly, the associated bundle is a vector bundle.

Remark. It is easy to check the local triviality of $P \times_G F = (P \times F)/G \to M$. Indeed, any local trivialization $\pi^{-1}(U) = U \times G$ gives rise to a linear local trivialization

$$(U \times G) \times_G F \to U \times F, \quad [(m, g), y] \mapsto (m, g^{-1} \cdot y)$$

Definition 1.22. Let G be a Lie group. $\pi : E \to M$ is called a G-equivariant vector bundle, if $\pi : E \to M$ is a vector bundle and G acts on E by vector bundle maps.

Remark. In particular, note that a G-action on M naturally induces the G-action on the tangent bundle TM by $g \cdot v = dL_g(v)$

Lemma 1.23. Let G be a compact Lie group and $\pi : E \to M$ be G-equivariant vector bundle. There exists G-invariant Riemannian metric on M

Proof. Since G is a compact Lie group, there exists normalized left and right invariant Haar measure. We denote it by dg. Let $\langle \cdot, \cdot \rangle$ be a metric on $\pi : E \to M$. We define the G-invariant metric $\langle \cdot, \cdot \rangle_G$ as follows: For $u, v \in E_x = \pi^{-1}(x)$,

$$\langle u, v \rangle_G := \int_G \langle (dL_g)_x(u), (dL_g)_x(v) \rangle dg$$

Since

$$\begin{aligned}\langle (dL_h)_x u, (dL_h)_x v\rangle_G &= \int_G \langle (dL_{gh})_x u, (dL_{gh})_x v\rangle dg \\ &= \int_G \langle (dL_g)_x u, (dL_g)_x v\rangle dg \\ &= \langle u, v\rangle_G,\end{aligned}$$

we see that $\langle \cdot, \cdot \rangle_G$ is G-invariant metric on E. □

Corollary 1.24 (Equivariant tubular neighborhood theorem)**.** Suppose G acts on M properly and $N \subset M$ a G-invariant embedded submanifold with normal bundle $\nu_N = TM|_N/TN$. Then there exists a G-equivariant diffeomorphism from ν_N to a neighborhood of N in M, restricting to the identity map from the zero section $N \subset \nu_N$ to $N \subset M$.

Proof. This follows from the standard proof of the tubular neighborhood theorem, by using a G-invariant Riemannian metric on M. □

Theorem 1.25 (The Slice Theorem)**.** Let G be a Lie group acting properly on a manifold M. Then, for any $m \in M$, there exists a G-equivariant diffeomorphism from the disk bundle $G \times_{G_m} D$ onto a G-invariant neighborhood of the orbit $G \cdot m \subset M$, where D is some G_m-invariant open subset in a vector space. Its restriction to the zero section $G \times_{G_m} \{0\} = G/G_m$ coincides with the diffeomorphism $G/G_m \to G \cdot m, \ \ G_m \cdot g \mapsto g \cdot m$

Proof. Identifying V with a complement of $T_m(G \cdot m)$ in T_mM (e.g. the orthogonal complement for an invariant inner product), we obtain an G_m-equivariant embedding $V \to M$, as a submanifold transversal to the orbit $G \cdot m$. It extends to a G-equivariant map $G \times_{G_m} V \to M$. As before, we see that the map $G \times_{G_m} V \to M$ is in fact a diffeomorphism over $G \times_{G_m} B_\varepsilon(0)$ for ε sufficiently small. Choosing an G_m-equivariant diffeomorphism $V \simeq B_\varepsilon(0)$, the slice theorem is proved. □

As an application of the slice theorem, we have the following theorem. This theorem is used to show that the isomorphism between a *equivariant de Rham cohomology* and a *equivariant Čech-de Rham cohomology.*

Theorem 1.26 (Invariant partition of unity)**.** Suppose G acts on M properly. For every G-invariant open covering, there exists an G-invariant partition of unity subordinate to the covering.

Lemma 1.27. Under the above situations, let S be an G-invariant subset of M. If S/G is compact, then S is closed in M.

Proof. Since M/G is Hausdorff by Prop1.12, the compact subset S/G is closed. Because the quotient map $\pi : M \to M/G$ is continuous, $S = \pi^{-1}(S/G)$ is closed. □

(proof of theorem1.26). Let us use the name *tube* for an open G-invariant subset of M which is equivariantly to $G \times_H D$ where H is a compact subgroup of G and D is an open subset of a vector space on which H acts. It follow from the slice theorem that every invariant open set is a union of tubes.

Suppose we are given an open covering of M by G-invariant open sets. Since M is locally compact and paracompact, we may take an open subsets $W_n'' \subset W_n' \subset W_n$ such that $\overline{W_n''}$ is a compact set in W' and $\overline{W_n'}$ is a compact set contained in W such that each W_n is contained in a tube and in some element of the given covering, and such that W'' also cover M. Let $V_n = (G \cdot W_n) \setminus \cup_{k<n}(G \cdot \overline{W_k''})$ and $C_n = (G \cdot \overline{W_n'}) \setminus \cup_{k<n}(G \cdot \overline{W_k'})$. Then $\{V_n\}_{n \in \mathbb{N}}$ form a locally finite refinement of the given covering and each V_n is G-invariant, which is contained in a tube and in an element of the given covering. Moreover, $\{C_n\}_{n \in \mathbb{N}}$ still cover M. V_n is isomorphic to $G \times_H D$ for some compact subgroup H of G and H-invariant open subset D of a vector space. This isomorphism takes C_n to a set of the form $G \times_H K$ for some compact set $K \subset D$. We take any smooth function on D which is positive on K and whose support is a compact subset $\tilde{K}$ of D. Its average with respect to H-action(which is possible since H is compact) has these same properties. This gives rise to a function ρ_n' that is supported on V_n. Indeed, the closure of the set where ρ_n' is positive in V_n is $G \times_H \tilde{K}$. Since its quotient with respect to G is compact, by 1.27, this set is closed in M. Therefore, the support of ρ_n' in V_n coincides with the support of ρ_n' in M. In particular, the support of ρ_n' is contained in V_n. Then the functions $\rho_n = \rho_n' / \sum_k \rho_k'$ form an G-invariant partition of unity on M, subordinate to the given covering. □

2 Equivariant Čech-de Rham cohomology

2.1 Equivariant de Rham cohomology

Let M be a smooth manifold and G a Lie group with Lie algebra $\mathfrak{g}$. We denote by $(\Omega^*(M), d)$ the $\mathbb{C}$-valued de Rham complex of M and by $\mathbb{C}[\mathfrak{g}]$ the algebra of polynomials on $\mathfrak{g}$ (which is isomorphic to the symmetric algebra $S(\mathfrak{g}^*)$ of $\mathfrak{g}^*$). For $X \in \mathfrak{g}$, we denote by L_X the Lie derivative

$$L_X := L(X_M) : \Omega^k(M) \to \Omega^k(M)$$

and by ι_X the contraction.

$$\iota_X := \iota(X_M) : \Omega^k(M) \to \Omega^{k-1}(M).$$

Then we may easily check the following identities: For $X \in \mathfrak{g}$,

$$L_X = \iota_X d + d\iota_X$$

$$L_X d = dL_X.$$

Suppose that G acts on M smoothly. If $X_1, ..., X_n$ is a basis of $\mathfrak{g}$, we will let $x^1, ..., x^n$ denote the corresponding dual basis. Then we naturally get the action of G on $\Omega^*(M)$ and $\mathbb{C}[\mathfrak{g}]$ as follows: For $g \in G$,

$$\omega \mapsto g \cdot \omega := L^*_{g^{-1}}\omega, \quad \omega \in \Omega^*(M)$$

$$x^I \mapsto g \cdot x^I := (\mathrm{Ad}^*_g x)^I, \quad x^I \in \mathbb{C}[\mathfrak{g}]$$

where $L^*_{g^{-1}}$ is the pull back of a left transformation $L_{g^{-1}}$ and $I = (i_1, ..., i_n)$ is a multi-index.

Definition 2.1. $\alpha = \sum_I x^I \otimes \omega_I \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ is called G-equivariant differential form, if it satisfies the following condition: For any $g \in G$

$$g \cdot \alpha := \sum_I g \cdot x^I \otimes g \cdot \omega_I = \sum_I x^I \otimes \omega_I = \alpha$$

The wedge product of two equivariant forms is defined as the usual wedge product of differential forms. We denoted by $\Omega^*_G(M) := (\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G$ the algebra of G-equivariant differential forms. The degree of an equivariant form $\alpha = x^I \otimes \omega_I(|I| = p, \omega_I \in \Omega^k(M))$ is defined by $\deg(\alpha) := 2p + k$.

Remark. In other words, $\alpha = \sum_I x^I \otimes \omega_I$ may be also regarded as a G-equivariant polynomial map $\alpha : \mathfrak{g} \mapsto \Omega^*(M)$, i.e.,

$$\alpha(\sum \xi^i X_i) = \sum \xi^I \omega_I, \quad \alpha(\mathrm{Ad}_g X) = g \cdot \alpha(X)$$

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\alpha} & \Omega^*(M) \\
{\scriptstyle Ad_g}\downarrow & & \downarrow{\scriptstyle L_{g^{-1}}} \\
\mathfrak{g} & \xrightarrow{\alpha} & \Omega^*(M).
\end{array}$$

Definition 2.2. The *twisted de Rham differential* d_{eq} is defined as follows: For $\alpha \in \Omega_G^*(M)$,

$$(d_{eq}\alpha)(X) := d(\alpha(X)) - \iota_X \alpha(X) \quad (X \in \mathfrak{g}).$$

Lemma 2.3. Suppose that $\alpha \in \Omega_G^*(M)$. Then, we have the following fundamental properties:

(1) $d_{eq}\alpha \in \Omega_G^*(M)$.

(2) $(d_{eq})^2 = 0$.

(3) The twisted de Rham differential d_{eq} increases the degree of an equivariant differential form by 1.

(4) $d_{eq}(\alpha \wedge \beta) = d_{eq}\alpha \wedge \beta + (-1)^k \alpha \wedge d_{eq}\beta$, where $\alpha \in \Omega_G^k(M)$.

Proof. (1) Since $\alpha : \mathfrak{g} \mapsto \Omega^*(M)$ is G-equivariant and de Rham differential d commutes with pull-backs, for $X \in \mathfrak{g}, g \in G$,

$$d(\alpha(\mathrm{Ad}_g X)) = d(g \cdot \alpha(X)) = g \cdot d(\alpha(X))$$

Thus, the first component $d(\alpha(X))$ is G-equivariant. To prove that $\iota_X(\alpha(X))$ is G-equivariant, we use the identity

$$\iota_{\mathrm{Ad}_g X} = g \cdot \iota_X \cdot g^{-1}$$

Indeed, since

$$\begin{aligned}
(\mathrm{Ad}_g X)_M(m) &= \left.\frac{d}{dt}\right|_{t=0} \exp(-t\mathrm{Ad}_g(X)) \cdot m \\
&= \left.\frac{d}{dt}\right|_{t=0} (g \cdot \exp(-tX) \cdot g^{-1}) \cdot m \\
&= dL_g dR_{g^{-1}} X_M(m),
\end{aligned}$$

we get the above identity. Therefore,

$$\iota_{\mathrm{Ad}_g X}(\alpha(\mathrm{Ad}_g X)) = (g \cdot \iota_X \cdot g^{-1})(g \cdot \alpha(X)) = g \cdot (\iota_X \alpha(X)).$$

Thus, we see that $d_{eq}\alpha$ is G-equivariant.

(2)Directly computing, we have

$$\begin{aligned}
((d_{eq})^2\alpha)(X) &= d^2(\alpha(X)) - (d\iota_X + \iota_X d)\alpha(X) + (\iota_X)^2\alpha(X) \\
&= -L_X\alpha(X) = -\left.\frac{d}{dt}\right|_{t=0} L^*_{\exp(-tX)^{-1}}\alpha(X) \\
&= -\left.\frac{d}{dt}\right|_{t=0} \exp(tX)\cdot\alpha(X) \\
&= -\left.\frac{d}{dt}\right|_{t=0} \alpha(\mathrm{Ad}_{\exp(-tX)}X)) \\
&= -\left.\frac{d}{dt}\right|_{t=0} \alpha(e^{-\mathrm{ad}(tX)}X) \\
&= -\left.\frac{d}{dt}\right|_{t=0} \alpha(X) \\
&= 0.
\end{aligned}$$

(3) For $\alpha = x^I \otimes \omega_I$, $\alpha(X) = \alpha(\sum \xi^i X_i) = \sum \xi^I \omega_I$. Thus,

$$\begin{aligned}
(d_{eq}\alpha)(X) &= d\alpha(X) - \iota_X\alpha(X) = \sum \xi^I d\omega_I - \sum \xi^I \iota_{\sum \xi^i X_i}\omega_I \\
&= \sum \xi^I d\omega_I - \sum_{I,i} \xi^I \xi^i \iota_X \omega_I \\
&= (x^I \otimes d\omega_I - \sum_i (x^i x^I \otimes \iota_{X_i}\omega_I))(X).
\end{aligned}$$

We see that with respect to the first component, d increases the differential form degree by 1 and leaves the polynomial degree unchanged and the second component, ι_X decreases the differential form degree by 1 and increases the polynomial degree by 1. Thus, the total degree is increased by 1.
(4) Note that, $(-1)^p = (-1)^{p+2q} = (-1)^k$. For $\alpha \in (\mathbb{C}[\mathfrak{g}]^q \otimes \Omega^p(M))^G$, since

$$d(\alpha(X)\wedge\beta(X)) = d\alpha(X)\wedge\beta(X) + (-1)^p\alpha(X)\wedge d\beta(X)$$

$$\iota_X(\alpha(X)\wedge\beta(X)) = \iota_X\alpha(X)\wedge\beta(X) + (-1)^p\alpha(X)\wedge\iota_X\beta(X),$$

we have $d_{eq}(\alpha\wedge\beta) = d_{eq}\alpha + (-1)^k\alpha\wedge d_{eq}\beta$. □

From the above lemma, we see that $(\Omega^*_G(M), d_{eq})$ is a $\mathbb{Z}$-graded complex.

Definition 2.4. The p-th *equivariant de Rham cohomology algebra* is defined by the p-th cohomology of the $\mathbb{Z}$-graded complex $(\Omega^*_G(M), d_{eq})$:

$$H^p_G(M) := \mathrm{Ker}d^p_{eq}/\mathrm{Im}d^{p-1}_{eq}$$

Proposition 2.5. Let M, N be G-manifolds. If $\phi : M \to N$ is G-morphism, then it induces a pull-back

$$\phi^* : \Omega_G^*(N) \to \Omega_G^*(M), \quad x^I \otimes \omega_I \mapsto x^I \otimes \phi^* \omega_I$$

and it satisfies that $d_{eq}\phi^* = \phi^* d_{eq}$. Therefore, we get a ring homomorphism

$$\phi^* : H_G^*(N) \to H_G^*(M).$$

Proof. For $X \in \mathfrak{g}$, since de Rham differential commutes with pull-backs, it is sufficient to show that

$$\iota_X^M \phi^* = \phi^* \iota_X^N,$$

where ϕ^* is a pull-back in de Rham complex and ι_X^M, ι_X^N is the contractions of M,N respectively. Since

$$\begin{aligned} d\phi_m(X_M(m)) &= \left.\frac{d}{dt}\right|_{t=0} \phi(\exp(-tX) \cdot m) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\exp(-tX) \cdot \phi(m)) \\ &= X_N(\phi(m)), \end{aligned}$$

then X_M and X_N is ϕ-related. Thus, we have $\iota_X^M \phi^* = \phi_* \iota_X^N$. □

Definition 2.6. Suppose M is n-dimensional oriented, compact G-manifold. The integral of an equivariant form $\alpha \in \Omega_G^*(M)$ over M is defined as follows:
For $X \in \mathfrak{g}$, letting $\alpha(X)_j$ denote the component of $\alpha(X)$ lying in $\Omega^j(M)$,

$$(\int_M \alpha)(X) := \int_M \alpha(X)_n.$$

Proposition 2.7. In addition to the above situation, we assume that any left transformation of G on M preserves the orientation of M. The integral of an equivariant form have the following properties:

(1) $\int_M \alpha \in \mathbb{C}[\mathfrak{g}]^G$.

(2) $H_G^*(M) \ni [\alpha] \mapsto \int_M \alpha \in \mathbb{C}[\mathfrak{g}]^G$ is well-defined.

Proof. (1) Since $\alpha(\mathrm{Ad}_g(X))_n = g \cdot \alpha(X)_n$ and a left transformation $L_{g^{-1}}$ preserve the orientation of M,

$$\begin{aligned} (\int_M \alpha)(\mathrm{Ad}_g X) &= \int_M \alpha(\mathrm{Ad}_g X)_n \\ &= \int_M g \cdot \alpha(X)_n \\ &= \int_{L_{g^{-1}} M} \alpha(X)_n \\ &= \int_M \alpha(X)_n = (\int_M \alpha)(X). \end{aligned}$$

Thus, $(\int_M \alpha) \in \mathbb{C}[\mathfrak{g}]^G$.

(2) Let α be an equivariant exact form, i.e., $\alpha = d_{eq}\beta$. For $\beta(X) = \sum_i^n \beta_i(X)$ ($\beta_i(X) \in \Omega^i(M)$),

$$(d_{eq}(\beta))(X) = d\beta(X)_{n-1} + (d\beta(X)_{n-2} - \iota_X \beta(X)_n) + \cdots .$$

Hence, we have $\alpha(X)_n = (d_{eq}\beta)(X)_n = d\beta(X)_{n-1}$. It follows from this and Stokes theorem that

$$(\int_M \alpha)(X) = \int_M d\beta(X)_{n-1} = 0.$$

□

2.2 Equivariant Čech-de Rham cohomology

Let G be a compact Lie group and M a G-manifold. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be a G-invariant open covering of M (i.e. for any $g \in G,\ g\cdot U_\alpha = U_\alpha$). We assume that I is an ordered set such that if $U_{\alpha_0\cdots\alpha_r} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_r} \neq \emptyset$, the induced order on the subset $\{\alpha_0, ..., \alpha_r\}$ is total. We set

$$I^{(r)} = \{(\alpha_0, ..., \alpha_r) \mid \alpha_0 < \cdots < \alpha_r,\ \alpha_\nu \in I\}.$$

Definition 2.8. We define $C^p(\mathcal{U}, \Omega_G^q)$ to be the follori ng direct product:

$$C^p(\mathcal{U}, \Omega_G^q) := \prod_{(\alpha_0,...,\alpha_p)\in I^{(p)}} \Omega_G^q(U_{\alpha_0\cdots\alpha_p}).$$

An element $\sigma \in C^p(\mathcal{U}, \Omega_G^q)$ assigns to each $(\alpha_0, ..., \alpha_p) \in I^{(p)}$ a form $\sigma_{\alpha_0\ldots\alpha_p} \in \Omega_G^q(U_{\alpha_0\cdots\alpha_p})$. The coboundary operator

$$\delta : C^p(\mathcal{U}, \Omega_G^q) \to C^{p+1}(\mathcal{U}, \Omega_G^q)$$

is defined by

$$(\delta\sigma)_{\alpha_0\ldots\alpha_{p+1}} := \sum_{\nu=0}^{p+1} (-1)^\nu \sigma_{\alpha_0\ldots\widehat{\alpha_\nu}\ldots\alpha_{p+1}},$$

where $\widehat{}$ means the letter under it is to be omitted and each form $\sigma_{\alpha_0\ldots\widehat{\alpha_\nu}\ldots\alpha_{p+1}}$ is to be restricted to $U_{\alpha_0\cdots\alpha_{p+1}}$. Since a restriction map is G-equivariant, it is well-defined. This together with the G-equivariant operator

$$d_{eq} : C^p(\mathcal{U}, \Omega_G^q) \to C^p(\mathcal{U}, \Omega_G^{q+1})$$

makes $C^*(\mathcal{U}, \Omega_G^*)$ a double complex. The simple complex associated to this is denoted by $(\Omega_G^*(\mathcal{U}), D_{eq})$, i.e.

$$\Omega_G^r(\mathcal{U}) := \bigoplus_{p+q=r} C^p(\mathcal{U}, \Omega_G^q)$$

$$D_{eq} := \delta + (-1)^p d_{eq}, \text{on } C^p(\mathcal{U}, \Omega_G^q).$$

We call $(\Omega_G^*(\mathcal{U}), D_{eq})$ the equivariant Čech-de Rham complex and its r-th cohomology $H_G^r(\mathcal{U})$ the r-th equivariant Čech-de Rham cohomology of $\mathcal{U}$.

Lemma 2.9. Consider a natural homomorphism

$$r : \Omega^p_G(M) \to C^p(\mathcal{U}, \Omega^0_G) = \bigoplus_\alpha \Omega^p_G(U_\alpha)$$

which assigns to an $\omega \in \Omega_G(M)$ the cochain ξ given by $\xi_\alpha = \omega|_{U_\alpha}$. The following sequence is exact:

$$0 \longrightarrow \Omega_G(M) \xrightarrow{r} C^p(\mathcal{U}, \Omega^0_G) \xrightarrow{\delta} C^p(\mathcal{U}, \Omega^1_G) \xrightarrow{\delta} C^p(\mathcal{U}, \Omega^2_G) \xrightarrow{\delta} \cdots$$

Proof. Clearly r is injective. Since an element of $C^p(\mathcal{U}, \Omega^0_G)$ defines a global form on M if and only if its components agree on the overlaps, we see that $\mathrm{Im} r = \mathrm{Ker}\delta$. By Theorem 1.26, there exists an G-equivariant partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. Take $\beta \in C^p(\mathcal{U}, \Omega^q_G)$ for $q > 0$ such that $\delta\beta = 0$. Then, we define a $(p-1)$-cochain γ by

$$\gamma_{\alpha_0 \cdots \alpha_{p-1}} := \sum_\alpha \rho_\alpha \beta_{\alpha\alpha_0\cdots\alpha_{p-1}}.$$

Since $(\delta\beta)_{\alpha\alpha_0\cdots\alpha_p} = 0$, we have the following equality:

$$\sum_i (-1)^i \beta_{\alpha\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_p} = \beta_{\alpha_0\cdots\alpha_p}$$

Then,

$$\begin{aligned}
(\delta\gamma)_{\alpha_0\cdots\alpha_p} &= \sum_i (-1)^i \gamma_{\alpha_0\cdots\widehat{\alpha}_i\cdots\alpha_p} \\
&= \sum_{i,\alpha} \rho_\alpha \beta_{\alpha\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_p} \\
&= \sum_\alpha \rho_\alpha \sum_i (-1)^i \beta_{\alpha\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_p} \\
&= \beta_{\alpha_0\cdots\alpha_p}.
\end{aligned}$$

It follows from this and $\delta^2 = 0$ that the sequence is exact. $\square$

Theorem 2.10. The natural homomorphism $r : \Omega^r_G(M) \to C^0(\mathcal{U}, \Omega^r_G) \subset \Omega^r_G(\mathcal{U})$ induces an isomorphism:

$$r : H^r_G(M) \xrightarrow{\sim} H^r_G(\mathcal{U})$$

Proof. Since $\mathrm{Im} r = \mathrm{Ker}\delta$,

$$D_{eq} r = (\delta + d_{eq}) r = d_{eq} r = r d_{eq}.$$

Thus r induces a homomorphism $H^r_G(M) \to H^r_G(\mathcal{U})$. First we show that r is surjective. Suppose $\phi \in \Omega^r_G(\mathcal{U})$ and $D_{eq}\phi = 0$. Furthermore, we may assume that $\phi \in C^0(\mathcal{U}, \Omega^r_G(M))$. Indeed, decomposing ϕ as follows

$$\phi = \phi_0 + \phi_1 \cdots + \phi_r \in C^0(\mathcal{U}, \Omega^r_G(M)) \oplus C^1(\mathcal{U}, \Omega^r_G(M)) \oplus \cdots \oplus C^r(\mathcal{U}, \Omega^0_G(M)),$$

we easily see that $\delta\phi_r = 0$ from $D_{eq}\phi_r = 0$. Then, by Lemma 2.9, there exists $(r-1,0)$-form ψ such that $\delta\psi = \phi_r$. Therefore, $\phi - D_{eq}\psi$ is D_{eq}-closed and still stay in the same cohomology class as ϕ and has no $(r,0)$-component. After iterating this procedure enough times, we get $\phi' \in C^0(\mathcal{U}, \Omega^r_G(M))$ such that its cohomology class is equal to that of ϕ. Thus we assume that $\phi \in C^0(\mathcal{U}, \Omega^r_G(M))$ such that $D_{eq}\phi = 0$. Since

$$\begin{aligned} D_{eq}\phi = 0 \quad &\Longleftrightarrow \quad \delta\phi + d_{eq}\phi = 0 \\ &\Longleftrightarrow \quad \delta\phi = 0, d_{eq}\phi = 0 \end{aligned}$$

and $\mathrm{Im} r = \mathrm{Ker}\delta$, there exists ψ such that $r(\psi) = \phi$. Furthermore, since

$$r(d_{eq}\psi) = d_{eq}r(\psi) = d_{eq}\phi = 0$$

and r is injective, we have $d_{eq}\psi = 0$. Hence, r is surjective as a homomorphism of cohomology. Now we show that r is injective as a homomorphism of cohomology. Suppose $r(\psi) = D_{eq}\phi$ and $\phi \in C^0(\mathcal{U}, \Omega^{r-1}_G)$ as the same reason above. Since $r(\psi) \in C^0(\mathcal{U}, \Omega^r_G)$, we see that $\delta\phi = 0$ and $r(\psi) = d_{eq}\phi$. Then, we have $r(d_{eq}\psi) = d_{eq}r(\psi) = d^2_{eq}\phi = 0$. Thus, we get $d_{eq}\psi = 0$ and from this r is injective as a homomorphism of cohomology. □

Now we consider a *relative equivariant Čech-de Rham cohomology* and its product structure as a special case of equivariant Čech-de Rham cohomology (In fact, we may also give a product structure of a general equivariant Čech-de Rham cohomology [17, 19])

Let $\mathcal{U} = \{U_0, U_1\}$ be an open covering of M with $0 < 1$. Then we have

$$\Omega^r_G(\mathcal{U}) = \Omega^r_G(U_0) \oplus \Omega^r_G(U_1) \oplus \Omega^{r-1}_G(U_{01}).$$

The differential of an element $\xi = (\xi_0, \xi_1, \xi_{01}) \in \Omega^r_G(\mathcal{U})$ is given by

$$D_{eq}\xi = (d_{eq}\xi_0, d_{eq}\xi_1, \xi_1 - \xi_0 - d_{eq}\xi_{01}).$$

We define a product

$$\Omega^r_G(\mathcal{U}) \times \Omega^{r'}_G(\mathcal{U}) \longrightarrow \Omega^{r+r'}_G(\mathcal{U})$$

by for $\xi = (\xi_0, \xi_1, \xi_{01}) \in \Omega^r_G(\mathcal{U})$ and $\eta = (\eta_0, \eta_1, \eta_{01}) \in \Omega^{r'}_G(\mathcal{U})$,

$$(\xi_0, \xi_1, \xi_{01}) \smile (\eta_0, \eta_1, \eta_{01}) = (\xi_0 \wedge \eta_0, \xi_1 \wedge \eta_1, (-1)^r \xi_0 \wedge \eta_{01} + \xi_{01} \wedge \eta_1).$$

Directly computing, we see that D_{eq} is antiderivation, i.e.,

$$D_{eq}(\xi \smile \eta) = D_{eq}\xi \smile \eta + (-1)^r \xi \smile D_{eq}\eta.$$

Thus it induces a product of cohomology:

$$H^r_G(\mathcal{U}) \times H^{r'}_G(\mathcal{U}) \longrightarrow H^{r+r'}_G(\mathcal{U}).$$

Definition 2.11. In the above situation, we set

$$\Omega^p_G(\mathcal{U}, U_0) = \{\xi = (\xi_0, \xi_1, \xi_{01}) \in \Omega^p_G(\mathcal{U}) \mid \xi_0 = 0\}.$$

Since

$$D_{eq}(0, \xi_1, \xi_{01}) = (0, d_{eq}\xi_1, \xi_1 - d_{eq}\xi_{01}) \in \Omega^{p+1}_G(\mathcal{U}, U_0),$$

$(\Omega^*_G(\mathcal{U}, U_0), D_{eq})$ is a subcomplex of $(\Omega^*_G(\mathcal{U}), D_{eq})$. Then the p-th cohomology of $(\Omega^*_G(\mathcal{U}, U_0), D_{eq})$ is called the *p-th relative equivariant Čech-de Rham cohomology of* $(\mathcal{U}, U_0)$ and we denote it by $H^p_G(\mathcal{U}, U_0)$.

In the cup product $\Omega^r_G(\mathcal{U}) \times \Omega^{r'}_G(\mathcal{U}) \longrightarrow \Omega^{r+r'}_G(\mathcal{U})$, we have

$$(\xi_0, \xi_1, \xi_{01}) \smile (\eta_0, \eta_1, \eta_{01}) = (\xi_0 \wedge \eta_0, \xi_1 \wedge \eta_1, (-1)^r \xi_0 \wedge \eta_{01} + \xi_{01} \wedge \eta_1).$$

If $\xi_0 = 0$, the right hand side depends only on ξ_1, ξ_{01} and η_1. Thus we have a following paring $\Omega^r_G(\mathcal{U}, U_0) \times \Omega^{r'}_G(U_1) \to \Omega^{r+r'}_G(\mathcal{U}, U_0)$;

$$(0, \xi_1, \xi_{01}) \smile \eta_1 = (0, \xi_1 \wedge \eta_1, \xi_{01} \wedge \eta_1).$$

2.3 Equivariant Fiber Integration

Before considering an equivariant case, we define the fiber integration of a usual differential form. General reference are e.g.,[11, 17, 19].

Let M be an oriented and compact smooth manifold of dimension m and F a compact oriented smooth manifold of dimension l possibly with boundary ∂F. Also, let $\pi : T \to M$ be an oriented smooth fiber bundle with fiber F.

Proposition 2.12. In the above situation, there exists a $\mathbb{C}$-linear map

$$\pi_* : \Omega^p(T) \to \Omega^{p-l}(M)$$

which has the following property and π_* is uniquely determined by this property.

$$\int_T \omega \wedge \pi^*\theta = \int_M \pi_*\omega \wedge \theta$$

for $\omega \in \Omega^p(T)$ and $\theta \in \Omega^{m+l-p}(M)$.

Proof. See [17, 19]. □

Definition 2.13. $\pi_* : \Omega^p(T) \to \Omega^{p-l}(M)$, defined above, is called the *fiber integration*.

Since the following lemma is easy to prove, we omit the proof.

Lemma 2.14. Let M be a oriented compact smooth manifold of dimension m. If $\omega \in \Omega^p(M)$ satisfies the following condition

$$\int_M \omega \wedge \nu = 0, \forall \nu \in \Omega^{m-p}(M),$$

then $\omega = 0$.

We consider the following fundamental formulae with respect to the fiber integration.

Proposition 2.15. Let M be a smooth compact oriented manifold of dimension m without boundary and $\pi : T \to M$ an oriented fiber bundle with fiber a compact oriented manifold F of dimension l possible with boundary. Then, we have the following equalities:

(1) For $\omega \in \Omega^p(T)$ and $\theta \in \Omega^q(M)$,

$$\pi_*(\omega \wedge \pi^*\theta) = \pi_*\omega \wedge \theta.$$

(2) Let ∂T be a boundary of T and $i : \partial T \hookrightarrow T$ be the inclusion. Then we have

$$\pi_* \circ d + (-1)^{l+1} d \circ \pi_* = (\partial\pi)_* \circ i^*,$$

where $\partial\pi = \pi \circ i$.

Proof. (1) For any $\alpha \in \Omega^{m+l-p-q}(M)$, we have

$$\begin{aligned} \int_M \pi_*(\omega \wedge \pi^*\theta) \wedge \alpha &= \int_T (\omega \wedge \pi^*\theta) \wedge \pi^*\alpha \\ &= \int_T \omega \wedge \pi^*(\theta \wedge \alpha) \\ &= \int_M \pi_*\omega \wedge (\theta \wedge \alpha) \\ &= \int_M (\pi_*\omega \wedge \theta) \wedge \alpha. \end{aligned}$$

Therefore, by Lemma 2.14, we have $\pi_*(\omega \wedge \pi^*\theta) = \pi_*\omega \wedge \theta$.
(2) For $\omega \in \Omega^p(T)$ and $\alpha \in \Omega^{m+l-p-1}(M)$, by using Stokes theorem, we have

$$\begin{aligned} \int_M (\pi_* d\omega + (-1)^{l+1} d\pi_*\omega) \wedge \alpha &= \int_M \pi_*(d\omega) \wedge \alpha \\ &\quad + (-1)^{l+1} \int_M \{d(\pi_*\omega \wedge \alpha) - (-1)^{p-l}\pi_*\omega \wedge d\alpha\} \\ &= \int_M \pi_*(d\omega) \wedge \alpha + (-1)^p \int_M \pi_*(\omega) \wedge d\alpha \\ &= \int_T d\omega \wedge \pi^*\alpha + (-1)^p \int_T \omega \wedge \pi^*(d\alpha) \\ &= \int_T d(\omega \wedge \pi^*\alpha) \\ &= \int_{\partial T} i^*\omega \wedge (\partial\pi)^*\alpha \\ &= \int_M (\partial\pi)_* i^*\omega \wedge \alpha. \end{aligned}$$

Thus, we have $\pi_* \circ d + (-1)^{l+1} d \circ \pi_* = (\partial\pi)_* \circ i^*$.

$\square$

Proposition 2.16. In the above situation, we obtain the following properties:

(1) Let $\phi : T \to T$ and $\psi : M \to M$ be a orientation-preserving diffeomorphism. If ϕ and ψ is π-related in the sense that $\pi \circ \phi = \psi \circ \pi$, then we have

$$\pi_* \circ \phi^* = \psi^* \circ \pi_*.$$

(2) If $Y \in \mathfrak{X}(T)$ and $X \in \mathfrak{X}(M)$ are π-related in the sense that $d\pi(Y) = X$, then we have

$$\pi_* \circ \iota(Y) = (-1)^l \iota(X) \circ \pi_*,$$

where ι is the contraction.

Proof. (1) Note that it follows that since ϕ and ψ is π-related, $\phi^* \circ \pi^* = \pi^* \circ \psi$. For $\omega \in \Omega^p(T), \alpha \in \Omega^{m-p+l}(M)$,

$$\begin{aligned}
\int_M \phi^*(\pi_*\omega \wedge \alpha) &= \int_M \pi_*\omega \wedge \alpha \\
&= \int_T \omega \wedge \pi^*\alpha \\
&= \int_T \phi^*(\omega \wedge \pi^*\alpha) \\
&= \int_T \phi^*\omega \wedge \phi^*\pi^*\alpha \\
&= \int_T \phi^*\omega \wedge \pi^*\psi^*\alpha \\
&= \int_M \pi_*\phi^*\omega \wedge \psi^*\alpha
\end{aligned}$$

Also, since $\int_M \psi^*(\pi_*\omega \wedge \alpha) = \int_M \psi^*\pi_*\omega \wedge \psi^*\alpha$, by Lemma **??**, we obtain $\pi_* \circ \phi^* = \psi^* \circ \pi_*$
(2)First, note that $\iota(Y) \circ \pi^* = \pi^* \circ \iota(X)$. Indeed, for $\omega \in \Omega^p(M), Y_2, \cdots Y_p \in \mathfrak{X}(T)$,

$$\begin{aligned}
(\iota(Y)\pi^*\omega)(Y_2, \cdots, Y_p) &= \pi^*\omega(Y, Y_1, \cdots, Y_p) \\
&= \omega(d\pi(Y), d\pi(Y_2), \cdots, d\pi(Y_p)) \\
&= \omega(X, d\pi(Y_2), \cdots, d\pi(Y_p)) \\
&= (\iota(X)\omega)(d\pi(Y_2), \cdots, d\pi(Y_p)) \\
&= (\pi^*\iota_X\omega)(Y_2, \cdots, Y_p).
\end{aligned}$$

Hence, we have $\iota(Y) \circ \pi^* = \pi^* \circ \iota(X)$. Now, take $\omega \in \Omega^p(T)$. For any $\theta \in \Omega^{m+l-p}(M)$, since $\omega \wedge \pi^*\theta = 0$,

$$0 = \iota(Y)(\omega \wedge \pi^*\theta) = \iota(Y)\omega \wedge \pi^*\theta + (-1)^{p+1}\omega \wedge \iota(Y)\pi^*\theta$$

$$\Longrightarrow \iota(Y)\omega \wedge \pi^*\theta = (-1)^p\omega \wedge \iota(Y)\pi^*\theta.$$

In the same fashion, we see also that $\iota(X)\pi_*\omega \wedge \theta = (-1)^{p-l}\pi_*\omega \wedge \iota(X)\theta$. By using these equalities, we have

$$\begin{aligned}
\int_M \pi_*(\iota(Y)\omega) \wedge \theta &= \int_T \iota(Y)\omega \wedge \pi^*\theta \\
&= (-1)^p \int_T \omega \wedge \iota(Y)\pi^*\theta \\
&= (-1)^p \int_T \omega \wedge \pi^*\iota(X)\theta \\
&= (-1)^p \int_M \pi_*\omega \wedge \iota(X)\theta \\
&= (-1)^l \int_M \iota(X)\pi_*\omega \wedge \theta.
\end{aligned}$$

It follow from Lemma 2.13 that $\pi_* \circ \iota(Y) = (-1)^l \iota(X) \circ \pi_*$. □

Based on the above discussion, we shall explain the equivariant fiber integration. Let G be a Lie group.

Definition 2.17. $\pi : T \to M$ is called an *equivariant fiber bundle*, if $\pi : T \to M$ is a fiber bundle and G acts on T by bundle maps (In other words, T,M are G-manifolds and $\pi : T \to M$ is G-morphism).

Definition 2.18. Suppose that M is an oriented compact manifold and $\pi : T \to M$ is an equivariant oriented fiber bundle with fiber F of dimension l, where F is compact oriented possibly with boundary. We define the equivariant fiber integration π_* as follows:

$$(\pi_*\alpha)(X) := \pi_*(\alpha(X)), \quad \text{for } \alpha \in \Omega_G^*(T), X \in \mathfrak{g},$$

where π_* on the right hand side is the usual fiber integration.

Lemma 2.19. Suppose G acts on T and M preserving the orientations. Then for any $\alpha \in \Omega_G^p(T)$, $\pi_*\alpha \in \Omega_G^{p-l}(M)$. Namely, we get a homomorphism

$$\pi_* : \Omega_G^p(T) \to \Omega_G^{p-l}(M).$$

Proof. For simplicity, we use the same notation L_g of a left transformation to M and T. Since $\alpha(\mathrm{Ad}_g(X)) = L_{g^{-1}}^*\alpha(X)$ for $X \in \mathfrak{g}$ and a left transformation $L_{g^{-1}}$ preserve the orientation of T and M, we have

$$\begin{aligned}
(\pi_*\alpha)(\mathrm{Ad}_g X) &= \pi_*(\alpha(\mathrm{Ad}_g X)) \\
&= \pi_*(L_{g^{-1}}^*\alpha(X)) \\
&= L_{g^{-1}}^*(\pi_*\alpha(X)).
\end{aligned}$$

□

In what follows, we always assume that G acts on a fiber bundle preserving orientation. The equivariant fiber integration has the same properties as Proposition2.22.

Proposition 2.20. Let M be a smooth compact oriented manifold of dimension m without boundary and $\pi : T \to M$ an G-equivariant oriented fiber bundle with fiber a compact oriented manifold F of dimension l possible with boundary.

(1) For $\alpha \in \Omega_G^p(T)$ and $\beta \in \Omega_G^q(M)$,

$$\pi_*(\alpha \wedge \pi^*\beta) = \pi_*\alpha \wedge \beta.$$

(2) Let ∂T be a boundary of T and $i : \partial T \hookrightarrow T$ be the inclusion. Then we have

$$\pi_* \circ d_{eq} + (-1)^{l+1} d_{eq} \circ \pi_* = (\partial\pi)_* \circ i^*.$$

Proof. (1) For $\alpha \in \Omega_G^p(T)$, $\beta \in \Omega_G^q(M)$ and $X \in \mathfrak{g}$,

$$\begin{aligned}
\pi_*(\alpha \wedge \pi^*\beta)(X) &= \pi_*(\alpha(X) \wedge \pi^*\beta(X)) \\
&= \pi^*\alpha(X) \wedge \pi^*\beta(X) \\
&= \pi_*(\alpha \wedge \pi^*\beta)(X).
\end{aligned}$$

Thus, we have $\pi_*(\alpha \wedge \pi^*\beta) = \pi_*\alpha \wedge \beta$.
(2)First, for $X \in \mathfrak{g}$, we show that X_M and X_T is π-related. For $y \in T$,

$$\begin{aligned}
d\pi_y(X_T)_y &= \left.\frac{d}{dt}\right|_{t=0} \pi(\exp(-tX) \cdot y) \\
&= \left.\frac{d}{dt}\right|_{t=0} \exp(-tX) \cdot \pi(y)) \\
&= (X_M)_{\pi(y)}.
\end{aligned}$$

Hence, X_M and X_T is π-related. From this and Proposition2.16(2), we see that for $\alpha \in \Omega_G^p(T)$ and $X \in \mathfrak{g}$,

$$\begin{aligned}
&(\pi_* d_{eq}\alpha)(X) + (-1)^{l+1}(d_{eq}\pi_*\alpha)(X) \\
&\quad = \pi_*(d\alpha(X) - \iota_X\alpha(X)) + (-1)^{l+1}(d\pi_*\alpha(X) - \iota_X\pi_*\alpha(X)) \\
&\quad = (\pi_* d\alpha(X) + (-1)^{l+1} d\pi_*\alpha(X)) - (\pi_*\iota_X\alpha(X) + (-1)^{l+1}\iota_X\pi_*\alpha(X)) \\
&\quad = \pi_* d\alpha(X) + (-1)^{l+1} d\pi_*\alpha(X) \\
&\quad = (\partial\pi)_* i^*\alpha(X).
\end{aligned}$$

Thus, we obtain $\pi_* \circ d_{eq} + (-1)^{l+1} d_{eq} \circ \pi_* = (\partial\pi)_* \circ i^*$. □

2.4 Equivariant Thom Form

In this section, we shall define the equivariant Thom form as an element of relative equivariant Čech-de Rham complex defined in Section 2.2.

To formulate the equivariant Thom form, we introduce the G-equivariant fiber integration on relative equivariant Čech-de Rham cochains. Let G be a compact Lie group and M a smooth manifold of dimension m and $\pi : E \to M$ a G-equivariant oriented vector bundle of rank l. We identify M with the image of the zero section of E. Setting $W_0 = E \setminus M$ and $W_1 = E$, $\mathcal{W} = \{W_0, W_1\}$ is an open covering of E. Let $T_1 \to M$ be a closed unit ball bundle in W_1 with respect to some G-invariant Riemannian metric on E and set $T_0 = E \setminus \mathrm{Int}T_1$. Then $\{T_0, T_1\}$ is honeycomb system adapted to $\mathcal{W}$ (for details, see [17, 19]). Let π_1 and π_{01} denote the restriction of π to T_1 and T_{01} respectively. Thus,

- $\pi_1 : T_1 \to M$ is a G-equivariant closed l-unit ball bundle
- $\pi_{01} : T_{01} \to M$ is a G-equivariant $(l-1)$-sphere bundle.

By the definition of honeycomb system, the orientation of T_{01} is opposite to that of the boundary ∂T_1 of T_1

Definition 2.21. The G-equivariant fiber integration on relative equivariant Čech-de Rham cochains

$$\pi_* : \Omega^p_G(\mathcal{W}, W_0) \to \Omega^{p-l}_G(M)$$

is defined by

$$\pi_*\alpha := (\pi_1)_*\alpha_1 + (\pi_{01})_*\alpha_{01},$$

where $\alpha = (0, \alpha_1, \alpha_{01}) \in \Omega^p_G(\mathcal{W}, W_0)$.

Proposition 2.22. For $\alpha \in \Omega^p_G(\mathcal{W}, W_0), \beta \in \Omega^q_G(W_1)$, we have

$$\pi_*(\alpha \smile \pi^*\beta) = \pi_*\alpha \wedge \beta.$$

Proof. Take $\alpha = (0, \alpha_1, \alpha_{01}) \in \Omega^p_G(\mathcal{W}, W_0)$. By using Proposition 2.22,

$$\begin{aligned} \pi_*(\alpha \smile \pi^*\beta) &= \pi_*(0, \alpha_1 \wedge \pi^*\beta, \alpha_{01} \wedge \pi^*\beta) \\ &= (\pi_1)_*(\alpha_1 \wedge \pi^*\beta) + (\pi_{01})_*(\alpha_{01} \wedge \pi^*\beta) \\ &= (\pi_1)_*\alpha_1 \wedge \beta + (\pi_{01})_*\alpha_{01} \wedge \beta \\ &= \pi_*\alpha \wedge \beta \end{aligned}$$

□

Proposition 2.23. In the above situation, we obtain the following formula

$$\pi_* \circ D_{eq} + (-1)^{l+1} d_{eq} \circ \pi_* = 0$$

Thus, $\pi_* : \Omega^p_G(\mathcal{W}, W_0) \to \Omega^{p-l}_G(M)$ induces a homomorphism $\pi_* : H^p_G(\mathcal{W}, W_0) \to H^{p-l}_G(M)$.

Proof. Noting that for $\alpha = (0, \alpha_1, \alpha_{01})$,$(\partial\pi_1)_* = -(\pi_{01})_*$, we have the following equalities:

- $(\pi_1)_* d_{eq}\alpha_1 + (-1)^{l+1} d_{eq}(\pi_1)_*\alpha_1 = (\partial\pi_1)_* i^*\alpha_1$
- $-((\pi_{01})_* d_{eq}\alpha_{01} + (-1)^l d_{eq}(\pi_{01})_*\alpha_{01}) = 0$
- $(\pi_{01})_* i^*\alpha_1 = -(\partial\pi_1) i^*\alpha_1$,

where $i : \partial T_1 \hookrightarrow T_1$ is the inclusion. Using these and directly computing, we get $\pi_* \circ D_{eq} + (-1)^{l+1} d_{eq} \circ \pi_* = 0$. □

The same Mayer-Vietoris argument in non-equivariant case (Theorem 5.3 in [17]) shows that $\pi_* : H^p_G(\mathcal{W}, W_0) \to H^{p-l}_G(M)$ is isomorphism. Then, there exists the inverse map

$$(\pi_*)^{-1} : H^{p-l}_G(M) \xrightarrow{\sim} H^p_G(\mathcal{W}, W_0)$$

and we denote it by T_E and call it the *G-equivariant Thom isomorphism.* Then, setting

$$\psi^E_{eq} := T_E([1]) \in H^l_G(\mathcal{W}, W_0) \quad (1 \in \Omega^0_G(M)),$$

we call it G-equivariant Thom class. It follows from Proposition 2.22 that

$$\pi_*(T_E([1]) \smile \pi^*\beta) = \pi_* T_E([1]) \wedge \beta = \beta$$

$$\Longrightarrow T_E(\beta) = \psi^E_{eq} \smile \pi^*\beta$$

Under the assumption above, we have the following theorem.

Theorem 2.24. The equivariant Thom class $\psi^E_{eq} \in H^l_G(\mathcal{W}, W_0)$ is represented by the following form

$$(0, \pi^*\varepsilon_{eq}, -\psi_{eq}) \in \Omega^l_G(\mathcal{W}, W_0),$$

where ε_{eq} is d_{eq}-closed G-equivariant l-form on M and ψ_{eq} is G-equivariant $(l-1)$-form on W_{01} such that

$$d_{eq}\psi_{eq} = -\pi^*\varepsilon_{eq} \text{ in } W_{01} \text{ and } -(\pi_{01})_*\psi_{eq} = 1$$

Proof. Suppose that $\psi_E = [(0, \psi_1, \psi_{01})] \in H^l_G(\mathcal{W}, W_0)$. Since

$$D_{eq}(0, \psi_1, \psi_{01}) = (0, d_{eq}\psi_1, \psi_1 - d_{eq}\psi_{01}) = 0,$$

we see that ψ_1 is closed l-form on W_1 and ψ_{01} is $(l-1)$-form such that $d_{eq}\psi_{01} = \psi_1$ on W_{01}. Note that $\pi : E = W_1 \to M$ induces an isomorphism equivariant de Rham cohomology, because π is G-equivariant deformation retract. Thus, there exists $\phi_1 \in \Omega^{l-1}_G(W_1)$ and $\varepsilon_{eq} \in \Omega^l_G(M)$ such that

$$\psi_1 = \psi^*\varepsilon_{eq} + d_{eq}\phi_1$$

Here, setting $\psi_{eq} := -\psi_{01} + \phi_1$ which is $(l-1)$-form on W_{01}, we see that

$$(0, \psi_1, \psi_{01}) = (0, \pi^*\varepsilon_{eq}, -\psi_{eq}) + D_{eq}(0, \phi_1, 0).$$

Thus, ψ_E is represented by $(0, \pi^*\varepsilon_{eq}, -\psi_{eq})$. Then we have

$$d_{eq}\psi_{eq} = -d_{eq}\psi_{01} + d\phi_1 = -\psi_1 + (\psi_1 - \pi^*\varepsilon_{eq}) = -\pi^*\varepsilon.$$

Moreover,we have

$$\begin{aligned}
(\pi_1)_*\psi_1 &= (\pi_1)_*\pi^*\varepsilon_{eq} + (pi_1)_* d_{eq}\phi_1 \\
&= (\pi_1)_* d_{eq}\phi_1 = (\partial\pi_1)_*\phi_1 + (-1)^l d_{eq}(\pi_1)_*\phi_1 \\
&= -(\pi_{01})_*\phi_1.
\end{aligned}$$

From this and $\pi_*(0, \psi_1, \psi_{01}) = 1$, it follow that

$$\begin{aligned}
(\pi_1)_*\psi_1 + (\pi_{01})_*\psi_{01} = 1 \quad &\Longleftrightarrow \quad (\pi_1)_*\psi_1 + (\pi_{01})_*\phi_1 - (\pi_{01})_*\psi_{eq} = 1 \\
&\Longleftrightarrow \quad -(\pi_{01})_*\psi_{eq} = 1.
\end{aligned}$$

□

3 Equivariant Chern-Weil theory

In this chapter, we will discuss equivariant Chern-Weil theory. In the process, we introduce the equivariant Bott's difference form. By using this, we may formulate the localized equivariant characteristic classes defined in the next chapter. General reference are e.g.,[2, 14].

3.1 Equivariant connection

We start by defining a usual connection of a vector bundle. Let $\pi : E \to M$ is a complex vector space and
$$\Omega^p(M, E) := \Gamma(M, \wedge^p T^*M \otimes E)$$
be the vector space of E-valued differential forms on M (in other words, the vector bundle of global sections of $\wedge^p T^*M \otimes E$ on M). Note that $\Omega^p(M, E)$ is the $C^\infty(M)$-module.

Definition 3.1.

(1) A *connection* for E is a $\mathbb{C}$-linear map
$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$$
satisfying the following Leibniz's rule: that is, if $s \in \Omega^0(M, E)$ and $f \in C^\infty(M)$, then
$$\nabla(fs) = df \otimes s + f\nabla s.$$

(2) Inductively, a connection extends in a unique way to a map
$$\nabla : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$$
satisfying the Leibniz's rule: if $\omega \in \Omega^k(M)$ and $\alpha \in \Omega^*(M, E)$, then
$$\nabla(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^k \omega \wedge \alpha.$$

Now let G be a compact Lie group and $\pi : E \to M$ be a complex G-equivariant vector bundle. We define the vector space of E-valued G-equivariant differential forms by
$$\Omega^*_G(M, E) := (\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M, E))^G,$$
where G acts on the section of E such that for $g \in G$ and $s \in \Omega^0(M, E)$
$$(g \cdot s)(m) := g \cdot s(g^{-1} \cdot m).$$

Definition 3.2. $\nabla : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$ is called a *G-invariant connection*, if ∇ commutes with G-action on $\Omega^*(M, E)$; that is, $g\nabla = \nabla g$.

$$\begin{array}{ccc}
\Omega^*(M,E) & \xrightarrow{\nabla} & \Omega^{*+1}(M,E) \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
\Omega^*(M,E) & \xrightarrow{\nabla} & \Omega^{*+1}(M,E)
\end{array}$$

Remark. We may show the existence of a G-invariant connection as follows. Let ∇ be a possibly non-invariant connection (its existence is easy to prove by using a partition of unity). Since G is compact, left and right invariant Haar measure dg exists. Setting

$$\nabla^G := \int_G (g^{-1} \cdot \nabla \cdot g) dg,$$

we see that it is a G-invariant connection.

Definition 3.3. The *equivariant connection* ∇_{eq} corresponding to a G-invariant connection ∇ is the operator on $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M, E)$ defined as follows; for $X \in \mathfrak{g}$, $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M, E)$,

$$(\nabla_{eq}\alpha)(X) := (\nabla - \iota_X)\alpha(X)$$

Lemma 3.4. Suppose that $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$ and $\beta \in \mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M, E)$. Then, we have the following properties:

(1) $\nabla_{eq}(\alpha \wedge \beta) = d_{eq}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \nabla_{eq}\beta$.

(2) If $\beta \in \Omega^*_G(M, E)$, then $\nabla_{eq}\beta \in \Omega^*_G(M, E)$.

Proof. (1) Note that for $X \in \mathfrak{g}$,

$$\nabla(\alpha(X) \wedge \beta(X)) = d\alpha(X) \wedge \beta(X) + (-1)^{\deg \alpha} \alpha(X) \wedge \nabla\beta(X)$$

$$\iota_X(\alpha(X) \wedge \beta(X)) = \iota_X\alpha(X) \wedge \beta(X) + (-1)^{\deg \alpha} \alpha(X) \wedge \iota_X\beta(X).$$

Thus, we have $\nabla_{eq}(\alpha \wedge \beta) = d_{eq}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \nabla_{eq}\beta$.
(2) Since $\alpha(\mathrm{Ad}_g X) = g \cdot \alpha(X)$ and $g \cdot \nabla = \nabla \cdot g$,

$$\begin{aligned}
(\nabla_{eq}\alpha)(\mathrm{Ad}_g X) &= (\nabla - \iota_{\mathrm{Ad}_g X})\alpha(\mathrm{Ad}_g X) \\
&= (\nabla - g \cdot \iota_X \cdot g^{-1}) g \cdot \alpha(X) \\
&= g \cdot (\nabla\alpha(X) - \iota_X\alpha(X)) \\
&= g \cdot (\nabla_{eq}\alpha)(X).
\end{aligned}$$

□

Lemma 3.5. A equivariant connection ∇_{eq} is a local operator, that is, a section $s \in \Omega^0_G(M, E)$ is identically 0 on a G-invariant open set U, then so is $\nabla_{eq}(s) = \nabla(s) = 0$ on U.

Proof. Take $m \in U$ and let U_m be a G-invariant neighborhood of $G \cdot m$ in U. Using Theorem 1.26, we may take a G-invariant partition of unity $\{\rho, \rho'\}$ subordinate to a covering $\{U, M \setminus \overline{U_m}\}$. Since $\rho s \equiv 0$ on M, so is $\nabla_{eq}(\rho s)(X) = d\rho \otimes s + \rho \nabla s$. Evaluating the both sides at m, we have $\nabla_{eq} s(m) = 0$. □

In the same fashion of Section 1, the G-action on E and $\Omega^*_G(M, E)$ induces infinitesimal actions of $\mathfrak{g}$ which we denote by L^E_X.

Definition 3.6. The equivariant curvature $K_{eq} : \Omega^0_G(M,E) \to \Omega^2_G(M,E)$ of an equivariant connection ∇_{eq} is defined by the formula:

$$K_{eq}(X) = \nabla_{eq}(X)^2 + L^E_X.$$

Lemma 3.7. For $f \in C^\infty(M)$ and $s \in \Omega^0_G(M,E)$, we have

$$K_{eq}(fs) = fK_{eq}(s).$$

Thus K_{eq} is an element of $\Omega^2_G(M, End(E))$.

Proof. For $X \in \mathfrak{g}$, we have

$$\begin{aligned} K_{eq}(s)(X) &= (\nabla^2_{eq}(s))(X) + L^E_X(s) \\ &= (\nabla_{eq}\nabla s)(X) + L^E_X(s) \\ &= \nabla^2 s - \iota_X \nabla s + L^E_X(s). \end{aligned}$$

Thus, by using $\nabla^2(fs) = f\nabla s$,

$$\begin{aligned} \Theta(fs)(X) &= \nabla^2(fs) + \iota_X\nabla(fs) + L^E_X(fs) \\ &= f\nabla^2 s - \iota_X(df \otimes s + f\nabla s) + (L_X f)s + fL^E_X s \\ &= f\nabla^2 s - (X_M f)s - f\iota_X\nabla s + (X_M f)s + fL^E_X s \\ &= f\nabla^2 s - f\iota_X \nabla s + fL^E_X s \\ &= f(K_{eq}s)(X). \end{aligned}$$

Therefore, we obtain $K_{eq}(fs) = fK_{eq}(s)$. □

3.2 Equivariant Characteristic Classes

We give some notations, before defining an equivariant characteristic classes.

Definition 3.8. A polynomial ϕ on $M(l,\mathbb{C})$ is called invariant if, for all P in $GL(l,\mathbb{C})$,

$$\phi(P^{-1}AP) = \phi(A).$$

(that is, $\phi \in \mathbb{C}[\mathfrak{gl}(l,\mathbb{C})]^{GL(l,\mathbb{C})}$)

It is known that for an invariant polynomial ϕ, homogeneous of degree k, there exists k-linear form

$$\tilde{\phi} : \underbrace{M(l,\mathbb{C}) \times \cdots \times M(l,\mathbb{C})}_{k} \to \mathbb{C},$$

called the polarization of ϕ such that $\phi(A) = \tilde{\phi}(A, ..., A)$. The invariance of ϕ implies that

$$\tilde{\phi}(P^{-1}A_1P, ..., P^{-1}A_kP) = \tilde{\phi}(A_1, ..., A_k) \quad \text{for all } P \in GL(l,\mathbb{C}).$$

Remark. The polarization of a polynomial ϕ homogeneous of degree k is given by

$$\tilde{\phi}(A_1, \dots, A_k) = \frac{1}{k!} \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k} \phi(\lambda_1 A_1 + \cdots + \lambda_k A_k)$$

Note that the right hand side does not depend on the λ_i's and is equal to the coefficient of $\lambda_1 \cdots \lambda_k$ in the expansion of $\phi(\lambda_1 A_1 + \cdots + \lambda_k A_k)$ divided by $k!$.

Lemma 3.9. For $B \in M(l, \mathbb{C})$ and k-linear form $\tilde{\phi}$, we have

$$\sum_{i=1}^{k} \tilde{\phi}(A_1, ..., [A_i, B], ..., A_k) = 0,$$

where $[\cdot, \cdot]$ is the usual commutator of matrices.

Proof. Since $\tilde{\phi}(P^{-1}A_1P, ..., P^{-1}A_kP) = \tilde{\phi}(A_1, ..., A_k)$, letting $P = e^{tB}$, and taking the derivative at $t = 0$ of the both sides,

$$\begin{aligned} \frac{d}{dt}\tilde{\phi}(P^{-1}A_1P, ..., P^{-1}A_kP)|_{t=0} &= \sum_{i=1}^{k} \tilde{\phi}(P^{-1}A_1P, ..., \frac{d}{dt}(P^{-1}A_iP), ..., P^{-1}A_kP)|_{t=0} \\ &= \sum_{i=1}^{k} \tilde{\phi}(A_1, ..., [A_i, B]..., A_k) \\ &= 0. \end{aligned}$$

□

Let U be an open set of M. Now we extend k-linear form $\tilde{\phi}$ on $M(l, \mathbb{C})$ to an invariant k-linear form on $M(l, \mathbb{C}) \otimes \Omega^*(U)$ as follows; we set

$$\tilde{\phi}(A_1 \otimes \omega_1, ..., A_k \otimes \omega_k) := \tilde{\phi}(A_1, ..., A_k)\omega_1 \wedge \cdots \wedge \omega_k$$

and then extend it linearly. Using this, we may extend an invariant polynomial ϕ to an invariant polynomial function on $M(l, \mathbb{C}) \otimes \Omega^*(U)$ by

$$\phi(\xi) = \tilde{\phi}(\xi, ..., \xi) \quad \text{for } \xi \in M(l, \mathbb{C}) \otimes \Omega^*(U).$$

The product of $M(l, \mathbb{C}) \otimes \Omega^*(U)$ is defined as follows; for decomposable elements, we set

$$(A_1 \otimes \omega_1) \wedge (A_2 \otimes \omega_2) := (A_1A_2) \otimes (\omega_1 \wedge \omega_2)$$

and extend it linearly. Also, we may define the bracket of $\xi \in M(l, \mathbb{C}) \otimes \Omega^p(U)$ and $\eta \in M(l, \mathbb{C}) \otimes \Omega^q(U)$ by

$$[\xi, \eta] = \xi \wedge \eta - (-1)^{pq}\eta \wedge \xi$$

In the following, we define an equivariant characteristic form and class. Let G be a compact Lie group and $\pi : E \to M$ be a G-equivariant vector bundle of rank l.

Furthermore, let ∇ be G-invariant connection and ∇_{eq} be its equivariant connection and K_{eq} be its equivariant curvature as defined in Section 3.1.

Take a G-invariant open set U in M such that E is trivial on U. If $s^{(l)} = (s_1, ..., s_l)$ is a local frame of E on U, we may write, for $i = 1, ..., l$ and $X \in \mathfrak{g}$,

$$\begin{aligned}(\nabla_{eq} s_i)(X) &= \sum_{j=1}^{l} \theta_{ji} \otimes s_j, \quad \theta_{ij} \in \Omega^1(U), \\ (K_{eq} s_i)(X) &= \sum_{j=1}^{l} \kappa_{ji}(X) \otimes s_j, \quad \kappa_{ij} \in (\mathbb{C}[\mathfrak{g}] \otimes \Omega^0(U)) \oplus \Omega^2(U).\end{aligned}$$

We call $\theta = (\theta_{ij})$ the connection matrix and $\kappa = (\kappa_{ij})$ the equivariant curvature with respect to $s^{(l)}$.

Lemma 3.10 (equivariant Bianchi identity)**.** In the above situation, the connection form θ and the equivariant curvature form κ satisfy the following identity:

$$d_{eq}\kappa = [\kappa, \theta].$$

Proof. Note that $[\kappa, \theta] = \kappa \wedge \theta - \theta \wedge \kappa$, since κ is of even differential degree. From the definition, κ_{ij} is computed explicitly as follows. Letting $L_X^E s_i = \sum_{j=1}^{l} \ell_{ji}(X) s_j$, we have

$$\begin{aligned}(K_{eq} s_i)(X) &= \nabla^2 s_i - \iota_X \nabla s_i + L_X^E s_i \\ &= \sum_{j=1}^{l} \{ d\theta_{ji} + \sum_{k=1}^{l} \theta_{jk} \wedge \theta_{ki} - \iota_X \theta_{ji} + \ell_{ji}(X) \} \otimes s_j \\ \Longrightarrow \kappa_{ij}(X) &= d\theta_{ij} + \sum_{k=1}^{l} \theta_{ik} \wedge \theta_{kj} - \iota_X \theta_{ij} + \ell_{ji}(X).\end{aligned}$$

Computing $d\kappa_{ij}$ and $\iota_X \theta_{ij}$ respectively,

$$\begin{aligned}d\kappa_{ij}(X) &= \sum_{k=1}^{l} (d\theta_{ik} \wedge \theta_{kj} - \theta_{ik} \wedge d\theta_{kj}) + d\iota_X(\theta_{ij}) + d\ell_{ji}(X), \\ \iota_X \theta_{ij} &= \iota_X(d\theta_{ij}) + \sum_{k=1}^{l} (\iota_X(\theta_{ik})\theta_{kj} - \iota_X(\theta_{kj})\theta_{ik}).\end{aligned}$$

Notice that $-L_X \theta_{ji} + d\ell(X)_{ji} = \sum_{k=1}^{l} (\ell_{jk}(X)\theta_{ki} - \theta_{jk}\ell_{ki}(X))$. Indeed, since $g \cdot \nabla = \nabla \cdot g$, we have

$$L_X^E \nabla s_i = \nabla L_X^E s_i.$$

Computing the left and right hand sides,

$$\begin{aligned} L_X^E \nabla s_i &= L_X^E(\sum_{j=1}^{l} \theta_{ji} \otimes s_j) \\ &= \sum_{j=1}^{l}(L_X\theta_{ji} \otimes s_j + \theta_{ji} L_X^E s_j) \\ &= \sum_{j=1}^{l} \left\{ (L_X\theta_{ji} \otimes s_j) + \theta_{ji} \otimes (\sum_{k=1}^{l} \ell_{kj}(X) s_k) \right\} \\ &= \sum_{j=1}^{l} \left\{ L_X\theta_{ji} + \sum_{k=1}^{l} \ell_{jk}(X)\theta_{ki} \right\} \otimes s_j, \end{aligned}$$

$$\begin{aligned} \nabla L_X^E s_i &= \nabla(\sum_{j=1}^{l} \ell_{ji}(X) s_j) \\ &= \sum_{j=1}^{l}(d\ell_{ji}(X) \otimes s_j + \ell_{ji}(X)\nabla s_j) \\ &= \sum_{j=1}^{l} \left\{ d\ell_{ji}(X) \otimes s_j + \ell_{ji}(X) \sum_{k=1}^{l}(\theta_{kj} \otimes s_k) \right\} \\ &= \sum_{j=1}^{l} \left\{ d\ell_{ji}(X) + \sum_{k=1}^{l} \theta_{ji}\ell_{ki}(X) \right\} \otimes s_j. \end{aligned}$$

By comparing the both sides, we have the desired equality. Thus, we have

$$\begin{aligned} d_{eq}\kappa_{ij}(X) &= d\kappa_{ij}(X) - \iota_X \kappa_{ij}(X) \\ &= \sum_{k=1}^{l}(d\theta_{ik} \wedge \theta_{kj} - \theta_{ik} \wedge d\theta_{kj}) - \sum_{k=1}^{l}(\iota_X(\theta_{ik})\theta_{kj} - \iota_X(\theta_{kj})\theta_{ik}) \\ &\quad -d\iota_X\theta_{ij} - \iota_X d\theta_{ij} + d\ell_{ji}(X) \\ &= \sum_{k=1}^{l}\{(d\theta_{ik} - \iota_X\theta_{ik}) \wedge \theta_{kj} - \theta_{ik} \wedge (d\theta_{kj} - \iota_X\theta_{kj})\} - L_X\theta_{ij} + d\ell_{ji}(X) \\ &= \sum_{k=1}^{l}\{(d_{eq}\theta_{ik} + \ell_{ik}(X)) \wedge \theta_{kj} - \theta_{ik} \wedge (d_{eq}\theta_{kj} + \ell_{kj}(X))\}. \end{aligned}$$

Therefore, letting $\ell(X) = (\ell_{ij}(X))$, the above equation may be written in terms of a matrix form as

$$\begin{aligned} d_{eq}\kappa &= (d_{eq}\theta + \ell(X)) \wedge \theta - \theta \wedge (d_{eq}\theta + \ell(X)) \\ &= (d_{eq}\theta + \theta \wedge \theta + \ell(X)) \wedge \theta - \theta \wedge (d_{eq}\theta + \theta \wedge \theta + \ell(X)) \\ &= \kappa \wedge \theta - \theta \wedge \kappa. \end{aligned}$$

□

Lemma 3.11. Let $s'^{(l)} = (s'_1, ..., s'_l)$ be another local frame of E on U' and $P = (p_{ij}) : U \cap U' \to GL(l, \mathbb{C})$ be a C^∞ map such that $s'_i = \sum_{j=1}^{l} p_{ji} s_j$. If we denote by κ' the curvature matrix with respect to $s'^{(r)}$,

$$\kappa' = P^{-1} \kappa P \quad \text{on} \quad U \cap U'.$$

Proof. Note that p_{ij} is also G-invariant. From the definition, we compute

$$\begin{aligned} (K_{eq} s'_i)(X) &= \nabla^2 s'_i - \iota_X \nabla s'_i + L^E_X s'_i \\ &= \nabla^2 (\sum_{k=1}^{l} p_{ji} s_j) - \iota_X \nabla (\sum_{k=1}^{l} p_{ji} s_j) + L^E_X (\sum_{k=1}^{l} p_{ji} s_j) \\ &= \sum_{k=1}^{l} p_{ji} \nabla^2 (s_j) - \sum_{k=1}^{l} \iota_X (dp_{ji} \otimes s_j + p_{ji} \nabla s_j) + \sum_{k=1}^{l} \{(L_X p_{ji}) s_j + p_{ji} L^E_X s_j\} \\ &= \sum_{k=1}^{l} p_{ji} (\nabla^2 (s_j) - \iota_X \nabla s_j + L^E_X s_j). \end{aligned}$$

□

Therefore, for an invariant polynomial ϕ, choosing a local frame of $E|_U$ (where U is G-invariant open set), an equivariant curvature K_{eq} is locally represented by a matrix $\kappa \in M(l, \mathbb{C}) \otimes ((\mathbb{C}[\mathfrak{g}] \otimes \Omega^0(U)) \oplus \Omega^2(U))$ and $\phi(K_{eq})$ is defined by $\phi(\kappa)$. By Lemma 3.11, this definition is independent of the choice of a local frame of E, so that $\phi(K_{eq})$ is well-defined. Moreover, it is easy to see that $\phi(K_{eq})$ is a G-equivariant form.

Proposition 3.12. For any homogeneous invariant polynomial ϕ, the form $\phi(K_{eq})$ is d_{eq}-closed.

Proof. It is enough to show that $\phi(\kappa)$ is d_{eq}-closed. Since $\tilde{\phi}$ is multilinear,

$$d_{eq}\phi(\kappa) = \sum_{i=1}^{k} \tilde{\phi}(\kappa, ..., d_{eq}\kappa, ..., \kappa).$$

By Lemma 3.9 and Lemma refequivariant Bianchi identity, we see that the right hand side is zero. □

Definition 3.13. The equivariant characteristic form of ∇_{eq} with respect to ϕ is defined by

$$\phi(\nabla_{eq}) := \phi(K_{eq})$$

In the following, we introduce the Bott's difference form for some connections.

Proposition 3.14 (Bott's difference form [3, 17, 19])**.** Suppose $\nabla_{eq}^{(0)}, ..., \nabla_{eq}^{(p)}$ are equivariant connections for E. For an homogeneous invariant polynomial ϕ of degree k, there exists a form $\phi(\nabla_{eq}^{(0)}, ..., \nabla_{eq}^{(p)}) \in \Omega_G^{2k-p}(M)$ satisfying the following properties:

(1) $\phi(\nabla_{eq}^{(0)}, ..., \nabla_{eq}^{(p)})$ is alternating in the $p+1$ entries.

(2) $\sum_{\nu=0}^{p}(-1)^{\nu}\phi(\nabla_{eq}^{(0)}, ..., \widehat{\nabla}_{eq}^{(\nu)}, ..., \nabla_{eq}^{(p)}) + (-1)^p d_{eq}\phi(\nabla_{eq}^{(0)}, ..., \nabla_{eq}^{(p)}) = 0.$

Proof. Let $\tilde{\nabla}$ be the connection on the G-equivariant bundle

$$\pi \times \mathrm{id} : \tilde{E} = E \times \mathbb{R}^p \to M \times \mathbb{R}^p$$

obtained by

$$\tilde{\nabla} = (1 - \sum_{\nu=1}^{p} t_\nu)\nabla_{eq}^{(0)} + \sum_{\nu=1}^{p} t_\nu \nabla_{eq}^{(\nu)},$$

where G acts on $\mathbb{R}^p$ trivially. To be more specific, on sections s which are constant in $\mathbb{R}^p$-directions, let

$$\tilde{\nabla}_{\frac{\partial}{\partial t_i}} s = (\tilde{\nabla}s)(\frac{\partial}{\partial t_i}) = 0 \quad (\text{for } i = 1, ..., p).$$

Every section is expressed by a function-linear combination of sections constant in $\mathbb{R}^p$. Hence $\tilde{\nabla}$ extends to all sections such that $\tilde{\nabla}$ satisfies the Leibniz's rule. It is easy to check $\tilde{\nabla}$ is a G-equivariant connection. Denoting by Δ^p the standard p-simplex in $\mathbb{R}^p$ and by $\rho : M \times \Delta^p \to M$ the projection to the first factor, we have the fiber integration

$$\rho_* : \Omega_G^*(M \times \Delta^p) \to \Omega_G^{*-p}(M).$$

We set

$$\phi(\nabla_{eq}^{(0)}, ..., \nabla_{eq}^{(p)}) := \rho_*\phi(\tilde{\nabla}).$$

It follows from the definition that $\phi(\nabla_{eq}^{(0)}, ..., \nabla_{eq}^{(p)})$ is alternating. Also, by Proposition 2.22 (2) and that $\phi(\tilde{\nabla})$ is d_{eq}-closed,

$$(\partial\rho)_* \circ i^*\phi(\tilde{\nabla}) + (-1)^p d_{eq}\rho_*\phi(\tilde{\nabla}) = -\rho_* d_{eq}\phi(\tilde{\nabla}) = 0.$$

By using the Stokes theorem with respect to a standard simplex, we have

$$(\partial\rho)_* \circ i^*\phi(\tilde{\nabla}) = \sum_{\nu=0}^{p}(-1)^{\nu}\phi(\nabla_{eq}^{(0)}, ..., \widehat{\nabla}_{eq}^{(\nu)}, ..., \nabla_{eq}^{(p)}).$$

Then, we get the property (2). □

In particular, by using this, if ∇_{eq} and ∇'_{eq} are equivariant connections for E, then we have

$$\phi(\nabla'_{eq}) - \phi(\nabla_{eq}) = d_{eq}\phi(\nabla_{eq}, \nabla'_{eq}).$$

Thus, we see that $[\phi(\nabla_{eq})] \in H^*_G(M)$ is independent of the choice of an equivariant connection ∇_{eq}.

Definition 3.15. Let $\pi : E \to M$ be G-equivariant vector bundle of rank l and ∇_{eq} be an equivariant connection and ϕ be homogeneous invariant polynomial on $M(l, \mathbb{C})$. The *equivariant characteristic class* of E for polynomial ϕ is defined by

$$\phi(E)_{eq} := [\phi(\nabla_{eq})] \in H^*_G(M).$$

In the same way, we may define an equivariant characteristic classes in an equivariant Čech-de Rham cohomology. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a G-invariant open covering of M as in Section 2.2. Also, let $\pi : E \to M$ be a complex vector bundle of rank l and ϕ an invariant polynomial homogeneous of degree k. For each α, we choose a connection $\nabla^{(\alpha)}_{eq}$ for $E|_{U_\alpha}$ and for the collection $\nabla^*_{eq} = (\nabla^{(\alpha)}_{eq})_{\alpha \in I}$, we define $\phi(\nabla^*_{eq}) \in \Omega^{2k}_G(\mathcal{U})$ by

$$\phi(\nabla^*_{eq})_{\alpha_0 \cdots \alpha_p} := \phi(\nabla^{(0)}_{eq}, ..., \nabla^{(p)}_{eq}) \in \Omega^{2k-p}_G(U_{\alpha_0 \cdots \alpha_p}).$$

Lemma 3.16. In the above situation, we have the following properties:

(1) $D_{eq}\phi(\nabla^*_{eq}) = 0$.

(2) For another collection $\tilde{\nabla}^*_{eq} = (\tilde{\nabla}^{(\alpha)}_{eq})_{\alpha \in I}$, there exists the element $\psi \in \Omega^{2k-1}_G(\mathcal{U})$ such that

$$\phi(\tilde{\nabla}^*_{eq}) - \phi(\nabla^*_{eq}) = D_{eq}\psi$$

.

Proof. (1) By Proposition 3.14, we obtain

$$\begin{aligned}
(D_{eq}\phi(\nabla^*_{eq}))_{\alpha_0 \cdots \alpha_p} &= \delta\phi(\nabla^*_{eq})_{\alpha_0 \cdots \alpha_p} + (-1)^p d_{eq}\phi(\nabla^*_{eq})_{\alpha_0 \cdots \alpha_p} \\
&= \sum_{\nu=0}^{p} (-1)^\nu \phi(\nabla^*_{eq})_{\alpha_0 \cdots \widehat{\alpha}_\nu \cdots \alpha_p} + (-1)^p d_{eq}\phi(\nabla^*_{eq})_{\alpha_0 \cdots \alpha_p} \\
&= \sum_{\nu=0}^{p} (-1)^\nu \phi(\nabla^{(0)}_{eq}, ..., \widehat{\nabla}^{(\nu)}_{eq}, ..., \nabla^{(p)}_{eq}) + (-1)^p d_{eq}\phi(\nabla^{(0)}_{eq}, ..., \nabla^{(p)}_{eq}) \\
&= 0.
\end{aligned}$$

(2) Setting

$$\psi = \sum_{\nu=0}^{p} \phi(\nabla^{(\alpha_0)}_{eq}, ..., \nabla^{(\alpha_\nu)}_{eq}, \tilde{\nabla}^{(\alpha_\nu)}_{eq}, ..., \tilde{\nabla}^{(\alpha_p)}_{eq}),$$

we easily see that $\phi(\tilde{\nabla}^*_{eq}) - \phi(\nabla^*_{eq}) = D_{eq}\psi$. □

It follows from this lemma that the element $\phi(\nabla^*_{eq})$ defines a cohomology class $[\phi(\nabla^*_{eq})] \in H^{2k}_G(\mathcal{U})$ which depends only on E but not on the choice of the collection of connections ∇^*_{eq}. Also, from the following theorem, we may naturally regard $[\phi(\nabla^*_{eq})]$ as a characteristic class in $H^*_G(\mathcal{U})$.

Theorem 3.17. The class $[\phi(\nabla^*_{eq})]$ in $H^*_G(\mathcal{U})$ corresponds to the class $\phi(E)_{eq}$ in $H^*_G(M)$ under the isomorphism of Theorem 2.10.

Proof. Take an equivariant connection ∇_{eq} on M. For each $U_\alpha \in \mathcal{U}$, defining $\nabla^{(\alpha)}_{eq}$ to be $\nabla_{eq}|_{U_\alpha}$, we see that it is an equivariant connection for $E|_{U_\alpha}$. Then for the collection $\nabla^*_{eq} = (\nabla^{(\alpha)}_{eq})_{\alpha\in I}$, by definition,

$$\phi(\nabla^*_{eq}) \in C^0(\mathcal{U}, \Omega^*_G).$$

Thus, $r(\phi(\nabla_{eq})) = \phi(\nabla^*_{eq})$ and $r([\phi(\nabla^*_{eq})]) = \phi(E)_{eq}$. □

3.3 Equivariant Chern Classes

In this subsection, we introduce the equivariant Chern form and the equivariant class and investigate its properties.

For $A \in M(r, \mathbb{C})$, $\det(I_l + A)$ is a homogeneous invariant polynomial and is decomposed into

$$\det(I_r + A) = 1 + \sigma_1(A) + \cdots + \sigma_r(A),$$

where $\sigma_i(A)$ is of degree i. We call σ_i the i-th *elementary invariant polynomial*. We set $c^i = (\frac{\sqrt{-1}}{2\pi})^i \sigma_i$ and call it the i-th *Chern polynomial*.

Definition 3.18. Let $\pi : E \to M$ be G-equivariant vector bundle of rank l and ∇_{eq} be an equivariant connection for E. The i-the *equivariant Chern form* of ∇_{eq} is $c^i(\nabla_{eq})$. Its class $[c^i(\nabla_{eq})] \in H^{2i}_G(M)$ is denoted by $c^i_{eq}(E)$ and call it the i-th equivariant Chern class of E. Also, if κ is the equivariant connection matrix of ∇_{eq} with respect to some frame of $E|_U$, the total equivariant Chern form is given by

$$c^*(\nabla_{eq}) := \det(I_r + \frac{\sqrt{-1}}{2\pi}\kappa) = 1 + c^i_{eq}(\nabla_{eq}) + \cdots + c^l_{eq}(\nabla_{eq}) \in \Omega^*_G(M)$$

and the total equivariant Chern class of E is defined by its cohomology class

$$c^*_{eq}(E) = 1 + c^1_{eq}(E) + \cdots + c^l_{eq}(E) \in H^*_G(M).$$

Note that the form $c^*(\nabla_{eq})$ and the class $c^*_{eq}(E)$ is invertible in $\Omega^*_G(M)$ and $H^*_G(M)$ respectively, since these 0-th terms is 1.

In what follows, we prove that the equivariant Chern form (or class) has functoriality with respect to a pull-back and additivity with respect to an exact sequence.

• **Pull-back:** Let $\pi : E \to M$ be G-equivariant vector bundle and ∇_{eq} an equivariant connection and $f : N \to M$ a G-equivariant map. Set

$$f^*E = \{(y, e) \in N \times E \mid f(y) = \pi(e)\}$$

and consider the projection

$$f^*E \to N, \quad (y, e) \mapsto y.$$

Suppose G acts on f^*E diagonaly. Then, we see that $f^*E \to N$ is G-equivariant vector bundle. For G-invariant open set U in M, f induces a homomorphism

$$f^* : \Omega^p_G(U, E) \to \Omega^p_G(f^{-1}(U), f^*E).$$

Since every section of f^*E is obtained from the pull-back of a section of E (i.e. $f^*s = s \circ f$, for $s \in \Omega^0_G(M, E)$), we may define the equivariant connection $f^*\nabla_{eq}$ of f^*E as follows:

$$f^*\nabla_{eq} : \Omega^0_G(N, f^*E) \to \Omega^1_G(N, f^*E).$$

$$f^*\nabla_{eq}(f^*s) := f^*(\nabla_{eq}s) \ \text{ for } \ s \in \Omega^0_G(M, E).$$

In this situation, we consider the relation between equivariant Chern classes of ∇_{eq} and $f^*\nabla_{eq}$. Let θ and κ be, respectively, the connection and equivariant curvature matrices of ∇_{eq} with respect to a frame $(s_1, ..., s_l)$ of $E|_U$. Then, since $(f^*s_1, ..., f^*s_l)$ is a frame of $f^*E|_{f^{-1}(U)}$, it follow from the definition of $f^*\nabla_{eq}$ that the connection and equivariant curvature matrices of $f^*\nabla_{eq}$ with respect to this frame are given by $f^*\theta$ and $f^*\kappa$. From this, in particular, we have

$$c^*(f^*\nabla_{eq}) = f^*c^*(\nabla_{eq}) \ \text{ and } \ c^*_{eq}(f^*E) = f^*c^*_{eq}(E).$$

• **Exact sequences:** Let

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0 \qquad (*)$$

be an exact sequence of G-equivariant vector bundles on M where f and g is a bundle map and a G-morphism. Also, let $\nabla'_{eq}, \nabla_{eq}$ and ∇''_{eq} be a G-equivariant connections for E', E and E'', respectively. The triple $(\nabla'_{eq}, \nabla_{eq}, \nabla''_{eq})$ is said to be *compatible* with the above sequence $(*)$, if the following diagram is commutative:

$$\begin{tikzcd}
\Omega^0_G(M, E') \arrow[r, "f"] \arrow[d, "\nabla'_{eq}"'] & \Omega^0_G(M, E) \arrow[r, "g"] \arrow[d, "\nabla_{eq}"'] & \Omega^0_G(M, E'') \arrow[d, "\nabla''_{eq}"'] \\
\Omega^1_G(M, E') \arrow[r, "1\otimes f"] & \Omega^1_G(M, E) \arrow[r, "1\otimes g"] & \Omega^1_G(M, E'')
\end{tikzcd}$$

Proposition 3.19. Given an equivariant connection ∇''_{eq} for E'', there exist connections $\nabla'_{eq}, \nabla_{eq}$ for E' and E such that the triple $(\nabla'_{eq}, \nabla_{eq}, \nabla''_{eq})$ is compatible with the above sequence.

Proof. Suppose E is isomorphic to $E' \oplus E''$ as an equivariant vector bundle. Letting ∇'_{eq} be an arbitrary connection for E' and choosing an equivariant connection ∇_{eq} for E corresponding to $\nabla'_{eq} \oplus \nabla''_{eq}$, we see that $(\nabla'_{eq}, \nabla'_{eq}, \nabla''_{eq})$ is compatible with $(*)$. Hence it is sufficient to show that the sequence $(*)$ has a G-equivariant splitting (i.e., a G-equivariant homomorphism $\eta : E'' \to E$ such that $\eta \circ g = 1_E$). Let $\mathcal{U} = \{U_\alpha\}$ be a locally finite G-invariant covering of M such that the bundle are trivial on each U_α. Choosing suitable G-invariant frames, we see that there exists a splitting η_α on each U_α. Let $\{\rho_\alpha\}$ be a G-invariant partition of unity subordinate to $\mathcal{U}$. Then $\eta = \sum_\alpha \rho_\alpha \eta_\alpha$ gives a splitting of $(*)$ $\square$

Proposition 3.20. For a compatible triple $(\nabla'_{eq}, \nabla_{eq}, \nabla''_{eq})$ of equivariant connections, we have the following:

$$c^*(\nabla_{eq}) = c^*(\nabla'_{eq}) \cdot c^*(\nabla''_{eq}).$$

Proof. There exists a local frame $s'^{(l')} = (s'_1, ..., s'_{l'})$ of E' such that E has a frame $s^{(l)}$ of the form

$$s^{(r)} = (f(s'_1), ..., f(s'_{l'}), s_{l'+1}, ..., s_l).$$

Then, $s''^{(l'')} = (g(s_{l'+1}), ..., g(s_l))$ is a local frame of E''. Let θ', θ and θ'' be the connection matrices of ∇', ∇ and ∇'' with respect to $s'^{(l')}, s^{(l)}$ and $s''^{(l'')}$ respectively. Then since $(\nabla'_{eq}, \nabla_{eq}, \nabla''_{eq})$ is compatible with $(*)$, we have

$$\begin{aligned} \nabla_{eq}(f(s'_i)) &= f(\nabla'_{eq} s'_i) \\ &= f(\sum_{j=1}^{l'} \theta'_{ji} \otimes s'_j) \\ &= \sum_{j=1}^{l'} \theta'_{ji} \otimes f(s'_j) \end{aligned}$$

and by $g \circ f = 0$

$$\begin{aligned} \nabla'_{eq}(g(s_k)) &= g(\nabla_{eq} s_k) \\ &= g(\sum_{j=l'+1}^{l} \theta_{jk} \otimes s_k) \\ &= \sum_{j=l'+1}^{l} \theta_{jk} \otimes g(s_k). \end{aligned}$$

Thus, we have

$$\theta = \begin{pmatrix} \theta' & * \\ 0 & \theta'' \end{pmatrix}$$

and it implies the desired proposition. $\square$

By Proposition 3.19 and 3.20, we have the additivity of Chern classes

$$c^*_{eq}(E) = c^*_{eq}(E') \cdot c^*_{eq}(E'').$$

• **Virtual bundle** Let E and F be G-equivariant vector bundles on M. We consider the formal difference $F - E$. Since $c^*(E)$ is invertible, we may define the total Chern class of $F - E$ by

$$c^*_{eq}(F - E) = c^*_{eq}(F)/c^*_{eq}(E).$$

For equivariant connections ∇^E_{eq} and ∇^F_{eq} for E and F respectively, we set $\nabla^\bullet_{eq} = (\nabla^E_{eq}, \nabla^F_{eq})$. Then the class $c^*_{eq}(F - E)$ is represented by the form

$$c^*(\nabla^\bullet_{eq}) = c^*(\nabla^F_{eq})/c^*(\nabla^E_{eq}).$$

More generally, since every invariant polynomial ϕ is a polynomial in the Chern polynomial, we write $\phi = P(c^1, c^2, ...)$ where P is a polynomial with coefficient in $\mathbb{C}$. From this, the equivariant characteristic class of $F - E$ for ϕ is defined by

$$\phi(F - E)_{eq} = P(c^1_{eq}(F - E), c^2_{eq}(F - E), ...)$$

and it is represented by the form

$$\phi(\nabla^\bullet_{eq}) = P(c^1(\nabla^\bullet_{eq}), c^2(\nabla^\bullet_{eq}), ...).$$

For $p + 1$ pairs of connections $\nabla^{\bullet(\nu)}_{eq} = (\nabla^{E(\nu)}_{eq}, \nabla^{F(\nu)}_{eq})$ $(\nu = 0, ..., p)$, we may define the difference form $\phi(\nabla^{\bullet(0)}_{eq}, ..., \nabla^{\bullet(p)}_{eq})$ in the same way of the usual difference form. To be more specific, let $\tilde{\nabla}^E_{eq}$ be the connection for the vector bundle $E \times \mathbb{R}^p \to M \times \mathbb{R}^p$ given by

$$\tilde{\nabla}^E_{eq} = (1 - \sum_{\nu=1}^{p} t_\nu)\nabla^{E(0)}_{eq} + \sum_{\nu=1}^{p} t_\nu \nabla^{E(\nu)}_{eq}$$

and let $\tilde{\nabla}^F_{eq}$ be defined similarly and set $\tilde{\nabla}^\bullet_{eq} = (\tilde{\nabla}^E_{eq}, \tilde{\nabla}^F_{eq})$ and define

$$\phi(\nabla^{\bullet(0)}_{eq}, ..., \nabla^{\bullet(p)}_{eq}) = \rho_*(\phi(\tilde{\nabla}^\bullet_{eq})),$$

where ρ is the projection $M \times \Delta^p \to M$. Then we have the following property as Proposition 3.14:

$$\sum_{\nu=0}^{p} (-1)^\nu \phi(\nabla^{\bullet(0)}_{eq}, ..., \widehat{\nabla}^{(\bullet\nu)}_{eq}, ..., \nabla^{\bullet(p)}_{eq}) + (-1)^p d_{eq}\phi(\nabla^{\bullet(0)}_{eq}, ..., \nabla^{\bullet(p)}_{eq}) = 0.$$

Proposition 3.21. Let ϕ be a homogeneous invariant polynomial and

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0$$

be an exact sequence of G-equivariant vector bundles on M. For triple equivariant connections $(\nabla_{eq}^{E'(\nu)}, \nabla_{eq}^{E(\nu)}, \nabla_{eq}^{E''(\nu)})$ $(\nu = 0, ..., p)$ compatible with the above sequence, we have

$$\phi(\nabla_{eq}^{\bullet(0)}, ..., \nabla_{eq}^{\bullet(p)}) = \phi(\nabla_{eq}^{E''(0)}, ..., \nabla_{eq}^{E''(p)})$$

where $\nabla_{eq}^{\bullet(\nu)}$ is the pair of connections $(\nabla_{eq}^{E'(\nu)}, \nabla_{eq}^{E(\nu)})$ for $E - E'$. In particular, $\phi(E - E') = \phi(E'')$.

Proof. From Proposition 3.20, we have the above proposition. $\square$

4 Localization of equivariant characteristic classes

In this subsection, we define the equivariant localized characteristic classes by referring to Suwa [17, 19]. Also, we show the vanishing theorem of a Chern class with respect to frames and by using this we will define a localized equivariant Chern class by frames. In Section4.4, we will check that an equivariant universal Thom form may be expressed by a localized equivariant Chern form. Finally, as applications, we show the equivariant version of the Riemann-Roch theorem for the zero locus of a section of a complex vector bundle and the Atiyah-Bott localization theorem in our setting.

4.1 Localized Characteristic Classes

Let M be a G-manifold and S a G-invariant closed set in M and $\pi : E \to M$ a complex G-equivariant vector bundle. Letting $U_0 = M \setminus S$ and U_1 a G-invariant neighborhood of S, we consider the G-invariant covering $\mathcal{U} = \{U_0, U_1\}$. In what follows, let $H_G(\mathcal{U}, U_0)$ denote $H_G(M, M \setminus S)$.

Suppose there is some "geometric object" γ on U_0, to which is associated a class $\mathcal{C}$ of equivariant connections for E on U_0 such that, for a certain homogeneous invariant polynomial ϕ,

$$\phi((\nabla^{(0)}_{eq})_0, ..., (\nabla^{(k)}_{eq})_0) \equiv 0 \ \text{ if every } (\nabla^{(i)}_{eq})_0 \text{ belongs } \mathcal{C}.$$

A equivariant connection $(\nabla_{eq})_0$ for E on U_0 is said to be *special*, if $(\nabla_{eq})_0$ belongs to $\mathcal{C}$ and the polynomial ϕ as above is said to be *adapted* to γ.

Lemma 4.1. In the above situation, suppose ∇^0_{eq} is special and ϕ is adapted to γ. we consider the cocycle

$$\phi(\nabla^*_{eq}) = (0, \phi(\nabla^1_{eq}), \phi(\nabla^0_{eq}, \nabla^1_{eq})) \in \Omega_G(\mathcal{U}, U_0).$$

Then its class in $H^*_G(M, M \setminus S)$ is independent of the choice of the special equivariant connection ∇^0_{eq} or the equivariant connection ∇^1_{eq}.

Proof. If ∇^0_{eq} and ${\nabla'}^0_{eq}$ are both special, by using $\phi(\nabla^0_{eq}, {\nabla'}^0_{eq}) = 0$ and Proposition 3.14 (2), we have

$$(0, \phi(\nabla^1_{eq}), \phi({\nabla'}^0_{eq}, \nabla^1_{eq})) - (0, \phi(\nabla^1_{eq}), \phi(\nabla^0_{eq}, \nabla^1_{eq})) = D_{eq}(0, 0, \phi(\nabla^0_{eq}, {\nabla'}^0_{eq}, \nabla^1_{eq})).$$

Similarly, for equivariant connections ∇^1_{eq} and ${\nabla'}^1_{eq}$ on U_1,

$$(0, \phi({\nabla'}^1_{eq}), \phi(\nabla^0_{eq}, {\nabla'}^1_{eq})) - (0, \phi(\nabla^1_{eq}), \phi(\nabla^0_{eq}, \nabla^1_{eq})) = D_{eq}(0, \phi({\nabla'}^1_{eq}, \nabla^1_{eq}), \phi(\nabla^0_{eq}, \nabla^1_{eq}, {\nabla'}^1_{eq})).$$

□

Definition 4.2. If $(\nabla_{eq})_0$ is special and ϕ (homogeneous of degree d) is adapted to γ, the class $\phi_S(E,\gamma) \in H^{2d}_G(M, M \setminus S)$ is defined by

$$\phi_S(E,\gamma) := [\phi(\nabla^*_{eq})].$$

and is called the *localized equivariant characteristic class* of $\phi(E)_{eq}$ at S by γ.

In the following, we consider a geometric object by frames and its localized equivariant Chern class. Suppose $\pi : E \to M$ is a complex G-equivariant vector bundle of rank l. Then, it follow from a way of definition of the Bott's difference form that for any k equivariant connections $\nabla^{(1)}_{eq}, ..., \nabla^{(k)}_{eq}$ for E,

$$c^i(\nabla^{(1)}_{eq}, ..., \nabla^{(k)}_{eq}) \equiv 0 \quad \text{for} \ \ i \geq l+1.$$

As a consequence, we have the following lemma.

Lemma 4.3. Let $s^{(r)} = (s_1, ..., s_r)$ be an r-frame of E on a G-invariant open set U in M. If $\nabla^{(1)}_{eq}, ..., \nabla^{(k)}_{eq}$ is $s^{(r)}$-trivial on U, then on U

$$c^i(\nabla^{(1)}_{eq}, ..., \nabla^{(k)}_{eq}) \equiv 0 \quad \text{for } i \geq l-r+1.$$

Proof. Let E' be the G-equivariant subbundle of E on U spanned by $s^{(r)}$ and $\nabla'^{(j)}_{eq}$ ($j = 0, ..., k$) be the restriction of $\nabla^{(j)}_{eq}$ on U (we easily see that $\nabla'^{(j)}_{eq}$ is an equivariant connections for E'). Then $\nabla^{(j)}_{eq}$ induces a G-equivariant connection $\nabla''^{(j)}_{eq}$ for the quotient bundle $E'' = E/E'$ on U so that $(\nabla'^{(j)}_{eq}, \nabla^{(j)}_{eq}, \nabla''^{(j)}_{eq})$ is compatible with the exact sequence

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0.$$

Setting $\nabla^{\bullet}_{eq} = (\nabla'^{(j)}_{eq}, \nabla^{(j)}_{eq})$, by Proposition 3.21, we have

$$c^i(\nabla^{\bullet(1)}_{eq}, ..., \nabla^{\bullet(k)}_{eq}) = c^i(\nabla''^{(1)}_{eq}, ..., \nabla''^{(k)}_{eq}).$$

Also, since the G-equivariant connection matrix of $\nabla'^{(j)}_{eq}$ with respect to the frame $s^{(r)}$ of E' is zero,

$$c^i(\nabla^{\bullet(1)}_{eq}, ..., \nabla^{\bullet(k)}_{eq}) = c^i(\nabla^{(1)}_{eq}, ..., \nabla^{(k)}_{eq}).$$

This implies that $c^i(\nabla^{(1)}_{eq}, ..., \nabla^{(k)}_{eq}) = c^i(\nabla''^{(1)}_{eq}, ..., \nabla''^{(k)}_{eq})$ and since the rank of E'' is $l-r$, $c^i(\nabla''^{(1)}_{eq}, ..., \nabla''^{(k)}_{eq})$ vanishes for $i \geq l-r+1$. Hence $c^i(\nabla^{(1)}_{eq}, ..., \nabla^{(k)}_{eq}) \equiv 0$ for $i \geq l-r+1$. □

Definition 4.4. Let $s^{(r)}$ be a local frame on $M \setminus S$. If ∇^0_{eq} is $s^{(r)}$-trivial, by Lemma 4.3,

$$c^i(\nabla^*_{eq}) = (0, c^i(\nabla^1_{eq}), c^i(\nabla^0_{eq}, \nabla^1_{eq})) \ \text{ for } i \geq l-r+1$$

and induce the class $[c^i(\nabla^*_{eq})] \in H^{2i}_G(M, M \setminus S)$. Since this class is independent of the choice of $s^{(r)}$-trivial G-equivariant connection ∇^0_{eq} on U_0 and a G-equivariant connection ∇^1_{eq} on U_1 by Lemma 4.1, we denote by

$$c^i_S(E, s^{(r)})_{eq} := [c^i(\nabla^*_{eq})]$$

and we call it *the localized Chern class* of $c^i(E)_{eq}$ by $s^{(r)}$ at S.

4.2 Equivariant Thom class via localized Chern class

By referring to the above argument (and Suwa [17, 19]), we give the equivariant universal Thom form by using the localized Chern class. Suppose the unitary group $U(l)$ acts on $\mathbb{C}^l$ naturally. Then

$$\pi : \mathbb{C}^l \to \{0\}$$

is clearly an $U(l)$-equivariant vector bundle. Setting

$$W_0 = \mathbb{C}^l \setminus \{0\}, \quad W_1 = \mathbb{C}^l,$$

we have an $U(l)$-invariant covering $\mathcal{W} = \{W_0, W_1\}$. We consider the pull-back of $\mathbb{C}^l$ by π, i.e.,

$$\pi^*\mathbb{C}^l = \{(z_1, z_2) \in \mathbb{C}^l \times \mathbb{C}^l \mid \pi(z_1) = \pi(z_2)\} = \mathbb{C}^l \times \mathbb{C}^l$$
$$\varpi : \pi^*\mathbb{C}^l = \mathbb{C}^l \times \mathbb{C}^l \to \mathbb{C}^l,$$

where ϖ is the projection to the second factor. From the definition of pull-back, $U(l)$ acts on $\mathbb{C}^l \times \mathbb{C}^l$ diagonaly ($A(z_1, z_2) = (Az_1, Az_2)$) and $\varpi : \pi^*\mathbb{C}^l = \mathbb{C}^l \times \mathbb{C}^l \to \mathbb{C}^l$ is an $U(l)$-equivariant vector bundle. Then the diagonal section

$$s_\Delta : \mathbb{C}^l \to \pi^*\mathbb{C}^l = \mathbb{C}^l \times \mathbb{C}^l, \quad z \mapsto (z, z)$$

is naturally $U(l)$-invariant frame on $\mathbb{C}^l \setminus \{0\}$. Thus, we may consider the localized Chern class of $c^l(\pi^*\mathbb{C}^l)_{eq}$ by s_Δ, that is,

$$c^l(\pi^*\mathbb{C}^l, s_\Delta)_{eq} \in H^{2l}_{U(l)}(\mathbb{C}^l, \mathbb{C}^l \setminus \{0\}).$$

This class is represented by the following form

$$(0, c^l_{eq}(D^1_{eq}), c^l_{eq}(D^0_{eq}, D^1_{eq})) \in \Omega^{2l}_G(\mathcal{W}, W_0),$$

where

- D^0_{eq} is an s_Δ-trivial $U(l)-$equivariant connection for $\pi^*\mathbb{C}^l$ on $W_0 = \mathbb{C}^l \setminus \{0\}$,
- D^1_{eq} is an $U(l)$-equivariant connection for $\pi^*\mathbb{C}^l$ on $W_1 = \mathbb{C}^l$.

On the other hand, as a real vector bundle of rank $2l$, we may consider the $U(l)$-equivariant Thom class $\psi^{\mathbb{C}^l}_{eq} \in H^{2l}_{U(l)}(\mathbb{C}^l, \mathbb{C}^l \setminus \{0\})$.

Theorem 4.5 (equivariant universal Thom class)**.** In the above situation, we have

$$c^l(\pi^*\mathbb{C}^l, s_\Delta)_{eq} = \psi_{eq}^{\mathbb{C}^l}.$$

Proof. Setting

$$T_1 = D^{2l} = \{z \in \mathbb{C}^l \mid \|z\| \leq 1\}, \quad T_0 = \mathbb{C}^l \setminus \mathrm{Int}T_1,$$

we have a honeycomb system $\{T_0, T_1\}$ adapted to $\mathcal{W}$. Note that $T_{01} = -S^{2l-1}$. By the definition of $\psi_{eq}^{\mathbb{C}^l}$, it suffices to find the equivariant connections D_{eq}^0, D_{eq}^1 satisfying

$$(\pi_1)_* c^l(D_{eq}^1) + (\pi_{01})_* c^l(D_{eq}^0, D_{eq}^1) = 1,$$

where $\pi_1 : T_1 \to \{0\}$, $\pi_{01} : T_{01} \to \{0\}$. Let sections $s_1, ..., s_l$ of $\varpi : \pi^*\mathbb{C}^l \to \mathbb{C}^l$ be

$$s_i(z) = (e_i, z) \quad (i = 1, ..., l),$$

where $\{e_i\}$ is the standard basis of $\mathbb{C}^l$. Now we define the connection D_1 for $\pi^*\mathbb{C}^l$ on $\mathbb{C}^l$ by

$$D_1(\sum_{i=1}^l f_i s_i) := \sum_{i=1}^l df_i \otimes s_i \quad (\text{for } f_i \in C^\infty(\mathbb{C}^l)),$$

which is $s^l = (s_1, ..., s_l)$-trivial. Also, we easily see that D_1 is a $U(l)$-invariant connection. Thus we may define the equivariant connection D_{eq}^1 corresponding to D_1. From the definition of D_{eq}^1, the form degree of its curvature form is 0 and $(\pi_1)_* c^l(D_{eq}^1) = 0$.

Next we define the connection D_0 for $\pi^*\mathbb{C}^l$ on $\mathbb{C}^l \setminus \{0\}$ by

$$D_0 s_i = -\frac{\bar{z}_i}{\|z\|^2} \sum_{j=1}^l dz_j \otimes s_j \quad (i = 1, ..., l)$$

For $f_i \in C^\infty(\mathbb{C}^l \setminus \{0\})(i = 1, ..., l), \quad g \in U(l)$, we have

$$g \cdot D_0(\sum_{i=1}^l f_i s_i) = \sum_{i=1}^l ((g \cdot df_i) \otimes (g \cdot s_i) + (g \cdot f_i)(g \cdot D_0 s_i))$$

$$D_0(g \cdot (\sum_{i=1}^l f_i s_i)) = \sum_{i=1}^l ((g \cdot df_i) \otimes (g \cdot s_i) + (g \cdot f_i)(D_0(g \cdot s_i))).$$

Therefore, to show that D_0 is $U(l)$-invariant connection, it suffices to check

$$g \cdot (D_0 s_i) = D_0(g \cdot s_i).$$

For $g = (g_{ij}) \in U(l)$, directly computing, we have

$$g \cdot \left(-\frac{\bar{z}_i}{\|z\|^2}\right) = -\sum_{k=1}^l g_{ki} \frac{\bar{z}_k}{\|z\|^2}$$

$$g\cdot \mathrm{d}z_j = \sum_{m=1}^{l} \overline{g_{mj}} dz_m, \quad g\cdot s_j = \sum_{n=1}^{l} g_{nj} s_n.$$

Thus, we have

$$\begin{aligned}
g\cdot(D_0 s_i) &= -\sum_{k=1}^{l} g_{ki}\frac{\bar{z}_k}{\|z\|^2}\sum_{j=1}^{l}\left\{(\sum_{m=1}^{l}\overline{g_{mj}}dz_m)\otimes(\sum_{n=1}^{l}g_{nj}s_n)\right\}\\
&= -\sum_{k=1}^{l} g_{ki}\frac{\bar{z}_k}{\|z\|^2}\sum_{m=1}^{l}\sum_{n=1}^{l}\left(\sum_{j=1}^{l}\overline{g_{mj}}g_{nj}\right)(dz_m\otimes s_n)\\
&= -\sum_{k=1}^{l} g_{ki}\frac{\bar{z}_k}{\|z\|^2}\sum_{m=1}^{l}\sum_{n=1}^{l}\delta_{mn}(dz_m\otimes s_n)\\
&= -\sum_{k=1}^{l} g_{ki}\frac{\bar{z}_k}{\|z\|^2}\sum_{j=1}^{l}dz_j\otimes s_j
\end{aligned}$$

and

$$D_0(g\cdot s_i) = D_0(\sum_{k=1}^{l} g_{ki}s_k) = \sum_{k=1}^{l} g_{ki} D_0 s_k = -\sum_{k=1}^{l} g_{ki}\frac{\bar{z}_k}{\|z\|^2}\sum_{j=1}^{l} dz_j\otimes s_j.$$

So we get $g\cdot(D_0 s_i) = D_0(g\cdot s_i)$. Also, for the diagonal section $s_\Delta = \sum_{i=1}^{l} z_i s_i$, we easily see that $D_0 s_\Delta = 0$. Hence we have an s_Δ-trivial $U(l)-$equivariant connection D^0_{eq} corresponding to D_0 for $\pi^*\mathbb{C}^l$ on $W_0 = \mathbb{C}^l\setminus\{0\}$. The rest of proof is to show that

$$-\int_{S^{2l-1}} c^l(D^0_{eq}, D^1_{eq}) = 1. \tag{1}$$

The connection matrix $\theta_0 = (\theta_{ij})$ of D^0_{eq} with respect to $(s_1,...,s_l)$ is express by

$$\theta_{ij} = -\frac{\bar{z}_j}{\|z\|^2}dz_i,$$

while the connection matrix θ_1 of D^1_{eq} with respect to $(s_1,...,s_l)$ is zero. For $t\in\mathbb{R}$ and the natural projection $\rho:\mathbb{R}\times(\mathbb{C}^l\setminus\{0\})\to\mathbb{C}^l\setminus\{0\}$, we set

$$\tilde{D}_{eq} = (1-t)\rho^* D^0_{eq} + t\rho^* D^1_{eq},$$

and denote $\rho^* D^0_{eq}, \rho^* D^1_{eq}$ by D^0_{eq}, D^1_{eq} for short. Then the connection matrix $\tilde{\theta}$ of $\tilde{D}_{eq}$ with respect to $(s_1,...,s_l)$ is given by $\tilde{\theta} = (1-t)\theta_0$, and thus by (κ) in subsection 2.1, the corresponding equivariant curvature matrix $\tilde{\kappa}$ is given by

$$\tilde{\kappa}(X) = d\tilde{\theta} + \tilde{\theta}\wedge\tilde{\theta} - \iota_X\tilde{\theta} + \ell(X)$$

for $X = (X_{ij}) \in \mathfrak{u}(l)$. Recall that $\ell(X) = (\ell_{ij}(X))_{ij}$ is defined by $L_X^E s_i = \sum_{j=1}^l \ell_{ji}(X) s_j$. For later use, we rewrite it as

$$\tilde{\kappa}(X) = -dt \wedge \theta_0 + \kappa_t(X),$$
$$\kappa_t(X) = (1-t)d\theta_0 + (1-t)^2\theta_0 \wedge \theta_0 - (1-t)\iota_X\theta_0 + \ell(X).$$

By the definition of the equivariant Bott-difference form,

$$\begin{aligned} c^l(D_{eq}^0, D_{eq}^1) &= \rho'_* c^l(\tilde{\kappa}) \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^l \rho'_* \det\tilde{\kappa} \\ &= -\left(\frac{\sqrt{-1}}{2\pi}\right)^l \sum_{j=1}^l \int_0^1 \det Q_j dt, \end{aligned}$$

where Q_j is the matrix obtained from κ_t by replacing the j-th column by that of θ_0. In the following, we compute $\det Q_j$. Computing the (i,j)-entry of $d\theta_0$, $\theta_0 \wedge \theta_0$, $\ell(X)$ and $\iota_X\theta_0$, we have

$$(d\theta_0)_{ij} = -\left(\frac{1}{\|z\|^2} d\bar{z}_j + \bar{z}_j d\left(\frac{1}{\|z\|^2}\right)\right) \wedge dz_i$$

$$(\theta_0 \wedge \theta_0)_{ij} = \frac{1}{\|z\|^4} \bar{z}_j dz_i \wedge (\sum_{k=1}^l \bar{z}_k dz_k)$$

$$(\iota_X\theta_0) = -\frac{\bar{z}_j}{\|z\|^2} \sum_{k=1}^l X_{ik} z_k.$$

$$\ell(X)_{ij} = X_{ij}.$$

We set the matrices $\tau(X)$ and $\eta(X)$ as follows;

$$\begin{aligned} \tau(X)_{ij} &:= -\frac{(1-t)}{\|z\|^2} d\bar{z}_j \wedge dz_i + X_{ij} \\ \eta(X)_{ij} &:= -(1-t)\bar{z}_j d\left(\frac{1}{\|z\|^2}\right) \wedge dz_i + (1-t)^2(\theta_0 \wedge \theta_0)_{ij} + (1-t)(\iota_X\theta_0)_{ij}. \end{aligned}$$

Then, $\kappa_t(X) = \tau(X) + \eta(X)$. Denoting k-th column of $\kappa_t(X)$ and $\tau(X), \eta(X)$ by $\kappa_t(X)^{(k)}$ and $\tau(X)^{(k)}, \eta(X)^{(k)}$ respectively, the matrix Q_j may be expressed as follows;

$$\det Q_j = \det[\kappa_t(X)^{(1)}, ..., \theta_0^{(j)}, ..., \kappa_t(X)^{(l)}].$$

We decompose this determinant with respect to the columns $\tau(X)^{(k)}, \eta(X)^{(k)}$ by using multilinearity of determinant. Note that, if more than two columns of $\eta(X)$ appear in the determinant obtained from the decomposed term, the term vanishes. Thus, we have

$$\det Q_j = \det R_j + \sum_{k \neq j} \det R_{jk},$$

where

$$R_j := [\tau(X)^{(1)}, ..., \theta_0^{(j)}, ..., \tau(X)^{(l)}]$$

$$R_{jk} := [\tau(X)^{(1)}, ..., \theta_0^{(j)}, ..., \eta(X)^{(k)}, ..., \tau(X)^{(l)}].$$

By the definition, we see that $\det R_{jk} = -\det R_{kj}$. Directly computing, we have

$$\begin{aligned} c^l(D^0_{eq}, D^1_{eq}) &= -\left(\frac{\sqrt{-1}}{2\pi}\right)^l \left\{\sum_{j=1}^{l}\int_0^1 \det R_j dt + \int_0^1 \sum_{j=1}^{l}\sum_{k\neq j} \det R_{jk} dt\right\} \\ &= -\left(\frac{\sqrt{-1}}{2\pi}\right)^l \sum_{j=1}^{l}\int_0^1 \det R_j dt \\ &= -C_l \frac{\sum_{j=1}^{l} \overline{\Phi_j(z)} \wedge \Phi(z)}{||z||^{2l}} + (\text{terms with } X_{ij}), \end{aligned} \tag{2}$$

where

$$\begin{aligned} \Phi(z) &= dz_1 \wedge \cdots dz_l \\ \Phi_i(z) &= (-1)^{i-1} z_i dz_i \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_l \end{aligned}$$

and

$$C_l = (-1)^{\frac{l(l-1)}{2}} \frac{(l-1)!}{(2\pi\sqrt{-1})^l}.$$

Thus, we have $-\int_{S^{2l-1}} c^l(D^0_{eq}, D^1_{eq}) = 1$, since $C_l \frac{\sum_{j=1}^{l} \overline{\Phi_j(z)} \wedge \Phi(z)}{||z||^{2l}}$ coinsides the Bochner-Martinelli kernel β_l on $\mathbb{C}^l$(see [17]). $\square$

4.3 Explicit formula of universal $U(l)$-equivariant Thom form

We give an explicit formula of universal $U(l)$-equivariant Thom form

$$(0, c^l_{eq}(D^1_{eq}), c^l_{eq}(D^0_{eq}, D^1_{eq})) \in \Omega^{2l}_{U(l)}(\mathcal{W}, W_0).$$

In particular, higher terms in (2) are precisely determined.

We provide some notations to simplify a calculation. Let V be a complex vector space of dimension l with a basis $e_1, \cdots, e_l$. For any anticommutative $\mathbb{Z}$-graded algebra $\mathcal{A}$, we consider the algebra $\mathcal{A} \otimes \wedge^* V$ with the following wedge product; $(\alpha \otimes \xi) \wedge (\beta \otimes \eta) := (\alpha \wedge \beta) \otimes (\xi \wedge \eta)$. It is easy to see the following lemma:

Lemma 4.6.

(1) Let $\omega_i = \sum_{k=1}^{l} \omega_{ik} \otimes e_k$, then

$$\omega_1 \wedge \cdots \wedge \omega_l = \sum_{\sigma \in \mathfrak{S}_l} \mathrm{sgn}(\sigma)(\omega_{1\sigma(1)} \wedge \cdots \wedge \omega_{l\sigma(l)}) \otimes (e_1 \wedge \cdots \wedge e_l).$$

(2) Let $\alpha = \sum_{k=1}^{l} \alpha_k \otimes e_k$ and $\beta = \sum_{k=1}^{l} \beta_k \otimes e_k \in \mathcal{A} \otimes \wedge^* V$ with $\deg(\alpha_k) = s$ and $\deg(\beta_k) = t$, then

$$\alpha \wedge \beta = -(-1)^{st} \beta \wedge \alpha.$$

We write $[l] := \{1, 2, \cdots, l\}$. If I is a subset of $[l]$, we denote by e_I the product $e_{i_1} \wedge \cdots \wedge e_{i_p}$ where we write $I = \{i_1, i_2, ..., i_p\}$ with $i_1 < i_2 < \cdots < i_p$. Denote by $|I| = p$, the cardinality of I. For $I = \{i_1, i_2, ..., i_p\}$ and $I' = \{i'_1, i'_2, ..., i'_p\}$ in $[l]$, we set $\epsilon_{(I,I')} := (-1)^{\sum_{s=1}^{p}(i_s + i'_s)}$. Let $X = [X_{ij}] \in \mathfrak{u}(l)$, and denote by $X_{I,I'}$ the retainer minor of X with respect to I and I': $X_{I,I'} = \det[X_{i_s i'_t}]_{1 \le s,t \le p}$. If $1 \le k \le l$ and $k \notin J$, we denote by $\epsilon(k, J)$ the sign such that $e_k \wedge e_J = \epsilon(\{k\}, J) e_{\{k\} \cup J}$. Put

$$\gamma_{(k,I,J)} = (-1)^{\frac{|J|(|J|-1)}{2}} \left(\frac{\sqrt{-1}}{2\pi} \right)^{l} |J|! \, \epsilon(k, J).$$

Theorem 4.7. For $X \in \mathfrak{u}(l)$, we have

$$\chi_{eq}(X) := c^l(D^1_{eq}) = \left(\frac{\sqrt{-1}}{2\pi} \right)^{l} \det X$$

$$\beta_{eq}(X) := c^l(D^0_{eq}, D^1_{eq}) = \sum_{k,I,J} \gamma_{(k,I,J)} \sum_{I',J'} \epsilon_{(I,I')} X_{I,I'} \frac{\bar{z}_k d\bar{z}_J \wedge dz_{J'}}{\|z\|^{2(|J|+1)}}$$

where for $1 \le k \le l$, the sets I, J vary over the subsets of $[l]$ such that $\{k\} \cup I \cup J$ is a partition of $[l]$, and I' and J' vary over the subsets of $[l]$ such that $|I| = |I'|$ and $I' \cup J'$ is a partition of $[l]$.

Proof. Let θ_1 and κ_1 be the connection matrix and the corresponding equivariant curvature matrix with respect to the frame $(s_1, ..., s_l)$. Since $\theta_1 = 0$,

$$\kappa_1(X) = d\theta_1 + \theta_1 \wedge \theta_1 - \iota_X \theta_1 + \ell(X) = \ell(X) = X.$$

Thus, $c^l(D^1_{eq}) = c^l(X) = \left(\frac{\sqrt{-1}}{2\pi} \right)^{l} \det X$. Next, we compute $c^l(D^0_{eq}, D^1_{eq})$. Set $Y = -\frac{\|z\|^2}{1-t} X$. Then

$$\begin{aligned} \det R_k &= \det\left[\tau(X)^{(1)}, ..., \theta_0^{(k)}, ..., \tau(X)^{(l)}\right] \\ &= (-1)^l \frac{(1-t)^{l-1}}{\|z\|^{2l}} P \end{aligned}$$

with

$$
\begin{aligned}
P &= \sum_{\sigma\in\mathfrak{S}_l} \mathrm{sgn}(\sigma)(d\bar{z}_1\wedge dz_{\sigma(1)} + Y_{1\sigma(1)})\wedge\cdots\cdots\wedge \bar{z}_k dz_{\sigma(k)}\wedge\cdots\wedge(d\bar{z}_l\wedge dz_{\sigma(l)}+Y_{l\sigma(l)})\\
&= \sum_{\sigma\in\mathfrak{S}_l} \mathrm{sgn}(\sigma)\sum_{I,J}(d\bar{z}_{j_1}\wedge dz_{\sigma(j_1)})\wedge\cdots\wedge(\bar{z}_k dz_{\sigma(k)})\wedge\cdots\wedge(d\bar{z}_{j_q}\wedge dz_{\sigma(j_q)})Y_{i_1\sigma(i_1)}\cdots Y_{i_p\sigma(i_p)}\\
&= \sum_{I,J}(-1)^{\frac{|J|(|J|-1)}{2}}\epsilon(k,J)\cdot \bar{z}_k d\bar{z}_J\cdot\\
&\qquad\qquad \sum_{\sigma\in\mathfrak{S}_l}\mathrm{sgn}(\sigma) dz_{\sigma(j_1)}\wedge\cdots\wedge dz_{\sigma(k)}\wedge\cdots\wedge dz_{\sigma(j_q)}Y_{i_1\sigma(i_1)}\cdots Y_{i_p\sigma(i_p)},
\end{aligned}
$$

where $\{k\}$, $I=\{i_1,i_2,...,i_p\}$ and $J=\{j_1,j_2,...,j_q\}$ is a partition of $[l]$. Set

$$
Z=\sum_{i=1}^{l} dz_i\otimes e_i,\quad Y_j=\sum_{i=1}^{l} Y_{ji}\otimes e_i.
$$

By Lemma 4.6, we see

$$
\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_l}\mathrm{sgn}(\sigma)\, dz_{\sigma(j_1)}\wedge\cdots\wedge dz_{\sigma(k)}\wedge\cdots\wedge dz_{\sigma(j_q)}Y_{i_1\sigma(i_1)}\cdots Y_{i_p\sigma(i_p)}\otimes(e_1\wedge\cdots\wedge e_l)\\
=\ & Z\wedge\cdots\wedge Y_{i_1}\wedge\cdots\wedge Z\wedge\cdots\wedge Y_{i_p}\wedge\cdots\wedge Z\\
=\ & (-1)^m(Y_{i_1}\wedge\cdots\wedge Y_{i_p})\wedge(Z\wedge\cdots\wedge Z)
\end{aligned}
$$

where $m=\sum_{s=1}^{p} i_s-\frac{1}{2}p(p+1)$. Note that

$$
Y_{i_1}\wedge\cdots\wedge Y_{i_p}=\sum_{I'}Y_{I,I'}\,e_{I'},\quad Z\wedge\cdots\wedge Z=\sum_{J'}(q+1)!\,dz_{J'}\otimes e_{J'},
$$

where I' runs over subsets of p elements in $[l]$, J' runs over subsets of $(l-p)$ elements in $[l]$, and $Y_{I,I'}$ is a retainer minor of $[Y_{ij}]$ with respect to I and I'. Then

$$
\begin{aligned}
&(-1)^m(Y_{i_1}\wedge\cdots\wedge Y_{i_p})\wedge(Z\wedge\cdots\wedge Z)\\
=\ & (-1)^m\sum_{I'}Y_{I,I'}\sum_{J'}(q+1)!\,dz_{J'}\otimes(e_{I'}\wedge e_{J'})\\
=\ & (q+1)!\sum_{I',J'}(-1)^{\sum_{s=1}^{p}(i_s+i'_s)}Y_{I,I'}dz_{J'}\otimes(e_1\wedge\cdots\wedge e_l).
\end{aligned}
$$

Since $Y_{ii'}=-\frac{\|z\|^2}{1-t}X_{ii'}$, we have

$$
\det R_k=\sum_{I,J}\epsilon(k,J)(-1)^{\frac{|J|(|J|-1)}{2}+l}(|J|+1)!\sum_{I',J'}\epsilon_{(I,I')}X_{I,I'}\frac{\bar{z}_k d\bar{z}_J\wedge dz_{J'}}{\|z\|^{2(|J|+1)}}(1-t)^{|J|}.
$$

Hence,

$$\begin{aligned} c^l(D^0_{eq}, D^1_{eq}) &= -\left(\frac{\sqrt{-1}}{2\pi}\right)^l \sum_{k=1}^{l} \int_0^1 \det R_k \, dt \\ &= \sum_{k,I,J} \gamma_{(k,I,J)} \sum_{I',J'} \epsilon_{(I,I')} X_{I,I'} \frac{\bar{z}_k d\bar{z}_J \wedge dz_{J'}}{\|z\|^{2(|J|+1)}}. \end{aligned}$$

□

Example 4.8. For small l, the equivariant Bochner-Martinelli kernel is computed as follows.

1. In the case of $l = 1$,
$$\beta_{eq}(X) = \frac{\sqrt{-1}}{2\pi} \frac{\bar{z} dz}{\|z\|^2} = \frac{\sqrt{-1}}{2\pi} \frac{dz}{z}.$$
This is nothing but the original (non-equivariant) kernel.

2. In the case of $l = 2$, for $X = (X_{ij}) \in \mathfrak{u}(2)$,
$$\begin{aligned} \beta_{eq}(X) \;\; = \;\; & \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \left\{ \frac{\bar{z}_1 d\bar{z}_2 \wedge dz_1 \wedge dz_2}{\|z\|^4} - \frac{\bar{z}_2 d\bar{z}_1 \wedge dz_1 \wedge dz_2}{\|z\|^4} \right. \\ & \left. + X_{1,1} \frac{\bar{z}_2 dz_2}{\|z\|^2} - X_{1,2} \frac{\bar{z}_2 dz_1}{\|z\|^2} - X_{2,1} \frac{\bar{z}_1 dz_2}{\|z\|^2} + X_{2,2} \frac{\bar{z}_1 dz_1}{\|z\|^2} \right\}. \end{aligned}$$
To be more specific, we see the real part of this form: Set $z_1 = x_1 + \sqrt{-1} y_1$, $z_2 = x_2 + \sqrt{-1} y_2$ and
$$X = \begin{pmatrix} \sqrt{-1}A & B + \sqrt{-1}C \\ -B + \sqrt{-1}C & \sqrt{-1}D \end{pmatrix},$$
where A,B,C,D are real numbers. Then, a simple computation shows
$$\begin{aligned} & Re(\beta_{eq}(X)) \\ &= \frac{1}{2\pi^2\|z\|^4}(x_1 dx_2 \wedge dy_1 \wedge dy_2 + x_2 dx_1 \wedge dy_1 \wedge dy_2 \\ &\qquad -y_1 dx_1 \wedge dx_2 \wedge dy_2 - y_2 dx_1 \wedge dx_2 \wedge dy_1) \\ &+ \frac{1}{4\pi^2\|z\|^2}(-Ax_2dy_2 + Ay_2dx_2 + Bx_1dx_2 + By_1dy_2 - Bx_2dx_1 - By_2dy_1 \\ &\qquad +Cx_1dy_2 - Cy_1dx_2 + Cx_2dy_1 - Cy_2dx_1 - Dx_1dy_1 + Dy_2dx_2). \end{aligned}$$
This form coincides with the angular form for $\mathfrak{so}(4)$ of Proposition 4.10 in Paradan-Vergne [16].

4.4 Explicit formula of G-equivariant Thom form

In this subsection, applying the equivariant Chern-Weil map [2, 16] to Theorem 4.7, we obtain a formula expressing the equivariant Thom form for general G-vector bundles.

Definition 4.9. Let M be a manifold with a Lie group G-action. $\alpha \in \Omega^*(M)$ is called *horizontal* if $\iota_X\alpha = 0$ for any $X \in \mathfrak{g}$. We denote by $\Omega^*(M)_{hor}$ the subalgebla formed by the differential form that are horizontal. Also we define the algebra of the basic differential forms as follows;

$$\Omega^*(M)_{basic} := (\Omega^*(M)_{hor})^G$$

Let $\pi : P \to B$ be a principal G-bundle. And suppose G acts on a manifold F. For the associated bundle $\mathcal{F} = P \times_G F$, The Chern-Weil map in non-equivariant case gives the following isomorphism;

$$\phi_\theta^F : \Omega_G^*(F) \xrightarrow{\sim} \Omega^*(\mathcal{F}),$$

where θ is a connection form of P. In more details, for a G-equivariant form α, $\phi_\theta^F(\alpha)$ is equal to the projection of $\alpha(\Omega) \in \Omega(P \times F)^G$ on the basic space $\Omega(P \times F)_{basic} \xrightarrow{\sim} \Omega(\mathcal{F})$, where Ω is the curvature of the connection θ. We give the equivariant version of this construction in the following.

Let K and G be two compact Lie groups and P be a smooth manifold. We assume that $K \times G$ acts on P as follows; $(k, g)(y) := kyg^{-1}$, for $k \in K,\ g \in G$. And G acts on P freely. Then, $B = P/G$ is a manifold provided with a left action of K. There is K-invariant connection θ of P, since K is compact. Then, for a K-invariant connection θ, K-equivariant curvature of P is defined as follows;

$$\tilde{\Omega} := d_K\theta + \frac{1}{2}[\theta \wedge \theta],$$

where d_K is K-equivariant differential. Using this, we consider the equivariant Chern-Weil map;

$$\phi_\theta^F : \Omega_G^*(F) \xrightarrow{\sim} \Omega_K^*(\mathcal{F}).$$

It is defined as follows. For a G-equivatiant form α on F, $\phi_\theta^F(\alpha)$ is equal to the projection of $\alpha(\tilde{\Omega}) \in \Omega_K^*(P \times F)^G$ onto the basic space $\Omega_K^*(P \times F)_{basicG} \xrightarrow{\sim} \Omega_K^*(\mathcal{F})$.

Proposition 4.10. The equivariant Chern-Weil map above satisfies the following condition;

$$\phi_\theta^F \circ d_G = d_K \circ \phi_\theta^F.$$

We construct the explicit formula of G-equivariant Thom form in the following. First, we consider a G-equivariant vector bundle $\pi : E \to M$ and take a G-invariant metric for E. Then, for any $x \in M$, set $P_x = \{\xi : \mathbb{C}^l \to E_x : \text{isometry}\}$ and $P = \cup_{x \in M} P_x$ is naturally $U(l)$-equivariant G-principal bundle. The above argument applying for this, we get the following Chern-Weil maps;

$$\phi_\theta^{\mathbb{C}^l} : \Omega_{U(l)}^*(\mathbb{C}^l) \xrightarrow{\sim} \Omega_G^*(E)$$

$$\phi_\theta^{\mathbb{C}^l\setminus\{0\}} : \Omega^*_{U(l)}(\mathbb{C}^l \setminus \{0\}) \xrightarrow{\sim} \Omega^*_G(E \setminus \Sigma)$$

By using this, we may give the G-equivariant Thom form as follows;

$$(0, \phi_\theta^{\mathbb{C}^l} c^l_{eq}(D^1_{eq}), \phi_\theta^{\mathbb{C}^l\setminus\{0\}} c^l_{eq}(D^0_{eq}, D^1_{eq})) \in \Omega^{2l}_G(\mathcal{W}, W_0)$$

It follow from Proposition 4.10 that this form is closed. Then, we denote by $c^l_\Sigma(\pi^*E, s_\Delta)_{eq}$ the class of this form, where $s_\Delta : E \to \pi^*E$ is the diagonal section and Σ is the zero section of E. It is not difficult to show that the equivariant fiber integration is compatible with the equivariant Chern-Weil map. Thus, we have the following formula:

Theorem 4.11. In the above situation, we have

$$\Psi^E_{eq} = c^l_\Sigma(\pi^*E, s_\Delta)_{eq},$$

where Ψ^E_{eq} is the G-equivariant Thom class for E.

4.5 Equivariant Riemann-Roch Theorem

Let M be as above, s a G-invariant section in M. Let S denote the zero set of s (note that S is also G-invariant). Letting $U_0 = M \setminus S$ and U_1 a G-invariant neighborhood of S, we consider the G-invariant covering $\mathcal{U} = \{U_0, U_1\}$. We set $\lambda_{E^*} = \sum_{i=0}^l (-1)^i \bigwedge^i E^*$. Let ∇_0 be an s-trivial G-equivariant connection for E on U_0 and ∇_1 an arbitrary G-equivariant connection for E on U_1. Consider the Koszul complex associated to (E, s) (for more details, see [19]);

$$0 \longrightarrow \bigwedge^l E^* \xrightarrow{d_s} \cdots \xrightarrow{d_s} \bigwedge^1 E^* \xrightarrow{d_s} \bigwedge^0 E^* \longrightarrow 0$$

which is exact on U_0. It is easy to show that the family $\bigwedge^\bullet(\nabla^*_{eq})_0 = (\bigwedge^l(\nabla^*_{eq})_0, ..., \bigwedge^0(\nabla^*_{eq})_0)$ is compatible with the above sequence on U_0. The fact that $ch^*(\bigwedge^\bullet(\nabla^*_{eq})_0) = 0$ follows from this. Then, we have the localization $\mathrm{ch}^*_S(\lambda_{E^*}, s)_{eq}$ in $H^{2i}_G(M, M \setminus S; \mathbb{C})$, which is represented by the cocycle

$$\mathrm{ch}^*(\bigwedge^\bullet(\nabla^*_{eq})_\star) = (0, \mathrm{ch}^*(\bigwedge^\bullet(\nabla^*_{eq})_1), \mathrm{ch}^*(\bigwedge^\bullet(\nabla^*_{eq})_0, \bigwedge^\bullet(\nabla^*_{eq})_1))$$

We also have the inverse equivariant Todd class $\mathrm{td}^{-1}(E)_{eq}$, which is represented by the cocycle

$$\mathrm{td}^{-1}((\nabla_{eq})_\star) = (\mathrm{td}^{-1}((\nabla_{eq})_0), \mathrm{td}^{-1}((\nabla_{eq})_1), \mathrm{td}^{-1}((\nabla_{eq})_0, (\nabla_{eq})_1))$$

We give some definitions for the theorem in the following. Let $\rho : \mathbb{R} \times U_{01} \to U_{01}$ be the projection and we consider the connection $\tilde{\nabla}_{eq} = (1-t)\rho^*(\nabla_{eq})_0 + t\rho^*(\nabla_{eq})_1$ for ρ^*E. Let $\bigwedge^\bullet(\nabla^*_{eq})_\nu$ denote the family of connections $(\bigwedge^l(\nabla^*_{eq})_\nu, ..., \bigwedge^0(\nabla^*_{eq})_\nu)$ on U_ν, for $\nu = 0, 1$. Also we denote by $\bigwedge^\bullet(\tilde{\nabla}^*_{eq})$ the family $(\bigwedge^l(\tilde{\nabla}^*_{eq}), ..., \bigwedge^0(\tilde{\nabla}^*_{eq})_\nu)$. Let $\rho' : [0,1] \times U_{01} \to U_{01}$ be the restriction of ρ.

Theorem 4.12. In the above situation, we have

$$\mathrm{ch}^*(\bigwedge^\bullet(\nabla_{eq}^*)_\star) = c^l((\nabla_{eq})_\star)) \smile \mathrm{td}^{-1}((\nabla_{eq})_\star) + D_{eq}\tau$$

where $\tau = (0, 0, \tau_{01}), \tau = \rho'_*(c^l(\rho^*(\nabla_{eq})_0), \tilde{\nabla}_{eq}) \cdot d_{eq}\mathrm{td}^{-1}(\rho^*(\nabla_{eq})_1), \tilde{\nabla}_{eq})$.

The following corollary follows immediately from this.

Corollary 4.13. We have

$$\mathrm{ch}_S^*(\lambda_{E^*}, s)_{eq} = c_S^l(E.s) \cdot \mathrm{td}^{-1}(E)_{eq}$$

Also, as an applications of the above, we may get the equivariant universal localized Riemann-Roch theorem for embeddings by using the result in the previous section. Let $\pi : E \to M$ be a G-equivariant vector bundle of rank l. We have the G-equivariant Thom class Ψ_{eq}^E and the Thom isomorphism

$$T_E : H_G^*(M) \to H_G^{*+2i}(E, E \setminus \Sigma)$$

which is given by $T_E(\alpha) = \Psi_{eq}^E \cdot \pi^*\alpha$. Since $\Psi_{eq}^E = c_\Sigma^l(\pi^*E, s_\Delta)_{eq}$, applying the above Corollary to π^*E and s_Δ, we have :

Theorem 4.14 (Equivariant universal localized RR for embeddings)**.**

$$\begin{aligned}
\mathrm{ch}_\Sigma^*(\lambda_{\pi^*E^*}, s_\Delta)_{eq} &= \Psi_{eq}^E \cdot \mathrm{td}^{-1}(\pi^*E)_{eq} \\
&= T_E(\mathrm{td}^{-1}(E)_{eq})
\end{aligned}$$

4.6 Atiyah-Bott localization theorem

In this subsection, we prove the Atiyah-Bott localization theorem. The idea of it is to express the global integral of an closed equivariant form over M in terms of its "residues" at the fixed points of G.

Let M be a n-dimensional oriented, compact manifold with a compact Lie group G-action. Moreover, we assume the action of G on M preserves the orientation of G. Let X be a connected components of M^G, and suppose we have chosen an orientation on X (note that X is a closed embedded submanifold of M by Proposition 1.15). Let $\pi : N \to X$ be a G-equivariant tubular neighborhood and suppose the rank $2l$. We may consider the following G-equivariant Thom form of N (refer Section2.4):

$$(0, \pi^*\varepsilon_{eq}, -\psi_{eq}) \in \Omega_G^{2l}(\mathcal{U}, U_0),$$

where $\mathcal{U} = \{U_0, U_1\} = \{N \setminus \Sigma_X, N\}$. Then we have:

Lemma 4.15. In the above situation, we have the following equality: for d_{eq}-closed form $\mu \in \Omega^*_G(\mathcal{U}, U_0)$,

$$\int_X i^*\mu = \int_N \mu \wedge \pi^*\varepsilon_{eq},$$

where $i : X \to N$ is the (equivariant) inclusion.

Proof. Since X is a G-equivariant deformation retract of N, id_N and $i \circ \pi$ is homotopic. Thus, the restriction of any d_{eq}-closed form $\mu \in \Omega^*_G(\mathcal{U}, U_0)$ is homologous to $\pi^*i^*\mu$: $[\mu] = [\pi^*i^*\mu] \in H^*_G(\mathcal{U}, U_0)$, i.e. $\mu = \pi^*i^*\mu + D_{eq}\gamma$. Hence

$$\begin{aligned}
\int_N (0, \pi^*\varepsilon_{eq}, -\psi_{eq}) \smile \mu &= \int_N (0, \pi^*\varepsilon_{eq}, -\psi_{eq}) \smile \pi^*i^*\mu + \int_N (0, \pi^*\varepsilon_{eq}, -\psi_{eq}) \smile D_{eq}\gamma \\
&= \int_N (0, \pi^*\varepsilon_{eq}, -\psi_{eq}) \smile \pi^*i^*\mu \quad (\because \text{ Leibniz rule}) \\
&= \int_X \pi_*(0, \pi^*\varepsilon_{eq}, -\psi_{eq}) \smile i^*\mu \quad (\because \text{ projection formula}) \\
&= \int_X i^*\mu.
\end{aligned}$$

On the other hands,

$$\begin{aligned}
\int_N (0, \pi^*\varepsilon_{eq}, -\psi_{eq}) \smile \mu &= \int_N \mu \smile (0, \pi^*\varepsilon_{eq}, -\psi_{eq}) \\
&= \int_N \mu \wedge \pi^*\varepsilon_{eq}
\end{aligned}$$

□

Since X is fixed points of G, G acts on X trivially. Thus,

$$H^*_G(X) = \mathbb{C}[\mathfrak{g}^*]^G \otimes H^*(X)$$

so

$$\varepsilon_{eq}(N) := \varepsilon_{eq} = f_l + f_{l-1}\alpha_1 + \cdots + \alpha_l, \quad f_i \in \mathbb{C}_i(\mathfrak{g}^*)^G, \alpha_i \in \Omega^{2i}(X).$$

Suppose that $f_l \neq 0$. Then $\varepsilon_{eq}(N)$ may be written as

$$\varepsilon_{eq}(N) = f_l(1 - \frac{\alpha}{f_l}), \quad \alpha := -(f_{l-1}\alpha_1 + \cdots + \alpha_l) = -\varepsilon_{eq}(N) + f_l$$

Taking the inverse of the both formally, we get

$$\frac{1}{\varepsilon_{eq}(N)} = \frac{1}{f_l}\left(1 + \frac{\alpha}{f_l} + \frac{\alpha^2}{f_l^2} + \cdots + \frac{\alpha^{q-1}}{f_l^{q-1}}\right),$$

where $q-1$ is the largest integer which is less than or equal to $\frac{1}{2}\dim X$. We multiply the both sides by f_l^q, we have

$$\beta(N) := f_l^{q-1} + f_l^{q-2}\alpha + \cdots + \alpha^{q-1}.$$

Since $d_{eq}\varepsilon_{eq}(N) = 0$,

$$0 = d_{eq}\varepsilon_{eq}(N) = d_{eq}(f_l - \alpha) = d_{eq}f_l - d_{eq}\alpha = -d_{eq}\alpha \iff d_{eq}\alpha = 0.$$

Then we see that $d_{eq}\beta(N) = 0$, and we have

$$\beta(N)\varepsilon_{eq}(N) = f_l^q$$

Thus, if we replace μ by $\mu \wedge \pi^*\beta(N)$ in Proposition 4.15, we get

$$\int_X i^*\mu \wedge \beta(N) = f_l^q \int_N \mu$$

We may rewrite this as follows:

$$\int_N \mu = \int_X \frac{i^*\mu}{\varepsilon_{eq}(N)} \qquad (\star)$$

The formula express an integral over N in terms of an integral over X. Until now, we assumed that the codimension of X is even and $f_l \neq 0$. In the following, we give some propositions to satisfy these conditions.

Proposition 4.16. Let T be maximal torus of G, $x_0 \in X$ and $N_0 = N|_{x_0}$. Then, $f_l \neq 0$ if and only if the weights of the isotropy representation of T on N_0 are all non-zero.

Proof. Since the dimension of N_0 is even, we may regard N_0 as a complex vector space with dimension l. Putting G-invariant metric on N_0, we have a homomorphism

$$G \to U(N_0) = U(l)$$

. So we get a homomorphism

$$h : \mathbb{C}[\mathfrak{u}(l)^*]^{U(l)} \to \mathbb{C}[\mathfrak{g}^*]^G$$

If j_0 denotes the inclusion of N_0 in N, then

$$f_l = j_0^*\varepsilon_{eq}(N) = \varepsilon_{eq}(N_0) = \left(\frac{\sqrt{-1}}{2\pi}\right)^l h(\det),$$

where det is the determinants with respect to $\mathfrak{u}(l)$. Let T be a maximal torus of G with Lie algebra $\mathfrak{t}$. Then, for any $g \in G$, there exists $h \in G$ such that $h^{-1}gh \in T$. Since

$h(\det)$ is invariant for the adjoint action, thus it is determined by its restriction to $\mathfrak{t}^*$. So it is sufficient to compute the image of $h(\det)$ under the restriction map

$$\lambda : \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[\mathfrak{t}^*].$$

We use the weight decomposition of N_0 under the isotropy action of the torus: We may identify N_0 with $\mathbb{C}^l$ in such a way that the action of T is given by

$$(\exp \xi)v = (e^{is_1(\xi)}z_1, \ldots, e^{is_m(\xi)}z_l)$$

Directly computing, we get

$$\lambda \circ h(\det) = s_1 \cdots s_l$$

□

Note that since X is a connected component of fixed points, the isotropy representation of T on the fiber of the normal bundle of X is the same at all points. And, even if we take another maximal torus T', since T' is a conjugacy of T, so the isotropy representations is equivalent. Thus, the hypothesis of the above proposition does not depend on the choice of x_0 and T. Since $f_l \neq 0$ if and only if $N^T = N^G$, so we may assume that $G = T$ without loss of generality, i.e., G is an torus group.

Lemma 4.17. Let Λ_G be the group lattice of G. Then, there exists a finite number of weights $\beta_1, \ldots, \beta_q$ in Λ_G^* with the property that if $\xi \in \mathfrak{g}$ satisfies

$$\beta_i(\xi) \neq 0 \quad (i = 1, \ldots, q)$$

the corresponding vector field ξ_M on M is non-zero except at points of M^G.

Corollary 4.18. Let ξ be in $\mathfrak{g}_{\mathbb{Q}} := \{\frac{\eta}{m} \mid \eta \in \mathfrak{g}, m \in \mathbb{Z}\}$ and satisfy the Lemma 4.17. Then the circle group S^1 generated by ξ has the property

$$M^G = M^{S^1}$$

Then, consider the set

$$\{\xi \in \mathfrak{g} \mid \beta_i(\xi) \neq 0, i = 1, \ldots, q\}$$

The connected components of this set are called the action chambers of $\mathfrak{g}$

Proposition 4.19. The connected components of M^G are of even codimension. Moreover, having fixed an action chamber, one can assign to each of these components a canonical orientation

Theorem 4.20 (Localization theorem for S^1 action [15])**.** If $\mu \in \Omega^k_{S^1}(M)$ $(k \geq d)$ is d_{eq}-closed

$$\int_M \mu = \sum_X \int_X \frac{i_X^* \mu}{\varepsilon_{eq}(N_X)},$$

where the sum runs over the connected components of M^{S^1}, i_X is the inclusion of X into M, and $\varepsilon_{eq}(N_X)$ is the equivariant Euler class of the normal bundle of X.

Proof. We set $U = M - M^{S^1}$. Since S^1 is one-dimensional, it acts on U locally free; thus $H^k_{S^1}(U) = H^k(U/S^1)$ and in particular, since $\dim U/S^1 = d-1, H^k_{S^1}(U) = 0$ for $k \geq d$. Thus, there exists $\nu \in \Omega^{k-1}_{S^1}(U)$ such that $\mu = d_{eq}\nu$. Let U_X be a tubular neighborhood of X and ρ_X be a S^1-invariant smooth function which is identically one in a neighborhood of X and zero outside of U_X. Letting $\nu' = \nu - \sum \rho_X \nu$,

$$\mu = d_{eq}\nu' + \sum \mu_X,$$

where $\mu_X \in \Omega^k_{S^1}(U_X)_0$ and $\mu_X = \mu$ on a neighborhood of X. Therefore,

$$\int_M \mu = \sum_X \int_M \mu_X$$

and by the equation $(\star)$,

$$\int_M \mu_X = \int_X \frac{i_X^* \mu}{\varepsilon_{eq}(N_X)}.$$

$\square$

We shall extend this result to torus actions. Recall that if K is a subtorus of G, there is a natural map

$$H^*_G(M) \to H^*_K(M)$$

induced by the restriction map on forms. To be more specific, given equivariant form in $\Omega^*_G(M)$,

$$\mu : \mathfrak{g} \to \Omega^*(M)$$

one gets an equivariant form in $\Omega^*_K(M)$ by restricting μ to $\mathfrak{k}$, and the restriction $\mu \to \mu_{\mathfrak{k}}$, induced the above map on cohomology. In particular, let $\xi \in \mathfrak{g}_Q$ and let K be the circle subgroup of G generated by ξ. If ξ satisfies the Lemma 4.17 $M^G = M^K$; hence, applying $H^*_G(M) \to H^*_K(M)$ to $\mu|_{\mathfrak{k}}$ and $(\varepsilon_{eq}(N_X))|_{\mathfrak{k}}$:

$$\int_M \mu(\xi) = \sum_X \int_X \frac{i_X^* \mu(\xi)}{\varepsilon_{eq}(N_X)(\xi)}.$$

5 Simplicial Čech-de Rham cohomology

Until now, we have considered the equivariant Čech-de Rham cohomology using the Cartan model. In this section, we recall another definition of equivariant cohomology from a topological viewpoint, i.e., the Borel construction via the simplicial method. Then we introduce the equivariant Čech-version, say *simplicial Čech-de Rham cohomology.*

5.1 Equivariant cohomology by Borel construction

Let G be a topology group and X a G-space. Fix a classifying bundle $EG \to BG$ (for example, if G is a compact Lie group, it exists. See [11])

Definition 5.1. Let R be a commutative ring. The equivariant cohomology $H^*_G(X;R)$ is defined by

$$H^*_G(X;R) := H^*(X_G;R)$$

where the right hand side is the singular cohomology with coefficient in R of the associated fiber bundle

$$X_G := EG \times_G X \to BG.$$

The space X_G is called the *Borel construction.*

Remark. It is well-defined. Indeed, given a different model for classifying bundle, since these are G-equivariant homotopy equivariant, hence define the same equivariant cohomology.

For example, let G be a Lie group and M be G-manifold. If G acts on M freely and properly, we get a fiber bundle with fiber EG on M/G;

$$M_G = EG \times_G M = M \times_G EG \to M/G.$$

Since EG is contractible, we have $H^*_G(M) = H^*(M_G) = H^*(M/G)$.

Theorem 5.2 (e.g.,[11])**.** Let G be a compact Lie group acting on a smooth compact manifold. Then we have

$$H^*_G(M, \mathbb{C}) \cong H^*(\Omega^*_G(M), d_{eq})$$

5.2 Simplicial manifolds

Let Δ be the simplicial category, whose objects are non empty ordered sets

$$[n] := \{0, 1, ..., n\} \subset \mathbb{N} \cup \{0\} \quad \text{for} \quad n \geq 0$$

and the set of morphisms is the set of order preserving maps

$$\Delta([n], [m]) := \{f : [n] \to [m] \mid f(i+1) \geq f(i) \text{ for all } i\}$$

The morphisms of this category is generated by the following coface maps ∂^i and codegeneracy maps σ^i (it is easy to check):

$$\partial^i : [n-1] \to [n], \ \partial^i(j) := \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

$$\sigma^i : [n+1] \to [n], \ \partial^i(j) := \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

One may think of $[n]$ as the vertices of an n-standard simplex Δ^n. Any order-preserving map determines a map of simplices,

$$\Delta(f) : \Delta^n \to \Delta^m, \ \ \sum_{i=0}^{n} t_i e_i \mapsto \sum_{i=0}^{m} t_i e_{f(i)}.$$

Abusing the notation, we use the notation $\partial^i = \Delta(\partial^i)$ and $\sigma^i = \Delta(\sigma^i)$.

Here, we denote by **Top** the category of topological spaces and by **Mfd** the category of smooth manifolds.

Definition 5.3. (1) A simplicial space (or manifold) X is a contravariant functor from the simplicial category Δ into the category **Top** (or **Mfd**), i.e.,

$$X_\bullet : \Delta \to \mathbf{Top} \text{ (or } \mathbf{Mfd})$$

(2) A simplicial map is a natural transformation between two simplicial spaces.

Given a simplicial space $X_\bullet$, we may identify X with the collection of topological space $\{X_n\}$ ($X_n := X([n])$) together with continuous maps $X(f) : X_m \to X_n$ for any order-preserving map $f : [n] \to [m]$ such that $X(f \circ g) = X(g) \circ X(f)$. Also, a simplicial map $F_\bullet : X_\bullet \to Y_\bullet$ is identified with the collection of continuous map $F_n : X_n \to Y_n$ such that for any order-preserving map $f : [n] \to [m]$, the following diagram commutes:

$$\begin{array}{ccc} X_m & \xrightarrow{X(f)} & X_n \\ {\scriptstyle F_m}\downarrow & & \downarrow{\scriptstyle F_n} \\ Y_m & \xrightarrow[Y(f)]{} & Y_m \end{array}$$

The maps $\partial_i := X(\partial^i)$ and $\sigma_i := X(\sigma^i)$ are called the face and degeneracy maps of the simplicial space.

Definition 5.4. Let $X_\bullet$ be a simplicial space. The geometric (fat) realization of $X_\bullet$ is the quotient space

$$\|X\| := \coprod_{n=0}^{\infty} \Delta^n \times X_n / \sim$$

under the equivalence relations

$$(\partial^i t, x) \sim (t, \partial_i x)$$

for any $x \in X_n, t \in \Delta^{n-1}, i = 0, ..., n$ and $n = 1, 2, ...$

From this definition, we easily see that a simplicial map $F_\bullet : X_\bullet \to Y_\bullet$ induces a continuous map

$$\|F\| : \|X\| \to \|Y\|, \quad [(t, x)] \mapsto [(t, F_n(x))],$$

where $[(t, x)]$ is the equivalent class of $(t, x) \in \Delta^n \times X_n$. In other words, the geometric realization is a covariant functor from the category of simplicial spaces into the category of topological spaces.

In the following, we introduce some examples of simplicial spaces this paper deal with.

5.2.1 Classifying Bundle

Let G be a Lie group. A simplicial space $E_\bullet G$ is the collection of $E_n G = G^{n+1}$ with

$$(EG)(f) : E_m G \to E_n G, \quad (x_0, ..., x_m) \mapsto (x_{f(0)}, ..., x_{f(m)})$$

for any order-preserving map $f : [n] \to [m]$.

Lemma 5.5 (Dupont [7]). The geometric realization of $E_\bullet G$, that is $\|EG\|$, is contractible.

Proof. We denote by $[g_0, ..., g_n]$ the quotient of $\Delta^n \times \{(g_0, ..., g_n)\}$.Then, we may regard $[g_0, ..., g_n]$ as an n-simplex with elements g_i as its vertices and thus $[g_0, ..., g_n]$ as the 0-face of the $n+1$-simplex $[e, g_0, ..., g_n]$. Given $x \in [g_0, ..., g_n]$, using the linear structure in $[e, g_0, ..., g_n]$, consider the continuous map

$$H : [e, g_0, ..., g_n] \times [0, 1] \to [e, g_0, ..., g_n], \quad H(x, t) = (1 - t)x + te.$$

For each $[g_0, ..., g_n]$, putting the above map all together, we obtain a homotopy H from $\|EG\|$ onto $[e]$. Hence, $\|EG\|$ is contractible. $\square$

Now we define the right G-action on $\|EG\|$ by

$$g \cdot [(t, (g_0, ..., g_n))] := [(t, g_0 g^{-1}, ..., g_n g^{-1})]$$

and let $\|BG\| = \|EG\|/G$ with the natural projection $\|EG\| \to \|BG\|$.

Lemma 5.6. $\|EG\| \to \|BG\|$ is a principal G-bundle.

From the two lemmas above, $\|EG\| \to \|BG\|$ is the model for the classifying bundle.

5.2.2 Borel construction X_G as a simplicial space

Suppose X is a G-space. A simplicial space $(X_G)_\bullet$ is the collection of $(X_G)_n = G^n \times X$ with face and degeneracy maps

$$\begin{aligned} \partial_i(g_1, ..., g_n, x) &= \begin{cases} (g_2, ..., g_n, x) & \text{if } \ i = 0 \\ (g_1, ..., g_i g_{i+1}, ..., g_n, x) & \text{if } \ 0 < i < n \\ (g_1, ..., g_{n-1}, g_n \cdot x) & \text{if } \ i = n \end{cases} \\ \sigma_i(g_1, ..., g_n, x) &= (g_1, ..., g_i, e, g_{i+1}, ..., g_n, x) \end{aligned}$$

This simplicial space is naturally identified with the simplicial space $E_n G \times_G X$ as follows: Consider the following simplicial map (this map is well-defined)

$$E_n G \times_G X \to G^n \times X, \quad [(g_0, ..., g_n, x)] \mapsto (g_0 g_1^{-1}, ..., g_{n-1} g_n^{-1}, g_n \cdot x)$$

We easily see that this simplicial map gives rise to an isomorphism between simplicial spaces $E_\bullet G \times_G X$ and $(X_G)_\bullet$. Hence the geometric realization of this simplicial space is the Borel construction X_G.

5.2.3 Nerve of a Čech cover

Let X be a topological space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ an open covering of X as in section 2.2. A simplicial space $\mathcal{U}_\bullet X$ is the collection of

$$\mathcal{U}_n X := \coprod_{(\alpha_0, ..., \alpha_n) \in I^{(n)}} U_{\alpha_0 ... \alpha_n}$$

with face maps induced by inclusions, and degeneracy maps induced by the natural bijections $U_{\alpha_0 ... \alpha_i \alpha_i ... \alpha_n} \to U_{\alpha_0 ... \alpha_n}$.

Theorem 5.7. If X a paracompact Hausdorff space, the geometric realization of $\mathcal{U}_\bullet X$ is homotopy equivalent to X. In more detail, the natural inclusion $\mathcal{U}_n X \to X$ induces a homotopy equivalence map

$$\|\mathcal{U} X\| \to X$$

Proof. Let $g : \|\mathcal{U} X\| \to X$ be a map induced by the natural inclusion $\mathcal{U}_n X \to X$ (i.e., $g([(t, x)]) = x$). We define a homotopy inverse $f : X \to \|\mathcal{U} X\|$ of g as follows. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. For any $x \in X$, since $\{\mathrm{supp}\rho_\alpha\}$ is locally finite, there exists $(\alpha_0, ..., \alpha_n) \in I^{(n)}$ such that

$$\sum_{i=0}^{n} \rho_{\alpha_i}(x) = 1, \quad x \in U_{\alpha_0 ... \alpha_n}.$$

Then, define $f(x)$ to be the image of

$$(\sum_{i=0}^{n} \rho_{\alpha_i}(x) e_i, x) \in \Delta^n \times U_{\alpha_0 ... \alpha_n}$$

This definition is independent of the choice of $(\alpha_0, ..., \alpha_n) \in I^{(n)}$ satisfying the above conditions (notice that we allow $\rho_i(x) = 0$). For each $U_{\alpha_0 \ldots \alpha_n}$ such that $\sum_{i=0}^n \rho_{\alpha_i}(x) = 1$ $(x \in U_{\alpha_0 \ldots \alpha_n})$, since $f|_{U_{\alpha_0 \cdots \alpha_n}}$ is the composition of a map $X \ni x \mapsto (\sum_{i=0}^n \rho_{\alpha_i}(x) e_i, x) \in \Delta^n \times U_{\alpha_0 \ldots \alpha_n}$ and the quotient map $\coprod_{n=0}^\infty \Delta^n \times \mathcal{U}_n X \to \|\mathcal{U} X\|$, we easily see that f is continuous. By definition, we have $g \circ f(x) = x$. Also, we see that $f \circ g$ is homotopic to the identity $id_{\|\mathcal{U}X\|}$ as follows: The required homotopy is induced by the homotopies $I \times (\Delta^n \times U_{\alpha_0 \ldots \alpha_n}) \to \Delta^n \times U_{\alpha_0 \ldots \alpha_n}$

$$(t, (\sum_{i=0}^n s_i e_i, x)) \mapsto (\sum_{i=0}^n ((1-t)s_i + t\rho_{\alpha_i}(x)) e_i, x)$$

□

From this theorem, all homotopy invariants of X may be studied in terms of $\mathcal{U}_\bullet X$.

5.3 Simplicial de Rham theory

First, we define a singular cohomology of simplicial spaces $X_\bullet$

Definition 5.8. A simplicial space $X_\bullet$ defines a double complex

$$(C^*(X_\bullet; R), d, \delta)$$

where $C^q(X_p; R)$ is singular n-cochains with coefficient in an abelian group R and $d : C^q(X_p; R) \to C^{q+1}(X_p; R)$ is the usual coboundary map and $\delta : C^q(X_p; R) \to C^q(X_{p+1}; R)$ is defined in terms of the face maps as

$$\delta = \sum_{i=0}^{q+1} (-1)^i (\partial_i)^*.$$

The total complex associated to this double complex is denoted by $(C^*(X; R), D)$ where

$$C^r(X) := \bigoplus_{p+q=r} C^q(X_p)$$

and $D = \delta + (-1)^p d$ on $C^q(X_p)$.

Note that a simplicial map $F_\bullet : X_\bullet \to Y_\bullet$ induces a map of double complexes $F^* : C^q(Y_p; R) \to C^q(X_p; R)$.

Proposition 5.9 (See [7]). Suppose $X_\bullet$ is a simplicial space. Then, there exists isomorphism

$$H^*(\|X\|; R) \overset{\sim}{\to} H^*(C^*(X; R))$$

which is natural, i.e., if $F : X_\bullet \to Y_\bullet$ is a simplicial map then the following diagram commutes

$$\begin{array}{ccc} H^*(\|Y\|;R) & \xrightarrow{\sim} & H^*(C^*(Y;R)) \\ {\scriptstyle \|F\|^*}\Big\downarrow & & \Big\downarrow{\scriptstyle F^*} \\ H^*(\|X\|;R) & \xrightarrow{\sim} & H^*(C^*(X;R)) \end{array}$$

Proof. We will give a sketch of proof (for more details, see, [7]). For a simplicial space $\{X_\bullet\}$, consider the double simplicial set $S(X) = \{S_q(X_p)\}$ (where $S_q(X_p)$ is the set of all singular q-simplices in X_p) with face maps

$$S_q(X_p) \to S_q(X_{p-1}), \quad S_q(X_p) \to S_{q-1}(X_p).$$

We have the geometric realization for this double simplicial set

$$\|S(X)\| := \coprod_{p,q \geq 0} \Delta^p \times \Delta^q \times S_q(X_p)/\sim$$

with suitable identifications. Then $\|S(X)\|$ is of a CW-complex structure. We may check that

- The celluler cochain complex of $\|S(X)\|$ is naturally identified with the total complex $C(X)$. Thus, we have $H^*(\|S(X)\|) \cong H^*(C^*(X))$.
- $\|S(X)\|$ is homeomorphic with the geometric realization of the simplicial space $\|S(X_\bullet)\|$.
- There exists a natural simplicial map $\rho_\bullet : \|S(X_\bullet)\| \to X_\bullet$ and $\|\rho\| : \|S(X_\bullet)\| \to \|X_\bullet\|$ induces an isomorphism in cohomology with coefficient in R.

This implies that $H^*(\|X\|;R) \overset{\sim}{\to} H(C^*(X;R))$. $\square$

Now we will prove the simplicial de Rham theorem. Namely, we will show that the cohomology of geometric realization of a simplicial manifold equals to the the cohomology of the double complex of simplicial differential forms. For the proof of the simplicial de Rham theorem, let us recall briefly the usual de Rham theorem (For more details, see, i.e., [7]). We denote by $C_k^{sm}(M)$ a complex of smooth singular chains of a manifold M. Since one proves that the inclusion map $C_k^{sm}(M) \subset C_k(M)$ is a chain homotopy equivalence, dually the map $C^k(M, \mathbb{R}) \to \mathrm{Hom}(C_k^{sm}(M), \mathbb{R})$ is a cochain homotopy equivalence. Integration over smooth chains defines a homomorphism

$$I : \Omega^n(M) \to \mathrm{Hom}(C_n^{sm}(M), \mathbb{R})$$

by the formula

$$(I(\omega))(\sigma) := \int_\sigma \omega = \int_{\Delta^n} \sigma^* \omega$$

and this map is a cochain homotopy equivalence. This is a summary of the usual de Rham theorem.

Let $M_\bullet$ be a simplicial manifold. As the same way above, we consider the double complex of differential forms,

$$(\Omega^*(M_\bullet), \mathrm{d}, \partial)$$

where $\mathrm{d} : \Omega^q(M_p) \to \Omega^{q+1}(M_p)$ is the usual de Rham differential and $\partial : \Omega^q(M_p) \to \Omega^q(M_{p+1})$ is again alternating sum of pull-backs of face maps. Let $\Omega^*(M)$ be the associated total complex. Furthermore, we have an integration map $I : \Omega^q(M_p) \to \mathrm{Hom}(C_q^{sm}(M_p); \mathbb{R})$ as above. Then we have

Theorem 5.10 (Simplicial de Rham theorem)**.** $I : \Omega^q(M_p) \to \mathrm{Hom}(C_q^{sm}(M_p); \mathbb{R})$ induces a canonical isomorphism from the cohomology of $\Omega^*(M)$ to the singular cohomology of $\|M\|$ with coefficients in $\mathbb{R}$.

Proof. $I : \Omega^q(M_p) \to \mathrm{Hom}(C_q^{sm}(M_p); \mathbb{R})$ is a homomorphism of double complex. In fact, for $\omega \in \Omega^q(M_p),\ \sigma \in C_q^{sm}(M_p)$,

$$(I(\mathrm{d}\omega))(\sigma) = \int_\sigma \mathrm{d}\omega = \int_{\partial\sigma} \omega = (I(\omega))(\partial\sigma)$$

and

$$(I(\partial\omega))(\sigma) = \int_\sigma \partial\omega = \sum_{i=0}^{q+1}(-1)^i \int_\sigma \partial_i^*\omega = \sum_{i=0}^{q+1}(-1)^i \int_{(\partial_i)_*\sigma} \omega = (I(\omega))(\sum_{i=0}^{q+1}(-1)^i(\partial_i)_*\sigma).$$

By usual de Rham theorem, the induced map in d-cohomology is an isomorphism. Thus, since the associated spectral sequences coincide at the E_1-term, we see that the above map induces an isomorphism of cohomology groups for the total complex. So is $C^q(M_p, \mathbb{R}) \to \mathrm{Hom}(C_q^{sm}(M_p), \mathbb{R})$. □

5.4 Simplicial Čech-de Rham cohomology

In what follows, we consider the simplicial counterpart of Čech Cartan complex and it is a sort of the Čech version of the Borel model of the equivariant cohomology.

Let G be a compact Lie group and M a G-manifold. Given a G-invariant open covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ as in section 2.2, we may define a simplicial manifold $(\mathcal{U}M_G)_\bullet$ which is the collection of

$$(\mathcal{U}M_G)_p := \coprod_{(\alpha_0,\dots,\alpha_p)\in I^{(p)}} (G^p \times U_{\alpha_0\dots\alpha_p})$$

with the face map for each $G^p \times U_{\alpha_0\dots\alpha_p}$,

$$\partial_i : G^p \times U_{\alpha_0\dots\alpha_p} \to G^{p-1} \times U_{\alpha_0\dots\widehat{\alpha_i}\dots\alpha_p}$$

$$\partial_i(g_1,\dots,g_p,x) \quad = \quad \begin{cases} (g_2,\dots,g_p,x) & \text{if } \ i=0 \\ (g_1,\dots,g_ig_{i+1},\dots,g_p,x) & \text{if } \ 0<i<p \\ (g_1,\dots,g_{p-1},g_p\cdot x) & \text{if } \ i=p. \end{cases}$$

A similar argument in the following can be found in Kübel [13].

Theorem 5.11. The inclusion $G^p \times U_\alpha \subset G^p \times M$ induce a simplicial map

$$(\mathcal{U}M_G)_\bullet \to (M_G)_\bullet$$

which induces an isomorphism

$$H^*(\Omega^*(\mathcal{U}M_G)) \cong H^*(\Omega^*(M_G))$$

Proof. Consider the following triple complex (it is a Čech resolution with respect to the covering $G^\bullet \times \mathcal{U} = \{G^p \times U_\alpha\}_p$):

$$\check{\mathrm{C}}^{p,q,r}(G^\bullet \times \mathcal{U}) := \prod_{(\alpha_0,\ldots,\alpha_q)\in I^{(q)}} \Omega^r(G^p \times U_{\alpha_0\ldots\alpha_q})$$

with boundary $\partial + (-1)^r \mathrm{d} + (-1)^{p+r}\delta$, where ∂ is an alternating sum of pull-backs of face maps and d is the usual de Rham differential and δ is the usual boundary in Čech cohomology(as in section 2.2). Since differential forms form a fine sheaf, i.e., admit a partition of unity, in the same way of Theorem 2.10, one may show that the natural restriction

$$\Omega^r(G^p \times M) \to \check{\mathrm{C}}^{p,0,r}(G^\bullet \times \mathcal{U}) = \prod_{\alpha\in I} \Omega^r(G^p \times U_\alpha)$$

induces the isomorphism between the cohomology of the de Rham double complex on $G^\bullet \times M$ and the above triple complex.

On the other hand, we have an equality of vector space

$$\Omega^{p,r}(\mathcal{U}M_G)(= \Omega^r((\mathcal{U}M_G)_p)) = \check{\mathrm{C}}^{p,p,r}(G^\bullet \times \mathcal{U}).$$

That is, $\Omega^{\bullet,r}(\mathcal{U}M_G)$ coincides on the set-level with the diagonal of the bisimplicial complex $\check{\mathrm{C}}^{\bullet,\bullet,r}(G^\bullet \times \mathcal{U})$. Furthermore, it is easy to check induced map by face maps in these complex coincides. Hence the cohomology of $\Omega^{\bullet,r}(\mathcal{U}M_G)$ is isomorphic to the cohomology of $\check{\mathrm{C}}^{\bullet,\bullet,r}(G^\bullet \times \mathcal{U})$.

By the generalized Eilenberg-Zilber theorem (see [9]), there exists a chain homotopy equivalence between the diagonal and the total complex of a bisimplicial complex (in this case $\bigoplus_{p+q+r=n} \check{\mathrm{C}}^{p,q,r}(G^\bullet \times \mathcal{U})$ is chain homotopic to $\bigoplus_{p+r=n} \check{\mathrm{C}}^{p,p,r}(G^\bullet \times \mathcal{U})$). Composing these maps defined above, we have the map

$$\bigoplus_{p+r=n} \Omega^{p,r}(\mathcal{U}M_G) \to \bigoplus_{p+r=n} \check{\mathrm{C}}^{p,p,r}(G^\bullet \times \mathcal{U}) \to \bigoplus_{p+q+r=n} \check{\mathrm{C}}^{p,q,r}(G^\bullet \times \mathcal{U}) \leftarrow \bigoplus_{p+r=n} \Omega^{p,r}(M_G)$$

which induces the desired isomorphism in cohomology. □

From this theorem and the simplicial de Rham theory, we have the following isomorphism:

$$H^*(\Omega^*(\mathcal{U}M_G)) \cong H^*(M_G) = H^*_G(M)$$

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