Non-Abelian discrete flavor symmetries from modular symmetry in string compactification (超弦理論におけるモジュラー対称性由来の非可換離散フレーバー対称性に関する研究)

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Abstract

Modular symmetry is a symmetry on two dimensional torus T^2 typically considered in string compactification. First, we study the modular symmetry in magnetized D-brane models on T^2. Zero-modes on T^2 with magnetic flux $M = 2$ (in a certain unit) are transformed as doublet of S_3 with certain identification under the transformation of the modulus. We also study the modular symmetry in heterotic orbifold models. The T^2/Z_4 orbifold model has the same modular symmetry as the magnetized D-brane model with $M = 2$. Next, we study lepton flavor models with S_3 and A_4 symmetries from the modular group. We consider S_3 model with flavons and A_4 models with no flavon. In A_4 model, we classify our neutrino models along with type I seesaw model, Weinberg operator model, and the Dirac neutrino model. In the normal hierarchy of neutrino masses, the seesaw model is available by taking account for recent experimental data of neutrino oscillations. In the case of inverted hierarchy, the Dirac neutrino model is consistent with experimental data.

Contents

1 Introduction

The standard model (SM) is a theory explaining electroweak and strong interactions and which is precisely confirmed by many experiments. On the other hand, there are discrepancies between theoretical and experimental values such as anomalous magnetic moment of muon (muon $g-2$) at BNL [1] and anomalies in semileptonic decays of B meson (B-anomalies) at BaBar [2] and at LHCb [3]. These discrepancies can be hints for new physics beyond the SM. The existence of neutrino oscillation phenomena, and hence non-vanishing mass of neutrinos, are also important hints for new physics, which are observed by Super-Kamiokande [4–6], SNO [7], Borexino [8–10], IceCube [11], KamLAND [12], T2K [13], and NOνA [14]. The flavor structure of leptons affects directly on neutrino oscillation, and possibly on muon $g-2$ and B-anomalies. Thus, it is worthwhile to understand the lepton flavor structure.

One of interesting ideas on the flavor structure is to impose non-Abelian discrete symmetries for flavors on a theory. Many models have been proposed by using S_3, A_4, S_4, A_5 and other groups with lager orders [15–19]. In particular, S_3 and A_4 symmetries are small and attractive for flavor model. S_3 is the permutation group of three elements and which is the smallest non-Abelian discrete group. This symmetry is used as permutation symmetry of three families of leptons to lead so-called tri-bimaximal type of the PMNS matrix [20]. A_4 is the minimal group which has a triplet as its irreducible representation and enable us to explain three families of quarks and leptons naturally [21–26]. However, variety of models is so wide that it is difficult to obtain clear clues of flavor symmetry. Indeed, symmetry breakings are required to reproduce realistic mixing angles [27]. The effective Lagrangian of a typical flavor model is given by introducing the gauge singlet scalars which are so-called flavons. Those vacuum expectation values (VEVs) determine the flavor structure of quarks and leptons. Finally, the breaking sector of flavor symmetry typically produces many unknown parameters.

The absence of gravity in the SM is also a hint for new physics, and superstring theory is a promising candidate of unified theory including gravity. Superstring theory with certain compactifications can lead to non-Abelian discrete flavor symmetries. For example, heterotic orbifold models lead to D_4, $\Delta(54)$, etc. [28]. (See also [29, 30].) Similar flavor symmetries are also derived in type II magnetized and intersecting D-brane models [31, 32]. On the other hand, string theory on tori or orbifolds has the modular symmetry which acts non-trivially on flavors of quarks and leptons [33–38]. In this sense, the modular symmetry is a non-Abelian discrete flavor symmetry.

It is interesting that the modular group includes S_3, A_4, S_4, and A_5 as its finite subgroups, $\Gamma(N)$. However, there is a difference between the modular symmetry and the usual flavor symmetry. Yukawa couplings are written as modular forms, functions of the modulus τ, and transform non-trivially under the modular symmetry as well as fields. On the other hand, Yukawa couplings are invariants in the usual flavor symmetries. In this aspect, an attractive ansatz was proposed by taking $\Gamma(3) \simeq A_4$ in Ref. [39] where Yukawa couplings are A_4 triplets of modular forms, and both left-handed leptons and right-handed neutrinos are A_4 triplets while right-handed charged leptons are A_4 singlets. (See also [40].) Along with this work, $\Gamma(4) \simeq S_4$ [41, 42] and $\Gamma(5) \simeq A_5$ [43] have been discussed.

In this paper, we have following two purpose. First, we study more how modular transformation is represented by zero-modes in magnetized D-brane models, and to discuss relations between modular transformation and non-Abelian flavor symmetries in magnetized D-brane models. Intersecting D-brane models have the same aspects as magnetized D-brane models, because they are T-dual to each other. Furthermore, intersecting D-brane models in type II superstring theory and heterotic string theory have similarities, e.g. in two-dimensional conformal field theory. Thus, we also study modular symmetry and non-Abelian discrete flavor symmetries in heterotic orbifold models. Next, we present a comprehensive study of $\Gamma(2) \simeq S_3$ and $\Gamma(3) \simeq A_4$ numerically by taking account of the recent experimental data of neutrino oscillations. The mass matrices of neutrinos and charged leptons are essentially given by the expectation value of the modulus τ, which is the only source of modular invariance breaking. However, there are freedoms for the assignments of irreducible representations and modular weights to leptons.

This thesis is organized as follows. In chapter 2, we study the relation between modular symmetry and string compactification. In sec 2.1, we introduce the modular group and its finite subgroups. We study magnetized D-brane models in section 2.2 and heterotic orbifold models in section 2.3. In chapter 3, we study lepton flavor models with non-Abelian discrete symmetry coming from the modular symmetry We study S_3 models in section 3.3, and A_4 models in section 3.4. Section 4 is devoted to a summary.

This paper is based on [44–46].

2 Modular symmetry and string compactification

Since superstring theory is ten-dimensional theory, the extra six-dimensional space must be sufficiently small so as not to be observed. Torus is a simple but interesting manifold for compact space because of its flatness and doubly periodic structure. There are well known two ways of compactification using torus to build realistic models from string theory, magnetized torus models and toroidal orbifold models. In the first way, we impose the vacuum expectation values of some gauge fields on the torus, and earn multi-generation chiral fermions. Magnetized D-brane models are models in this class. In the second way, we consider quotient of torus by rotational symmetry Z_N such as T^2/Z_N, T^4/Z_N and T^6/Z_N . Heterotic orbifold models are models in this class.

In this chapter, we study the relation between modular symmetry and string compactification.

2.1 Modular group and its finite subgroups

In this section, we introduce the modular transformation, or the modular group, and its finite subgroups.

Two dimensional torus T^2 is constructed by $\mathbb{C}/\Lambda$, where Λ is a lattice spanned by basis vectors (α_1, α_2). We can take α_i as $\alpha_1 = 2\pi R$ and $\alpha_2 = 2\pi R\tau$ by using $R > 0$ and $\tau \in \mathbb{C}$ defined in upper half-plane $\mathrm{Im}(\tau) > 0$ without loss of generality. τ is called the modulus which represents the complex structure of the T^2. There are specific transformations of the basis vectors keeping the lattice unchanged, and denoted by

$$\begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{2.1}$$

The transformation (2.1) transform the modulus $\tau = \alpha_2/\alpha_1$ as

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{2.2}$$

This transformation is called the modular transformation, and this group is called the modular group Γ. Note that the transformation (2.2) keeps τ

on upper half-plane $\mathrm{Im}(\tau) > 0$. The modular transformation includes two important generators, S and T,

$$\begin{aligned} S &: \tau \to -\frac{1}{\tau}, \\ T &: \tau \to \tau + 1, \end{aligned} \tag{2.3}$$

which satisfy

$$S^2 = \mathbb{I}, \quad (ST)^3 = \mathbb{I}. \tag{2.4}$$

Thus, the modular group is represented by

$$\Gamma \simeq \left\{ S, T \,\middle|\, S^2 = \mathbb{I}, (ST)^3 = \mathbb{I} \right\}. \tag{2.5}$$

When we impose an additional algebraic relation

$$T^N = \mathbb{I} \tag{2.6}$$

on Γ, we earn subgroups of Γ represented by

$$\Gamma(N) = \left\{ S, T \,\middle|\, S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, (T)^N = \mathbb{I} \right\}. \tag{2.7}$$

The $\Gamma(N)$ is a finite Non-Abelian discrete group, and it is found that $\Gamma(2) \simeq S_3$, $\Gamma(3) \simeq A_4$, $\Gamma(4) \simeq S_4$, and $\Gamma(5) \simeq A_5$.

The group A_4 is the symmetry of tetrahedron, and which is often called the tetrahedral group $T = A_4$. It may also be useful to mention $\Delta(3N^2) \simeq (Z_N \times Z_N) \rtimes Z_3$ and $\Delta(6N^2) \simeq (Z_N \times Z_N) \rtimes S_3$, and $S_3 \simeq \Delta(6)$, $A_4 \simeq \Delta(12)$, and $S_4 \simeq \Delta(24)$.

2.2 Modular transformation in magnetized D-brane models

In this section, we study modular transformation of zero-mode wavefunctions in magnetized D-brane models.

2.2.1 Zero-mode wavefunction

First, we give a brief review on zero-mode wavefunctions on torus with magnetic flux [37]. We concentrate on T^2 with $U(1)$ magnetic flux for simplicity.

The complex coordinate on T^2 is denoted by $z = x^1 + \tau x^2$, where τ is the complex modular parameter, and x^1 and x^2 are real coordinates. The metric on T^2 is given by

$$g_{\alpha\beta} = \begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} = (2\pi R)^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{2.8}$$

We identify $z \sim z+1$ and $z \sim z+\tau$ on T^2.

On T^2, we introduce the $U(1)$ magnetic flux F,

$$F = i\frac{\pi M}{\mathrm{Im}\tau}(dz \wedge d\bar{z}), \tag{2.9}$$

which corresponds to the vector potential,

$$A(z) = \frac{\pi M}{\mathrm{Im}\tau}\mathrm{Im}(\bar{z}dz). \tag{2.10}$$

Here we concentrate on vanishing Wilson lines.

On the above background, we consider the zero-mode equation for the spinor field with the $U(1)$ charge $q = 1$,

$$i\not{D}\Psi = 0. \tag{2.11}$$

The spinor field on T^2 has two components,

$$\Psi(z, \bar{z}) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{2.12}$$

The magnetic flux should be quantized such that M is integer. Either ψ_+ or ψ_- has zero-modes exclusively for $M \neq 0$. For example, we set M to be positive. Then, ψ_+ has M zero-modes, while ψ_- has no zero-mode. Hence, we can realize a chiral theory. Their zero-mode profiles are given by

$$\psi^{j,M}(z) = \mathcal{N} e^{i\pi M z \frac{\mathrm{Im}z}{\mathrm{Im}\tau}} \cdot \vartheta \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (Mz, M\tau), \tag{2.13}$$

with $j = 0, 1, \cdots, (M-1)$, where ϑ denotes the Jacobi theta function,

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu, \tau) = \sum_{l \in \mathbf{Z}} e^{\pi i (a+l)^2 \tau} e^{2\pi i (a+l)(\nu+b)}. \tag{2.14}$$

Here, $\mathcal{N}$ denotes the normalization factor given by

$$\mathcal{N} = \left(\frac{2\mathrm{Im}\tau M}{\mathcal{A}^2}\right)^{1/4}, \tag{2.15}$$

with $\mathcal{A} = 4\pi^2 R^2 \mathrm{Im}\tau$.

The ground states of scalar fields also have the same profiles as $\psi^{j,M}$. Thus, the Yukawa coupling including one scalar and two spinor fields can be computed by using these zero-mode waverfunctions. Zero mode wavefunctions satisfy the following relation,

$$\begin{aligned}\psi^{i,M}\psi^{j,M} &= \mathcal{A}^{-1/2}(2\mathrm{Im}\tau)^{1/4}\left(\frac{MN}{M+N}\right)^{1/4} \\ &\quad \times \sum_m \psi^{i+j+Mm,M+N} \cdot \vartheta \begin{bmatrix} \frac{Ni-Mj+MNm}{MN(M+N)} \\ 0 \end{bmatrix} (0, MN(M+N)\tau)\,.\end{aligned} \tag{2.16}$$

By use of this relation, their Yukawa couplings are given by the wavefunction overlap integral,

$$\begin{aligned}Y_{ijk} &= y\int d^2z \psi^{i,M}\psi^{j,N}(\psi^{k,M'})^* \\ &= y\left(\frac{2\mathrm{Im}\tau}{\mathcal{A}^2}\right)^{1/4} \sum_{m\in Z_{M'}} \delta_{k,i+j+Mm} \cdot \vartheta \begin{bmatrix} \frac{Ni-Mj+MNm}{MNM'} \\ 0 \end{bmatrix} (0, MNM'\tau)\,,\end{aligned} \tag{2.17}$$

where y is constant. This Yukawa coupling vanishes for $M' \neq M+N$. Similarly, we can compute higher order couplings using the relation (2.16) [47]. In the above equation, the Kronecker delta $\delta_{k,i+j+Mm}$ implies the coupling selection rule. For $g = \gcd(M, N, M')$, non-vanishing Yukawa couplings appear only if

$$i + j = k \qquad (\mathrm{mod}\ g). \tag{2.18}$$

Hence, we can definite Z_g charges in these couplings [31].

2.2.2 Modular transformation of zero-mode

The τ in this context is the same thing as the modulus τ in chapter 2.1. Since the zero-mode wave functions $\psi^{j,M}(z)$ in (2.13) and hence the Yukawa couplings in (2.17) depend on the modulus τ, they transform under the modular transformation (2.2). To investigate the transformation row of the zero-modes $\psi^{j,M}(z)$ under the modular transformation, we check their behavior along to the generators of the modular transformation, S- and T-transformation (2.3).

Following [38], we restrict ourselves to even magnetic fluxes M ($M > 0$).

The zero-mode wavefunctions transform as

$$\psi^{j,M} \to \frac{1}{\sqrt{M}} \sum_k e^{2\pi i jk/M} \psi^{k,M} \tag{2.19}$$

under S-transformation according to [37, 38], and transform as

$$\psi^{j,M} \to e^{\pi i j^2/M} \psi^{j,M} \tag{2.20}$$

under T-transformation according to [38]. Generically, the T-transformation satisfies

$$T^{2M} = \mathbb{I}, \tag{2.21}$$

on the zero-modes, $\psi^{j,M}$. Furthermore, in Ref. [38] it is shown that

$$(ST)^3 = e^{\pi i/4}, \tag{2.22}$$

on the zero-modes, $\psi^{j,M}$.

In what follows, we study more concretely.

2.2.3 Magnetic flux $M = 2$

Let us study the case with the magnetic flux $M = 2$ concretely. There are two zero-modes, $\psi^{0,2}, \psi^{1,2}$. The S-transformation acts on these zero-modes as

$$\begin{pmatrix} \psi^{0,2} \\ \psi^{1,2} \end{pmatrix} \longrightarrow S_{(2)} \begin{pmatrix} \psi^{0,2} \\ \psi^{1,2} \end{pmatrix}, \qquad S_{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.23}$$

The T-transformation acts as

$$\begin{pmatrix} \psi^{0,2} \\ \psi^{1,2} \end{pmatrix} \longrightarrow T_{(2)} \begin{pmatrix} \psi^{0,2} \\ \psi^{1,2} \end{pmatrix}, \qquad T_{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{2.24}$$

They satisfy the following algebraic relations,

$$S_{(2)}^2 = \mathbb{I}, \qquad T_{(2)}^4 = \mathbb{I}, \qquad (S_{(2)} T_{(2)})^3 = e^{\pi i/4} \mathbb{I}. \tag{2.25}$$

They construct a closed algebra with the order 192, which we denote here by $G_{(2)}$. By such an algebra, modular transformation is represented by two zero-modes, $\psi^{0,2},\psi^{1,2}$. We find that $(S_{(2)}T_{(2)})^3$ is a center. Indeed, there are eight center elements and their group is Z_8. Other diagonal elements correspond to Z_4, which is generated by $T_{(2)}$. Here, we denote

$$a=(S_{(2)}T_{(2)})^3, \qquad a'=T_{(2)}. \tag{2.26}$$

The diagonal elements are represented by $a^m a'^n$, i.e. $Z_8\times Z_4$.

Here, we examine the right coset Hg for $g\in G_{(2)}$, where H is the above $Z_8\times Z_4$, i.e. $H=\{a^m a'^n\}$. There would be $6(=192/(8\times 4))$ cosets. Indeed, we obtain the following six cosets:

$$H, \quad HS_{(2)}, \quad HS_{(2)}T^k_{(2)}, \quad HS_{(2)}T^2_{(2)}S_{(2)}, \tag{2.27}$$

with $k=1,2,3$. By simple computations, we find

$$HS_{(2)}T^k_{(2)}S_{(2)}\sim HS_{(2)}T^{4-k}_{(2)}S_{(2)}, \quad HS_{(2)}T^2_{(2)}S_{(2)}T\sim HS_{(2)}T^2_{(2)}S_{(2)}. \tag{2.28}$$

Furthermore, we would make a (non-Abelian) subgroup with the order 6 by choosing properly six elements such that we pick one element up from each coset and their algebra is closed. The non-Abelian group with the order 6 is unique, i.e. S_3. For example, we may be able to obtain the Z_3 generator from $HS_{(2)}T_{(2)}$ because $(S_{(2)}T_{(2)})^3\in H$. That is, we define

$$b=a^m a'^n S_{(2)}T_{(2)}. \tag{2.29}$$

Then, we require $b^3=\mathbb{I}$. There are three solutions, $(m,n)=(3,2)$, (5,0) mod (8,4). Similarly, we can obtain the Z_2 generator e.g. form $HS_{(2)}T^2_{(2)}S_{(2)}$ because $(S_{(2)}T^2_{(2)}S_{(2)})^2\in H$. Then, we define

$$c=a^{m'}a'^{n'}S_{(2)}T^2_{(2)}S_{(2)}. \tag{2.30}$$

We find $c^2=\mathbb{I}$ when $n'=-m'$ mod 4. On top of that, we require $(bc)^2=\mathbb{I}$, and that leads to the conditions, $n=-m'-1$ mod 4 and $m=m'+2$ mod 8. As a result, there are six solutions, $(m,n,m')=(3,2,1)$, (3,2,5), (5,0,3), (5,0,7) with $n'=-m'$ mod 4.

For example, for $(m,n,m')=(3,2,5)$ we write

$$b=\frac{1}{\sqrt{2}}\begin{pmatrix}\rho^3 & \rho^{-3}\\ \rho^{-1} & \rho^{-3}\end{pmatrix}, \qquad c=\begin{pmatrix}0 & \rho^{-3}\\ \rho^3 & 0\end{pmatrix}. \tag{2.31}$$

The six elements of the subgroup are written explicitly,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \rho^{-3} \\ \rho^3 & 0 \end{pmatrix},$$
$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} \rho^3 & \rho^{-3} \\ \rho^{-1} & \rho^{-3} \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} \rho^{-3} & \rho \\ \rho^3 & \rho^3 \end{pmatrix} \tag{2.32}$$

where $\rho = e^{2\pi i/8}$. They correspond to $S_3 \simeq \Gamma(2) \simeq \Delta(6)$ because they satisfy the following algebraic relations,

$$c^2 = b^3 = (bc)^2 = \mathbb{I}. \tag{2.33}$$

Moreover, they satisfy the following algebraic relation with $Z_8 \times Z_4$,

$$b^{-1}ab^1 = a, \qquad cac = a, \qquad b^{-1}a'b = a, \qquad ca'c^{-1} = a^2a'^3. \tag{2.34}$$

Thus, the algebra of $G_{(2)}$ is isomorphic to $(Z_8 \times Z_4) \rtimes S_3$.

We have started by choosing $HS_{(2)}T^2_{(2)}S_{(2)}$ for a candidate of the Z_2 generator. We can obtain the same results by starting with $HS_{(2)}$ for a candidate of the Z_2 generator.

2.2.4 Magnetic flux $M = 4$

Similarly, we study the case with the magnetic flux $M = 4$. There are four zero-modes, $\psi^{i,M}$ with $i = 0, 1, 2, 3$. The S and T-transformations are represented by $\psi^{i,M}$,

$$S_{(4)} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}, \qquad T_{(4)} = \begin{pmatrix} 1 & & & \\ & e^{\pi i/4} & & \\ & & -1 & \\ & & & e^{\pi i/4} \end{pmatrix}. \tag{2.35}$$

This is a reducible representation. In order to obtain irreducible representations, we use the flowing basis,

$$\begin{pmatrix} \psi^{0.4} \\ \psi^{1,4}_+ \\ \psi^{2,4} \end{pmatrix} = \begin{pmatrix} \psi^{0.4} \\ \frac{1}{\sqrt{2}}(\psi^{1,4} + \psi^{3,4}) \\ \psi^{2,4} \end{pmatrix}, \qquad \psi^{1,4}_- = \frac{1}{\sqrt{2}}(\psi^{1,4} - \psi^{3,4}). \tag{2.36}$$

This is nothing but zero-modes on the T^2/Z_2 orbifold [48]. The former corresponds to Z_2 even states, while the latter corresponds to the Z_2 odd state.

Note that $(ST)^3$ transforms the lattice basis $(\alpha_1, \alpha_2) \to (-\alpha_1, -\alpha_2)$. Thus, it is reasonable that the zero-modes on the T^2/Z_2 orbifold correspond to the irreducible representations.

The S and T-representations by the Z_2 odd zero-mode are quite simple, and these are represented by

$$S_{(4-)} = i, \qquad T_{(4)-} = e^{\pi i/4}. \tag{2.37}$$

Their closed algebra is Z_8.

On the other hand, the S and T-transformations are represented by the Z_2 even zero-modes,

$$S_{(4)+} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \qquad T_{(4)+} = \begin{pmatrix} 1 & & \\ & e^{\pi i/4} & \\ & & -1 \end{pmatrix}. \tag{2.38}$$

They satisfy the following algebraic relation,

$$(S_{(4)+})^2 = \mathbb{I}, \qquad (T_{(4)+})^8 = \mathbb{I}, \qquad (S_{(4)+}T_{(4)+})^3 = e^{\pi i/4}\mathbb{I}. \tag{2.39}$$

We denote the closed algebra of $S_{(4)+}$ and $T_{(4)+}$ by $G_{(4)+}$. Its order is equal to 768, and it includes the center element $(S_{(4)+}T_{(4)+})^3$, i.e. Z_8. Other diagonal elements correspond to Z_8, which is generated by $T_{(4)+}$. Again, we denote $a = (S_{(4)+}T_{(4)+})^3$ and $a' = T_{(4)+}$, and the diagonal elements are written by $a^m a'^n$, i.e. $Z_8 \times Z_8$.

Similar to the case with $M = 2$, we examine the coset structure, Hg. Indeed, there are the following 12 cosets:

$$H, \quad HS_{(4)+}, \quad HS^k_{(4)+}, \quad HS_{(4)+}T^\ell_{(4)+}S_{(4)+}, \tag{2.40}$$

where $k = 1, \cdots, 7$ and $\ell = 2, 4, 6$. By simple computation, we find that

$$\begin{aligned} &HS_{(4)+}T^k_{(4)+}S_{(4)+} \sim HS_{(4)+}T^{8-k}_{(4)+}, \\ &HS_{(4)+}T^\ell_{(4)+}S_{(4)+}T \sim HS_{(4)+}T^{8-\ell}_{(4)+}S_{(4)+} \end{aligned} \tag{2.41}$$

for $k =$ odd and $\ell =$ even.

We make a subgroup with the order 12 by choosing properly 12 elements such that we pick one element up from each coset and their algebra is closed. The non-Abelian group with the order 12 are D_6, Q_6 and A_4. Among them,

A_4 would be a good candidate. Indeed, we can obtain the Z_3 generator from $HS_{(4)+}T_{(4)+}$, gain. That is, we define

$$t = a^m a'^n S_{(4)+} T_{(4)+}. \tag{2.42}$$

The solutions for $t^3 = \mathbb{I}$ are obtained by $(m, n) = (1, 4)$, (3,6), (5,0), and (7,2). We also define

$$s = a^{m'} a'^{n'} S_{(4)+} T^4_{(4)+} S_{(4)+}. \tag{2.43}$$

The solutions for $s^2 = \mathbb{I}$ are obtained by $(m', n') = (0, 0)$, (0,4), (4,0) and (4,4). These two generators satisfy $(st)^3 = \mathbb{I}$ if $(m', n') = (0, 4)$, and (4,0), i.e.

$$s = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & -1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}. \tag{2.44}$$

As a result, they satisfy

$$s^2 = t^3 = (st)^3 = \mathbb{I}. \tag{2.45}$$

That is the A_4 algebra.

2.2.5 Large magnetic flux M

For larger magnetic fluxes, S and T-transformations are represented by zero-modes $\psi^{j,M}$, but those are reducible representations. The irreducible representations are obtained in the T^2/Z_2 orbifold basis,

$$\psi^{j,M}_{\pm} = \frac{1}{\sqrt{2}} \left(\psi^{j,M} \pm \psi^{M-j,j} \right). \tag{2.46}$$

The representations of $T_{(M)}$ are simply obtained by

$$T_{(M)+} \begin{pmatrix} \psi^{0,M}_{+} \\ \psi^{1,M}_{+} \\ \vdots \\ \psi^{j,M}_{+} \\ \vdots \\ \psi^{M/2,M}_{+} \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ & e^{\pi i/M} & & & & \\ & & \ddots & & & \\ & & & e^{\pi i j^2/M} & & \\ & & & & \ddots & \\ & & & & & e^{\pi i M/4} \end{pmatrix} \begin{pmatrix} \psi^{0,M}_{+} \\ \psi^{1,M}_{+} \\ \vdots \\ \psi^{j,M}_{+} \\ \vdots \\ \psi^{M/2,M}_{+} \end{pmatrix}, \tag{2.47}$$

and

$$T_{(M)-}\begin{pmatrix}\psi_-^{1,M}\\ \vdots\\ \psi_-^{j,M}\\ \vdots\\ \psi_-^{M/2-1,M}\end{pmatrix}=\begin{pmatrix}e^{\pi i/M} & & & & \\ & \ddots & & & \\ & & e^{\pi ij^2/M} & & \\ & & & \ddots & \\ & & & & e^{\pi(M/2-1)^2/M}\end{pmatrix}\begin{pmatrix}\psi_-^{1,M}\\ \vdots\\ \psi_-^{j,M}\\ \vdots\\ \psi_-^{M/2-1,M}\end{pmatrix}. \tag{2.48}$$

Both correspond to Z_{2M}.

On the other hand, the $S_{(M)\pm}$ transforms

$$S_{(M)\pm}\psi_\pm^{j,M}=\frac{1}{\sqrt{2M}}\sum_k\left(e^{2\pi jk/M}\pm e^{2\pi i(M-j)k/M,M}\right)\psi^{k,M}. \tag{2.49}$$

This representation is also written by

$$S_{(M)\pm}\psi_\pm^{j,M}=\frac{1}{\sqrt{2M}}\sum_k\left(e^{2\pi(M-j)(M-k)/M}\pm e^{2\pi ij(M-k)/M,M}\right)\psi^{M-k,M}. \tag{2.50}$$

Thus, the S-transformation is represented on the T^2/Z_2 orbifold basis by

$$S_{(M)\pm}\psi_\pm^{j,M}=\frac{1}{\sqrt{M}}\sum_{k\le M/2}\left(e^{2\pi jk/M}\pm e^{2\pi i(M-j)k/M}\right)\psi_\pm^{k,M}. \tag{2.51}$$

These are written by

$$\begin{aligned}S_{(M)+}\psi_+^{j,M}&=\frac{2}{\sqrt{M}}\sum_{k\le M/2}\cos(2\pi jk/M)\psi_+^{j,M},\\ S_{(M)-}\psi_-^{j,M}&=\frac{2i}{\sqrt{M}}\sum_{k\le M/2}\sin(2\pi jk/M)\psi_-^{j,M}.\end{aligned} \tag{2.52}$$

For example, for $M=6$, S and T are represented by Z_2 even zero-modes,

$$S_{(6)+}\begin{pmatrix}\psi^{0,6}\\ \psi_+^{1,6}\\ \psi_+^{2,6}\\ \psi^{3,6}\end{pmatrix}=\frac{1}{\sqrt{6}}\begin{pmatrix}1 & \sqrt{2} & \sqrt{2} & 1\\ \sqrt{2} & 1 & -1 & -\sqrt{2}\\ \sqrt{2} & -1 & -1 & \sqrt{2}\\ 1 & -\sqrt{2} & \sqrt{2} & -1\end{pmatrix}\begin{pmatrix}\psi^{0,6}\\ \psi_+^{1,6}\\ \psi_+^{2,6}\\ \psi^{3,6}\end{pmatrix}, \tag{2.53}$$

$$T_{(6)+}\begin{pmatrix} \psi^{0,6} \\ \psi_+^{1,6} \\ \psi_+^{2,6} \\ \psi^{3,6} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & e^{\pi i/6} & & \\ & & e^{2\pi i/3} & \\ & & & e^{3\pi i/2} \end{pmatrix} \begin{pmatrix} \psi^{0,6} \\ \psi_+^{1,6} \\ \psi_+^{2,6} \\ \psi^{3,6} \end{pmatrix}, \tag{2.54}$$

while S and T are represented by Z_2 odd zero-mode,

$$S_{(6)-}\begin{pmatrix} \psi_-^{1,6} \\ \psi_-^{2,6} \end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \psi_-^{1,6} \\ \psi_-^{2,6} \end{pmatrix}, \tag{2.55}$$

$$T_{(6)-}\begin{pmatrix} \psi_-^{1,6} \\ \psi_-^{2,6} \end{pmatrix} = \begin{pmatrix} e^{\pi i/6} & 0 \\ 0 & e^{2\pi i/3} \end{pmatrix}\begin{pmatrix} \psi_-^{1,6} \\ \psi_-^{2,6} \end{pmatrix}. \tag{2.56}$$

2.2.6 Non-Abelian discrete flavor symmetries

In Ref. [31], it is shown that the models with $M = 2$ as well as even magnetic fluxes have the D_4 flavor symmetry. See Appendix A. One of the Z_2 elements in D_4 corresponds to $(T_{(2)})^2$ on the zero-modes, $\psi^{0,2}$ and $\psi^{1,2}$, i.e.

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (T_{(2)})^2. \tag{2.57}$$

In addition, the permutation Z_2^C element in D_4 corresponds to $S_{(2)}T_{(2)}T_{(2)}S_{(2)}$, i.e.

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S_{(2)}T_{(2)}T_{(2)}S_{(2)}. \tag{2.58}$$

Thus, the D_4 group, which includes the eight elements (A.8), is subgroup of $G_{(2)} \simeq (Z_8 \times Z_4) \rtimes S_3$.

However, there is the difference between the modular symmetry and the D_4 flavor symmetry, which studied in Ref. [31]. The modular symmetry transforms the Yukawa couplings, while the Yukawa couplings are invariant under the D_4 flavor symmetry. In order to study this point, here we examine the Yukawa couplings among $\psi^{i,2}$, $\psi'^{j,2}$ and $\psi^{k,4}$. Both $\psi^{i,2}$ and $\psi'^{j,2}$ are D_4 doublets, and their tensor product $\mathbf{2} \times \mathbf{2}$ is expanded by

$$\mathbf{2} \times \mathbf{2} = \mathbf{1}_{++} + \mathbf{1}_{+-} + \mathbf{1}_{-+} + \mathbf{1}_{--}. \tag{2.59}$$

Thus, the products $\psi^{i,2}\psi'^{j,2}$ correspond to four singlets,

$$\mathbf{1}_{+\pm} : \psi^{0,2}\psi'^{0,2} \pm \psi^{1,2}\psi'^{1,2}, \qquad \mathbf{1}_{-\pm} : \psi^{0,2}\psi'^{1,2} \pm \psi^{1,2}\psi'^{0,2}. \tag{2.60}$$

On the other hand, by use of Eq.(2.16), the products $\psi^{i,2}\psi'^{j,2}$ are expanded by $\psi^{k,4}$. For example, we can expand as

$$\begin{aligned}&\psi^{0,2}\psi'^{0,2} \pm \psi^{1,2}\psi'^{1,2} \\ &\sim \left(Y^{(0)}(16\tau) + Y^{(8/16)}(16\tau) \pm \left(Y^{(4/16)}(16\tau) + Y^{(12/16)}(16\tau)\right)\right) \\ &\quad\times \left(\psi^{0,4} \pm \psi^{2,4}\right)\end{aligned} \tag{2.61}$$

up to constant factors, where

$$Y^{(j/M)}(M\tau) = \mathcal{N} \cdot \vartheta \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (0, M\tau)\,. \tag{2.62}$$

Note that $Y^{(j/M)}(M\tau) = Y^{(1-j/M)}(M\tau)$. It is found that

$$\begin{aligned}&(T_{(4)})^2 \left(\psi^{0,4} \pm \psi^{2,4}\right) = \left(\psi^{0,4} \pm \psi^{2,4}\right), \\ &(S_{(4)}T_{(4)}T_{(4)}S_{(4)}) \left(\psi^{0,4} \pm \psi^{2,4}\right) = \pm \left(\psi^{0,4} \pm \psi^{2,4}\right).\end{aligned} \tag{2.63}$$

Thus, the zero-modes $\psi^{0,4} \pm \psi^{2,4}$ are indeed D_4 singlets, $\mathbf{1}_{+\pm}$ when we identify $(T_{(4)})^2$ and $(S_{(4)}T_{(4)}T_{(4)}S_{(4)})$ as Z_2 and Z_2^C of D_4. In this sense, the D_4 flavor symmetry is a subgroup of the modular symmetry. Also, it is found that the above Yukawa couplings, $Y^{(m/4)}(16\tau)$, with $m = 0, 1, 2, 3$ are invariant under T^2 and $STTS$ transformation.

Similarly, we can expand

$$\begin{aligned}&\psi^{0,2}\psi'^{1,2} + \psi^{1,2}\psi'^{0,2} \\ &\sim \left(Y^{(2/16)}(16\tau) + Y^{(6/16)}(16\tau)\right) \times \left(\psi^{1,4} + \psi^{3,4}\right),\end{aligned} \tag{2.64}$$

up to constant factors. It is found that

$$(T_{(2)})^2 \left(\psi^{0,2}\psi'^{1,2} + \psi^{1,2}\psi'^{0,2}\right) = -\left(\psi^{0,2}\psi'^{1,2} + \psi^{1,2}\psi'^{0,2}\right). \tag{2.65}$$

On the other hand, we obtain

$$(T_{(4)})^2 \left(\psi^{1,4} + \psi^{3,4}\right) = i \left(\psi^{1,4} + \psi^{3,4}\right). \tag{2.66}$$

In addition, we find

$$T^2 : \left(Y^{(2/16)}(16\tau) + Y^{(6/16)}(16\tau)\right) \to i \left(Y^{(2/16)}(16\tau) + Y^{(6/16)}(16\tau)\right). \tag{2.67}$$

Thus, the T^2 transformation is consistent between left and right hand sides in (2.64). However, when we interpret T^2 as Z_2 of the D_4 flavor symmetry, we face inconsistency, because Yukawa couplings are not invariant and

$(\psi^{1,4}+\psi^{3,4})$ has transformation behavior different from $(\psi^{0,2}\psi'^{1,2}+\psi^{1,2}\psi'^{0,2})$. We can make this consistent by defining Z_2 of the D_4 on $(\psi^{1,4}+\psi^{3,4})$ such that its transformation absorbs the phase of Yukawa couplings under T^2 transformation. Then, the mode $(\psi^{1,4}+\psi^{3,4})$ exactly corresponds to the D_4 singlet, $\mathbf{1}_{-+}$. We find that $(\psi^{0,2}\psi'^{1,2}+\psi^{1,2}\psi'^{0,2})$ is invariant under $S_{(2)}T_{(2)}T_{(2)}S_{(2)}$, and $(\psi^{1,4}+\psi^{3,4})$ is also invariant under $S_{(4)}T_{(4)}T_{(4)}S_{(4)}$. That is consistent. Therefore, the D_4 flavor symmetry is a subgroup of the modular symmetry on $\psi^{j,2}$ $(j=0,1)$. However, when the model includes couplings to zero-modes with larger M, we have to modify their modular symmetries such that coupling constants are invariant under the flavor symmetry. Then, we can define the D_4 flavor symmetry.

Here, we give a comment on the T^2/Z_2 orbifold. The T^2/Z_2 orbifold basis gives the irreducible representations of the modular symmetry. The D_4 flavor symmetry is defined through the modular symmetry, as above. That is the reason why the D_4 flavor symmetry remains on the T^2/Z_2 orbifold [49, 50].

2.3 Heterotic orbifold models

Intersecting D-brane models in type II superstring theory is T-dual to magnetized D-brane models. Thus, intersecting D-brane models also have the same behavior under modular transformation as magnetized D-brane models. Furthermore, intersecting D-brane models in type II superstring theory and heterotic string theory on orbifolds have similarities, e.g. in two-dimensional conformal field theory. For example, computations of 3-point couplings as well as n-point couplings are similar to each other. Here, we study modular symmetry in heterotic orbifold models. Using results in Ref. [33, 35, 36], we compare the modular symmetries in heterotic orbifold models with non-Abelian flavor symmetries and also the modular symmetries in the magnetized D-brane models, which have been derived in the previous section.

2.3.1 Twisted sector

Here, we give a brief review on heterotic string theory on orbifolds. The orbifold is the division of the torus T^n by the Z_N twist θ, i.e. T^n/Z_N. Since the T^n is constructed by $\mathbb{R}^n/\Lambda$, the Z_N twist θ should be an automorphism of the lattice Λ. Here, we focus on two-dimensional orbifolds, T^2/Z_N. The six-dimensional orbifolds can be constructed by products of two-dimensional ones. All of the possible orbifolds are classified as T^2/Z_N with $N = 2, 3, 4, 6$.

On orbifolds, there are fixed points, which satisfy the following condition,

$$x^i = (\theta^n x)^i + \sum_k m_k \alpha_k^i, \tag{2.68}$$

where x^i are real coordinates, α_k^i are two lattice vectors, and m_k are integer for $i, k = 1, 2$. Thus, the fixed points can be represented by corresponding space group elements $(\theta^n, \sum_k m_k \alpha_k^i)$, or in short $(\theta^n, (m_1, m_2))$.

The heterotic string theory on orbifolds has localized modes at fixed points, and these are the so-called twisted strings. These twisted states can be labeled by use of fixed points, $\sigma_{\theta,(m_1,m_2)}$. All of the twisted states $\sigma_{\theta,(m_1,m_2)}$ have the same spectrum, if discrete Wilson lines vanish. Thus, the massless modes are degenerate by the number of fixed points.

On the T^2/Z_2 orbifold, there are four fixed points, which are denoted by $(\theta, (0, 0))$, $(\theta, (1, 0))$, $(\theta, (0, 1))$, $(\theta, (1, 1))$. The corresponding twisted states are denoted by $\sigma_{\theta,(m,n)}$ for $m, n = 0, 1$.

On the T^2/Z_3 orbifold, α_1 and α_2 correspond to the $SU(3)$ simple roots and they are identified each other by the Z_3 twist. Thus, three fixed points on the T^2/Z_3 orbifold are represented by the space group elements, $(\theta, m\alpha_1)$ for $m = 0, 1, 2$, or in short (θ, m). The corresponding twisted states are denoted by $\sigma_{\theta,m}$ for $m = 0, 1, 2$.

Similarly, we can obtain the fixed points and twisted states on the T^2/Z_4, where α_1 and α_2 correspond to the $SO(4)$ simple roots and they are identified each other by the Z_4 twist. For the Z_4 twist θ, two fixed points satisfy Eq.(2.68), and these can be represented by $(\theta, m\alpha_1)$ for $m = 0, 1$, or in short (θ, m). Then, the first twisted states are denoted by $\sigma_{\theta,m}$ for $m = 0, 1$. In addition, for θ^2, there are four points, which satisfy Eq.(2.68), and these can denoted by $(\theta^2, (m, n))$ for $m, n = 0, 1$. Indeed, these correspond to the four fixed points on the T^2/Z_2 orbifold. Then, the second twisted states are denoted by $\sigma_{\theta^2,(m,n)}$ for $m, n = 0, 1$. However, the fixed points $(\theta^2, (1, 0))$ and $(\theta^2, (0, 1))$ transform each other under the Z_4 twist θ. Thus, the Z_4 invariant states are written by [51]

$$\sigma_{\theta^2,(0,0)}, \qquad \sigma_{\theta^2,+}, \qquad \sigma_{\theta^2,(1,1)}, \tag{2.69}$$

while $\sigma_{\theta^2,-}$ transforms to $-\sigma_{\theta^2,-}$ under the Z_4 twist, where

$$\sigma_{\theta^2,\pm} = \frac{1}{\sqrt{2}} \left(\sigma_{\theta^2,(1,0)} \pm \sigma_{\theta^2,(0,1)} \right). \tag{2.70}$$

Similarly, we can obtain the fixed points on T^2/Z_6. There is a fixed point $(\theta, 0)$ for the Z_6 twist θ, and a single twisted state $\sigma_{\theta,0}$. The second twisted sector has three fixed points (θ^2, m) $(m = 0, 1, 2)$, which correspond to the three fixed points on the T^2/Z_3 orbifold. The two fixed points $(\theta^2, 1)$ and $(\theta^2, 2)$ transform each other by the Z_6 twist, while $(\theta^2, 0)$ is invariant. Thus, we can write the Z_6-invariant θ^2-twisted states by

$$\sigma_{\theta^2,0}, \qquad \sigma_{\theta^2,+}, \tag{2.71}$$

while $\sigma_{\theta^2,-}$ transforms to $-\sigma_{\theta^2,-}$ under the Z_6 twist, where

$$\sigma_{\theta^2,\pm} = \frac{1}{\sqrt{2}} \left(\sigma_{\theta^2,1} \pm \sigma_{\theta^2,2}\right). \tag{2.72}$$

The third twisted sector has four fixed points, which correspond to the fixed points on T^2/Z_2, and the corresponding θ^3 twisted states. Their linear combinations are Z_6 eigenstates similar to the second twisted states. Since the first twisted sector has the single fixed point and twisted state, the modular symmetry as well as non-Abelian discrete flavor symmetry is rather trivial. We do not discuss the T^2/Z_6 orbifold itself.

2.3.2 Modular symmetry

In Ref. [33], modular symmetry in heterotic string theory on orbifolds was studied in detail. Here we use those results.

T^2/Z_4 orbifold

The S and T transformations are represented by the first twisted sectors of T^2/Z_4 orbifold as [33],

$$\begin{aligned}
\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix} &\longrightarrow S_{Z_4} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}, \qquad S_{Z_4} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\
\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix} &\longrightarrow T_{Z_4} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}, \qquad T_{Z_4} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.
\end{aligned} \tag{2.73}$$

These are exactly the same as representations of $S_{(2)}$ and $T_{(2)}$ on two-zero modes, $\psi^{0,2}$ and $\psi^{1,2}$ in the magnetized model with magnetic flux $M = 2$. Hence, the twisted sectors on the T^2/Z_4 orbifold has the same behavior of modular symmetry as the magnetized model with magnetic flux $M = 2$. Indeed, the twisted sectors have the D_4 flavor symmetry and two twisted states, $\sigma_{\theta,0}$ and $\sigma_{\theta,1}$ correspond to the D_4 doublet [28]. The whole flavor symmetry of the T^2/Z_4 orbifold model is slightly larger than D_4. (See Appendix B.) The T^2/Z_4 orbifold model has the Z_4 symmetry, which transforms the first twisted sector,

$$\sigma_{\theta,m} \longrightarrow e^{\pi i/2}\sigma_{\theta,m}, \tag{2.74}$$

for $m = 0, 1$ and the second twisted sector,

$$\sigma_{\theta^2,(m,n)} \longrightarrow e^{\pi i}\sigma_{\theta^2,(m,n)}, \tag{2.75}$$

for $m, n = 0, 1$. The above Z_4 transformation (2.74) is nothing but $(S_{Z_4}T_{Z_4})^6$ as clearly seen from Eq. (2.25). Thus, the whole flavor symmetry originates from the modular symmetry.

The second twisted sectors correspond to D_4 singlets, $\mathbf{1}_{\pm1,\pm}$ [28] as

$$\mathbf{1}_{+\pm} : \sigma_{\theta^2,(0,0)} \pm \sigma_{\theta^2,(1,1)}, \qquad \mathbf{1}_{-\pm} : \sigma_{\theta^2,\pm}, \tag{2.76}$$

up to coefficients. Compared with the results in section 2.2.6, the D_4 behavior of the second twisted states correspond to one of the zero-modes $\psi^{m,4}$ with magnetic flux $M = 4$. Their correspondence can be written as

$$\begin{aligned} \sigma_{\theta^2,(0,0)} &\sim \psi^{0,4}, \qquad \sigma_{\theta^2,(1,1)} \sim \psi^{2,4}, \\ \sigma_{\theta^2,(1,0)} &\sim \psi^{1,4}, \qquad \sigma_{\theta^2,(1,0)} \sim \psi^{3,4}. \end{aligned} \tag{2.77}$$

The above correspondence can also been seen from the Yukawa couplings. By use of operator product expansion, we obtain the following relations [33],

$$\begin{aligned} &\sigma_{\theta,0}\sigma_{\theta,0} \sim Y_{0,0}\left(\sigma_{\theta^2,(0,0)} + \sigma_{\theta^2,(1,1)}\right), \\ &\sigma_{\theta,1}\sigma_{\theta,1} \sim Y_{1,1}\left(\sigma_{\theta^2,(0,0)} + \sigma_{\theta^2,(1,1)}\right), \\ &\sigma_{\theta,0}\sigma_{\theta,1} + \sigma_{\theta,1}\sigma_{\theta,0} \sim Y_{0,1}\sigma_{\theta^2,+} \end{aligned} \tag{2.78}$$

up to constants. The second twisted state $\sigma_{\theta^2,-}$ can not couple with the first twisted sectors. Using results in Ref. [33], it is found that

$$(T_{Z_4})^2 \begin{pmatrix} Y_{0,0} \\ Y_{1,1} \\ Y_{0,1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} Y_{0,0} \\ Y_{1,1} \\ Y_{0,1} \end{pmatrix}. \tag{2.79}$$

This is the same as behavior of the Yukawa couplings under T^2 studied in section 2.2.6.

T^2/Z_2 orbifold

Here, let us study the T^2/Z_2 orbifold in a way to similar to the previous section on the T^2/Z_4. The S transformation is represented by the four twisted states on the T^2/Z_2 orbifold [33],

$$\begin{pmatrix} \sigma_{\theta,(0,0)} \\ \sigma_{\theta,(0,1)} \\ \sigma_{\theta,(1,0)} \\ \sigma_{\theta,(1,1)} \end{pmatrix} \longrightarrow S_{Z_2} \begin{pmatrix} \sigma_{\theta,(0,0)} \\ \sigma_{\theta,(0,1)} \\ \sigma_{\theta,(1,0)} \\ \sigma_{\theta,(1,1)} \end{pmatrix}, \quad S_{Z_2} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{2.80}$$

Also the T transformation is represented as

$$\begin{pmatrix} \sigma_{\theta,(0,0)} \\ \sigma_{\theta,(0,1)} \\ \sigma_{\theta,(1,0)} \\ \sigma_{\theta,(1,1)} \end{pmatrix} \longrightarrow T_{Z_2} \begin{pmatrix} \sigma_{\theta,(0,0)} \\ \sigma_{\theta,(0,1)} \\ \sigma_{\theta,(1,0)} \\ \sigma_{\theta,(1,1)} \end{pmatrix}, \qquad T_{Z_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.81}$$

The representation S_{Z_2} is similar to S_{Z_4} and $S_{(2)}$. Indeed, we find that $S_{Z_2} = S_{(2)} \otimes S_{(2)}$. However, the representation T_{Z_2} is different from T_{Z_4} and $T_{(2)}$.

The matrices S_{Z_2} and T_{Z_2} satisfy the following relations,

$$(S_{Z_2})^2 = (T_{Z_2})^2 = (S_{Z_2} T_{Z_2})^6 = \mathbb{I}. \tag{2.82}$$

These correspond to the D_6. Indeed, the order of closed algebra including S_{Z_2} and T_{Z_2} is equal to 12. At any rate, these matrices are reducible. We change the basis in order to obtain irreducible representations,

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sigma_{\theta,(0,0)} \\ \sigma_{\theta,(1,0)} \\ \sigma_{\theta,(0,1)} \\ \sigma_{\theta,(1,1)} \end{pmatrix}. \tag{2.83}$$

Then, σ_1 and σ_2 correspond to the D_6 doublet, while σ_3 and σ_4 correspond to the D_6 singlets. For example, $S_{Z_2}T_{Z_2}$ and T_{Z_2} are represented by

$$S_{Z_2}T_{Z_2}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix}=\begin{pmatrix}\cos(2\pi/6) & -\sin(2\pi/6) & 0 & 0\\ \sin(2\pi/6) & \cos(2\pi/6) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix},$$

$$T_{Z_2}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\end{pmatrix}. \qquad (2.84)$$

It is found that σ_3 and σ_4 correspond to $\mathbf{1}_{--}$ and $\mathbf{1}_{-+}$.

The twisted sector on the T^2/Z_2 orbifold has the flavor symmetry $(D_4 \times D_4)/Z_2$. However, this flavor symmetry seems independent of the above D_6, because they do not include any common elements. The twisted sector on the S^1/Z_2 orbifold has the flavor symmetry D_4. The flavor symmetry of T^2/Z_2 orbifold is obtained as a kind of product, $D_4 \times D_4$, although two D_4 groups have a common Z_2 element. Thus, the flavor symmetry of T^2/Z_2 originates from the product of symmetries of the one-dimensional orbifold. On the other hand, the modular symmetry appears in two or more dimensions, but not in one dimension. Hence, these symmetries would be independent. When we include the above D_6 as low-energy effective field theory in addition to the flavor symmetry $(D_4 \times D_4)/Z_2$, low-energy effective field theory would have larger symmetry including D_6 and $(D_4 \times D_4)/Z_2$, although Yukawa couplings as well as higher order couplings transform non-trivially under D_6.

T^2/Z_3 orbifold

The S and T transformations are represented by the first twisted sectors of T^2/Z_3 orbifold as [33],

$$\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\\ \sigma_{\theta,2}\end{pmatrix}\longrightarrow S_{Z_3}\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\\ \sigma_{\theta,2}\end{pmatrix}, \quad S_{Z_3}=\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & e^{2\pi i/3} & e^{-2\pi i/3}\\ 1 & e^{-2\pi i/3} & e^{2\pi i/3}\end{pmatrix},$$

$$\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\\ \sigma_{\theta,2}\end{pmatrix}\longrightarrow T_{Z_3}\begin{pmatrix}\sigma_{\theta,0}\\ \sigma_{\theta,1}\\ \sigma_{\theta,2}\end{pmatrix}, \quad T_{Z_3}=\begin{pmatrix}1 & 0 & 0\\ 0 & e^{2\pi i/3} & 0\\ 0 & 0 & e^{2\pi i/3}\end{pmatrix}. \qquad (2.85)$$

These forms look similar to S and T transformations in magnetized models (2.19) and (2.20). Indeed, they correspond to submatrices of $S_{(6)}$ and $T_{(6)}$

in the magnetized models with the magnetic flux $M = 6$. Alternatively, in Ref. [35] the following S and T representations were studied[1]

$$S'_{Z_3} = -\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{pmatrix}, \quad T'_{Z_3} = \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.86}$$

At any rate, the above representations are reducible representations. Thus, we use the flowing basis,

$$\begin{pmatrix} \sigma_+ \\ \sigma_0 \\ \sigma_- \end{pmatrix}, \tag{2.87}$$

where $\sigma_\pm = (\sigma_1 \pm \sigma_-)/\sqrt{2}$. The (σ_+, σ_0) is a doublet, while σ_- is a singlet. The former corresponds to the Z_6 invariant states among the θ^2 twisted sector on the T^2/Z_6 orbifold. Similarly, σ_- is the θ^2 twisted state, which transforms $\sigma_- \to -\sigma_-$ under the Z_6 twist. Alternatively, we can say that the doublet (σ_+, σ_0) corresponds to Z_2 even states and the singlet σ_- is the Z_2 odd states, where the Z_2 means the π rotation of the lattice vectors, $(\alpha_1, \alpha_2) \to (-\alpha_1, -\alpha_2)$. This point is similar to the aspect in magnetized D-brane models, where irreducible representations correspond to the T^2/Z_2 orbifold basis. Also, note that the first twisted states of the T^2/Z_4 orbifold correspond already to the Z_2-invariant basis.

For example, we represent S'_{Z_3} and T'_{Z_3} on the above basis [35] ,

$$S'_{Z_3} = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}, \quad T'_{Z_3} = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.88}$$

on the doublet $(\sigma_+, \sigma_0)^T$, while σ_- is the trivial singlet. Here, we define

$$Z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{T}_{Z_3} = Z T'_{Z_3}. \tag{2.89}$$

Then, they satisfy the following algebraic relations [35, 36],

$$(S'_{Z_3})^2 = (\tilde{T}_{Z_3})^3 = (S'_{Z_3} \tilde{T}_{Z_3})^3 = Z, \qquad Z^2 = \mathbb{I}. \tag{2.90}$$

This group is the so-called T', which is the binary extension of $A_4 = T$.

[1] See also Ref. [36].

The non-Abelian discrete flavor symmetry on the T^2/Z_3 orbifold is $\Delta(54)$, and the three twsisted states correspond to the triplet of $\Delta(54)$. Thus, this modular symmetry seems independent of the $\Delta(54)$ flavor symmetry.

Two representations are related as

$$S'_{Z_3} = -iS_{Z_3}, \qquad T'_{Z_3} = e^{2\pi i/3}(T_{Z_3})^{-1}. \tag{2.91}$$

When we change phases of S, T and ST, the group such as $(Z_N \times Z_M) \rtimes H$ in sections 2.2 2.3 and would change to $(Z_{N'} \times Z_{M'}) \rtimes H$.

3 Flavor models with modular symmetry

3.1 Modular forms

String theory on T^2 as well as orbifolds T^2/Z_N has the modular symmetry. Furthermore, four-dimensional low-energy effective field theory on the compactification $T^2 \times X_4$ as well as $(T^2/Z_N) \times X_4$ also has the modular symmetry, where X_4 is a four-dimensional compact space.

In this section and following two sections, we study lepton flavor models with S_3 or A_4 symmetry from the modular group. Here, we do not specify explicit string model but assume effective theory having S_3 or A_4 symmetry.

In effective theories having modular invariance, coupling constants and fields should be obey specific transformation under transformation of the modulus τ. Coupling constants are written by using holomorphic functions which transform as

$$f(\tau) \to (c\tau + d)^k f(\tau) \tag{3.1}$$

under the modular transformation Eq.(2.2) called modular forms of weight k. Similarly, a set of chiral superfields $\phi^{(I)}$ transform under the modular transformation (2.2) as a multiplet [52],

$$\phi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \phi^{(I)}, \tag{3.2}$$

where $-k_I$ is the so-called modular weight and $\rho^{(I)}$ denotes a representation matrix. Modular invariant kinetic terms expanded around a VEV of the modulus τ are written by

$$\frac{|\partial_\mu \tau|^2}{\langle -i\tau + i\bar{\tau}\rangle^2} + \sum_I \frac{|\partial_\mu \phi^{(I)}|^2}{\langle -i\tau + i\bar{\tau}\rangle^{k_I}}. \tag{3.3}$$

Also, the superpotential should be invariant under the modular symmetry. That is, the superpotential should have vanishing modular weight in global supersymmetric models. Indeed, Yukawa coupling constants as well as higher-order couplings constants are modular functions of τ [37,47,53,54]. In the framework of supergravity theory, the superpotential must be invariant up to the Kähler transformation [52]. That implies that the superpotential of supergravity models with the above kinetic term should have modular weight one. In sections 3.3 and 3.4, we consider the global supersymmetric models, and require that the superpotential has vanishing modular weight, although it is straightforward to arrange modular weights of chiral superfields for supergravity models.

The Dedekind eta-function $\eta(\tau)$ is one of famous modular functions, which is written by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \tag{3.4}$$

where $q = e^{2\pi i\tau}$. The $\eta(\tau)$ function behaves under S and T transformations as

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau). \tag{3.5}$$

The former transformation implies that the $\eta(\tau)^{24}$ function has the modular weight 12.

The modular functions (Y_1, Y_2, Y_3) with weight 2, which behave as an A_4 triplet, are obtained as

$$\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right), \\
Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right), \\
Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),
\end{aligned} \tag{3.6}$$

in Ref. [39], where $\omega = e^{2\pi i/3}$. (See Appendix C.)

We can obtain the modular functions with weight 2, which behave as an S_3 doublet,

$$\begin{aligned}
Y_1(\tau) &= \frac{i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - \frac{8\eta'(2\tau)}{\eta(2\tau)}\right), \\
Y_2(\tau) &= \frac{\sqrt{3}i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}\right),
\end{aligned} \tag{3.7}$$

by a similar technique. (See Appendix C.)

3.2 Experimental values

Flavor eigenstates of neutrino $(\nu_e, \nu_\mu, \nu_\tau)$ are linear combinations of mass eigenstates (ν_1, ν_2, ν_3). Their mixing matrix U, i.e. the so-called PMNS matrix can be written by

$$\begin{aligned} U &= \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{CP}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & e^{i\alpha_3/2} \end{pmatrix}, \end{aligned} \tag{3.8}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ for mixing angles θ_{ij}, δ_{CP} is the Dirac CP phase, and α_i are Majorana CP phases. The mass-squared differences are defined by

$$\delta m^2 = m_2^2 - m_1^2, \tag{3.9}$$

$$\Delta m^2 = m_3^2 - \frac{m_1^2 + m_2^2}{2}, \tag{3.10}$$

where m_i is the mass eigenvalue of ν_i. We also define the ratio between the mass-squared differences as

$$r = \frac{\delta m^2}{|\Delta m^2|}. \tag{3.11}$$

Experimental values with normal ordering (NO) and inverted ordering (IO) are shown in Table 3.1.

3.3 S_3 models

3.3.1 Models with S_3 symmetry

In this section, we construct the models with the flavor symmetry $\Gamma(2) \simeq S_3$ and study them systematically.

Parameter	Normal Ordering	Inverted Ordering
$\delta m^2/10^{-5}\text{eV}^2$	$7.37^{+0.17}_{-0.16}$	$7.37^{+0.17}_{-0.16}$
$\lvert\Delta m^2\rvert/10^{-3}\text{eV}^2$	$2.525^{+0.042}_{-0.030}$	$2.505^{+0.034}_{-0.032}$
$\sin^2\theta_{12}/10^{-1}$	$2.97^{+0.17}_{-0.16}$	$2.97^{+0.17}_{-0.16}$
$\sin^2\theta_{13}/10^{-2}$	$2.15^{+0.07}_{-0.07}$	$2.16^{+0.08}_{-0.07}$
$\sin^2\theta_{23}/10^{-1}$	$4.25^{+0.21}_{-0.15}$	$5.89^{+0.16}_{-0.22} \oplus 4.33^{+0.15}_{-0.16}$
δ_{CP}/π	$1.38^{+0.23}_{-0.20}$	$1.31^{+0.31}_{-0.19}$
r	$2.92^{+0.10}_{-0.11} \times 10^{-2}$	$2.94^{+0.11}_{-0.10} \times 10^{-2}$

Table 3.1: The best-fit values and 1σ-ranges in experiments with NO and IO from Ref. [55].

	$SU(2)_L \times U(1)_Y$	S_3	k_I
$e^c_{R_a}$	$(\mathbf{1}, +1)$	$\mathbf{1}$	-3
$e^c_{R_b}$	$(\mathbf{1}, +1)$	$\mathbf{1}$	-4
$e^c_{R_c}$	$(\mathbf{1}, +1)$	$\mathbf{1}'$	-4
$L^{(1)}$	$(\mathbf{2}, -1/2)$	$\mathbf{1}$	1
$L^{(2)}$	$(\mathbf{2}, -1/2)$	$\mathbf{2}$	1
H_u	$(\mathbf{2}, +1/2)$	$\mathbf{1}$	0
H_d	$(\mathbf{2}, -1/2)$	$\mathbf{1}$	0
$\phi^{(1)}$	$(\mathbf{1}, 0)$	$\mathbf{1}$	2
$\phi^{(2)}$	$(\mathbf{1}, 0)$	$\mathbf{2}$	3

Table 3.2: S_3 representations and k_I in the S_3 models.

Table 3.2 shows the S_3 representations and k_I of lepton and Higgs superfields. The $\phi^{(1)}$ and $\phi^{(2)}$ fields are flavon fields, and $\phi^{(1)}$ and $\phi^{(2)}$ are S_3 singlet and doublet, respectively. In order to distinguish e_{R_a} and e_{R_b}, we assign k_I different from each other. For such a purpose, we can impose an additional symmetry, e.g. Z_2. We assign k_I such that we can realize the diagonal charged lepton mass matrix similar to the A_4 model. Indeed, the superpotential terms in the charged lepton sector can be written by

$$W_e = \beta_a e^c_{R_a} H_d (L^{(1)}\phi^{(1)})_1 + \beta_b e^c_{R_b} H_d (L^{(2)}\phi^{(2)})_1 - \beta_c e^c_{R_c} H_d (L^{(2)}\phi^{(2)})_{1'}, \tag{3.12}$$

where the β_i are constant coefficients. We assume that the flavon fields develop their VEVs as

$$\langle\phi^{(1)}\rangle = u_1, \quad \langle\phi^{(2)}\rangle = (u_2, 0). \tag{3.13}$$

Then, we can realize the diagonal charged lepton mass matrix when the neutral component of H_d develops its VEV. Similar to the A_4 model, we can realize the experimental values of the charged lepton masses, $m_{e,\mu,\tau}$ by choosing proper values of couplings β_a. Note that the assignment of generations to e_{R_i}, $i = a, b, c$ is not fixed yet.

Modular invariant Weinberg operators in the superpotential can be written by

$$\begin{aligned}\mathcal{L}^\nu_{\text{eff}} = \frac{1}{\Lambda} \big[& dHH \left(L^{(2)}L^{(2)}\right)_2 Y^{(2)} + 2aHH \left(L^{(1)}L^{(2)}\right)_2 Y^{(2)} \\ & + bHH \left(L^{(1)}L^{(1)}\right)_1 Y^{(1)} + cHH \left(L^{(2)}L^{(2)}\right)_1 Y^{(1)} \big], \end{aligned} \tag{3.14}$$

where $a, b, c, d \in \mathbb{C}$ are constant coefficients. $Y^{(1)}$ and $Y^{(2)}$ are modular forms with modular weight 2, and $Y^{(1)}$ and $Y^{(2)}$ are S_3 singlet and doublet [1], respectively. Note that since $\mathbf{1'}$ in $\mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{1'} + \mathbf{2}$ is antisymmetric, $\left(L^{(2)}L^{(2)}\right)_{1'} = 0$. We denote $Y^{(1)} = Y$ and $Y^{(2)} = (Y_1, Y_2)$. There are 6 ways to assign 3 generations of lepton doublets L_i to S_3 singlet $L^{(1)}$ and doublet $L^{(2)}$. When we assign leptons (L_1, L_2, L_3) as $L^{(1)} = L_1$ and $L^{(2)} = (L_2, L_3)$, we obtain the following mass matrix

$$M_\nu \propto d \begin{pmatrix} 0 & 0 & 0 \\ 0 & -Y_1 & -Y_2 \\ 0 & -Y_2 & Y_1 \end{pmatrix} + a \begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & 0 & 0 \\ Y_2 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Y \end{pmatrix}. \tag{3.15}$$

[1] There are two independent modular forms with weight 2 and $\Gamma(2)$ [39,56]. Thus, there is only one independent modular form doublet $Y^{(2)}$ in (3.14).

Furthermore, we rewrite $y(\tau) = Y_2(\tau)/Y_1(\tau)$, and $(B, C, D) = (bY, cY, d)/a$ and we obtain

$$M_\nu \propto \begin{pmatrix} B & 1 & y \\ 1 & C-D & -Dy \\ y & -Dy & C+D \end{pmatrix}. \tag{3.16}$$

3.3.2 Numerical results

There are four complex parameters (y, B, C, D) except overall coefficient in the S_3 model (3.14). Since there are eight real free parameters and also six ways to assign three generations of leptons, it is difficult to search whole region of parameters. In this subsection, we only study the case of $L^{(1)} = L_1$ and $L^{(2)} = (L_2, L_3)$ leading the mass matrix

$$M_\nu = \begin{pmatrix} B & 1 & y \\ 1 & C-D & -Dy \\ y & -Dy & C+D \end{pmatrix}. \tag{3.17}$$

First, we treat the function $y(\tau)$ as a free complex parameter instead of its argment τ. Next, we fit the value of y by using concrete form of modular form in (C.24).

We search parameters under following conditions:

$$|y| < 2.0, \quad |\mathrm{Re}(B, C, D)| < 10 \quad , |\mathrm{Im}(B, C, D)| < 10. \tag{3.18}$$

There are many sets of parameters consistent with 3σ of experimental results in both of NH and IH case. In the case of NH, predicted values of mixing angles in this model cover whole region of experimental bound with 3σ deviation. Thus, we cannot find any meaningful correlation between mixing angles. On the other hand, we find some correlation between Dirac and Majorana CP phases as in Figure 3.1.

All points in Figure 3.1 are consistent with 3σ of experimental results. Hare, we pick up one point in Figure 3.1 as an example of solutions for NH. Input values are

$$y = 0.97 + 0.70i, \quad B = -10.0 + 4.0i, \quad C = -2.0 - 4.0i, \quad D = 8.0 - 6.0i \tag{3.19}$$

and predicted values of observables are

$$s_{12}^2 = 3.23 \times 10^{-1}, \quad s_{13}^2 = 2.17 \times 10^{-2}, \quad s_{23}^2 = 4.47 \times 10^{-1}, \quad r = 3.11 \times 10^{-2}, \tag{3.20}$$

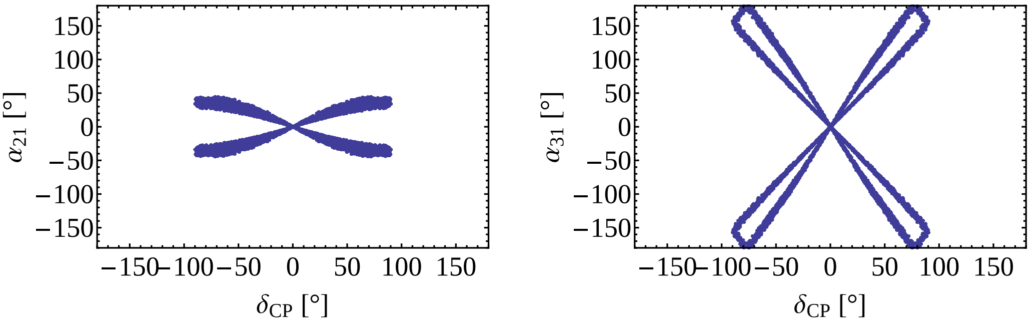

Figure 3.1: The predictions of (Left) δ_{CP} versus α_{21} and (Right) δ_{CP} versus α_{31} for NH in S_3 model.

and predicted values of Dirac and Majorana CP phases are

$$d(\delta_{CP}, \alpha_{21}, \alpha_{31}) = (52.0, -34.2, -128.4)[\text{deg}]. \tag{3.21}$$

The value of the modulus τ leading $y(\tau) = Y_2(\tau)/Y_1(\tau)$ is $\tau = 0.247 + 0.774i$.

In the case of IH, the number of realistic solutions is much less than those in NH.

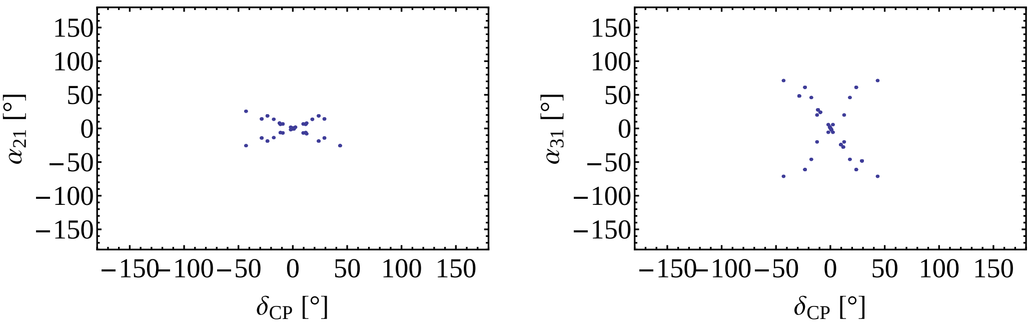

Figure 3.2: The predictions of (Left) δ_{CP} versus α_{21} and (Right) δ_{CP} versus α_{31} for IH in S_3 model.

Figure 3.2 shows the correlations between Dirac and Majorana CP phases. All points in this figure are consistent with 3σ of experimental results. Hare, we pick up one point in Figure 3.1 as an example of solutions for NH. Input values are

$$y = 0.70 + 0.97i, \quad B = -9.0 + 6.0i, \quad C = 2.0 - 5.0i, \quad D = -4.0 - 1.0i \tag{3.22}$$

and predicted values of observables are

$$s_{12}^2 = 3.39 \times 10^{-1}, \quad s_{13}^2 = 2.16 \times 10^{-2}, \quad s_{23}^2 = 4.41 \times 10^{-1}, \quad r = 3.04 \times 10^{-2} \tag{3.23}$$

and predicted values of Dirac and Majorana CP phases are

$$(\delta_{CP}, \alpha_{21}, \alpha_{31}) = (-23.2, 18.0, -59.6)[\mathrm{deg}]. \tag{3.24}$$

The value of the modulus τ leading the value of $y(\tau)$ is $\tau = 0.340 + 0.800i$.

3.4 A_4 models

3.4.1 Models with A_4 symmetry

Let us consider a modular invariant flavor model with the A_4 symmetry for leptons. At first, we discuss the type I seesaw model where neutrinos are Majorana particles. There are freedoms for the assignments of irreducible representations and modular weights to leptons. We suppose that three left-handed lepton doublets are compiled in a triplet of A_4. The three right-handed neutrinos are also of a triplet of A_4. On the other hand, the Higgs doublets are supposed to be singlets of A_4. The generic assignments of representations and modular weights to the MSSM fields and right-handed neutrino superfields are presented in Table 3.3. In order to build a model with minimal number of parameters, we introduce no flavons.

For the charged leptons, we assign three right-handed charged leptons for three different singlets of A_4, $(1, 1'', 1')$. Therefore, there are three independent couplings in the superpotential of the charged lepton sector. Those coupling constants can be adjusted to the observed charged lepton masses. Since there are three singlets in the A_4 group, there are six cases for the assignment of three right-handed charged leptons. However, the freedom of these assignments for right-handed neutrinos do not affect the results for lepton mixing angles.

It may be helpful to comment that if the right-handed charged leptons are of a A_4 triplet, we cannot reproduce the well known charged lepton mass hierarchy $1 : \lambda^2 : \lambda^5$, where $\lambda \simeq 0.2$.

The modular invariant mass terms of the leptons are given as the following

	L	e_R, μ_R, τ_R	ν_R	H_u	H_d	Y
$SU(2)$	2	1	1	2	2	1
A_4	3	1, 1″, 1′	3	1	1	3
$-k_I$	-1 (1)	-1 (-3)	-1	0	0	$k=2$

Table 3.3: The charge assignment of $SU(2)$, A_4, and the modular weight ($-k_I$ for fields and k for coupling Y) in the type I seesaw model. The right-handed charged leptons are assigned three A_4 singlets, respectively. Values of $-k_I$ in the parentheses are alternative assignments of the modular weight.

superpotentials:

$$w_e = \alpha e_R H_d(LY) + \beta\mu_R H_d(LY) + \gamma\tau_R H_d(LY) \ , \tag{3.25}$$

$$w_D = g(\nu_R H_u LY)_{\mathbf{1}} \ , \tag{3.26}$$

$$w_N = \Lambda(\nu_R\nu_R Y)_{\mathbf{1}} \ , \tag{3.27}$$

where sums of the modular weights vanish. The parameters α, β, γ, g, and Λ are constant coefficients. The functions $Y_i(\tau)$ are A_4 triplet modular forms and they consist of the modulus parameter τ:

$$Y = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix} , \qquad q = e^{2\pi i\tau}, \tag{3.28}$$

where the q-expansion of $Y_i(\tau)$ is used. The $Y_i(\tau)$ satisfy the constraint [39]:

$$Y_2^2 + 2Y_1Y_3 = 0 \ . \tag{3.29}$$

Since the dimension of the space of modular forms of weight 2 for $\Gamma(3) \simeq A_4$ is 3 (see, e.g. [39, 56]), all Y's in Eqs.(3.25)-(3.27) are the same modular forms.

There is an alternative assignment of the modular weight for the left-handed lepton and the right-handed charged leptons as presented in parentheses of Table 3.3 [40]. For the alternative assignment, the modular invariant superpotential w_D is given with constant parameters without the modular coupling Y as:

$$w_D = g(\nu_R H_u L)_{\mathbf{1}} \ . \tag{3.30}$$

Next, we discuss the case where neutrino masses originate from the Weinberg operator. We have the unique possibility of the superpotential

$$w_\nu = -\frac{1}{\Lambda}(H_u H_u L L Y)_{\mathbf{1}} \ , \tag{3.31}$$

where both modular weights of L and right-handed charged leptons are -1 as shown in Table 3.3.

There is another possibility for neutrinos, that is, neutrinos are Dirac particles. In this case, the neutrino mass matrix is derived only from w_D in Eq.(3.26).

3.4.2 Charged lepton mass matrix

Let us consider an assignment of A_4 for the right-handed charged leptons as $(e_R, \mu_R, \tau_R) = (1, 1'', 1')$ in Table 3.3. By using the decomposition rule of a A_4 tensor product in Appendix A, we obtain the mass matrix of charged leptons as follows [2]:

$$M_E = \mathrm{diag}[\alpha, \beta, \gamma] \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL} . \tag{3.32}$$

The coefficients α, β, and γ are taken to be real positive by rephasing right-handed charged lepton fields without loss of generality. Those parameters can be written in terms of the modulus parameter τ and the charged lepton masses as seen in Appendix B.

3.4.3 Neutrino mass matrix

Since the tensor product of $3 \otimes 3$ is decomposed into a symmetric triplet and an antisymmetric triplet as seen in Appendix A, the superpotential of the Dirac neutrino mass in Eq.(3.26) is expressed with additional two parameters

[2] There are six cases to assign A_4 singlets for the right-handed charged leptons as $(e_R, \mu_R, \tau_R) = (1, 1'', 1')$, $(1, 1', 1'')$, $(1', 1, 1'')$, $(1', 1'', 1)$, $(1'', 1', 1)$, $(1'', 1, 1')$. The mass matrices are obtained by permutations of rows each other. Then, the combinations $M_E^\dagger M_E$ are same ones up to re-labeling of parameters α, β, and γ for all cases.

g_1 and g_2 as:

$$
\begin{aligned}
w_D = v_u \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \end{pmatrix} \otimes \left[g_1 \begin{pmatrix} 2\nu_e Y_1 - \nu_\mu Y_3 - \nu_\tau Y_2 \\ 2\nu_\tau Y_3 - \nu_e Y_2 - \mu Y_1 \\ 2\nu_\mu Y_2 - \nu_\tau Y_1 - \nu_e Y_3 \end{pmatrix} \oplus g_2 \begin{pmatrix} \nu_\mu Y_3 - \nu_\tau Y_2 \\ \nu_e Y_2 - \nu_\mu Y_1 \\ \nu_\tau Y_1 - \nu_e Y_3 \end{pmatrix} \right] \\
= v_u g_1 \left[\nu_{R1}(2\nu_e Y_1 - \nu_\mu Y_3 - \nu_\tau Y_2) + \nu_{R2}(2\nu_\mu Y_2 - \nu_\tau Y_1 - \nu_e Y_3) \right. \\
\left. + \nu_{R3}(2\nu_\tau Y_3 - \nu_e Y_2 - \nu_\mu Y_1) \right] \\
+ v_u g_2 \left[\nu_{R1}(\nu_\mu Y_3 - \nu_\tau Y_2) + \nu_{R2}(\nu_\tau Y_1 - \nu_e Y_3) + \nu_{R3}(\nu_e Y_2 - \nu_\mu Y_1) \right] .
\end{aligned}
\tag{3.33}
$$

The Dirac neutrino mass matrix is given as

$$
M_D = v_u \begin{pmatrix} 2g_1 Y_1 & (-g_1 + g_2) Y_3 & (-g_1 - g_2) Y_2 \\ (-g_1 - g_2) Y_3 & 2g_1 Y_2 & (-g_1 + g_2) Y_1 \\ (-g_1 + g_2) Y_2 & (-g_1 - g_2) Y_1 & 2g_1 Y_3 \end{pmatrix}_{RL} . \tag{3.34}
$$

For the alternative case in Eq.(3.30), the superpotential of the Dirac neutrino is written as:

$$
w_D = v_u g \begin{pmatrix} \nu_{R1} \\ \nu_{R2} \\ \nu_{R3} \end{pmatrix} \otimes \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = v_u g \left(\nu_{R1}\nu_e + \nu_{R2}\nu_\tau + \nu_{R3}\nu_\mu \right) . \tag{3.35}
$$

The Dirac neutrino mass matrix is simply given as

$$
M_D = v_u g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{RL} . \tag{3.36}
$$

On the other hand, since the Majorana neutrino mass terms are symmetric, the superpotential in Eq.(3.27) is expressed simply as

$$
\begin{aligned}
w_N = & \Lambda \begin{pmatrix} 2\nu_{R1}\nu_{R1} - \nu_{R2}\nu_{R3} - \nu_{R3}\nu_{R2} \\ 2\nu_{R3}\nu_{R3} - \nu_{R1}\nu_{R2} - \nu_{R2}\nu_{R1} \\ 2\nu_{R2}\nu_{R2} - \nu_{R3}\nu_{R1} - \nu_{R1}\nu_{R3} \end{pmatrix} \otimes \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \\
= & \Lambda \left[(2\nu_{R1}\nu_{R1} - \nu_{R2}\nu_{R3} - \nu_{R3}\nu_{R2}) Y_1 + (2\nu_{R3}\nu_{R3} - \nu_{R1}\nu_{R2} - \nu_{R2}\nu_{R1}) Y_3 \right. \\
& \left. + (2\nu_{R2}\nu_{R2} - \nu_{R3}\nu_{R1} - \nu_{R1}\nu_{R3}) Y_2 \right] .
\end{aligned}
\tag{3.37}
$$

Then, the right-handed Majorana neutrino mass matrix is given as

$$M_N = \Lambda \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}_{RR} . \tag{3.38}$$

Finally, the effective neutrino mass matrix is obtained by the type I seesaw as follows:

$$M_\nu = -M_D^{\mathrm{T}} M_N^{-1} M_D . \tag{3.39}$$

Models	Mass Matrices
I (a): Seesaw	$M_D \sim \begin{pmatrix} 2g_1Y_1 & (-g_1+g_2)Y_3 & (-g_1-g_2)Y_2 \\ (-g_1-g_2)Y_3 & 2g_1Y_2 & (-g_1+g_2)Y_1 \\ (-g_1+g_2)Y_2 & (-g_1-g_2)Y_1 & 2g_1Y_3 \end{pmatrix}$, $M_N \sim \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}$
I (a): Seesaw	$M_D \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, M_N \sim \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}$
II: Weinberg Operator	$M_\nu \sim \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}$
III: Dirac Neutrino	$M_\nu \sim \begin{pmatrix} 2g_1Y_1 & (-g_1+g_2)Y_3 & (-g_1-g_2)Y_2 \\ (-g_1-g_2)Y_3 & 2g_1Y_2 & (-g_1+g_2)Y_1 \\ (-g_1+g_2)Y_2 & (-g_1-g_2)Y_1 & 2g_1Y_3 \end{pmatrix}$

Table 3.4: The classification of the modular invariant mass matrices for neutrino models.

For the case where neutrino masses originate from the Weinberg operator, the superpotential in Eq.(3.31) is written as:

$$
\begin{aligned}
w_\nu &= -\frac{v_u^2}{\Lambda}\begin{pmatrix}2\nu_e\nu_e - \nu_\mu\nu_\tau - \nu_\tau\nu_\mu \\ 2\nu_\tau\nu_\tau - \nu_e\nu_\mu - \nu_\mu\nu_\tau \\ 2\nu_\mu\nu_\mu - \nu_\tau\nu_e - \nu_e\nu_\tau\end{pmatrix} \otimes \begin{pmatrix}Y_1 \\ Y_2 \\ Y_3\end{pmatrix} \\
&= -\frac{v_u^2}{\Lambda}\left[(2\nu_e\nu_e - \nu_\mu\nu_\tau - \nu_\tau\nu_\mu)Y_1 + (2\nu_\tau\nu_\tau - \nu_e\nu_\mu - \nu_\mu\nu_e)Y_3 \right. \\
&\qquad \left. +(2\nu_\mu\nu_\mu - \nu_\tau\nu_e - \nu_e\nu_\tau)Y_2\right].
\end{aligned} \tag{3.40}
$$

The neutrino mass matrix is given as follows:

$$
M_\nu = -\frac{v_u^2}{\Lambda}\begin{pmatrix}2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3\end{pmatrix}_{LL}. \tag{3.41}
$$

For the case where the neutrino is the Dirac particle, we use the mass matrix in Eq.(3.34).

It is important to address the transformation needed to put kinetic terms of matter superfields in the canonical form because kinetic terms are given in Eq.(3.3). The canonical form is realized by the overall normalization of the lepton mass matrices, which shifts our parameters such as

$$
\begin{aligned}
&\alpha \to \alpha' = \alpha(K_L K_{e_R})^{-1/2}, \quad \beta \to \beta' = \beta(K_L K_{\mu_R})^{-1/2}, \\
&\gamma \to \gamma' = \gamma(K_L K_{\tau_R})^{-1/2}, \\
&g_i \to g_i' = g_i(K_L K_{\nu_R})^{-1/2} \ (i = 1, 2), \quad \Lambda \to \Lambda' = \Lambda {K_{\nu_R}}^{-1},
\end{aligned} \tag{3.42}
$$

where K_ϕ denotes a coefficient of the kinetic term of Eq.(3.3). Hereafter, we rewrite α, β, γ, g_i, and Λ for α', β', γ', g_i', and Λ' in our convention.

Finally, we summarize the classification of mass matrices for neutrino models in Table 3.4.

3.4.4 Numerical results

We discuss numerical results for neutrino models in Table 3.4. The lepton mass matrices in the previous section are given by modulus parameter τ. By fixing τ, the modular invariance is broken, and then the lepton mass matrices give the mass eigenvalues and flavor mixing numerically. In order to fix the value of τ, we use the result of NuFIT 3.2 with the $3\,\sigma$ error-bar [57,58]. We consider both the normal hierarchy (NH) of neutrino masses $m_1 < m_2 < m_3$ and the inverted hierarchy (IH) of neutrino masses $m_3 < m_1 < m_2$, where m_1, m_2, and m_3 denote three light neutrino masses. The sum of neutrino masses are restricted by the cosmological observations [59, 60]. Planck 2018 results provide us its cosmological upper bound for sum of neutrino masses; 120-160 meV [61] at the 95% C.L. depending on the combined data. We have used the upper bound of 160 meV as a conservative constraint of our models. By inputting the data of $\Delta m^2_{\rm atm} \equiv m_3^2 - m_1^2$, $\Delta m^2_{\rm sol} \equiv m_2^2 - m_1^2$, and three mixing angles θ_{23}, θ_{12}, and θ_{13} with $3\,\sigma$ error-bar given in Table 3.5, we fix the modulus τ and the other parameters. Then we can predict the CP violating Dirac phases δ_{CP} and Majorana phases α_{31}, α_{21}, which are defined in Appendix C.

observable	$3\,\sigma$ range for NH	$3\,\sigma$ range for IH
$\Delta m^2_{\rm atm}/10^{-3}{\rm eV}^2$	(2.399 - 2.593)	(-2.562 - -2.369)
$\Delta m^2_{\rm sol}/10^{-5}{\rm eV}^2$	(6.80 - 8.02)	(6.80 - 8.02)
$\sin^2\theta_{23}$	0.418 - 0.613	0.435 - 0.616
$\sin^2\theta_{12}$	0.272 - 0.346	0.272 - 0.346
$\sin^2\theta_{13}$	0.01981 - 0.02436	0.02006 - 0.02452

Table 3.5: The $3\,\sigma$ ranges of neutrino oscillation parameters from NuFIT 3.2 for NH and IH [57, 58].

Model I(a): Seesaw

The coefficients α/γ and β/γ in the charged lepton mass matrix are given only in terms of τ after inputting the observed values m_e/m_τ and m_μ/m_τ as shown in Appendix B. Then, we have two free parameters, g_1/g_2 and the modulus τ apart from the overall factors in the neutrino sector. Since these are complex, we set

$$\tau = {\rm Re}[\tau] + i\ {\rm Im}[\tau]\ , \qquad \frac{g_2}{g_1} = g\ e^{i\phi_g}\ . \tag{3.43}$$

The fundamental domain of τ is presented in Ref. [39]. In practice, we restrict our parametric search in ${\rm Re}[\tau] \in [-1.5, 1.5]$ and ${\rm Im}[\tau] > 0.6$. We also take

$\phi_g \in [-\pi, \pi]$. These four parameters are fixed by the observed $\Delta m^2_{\rm sol}/\Delta m^2_{\rm atm}$ and three mixing angles θ_{23}, θ_{12} and θ_{13}.

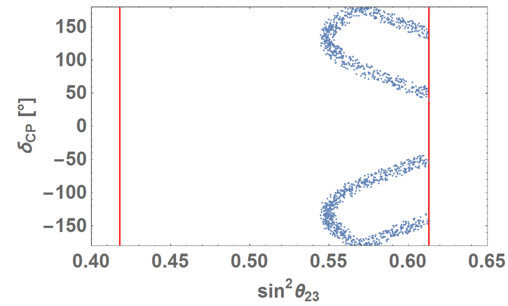

Figure 3.3: The prediction of δ_{CP} versus $\sin^2\theta_{23}$ for NH in model I(a). The vertical red lines represent the upper and lower bounds of the experimental data with 3 σ.

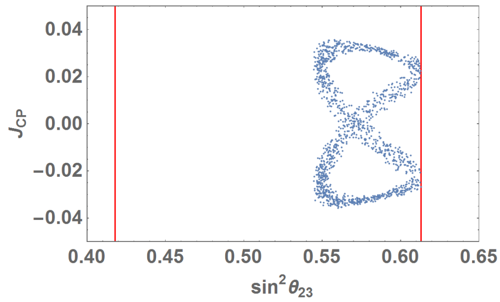

Figure 3.4: The prediction of J_{CP} versus $\sin^2\theta_{23}$ for NH in model I(a). The vertical red lines represent the upper and lower bounds of the experimental data with 3 σ.

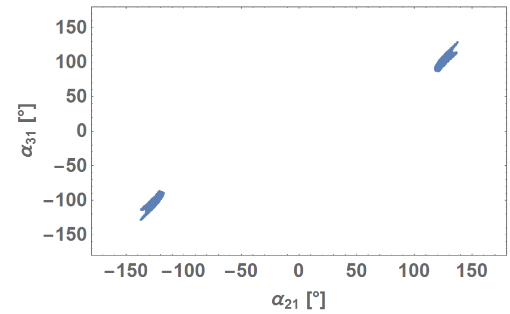

Figure 3.5: The prediction of Majorana phases α_{21} and α_{31} for NH in model I(a).

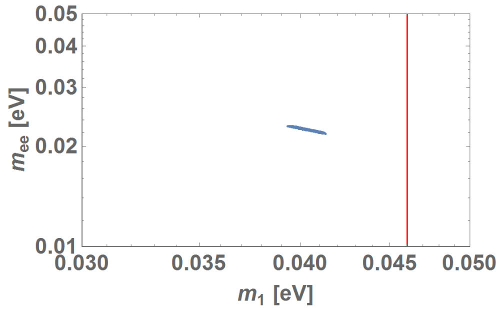

Figure 3.6: The prediction of m_{ee} versus m_1 for NH in model I(a). The red vertical line denotes the upper-bound of m_1.

At first, we present the prediction of the Dirac CP violating phase δ_{CP} versus $\sin^2\theta_{23}$ for NH of neutrino masses in Fig.1. It is emphasized that $\sin^2\theta_{23}$ is restricted to be larger than 0.54, and $\delta_{CP} = \pm(50°–180°)$. Since the correlation of $\sin^2\theta_{23}$ and δ_{CP} is characteristic, this prediction is testable in the future experiments of neutrinos. On the other hand, predicted $\sin^2\theta_{12}$

parameter	Im[τ]	Re[τ]		
region	$(0.66, 0.73)$ $\oplus(1.17, 1.32)$	$\pm(0.25, 0.31) \oplus \pm(0.46, 0.54) \oplus \pm(0.66, 0.75)$ $\oplus \pm (1.25, 1.31) \oplus \pm(1.46, 1.50)$		
parameter	g	ϕ_g	α/γ	β/γ
region	$(1.20, 1.22)$	$\pm(84, 88)^\circ \oplus \pm(92, 93)^\circ$	$(202, 203)$	$(3286, 3306)$

Table 3.6: The parameter regions consistent with the experimental data of Table 3.5 for model I(a). Results do not change under the exchange of α/γ and β/γ.

and $\sin^2\theta_{13}$ cover observed full region with $3\,\sigma$ error-bar, and there are no correlations with δ_{CP}.

We also show the predicted Jarlskog invariant J_{CP} [62], characterizing the magnitude of CP violation in neutrino oscillations, versus $\sin^2\theta_{23}$ for NH of neutrino masses in Fig.2. The magnitude of J_{CP} and predicted to be $0-0.035$ depending on θ_{23}.

We show the prediction of Majorana phases α_{21} and α_{31} in Fig.3. The predicted regions are restricted, $\alpha_{21} \simeq \alpha_{31} = \pm(90^\circ\text{–}140^\circ)$. This result is used in the calculation of neutrinoless double beta decay.

Let us show the prediction of the effective mass m_{ee} which is the measure of the neutrinoless double beta decay as seen in Appendix C. The prediction of m_{ee} is presented versus m_1 in Fig.4. It is remarkable that m_{ee} is around 22 meV while m_1 is 40 meV. The red vertical line in Fig.4 denotes the upper bound of m_1, which is derived from the cosmological bound $\sum m_i < 160\,\text{meV}$. The obtained value of m_1 indicates near degenerate neutrino mass spectrum, $m_1 \simeq m_2 \simeq 40\,\text{meV}$ and $m_3 \simeq 60\,\text{meV}$. The prediction of $m_{ee} \simeq 22\,\text{meV}$ is testable in the future experiments of the neutrinoless double beta decay. We predict the rather large sum of neutrino masses as $\sum m_i \simeq 145\,\text{meV}$, which is required by consistency with the observed value of $\sin^2\theta_{13}$.

The parameters of our model are determined by the input data of Table 3.5. Numerical values are listed in Table 3.6.

We have also scanned the parameter space for the case of IH of neutrino masses. We have found parameter sets which fit the data of $\Delta m^2_{\rm sol}$ and $\Delta m^2_{\rm atm}$ reproduce the observed three mixing angles $\sin^2\theta_{23}$, $\sin^2\theta_{12}$, and $\sin^2\theta_{13}$. However, the predicted $\sum m_i$ is around 190–200 meV. Therefore, we also omit to show numerical results.

Model I(b): Seesaw

There is another assignment of the modular weight for the left-handed lepton and the right-handed charged leptons as presented in parentheses of Table 3.3 [40]. Then, the Dirac neutrino mass matrix is given by the constant parameter as seen in Eq.(3.36). We have scanned the parameter space for both NH and IH of neutrino masses. The parameters to reproduce the observed $\Delta m^2_{\rm sol}$ and $\Delta m^2_{\rm atm}$ cannot give the large mixing angle of θ_{23}. The predicted value $\sin^2\theta_{23} \simeq 0.18$ for NH. We also obtain $\sin^2\theta_{12} \simeq 0.8$ and $\sin^2\theta_{13} \simeq 0.15$. On the other hand, the predicted value $\sin^2\theta_{23} \simeq 0$, $\sin^2\theta_{12} \simeq 0.5$, and $\sin^2\theta_{13} \simeq 0$ for IH. In conclusion, the model I(b) is inconsistent with the experimental data of Table 3.5.

It may be useful to add the discussion on the model by Criado and Feruglio [40], where the charged lepton mass matrix is different from ours in Eq.(3.32), but given by a flavon while the neutrino mass matrix is just same one in model I(b). We have reproduced the numerical results of Ref. [40], in which the three mixing angles and masses are consistent with the experimental data and the cosmological bound, respectively, for NH of neutrino masses. The predicted CP violating phase is $\delta_{CP} \simeq \pm 100°$.

Model II: Weinberg Operator

In this case, the modulus τ is the only parameter in the neutrino mass matrix apart from the overall factors. We can find the parameter space to be consistent with the observed $\sin^2\theta_{12}$ as well as $\Delta m^2_{\rm sol}$ and $\Delta m^2_{\rm atm}$ for both NH and IH. However, the predicted $\sin^2\theta_{23}$ is around 0.8 and $\sin^2\theta_{13}$ is very large as 0.45 for NH. On the other hand, for IH, the predicted $\sin^2\theta_{23}$ is rather small as 0.35 and $\sin^2\theta_{13}$ is around 0.04, which is larger than 1.6 times of the observed value. Thus, the neutrino mass matrix by the Weinberg operator do not lead to the realistic flavor mixing.

Model III: Dirac Neutrino

There is still a possibility of the neutrino being the Dirac particle. Then, the neutrino mass matrix is different from the Majorana one as shown in Table 3.4 although parameters are τ and g likewise in the case of the seesaw model I(a).

We have found the parameter space to be consistent with both observed $\sin^2\theta_{23}$ and $\sin^2\theta_{12}$ as well as $\Delta m^2_{\rm sol}$ and $\Delta m^2_{\rm atm}$ for NH. However, the predicted $\sin^2\theta_{13}$ is much smaller than the observed value of $\mathcal{O}(10^{-3})$.

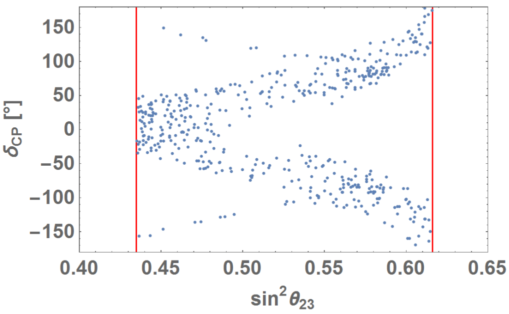

Figure 3.7: The prediction of δ_{CP} versus $\sin^2\theta_{23}$ for IH in model III. The vertical red lines represent the upper and lower bounds of the experimental data with 3 σ.

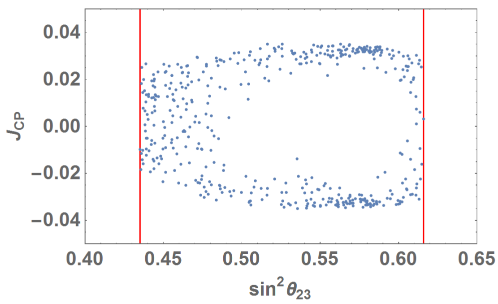

Figure 3.8: The prediction of J_{CP} versus $\sin^2\theta_{23}$ for IH in model III. The vertical red lines represent the upper and lower bounds of the experimental data with 3 σ.

$\mathrm{Im}[\tau]$	$\mathrm{Re}[\tau]$	g	ϕ_g	α/γ	β/γ
$0.90-1.12$	$\pm(0.01-0.07)$ $\pm(0.94-1.10)$	$1.43-2.12$	$\pm(76-104)^\circ$	$59-88$	$857-1302$

Table 3.7: The parameter regions consistent with the experimental data of Table 3.5 for model III. Results do not change under the exchange of α/γ and β/γ.

On the other hand, the $\sin^2\theta_{13}$ is completely consistent with the observed value for IH of neutrino masses. We present the prediction of the Dirac CP violating phase δ_{CP} versus $\sin^2\theta_{23}$ for IH in Fig.5. The predicted δ_{CP} is still allowed in $[-\pi,\pi]$ depending on the magnitude of $\sin^2\theta_{23}$. Since there are no correlations among $\sin^2\theta_{12}$, $\sin^2\theta_{13}$, and δ_{CP}, we omit figures of $\sin^2\theta_{12}$ and $\sin^2\theta_{13}$.

We also show the predicted Jarlskog invariant J_{CP} versus $\sin^2\theta_{23}$ for IH of neutrino masses in Fig.6. The magnitude of J_{CP} is predicted to be $0-0.035$.

In order to see the neutrino mass dependence of $\sin^2\theta_{13}$, we plot $\sin^2\theta_{13}$ versus $\sum m_i$ in Fig.6. The $\sum m_i$ is required in $102-150$ meV to be consistent with the observed value of $\sin^2\theta_{13}$.

We summarize numerical values of parameters in Table 5. In the Dirac neutrino model, the neutrinoless double beta decay is forbidden.

4 Conclusion

Modular symmetry is the symmetry of T^2 and which has subgroups such as S_3 and A_4. In chapter 2, we have studied the modular symmetry in magnetized D-brane models and heterotic orbifold models. In magnetized D-brane models, representations due to zero-modes on T^2 are reducible except the models with the magnetic flux $M = 2$. Irreducible representations are provided by zero-modes on the T^2/Z_2, i.e. Z_2 even states and odd states. It is reasonable because $(ST)^3$ transforms the lattice vectors (α_1, α_2) to $(-\alpha_1, -\alpha_2)$. The orders of modular groups are large, and in general, they include the Z_8 symmetry as the center. In the case of $M = 2$, the forth power of T-transformation becomes identity $(T_{(2)})^4 = \mathbb{I}$ and zero-modes form doublet of S_3. After all, $(Z_8 \times Z_4) \rtimes S_3$ appears as whole symmetry. In the case of $M = 4$, the eighth power of T-transformation becomes identity $(T_{(4)\pm})^8 = \mathbb{I}$. Zero-modes are divided into Z_2 even mods forming triplet of A_4 and odd mode forming singlet. After all, $(Z_8 \times Z_8) \rtimes A_4$ appears as whole symmetry. The D_4 flavor symmetry is a subgroup of the modular group, which is represented in the models with the magnetic flux $M = 2$. The system including zero-modes with $M = 2$, $M = 4$ and larger even M, also includes the D_4 flavor symmetry, when we define transformations of couplings in a proper way.

In heterotic orbifold models, the similarity with magnetized D-brane models on T^2 can be seen in the behavior of their zero-modes. The heterotic model on the T^2/Z_4 has exactly the same representation as the magnetized model with $M = 2$, and the modular symmetry includes the D_4 flavor symmetry. The representation due to the twisted states on the T^2/Z_3 orbifold is reducible, and their irreducible representations correspond to Z_2 even and odd states, similar to those in magnetized models. Thus, the $\Delta(54)$ flavor symmetry seems independent of the modular symmetry in the T^2/Z_3 orbifold models. Note that the first twisted states on the T^2/Z_4 are Z_2-invariant states. In this sense, we find that the modular symmetry is the symmetry on the Z_2 orbifold in both heterotic orbifold models and magnetized D-brane models. The symmetries, which remain under the Z_2 twist, can be realized as the modular symmetry.

In chapter 3, we have studied lepton flavor models with S_3 and A_4 symmetries. At this stage, we assume effective theories with S_3 and A_4 flavor symmetries from the modular symmetry without specific model building of string compactification. We study the phenomenological implications of the modular symmetry $\Gamma(2) \simeq S_3$ and $\Gamma(3) \simeq A_4$ facing recent experimental data of neutrino oscillations. The mass matrices of neutrinos and charged leptons are essentially given by fixing the expectation value of the modulus τ. In the case of no flavon, this modulus is the only source of modular invariance breaking. In S_3 models, we introduce flavons and only consider Weinberg operator for effective Majorana mass term of left-handed neutrino. As the result of numerical study, we have found realistic value of parameters consistent with experimental results within 3σ-range for NH and IH.

In A_4 models, we introduce no flavons in contrast with conventional flavor models with the A_4 symmetry. We classify the neutrino models along with type I seesaw (model I(a) and I(b)), Weinberg operator (model Ⅱ), and Dirac neutrino (model Ⅲ). For the charged lepton mass matrix, three right-handed charged leptons e_R, μ_R, and τ_R are assigned to three different singlets 1, $1''$, and $1'$ of A_4, respectively. For NH of neutrino masses, we have found that the seesaw model I(a) is available facing recent experimental data of NuFIT 3.2 [57,58] and the cosmological bound of the sum of neutrino masses [61]. The predicted $\sin^2\theta_{23}$ is restricted to be larger than 0.54 and $\delta_{CP} = \pm(50^\circ - 180^\circ)$. The sharp correlation between $\sin^2\theta_{23}$ and δ_{CP} is testable in the future experiments of the neutrino oscillations. It is remarkable that m_{ee} is around 22 meV while the sum of neutrino masses is 145 meV. For IH of neutrino masses, the Dirac neutrino model Ⅲ is completely consistent with the experimental data of NuFIT 3.2 and the cosmological bound of the sum of neutrino masses. The predicted δ_{CP} is still allowed in $[-\pi, \pi]$ depending on the magnitude of $\sin^2\theta_{23}$. The $\sum m_i = 102 - 150\,\mathrm{meV}$ is required by consistency with the observed value of $\sin^2\theta_{13}$. The seesaw model I(b) and the Weinberg operator model Ⅱ cannot reproduce the observed mixing angles after inputting the data of $\Delta m^2_{\rm sol}$ and $\Delta m^2_{\rm atm}$ for both NH and IH.

It would be interesting to try to explain baryon asymmetry universe by using lepton number violating effect in our A_4 models. Since almost all of parameters in model I(a) of A_4 are determined and the region of Dirac and two Majorana CP violating phases are predicted, the magnitude of baryon asymmetry and the energy scale of right-handed neutrinos would be predictable. It would be also interesting to study flavon-less models of S_3 symmetry for leptons and quarks to construct minimal unification models of quarks and leptons.

As seen in chapter 2, non-Abelian discrete symmetries for flavor possibly appear from string theory in natural ways. Thus, flavor models with finite modular groups are worth studying in the sense of not only model building for flavor physics, but also exploration of underlying theory of the SM.

A Non-Abelian discrete flavor symmetry in magnetized D-brane models

In this Appendix, we give a brief review on non-Abelian discrete flavor symmetries in magnetized D-brane models [31].

As mentioned in section 2.2.1, the Yukawa coulings as well as higher order couplings have the coupling selection rule (2.18). That is, we can define Z_g charges for zero-modes. Such Z_g transformation is represented on $\psi^{i,M=g}$ by

$$Z = \begin{pmatrix} 1 & & & & \\ & \rho & & & \\ & & \rho^2 & & \\ & & & \ddots & \\ & & & & \rho^{g-1} \end{pmatrix}, \tag{A.1}$$

where $\rho = e^{2\pi i/g}$. Furthermore, their effective field theory has the following permutation symmetry,

$$\psi^{i,g} \to \psi^{i+1,g}, \tag{A.2}$$

and such permutation can be represented by

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{A.3}$$

This is another Z_g^C symmetry. However, these two generators do not commute each other,

$$CZ = \rho Z C. \tag{A.4}$$

Thus, the flavor symmetry corresponds to the closed algebra including Z and C. Its diagonal elements are given by $Z^m Z'^n$, i.e. $Z_g \times Z'_g$ where

$$Z' = \begin{pmatrix} \rho & & \\ & \ddots & \\ & & \rho \end{pmatrix}, \tag{A.5}$$

and the full group corresponds to $(Z_g \times Z'_g) \rtimes Z_g^C$.

Furthermore, the zero-modes $\psi^{i,M=gn}$ with the magnetic flux $M = gn$ also represent $(Z_g \times Z_g') \rtimes Z_g^C$. The zero-modes, $\psi^{i,M=gn}$ have Z_g charges (mod g). Under C, they transform as

$$\psi^{i,M=gn} \to \psi^{i+n,M=gn}. \tag{A.6}$$

For example, the model with $g = 2$ has the D_4 flavor symmetry. The zero-modes,

$$\begin{pmatrix} \psi^{0,2} \\ \psi^{1,2} \end{pmatrix}, \tag{A.7}$$

correspond to the D_4 doublet $\mathbf{2}$, where eight D_4 elements are represented by

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.8}$$

In addition, when the model has the zero-modes $\psi^{i,4}$ $(i = 0, 1, 2, 3)$, the zero-modes, $\psi^{0,4}$ and $\psi^{2,4}$ ($\psi^{1,4}$ and $\psi^{3,4}$) transform each other under C, and they have Z_2 charge even (odd). Thus, $\psi^{0,4} \pm \psi^{2,4}$ correspond to $\mathbf{1}_{+\pm}$ of D_4 representations, while $\psi^{1,4} \pm \psi^{3,4}$ correspond to $\mathbf{1}_{-\pm}$. Furthermore, among the zero-modes $\psi^{i,6}$ $(i = 0, 1, 2, 3, 4, 5)$, the zero-modes $\psi^{i,6}$ and $\psi^{i+3,6}$ transform each other under C. Hence, three pairs of zero-modes,

$$\begin{pmatrix} \psi^{0,6} \\ \psi^{3,6} \end{pmatrix}, \quad \begin{pmatrix} \psi^{1,6} \\ \psi^{4,6} \end{pmatrix}, \quad \begin{pmatrix} \psi^{2,6} \\ \psi^{5,6} \end{pmatrix}, \tag{A.9}$$

correspond to three D_4 doublets. These results are shown in Table A.1.

Magnetic flux M	D_4 representations
2	$\mathbf{2}$
4	$\mathbf{1}_{++}$, $\mathbf{1}_{+-}$, $\mathbf{1}_{-+}$, $\mathbf{1}_{--}$
6	$3 \times \mathbf{2}$

Table A.1: D_4 representation

B Non-Abelian discrete flavor symmetry in heterotic orbifold models

Here, we give a brief review on non-Abelian discrete flavor symmetries in heteotic orbifold models [28].

The twisted string x^i on the orbifold satisfy the following boundary condition:

$$x^i(\sigma = 2\pi) = (\theta^n x(\sigma = 0))^i + \sum_k m_k \alpha_k^i, \tag{B.1}$$

similar to Eq. (2.68). Thus, the twisted string can be characterized by the space group element $g = (\theta^n, \sum_k m_k \alpha_k^i)$. The product of the two space group elements (θ^{n_1}, v_1) and (θ^{n_2}, v_2) is computed as

$$(\theta^{n_1}, v_1)(\theta^{n_2}, v_2) = (\theta^{n_1}\theta^{n_2}, v_1 + \theta^{n_1} v_2). \tag{B.2}$$

The space group element g belongs to the same conjugacy class as hgh^{-1}, where h is any space group element on the same orbifold.

Now, let us consider the couplings among twisted strings corresponding to space group elements (θ^{n_k}, v_k). Their couplings are allowed by the space group invariance if the following condition:

$$\prod_k (\theta^{n_k}, v_k) = (1, 0), \tag{B.3}$$

is satisfied up to the conjugacy class. That includes the point group selection rule, $\prod_k \theta^{n_k} = 1$, which is the Z_N invariance on the Z_N orbifold. We can define discrete Abelian symmetries from the space group invariance as well as the point group invariance. These symmetries together with geometrical symmetries of orbifolds become non-Abelian discrete flavor symmetries in heterotic orbifold models. We show them explicitly on concrete orbifolds.

B.1 S^1/Z_2 orbifold

The S^1/Z_2 orbifold has two fixed points, which are denoted by the space group elements, $(\theta, m\alpha)$ with $m = 0, 1$, where α is the lattice vector. In short, we denote them by (θ, m) and the corresponding twisted states are denoted by $\sigma_{(\theta,m)}$. These states transform

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}, \tag{B.4}$$

under the Z_2 twist. In addition, the space group invariance requires $\sum_k m_k = 0 \pmod 2$ for the couplings corresponding to the product of the space group elements $\prod_k(\theta, m_k)$ with $m_k = 0, 1$. Hence, we can define another Z_2 symmetry, under which $\sigma_{(\theta,0)}$ is even, while $\sigma_{(\theta,1)}$ is odd. That is, another Z_2 transformation can be written by

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}. \tag{B.5}$$

Furthermore, there is the geometrical permutation symmetry, which exchange two fixed points each other. Such a permutation is represented by

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}. \tag{B.6}$$

The closed algebra including Eqs.(B.4), (B.5) and (B.6) is $D_4 \simeq (Z_2 \times Z_2) \rtimes Z_2$.

B.2 T^2/Z_3 orbifold

As shown in Section 2.3, the T^2/Z_3 orbifold has three fixed points denoted by (θ, m) with $m = 0, 1, 2$, and the corresponding twisted states are denote by $\sigma_{(\theta,m)}$. The Z_3 twist transforms

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix} \longrightarrow \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix}. \tag{B.7}$$

The space group invariance requires $\sum_k m_k = 0 \pmod 3$ for the couplings corresponding to the product of the space group elements $\prod_k(\theta, m_k)$ with $m_k = 0, 1, 2$. Then, we can define another Z_3 symmetry, under which $\sigma_{(\theta,m)}$ transform

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix}. \tag{B.8}$$

There is also the permutation symmetry of the three fixed points, that is, S_3. Thus, the flavor symmetry is $\Delta(54) \simeq (Z_3 \times Z_3) \rtimes S_3$.

B.3 T^2/Z_4 orbifold

As shown in Section 2.3, the T^2/Z_4 orbifold has two θ fixed points denoted by (θ, m) with $m = 0, 1$, and the corresponding twisted states are denote by $\sigma_{(\theta,m)}$. The Z_4 twist transforms

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix} \longrightarrow \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}. \tag{B.9}$$

The space group invariance requires $\sum_k m_k = 0 \pmod 2$ for the couplings corresponding to the product of the space group elements $\prod_k(\theta, m_k)$ with $m_k = 0, 1$. Then, we can define another Z_2 symmetry, under which $\sigma_{(\theta,m)}$ transform

$$\begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \\ \sigma_{\theta,2} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sigma_{\theta,0} \\ \sigma_{\theta,1} \end{pmatrix}. \tag{B.10}$$

There is also the permutation symmetry of the two fixed points. Thus, the flavor symmetry is almost the same as one on the S^1/Z_2 orbifold. The difference is the Z_4 twist, although its squire is nothing but the Z_2 twist. Hence, the flavor symmetry can be written as $(D_4 \times Z_4)/Z_2$.

B.4 T^2/Z_2 orbifold

As shown in Section 2.3, the T^2/Z_4 orbifold has two θ fixed points denoted by $(\theta, (m, n))$ with $m, n = 0, 1$, and the corresponding twisted states are denote by $\sigma_{\theta,(m,n)}$. The space group invariance requires $\sum_k m_k = \sum_j n_j = 0 \pmod 2$ for the couplings corresponding to the product of the space group elements $\prod_k(\theta, (m_k, n_j))$ with $m_k, n_j = 0, 1$. There are two independent permutation symmetries between $(\theta, (0, n))$ and $(\theta, (1, n))$, and $(\theta, (m, 0))$ and $(\theta, (m, 1))$. Thus, this structure seems be a product of two one-dimensional orbifolds, S^1/Z_2. However, the Z_2 twist is comment such as $\sigma_{\theta,(m,n)} \longrightarrow -\sigma_{\theta,(m,n)}$. Thus, the flavor symmetry can be written by $(D_4 \times D_4)/Z_2$.

C Modular forms

Here, following [39], we derive modular functions with modular weight 2, which behave as an A_4 triplet and an S_3 doublet.

Suppose that the function $f_i(\tau)$ has modular weight k_i. That is, it transforms under the modular transformation (2.2),

$$f_i(\tau) \to (c\tau + d)^{k_i} f_i(\tau). \tag{C.1}$$

Then, it is found that

$$\frac{d}{d\tau}\sum_i \log f_i(\tau) \to (c\tau+d)^2 \frac{d}{d\tau}\sum_i \log f_i(\tau) + c(c\tau+d)\sum_i k_i. \tag{C.2}$$

Thus, $\frac{d}{d\tau}\sum_i \log f_i(\tau)$ is a modular function with the weight 2 if

$$\sum_i k_i = 0. \tag{C.3}$$

We find the following transformation behaviors under T,

$$\begin{aligned}
&\eta(3\tau) \to e^{i\pi/4}\eta(3\tau), \\
&\eta(\tau/3) \to \eta((\tau+1)/3), \\
&\eta((\tau+1)/3) \to \eta((\tau+2)/3), \\
&\eta((\tau+2)/3) \to e^{i\pi/12}\eta(\tau/3),
\end{aligned} \tag{C.4}$$

and the following transformations under S,

$$\begin{aligned}
&\eta(3\tau) \to \sqrt{\frac{-i\tau}{3}}\eta(\tau/3), \\
&\eta(\tau/3) \to \sqrt{-i3\tau}\eta(3\tau), \\
&\eta((\tau+1)/3) \to e^{-i\pi/12}\sqrt{-i\tau}\eta((\tau+2)/3), \\
&\eta((\tau+2)/3) \to e^{i\pi/12}\sqrt{-i\tau}\eta((\tau+1)/3).
\end{aligned} \tag{C.5}$$

Using them, we can construct the modular functions with weight 2 by

$$\begin{aligned}
&Y(\alpha,\beta,\gamma,\delta|\tau) \\
&= \frac{d}{d\tau}\left(\alpha \log\eta(\tau/3) + \beta\log\eta((\tau+1)/3) + \gamma\log\eta((\tau+2)/3) + \delta\log\eta(3\tau)\right),
\end{aligned} \tag{C.6}$$

with $\alpha+\beta+\gamma+\delta=0$ because of Eq.(C.3). These functions transform under S and T as

$$\begin{aligned}S: &\quad Y(\alpha,\beta,\gamma,\delta|\tau)\to\tau^2 Y(\delta,\gamma,\beta,\alpha|\tau),\\ T: &\quad Y(\alpha,\beta,\gamma,\delta|\tau)\to Y(\gamma,\alpha,\beta,\delta|\tau).\end{aligned}\tag{C.7}$$

Now let us construct an A_4 triplet by the modular functions $Y(\alpha,\beta,\gamma,\delta|\tau)$. We use the (3×3) matrix presentations of S and T as

$$\rho(S)=\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix},\qquad \rho(T)=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix},\tag{C.8}$$

where $\omega=e^{2\pi i/3}$. They satisfy

$$(\rho(S))^2=\mathbb{I},\qquad (\rho(S)\rho(T))^3=\mathbb{I},\qquad (\rho(T))^3=\mathbb{I},\tag{C.9}$$

that is, $\Gamma(3)\simeq A_4$. Using these matrices and $Y(\alpha,\beta,\gamma,\delta|\tau)$, we search an A_4 triplet, which satisfy,

$$\begin{aligned}\begin{pmatrix}Y_1(-1/\tau)\\ Y_2(-1/\tau)\\ Y_3(-1/\tau)\end{pmatrix}&=\tau^2\rho(S)\begin{pmatrix}Y_1(\tau)\\ Y_2(\tau)\\ Y_3(\tau)\end{pmatrix},\\ \begin{pmatrix}Y_1(\tau+1)\\ Y_2(\tau+1)\\ Y_3(\tau+1)\end{pmatrix}&=\rho(T)\begin{pmatrix}Y_1(\tau)\\ Y_2(\tau)\\ Y_3(\tau)\end{pmatrix}.\end{aligned}\tag{C.10}$$

Their solutions are written by

$$\begin{aligned}Y_1(\tau)&=3cY(1,1,1,-3|\tau),\\ Y_2(\tau)&=-6cY(1,\omega^2,\omega,0|\tau),\\ Y_3(\tau)&=-6cY(1,\omega,\omega^2,0|\tau),\end{aligned}\tag{C.11}$$

up to the constant c. They are explicitly written by use of eta-function as

$$\begin{aligned}Y_1(\tau)&=\frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}-\frac{27\eta'(3\tau)}{\eta(3\tau)}\right),\\ Y_2(\tau)&=\frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),\\ Y_3(\tau)&=\frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),\end{aligned}\tag{C.12}$$

where we set $c = i/(2\pi)$. They can be expanded as

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots , \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots). \end{aligned} \tag{C.13}$$

Similarly, we can construct the modular functions, which behave as an S_3 doublet. Under T, we find the following transformation behaviors,

$$\begin{aligned} &\eta(2\tau) \to e^{i\pi/6}\eta(2\tau), \\ &\eta(\tau/2) \to \eta((\tau+1)/2), \\ &\eta((\tau+1)/2) \to e^{i\pi/12}\eta(\tau/2). \end{aligned} \tag{C.14}$$

(C.15)

Also, S transformation is represented by

$$\begin{aligned} &\eta(2\tau) \to \sqrt{\frac{-i\tau}{2}}\eta(\tau/2), \\ &\eta(\tau/2) \to \sqrt{-i3\tau}\eta(2\tau), \\ &\eta((\tau+1)/2) \to e^{-i\pi/12}\sqrt{-i\tau}\eta((\tau+1)/2). \end{aligned} \tag{C.16}$$

(C.17)

Then, we consider

$$Y(\alpha,\beta,\gamma|\tau) = \frac{d}{d\tau}\left(\alpha\log\eta(\tau/2) + \beta\log\eta((\tau+1)/2) + \gamma\log\eta(2\tau)\right). \tag{C.18}$$

These functions are the modular functions with the weight 2 if $\alpha+\beta+\gamma = 0$. They transform under S and T as

$$\begin{aligned} S: &\quad Y(\alpha,\beta,\gamma|\tau) \to \tau^2 Y(\gamma,\beta,\alpha|\tau), \\ T: &\quad Y(\alpha,\beta,\gamma|\tau) \to Y(\gamma,\alpha,\beta|\tau). \end{aligned} \tag{C.19}$$

Using $Y(\alpha,\beta,\gamma|\tau)$, we construct the S_3 doublet. For example, we use the (2×2) matrix representations of S and T as

$$\rho(S) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.20}$$

They satisfy

$$(\rho(S))^2 = \mathbb{I}, \qquad (\rho(S)\rho(T))^3 = \mathbb{I}, \qquad (\rho(T))^2 = \mathbb{I}, \tag{C.21}$$

that is, $\Gamma(3) \simeq S_3$. Using these matrices and $Y(\alpha,\beta,\gamma|\tau)$, we search an S_3 doublet, which satisfy,

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \end{pmatrix} = \tau^2\rho(S)\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \end{pmatrix} = \rho(T)\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}. \tag{C.22}$$

Their solutions are written by

$$Y_1(\tau) = cY(1,1,-2|\tau), \qquad Y_2(\tau) = \sqrt{3}cY(1,-1,0|\tau), \tag{C.23}$$

up to the constant c. They are explicitly written by use of eta-function as

$$\begin{aligned} Y_1(\tau) &= \frac{i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - \frac{8\eta'(2\tau)}{\eta(2\tau)}\right), \\ Y_2(\tau) &= \frac{\sqrt{3}i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}\right), \end{aligned} \tag{C.24}$$

where we set $c = i/(2\pi)$. Moreover, they can be expanded as

$$\begin{aligned} Y_1(\tau) &= \frac{1}{8} + 3q + 3q^2 + 12q^3 + 3q^4 \cdots, \\ Y_2(\tau) &= \sqrt{3}q^{1/2}(1 + 4q + 6q^2 + 8q^3 \cdots). \end{aligned} \tag{C.25}$$

D Multiplication rule of S_3 and A_4 group

D.1 S_3 group

We use the multiplication rule of the S_3 doublet as follows:

$$
\begin{aligned}
\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}_{\mathbf{2}} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}_{\mathbf{2}} &= (a_1 b_1 + a_2 b_2)_{\mathbf{1}} \oplus (a_1 b_2 - a_2 b_1)_{\mathbf{1}'} \\
&\oplus \begin{pmatrix} a_1 b_1 - a_2 b_2 \\ -a_1 b_2 - a_2 b_1 \end{pmatrix}_{\mathbf{2}} \ , \\
\mathbf{1} \otimes \mathbf{1} = \mathbf{1} \ , \qquad \mathbf{1}' \otimes \mathbf{1}' &= \mathbf{1} \ . \qquad \text{(D.1)}
\end{aligned}
$$

D.2 A_4 group

We use the multiplication rule of the A_4 triplet as follows:

$$
\begin{aligned}
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} &= (a_1 b_1 + a_2 b_3 + a_3 b_2)_{\mathbf{1}} \oplus (a_3 b_3 + a_1 b_2 + a_2 b_1)_{\mathbf{1}'} \\
&\oplus (a_2 b_2 + a_1 b_3 + a_3 b_1)_{\mathbf{1}''} \\
&\oplus \frac{1}{3} \begin{pmatrix} 2a_1 b_1 - a_2 b_3 - a_3 b_2 \\ 2a_3 b_3 - a_1 b_2 - a_2 b_1 \\ 2a_2 b_2 - a_1 b_3 - a_3 b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2} \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \end{pmatrix}_{\mathbf{3}} \ , \\
\mathbf{1} \otimes \mathbf{1} = \mathbf{1} \ , \qquad \mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}'' \ , &\qquad \mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}' \ , \qquad \mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1} \ . \qquad \text{(D.2)}
\end{aligned}
$$

More details are shown in the review [16, 17].

E Determination of α/γ and β/γ

The coefficients α, β, and γ in Eq.(3.32) are taken to be real positive without loss of generality. We show these parameters are described in terms of the modular parameter τ and the charged lepton masses. We rewrite the mass matrix of Eq.(3.32) as

$$M_E^{(1)} = \gamma Y_3 \mathrm{diag}[\hat{\alpha}, \hat{\beta}, 1] \begin{pmatrix} \hat{Y_1} & \hat{Y_2} & 1 \\ 1 & \hat{Y_1} & \hat{Y_2} \\ \hat{Y_2} & 1 & \hat{Y_1} \end{pmatrix}_{RL} , \tag{E.1}$$

where $\hat{\alpha} \equiv \alpha/\gamma$, $\hat{\beta} \equiv \beta/\gamma$, $\hat{Y_1} \equiv Y_1/Y_3$, and $\hat{Y_2} \equiv Y_2/Y_3$. We use the relation $Y_2^2 + 2Y_1Y_3 = 0$ to eliminate Y_1 in the equation. Then, we obtain the following three equations:

$$\mathrm{Tr}[M_E^{(1)\dagger} M_E^{(1)}] = \sum_{i=e}^{\tau} m_i^2 = \frac{|\gamma Y_3|^2}{4}(1 + \hat{\alpha}^2 + \hat{\beta}^2) C_1 , \tag{E.2}$$

$$\mathrm{Det}[M_E^{(1)\dagger} M_E^{(1)}] = \prod_{i=e}^{\tau} m_i^2 = \frac{|\gamma Y_3|^6}{64}\hat{\alpha}^2\hat{\beta}^2 C_2 , \tag{E.3}$$

$$\frac{\mathrm{Tr}[M_E^{(1)\dagger} M_E^{(1)}]^2 - \mathrm{Tr}[(M_E^{(1)\dagger} M_E^{(1)})^2]}{2} = \chi = \frac{|\gamma Y_3|^4}{16}(\hat{\alpha}^2 + \hat{\alpha}^2\hat{\beta}^2 + \hat{\beta}^2) C_3 , \tag{E.4}$$

where $\chi \equiv m_e^2 m_\mu^2 + m_\mu^2 m_\tau^2 + m_\tau^2 m_e^2$. The coefficients C_1, C_2, and C_3 depend only on $\hat{Y_2} \equiv Y e^{i\phi_Y}$, where Y is real positive and ϕ_Y is a phase parameter,

$$\begin{aligned} C_1 &= (2 + Y^2)^2, \\ C_2 &= 64 + 400Y^6 + Y^{12} - 40Y^3(Y^6 - 8)\cos(3\phi_Y) - 16Y^6\cos(6\phi_Y) , \\ C_3 &= 16 + 16Y^2 + 36Y^4 + 4Y^6 + Y^8 - 8Y^3(Y^2 - 2)\cos(3\phi_Y) . \end{aligned} \tag{E.5}$$

These values are determined if the value of modulus τ is fixed. Then, we obtain the general equations which describe $\hat{\alpha}$ and $\hat{\beta}$ as functions of charged lepton masses and τ:

$$\frac{(1+s)(s+t)}{t} = \frac{(\sum m_i^2/C_1)(\chi/C_3)}{\prod m_i^2/C_2} , \qquad \frac{(1+s)^2}{s+t} = \frac{(\sum m_i^2/C_1)^2}{\chi/C_3} , \tag{E.6}$$

where we redefine the parameters $\hat{\alpha}^2+\hat{\beta}^2 = s$ and $\hat{\alpha}^2\hat{\beta}^2 = t$. They are related as follows,

$$\hat{\alpha}^2 = \frac{s \pm \sqrt{s^2 - 4t}}{2} , \qquad \hat{\beta}^2 = \frac{s \mp \sqrt{s^2 - 4t}}{2} . \tag{E.7}$$

F Lepton mixing and neutrinoless double beta decay

Supposing neutrinos to be Majorana particles, the PMNS matrix $U_{\rm PMNS}$ [63, 64] is parametrized in terms of the three mixing angles θ_{ij} $(i,j=1,2,3;\ i<j)$, one CP violating Dirac phase $\delta_{\rm CP}$, and two Majorana phases α_{21}, α_{31} as follows:

$$
\begin{aligned}
U_{\rm PMNS} = &\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\rm CP}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & c_{23}c_{13} \end{pmatrix} \\
&\times \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix},
\end{aligned} \tag{F.1}
$$

where c_{ij} and s_{ij} denote $\cos\theta_{ij}$ and $\sin\theta_{ij}$, respectively.

The rephasing invariant CP violating measure, the Jarlskog invariant [62], is defined by the PMNS matrix elements $U_{\alpha i}$. It is written in terms of the mixing angles and the CP violating Dirac phase as:

$$
J_{CP} = {\rm Im}\left[U_{e1}U_{\mu 2}U_{e2}^{*}U_{\mu 1}^{*}\right] = s_{23}c_{23}s_{12}c_{12}s_{13}c_{13}^{2}\sin\delta_{\rm CP}\ . \tag{F.2}
$$

There are also other invariants I_1 and I_2 associated with Majorana phases [65–68],

$$
\begin{aligned}
I_1 &= {\rm Im}\left[U_{e1}^{*}U_{e2}\right] = s_{12}c_{12}c_{13}^{2}\sin\left(\frac{\alpha_{21}}{2}\right), \\
I_2 &= {\rm Im}\left[U_{e1}^{*}U_{e3}\right] = c_{12}s_{13}c_{13}\sin\left(\frac{\alpha_{31}}{2}-\delta_{\rm CP}\right).
\end{aligned} \tag{F.3}
$$

We calculate $\delta_{\rm CP}$, α_{21}, and α_{31} with these relations.

In terms of these parametrization, the effective mass for the $0\nu\beta\beta$ decay is given as follows:

$$
m_{ee} = \left|m_1c_{12}^{2}c_{13}^{2} + m_2s_{12}^{2}c_{13}^{2}e^{i\alpha_{21}} + m_3s_{13}^{2}e^{i(\alpha_{31}-2\delta_{CP})}\right|\ . \tag{F.4}
$$

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