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Edited by Y. Giga and T. Ozawa

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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on July 26 through July 28 in 2000 at Faculty of Science, Hokkaido University.

This is the 25th time of the symposium and also commemorates a successful retirement of Professor Rentaro Agemi, who made a large contribution to its organization for many years.

We wish to dedicate this volume to Professor Agemi who is going to leave Hokkaido University in March of 2001.

Y. Giga

CONTENTS

Programme

第２５回偏微分方程式論札幌シンポジウム

下記の要領でシンポジウムを行ないますのでご案内申し上げます。

代表者　　儀我　美一，小澤　徹
Organizers:　Y. Giga and T. Ozawa

記

1. 日時　2000 年 7 月 26 日（水）～7 月 28 日（金）
2. 場所　北海道大学大学院 理学研究科 5 号館 301 号室，304 号室 (数学教室の南向かい)
3. 講演

July 26, 2000 (水)

9:30-10:30	H. J. Choe (KAIST) Asymptotic properties of stationary axis-symmetric solutions of the Navier-Stokes equations
11:00-12:00	宮川 鉄朗 (T. Miyakawa), 神戸大 (Kobe U.) Asymtotic profiles of Navier-Stokes flows in $\mathbf{R}^n$ and $\mathbf{R}^n_+$
13:30-14:00	*
14:00-14:30	加藤　淳 (J. Kato), 北大理院 (Hokkaido U.) On the uniquenss of nondecaying solutions of the Navier-Stokes equations
15:00-15:30	沢田 宙広 (O. Sawada), 北大理院 (Hokkaido U.) Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity
15:45-16:15	後藤 俊一 (S. Goto), 金沢大 (Kanazawa U.) On the viscosity solution method for a dynamic boundary value problem arising in the theory of superconductivity
16:30-17:00	*

July 27, 2000 (木)

9:30-10:30	久保 英夫 (H. Kubo), 静岡大 (Shizuoka U.) Nonlinear wave equations with large potential

11:00-12:00　望月　清 (K. Mochizuki), 都立大 (Tokyo Metropolitan U.)
An improvement of a result of Jäger and Rejto on oscillating long-range potentials

13:30-14:00　*

14:00-15:00　高村 博之 (H. Takamura), 筑波大 (Tsukuba U.)
Weighted pointwise estimates for the wave equation

15:30-16:30　荻原 俊子 (T. Ogiwara), 城西大 (Josai U.)
Spiral traveling wave solutions for some nonlinear diffusion equations

16:45-17:30　*

18:00-20:00　懇親会　BANQUET　エンレイソウ

July 28, 2000 (金)

9:30-10:30　溝口 紀子 (N. Mizoguchi), 学芸大 (Tokyo Gakugei U.)
On the blowup of solutions in a semilinear parabolic equation with supercritical nonlinearity

11:00-12:00　堤　誉志雄 (Y. Tsutsumi), 東北大 (Tohoku U.)
Counterexamples to bilinear estimates related with the KdV equation and the nonlinear Schrödinger equation

12:00-12:30　*

*この時間は講演者を囲んで自由な質問の時間とする予定です。
* indicates discussion time. Lecturers in each session are invited to stay in the coffee-tea room during discussion time.

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例年と会場が異なりますのでご注意下さい。

Asymptotic properties of stationary axis-symmetric solutions of the Navier-Stokes equations

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Abstract

We let Ω be an exterior domain in $\mathbf{R}^3$, which is the complement of a compact axis-symmetric body B with respect to z-axis. Without loss of generality we assume $0 \in B$ and $B \subset B_1(0)$, where $B_1(0)$ is the unit ball. We study the asymptotic properties of an axis-symmetric solution $(\mathbf{u}, p)$ of the three dimensional Navier-Stokes equations

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \\ \mathrm{div}\mathbf{u} &= 0 \end{aligned} \tag{0.1}$$

satisfying the boundary condition

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \partial\Omega \qquad \text{and} \qquad \lim_{|\mathbf{x}|\to 0} \mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty \tag{0.2}$$

with bounded Dirichlet integral

$$\int_\Omega |\nabla\mathbf{u}|^2 dxdydz < +\infty. \tag{0.3}$$

We assume $\partial\Omega$ is smooth and $\mathbf{u}_0$ is smooth since we are only interested in the asymptotic behavior of $\mathbf{u}$ near infinity. Here $\mathbf{f}$ is also an axis-symmetric external force with respect to $z-$ axis in $\mathbf{R}^3$ and ν is a positive constant denoting viscosity.

In 1933, Leray [13] constructed a weak solution of (0.1) and (0.2) for a zero velocity at the boundary and constant velocity $\mathbf{u}_\infty$ at the infinity given. Unfortunately, Leray's construction could give little information for

the asymptotic behavior of the solution. In 1959, Finn [3] showed that for the Leray's solution in three-dimensional exterior domain there is a constant vector $\mathbf{u}_1$ such that $\mathbf{u}(x) \to \mathbf{u}_1$ as $|x| \to \infty$. But in that paper, it was not shown that $\mathbf{u}_\infty = \mathbf{u}_1$. In 1961 Fujita [6] constructed a two and three dimensional weak solutions of (0.1) and (0.2) by Galerkin method satisfying (0.3). In [6] he also showed that his solution converges to the prescribed $\mathbf{u}_\infty$ as $|x| \to \infty$. Still Fujita's result could not give any information of the asymptotic behavior of Leray's solution for large data since the uniqueness of the weak solutions of the Navier-Stokes equations with finite Dirichlet integral holds only for sufficiently small enough data compared to the viscosity. In [3] Finn suggested a class of physically reasonable functions in three dimensional exterior domain satisfying

$$\mathbf{u}(x) = \mathcal{O}(|x|^{-1}). \tag{0.4}$$

(See [4] for the concept of physically reasonable solution in three dimensional exterior domain.) In 1965 Finn [5] constructed a physically reasonable solution for a small data, small $\mathbf{u}_\infty$ and sufficiently small summable force $\mathbf{f}$, in a three dimensional exterior domain. (See also [7] and [14] for similar result in two dimensional exterior domain when the data are small enough and $\mathbf{u}_\infty \neq 0$.) It is easy to show that any physically reasonable solution has finite Dirichlet integral and hence it is also a Leray's solution for a small data because the Leray's solution is unique for a small data. It is then natural question if any Leray's solution is also a physically reasonable solution for large data. This will give an answer to the question whether there is a physically reasonable solution for large data. In 1973 Babenko [2] showed that the Leray's solution is also a physically reasonable solution for arbitrary nonzero $\mathbf{u}_\infty$ and properly decaying $\mathbf{f}$ in three dimensional exterior domain. Later, in 1992 Galdi [10] showed that the same result holds for $\mathbf{u}_\infty = 0$ provided that $\mathbf{u}$ obey a certain relation called "energy inequality" and the viscosity is sufficiently large compared to $\mathbf{f}$. In 1973 Gilbarg and Weinberger [11] developed similar result in two dimensional exterior domain. More precisely they showed that for any Leray's solution, the pressure has a finite limit, the velocity has a limit $\mathbf{u}_1$ in mean at infinity for some $\mathbf{u}_1$ provided the solution is bounded. In 1988 Amick [1]showed that $\mathbf{u} \in L^\infty(\Omega)$ if $\mathbf{f} \equiv 0$ in two dimensional exterior domain. In 1975 Gilbarg and Weinberger [12] showed that the Leray's solution in two dimensional exterior domain grows more slowly than $(\log r)^{1/2}$ and that the pressure has a finite limit at infinity, the velocity $\mathbf{u}$ has a limit $\mathbf{u}_1$ in the mean for some $\mathbf{u}_1$ or $\int_0^{2\pi} |\mathbf{u}(r,\theta)|^2 d\theta$ approaches infinity as $r \to \infty$.

In this paper, we find a decay rate of axis-symmetric Leray's solution for given homogeneous data $\mathbf{u}_\infty = 0$ at infinity. In fact, we have a good understanding when $\mathbf{u}_\infty$ is nonzero constant vector by the results of Babenko and there are few results for the homogeneous condition at infinity. Now we want to state a decay result along r-direction when the flow is axis-symmetric.

Theorem 0.1 *Suppose that* $(\mathbf{u}, p)$ *is axis-symmetric solution to* (0.1) *and* (0.2) *for the homogeneous data* $\mathbf{u}_\infty = 0$ *satisfying*

$$\int |\nabla \mathbf{u}|^2 d\mathbf{x} \leq M$$

for some constant M. *Then there is a constant* $c(M)$ *depending only on* M *such that for* $r > \max(2, |z|)$

$$|u^r(\mathbf{x})| + |u^\theta(\mathbf{x})| \leq c(M) \left(\frac{\log r}{r} \right)^{\frac{1}{2}},$$

and for any positive constant $\delta > 0$, *there is* $c(M, \delta)$ *depending only on* M, δ *such that for* $r > \max(2, |z|)$

$$|u^3(r,z)| \leq c(M,\delta) \frac{1}{r^{\frac{3}{8}-\delta}}$$

where u^r, u^θ, u^3 *are the components of* $\mathbf{u}$ *along the direction* $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3$, *in cylindrical coordinate, respectively.*

For the proof, we decouple our Navier-Stokes equations with u^θ and $(u^r, u^3, \omega^\theta)$ parts. The equation for u^θ does not include any pressure terms. Hence, we can apply the standard elliptic theory. Moreover, we find the cylindrical fundamental solution involves with elliptic integrals and it has a better decay property along r-direction. For the $(u^r, u^3, \omega^\theta)$ part, the potential expression using Biot-Savart law with vorticity field provides the necessary informations. To handle surface potentials, it is cruicial to observe

$$\int_{-\infty}^{\infty} |u^r(r,z)|^2 + |u^\theta(r,z)|^2 dz \leq c(M)$$

for all r.

It seems open to find a decay property of power type along z-direction. Our theorem only touches r-direction.

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Asymptotic profiles of Navier-Stokes flows in $\mathbb{R}^n$ and $\mathbb{R}^n_+$

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Statement of the Main Results

We are interested in space-time behavior as $t \to \infty$ of weak and strong solutions of the Navier-Stokes initial value problem :

$$\text{(NS)} \qquad \partial_t u + u \cdot \nabla u = \Delta u - \nabla p, \qquad \nabla \cdot u = 0, \qquad u|_{t=0} = a,$$

for unknown velocity u, unknown pressure p and a prescribed initial velocity a. Problem (NS) will be discussed on the whole space $\mathbb{R}^n$ and on the upper half-space$\mathbb{R}^n_+$; and in the latter case the boundary condition

$$\text{(BC)} \qquad u|_{\partial R^n_+} = 0$$

is added. First we consider problem (NS) on $\mathbb{R}^n$, assuming that a satisfies

$$\text{(1)} \qquad \int (1 + |y|)|a(y)|dy < \infty.$$

Our result is the following

Theorem 1. (i) *Suppose $a \in \boldsymbol{L}^1(\mathbb{R}^n) \cap \boldsymbol{L}^n_\sigma(\mathbb{R}^n)$ satisfies (1), with $\|a\|_n$ sufficiently small. Then there is a strong solution u defined for all $t \geq 0$ such that*

$$\text{(2)} \qquad \lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| u_j(t) + \partial_k E_t(\cdot) \int y_k a_j(y)dy + F_{\ell,jk}(\cdot, t) \int_0^\infty \int (u_k u_\ell)(y, s)dyds \right\|_q = 0 .$$

for all $1 \leq q \leq \infty$ and $j = 1, \ldots, n$. Here,

$$E_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-|x|^2/4t}, \qquad F_{\ell,jk}(x, t) = \partial_\ell E_t(x)\delta_{jk} + \int_0^\infty \partial_j \partial_k \partial_\ell E_{\tau+t}(x)d\tau.$$

(ii) *For each $a \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n) \cap \boldsymbol{L}^1(\mathbb{R}^n)$ satisfying (1), there exists a weak solution u which admits expansion (2) for all $1 \leq q \leq 2$.*

Condition (1) implies

$$\text{(3)} \qquad \|u(t)\|_q \leq \begin{cases} C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} & (1 \leq q \leq n) \\ Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} & (n < q \leq \infty) \end{cases}$$

if u is a strong solution, and

$$\|u(t)\|_q \leq C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad (1 \leq q \leq 2) \tag{4}$$

if u is a weak solution. In both cases we have

$$\|u(t)\|_2^2 \leq C(1+t)^{-1-\frac{n}{2}} \qquad \text{and so} \quad \int_0^\infty \int (u_k u_\ell)(y,s)\,dy\,ds \quad \text{is finite.} \tag{5}$$

See [3] for the proof. The results improve those of Carpio [2].

The exponent $1+n/2$ in (5) is expected to be optimal in general (see [8]). Indeed, we can apply (the proof of) Theorem 1 to deduce

Theorem 2. (i) *Let* $a \in \mathbb{L}_\sigma^2(\mathbb{R}^n) \cap \mathbb{L}^1(\mathbb{R}^n)$ *satisfy* (1), *so there exists a weak solution* u *satisfying* (5). *Then we have*

$$\|u(t)\|_2^2 \geq c^{-1-\frac{n}{2}} \qquad \textit{for large } t > 0 \tag{6}$$

if and only if

$$\textit{either} \quad \Big(\int y_k a_j(y)\,dy\Big)_{j,k=1}^n \neq 0, \quad \textit{or} \quad \int_0^\infty \int (u_k u_\ell)(y,s)\,dy\,ds \neq c\delta_{k\ell} \quad \textit{for all } c \geq 0. \tag{7}$$

(ii) *Suppose* $a \in \mathbb{L}_\sigma^2(\mathbb{R}^n)$ *satisfies*

$$\|e^{t\Delta}a\|_2^2 \leq C(1+t)^{-\frac{n}{2}} \tag{8}$$

and so there exists a weak solution u *such that* $\|u(t)\|_2^2 \leq C(1+t)^{-\frac{n}{2}}$. *Then*

$$\|u(t)\|_2^2 \geq ct^{-\frac{n}{2}} \qquad \textit{for large } t > 0 \tag{9}$$

if and only if

$$\|e^{t\Delta}a\|_2^2 \geq c' t^{-\frac{n}{2}} \qquad \textit{for large } t > 0. \tag{10}$$

Here $e^{t\Delta}$ *means convolution with the heat kernel.*

Remarks. (i) A full proof of Theorem 2 is given in [6]. Condition (7) equally involves all coordinate directions, and this reflects the isotropy of the space $\mathbb{R}^n$.

(ii) If $a \in \mathbb{L}_\sigma^2(\mathbb{R}^n) \cap \mathbb{L}^1(\mathbb{R}^n)$, then (8) is fulfilled. But (10) is not fulfilled, since

$$a \in \mathbb{L}^1(\mathbb{R}^n), \quad \nabla \cdot a = 0 \qquad \text{implies} \qquad \int a(x)\,dx = 0.$$

An example of initial data satisfying (8) and (10) is given by $a \in \mathbb{L}_\sigma^2(\mathbb{R}^n)$ such that

$$\int_{S^{n-1}} |\hat{a}(r,\omega)|^2 d\omega \in L^\infty(\mathbb{R}_+), \qquad \liminf_{r\to 0} \int_{S^{n-1}} |\hat{a}(r,\omega)|^2 d\omega > 0.$$

Here, $\hat{a}(\xi)$ is the Fourier transform of a, and $\xi = (r,\omega)$ in the polar coordinates.

We next consider problem (NS) – (BC) on the upper half-space $\mathbb{R}^n_+$. Using the Helmholtz decomposition ([1])

$$\boldsymbol{L}^q(\mathbb{R}^n_+) = \boldsymbol{L}^q_\sigma(\mathbb{R}^n_+) \oplus \boldsymbol{L}^q_\pi(\mathbb{R}^n_+), \qquad 1 < q < \infty,$$

with

$$\boldsymbol{L}^q_\sigma(\mathbb{R}^n_+) = \{u \in \boldsymbol{L}^q(\mathbb{R}^n_+) \ : \nabla \cdot u = 0, \quad u_n|_{\partial R^n_+} = 0\},$$

$$\boldsymbol{L}^q_\pi(\mathbb{R}^n_+) = \{\nabla p \in \boldsymbol{L}^q(\mathbb{R}^n_+) \ : p \in L^q_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})\},$$

and the associated bounded projection P to $\boldsymbol{L}^q_\sigma(\mathbb{R}^n)$, we reformulate the problem in the form of the integral equation:

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P(u \cdot \nabla u)(s)ds. \tag{11}$$

Here $A = -P\Delta$ is the Stokes operator and $\{e^{-tA}\}_{t \geq 0}$ is the associated bounded analytic semigroup in $\boldsymbol{L}^q_\sigma(\mathbb{R}^n_+)$. In what follows we assume

$$\int_{R^n_+} (1 + y_n)|a(y)|dy < \infty. \tag{12}$$

Let

$$e^{-tA}a = v = (v', v_n).$$

We systematically use Ukai's formula ([7]):

$$v_n(t) = Ue^{-tB}[a_n - S \cdot a'], \qquad v'(t) = e^{-tB}[a' + Sa_n] - Sv_n, \tag{13}$$

where $B = -\Delta$ is the Dirichlet-Laplacian on $\mathbb{R}^n_+$, $S = (S_1, \cdots, S_{n-1})$ are the Riesz transforms on $\mathbb{R}^{n-1}$, and U is the bounded linear operator on $\boldsymbol{L}^q(\mathbb{R}^n_+)$, $1 < q < \infty$, defined via the Fourier transform with respect to $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$ by

$$\widehat{Uf}(\xi', x_n) = |\xi'| \int_0^{x_n} e^{-(x_n - y)|\xi'|} \hat{f}(\xi', y)dy.$$

Using (13) we expand the right-hand side of (11), to get

Theorem 3. (i) *Let $a \in \boldsymbol{L}^1(\mathbb{R}^n_+) \cap \boldsymbol{L}^n_\sigma(\mathbb{R}^n_+)$ satisfy* (12), *with $\|a\|_n$ sufficiently small. Then there exists a strong solution u such that, for all $1 < q < \infty$,*

$$\begin{aligned} \lim_{t \to \infty} t^{\frac{1}{2} + \frac{n}{2}(1 - \frac{1}{q})} \Bigg\| & u_n(t) + 2U\partial_n E_t(\cdot) \int_{R^n_+} y_n a_n(y)dy \\ & - 2U\partial_n F_t(\cdot) \cdot \left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty \int_{R^n_+} (u_n u')(y,s)dyds \right) \Bigg\|_q = 0 \end{aligned} \tag{14}$$

and

$$\begin{aligned} \lim_{t \to \infty} t^{\frac{1}{2} + \frac{n}{2}(1 - \frac{1}{q})} \Bigg\| & u'(t) + 2(\partial_n F_t(\cdot) - SU\partial_n E_t(\cdot)) \int_{R^n_+} y_n a_n(y)dy \\ & + 2\partial_n E_t(\cdot) \left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty \int_{R^n_+} (u_n u')(y,s)dyds \right) \\ & + 2SU\partial_n F_t(\cdot) \cdot \left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty \int_{R^n_+} (u_n u')(y,s)dyds \right) \Bigg\|_q = 0. \end{aligned} \tag{15}$$

(ii) *Let* $n = 2,3,4$. *For each* $a \in L^2_\sigma(\mathbb{R}^n_+) \cap L^1(\mathbb{R}^n_+)$ *satisfying* (12), *there exists a weak solution* u *which satisfies* (14) *and* (15) *for all* $1 < q \leq 2$.

Here, E_t *stands for the heat kernel in various space dimensions and*

$$F_t(x) = \pi^{-\frac{1}{2}} E_t(x_n) \int_0^\infty \eta^{-\frac{1}{2}} \nabla' E_{\eta+t}(x')d\eta$$

is the kernel function of the convolution operator $e^{t\Delta}S = Se^{t\Delta}$.

See [4] for the proof. Condition (12) implies $\|u(t)\|_2^2 \leq C(1+t)^{-1-\frac{n}{2}}$ for weak and strong solutions treated in Theorem 3. Therefore, the integrals $\int_0^\infty \int_{R^n_+} (u_n u')(y,s)dyds$ are finite. Note also that the functions ∇E_t, $F_{\ell,jk}$ and $\partial_n F_t$ employed in Theorems 1 and 3 are all written in the form

$$t^{-\frac{n+1}{2}} K(xt^{-\frac{1}{2}})$$

in terms of some specific functions K which are bounded and L^p-integrable, $1 \leq p < \infty$ ($1 < p < \infty$ for $\partial_n F_t$). We should also mention that Theorem 3 exhibits no boundary effects. This is probably because we are dealing only with solutions decaying very rapidly (like $|x|^{-n-1}$) as $|x| \to \infty$.

We can apply Theorem 3 to prove an analogue of Theorem 2 for flows in $\mathbb{R}^n_+$.

Theorem 4. *Let* $n = 2,3,4$.

(i) *Let* $a \in L^2_\sigma(\mathbb{R}^n_+) \cap L^1(\mathbb{R}^n_+)$ *satisfy* (12); *so there exists a weak solution* u *such that* $\|u(t)\|_2^2 \leq C(1+t)^{-1-\frac{n}{2}}$. *Then*

$$\|u(t)\|_2^2 \geq Ct^{-1-\frac{n}{2}} \qquad \text{for large } t > 0 \tag{16}$$

if and ony if

$$\left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty \int_{R^n_+} (u_n u')(y,s)dyds, \ \int_{R^n_+} y_n a_n(y)dy \right) \neq (0,0). \tag{17}$$

(ii) *Let* $a \in L^2_\sigma(\mathbb{R}^n_+)$ *satisfy* $\|e^{-tA}a\|_2 \leq C(1+t)^{-\frac{n}{4}}$; *and so there exists a weak solution* u *such that* $\|u(t)\|_2 \leq C(1+t)^{-\frac{n}{4}}$. *Then*

$$\|u(t)\|_2 \geq ct^{-\frac{n}{4}} \qquad \text{for large } t > 0 \tag{18}$$

if and only if

$$\|e^{-tA}a\|_2 \geq c't^{-\frac{n}{4}}. \tag{19}$$

See [4] for the proof. In contrast to the case of flows in $\mathbb{R}^n$, condition of (17) represents an interaction between the initial velocity a and the corresponding solution u. Moreover, the integrals

$$\int_0^\infty \int_{R^n_+} (u_j u_k)(y,s)dyds, \qquad 1 \leq j,k \leq n-1, \quad j = k = n,$$

play no role in (17). These facts and condition (12) reflect some geometric properties of the half-space $\mathbb{R}^n_+$ that affect the behavior of flows therein. We further remark that

$$\|e^{-tA}a\|_2^2 \geq ct^{-1-\frac{n}{2}} \quad \text{if and only if} \quad \left(\int_{R^n_+} y_n a'(y)dy,\ \int_{R^n_+} y_n a_n(y)dy\right) \neq (0,0).$$

Further Results

For flows in $\mathbb{R}^n$, we can also prove the following:

(i) Let a satisfy $\nabla \cdot a = 0$ and

$$|a(y)| \leq C(1+|y|)^{-n-1}, \qquad a_j(y) = \sum_{k=1}^n \partial_k b_{jk}(y),$$
$$|b(y)| \leq C(1+|y|)^{-n}, \qquad \int |b(y)|dy < \infty.$$

Then, under some smallness assumptions on a there is a strong solution u such that

$$|u(x,t)| \leq C_\gamma(1+|x|)^{-\gamma}(1+t)^{-(n+1-\gamma)/2}, \qquad 0 \leq \gamma \leq n+1, \tag{20}$$

and, with $j = 1, \ldots, n$,

$$\begin{aligned} \lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| u_j(t) - \sum_{1\leq|\alpha|\leq m} \frac{(-1)^{|\alpha|}}{\alpha!}\partial_x^\alpha E_t(\cdot) \int y^\alpha a_j(y)dy \\ + \sum_{2p+|\beta|\leq m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!}\partial_t^p\partial_x^\beta F_{\ell,jk}(\cdot,t) \int_0^\infty\!\!\int s^p y^\beta (u_k u_\ell)(y,s)dyds \Bigg\|_q = 0 \end{aligned} \tag{21}$$

whenever $1 \leq q \leq \infty$ and $1 \leq m \leq n$.

(ii) Suppose $a \in \boldsymbol{L}^2_\sigma(\mathbb{R}^n) \cap \boldsymbol{L}^1(\mathbb{R}^n)$ satisfies

$$\int (1+|y|)^{n-1}|a(y)|dy < \infty, \qquad \int (1+|y|)^n |a(y)|^2 dy < \infty. \tag{22}$$

Then there exists a weak solution u which satisfies (21) for $1 \leq q \leq 2$ and $1 \leq m \leq n-1$.

(iii) In general, we do not know whether m can be chosen as $m > n$ in (i) and as $m > n-1$ in (ii), respectively. But, when $n = 2$, there exists a smooth Navier-Stokes flow u which satisfies

$$\begin{aligned} &\|u(t)\|_q \leq C_q e^{-\gamma_q t} \quad (1 \leq q \leq \infty), \\ &|u(x,t)| \leq C_m e^{-\gamma_m t}(1+|x|)^{-m}, \quad (m = 0,1,2,\ldots). \end{aligned} \tag{23}$$

We do not know whether some results corresponding to (i) – (iii) above hold for flows in $\mathbb{R}^n_+$.

Estimate (20) is obtained in [5]. The other results are all proved in [3].

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ON THE UNIQUENESS OF NONDECAYING SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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1. INTRODUCTION

In this talk, we consider a uniqueness probrem for the Navier-Stokes equations:

$$
\text{(NS)} \quad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 & \text{in } \mathbf{R}^n \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbf{R}^n \times (0, \infty), \\ u|_{t=0} = u_0 & \text{in } \mathbf{R}^n, \end{cases}
$$

where $u = u(t, x) = (u_1(t, x), \cdots, u_n(t, x))$ and $p = p(t, x)$ stand for the unknown velocity vector field of the fluid and its pressure respectively, while $u_0 = u_0(x) = (u_0^1(x), \cdots, u_0^n(x))$ is given initial velocity vector field.

The purpose of this talk is to show a uniqueness of solutions for (NS) that have no decay at space infinity. There is many results to treat a uniqueness of solutions to the Navier-Stokes equations. However, most of them are considered in the class of the solutions that have decay at space infinity.

When we don't assume the decay, we have a simple example of solutions that are not unique. If we set $u(t, x) = g(t)$ a spacially constant function and $p(t, x) = -g'(t) \cdot x$, then (u, p) evidently solves (NS) no matter how g is taken with $g(0) = u_0$. These solutions are not uniquely determined if we fix initial data $g(0)$. So we must impose some condition on p to get uniqueness.

2. KNOWN RESULTS

On the uniqueness without assuming the decay, there are some known results for classical solutions. Their sufficient conditions are the following.

(i) [1] $n = 3$, $|u| = |\nabla u| = O(1)$, $|p| = O(|x|^{1-\varepsilon})$ $(|x| \to \infty)$ for $\varepsilon > 1$,

(ii) [4], [3] $n = 2,\ 3$, $|\nabla u| = O(1)$, $|p| = O(|x|^{1-n/2})$ $(|x| \to \infty)$.

Our main result concerns (i) and gives the another characterization for the sufficient condition on the pressure term p.

3. MAIN RESULT

Before stating our main result, we introduce some function spaces. Let BUC denote the space of all bounded uniformly continuous functions and BMO the space of all functions of bounded mean oscillation. Note that BMO is a dual space of Hardy space $\mathcal{H}^1$. $\mathcal{S}$ denotes the Schwartz class, so that $\mathcal{S}'$ is the space of tempered distributions.

The solution (u,p) of (NS) in the distribution sense which we treat is as follows.

Definition 1. We call (u,p) the solution of (NS) on $(0,T)\times\mathbf{R}^n$ in the distribution sense if (u,p) satisfy the following conditions:

(i) $u,\ p\in L^1_{\mathrm{loc}}((0,T)\times\mathbf{R}^n)$ *and* $u(t),p(t)\in\mathcal{S}'$ for a.e. t,

(ii) $\operatorname{div}u=0$ in $\mathcal{S}'$ for a.e. t,

(iii)

$$\begin{aligned}\int_0^T\{\langle u(s),\partial_s\varphi(s)\rangle+\langle u(s),\Delta\varphi(t)\rangle+\langle(u\otimes u)(s),\nabla\varphi(s)\rangle\\ +\langle p(s),\operatorname{div}\varphi(s)\rangle\}\,ds=-\langle u_0,\varphi(0)\rangle,\end{aligned}\tag{3.1}$$

for all $\varphi\in C^1([0,T]\times\mathbf{R}^n)$ with $\varphi(s,\cdot)\in\mathcal{S}$ $(s\in[0,T])$ and $\varphi(T,\cdot)\equiv 0$, where $\langle u\otimes u,\nabla\varphi\rangle=\sum_{i,j=1}^n\langle u_iu_j,\partial_i\varphi_j\rangle$.

Now we are in a position to stata our main result.

Theorem 1. *Let* $u_0\in BUC$ *with* $\operatorname{div}u_0=0$. *Suppose that* (u,p) *is the solution of* (NS) *in the distribution sense satisfying*

$$u\in C([0,T];BUC),\quad p\in L^1_{\mathrm{loc}}((0,T);BMO).$$

Then (u,p) *is unique.*

Moerover, we have

$$\nabla p=\sum_{i,j=1}^n\nabla R_iR_ju^iu^j\quad\text{in }\mathcal{S}'\tag{3.2}$$

for a.e. t.

Remark 1. (i) The existence of such a solution was proved in [2].

(ii) Our condition $p(t,\cdot)\in BMO$ is not included by the condition $|p(t,\cdot)|=O(|x|^{1-\varepsilon})$ $(|x|\to\infty)$ of [1].

(iii) The uniqueness result assuming p is of the form $p=\pi+\sum_{i,j=1}^n R_iR_j\pi_{ij}$ for $\pi,\ \pi_{ij}\in L^\infty$ has been proved by the joint work with Y.Giga, K.Inui, and S.Matsui (unpublished).

4. PRELIMINARIES

In this section we prepare some lemmas.

Let k denote the fundamental solution of $-\Delta$, i.e. $-\Delta k=\delta$. Its explicit form is

$$k(x)=\begin{cases}C_n|x|^{2-n} & (n\ge 3)\\ C_2\log|x| & (n=2),\end{cases}$$

where $1/C_n=(n-2)|S^{n-1}|$ $(n\ge 3)$ and $1/C_2=-2\pi$.

Let $\psi\in C_0^\infty(R^n)$ be radial with $\psi(x)=0$ for $|x|\le 1$ and $\psi(x)=1$ for $|x|\ge 2$. We set $\lambda=1-\psi$. For any ε with $0<\varepsilon<1/2$ we define $\psi_\varepsilon(x)=\psi(x/\varepsilon)$, $\lambda_\varepsilon(x)=\lambda(\varepsilon x)$ and $k_\varepsilon=\psi_\varepsilon\lambda_\varepsilon k$ so that $\operatorname{supp}k_\varepsilon\subset\{x\ ;\ \varepsilon\le|x|\le 2/\varepsilon\}$.

Definition 2. For $f\in\mathcal{S}'$, we define the operator R^ε_{ij} by

$$R^\varepsilon_{ij}f(x)=(\partial_i\partial_jk_\varepsilon)*f(x)\quad(x\in\mathbf{R}^n).$$

Remark 2. We have

$$R_iR_jf(x) = (\text{p.v.}\, \partial_i\partial_j k) * f(x) - \frac{\delta_{ij}}{n} f(x) \quad (x \in \mathbf{R}^n)$$

for $f \in \mathcal{S}$, since it is known that

$$\mathcal{F}^{-1}\left[-\frac{\xi_i\xi_j}{|\xi|^2}\right] = \text{p.v.}\, \partial_i\partial_j k - \frac{\delta_{ij}}{n}\delta \quad \textit{in } \mathcal{S}',$$

where R_j denotes the Riesz transform that is defined by $\widehat{R_jf} = \sqrt{-1}(\xi_j/|\xi|)\widehat{f}$ and δ is Dirac's delta function.

The following lemma says that R^{ε}_{ij} approximates the operator R_iR_j in some sence.

Lemma 4.1. *Let* $1 \le i, j \le n$. *For* $f \in L^\infty$ *we have*

$$\lim_{\varepsilon\downarrow 0}\langle R^{\varepsilon}_{ij}f, \varphi\rangle = \langle R_iR_jf, \varphi\rangle \qquad \text{for all } \varphi \in \mathcal{S} \text{ with } \int \varphi = 0.$$

R^{ε}_{ij} satisfies the following properties.

Lemma 4.2. (i) *For* $f \in \mathcal{S}'$ *with* $\operatorname{div} f = 0$ *in* $\mathcal{S}'$ *we have*

$$\sum_{j=1}^{n} R^{\varepsilon}_{ij} f_j = 0 \quad \textit{in } \mathcal{S}' \quad (1 < \varepsilon < 1/2,\ 1 \le i \le n).$$

(ii) *For* $g \in BMO$ *we have*

$$\lim_{\varepsilon\downarrow 0}\sum_{j=1}^{n} R^{\varepsilon}_{ij}\partial_j g = -\partial_i g \quad \textit{in } \mathcal{S}' \quad (1 \le i \le n).$$

Lemma4.2(i) is immediately follows from the deffinition of R^{ε}_{ij}. On the other hand, (ii) is shown using the following lemma and the duality of $\mathcal{H}^1$ and BMO, since $\sum_{j=1}^{n} R^{\varepsilon}_{ij}\partial_j g = \partial_i\Delta k_\varepsilon * g$.

Lemma 4.3. *Let* $\varphi \in \mathcal{S}$ *with* $\int \varphi = 0$. *Then*

$$\lim_{\varepsilon\downarrow 0}(-\Delta)k_\varepsilon * \varphi = \varphi \quad \textit{in } \mathcal{H}^1.$$

For the proof of Lemma4.3 the following fact which generalizes the result in [2] plays an important role. For $0 < \alpha < 1$,

$$\|f\|_{\mathcal{H}^1} \le C(\||x|^\alpha f\|_{L^1} + \|(1+|x|)^{n+\alpha} f\|_{L^\infty})$$

holes if $\int f = 0$.

5. PROOF OF THEOREM1

Theorem1 is proved using the uniqueness of solutions of the corresponding integral equation

$$u(t) = e^{t\Delta}u_0 + \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u \otimes u)(s)ds, \tag{5.1}$$

in $C([0,T]; BUC)$, where $\mathbf{P} = (\delta_{ij} + R_iR_j)_{i,j=1,\cdots,n}$.

The uniqueness of (5.1) essentially comes from the boundedness of operator $\nabla \cdot e^{(t-s)\Delta}\mathbf{P}$ on L^∞, more pricisely

$$\|\nabla \cdot e^{(t-s)\Delta}\mathbf{P}f\|_{L^\infty} \le C(t-s)^{-\frac{1}{2}}\|f\|_{L^\infty} \quad (0 < s < t),$$

as is shown in [2]. Hence our main task is to show that u satisfies (5.1).

We first take $(R^\varepsilon_{l1}\varphi(s), \cdots, R^\varepsilon_{ln}\varphi(s))$ as a test function of (3.1), then applying Lemma4.2(i) we obtain

$$\int_0^T \Big\{\sum_{i,j=1}^n \langle \partial_i R^\varepsilon_{lj} u_i(s)u_j(s), \varphi(s)\rangle + \sum_{j=1}^n \langle \partial_j R^\varepsilon_{lj} p(s), \varphi(s)\rangle\Big\} ds = 0,$$

since $\operatorname{div} u = 0$, $\operatorname{div} u_0 = 0$ in $\mathcal{S}'$ for a.e. t. Therefore taking limit as $\varepsilon \downarrow 0$ we have

$$\int_0^T \Big\{\sum_{i,j=1}^n \langle \partial_i R_l R_j u_i(s)u_j(s), \varphi(s)\rangle - \langle \partial_l p(s), \varphi(s)\rangle\Big\} ds = 0, \tag{5.2}$$

by Lemma4.1, Lemma4.2(ii). Combining (5.2) and the lth component of (3.1) we obtain

$$\int_0^T \{\langle u_l(s), \partial_s\varphi(s)\rangle + \langle u_l(s), \Delta\varphi(s)\rangle - \langle (\nabla \cdot \mathbf{P}(u\otimes u)(s))_l, \varphi(s)\rangle\} ds$$
$$= -\langle u_0^l, \varphi(0)\rangle \quad (1 \le l \le n).$$

Next, we take $\eta(s)e^{(t-s)\Delta}\phi$ as a test function instead of $\varphi(s)$ of the equation above for each fixed $t \in [0,T]$, where $\eta \in C^1([0,T])$ satisfies $\operatorname{supp}\eta \subset [0,t)$, $\eta(0) = 1$, and $\phi \in \mathcal{S}$. Then we obtain

$$\int_0^T \{\langle u_l(s), (\partial_s\eta)(s)e^{(t-s)\Delta}\phi\rangle - \langle (\nabla \cdot \mathbf{P}(u\otimes u)(s))_l, \eta(s)e^{(t-s)\Delta}\phi\rangle\} ds$$
$$= -\langle u_0^l, e^{t\Delta}\phi\rangle.$$

Therefore, approximating η to $\chi_{(-\infty,t)}$ and hence $\partial_s\eta$ to $-\delta_t$, where $\chi_{(-\infty,t)}$ is a characteristic function of the interval $(-\infty, t)$, we have

$$\Big\langle u_l(t) - e^{t\Delta}u_0^l + \int_0^t (\nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u\otimes u)(s))_l ds, \phi\Big\rangle = 0$$

for all $\phi \in \mathcal{S}$. Therefore we observe that u satisfies the integral equation (5.1).

Finaly, the representation of ∇p (3.2) follows immediately from (5.2). □

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GLOBAL EXISTANCE OF TWO-DIMENSIONAL NAVIER-STOKES FLOW WITH NONDECAYING INITIAL VEROCITY

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This is a joint work with Y. Giga and S. Matsui.

1. Introduction

We consider the nonstationary Navier-Stokes equations in the plane:

$$\text{(NS)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 & \text{in } (0,T) \times \mathbb{R}^2, \\ \operatorname{div} u = 0 \quad \text{in } (0,T) \times \mathbb{R}^2, & \\ u|_{t=0} = u_0 \quad \text{in } \mathbb{R}^2, & (\text{with } \operatorname{div} u_0 = 0) \end{cases}$$

where $u = u(x,t) = (u^1(x,t), u^2(x,t))$ and $p = p(x,t)$ stand for the unknown velocity vector field of the fluid and unknown scalar function of its pressure; $x = (x_1, x_2)$ stands for a point of the plane $\mathbb{R}^2$ and $t(\geq 0)$ stands for the time.

We are interested in the global existance of smooth solution when initial data u_0 is merely bounded or bounded uniformly continuous.

We define several function spaces. We denote by BUC = BUC ($\mathbb{R}^2$), the space of all functions (or vector fields) which are bounded and uniformly continuous in $\mathbb{R}^2$, with $L^\infty(\mathbb{R}^2)$ norm denoted by $||\cdot||_\infty$. For a Banach space X and an interval $I \subset \mathbb{R}$, $C(I; X)$ denotes the space of all continuous functions from I to X.

Theorem 1. *Assume that the initial data* $u_0 \in$ BUC *satisfies* $\operatorname{div} u_0 = 0$. *Then there exists* $u \in C([0,\infty); \mathrm{BUC})$ *such that* $u(0) = u_0$ *and* $(u(t), \nabla p(t))$ *with* $p(t) = \sum_{i,j=1}^{2} R_i R_j u^i(t) u^j(t)$ *is a unique classical solution of (NS) globally in time.*

There is a large literature on local solvability of (NS) even in a various domain of $\mathbb{R}^n$ $(n \geq 2)$. In particular Leray [Le] has already obtained the time global solutions if $u_0 \in L^2(\mathbb{R}^2)$. The method of his proof is based on the energy estimate. This method does not apply directly to our situation because the energy is infinite, neverthless we have the above result.

In our result we do not assume $u_0 \in L^2(\mathbb{R}^2)$. We consider that $u_0 \in \mathrm{BUC}$ or L^∞. In this case, the initial data does not decay at space infinity. The time local solution (u, p) is constructed by Cannon-Knightly ([C-K] 1970), Cannone ([Ca] 1995), and Giga-Inui-Matsui ([G-I-M] 1999) inclusing higher dimensional problems.

2. SKETCH OF THE PROOF OF THEOREM1

Let us briefly explain main ideas of proving Theorem1.

We use the integral equations

$$\text{(INT)} \qquad u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u \otimes u)(s)ds,$$

where $\mathbf{P} = (P_{ij})_{i,j=1,2}$, $P_{ij} = \delta_{ij} + R_iR_j$ is formaly the orthogonal projector on the divergence-free subspace; R_j denotes the Riesz transform with a symbol $\sqrt{-1}\,\xi_j/|\xi|$. The operator $e^{t\Delta}$ is the solution operator of the heat equation. We call the solution of (INT) as the *mild solution.*

In [C-K] and [G-I-M] the maximal existance interval $(0, T_0)$ where the solution exists in $(0, T_0)$ is estimated by

$$T_0 \geq C/\|u_0\|_\infty^2.$$

The main trick is to establish a priori bounded for $\|u(t)\|_\infty$. Once we obtain it the solution can be extended globally.

Theorem 2. *Assume that the initial data* $u_0 \in \mathrm{BUC}$. *Assume that* u *is the mild solution in time* $[0, T]$. *Then there exists a positive constant* K *(independent of time) satisfies*

$$\|u(t)\|_\infty \leq K\exp(Ke^{Kt}) \qquad \text{for} \quad t \in [0, T],$$

provided that $\|\mathrm{curl}\, u_0\|_\infty < \infty$.

Since there is a regularizing effect (shown by [G-I-M]) so that $\nabla u(t_0) \in \mathrm{BUC}$ for $t_0 > 0$, Theorem2 implies Theorem1.

We give a sketch of the proof of Theorem2.

Step1 (Maximum principle of vorticity equation)

We consider the rotation of $u(t)$ for $t \in [0, T]$.

$$\omega_t - \Delta\omega + (u, \nabla)\omega = 0.$$

We denote by $\omega(t) = \mathrm{curl}\, u(t) = \partial u^2(t)/\partial x_1 - \partial u^1(t)/\partial x_2$.

Assume that $u_0 \in \mathrm{BUC}$, and $u(t)$ is the solution of (NS) in time $[0,T]$. Since we can apply the maximum principle to the vorticity equation, then the following inequality holds;

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty \quad \text{for } t>0,$$

where $\omega_0 = \operatorname{curl} u_0$.

We note that this estimate is not available for higher dimensional space since there is a vorticity stretching term.

Step2 (Estimate of bilinear terms)
There exists a numerical positive constant C satisfies that

$$\|\nabla\cdot e^{t\Delta}\mathbb{P}(u\otimes u)\|_\infty \leq C\{(1+1/\sqrt{t}+\log R)\|u\|_\infty\|\omega\|_\infty + (1/R)\|u\|_\infty^2\},$$

for all $t>0$, $R>1$.

For the proof we estimate the Riesz transform by using duality but we skip the detail.
We treat the mild solution. Using this inequality to $\|\cdot\|_\infty$-estimate the integrant of (INT) with $R=1+\|u(s)\|_\infty$, we obtain

$$\begin{aligned}\|u(t)\|_\infty \leq&\|u_0\|_\infty\\ &+C\|\omega_0\|_\infty\int_0^t\{(t-s)^{-1/2}+1+\log(1+\|u(s)\|_\infty)\}\|u(s)\|_\infty ds,\end{aligned}$$

by the maximum principle to the vorticity.

In the same way to prove the Gronwall type inequality, we obtain Theorem2. The constant K in Theorem2 depends only on $\|\omega_0\|_\infty$.

A similar Gronwall type inequality without singularity $(t-s)^{-1/2}$ is found in [Wo] (1933) and [B-G] (1980).

Lemma 1. *Let α and β be non-negative constants. Assume that a non-negative function $a(t,s)$ satisfies $a(\cdot,\cdot)\in C(0\leq s<t\leq T)$, $a(t,\cdot)\in L^1(0,t)$ for all $t\in(0,T]$. Furthermore, we assume that there exists a positive constant ϵ_0 such that*

$$\sup_{0\leq t\leq T}\int_{t-\epsilon_0}^t a(t,s)ds \leq 1/2.$$

If a non-negative function $f\in C([0,T])$ satisfies

$$f(t)\leq\alpha+\int_0^t a(t,s)ds+\beta\int_0^t\{1+\log(1+f(s))\}\cdot f(s)ds,$$

for all $t \in [0, T]$. *Then we have*

$$f(t) \leq \frac{\exp\{[1+\gamma/\beta+\log(1+2\alpha)]e^{2\beta t}\}}{\exp(1+\gamma/\beta)},$$

for all $t \in [0, T]$. *Here we put* $\gamma = \sup_{0\leq t\leq T}\{\sup_{0\leq s\leq t-\epsilon_0} a(t, s)\}$.

Remark 1. *In the case of* $a(t, s) = (t-s)^{-\delta}$ *with* $0 < \delta < 1$, *it is easy to show that the assumptions of Lemma1 are satisfied for*

$$\epsilon_0 = \left(\frac{1-\delta}{2}\right)^{1/(1-\delta)}, \quad \gamma = \left(\frac{2}{1-\delta}\right)^{\delta/(1-\delta)}.$$

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On the viscosity solution method for a dynamic boundary value problem arising in the theory of superconductivity*

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We will talk about the comparison principle of viscosity solutions for a dynamic boundary value problem, which comes from a model arising in the mean field theory of superconductivity (see [ESS]).

Let Ω be a bounded interval $(0, L) \subset \mathbb{R}$ and $T > 0$. Let u be a real valued function on $Q = \overline{\Omega} \times (0, T)$. For a function $k = k(x, t, \tau, p) : Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ we recall the definitions of viscosity sub- and supersolutions for $k(x, t, u_t, u_x) = 0$.

(1) u is called a viscosity subsolution of $k = 0$ if $u^* < \infty$ on $\overline{Q}$ and, for any $(x, t, \phi) \in Q \times C^1(\overline{Q})$ such that $(u^* - \phi)(x, t) = \max_{\overline{Q}}(u^* - \phi)$, the inequality $k(x, t, \phi_t(x, t), \phi_x(x, t)) \leqslant 0$ holds.

(2) u is called a viscosity supersolution of $k = 0$ if $u_* > -\infty$ on $\overline{Q}$ and, for any $(x, t, \phi) \in Q \times C^1(\overline{Q})$ such that $(u_* - \phi)(x, t) = \min_{\overline{Q}}(u_* - \phi)$, the inequality $k(x, t, \phi_t(x, t), \phi_x(x, t)) \geqslant 0$ holds.

Here u^* and u_* are the upper and lower semicontinuous envelopes of u, respectively.

Let $F(x, t)$ and $\alpha(x, t)$ be given functions in $C(\overline{Q})$ and $\gamma \geqslant 0$ be a given constant. We consider the initial boundary value problem

$$\text{(P)} \qquad \begin{aligned} &u_t - F(x,t)\left(u_x^2 + \gamma^2\right)^{1/2} = 0, \quad (x,t) \in \Omega \times (0,T), \\ &u_t - F(x,t)\alpha(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \overline{\Omega}. \end{aligned}$$

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*This is a joint work with C. M. Elliott and Y. Giga.

Unusual feature of this problem is that the time derivative appears in the boundary condition. Such a dynamic boundary condition is studied for parabolic equations, for example in [E1, E2], but not for degenerate equations. Since the equation is not parabolic, to solve the problem globally we need to establish the theory of the viscosity solution. However, the conventional approach for comparison principle does not work. In order to formulate the definition of a viscosity solution of (P) we define, for $(x,t,\tau,p) \in Q \times \mathbb{R} \times \mathbb{R}$,

$$F_{\min}(x,t,\tau,p) = \begin{cases} E(x,t,\tau,p) = \tau - F(x,t)\left(p^2+\gamma^2\right)^{1/2}, & x \in \Omega, \\ \min\left(\tau - F(x,t)\alpha(x,t), E(x,t,\tau,p)\right), & x \in \partial\Omega, \end{cases}$$

$$F_{\max}(x,t,\tau,p) = \begin{cases} E(x,t,\tau,p), & x \in \Omega, \\ \max\left(\tau - F(x,t)\alpha(x,t), E(x,t,\tau,p)\right), & x \in \partial\Omega. \end{cases}$$

DEFINITION. *We say that* $u \in C(\overline{Q})$ *is a viscosity solution of (P) if* $u(x,0) = u_0(x)$ *for any* $x \in \overline{\Omega}$, *u is a viscosity subsolution of* $F_{\min} = 0$ *and a viscosity supersolution of* $F_{\max} = 0$.

This is the usual notion of viscosity solutions for boundary value problems (see [CIL]). To prove the comparison theorem of viscosity solutions we give an equivalent notion of solutions by introducing

$$G(x,t,\tau,p) = \begin{cases} E(x,t,\tau,p), & x \in \Omega, \\ \tau - F(x,t)\max\left(\alpha(x,t), \left([p\nu(x)\,\mathrm{Sign}\,F(x,t)]_-{}^2+\gamma^2\right)^{1/2}\right), & x \in \partial\Omega. \end{cases}$$

Here $\nu(x)$ is the outer normal at $x \in \partial\Omega$ and $[f]_- = \min\{f,0\}$.

PROPOSITION (Equivalence). *A function u is a viscosity solution of (P) if and only if* $u \in C(\overline{Q})$, $u(x,0) = u_0(x)$ *for any* $x \in \Omega$, *and u is both a viscosity subsolution and a viscosity supersolution of* $G = 0$.

To develop the theory of viscosity solutions for (P) a key step is to prove the comparison theorem.

THEOREM (Comparison). *Suppose that there is a constant* $C > 0$ *such that*

$$|F(x,t) - F(y,t)| \leqslant C|x-y| \quad \textit{for any } (x,t),\ (y,t) \in Q.$$

Let u *be a viscosity subsolution of* $G = 0$ *and* v *be a viscosity supersolution of* $G = 0$. *If* $u^*(x,0) \leqslant v_*(x,0)$ *for any* $x \in \overline{\Omega}$, *then* $u^* \leqslant v_*$ *on* $\overline{Q}$.

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Nonlinear wave equations with large potential

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1 はじめに

$V(x)$ を $\mathbf{R}^3$ 上の関数, p を $p>1$ なる実数として, 次のようなポテンシャル項をもつ半線型波動方程式に対する初期値問題について考える:

$$\partial_t^2 u - \Delta u + V(x)u = |u|^p \quad \text{in } [0,\infty)\times\mathbf{R}^3. \tag{1.1}$$

ここでは, 右辺の $|u|^p$ は小さな摂動項とみるが, ポテンシャル項は主要部の一部とみなす立場をとることにする. 実際, 初期データがある意味で十分小さければ, 解も小さいことが期待されるので, ポテンシャル $V(x)$ に特に制限を課さない限り, $|u|^p$ は (1.1) の他の項と比べてより小さいと思うことができる.

まず, ポテンシャル項による摂動がない場合を考える. $u(t,x)=L_0(F)(t,x)$ を

$$\partial_t^2 u - \Delta u = F \quad \text{in } [0,\infty)\times\mathbf{R}^3, \tag{1.2}$$

$$u(0,x) = \partial_t u(0,x) = 0 \quad \text{for } x\in\mathbf{R}^3 \tag{1.3}$$

の解とする. 空間次元が 3 次元であるという特殊性から, $L_0(F)(t,x)$ は次のような性質を持つことが知られている. (F. John [5], W. Strauss and K. Tsutaya [7] などを参照のこと)

滑らかさ k を非負整数として

$$F\in C^k([0,\infty)\times\mathbf{R}^3) \implies L_0(F)\in C^k([0,\infty)\times\mathbf{R}^3)$$

が成り立つ. これは $L_0(F)$ の表示式

$$L_0(F)(t,x) = \frac{1}{4\pi}\int_0^t\int_{|\omega|=1}(t-s)F(s,x+(t-s)\omega)\,dS_\omega ds \tag{1.4}$$

から分かる.

ア・プリオリ評価 簡単のため, $T>0$ に対して $S_T=[0,\infty)\times\mathbf{R}^3$ と書くことにする. $L_0(F)(t,x)\in C^2(S_T)$ のとき

$$0<\lambda<1, \quad \mu>2+\lambda, \quad m>2 \tag{1.5}$$

を満たす任意の実数 λ, μ, m に対して

$$\|\tau_+\tau_-^\lambda L_0(F)\|_{L^\infty(S_T)} \le C_1\|\tau_+^\mu\tau_- F\|_{L^\infty(S_T)}, \tag{1.6}$$

$$\|\tau_+\tau_-^\lambda L_0(F)\|_{L^\infty(S_T)} \le C_1\|\langle x\rangle^m\tau_+\tau_-^\lambda F\|_{L^\infty(S_T)} \tag{1.7}$$

が成り立つ. ただし, C_1 は λ, μ, m のみに依存し T には関係しない正定数である. また $\tau_\pm=\tau_\pm(t,x)$ は次の式で定義される重み関数である:

$$\tau_+(t,x)=1+t+|x|, \quad \tau_-(t,x)=1+|t-|x||.$$

さて, $u(t,x) = L(F)(t,x)$ を

$$\partial_t^2 u - \Delta u + V(x)u = F \quad \text{in } [0,\infty) \times \mathbf{R}^3, \tag{1.8}$$

$$u(0,x) = \partial_t u(0,x) = 0 \quad \text{for } x \in \mathbf{R}^3 \tag{1.9}$$

の解とする. この $L(F)(t,x)$ についても, $L_0(F)(t,x)$ が満たすような滑らかさに関する命題や減衰評価 (1.6), (1.7) を導くことが可能だろうか. 本講演では, ポテンシャル $V(x)$ が滑らかな関数で, 空間無限遠方で十分速く減衰し, かつ

$$V(x) \geq 0 \quad \text{for } x \in \mathbf{R}^3 \tag{1.10}$$

を満たすとき, 上で述べたような $L_0(F)(t,x)$ の持つ性質が $L(F)(t,x)$ に対しても成り立つことを報告したい. 特に, (1.6) に対応する評価式を導くことは大切だと思われる. 実際, その様な評価式が得られれば, (1.1) に対する小さな初期値を課す初期値問題について, 時間大域解の存在を $p > p_c = 1 + \sqrt{2}$ なる全ての p に対して示すことができる. ここで, p_c は $V(x) \equiv 0$ の場合に, F. John [5] によって導かれた非線型項の冪に対する臨界値である.

2 主結果

初期値問題 (1.8)–(1.9) の解 $L(F)(t,x)$ に対して次の重みつき L^∞–L^∞ 評価が成り立つ.

定理 2.1 $V(x) \in C^2(\mathbf{R}^3)$ は (1.10) および

$$\|\langle\cdot\rangle^l V\|_{L^\infty(\mathbf{R}^3)} < +\infty, \quad l \geq 10 \tag{2.1}$$

を満たし, $F \in C([0,T] \times \mathbf{R}^3)$ は

$$\mathit{supp}F \subset K_M := \{(t,x) \in [0,T] \times \mathbf{R}^3 : |x| \leq t + M\} \tag{2.2}$$

を満たすとする. ただし, T, M は正の定数.

このとき, (1.5) を満たす任意の実数 λ, μ に対して

$$\|\tau_+\tau_-^\lambda L(F)\|_{L^\infty(S_T)} \leq C_2\|\tau_+^\mu\tau_- F\|_{L^\infty(S_T)} \tag{2.3}$$

が成り立つ. ここで, C_2 は λ, μ および V のみに依存する正定数である.

注意: 評価式 (2.3) は, 台コンパクトなポテンシャルに対して, V. Georgiev, C. Heiming and H. Kubo [2] において示されており, 今回の結果はその拡張にあたる.

3 証明の概略

任意に $(t,x) \in S_T$ を固定して, $L(F)(t,x) \in C^2(S_T)$ の絶対値を評価すればよい.

Step 1 有界な x への帰着

この場合には, $V(x)$ の仮定は次のように定理おける仮定より弱めることができる:

(3.1) $V(x) \in C(\mathbf{R}^3)$ であり, $M > 2, V_0 > 0$ に対して
$$\|\langle\cdot\rangle^M V\|_{L^\infty(\mathbf{R}^3)} \le V_0 \tag{3.1}$$
を満たす.

実際, $u(t,x) = L(F)(t,x)$ とすると

$$u(t,x) = L_0(F)(t,x) - L_0(Vu)(t,x) \tag{3.2}$$

と書ける. (1.5) を満たす任意の実数 λ, μ, m に対して (1.6), (1.7) より

$$\|\tau_+\tau_-^\lambda u\|_{L^\infty(S_T)} \le C_1\|\tau_+^\mu\tau_- F\|_{L^\infty(S_T)} + C_1\|\langle x\rangle^m \tau_+\tau_-^\lambda Vu\|_{L^\infty(S_T)}.$$

特に, $2 < m < M$ なる m を選ぶと, (3.1) より, ある $R \ge 1$ があって

$$C_1\langle x\rangle^m |V(y)| \le 1/2 \quad \text{for } |x| \ge R$$

が成り立つ. よって

$$\|\tau_+\tau_-^\lambda u\|_{L^\infty(S_T)} \le 2C_1\|\tau_+^\mu\tau_- F\|_{L^\infty(S_T)} + C_V'\|\tau_+\tau_-^\lambda u\|_{L^\infty([0,T]\times\{|y|\le R\})}.$$

従って, (2.3) の代わりに

$$\|\tau_+\tau_-^\lambda u\|_{L^\infty([0,T]\times\{|y|\le R\})} \le C_2\|\tau_+^\mu\tau_- F\|_{L^\infty(S_T)} \tag{3.3}$$

を示せばよいことが分かる.

注意: (3.1) において V_0 が十分小さければ, 上の論法から直接 (2.3) を得る. ([7] を参照)

Step 2 t が有界のとき

この場合は, (3.2) の右辺第一項には (1.6) を適用し, 右辺第二項については, L_0 の表示式 (1.4) に戻って評価を行い, Gronwall の不等式を使って左辺に吸収させることで, 所要の評価式を得ることができる.

Step 3 時間大域的な評価

C^* を与えられた正定数として, $t_0 \ge C^*R$, $|x_0| \le R$ なる点 $(t_0, x_0) \in S_T$ を固定する. (1.8) の右辺の F について, 次の二つの場合を考える. 一般の F に対しては, 適当なカット・オフ関数を使ってこれらの場合に帰着すればよい.

Case 1: $\mathrm{supp} F \subset \{(s,y) : \ |y - x_0| \ge t_0 - s - 2C^*R\}$
このとき有限伝播性から

$$u(t,x) = 0 \quad \text{if } \ |x - x_0| \le t_0 - t - 2C^*R$$

が成り立つので,

$$(t,x) \in \Sigma(t_0,x_0) := \{(s,y) : \ t_0 - s \ge |y - x_0| \ge t_0 - s - 2C^*R\}$$

なる (t,x) について考えれば十分である. Step 2 の場合と同様, $L_0(Vu)(t,x)$ の評価が本質的である.

$$h(s) = \sup_{y\in D(s)} \{\tau_+\tau_-^\lambda |u(s,y)|\}, \quad D(s) = \{y:\ |y-x_0| \le t_0 - s\}$$

とおくと, (1.4) から

$$|L_0(Vu)(t,x)| \le C_R \|\langle\cdot\rangle^M V\|_{L^\infty(\mathbf{R}^3)} \langle t_0\rangle^{-1-\lambda} \int_0^t \frac{h(s)}{\langle t_0 - s\rangle^{M-1-\lambda}} ds$$

を得る. 故に $M > 2+\lambda$ となるように λ を選べば, Gronwall の不等式から所要の評価式が従う.

Case 2: $\operatorname{supp} F \subset \{(s,y):\ |y-x_0| \le t_0 - s - C^*R\}$

この場合においてのみ, $V(x)$ に対して定理で述べられているような強い仮定が必要となる. まず, $L(F)$ を形式的に表現するために, 一般化されたフーリエ変換 $\mathcal{F}$ を導入する. 即ち, $R_0(z) = (-\Delta - z)^{-1}$ を $-\Delta$ のリゾルベント作用素, $F(f)(\xi) = \int \mathrm{e}^{-ix\xi} f(x)dx$ を標準的なフーリエ変換とするとき

$$\mathcal{F}(f)(\xi) = F[(I + VR_0(|\xi|^2 + i0))^{-1}(f)](\xi) \tag{3.4}$$

により $\mathcal{F}(f)(\xi)$ を定義する. (詳しくは [3], [6] vol.II, Def.14.6.3 などを参照のこと)

さらに, リゾルベント恒等式

$$R(z) - R_0(z) = -R_0(z)VR(z), \quad R(z) = (-\Delta + V - z)^{-1}. \tag{3.5}$$

を用いると

$$\mathcal{F}(f)(\xi) = \int [\mathrm{e}^{-ix\xi} + \varphi(x,\xi)] f(x)dx, \tag{3.6}$$

と変形できる. ただし, $\varphi(x,\xi)$ は

$$\varphi(x,\xi) = R(|\xi|^2 + i0)(V_\xi)(x), \quad V_\xi(x) = -V(x)e^{-ix\xi}. \tag{3.7}$$

により定義される関数で

$$(-\Delta_x + V - |\xi|^2)\varphi(x,\xi) = V_\xi(x) \tag{3.8}$$

を満たす. よって

$$\mathcal{F}(-\Delta_x u + Vu) = |\xi|^2 \mathcal{F}u$$

が成り立つ.

以上の準備のもと, $L(F)$ は形式的に

$$L(F)(t,x) = \int_0^t \int\int \left(\mathrm{e}^{ix\xi} + \overline{\varphi(x,\xi)}\right) \frac{\sin(t-s)|\xi|}{|\xi|} \left(\mathrm{e}^{-iy\xi} + \varphi(y,\xi)\right) F(s,y)dyd\xi ds$$

と表せることが分かる. さらに

$$L(F)(t,x) - L_0(F)(t,x) = \int_0^t \int K(t-s,x,y)F(s,y)dyds \tag{3.9}$$

と書くとき，この積分作用素の核 $K(t-s,x,y)$ は

$$|x| \leq R, \quad t-s-|y| \geq (C^*+1)R$$

のとき通常の関数となり，

$$|K(t-s,x,y)| \leq \frac{C_{R,V}\log(2+t-s-|y|)}{\langle t-s\rangle\langle t-s-|y|\rangle} \tag{3.10}$$

と評価される．この証明では，$\varphi(x,\xi)$ に対する精密な評価 ([1], Proposition 6.3) や，リゾルベント作用素に対する重みつき L^2 評価 ([4], Theorem 1.9) および $R_0(|\xi|^2+i0)(f)(x)$ の具体的な表示式をフルに使う．

最後に，(3.10) を用いて

$$|L(F)(t_0,x_0)-L_0(F)(t_0,x_0)| \leq C_2\langle t_0\rangle^{-1-\lambda}\|\tau_+^{\mu}\tau_- F\|_{L^\infty(S_T)}$$

を導けば証明が完成する．

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AN IMPROVEMENT OF A RESULT OF JÄGER AND REJTO ABOUT OSCILLATING LONG-RANGE POTENTIALS

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In the recent work [4], Jäger and Rejto have studied the Schrödinger operators with short-range perturbations of von Neumann-Wigner potentials:

$$L = -\Delta + \frac{c \sin b|x|}{|x|} + V_3(x), \tag{1}$$

where $V_3(x)$, $x \in \mathbf{R}^3$, is a real valued function behaving like $O(|x|^{-1-\delta})$ ($\delta > 0$) as $|x| \to \infty$. They illustrated how to establish the principle of limiting absorption for this potential and for a compact subinterval J of the positive real axis $\mathbf{R}_+$ such that

$$\mathrm{dist}\left\{J, \frac{b^2}{4}\right\} > \frac{3}{4}|bc|. \tag{2}$$

This work is closely related to the earlier one of Mochizuki-Uchiyama [7]. The principle of limiting absorption is proved in [7] for more general oscillating long-range potentials, and if the operator is restricted to (1), the condition on the interval J becomes

$$\inf J > \frac{b^2}{4} + \frac{1}{\min\{2, 4\delta\}}|bc|. \tag{3}$$

Note that in [4] is not proved the principle of limiting absorption itself, but a most important ingredient of its proof is given. It is the so called uniqueness or growth estimate of the generalized eigenfunctions. As in the previous results (see eg., Kato [5], Eidus [1], Mochizuki-Uchiyama [8]), the growth estimates are proved by formulating a differential inequality for a functional of solutions. A new point of the proof of [4] (see also [3]) is that they adopted a functional which includes an approximate phase of the operator (1). Their approximate phase is the same one as is introduced in [7], where a similar functional identity is formulated to establish the principle of limiting absorption. But it is not used there to show the growth estimate of generalized eigenfunctions.

The purpose of this paper is to make an improvement of the result of [4], and apply it to show the principle of limiting absorption. More specifically, we shall establish the growth estimate for the interval J satisfying

$$\text{dist}\left\{J, \frac{b^2}{4}\right\} > \frac{1}{2}|bc|, \tag{4}$$

and the principle for J satisfying

$$\text{dist}\left\{J, \frac{b^2}{4}\right\} > \frac{1}{\min\{2, 4\delta\}}|bc|. \tag{5}$$

In this paper we do not restrict ourselves to the 3 dimensional case. Our results are stated under some abstract conditions on the potentials, and it is also applicable to the operators

$$L = -\Delta + \frac{c\sin(\log|x|)}{\log|x|} + V_3(x), \tag{6}$$

where $V_3(x) = O(|x|^{-1}\{\log|x|\}^{-1-\delta})$ as $|x| \to \infty$. In this case the growth estimate holds for each $J \subset \mathbf{R}_+$, and the principle is obtained for J satisfying

$$\inf J > \frac{1}{4\delta}|c|. \tag{7}$$

We should mention that condition (5) or (7) does not cover all the compact subintervals of $\mathbf{R}_+$, and the principle still remains unsolved for some intervals. As is already shown in [7], Example I-2, condition (7) can be replaced by "each $J \subset \mathbf{R}_+$" if $V_3(x) = O(|x|^{-1-\delta})$ in (6).

In the next §2 we formulate a functional identity of solutions to the Schrödinger equation (Proposition 1). It slightly modifies the one used in [7], and the growth estimate of generalized eigenfunctions (Theorem 1) and the principle of limiting absorption (Theorem 2) are both proved based on this identy.

In §3 we shall follow the argument of [4] to show Theorem 1. The functional identity used in [4] is obtained by considering the equation as an operator-valued ordinary differential equation. So, the expression is apparently much different from ours. But the main part of difference between these two identities is the weight function. Roughly speaking, their weight is $\psi = |x|^{4/3}$. On the other hand ours is $\psi = |x|^2\sqrt{\lambda - \eta V_1(|x|)}$, where $\lambda \in J$, $\eta = 4\lambda/(4\lambda - a)$ with $a \geq 0$ and $V_1(|x|)$ is the oscillating long-range potential. If the potentials are as given in (1), then $a = b^2$ and $V_1(|x|) = c\sin b|x|/|x|$. Thus, in our case, the weight function is chosen also to relate with the approximate phase. This is the reason that we are able to improve the condition on J.

Once the growth estimate is established, the proof of Theorem 2 in §4 is standard (eg., [1] , [7]). If we follow the argument, however, it seems necessary to add some more condtions on J, as in (5) and (7), depending on the decay estimate of $V_3(x)$. Thus, our result generalizes [7], but [4] in part.

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Weighted pointwise estimates for the wave equation

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Abstract

All results which will be given in my talk are joint works with Yuki Kurokawa (Institute of Mathematics, University of Tsukuba, Tsukuba 305-8571, Japan. e-mail : kurokawa@math.tsukuba.ac.jp) I will talk about some weighted pointwise estimate for solutions of two dimensional wave equations. It gives a simple proof to obtain the best order of the lower bound of the lifespan of classical solutions to nonlinear systems. The radially symmetric solution in higher dimensional case can be also discussed if I have enough time.

1 Introduction

We are first concerned with pointwise estimates of a classical solution of the following initial value problem for inhomogeneous wave equations in low space dimenions.

$$\begin{cases} \Box u = H & \text{in } \mathbf{R}^n \times [0,\infty), \\ u|_{t=0} = f, \ u_t|_{t=0} = g, \end{cases} \tag{1.1}$$

where $\Box$ is a usual D'Alembertian and f, g are given smooth functions of compact support in $\mathbf{R}^n$. $H = H(x,t)$ is a smooth function in $\mathbf{R}^n \times [0,\infty)$ whose support is admissible to the initial data. Our attention goes to pointwise estimates, so that we consider the case of $n = 2, 3$ only, in which a fundamental solution of $\Box$ is positive.

Most of weighted L^∞ estimates of a solutions of (1.1) are global type. Actually, one can prove

$$\|wu\|_{L^\infty(\mathbf{R}^n\times[0,\infty))} \leq C_{f,g} + C\|w^p H\|_{L^\infty(\mathbf{R}^n\times[0,\infty))} \tag{1.2}$$

for a suitable weight w and some power p, where $C_{f,g}$ is a positive constant depending on the initial data. This inequality is often applied to nonlinear problem. For example, putting $H = |u|^{p-1}u$, or $|u|^p$ $(p \geq 2)$, we have a global existence of a classical solution of (1.1) with small initial data by (1.2) together with a contraction mapping argument, namely, a constructive method. See early works on this problem, F.John [6] and R.T.Glassey [4]. In this sense, the global estimate (1.2) is enough.

However, in order to investigate the lower bound of the lifespan of the solution when a blow-up occurs in the nonlinear problem, more precise estimate will be required rather than (1.2). To see this, we put a small parameter $\varepsilon > 0$ into the initial data, i.e.

$$f(x) = \varepsilon\varphi(x), \ g(x) = \varepsilon\psi(x) \tag{1.3}$$

for arbitrarily fixed functions φ, ψ. Then we have a similar estimate to (1.2) such that

$$\|wu\|_{L^\infty(\mathbf{R}^n\times[0,T])} \leq C_{\varphi,\psi}\varepsilon + C\|w^pH\|_{L^\infty(\mathbf{R}^n\times[0,T])}F(T,p), \tag{1.4}$$

where F is a function of p and a time T. But we have to improve this. Because a function space in which we will make a contraction has a bad order of ε such as

$$\|wu\|_{L^\infty(\mathbf{R}^n\times[0,T])} = O(\varepsilon). \tag{1.5}$$

Hence the local in time existence of a solution is quaranteed by

$$\varepsilon^{p-1}F(T,p) \ll 1 \tag{1.6}$$

in the contraction. Unfortunately, this condition does not make a optimal order of ε of the lifespan, $\sup T$. For example, see R.Agemi & H.Takamura [2] for $n = 2$.

In three space dimensions, $n = 3$, one can improve (1.4) easily by making use of strong Huygens' principle. Actually, there is a space-time domain A in which we find $C_{\varphi,\psi} \equiv 0$ by compactness of the support of the initial data. Therefore we have new estimates

$$\begin{aligned} \|wu\|_{L^\infty(\overline{A})} &\leq C\{\|w^pH\|_{L^\infty(\overline{A})}F(T,p) + \|w^pH\|_{L^\infty(\overline{B})}\}, \\ \|wu\|_{L^\infty(\overline{B})} &\leq C_{\varphi,\psi}\varepsilon + C\|w^pH\|_{L^\infty(\overline{B})}G(T,p), \end{aligned} \tag{1.7}$$

where $B = \mathbf{R}^3 \times [0,T] \setminus A$ and F, G are functions of T, p. Then the good orders of ε in function spaces are obtained such as

$$\|wu\|_{L^\infty(\overline{A})} = O(\varepsilon^p),\ \|wu\|_{L^\infty(\overline{B})} = O(\varepsilon). \tag{1.8}$$

Hence the local in time existence of a solution is quaranteed by

$$\varepsilon^{p(p-1)}F(T,p),\ \varepsilon^{p-1}G(T,p) \ll 1 \tag{1.9}$$

in the contraction. The first quantity is always bigger than the second one, and makes the optimal lifespan. See F.John [6] for a quadratic nonlinearity, or Zhou Yi [10] and R.Agemi & Y.Kurokawa & H.Takamura [1] for any power.

On the contrary, it is hard to prove a suitable weighted L^∞ estimate in two space dimensions like (1.7) by lack of the strong Huygens' principle. However, other clever proofs overcome the difficulity, which can be found in H.Lindblad [9] and Zhou Yi [11]. They made pointwise estimates and extended a local in time solution to the longest time by continuation principle, namely the contradiction argument. This method is sharper than constructive one. The lower and upper bounds of the lifespan coincide with each other and can be written by known quantity when ε goes to 0. But it works only for the sub-critical case of the nonlinear problem because a scaling argument is essential. For the critical case, the optimal lower bound of the lifespan has been obtained. We also remark that the pointwise estimate with some special function is required in two space dimensions.

The aim of this talk is to give some weighted pointwise estimate in two space dimensions by simple way as in three dimensional case. Moreover, as an application, we easily obtain the weighted L^∞ estimates like (1.7). This gives us the two dimensional version of R.Agemi

& Y.Kurokawa & H.Takamura [1] in which the lifespan of a classical solution to the initial value problem for

$$\begin{cases} \square u = |v|^p, \\ \square v = |u|^q \end{cases} \tag{1.10}$$

is precisely estimated from below and above in three space dimensions. This system was first studied by D.Del Santo & V.Georgiev & E.Mitidieri [3].

2 Main result

We shall consider the linear problem (1.1) with a scaled data (1.3). A key assumption on the present problem is

$$\mathrm{supp}\varphi, \mathrm{supp}\psi \subset \{x \in \mathbf{R}^2 \ ; \ |x| \le k\}. \tag{2.1}$$

Without loss of the generality, we may assume that $k > 1$. By finite propagation speed of the wave, the admissible support of H is

$$\mathrm{supp}H \subset \{(x,t) \in \mathbf{R}^2 \times [0,\infty) \ ; \ |x| \le t+k\}. \tag{2.2}$$

The weight function w is defined by

$$w(|x|,t) = \left(\frac{t+|x|+2k}{k}\right)^{1/2} N(t-|x|), \tag{2.3}$$

where $N(s)$ is a given function of s.

Our main result is the following theorem.

Theorem 1 *Let u be a classical solution of the initial value problem (1.1) in $\mathbf{R}^2 \times [0,T]$ with a scaled data (1.3) under assumptions on the compactness of support of the initial data, (2.1), and H, (2.2). Then, for $0 < \varepsilon < 1$ and $p \in \mathbf{R}$, there exist positive constans $C_{\varphi,\psi}$ depending on the initial data not on ε, and C independent of k, ε such that the following inequality holds for any $(x,t) \in \mathbf{R}^2 \times [0,T]$.*

$$\sqrt{\frac{t+|x|+2k}{k}}\sqrt{\frac{t-|x|+2k}{k}}|u(x,t)| \le C_{\varphi,\psi}\varepsilon + CkI(|x|,t), \tag{2.4}$$

where, when $p > 3$,

$$I(r,t) = \int_{-k}^{t-r} \left(1+\sqrt{\frac{t-r+2k}{t-r-\beta}}\chi_{[0,\infty)}(t-r)\right) \frac{F_{(p-2)/2}(t-r,\beta)}{N(\beta)^p}\|w^pH\|(\beta)d\beta. \tag{2.5}$$

Here $F_P(b,a)$ and $\|\cdot\|$ are defined by

$$F_P(b,a) = \begin{cases} \left(\dfrac{a+2k}{k}\right)^{1-P} & \text{when } P > 1, \\ \log 2\dfrac{b+2k}{a+2k} & \text{when } P = 1, \\ \left(\dfrac{b+2k}{k}\right)^{1-P} & \text{when } P < 1 \end{cases} \tag{2.6}$$

and

$$\|u\|(\tau-\lambda)=\sup_{\tau+\lambda\le 2T+k}\ \sup_{\omega\in S^1}|u(\lambda\omega,\tau)|. \tag{2.7}$$

For the case $p\le 3$, $F_{(p-2)/2}(t-r,\beta)$ *in* I *is replaced by* $G_p(t+r,t-r)$, *where*

$$G_p(\alpha,\beta)=\begin{cases}\left(\dfrac{\beta+2k}{k}\right)^{1/2}\log 2\dfrac{\alpha+2k}{\beta+2k} & \text{when } p=3,\\ \left(\dfrac{\beta+2k}{k}\right)^{1/2}\left(\dfrac{\alpha+2k}{k}\right)^{(3-p)/2} & \text{when } p<3.\end{cases} \tag{2.8}$$

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Spiral traveling wave solutions for some nonlinear diffusion equations

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1 Introduction

In this talk we consider a nonlinear diffusion equation on a two-dimensional annulus $\Omega = \{x \in \mathbb{R}^2 \mid a < |x| < b\}$:

$$\begin{cases} u_t = \Delta u + g(u - \theta), & x \in \Omega,\ t > 0, \\ u_r = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

where (r, θ) denotes the polar coordinates of $x \in \overline{\Omega}$ and g is the derivative of a multi-well potential.

Our motivation for studying problem (1.1) originates from crystallization processes in material sciences ([12], [5], [2]). Screw dislocations are observed on the surface of actual crystals such as silicon carbide, calcogen, paraffin and polyethylene. Recently Kobayashi [7] has proposed the following reaction-diffusion equation as a model of the motion of screw dislocations:

$$\begin{cases} \tau u_t = \varepsilon^2 \Delta u + g(u - \theta), & x \in \Omega,\ t > 0, \\ u_r = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.2}$$

where τ and ε are positive parameters. The unknown function $u(x,t)$ represents the normalized height of the crystal. Some numerical experiments imply that equation (1.2) has a rotating and growing solution with a spiral shape. In Section 2 we show the existence, uniqueness and stability of such a solution, which we call a *spiral traveling wave solution.*

Since the diffusion term is very small, equation (1.2) gives rise to sharp internal layers (or interfaces). In the singular limit as $\varepsilon \to 0$, by using matched asymptotic expansions, we obtain the interface equation

$$V = c - \kappa, \tag{1.3}$$

where V and κ denote the normal velocity and the curvature of the interface respectively, and c is a positive constant determined by the nonlinearity g. In Section 3 we study the properties of spiral solutions for (1.3).

2 Main results

Throughout the talk, we assume that the nonlinearity $g(v)$ satisfies the following:

(A1) g is a smooth, 2π-periodic function on $\mathbb{R}$;

(A2) g has three zeroes $0 < \zeta < 2\pi$ in the interval $[0, 2\pi]$;

(A3) $\displaystyle\int_0^{2\pi} g(v)dv > 0.$

It is known that, for any $u_0 \in C(\overline{\Omega})$, there exists a solution $u(x,t)$ of (1.1) with initial data $u(\cdot,0) = u_0$ (see [8]). Here $C(\overline{\Omega})$ denotes Banach space of continuous functions on $\overline{\Omega}$ endowed with the norm $\|u_0\|_{C(\overline{\Omega})} = \sup\{|u_0(x)| \mid x \in \overline{\Omega}\}$.

Definition 2.1 A solution $\overline{u}(x,t)$ of (1.1) is called a *spiral traveling wave solution* if it is written in the form

$$\overline{u}(x,t) = \varphi(r, \theta - \omega t) + \omega t, \quad x \in \Omega,\ t > 0$$

for some function $\varphi(r,\xi)$ and some constant ω. We call the constant ω the *growth speed* of the spiral traveling wave solution $\overline{u}$.

Remark 2.2 Clearly, if $\overline{u}(x,t) = \overline{u}(r,\theta,t)$ is a spiral traveling wave solution of (1.1), then $\overline{u}(x, t+\tau)$ is also a spiral traveling wave solution for any constant τ. Further, $\overline{u}(r, \theta - \alpha, t) + \alpha$ is also a spiral traveling wave solution for any constant α.

Definition 2.3 A spiral traveling wave solution $\overline{u}$ of (1.1) is called *stable* if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\|u(\cdot,t) - \overline{u}(\cdot,t)\|_{C(\overline{\Omega})} < \varepsilon, \qquad t > 0$$

holds for any solution u of (1.1) satisfying $\|u(\cdot,0) - \overline{u}(\cdot,0)\|_{C(\overline{\Omega})} < \delta$.

Concerning the existence, stability and uniqueness of spiral traveling wave solutions, we obtain the following:

Theorem A ([11]) *For any $b > a > 0$, (1.1) possesses a spiral traveling wave solution with positive growth speed.*

Theorem B ([11])

(i) *A spiral traveling wave solution $\overline{u}$ of (1.1) is stable and is monotone increasing in t, that is, $\overline{u}_t(x,t) > 0$ for all $x \in \Omega$, $t > 0$. Further it is unique up to translation to the t-direction, namely, if u is a spiral traveling wave solution of (1.1) then there exists some $\tau_0 \in \mathbb{R}$ such that $u(\cdot,t) = \overline{u}(\cdot, t+\tau_0)$ for $t > 0$.*

(ii) *For any solution* u *of* (1.1), *there exists some* τ_0 *such that*

$$\lim_{t\to\infty} \|u(\cdot,t) - \overline{u}(\cdot, t+\tau_0)\|_{C(\overline{\Omega})} = 0. \tag{2.1}$$

Remark 2.4 From Theorem B, we see that a spiral traveling wave solution $\overline{u}$ of (1.1) is stable with asymptotic phase, namely, it is stable and, for any solution u of (1.1) with initial data sufficiently close to $\overline{u}$, there exists some τ_0 such that (2.1) holds.

3 Existence of spirals for the interface equation

In this section, we consider equation (1.2):

$$\begin{cases} \tau u_t = \varepsilon^2 \Delta u + g(u-\theta), & x \in \Omega,\, t>0 \\ u_r = 0, & x\in\partial\Omega,\, t>0. \end{cases}$$

We assume that $\tau = \varepsilon^2$ and $g(v) = f(v\,;\varepsilon) := -\frac{\partial W}{\partial v}(v\,;\varepsilon)$ is a smooth function derived from a multi-well potential $W(v\,;\varepsilon)$ whose local minima lie at $v = 2m\pi$ $(m\in\mathbb{Z})$ for all $\varepsilon \geq 0$. More precisely, $f(v\,;\varepsilon)$ satisfies the following conditions:

(F1) $f(v\,;\varepsilon)$ is 2π-periodic in v for each $\varepsilon \geq 0$,

(F2) $f(\cdot\,;\varepsilon)$ has exactly three zeroes $0 < \zeta(\varepsilon) < 2\pi$ in $[0,2\pi]$ for each $\varepsilon\geq 0$,

(F3) $\dfrac{\partial f}{\partial v}(0\,;\varepsilon) < 0$ for each $\varepsilon \geq 0$,

(F4) $\displaystyle\int_0^{2\pi} f(v\,;0)dv = 0,\ \int_0^{2\pi} \frac{\partial f}{\partial \varepsilon}(v\,;0)dv > 0.$

By Theorems A and B, under the conditions (F1)–(F4), there exists a unique spiral traveling wave solution for each $\varepsilon > 0$. Roughly speaking, condition (F4) means that the difference of well-depth $W(2\pi\,;\varepsilon) - W(0\,;\varepsilon)$ is negative and of order ε as $\varepsilon\to 0$. It follows from (F1)–(F4) that there exists a unique solution $(\psi_\varepsilon(z), c(\varepsilon))$ of

$$\begin{cases} \psi_{zz} + \varepsilon c(\varepsilon)\psi_z + f(\psi\,;\varepsilon) = 0, & z\in\mathbb{R}, \\ \psi(-\infty) = 2\pi,\ \psi(0) = \zeta(\varepsilon),\ \psi(+\infty) = 0, \end{cases} \tag{3.1}$$

for each $\varepsilon \geq 0$ ([4]). Note that $c(\varepsilon) > 0$ for $\varepsilon > 0$ and

$$c = \lim_{\varepsilon\to 0} c(\varepsilon) = \frac{\displaystyle\int_0^{2\pi} \frac{\partial f}{\partial \varepsilon}(v\,;0)dv}{\displaystyle\int_{\mathbb{R}} \{\psi_0'(z)\}^2 dz}. \tag{3.2}$$

Let u^ε be a solution of (1.2). Since the diffusion term is very small and the potential W is multi-well type, u^ε approaches $\theta + 2m\pi$ for some $m \in \mathbb{Z}$ if $\theta + \zeta(\varepsilon) + 2(m-1)\pi < u^\varepsilon(x,0) < \theta + \zeta(\varepsilon) + 2m\pi$. Accordingly, a sharp internal layer appears between the regions $\{u^\varepsilon \approx \theta + 2m\pi\}$ and $\{u^\varepsilon \approx \theta + 2(m+1)\pi\}$ for each $m \in \mathbb{Z}$. Suppose that for each $t > 0$ the internal layer approaches a smooth curve Γ_t, which we call the interface, as $\varepsilon \to 0$. By using the so-called matched asymptotic expansions ([1, 3, 9]), we obtain the following law of the interface motion:

$$\begin{cases} V = c - \kappa & \text{on } \Gamma_t, \\ \langle \nu(x), \mathbf{n} \rangle = 0 & \text{on } \partial\Omega \cap \overline{\Gamma_t}, \end{cases} \tag{3.3}$$

where V is the normal velocity of the interface Γ_t, κ is the curvature of Γ_t, $\mathbf{n} = \mathbf{n}(x,t)$ is the outward unit normal vector on Γ_t, and $\nu(x)$ is the outward unit normal vector on $\partial\Omega$. We seek for a solution of (3.3) which is written in the form

$$\overline{\Gamma}(t) = \{(r\cos(\theta(r) + \omega t), r\sin(\theta(r) + \omega t)) \mid a \le r \le b,\ t \ge 0\}$$

for some function $\theta(r)$ and some constant ω. We call such $\overline{\Gamma}(t)$ a spiral with angular speed ω. One can easily see that $\overline{\Gamma}(t)$ is a solution of (3.3) if and only if $q(r) = r\theta'(r)$ satisfies

$$\begin{cases} \dfrac{dq}{dr} = h(r, q\,; \omega), & r > a, \\ q(a) = q(b) = 0, \end{cases}$$

where

$$h(r, q\,; \omega) = (1 + q^2)\left(-c\sqrt{1+q^2} - \frac{q}{r} + \omega r\right).$$

Theorem C ([11]) *Fix $a > 0$ arbitrarily.*

(i) *For any $b > a$, there exists a spiral with angular speed $\omega(b) > 0$. In addition, the spiral is unique up to rotation.*

(ii) *The angular speed $\omega(b)$ is strictly monotone decreasing in b and there exists $\omega_\infty > 0$ such that $\lim_{b\to\infty} \omega(b) = \omega_\infty$.*

(iii) *In the case where $\Omega = \{x \in \mathbb{R} \mid |x| > a\}$, there exists a spiral with speed ω_∞ such that $\lim_{r\to\infty} \theta'(r) = -\omega_\infty / c$.*

Remark 3.1

(a) The statement (i) of Theorem C is found in the paper of Keener and Tyson [6] without proof.

(b) The statement (iii) of Theorem C shows that the shape of the spiral for (3.3) looks like Archimedean spiral as $r \to \infty$ in the case where $b = +\infty$.

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On the blowup of solutions in a semilinear parabolic equation with supercritical nonlinearity

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We are concerned with a Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \mathbf{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N, \end{cases}$$

where $p > 1$ and u_0 is a nonnegative function in $L^\infty(\mathbf{R}^N)$. Here we say that a solution u blows up in finite time if $|u(t)|_\infty \to \infty$ as $t \to T$ for some $T < +\infty$, where $|\cdot|_\infty$ denotes the supremum norm in $\mathbf{R}^N$.

When a solution u blows up in finite time, the blowup is called complete if the continuation of the solution is trivial, that is, $u(x,t) \equiv \infty$ for $t > T$ with some $T < +\infty$ and incomplete otherwise ([1], [2]).

Our purpose is to obtain solutions blowing up *incompletely* for a large class of initial data and to show that such a class forms a separatrix between global classical solutions converging to zero locally uniformly in $\mathbf{R}^N$ as $t \to \infty$ and *completely* blowing-up solutions in finite time. We call a function u a global solution in the sense of L^1_{loc} if $u \in C([0,\infty); L^1_{loc}(\mathbf{R}^N))$ satisfies

$$\int_s^t \int_{\mathbf{R}^N} \{u\rho_\tau + u\Delta\rho + u^p\rho\}\, dx d\tau - \left[\int_{\mathbf{R}^N} u(\tau)\rho dx\right]_s^t = 0$$

for any $0 \le s < t < \infty$ and $\rho \in C^2(\mathbf{R}^N \times [0,\infty))$ with compact support in $\mathbf{R}^N \times [0,\infty)$, where $L^1_{loc}(\mathbf{R}^N)$ denotes the space of locally integrable functions on $\mathbf{R}^N$. Denote by $\mathcal{D}$ the set of nonnegative radially symmetric functions $f(r)$ of class C^1 with compact support in $[0,\infty)$ such that the set of local minima of $f(r)$ is bounded away from zero. Here $f(r_0)$ is called a local minimum of $f(r)$ if $f(r_0) \leq f(r)$ in U and $f(r_0) < f(r)$ on ∂U for some bounded neighborhood of r_0 in $[0,\infty)$.

Theorem Let $p_* = (N+2)/(N-2)$, and $p^* = \infty$ if $3 \leq N \leq 10$ and $p^* = 1 + 6/(N-10)$ if $N \geq 11$. If $p_* < p < p^*$, then for each $\varphi \in \mathcal{D}$ there exists $\lambda_\varphi > 0$ such that:

(i) If $\lambda < \lambda_\varphi$, then u_λ exists globally in time in the classical sense and $u_\lambda(t)$ converges to zero locally uniformly in $\mathbf{R}^N$ as $t \to \infty$.

(ii) If $\lambda = \lambda_\varphi$, then u_λ is a global solution in the sense of L^1_{loc} and blows up *incompletely* in finite time.

(iii) If $\lambda > \lambda_\varphi$, then u_λ blows up *completely* in finite time,

where u_λ denotes the solution with initial data $\lambda\varphi$.

Furthermore in the case of $\lambda \leq \lambda_\varphi$, there is $t_\lambda > 0$ such that for $t \geq t_\lambda$ the solution $u_\lambda(x,t)$ is nonincreasing in $|x|$ and satisfies

$$u_\lambda(x,t) \leq C|x|^{-2/(p-1)} \quad \text{for } x \in \mathbf{R}^N \setminus \{0\},$$

where C is a positive constant depending only on N if the first derivative φ_r with respect to $r = |x|$ changes its sign at most finitely many times.

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COUNTEREXAMPLES TO BILINEAR ESTIMATES RELATED WITH THE KDV EQUATION AND THE NONLINEAR SCHRÖDINGER EQUATION

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§1. Introduction and main results

The solvability of nonlinear evolution equations with quadratic nonlinearity is often reduced to the bilinear estimates corresponding to their nonlinearity. In other words, we can prove a new existence theorem of solution for the Cauchy problem of the quadratic nonlinear evolution equation, if we have a new bilinear estimate to control the nonlinearity. For nonlinear dispersive wave equations such as the nonlinear Schrödinger equation and the KdV equation, there has been a great progress in this direction. In [23], Strichartz did not only prove new space-time integrability estimates for the Schrödinger and the wave equations but also he pointed out that they were equivalent to the Fourier restriction theorem. Nowadays these space-time estimates are called the Strichartz estimates and it is well known that the Strichartz estimates are useful for the study of nonlinear dispersive wave equations. In [3] and [4], Bourgain further developed the Strichartz estimate and introduced the Fourier restriction norm method to solve the Cauchy problem of the nonlinear Schrödinger equation and the KdV equation in much weaker spaces than before. He directly evaluated the nonlinear term by using the argument in the proof of the Fourier restriction theorem (for the Fourier restriction theorem, see, e.g., Tomas [27]). One of the new ingredients in his papers [3] and [4] is a set of new bilinear estimates corresponding to the nonlinearity (see, e.g., Lemmas 7.41 and 7.42 in [4]). These bilinear estimates effectively represent a kind of smoothing property produced by the oscillation of wave. For the quadratic nonlinearity, Bourgain's argument is considered as a more powerful machine than the Strichartz estimate. In [9], [10] and [11], Kenig, Ponce and Vega simplified Bourgain's proof and improved the bilinear estimates. Recently, this method has been applied to various nonlinear evolution equations (see, e.g., [1], [2], [5], [7], [8], [15], [22], [24]-[26] and [28]-[30]). The related method for nonlinear wave equations was developed by Klainerman and Machedon [12]-[14] (see also Rauch and Reed [18]). In the present paper, we consider the optimality of the bilinear estimates obtained in [10] and [11].

We now list the notations, which will be used throughout this paper. For $f(t,x) \in \mathcal{S}'(\mathbf{R}^2)$, we denote the Fourier transform in t and x of f by $\hat{f}$. For $b, s \in \mathbf{R}$, we define the spaces $X_{b,s}$ and $Y_{b,s}$ as follows.

$$X_{b,s} = \{f \in \mathcal{S}'(\mathbf{R}^2)\ ;\quad \|f\|_{X_{b,s}} < \infty\},$$
$$Y_{b,s} = \{f \in \mathcal{S}'(\mathbf{R}^2)\ ;\quad \|f\|_{Y_{b,s}} < \infty\},$$

where

$$\|f\|_{X_{b,s}} = \Big(\int_{\mathbf{R}^2} (1+|\tau-\xi^3|)^{2b}(1+|\xi|)^{2s}|\hat{f}(\tau,\xi)|^2\ d\tau d\xi\Big)^{1/2},$$
$$\|f\|_{Y_{b,s}} = \Big(\int_{\mathbf{R}^2} (1+|\tau-\xi^2|)^{2b}(1+|\xi|)^{2s}|\hat{f}(\tau,\xi)|^2\ d\tau d\xi\Big)^{1/2}.$$

For $z \in \mathbf{C}$, we denote the complex conjugate of z by $\bar{z}$. We put

$$\langle f, g\rangle = \int_{\mathbf{R}^2} f(t,x)\ \overline{g(t,x)}\ dtdx.$$

For two functions $f(t,x)$ and $g(t,x)$, let $f * g$ denote the convolution with respect to the time and space variables. Let $D = \mathcal{F}^{-1}|\xi|\mathcal{F}$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively.

In [10] and [11], Kenig, Ponce and Vega proved the following theorem (see Theorems 1.1 and 1.3 in [10] and Theorems 1.1-1.4 in [11]).

Theorem 0. *(i) For any $s \in (-3/4, 0]$, there exist $b \in (1/2, 1)$ and $C > 0$ such that*

$$\|D(uv)\|_{X_{b-1,s}} \le C\|u\|_{X_{b,s}}\|v\|_{X_{b,s}}. \tag{1.1}$$

Furthermore, for any $s < -3/4$ and any $b \in \mathbf{R}$, the estimate (1.1) fails.

(ii) For any $s \in (-3/4, 0]$, there exist $b \in (1/2, 1)$ and $C > 0$ such that

$$\|uv\|_{Y_{b-1,s}} \le C\|u\|_{Y_{b,s}}\|v\|_{Y_{b,s}}, \tag{1.2}$$
$$\|\bar{u}\bar{v}\|_{Y_{b-1,s}} \le C\|u\|_{Y_{b,s}}\|v\|_{Y_{b,s}}. \tag{1.3}$$

For any $s \in (-1/4, 0]$, there exist $b \in (1/2, 1)$ and $C > 0$ such that

$$\|u\bar{v}\|_{Y_{b-1,s}} \le C\|u\|_{Y_{b,s}}\|v\|_{Y_{b,s}}. \tag{1.4}$$

Furthermore, for any $s < -3/4$ and any $b \in \mathbf{R}$, the estimates (1.2) and (1.3) fail, and for any $s < -1/4$ and any $b \in \mathbf{R}$, the estimate (1.4) fails.

The estimate (1.1) leads to the local well-posedness of the Cauchy problem for the KdV equation in H^s with $s > -3/4$:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = 0, \quad t \in [-T, T], \quad x \in \mathbf{R}, \tag{1.5}$$

$$u(0, x) = u_0(x), \quad x \in \mathbf{R}. \tag{1.6}$$

The estimates (1.2), (1.3) and (1.4) also lead to the local well-posedness of the Cauchy problem for the following quadratic nonlinear Schrödinger equations in H^s with $s > -3/4$, $s > -3/4$ and $s > -1/4$, respectively:

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = F_j(u, u), \quad t \in [-T, T], \quad x \in \mathbf{R}, \quad j = 1, 2, 3, \tag{1.7}$$

$$u(0, x) = u_0(x), \quad x \in \mathbf{R}, \tag{1.8}$$

where

$$F_1(u, u) = u^2, \quad F_2(u, u) = \bar{u}^2, \quad F_3(u, u) = u\bar{u}.$$

However, it has been open whether the estimates (1.1)-(1.4) hold for critical indices $s = -3/4$, $s = -3/4$, $s = -3/4$ and $s = -1/4$, respectively. This problem is interesting and important from both viewpoints of partial differential equations and harmonic analysis. In this paper, we show that the estimates (1.1)-(1.4) break down for these critical indices s. We have the following theorem.

Theorem 1. *(i) Let $s = -3/4$. For any $b \in \mathbf{R}$, the estimate (1.1) fails.*

(ii) Let $s = -3/4$. For any $b \in \mathbf{R}$, the estimates (1.2) and (1.3) fail.

(iii) Let $s = -1/4$. For any b with $b \geq 1/2$, the estimate (1.4) fails.

Remark 1. (i) Theorem 1 (i) shows that the $X_{b,s}$ space is not sufficient for the proof of the local well-posedness in H^s, $s = -3/4$ of the KdV equation. But it does not necessarily imply the ill-posedness in H^s, $s = -3/4$ of the KdV equation. This is also the case with Theorem 1 (ii), when we consider (1.7)-(1.8) with $j = 1, 2$ in H^s, $s = -3/4$.

(ii) In fact, a stronger result holds than Theorem 1 (iii). That is, if $a \geq -1/2$, for any $\alpha, \beta \in \mathbf{R}$, the following estimate fails:

$$\|u\bar{v}\|_{Y_{a,-1/4}} \leq C\|u\|_{Y_{\alpha,-1/4}}\|v\|_{Y_{\beta,-1/4}}. \tag{1.5}$$

There may be a chance that the estimate (1.4) holds with $s = -1/4$ and $b < 1/2$. But, even if this were true, it would be difficult that we apply this estimate to (1.7)-(1.8) with $j = 3$.

We conclude this section by giving a few remarks on the results implying directly the ill-posedness. For nonlinear wave equations, the time local well-posedness in minimal regularity has been extensively studied (see, e.g., [8], [12]-[21], [26], [29] and [30]). In this context, not only have the bilinear estimates corresponding to nonlinearity been studied, but also sharp counterexamples are constructed in [16] and [17], which show actually the ill-posedness of the Cauchy problem in critical regularity. This kind of ill-posedness results for the KdV and the nonlinear Schrödinger equations have recently obtained by Kenig, Ponce and Vega [31]. But their proof is based on the existence of explicit solutions such as soliton-like solutions and it does not seem applicable to the quadratic nonlinear Schrödinger equation (1.7). For the KdV equation, their paper [31] does not handle the critical regularity case.

The plan of this paper is the following. In Section 2, we give a proof of Theorem 1 (i). In Section 3, we show Theorem 1 (ii) and (iii).

§2. Proof of Theorem 1 (i)

In this section we give a proof of Theorem 1 (i). The proof of Theorem 1.3 in [10] implies that if the estimate (1.1) holds with $s=-3/4$, we must have $b=1/2$ (see the relations (4.8) and (4.15) in [10]). Therefore, we have only to consider the case of $s=-3/4$ and $b=1/2$.

Let N and m be sufficiently large positive integers such that $4^{m+1} \ll N$. We put A_j, $j=0,1,\cdots,m$ as follows.

$$\begin{aligned}
A_m = \{(\tau,\xi) \in \mathbf{R}^2; (\tau,\xi) &\text{ belongs to the inside of two parallelograms with} \\
&\text{vertices } (N^3-4^m,N),\ (N^3-4^{m+1},N), \\
&(N^3-4^{m+1}+3N^{3/2}4^{(m+1)/2}, N+4^{(m+1)/2}N^{-1/2}), \\
&(N^3-4^m+3N^{3/2}4^{(m+1)/2}, N+4^{(m+1)/2}N^{-1/2}) \\
\text{and } &(-N^3+4^m,-N),\ (-N^3+4^{m+1},-N), \\
&(-N^3+4^{m+1}-3N^{3/2}4^{(m+1)/2}, -N-4^{(m+1)/2}N^{-1/2}), \\
&(-N^3+4^m-3N^{3/2}4^{(m+1)/2}, -N-4^{(m+1)/2}N^{-1/2})\}, \\
A_{m-1} = \{(\tau,\xi);\ N \le |\xi| \le N+4^{(m+1)/2}N^{-1/2},&\ 4^{m-1} \le |\tau-\xi^3| < 4^m, \\
&(\tau,\xi) \notin A_m\}, \\
A_j = \{(\tau,\xi);\ N \le |\xi| \le N+4^{(m+1)/2}N^{-1/2},&\ 4^j \le |\tau-\xi^3| < 4^{j+1}\}, \\
&0 \le j \le m-2.
\end{aligned}$$

Let R denote the region consisting of two parallelograms similar to the parallelograms in the definition of A_m with one fourth area, which are centered at the points $(0,-\frac{7}{12}4^{(m+1)/2}N^{-1/2})$ and $(0,\frac{7}{12}4^{(m+1)/2}N^{-1/2})$, respectively, and whose longest sides are parallel to the vector $(3N^{3/2}4^{(m+1)/2}, 4^{(m+1)/2}N^{-1/2})$. Here we note that the set A_m translated by a vector of R intersects effectively each A_j, $0 \le j \le m$.

For a measurable set $A \subset \mathbf{R}^2$, we denote the area of A and the characteristic function of A by $|A|$ and χ_A, respectively. By the definitions of the sets A_j and R, we have

$$\begin{aligned} |A_j| &= 2\int_N^{N+4^{m/2}N^{-1/2}} \left[(\xi^3 - 4^j) - (\xi^3 - 4^{j+1})\right]\, d\xi \\ &\sim 4^j 4^{m/2} N^{-1/2}, \quad 0 \le j \le m-1, \end{aligned} \tag{2.1}$$

$$|A_m| \sim 4^m \sqrt{\frac{4^{m+1}}{N}} \sim 4^{3m/2} N^{-1/2}, \tag{2.2}$$

$$|R| \sim 4^{3m/2} N^{-1/2}, \tag{2.3}$$

$$\chi_{A_j} * \chi_{A_m} \ge C|A_j|\chi_R, \quad 0 \le j \le m, \tag{2.4}$$

$$j \ne k \implies A_j \cap A_k = \emptyset. \tag{2.5}$$

Let $\{a_j\}_{j=0}^m$ be an arbitrary sequence such that $a_j \ge 0$ for $0 \le j \le m$. We put

$$\hat{u} = N \sum_{0 \le j \le m} 4^{-j-m/4} a_j \chi_{A_j}.$$

Then, a simple calculation yields

$$\begin{aligned} \hat{u} * \hat{u} &= N^2 \Big(\sum_{0 \le j \le m} 4^{-j-m/4} a_j \chi_{A_j} \Big) * \Big(\sum_{0 \le k \le m} 4^{-k-m/4} a_k \chi_{A_k} \Big) \\ &= N^2 \sum_{0 \le k \le m} \sum_{0 \le j \le k} 4^{-j-k-m/2} a_j a_k \chi_{A_j} * \chi_{A_k} \\ &\quad + N^2 \sum_{0 \le k \le m} \sum_{k < j \le m} 4^{-j-k-m/2} a_j a_k \chi_{A_j} * \chi_{A_k} \\ &\ge N^2 a_m \Big(\sum_{0 \le j \le m} 4^{-j-3m/2} a_j \chi_{A_j} \Big) * \chi_{A_m}. \end{aligned} \tag{2.6}$$

By (2.1)-(2.6), we obtain

$$\begin{aligned} &\| |\xi| (1+|\xi|)^{-3/4} (1+|\tau-\xi^3|)^{-1/2} \hat{u} * \hat{u} \|_{L^2(\mathbf{R}^2)} \\ &\ge C \Big[(4^{m+1}N^{-1})(4^{-(m+1)/2}N^{-3/2}) N^4 a_m^2 \\ &\quad \times \int_{\mathbf{R}^2} \Big(\sum_{0 \le j \le m} 4^{-j-3m/2} a_j (\chi_{A_j} * \chi_{A_m}) \Big)^2 \, d\tau d\xi \Big]^{1/2} \\ &\ge C a_m \Big(\sum_{0 \le j \le m} a_j \Big), \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\| (1+|\xi|)^{-3/4} (1+|\tau-\xi^3|)^{1/2} \hat{u} \|_{L^2(\mathbf{R}^2)} \\ &= \Big(\int_{\mathbf{R}^2} (1+|\xi|)^{-3/2} (1+|\tau-\xi^3|) N^2 \sum_{0 \le j \le m} 4^{-2j-m/2} a_j^2 \chi_{A_j} \, d\tau d\xi \Big)^{1/2} \end{aligned} \tag{2.8}$$

$$\leq C(\sum_{0\leq j\leq m} a_j^2)^{1/2}.$$

Therefore, if the estimate (1.1) holds with $s = -3/4$ and $b = 1/2$, then we conclude by (2.7) and (2.8) that the following inequality must hold.

$$a_m(\sum_{0\leq j\leq m} a_j) \leq C(\sum_{0\leq j\leq m} a_j^2), \tag{2.9}$$

where C is a positive constant independent of m and N. We now choose sequences $\{a_j^{(m)}\}_{j=0}^m$, $m = 1, 2, \cdots$ such that $a_j^{(m)} \geq 0$, $a_m^{(m)} = 1$ and

$$\sup_{m\geq 1} \sum_{0\leq j\leq m} (a_j^{(m)})^2 < \infty,$$
$$\sum_{0\leq j\leq m} a_j^{(m)} \longrightarrow \infty \quad (m \to \infty).$$

For example, we can choose $a_j^{(m)} = (1+j)^{-1}$, $j = 0, 1, \cdots, m-1$ and $a_m^{(m)} = 1$. We insert $\{a_j^{(m)}\}$ into the inequality (2.9) and let $m \to \infty$ in the resulting inequality to obtain a contradiction. This completes the proof of Theorem 1 (i). □

Remark 2. The above proof shows that when $s = -3/4$ and $b = 1/2$, the breakdown of the estimate (1.1) is caused by the function $\hat{u}$ with its support in the region distant from the curve $\{\tau = \xi^3\}$ in contrast to the other case (see the proof of Theorem 1.3 in [10]).

§3. Proof of Theorem 1 (ii) and (iii)

In this section, we show Theorem 1 (ii) and (iii).

(Proof of Theorem 1 (ii)). We only prove that when $s = -3/4$, the estimate (1.2) fails for any $b \in \mathbf{R}$. Because the failure of (1.3) with $s = -3/4$ can be proved in the same way. By the duality argument and the Plancherel theorem, the estimate (1.2) is equivalent to the following:

$$\langle \hat{w}, \hat{u} * \hat{v} \rangle \leq C\|w\|_{Y_{1-b,-s}} \|u\|_{Y_{b,s}} \|v\|_{Y_{b,s}}. \tag{3.1}$$

The proof of Theorem 1.4 (i) in [11] implies that if the estimate (3.1) holds with $s = -3/4$, we must have $b = 1/2$ (see the relations (2.59) and (2.62) in [11]). Therefore, we have only to consider the case of $s = -3/4$ and $b = 1/2$. We can construct a counterexample similar to that in Section 2, but we here present a counterexample of slightly different type.

Let N be a sufficiently large positive integer and let η be a sufficiently small positive number independent of N. We define three functions $\hat{u}$, $\hat{v}$ and $\hat{w}$ as follows.

$$\hat{u}(\tau, \xi) = \begin{cases} 1, & -N \leq \xi \leq -N-1, \quad |\tau - \xi^2| \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\hat{v}(\tau,\xi) = \begin{cases} 1, & N \le \xi \le N+1, \quad |\tau + \xi^2| \le 1, \\ 0, & \text{otherwise}, \end{cases}$$
$$\hat{w}(\tau,\xi) = \begin{cases} (1+\tau)^{-1}, & |\tau + 2N\xi| \le \eta, \quad 1 \le \tau \le \eta N, \\ 0, & \text{otherwise}. \end{cases}$$

By the definitions of $\hat{u}$ and $\hat{v}$, we have

$$\text{supp }(\hat{u} * \hat{v}) \supset \{(\tau,\xi);\ |\tau + 2N\xi| \le \eta, \quad 1 \le \tau \le \eta N\}.$$

Therefore,

$$\begin{aligned} \langle \hat{w}, \hat{u} * \hat{v} \rangle &\ge C \int_1^{\eta N} (1+\tau)^{-1} \int_{-\tau/(2N)-\eta/(2N)}^{-\tau/(2N)+\eta/(2N)} d\xi d\tau \\ &\sim N^{-1} \int_1^{\eta N} (1+\tau)^{-1}\, d\tau \sim N^{-1} \log N. \end{aligned} \tag{3.2}$$

On the other hand, since $|\xi| \le C$ for $(\tau,\xi) \in \text{supp } \hat{w}$, simple calculations yield

$$\begin{aligned} \|w\|_{Y_{1/2,3/4}} \sim& \Big(\int_1^{\eta N} \int_{-\tau/(2N)-\eta/(2N)}^{-\tau/(2N)+\eta/(2N)} (1+\tau)^{-2}(1+|\xi|)^{3/2} \\ & \times (1+|\tau-\xi^2|)\, d\xi d\tau \Big)^{1/2} \\ \sim& N^{-1/2} \Big(\int_1^{\eta N} (1+\tau)^{-1}\, d\tau \Big)^{1/2} \\ \sim& N^{-1/2} (\log N)^{1/2}, \end{aligned} \tag{3.3}$$

$$\|u\|_{Y_{1/2,-3/4}} \sim N^{-3/4}, \tag{3.4}$$

$$\|v\|_{Y_{1/2,-3/4}} \sim N^{-3/4} N \sim N^{1/4}. \tag{3.5}$$

If the estimate (3.1) holds with $s = -3/4$ and $b = 1/2$, we must obtain by (3.2)-(3.5)

$$\begin{aligned} N^{-1} \log N &\le C N^{-1/2} (\log N)^{1/2} \times N^{-3/4} \times N^{1/4} \\ &= C N^{-1} (\log N)^{1/2}, \end{aligned} \tag{3.6}$$

where C is a positive constant independent of N. Hence, we let $N \to \infty$ in (3.6) to obtain a contradiction. This shows that when $s = -3/4$ and $b = 1/2$, the estimate (3.1) fails.

In the same way as above, we can prove the rest of Theorem 1 (ii). In fact, the proof of Theorem 1.4 (iii) in [11] implies that if the estimate (1.3) holds with $s = -3/4$, we must have $b = 1/2$ (see the relations (4.32) and (4.35) in [11]). Therefore, we have only to consider the case of $s = -3/4$ and $b = 1/2$. We next note by the duality argument and the Plancherel theorem that the estimate (1.3) is equivalent to the following:

$$\langle \hat{w}, \hat{\bar{u}} * \hat{\bar{v}} \rangle \le C \|w\|_{Y_{1-b,-s}} \|u\|_{Y_{b,s}} \|v\|_{Y_{b,s}}. \tag{3.7}$$

In this case, if we put $\hat{\bar{u}}_1 = \hat{u}$, $\hat{\bar{v}}_1 = \hat{v}$ and $\hat{w}_1 = \hat{w}$ for functions $\hat{u}$, $\hat{v}$ and $\hat{w}$ defined as above, these three functions $\hat{\bar{u}}_1$, $\hat{\bar{v}}_1$ and $\hat{w}_1$ yield a contradiction to the estimate (3.7) with $s = -3/4$ and $b = 1/2$. □

(Proof of Theorem 1 (iii)). We show that if $a \geq -1/2$, then for any α, $\beta \in \mathbf{R}$ the following estimate fails:

$$\|u\bar{v}\|_{Y_{a,-1/4}} \leq C\|u\|_{Y_{\alpha,-1/4}}\|v\|_{Y_{\beta,-1/4}}, \tag{3.8}$$

which is stronger than Theorem 1 (iii) as is stated in Remark 1 (ii). We first note by the duality argument and the Plancherel theorem that the estimate (3.8) is equivalent to the following:

$$\langle \hat{w}, \hat{u} * \hat{\bar{v}} \rangle \leq C\|w\|_{Y_{-a,1/4}}\|u\|_{Y_{\alpha,-1/4}}\|v\|_{Y_{\beta,-1/4}}. \tag{3.9}$$

Let N be a sufficiently large positive number again. Let $\hat{u}$ and $\hat{v}$ be defined as in the above proof of Theorem 1 (i)-(ii). We choose $\hat{u}_1 = \hat{u}$ and $\hat{\bar{v}}_1 = \hat{v}$. We put

$$\hat{w}_1(\tau,\xi) = \begin{cases} 1, & |\tau + 2N\xi| \leq 1/2, \quad |\xi| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $A = \{(\tau,\xi);\ |\tau + 2N\xi| \leq 1/10, \quad |\xi| \leq 1/10\}$. By the definitions of $\hat{u}_1$ and $\hat{\bar{v}}_1$, we easily see that for each $(\tau',\xi') \in A$, the support of $\hat{u}_1(\tau'-\tau,\xi'-\xi)$ intersects effectively the support of $\hat{\bar{v}}_1(\tau,\xi)$. Therefore, we have

$$(\hat{u}_1 * \hat{\bar{v}}_1)(\tau,\xi) \sim 1, \quad (\tau,\xi) \in A.$$

Accordingly, since supp $\hat{w}_1 \supset A$, we obtain

$$\langle \hat{w}_1, \hat{u}_1 * \hat{\bar{v}}_1 \rangle \geq C|A| \sim 1. \tag{3.10}$$

On the other hand, simple calculations yield

$$\|w_1\|_{Y_{-a,1/4}} \sim \left[|\text{supp } \hat{w}_1| \times N^{-2a}\right]^{1/2} \sim N^{-a}, \tag{3.11}$$

$$\|u_1\|_{Y_{\alpha,-1/4}}, \quad \|v_1\|_{Y_{\beta,-1/4}} \sim N^{-1/4}. \tag{3.12}$$

Here we note that the power of N in the right hand side of (3.12) does not depend on α and β. If the estimate (3.9) holds, then we must have by (3.10)-(3.12)

$$1 \leq CN^{-a} \times (N^{-1/4})^2 = CN^{-a-1/2} \tag{3.13}$$

where C is a positive constant independent of N. Since N is an arbitrary large positive integer, we conclude by (3.13) that the following relation must hold:

$$a \leq -1/2.$$

Thus, it remains only to exclude the case of $a = -1/2$. We change the function w_1 to another one, while we leave u_1 and v_1 as they are. Let η be a sufficiently small positive number independent of N. We next put

$$\hat{w}_2(\tau, \xi) = \begin{cases} (1+\tau)^{-1}, & |\tau + 2N\xi| \leq \eta, \quad 1 \leq \tau \leq \eta N, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have by the definitions of $\hat{w}_2$, $\hat{u}_1$ and $\hat{\bar{v}}_1$

$$\begin{aligned} \langle \hat{w}_2, \hat{u}_1 * \hat{\bar{v}}_1 \rangle &\sim \int_1^{\eta N} (1+\tau)^{-1} \int_{-\tau/(2N)-\eta/(2N)}^{-\tau/(2N)+\eta/(2N)} d\xi d\tau \\ &\sim N^{-1} \int_1^{\eta N} (1+\tau)^{-1}\, d\tau \sim N^{-1} \log N. \end{aligned} \tag{3.14}$$

On the other hand, since $|\xi| \leq C$ for $(\tau, \xi) \in \operatorname{supp} \hat{w}_2$, a simple calculation yields

$$\begin{aligned} \|w_2\|_{Y_{1/2,1/4}} &\sim \Big(\int_1^{\eta} \int_{-\tau/(2N)-\eta/(2N)}^{-\tau/(2N)+\eta/(2N)} (1+\tau)^{-2}(1+|\xi|)^{1/2} \\ &\qquad \times (1 + |\tau - \xi^2|)\, d\xi d\tau\Big)^{1/2} \\ &\sim N^{-1/2}\Big(\int_1^{\eta N} (1+\tau)^{-1}\, d\tau\Big)^{1/2} \\ &\sim N^{-1/2}(\log N)^{1/2}, \end{aligned} \tag{3.15}$$

Therefore, if the estimate (3.9) holds with $a = -1/2$, we must have by (3.12), (3.14) and (3.15)

$$\begin{aligned} N^{-1} \log N &\leq CN^{-1/2}(\log N)^{1/2} \times (N^{-1/4})^2 \\ &= CN^{-1}(\log N)^{1/2}, \end{aligned} \tag{3.16}$$

where C is a positive constant independent of N. We let $N \to \infty$ in (3.16) to obtain a contradiction. This completes the proof of Theorem 1 (iii). ☐

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