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第７回 関数空間セミナー 報告集

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(会場： 北大理学部)

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SOME APPLICATIONS OF TANAHASHI'S BEST POSSIBILITY OF FURUTA INEQUALITY

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ABSTRACT. We shall give some applications of Tanahashi's result which states the best possibility of Furuta inequality. Firstly, we shall discuss the best possibility of a well-known characterization of chaotic order: $\log A \geq \log B$ *if and only if* $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *holds for all* $p \geq 0$ *and* $r \geq 0$. Secondly, we shall discuss the best possibility of p-hyponormality of generalized Aluthge transformation $\tilde{T}_{s,t} = |T|^s U |T|^t$ for p-hyponormal or log-hyponormal operator T whose polar decomposition is $T = U|T|$.

1. Introduction

This report is based on the following preprint:

M.Yanagida, *Some applications of Tanahashi's result on the best possibility of Furuta inequality*, to appear in Mathematical Inequalities and Applications.

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The following Theorem A is an extension of the celebrated Löwner-Heinz theorem: $A \geq B \geq 0$ *ensures* $A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0, 1]$.

Theorem A (Furuta inequality [8]).

If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.

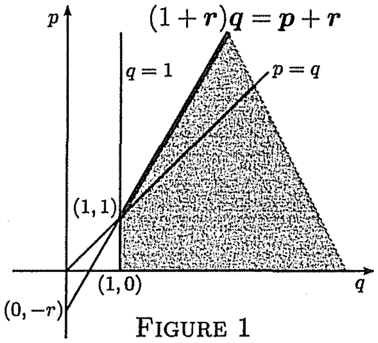

FIGURE 1

We remark that Theorem A yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem A are given in [3] and [13], and also an elementary one-page proof in [9]. Associated with Theorem A, Tanahashi [14] shows the following result.

Theorem B ([14]). *Let* $p > 0$, $q > 0$ *and* $r > 0$. *If* $0 < q < 1$ *or* $(1+r)q < p+r$, *there exist positive and invertible operators* A *and* B *on* $\mathbb{R}^2$ *such that* $A \geq B > 0$ *and*

$$A^{\frac{p+r}{q}} \not\geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}. \tag{1.1}$$

Theorem B states that the domain of the parameters p, q and r in Theorem A is the best possible for the inequalities (i) and (ii) under the assumption $A \geq B \geq 0$. We remark that invertibility of A and B are not mentioned in [14] but it is obvious by scrutinizing the proof of Theorem B in [14].

For positive and invertible operators A and B, chaotic order is defined by $\log A \geq \log B$. Chaotic order is weaker than usual order $A \geq B$ since $\log t$ is an operator monotone function. Ando [2] shows that $\log A \geq \log B$ if and only if $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ for all $p \geq 0$. As a generalization of this result, the following characterization of chaotic order is given by using Theorem A.

Theorem C' ([4][5][10]). *For positive and invertible operators* A *and* B, $\log A \ge \log B$ *if and only if*

$$A^r \ge (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}} \tag{1.2}$$

holds for all $p \ge 0$ *and* $r \ge 0$.

For positive operators A and B, we consider an order $A^\delta \ge B^\delta$ for $\delta > 0$. By Löwner-Heinz theorem, $A^{\delta_1} \ge B^{\delta_1}$ implies $A^{\delta_2} \ge B^{\delta_2}$ for $\delta_1 \ge \delta_2 > 0$. And we remark that the order $A^\delta \ge B^\delta$ coincides with usual order $A \ge B$ in case $\delta = 1$, and approaches chaotic order $\log A \ge \log B$ as letting $\delta \to +0$. The following result is obtained by applying Theorem A to positive operators A and B which satisfy $A^\delta \ge B^\delta$ for $\delta > 0$.

Theorem D ([6][7]). *For positive operators* A *and* B, $A^\delta \ge B^\delta$ *for a fixed* $\delta > 0$ *if and only if*

$$A^{\frac{p+r}{q}} \ge (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for all $p \ge 0$, $r \ge 0$ *and* $q \ge 1$ *with* $(\delta + r)q \ge p + r$.

Theorem D can be considered as a connection with Theorem A and Theorem C' via the order $A^\delta \ge B^\delta$ since Theorem C' can be rewritten as follows.

Theorem C. *For positive and invertible operators* A *and* B, $\log A \ge \log B$ *if and only if*

$$A^{\frac{p+r}{q}} \ge (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for all $p \ge 0$ *and* $r \ge 0$ *with* $rq \ge p + r$.

Figure 2 [7] shows the domains of the parameters p, q and r on which the inequality $A^{\frac{p+r}{q}} \ge (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$ holds under the assumptions $A \ge B$, $A^\delta \ge B^\delta$ for $\delta \in (0, 1)$ and $\log A \ge \log B$, respectively. We remark that the domain drawn for p, q and r in Figure 2 gets smaller as the order gets weaker.

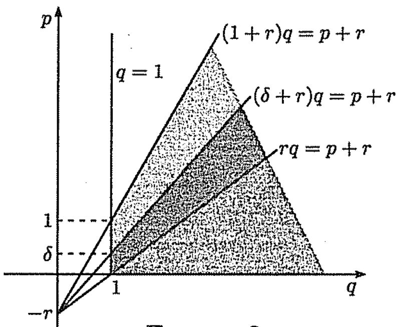

FIGURE 2

On the other hand, an operator T is said to be p-hyponormal for $p > 0$ if $(T^*T)^p \ge (TT^*)^p$ and an operator T is said to be log-hyponormal if T is invertible and $\log T^*T \ge \log TT^*$. p-hyponormal and log-hyponormal operators are defined as extensions of hyponormal one, i.e., $T^*T \ge TT^*$. It is easily obtained that every p-hyponormal operator is q-hyponormal for $p > q > 0$ by Löwner-Heinz theorem, and every p-hyponormal operator is log-hyponormal since $\log t$ is an operator monotone function.

Let T be a p-hyponormal operator whose polar decomposition is $T = U|T|$. Aluthge [1] introduced the operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, which is called Aluthge transformation, and also showed the following result by applying Theorem A.

Theorem E ([1]). *Let* $T = U|T|$ *be the polar decomposition of a* p*-hyponormal operator for* $0 < p < 1$ *and* U *be unitary. Then the following assertions hold:*

(i) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is* $(p + \frac{1}{2})$*-hyponormal if* $0 < p < \frac{1}{2}$.
(ii) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is hyponormal if* $\frac{1}{2} \le p < 1$.

As a natural generalization of Aluthge transformation, the operator $\tilde{T}_{s,t} = |T|^sU|T|^t$ for $s > 0$ and $t > 0$ can be considered. The following Theorem F on $\tilde{T}_{s,t}$ is a generalization of Theorem E on $\tilde{T}$.

Theorem F ([11][12][17]). *Let* $T = U|T|$ *be the polar decomposition of a* p*-hyponormal operator for* $p > 0$. *Then the following assertions hold:*

(i) $\tilde{T}_{s,t} = |T|^sU|T|^t$ *is* $\frac{p+\min\{s,t\}}{s+t}$*-hyponormal for* $s > 0$ *and* $t > 0$ *such that* $\max\{s,t\} \ge p$.
(ii) $\tilde{T}_{s,t} = |T|^sU|T|^t$ *is hyponormal for* $s > 0$ *and* $t > 0$ *such that* $p \ge \max\{s,t\}$.

We remark that Theorem F yields Theorem E when putting $s = t = \frac{1}{2}$ and the proof of [11] is cited under the condition $N(T) = N(T^*)$. As a parallel result to Theorem F for log-hyponormal operators, the following Theorem G is given in [16].

Theorem G ([16]). *Let $T = U|T|$ be the polar decomposition of a log-hyponormal operator. Then $\tilde{T}_{s,t} = |T|^sU|T|^t$ is $\frac{\min\{s,t\}}{s+t}$-hyponormal for $s > 0$ and $t > 0$.*

We remark that Theorem G is a parallel result to Theorem F. In fact, Theorem G corresponds to the case $p \to +0$ of Theorem F since p-hyponormality of T (i.e., $(T^*T)^p \geq (TT^*)^p$) approaches log-hyponormality of T (i.e., $\log T^*T \geq \log TT^*$) as $p \to +0$.

In this paper, we shall show some applications of Theorem B which states the best possibility of Theorem A. In fact, we shall discuss the best possibilities of some applications of Theorem A.

Firstly, we shall discuss a characterization of chaotic order. In fact, we shall prove the best possibilities of Theorem D and Theorem C, which are parallel results to Theorem B.

Secondly, we shall discuss generalized Aluthge transformation for p-hyponormal operators and log-hyponormal operators. In fact, we shall prove the best possibilities of Theorem F and Theorem G by using the best possibilities of Theorem D and Theorem C, respectively.

2. On a characterization of chaotic order

We show the following Theorem 1 and Theorem 2, which state the best possibilities of Theorem D and Theorem C, respectively. We remark that the fact of Theorem 1 was pointed out in [15].

Theorem 1. *Let $p > 0$, $q > 0$, $r > 0$ and $\delta > 0$. If $0 < q < 1$ or $(\delta + r)q < p + r$, there exist positive and invertible operators A and B on $\mathbb{R}^2$ such that $A^\delta \geq B^\delta$ and*

$$A^{\frac{p+r}{q}} \not\geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}. \tag{1.1}$$

Theorem 2. *Let $p > 0$, $q > 0$ and $r > 0$. If $rq < p + r$, there exist positive and invertible operators A and B on $\mathbb{R}^2$ such that $\log A \geq \log B$ and*

$$A^{\frac{p+r}{q}} \not\geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}. \tag{1.1}$$

We remark that Theorem 1 and Theorem 2 are parallel results to Theorem B. We also remark that Theorem 2 can be rewritten in the following form.

Theorem 2'. *Let $p > 0$ and $r > 0$. If $\alpha > 1$, there exist positive and invertible operators A and B on $\mathbb{R}^2$ such that $\log A \geq \log B$ and*

$$A^{r\alpha} \not\geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r\alpha}{p+r}}. \tag{2.1}$$

Theorem 2' states that the outside exponents on both sides of (1.2) in Theorem C' are the best possible.

Proof of Theorem 1. Assume $0 < q < 1$ or $(\delta + r)q < p + r$. Put $p_1 = \frac{p}{\delta} > 0$ and $r_1 = \frac{r}{\delta} > 0$, then $(\delta + r)q < p + r$ is equivalent to $(1 + r_1)q < p_1 + r_1$. By Theorem B, there exist positive and invertible operators A_1 and B_1 such that $A_1 \geq B_1 > 0$ and

$$A_1^{\frac{p_1+r_1}{q}} \not\geq (A_1^{\frac{r_1}{2}}B_1^{p_1}A_1^{\frac{r_1}{2}})^{\frac{1}{q}}. \tag{2.2}$$

Here we put $A = A_1^{\frac{1}{\delta}} > 0$ and $B = B_1^{\frac{1}{\delta}} > 0$, then $A_1 = A^\delta$ and $B_1 = B^\delta$, so that $A_1 \geq B_1$ is equivalent to $A^\delta \geq B^\delta$ and (2.2) is equivalent to the following (1.1):

$$A^{\frac{p+r}{q}} \not\geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}, \tag{1.1}$$

therefore A and B satisfy both $A^\delta \geq B^\delta$ and (1.1). Hence the proof of Theorem 1 is complete. □

Proof of Theorem 2. Assume $rq < p+r$. Since $0 < \frac{p+r}{q} - r$, there exists a $\delta > 0$ such that $0 < \delta < \frac{p+r}{q} - r$, that is, $(\delta + r)q < p + r$. By Theorem 1, there exist positive and invertible operators A and B such that $A^\delta \geq B^\delta$ and

$$A^{\frac{p+r}{q}} \not\geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}. \tag{1.1}$$

$A^\delta \geq B^\delta$ ensures $\log A \geq \log B$ since $\log t$ is operator monotone, so that A and B satisfy both $\log A \geq \log B$ and (1.1). Hence the proof of Theorem 2 is complete. $\square$

3. On generalized Aluthge transformation

By using Theorem 1 and Theorem 2 in the previous section, we show the best possibility of Theorem F and Theorem G, respectively.

Theorem 3. *Let $p > 0$, $s > 0$ and $t > 0$. And let $T = U|T|$ be the polar decomposition of T. Then the following assertions hold:*

- (i) *In case $\max\{s,t\} \geq p$, if $\alpha > \frac{p+\min\{s,t\}}{s+t}$, there exists a p-hyponormal operator T such that $\tilde{T}_{s,t} = |T|^sU|T|^t$ is not α-hyponormal.*
- (ii) *In case $p \geq \max\{s,t\}$, if $\alpha > 1$, there exists a p-hyponormal operator T such that $\tilde{T}_{s,t} = |T|^sU|T|^t$ is not α-hyponormal.*

Theorem 4. *Let $s > 0$ and $t > 0$. If $\alpha > \frac{\min\{s,t\}}{s+t}$, there exists a log-hyponormal operator T such that $\tilde{T}_{s,t} = |T|^sU|T|^t$ is not α-hyponormal, where $T = U|T|$ is the polar decomposition of T.*

In order to prove Theorem 3 and Theorem 4, we prepare the following lemma.

Lemma 1. *For positive operators A and B on a Hilbert space H, define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as follows:*

$$T = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & B^{\frac{1}{2}} & 0 & & & & \\ & & B^{\frac{1}{2}} & \boxed{0} & & & \\ & & & A^{\frac{1}{2}} & 0 & & \\ & & & & A^{\frac{1}{2}} & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix}, \tag{3.1}$$

where $\boxed{}$ shows the place of the $(0,0)$ matrix element. Then the following assertions hold:

- (i) *T is p-hyponormal for $p > 0$ if and only if $A^p \geq B^p$.*
- (ii) *T is log-hyponormal if and only if A and B are invertible and $\log A \geq \log B$.*

Furthermore, define $\tilde{T}_{s,t} = |T|^sU|T|^t$ for $s > 0$ and $t > 0$ where $T = U|T|$ is the polar decomposition of T. Then the following assertion holds:

- (iii) *$\tilde{T}_{s,t} = |T|^sU|T|^t$ is α-hyponormal for $\alpha > 0$ if and only if both*

$$(B^{\frac{t}{2}}A^sB^{\frac{t}{2}})^\alpha \geq B^{(s+t)\alpha}$$

and

$$A^{(s+t)\alpha} \geq (A^{\frac{s}{2}}B^tA^{\frac{s}{2}})^\alpha$$

hold.

Lemma 1 can be proved by easy calculation, so that we omit to discribe the proof of Lemma 1.

Proof of Theorem 3. In the process of the proof of (i), we divide the case (i) into (i-a) and (i-b) as follows.

(i-a) Case $t \geq s > 0$ and $t = \max\{s,t\} \geq p > 0$. Assume $\alpha > \frac{p+\min\{s,t\}}{s+t} = \frac{p+s}{s+t}$. Put $q = \frac{1}{\alpha} > 0$ and $\delta = p > 0$, then $(\delta + s)q < t + s$. By Theorem 1, there exist positive operators A and B on a Hilbert space H such that $A^{\delta} \geq B^{\delta}$ and

$$A^{\frac{t+s}{q}} \ngeq (A^{\frac{s}{2}} B^t A^{\frac{s}{2}})^{\frac{1}{q}}. \tag{3.2}$$

Since $q = \frac{1}{\alpha}$ and $\delta = p$, $A^{\delta} \geq B^{\delta}$ is equivalent to $A^p \geq B^p$ and (3.2) is equivalent to

$$A^{(s+t)\alpha} \ngeq (A^{\frac{s}{2}} B^t A^{\frac{s}{2}})^{\alpha}.$$

Here we define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1) in Lemma 1. Then T is p-hyponormal and $\tilde{T}_{s,t}$ is not α-hyponormal by (i) and (iii) of Lemma 1.

(i-b) Case $s \geq t > 0$ and $s = \max\{s,t\} \geq p > 0$. Assume $\alpha > \frac{p+\min\{s,t\}}{s+t} = \frac{p+t}{s+t}$. Put $q = \frac{1}{\alpha} > 0$ and $\delta = p > 0$, then $(\delta + t)q < s + t$. By Theorem 1, there exist positive and invertible operators A_1 and B_1 on a Hilbert space H such that $A_1^{\delta} \geq B_1^{\delta}$ and

$$A_1^{\frac{s+t}{q}} \ngeq (A_1^{\frac{t}{2}} B_1^s A_1^{\frac{t}{2}})^{\frac{1}{q}}. \tag{3.3}$$

Put $A = B_1^{-1}$ and $B = A_1^{-1}$, then $A_1^{\delta} \geq B_1^{\delta}$ is equivalent to $A^{\delta} \geq B^{\delta}$ and (3.3) is equivalent to the following (3.4):

$$(B^{\frac{t}{2}} A^s B^{\frac{t}{2}})^{\frac{1}{q}} \ngeq B^{\frac{s+t}{q}}. \tag{3.4}$$

Since $q = \frac{1}{\alpha}$ and $\delta = p$, $A^{\delta} \geq B^{\delta}$ is equivalent to $A^p \geq B^p$ and (3.4) is equivalent to

$$(B^{\frac{t}{2}} A^s B^{\frac{t}{2}})^{\alpha} \ngeq B^{(s+t)\alpha}.$$

Here we define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1) in Lemma 1, then T is p-hyponormal and $\tilde{T}_{s,t}$ is not α-hyponormal by (i) and (iii) of Lemma 1.

(ii) Case $p \geq \max\{s,t\}$, i.e., $p \geq s > 0$ and $p \geq t > 0$. Assume $\alpha > 1$. Put $q = \frac{1}{\alpha} > 0$ and $\delta = p > 0$, then $0 < q < 1$. By Theorem 1, there exist positive operators A and B on a Hilbert space H such that $A^{\delta} \geq B^{\delta}$ and

$$A^{\frac{t+s}{q}} \ngeq (A^{\frac{s}{2}} B^t A^{\frac{s}{2}})^{\frac{1}{q}}. \tag{3.5}$$

Since $q = \frac{1}{\alpha}$ and $\delta = p$, $A^{\delta} \geq B^{\delta}$ is equivalent to $A^p \geq B^p$ and (3.5) is equivalent to

$$A^{(s+t)\alpha} \ngeq (A^{\frac{s}{2}} B^t A^{\frac{s}{2}})^{\alpha}.$$

Here we define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1) in Lemma 1. Then T is p-hyponormal and $\tilde{T}_{s,t}$ is not α-hyponormal by (i) and (iii) of Lemma 1.

Consequently the proof of Theorem 3 is complete. □

Proof of Theorem 4.

(a) Case $t \geq s > 0$. Assume $\alpha > \frac{\min\{s,t\}}{s+t} = \frac{s}{s+t}$. Put $q = \frac{1}{\alpha} > 0$, then $sq < t + s$. By Theorem 2, there exist positive and invertible operators A and B on a Hilbert space H such that $\log A \geq \log B$ and

$$A^{\frac{t+s}{q}} \ngeq (A^{\frac{s}{2}} B^t A^{\frac{s}{2}})^{\frac{1}{q}}. \tag{3.6}$$

Since $q = \frac{1}{\alpha}$, (3.6) is equivalent to

$$A^{(s+t)\alpha} \ngeq (A^{\frac{s}{2}} B^t A^{\frac{s}{2}})^{\alpha}.$$

Here we define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1) in Lemma 1, then T is log-hyponormal and $\tilde{T}_{s,t}$ is not α-hyponormal by (ii) and (iii) of Lemma 1.

(b) Case $s \geq t > 0$. Assume $\alpha > \frac{\min\{s,t\}}{s+t} = \frac{t}{s+t}$. Put $q = \frac{1}{\alpha} > 0$, then $tq < s+t$. By Theorem 2, there exist positive and invertible operators A_1 and B_1 on a Hilbert space H such that $\log A_1 \geq \log B_1$ and

$$A_1^{\frac{s+t}{q}} \ngeq (A_1^{\frac{t}{2}} B_1^s A_1^{\frac{t}{2}})^{\frac{1}{q}}. \tag{3.7}$$

Put $A = B_1^{-1}$ and $B = A_1^{-1}$, then $\log A_1 \geq \log B_1$ is equivalent to $\log A \geq \log B$ and (3.7) is equivalent to the following (3.7):

$$(B^{\frac{t}{2}} A^s B^{\frac{t}{2}})^{\frac{1}{q}} \ngeq B^{\frac{s+t}{q}}. \tag{3.8}$$

Since $q = \frac{1}{\alpha}$, (3.8) is equivalent to

$$(B^{\frac{t}{2}} A^s B^{\frac{t}{2}})^{\alpha} \ngeq B^{(s+t)\alpha}.$$

Here we define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1) in Lemma 1, then T is log-hyponormal and $\tilde{T}_{s,t}$ is not α-hyponormal by (ii) and (iii) of Lemma 1.

Consequently the proof of Theorem 4 is complete. ☐

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Equivalence relation between an order preserving operator inequality and related operator functions

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Abstract

This report is based on the following two papers:

[FHI] T.Furuta, M.Hashimoto and M.Ito, *Equivalence relation between generalized Furuta inequality and related operator functions*, Scientiae Mathematicae, **1** (1998), 257–259.

[F] T.Furuta, *Simplified proof of an order preserving operator inequality*, Proc. Japan. Acad., **74** (1998), 114.

In this paper, we shall show equivalence relation between an order preserving operator inequality and related operator functions.

1 Introduction

第１章から第３章は [FHI] に基づいている。

ここではヒルベルト空間 H 上の有界線形作用素について考える。以下、単に作用素と呼ぶことにする。その中でも特に positive な作用素について考えるが、ここで作用素 T が positive であるとは positive definite、即ち $(Tx,x) \geq 0$ for all $x \in H$ と定義し、$T \geq 0$ と表す。また、T が positive かつ invertible であるとき、T は strictly positive であるといい、$T > 0$ と表す。positive operator の順序を保存する不等式として有名な Löwner-Heinz の定理: $A \geq B \geq 0$ *ensures* $A^{\alpha} \geq B^{\alpha}$ *for any* $\alpha \in [0,1]$ があるが、$\alpha > 1$ の時は必ずしも成立しないので応用上不便であった。そこで応用上便利なように次の定理が確立された。

Theorem F ([4]).

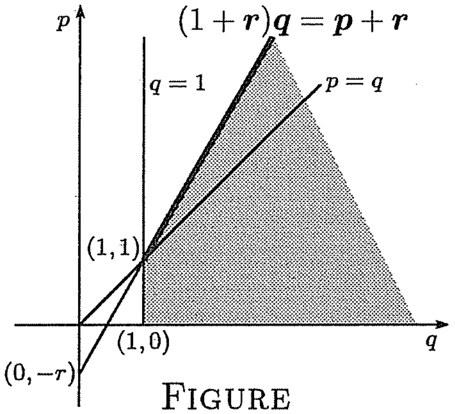
If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.



FIGURE

Theorem F の (i) または (ii) において $r=0$ とおくことにより Löwner-Heinz の定理が得られる。Theorem F の別証明は [2][11] で与えられており、また [5] では one-page proof が与えられている。また、Theorem F のパラメータ p,q,r の範囲を示したのが上図であるが、この領域は best possible であることが [12] で示された。

Theorem F の拡張として [6] で次の Theorem G が確立された。

Theorem G ([6]). *If* $A \geq B \geq 0$ *with* $A > 0$*, then for each* $t \in [0,1]$ *and* $p \geq 1$*,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ *and* $s \geq 1$*, and* $F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s)$*, that is, for each* $t \in [0,1]$ *and* $p \geq 1$*,*

$$(*) \qquad A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$ *and* $r \geq t$*.*

Ando-Hiai による [1] では log majorization に関する主定理と共に、それと同値な作用素不等式として次もまた紹介されている:
If $A \geq B \geq 0$ *with* $A > 0$*, then*

$$A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}}$$

holds for any $p \geq 1$ *and* $r \geq 1$*.*
Theorem G は Ando-Hiai による上の不等式と Theorem F 自身を interpolate するものである。Theorem G の別証明は [3] で示され、作用素不等式 $(*)$ における左辺の指数 $\frac{1-t+r}{(p-t)s+r}$ が best possible であることが [13] で示された。最近、Theorem G の拡張として [7] で次の結果が紹介された。

Theorem H ([7]). *Let* $A \geq B \geq 0$ *with* $A > 0$*. For each* $t \in [0,1]$*,* $q \geq 0$ *and* $p \geq \max\{q,t\}$*,*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ *and* $s \geq 1$*.*

Theorem H の簡単な証明は [8]、[9] で示され、また更なる発展が [10] で示された。

2 Result

今回、我々は Theorem G、Theorem H に関連して次の結果を得ることができた。

Theorem 1. *The following statements hold and follow from each other.*

(i) *If* $A \geq B \geq 0$ *with* $A > 0$*, then for each* $t \in [0,1]$ *and* $p \geq 1$*,*
$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ *holds for any* $s \geq 1$ *and* $r \geq t$*.*

(ii) *If* $A \geq B \geq 0$ *with* $A > 0$*, then for each* $1 \geq q \geq t \geq 0$ *and* $p \geq q$*,*
$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}$ *holds for any* $s \geq 1$ *and* $r \geq t$*.*

(iii) *If* $A \geq B \geq 0$ *with* $A > 0$*, then for each* $t \in [0,1]$ *and* $p \geq 1$*,*
$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$
is decreasing for $r \geq t$ *and* $s \geq 1$*.*

(iv) *If* $A \geq B \geq 0$ *with* $A > 0$, *then for each* $t \in [0,1]$, $q \geq 0$ *and* $p \geq t$,

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ *and* $s \geq 1$ *such that* $(p-t)s \geq q-t$.

(i), (iii) は Theorem G として、(ii), (iv) は Theorem H 及びその拡張としてそれぞれ既に得られている。また、(ii), (iv) は、今まで (i), (iii) の拡張という見方をされ、それぞれ別々に証明されてきた。しかし、実は (i) から (iv) は全て互いに導きあえる、つまり (i) から (iv) のどれかひとつさえわかれば、他を示すのに複雑な証明を経なくても良い、ということを Theorem 1 は意味している。そこで、本文中では (ii) の simplified proof を併せて紹介する。

以下の証明中、次の補題が重要になる。

Lemma F **([6]).** *Let* $A > 0$ *and* B *be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

3 Proof of Theorem 1

B が invertible であると仮定しても一般性は失われない。(i) から (iv) の関係を (iv) → (iii) → (i) → (ii) → (iv) の順に示していく。

Proof of (iv) $\Longrightarrow$ (iii).
(iv) で $q = 1$ と置き、$p \geq 1$ とすると (iii) を得る。

Proof of (iii) $\Longrightarrow$ (i).
(iii) において、関数値と最大値を比べることにより次の不等式を得る:
For each $t \in [0,1]$ and $p \geq 1$,

$$\begin{aligned} F_{p,t}(A,B,r,s) &\leq F_{p,t}(A,B,t,1) \\ &= A^{\frac{-t}{2}}BA^{\frac{-t}{2}} \\ &\leq A^{1-t} \qquad \text{by } A \geq B \end{aligned} \tag{3.1}$$

holds for any $r \geq t$ and $s \geq 1$.
(3.1) の両側から $A^{\frac{r}{2}}$ を掛けることにより (i) が得られる。

Proof of (i) $\Longrightarrow$ (ii).
$q \in [0,1]$ だから $A \geq B \geq 0$ に Löwner-Heinz theorem を適用すると $A^q \geq B^q$ が成り立つ。
(i) において、$A_1 = A^q$, $B_1 = B^q$, $t_1 = \frac{t}{q} \in [0,1]$, $p_1 = \frac{p}{q} \geq 1$, $r_1 = \frac{r}{q} \geq t_1$ と置き換えることにより、それぞれ

$$A_1^{1-t_1+r_1} = A^{q-t+r},\ A_1^{\frac{r_1}{2}} = A^{\frac{r}{2}},\ A_1^{\frac{-t_1}{2}} = A^{\frac{-t}{2}},\ B_1^{p_1} = B^p$$

となり、右辺の外側の指数は

$$\frac{1-t_1+r_1}{(p_1-t_1)s+r_1} = \frac{q-t+r}{(p-t)s+r}$$

と変形でき、次の (3.2) を得る。
For each $1 \geq q \geq t \geq 0$ and $p \geq q$,

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \tag{3.2}$$

holds for any $r \geq t$ and $s \geq 1$.
つまり (i) から (ii) を得ることができた。
Proof of (ii) $\Longrightarrow$ (iv).
(ii) において $D = A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}}$, $q = t$ と置くと次の (3.3) を得る。
For each $t \in [0,1]$ and $p \geq t$,

$$A^r \geq (A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{r}{(p-t)s+r}} \tag{3.3}$$

holds for any $r \geq t$ and $s \geq 1$.
(3.3) は Lemma F により次の (3.4) と同値である。

$$D^s \leq (D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{(p-t)s}{(p-t)s+r}}. \tag{3.4}$$

(3.3) と (3.4) に Löwner-Heinz theorem を適用すると、

$$A^u \geq (A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{u}{(p-t)s+r}} \qquad \text{for } r \geq u \geq 0, \tag{3.5}$$

$$(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{(p-t)w}{(p-t)s+r}} \geq D^w \qquad \text{for } s \geq w \geq 0 \tag{3.6}$$

の２式が得られる。
ここで、$G_{p,q,t}(A,B,r,s)$ が r, s に関してそれぞれ単調減少であることを示したいのだが、

$$\begin{aligned}
G_{p,q,t}(A,B,r,s) &= A^{\frac{-r}{2}}(A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}} \\
&= A^{\frac{-r}{2}}f(s)A^{\frac{-r}{2}} \\
&= D^{\frac{s}{2}}(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r}}D^{\frac{s}{2}} \quad \text{by Lemma F} \\
&= D^{\frac{s}{2}}g(r)D^{\frac{s}{2}}
\end{aligned}$$

と書き換えておく。ここで、

$$f(s) = (A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)s+r}}$$

$$g(r) = (D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r}}$$

とする。以下 $f(s)$, $g(r)$ がそれぞれ r, s に関して単調減少であることを示す。
(a) Proof of decreasing for $s \geq 1$ such that $(p-t)s \geq q-t$.

$$\begin{aligned}
f(s) &= (A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)s+r}} \\
&= \{(A^{\frac{r}{2}}D^sA^{\frac{r}{2}})^{\frac{(p-t)(s+w)+r}{(p-t)s+r}}\}^{\frac{q-t+r}{(p-t)(s+w)+r}} && \text{for } s \geq w \geq 0 \\
&= \{A^{\frac{r}{2}}D^{\frac{s}{2}}(D^{\frac{s}{2}}A^rD^{\frac{s}{2}})^{\frac{(p-t)w}{(p-t)s+r}}D^{\frac{s}{2}}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)(s+w)+r}} && \text{by Lemma F} \\
&\geq (A^{\frac{r}{2}}D^{\frac{s}{2}}D^wD^{\frac{s}{2}}A^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)(s+w)+r}} \\
&= (A^{\frac{r}{2}}D^{s+w}A^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)(s+w)+r}} \\
&= f(s+w).
\end{aligned}$$

最後の不等式は (3.6) による。なぜなら (ii) の条件より $\frac{q-t+r}{(p-t)(s+w)+r} \in [0,1]$ だから Löwner-Heinz theorem より成り立つ。
よって $G_{p;q,t}(A,B,r,s) = A^{\frac{-r}{2}} f(s) A^{\frac{-r}{2}}$ は $(p-t)s \geq q-t$ であるような $s \geq 1$ について単調減少である。
(b) Proof of decreasing for $r \geq t$.

$$
\begin{aligned}
g(r) &= (D^{\frac{s}{2}} A^r D^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r}} \\
&= \{(D^{\frac{s}{2}} A^r D^{\frac{s}{2}})^{\frac{(p-t)s+r+u}{(p-t)s+r}}\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}} && \text{for } r \geq u \geq 0 \\
&= \{D^{\frac{s}{2}} A^{\frac{r}{2}} (A^{\frac{r}{2}} D^s A^{\frac{r}{2}})^{\frac{u}{(p-t)s+r}} A^{\frac{r}{2}} D^{\frac{s}{2}}\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}} && \text{by Lemma F} \\
&\geq (D^{\frac{s}{2}} A^{\frac{r}{2}} A^u A^{\frac{r}{2}} D^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r+u}} \\
&= (D^{\frac{s}{2}} A^{r+u} D^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r+u}} \\
&= g(r+u).
\end{aligned}
$$

最後の不等式は (3.5) による。なぜなら (ii) の条件より $\frac{q-t-(p-t)s}{(p-t)s+r+u} \in [-1,0]$ だから Löwner-Heinz theorem と両辺の inverse を取ることにより成り立つ。
よって $G_{p,q,t}(A,B,r,s) = D^{\frac{s}{2}} g(r) D^{\frac{s}{2}}$ は $r \geq t$ について単調減少である。
(a), (b) により (ii) から (iv) が得られた。
以上より Theorem 1 は証明された。

4 Simplified proof of (ii)

第4章は [F] に基づいている。

まず初めに次の (4.1)、つまり $r = t$ の場合を証明する。
If $A \geq B \geq 0$ with $A > 0$, then

$$
A^q \geq \{A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}} \tag{4.1}
$$

holds for $1 \geq q \geq t \geq 0$, $p \geq q$ and $s \geq 1$.
$2 \geq s \geq 1$ の場合、条件より $s-1$, $\frac{q}{(p-t)s+t} \in [0,1]$、また Löwner-Heinz theorem より $A^t \geq B^t \cdots (**)$ だから次が成り立つ。

$$
\begin{aligned}
B_1 &= \{A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}} \\
&= \{B^{\frac{p}{2}} (B^{\frac{p}{2}} A^{-t} B^{\frac{p}{2}})^{s-1} B^{\frac{p}{2}}\}^{\frac{q}{(p-t)s+t}} && \text{by Lemma F} \\
&\leq \{B^{\frac{p}{2}} (B^{\frac{p}{2}} B^{-t} B^{\frac{p}{2}})^{s-1} B^{\frac{p}{2}}\}^{\frac{q}{(p-t)s+t}} && \text{by } (**) \\
&= B^q \leq A^q = A_1
\end{aligned} \tag{4.2}
$$

for $1 \geq q \geq t \geq 0$, $p \geq q$ and $2 \geq s \geq 1$.
ここで $A_1 \geq B_1 \geq 0$ に対して (4.2) を適用すると、

$$
{A_1}^{q_1} \geq \{{A_1}^{\frac{t_1}{2}} ({A_1}^{\frac{-t_1}{2}} {B_1}^{p_1} {A_1}^{\frac{-t_1}{2}})^{s_1} {A_1}^{\frac{t_1}{2}}\}^{\frac{q_1}{(p_1-t_1)s_1+t_1}} \tag{4.3}
$$

holds for $1 \geq q_1 \geq t_1 \geq 0$, $p_1 \geq q_1$ and $2 \geq s_1 \geq 1$.
(4.3) で $1 = q_1 \geq t_1 = \frac{t}{q} \geq 0$, $p_1 = \frac{(p-t)s+t}{q} \geq q_1 = 1$ と置き、A_1, B_1 を元に戻すと次のようになる。

$$
A^q \geq \{A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{ss_1} A^{\frac{t}{2}}\}^{\frac{q}{(p-t)ss_1+t}} \tag{4.4}
$$

holds for $1 \geq q \geq t \geq 0$, $p \geq q$ and $4 \geq ss_1 \geq 1$.
以下 (4.2) から (4.4) を導いた手順を繰り返すことにより、(4.1) は全ての $s \geq 1$ で成立する。

ここで、

$$A_2 = A^q,\ B_2 = \{A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}}$$

とすると、(4.1) より $A_2 \geq B_2 \geq 0$ を満たすので、Theorem F により

$$A_2{}^{1+r_2} \geq (A_2{}^{\frac{r_2}{2}} B_2{}^{p_2} A_2{}^{\frac{r_2}{2}})^{\frac{1+r_2}{p_2+r_2}} \tag{4.5}$$

holds for $p_2 \geq 1$ and $r_2 \geq 0$.
最後に (4.5) で $p_2 = \frac{(p-t)s+t}{q} \geq 1$, $r_2 = \frac{r-t}{q} \geq 0$ と置き、A_2, B_2 を元に戻すと、
for each $1 \geq q \geq t \geq 0$ and $p \geq q$,

$$A^{q-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}$$

holds for any $r \geq t$ and $s \geq 1$.
よって (ii) が証明された。

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SIMULTANEOUS EXTENSIONS OF SELBERG INEQUALITY AND HEINZ-KATO-FURUTA INEQUALITY

Masatoshi Fujii

0. Introduction.

Throughout this note, an operator means a bounded linear one acting on a Hilbert space. An operator A is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. We first cite the Heinz-Kato-Furuta inequality, [5]:

The Heinz-Kato-Furuta inequality. *Let A and B be positive operators on H. If an operator T on H satisfies $T^*T \leq A^2$ and $TT^* \leq B^2$, then*

$$|(T|T|^{\alpha+\beta-1}x, y)| \leq \|A^\alpha x\| \|B^\beta y\| \tag{1}$$

for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geq 1$ and $x, y \in H$.

Based on (1), we have the following extension of a recent Lin's refinement [7] of the generalized Schwarz inequality. Let $T = U|T|$ be the polar decomposition of an operator T on H in the below.

Theorem A. *Let T be an operator on H and $0 \neq y \in H$. For $z \in H$ satisfying $T|T|^{\alpha+\beta-1}z \neq 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$,*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2 (|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y) \tag{2}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and $x \in H$.

We here remark that Lin's theorem is just the case $\alpha + \beta = 1$ in Theorem A. As a consequence, we have the following improvement of the Heinz-Kato-Furuta inequality via the Löwner-Heinz inequality, i.e., $A \geq B \geq 0$ implies $A^\alpha \geq B^\alpha$ for $\alpha \in [0,1]$:

Theorem B. *Let A and B be positive operators on H. If an operator T on H satisfies $T^*T \leq A^2$ and $TT^* \leq B^2$, then*

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2 (|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \leq \|A^\alpha x\|^2 \|B^\beta y\|^2 \tag{3}$$

for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geq 1$ and $x, y, z \in H$ such that $T|T|^{\alpha+\beta-1}z \neq 0$ and $(T|T|^{\alpha+\beta-1}z, y) = 0$.

On the other hand, Kubo [6] informed us of the Selberg inequality which is a generalization of the Bessel inequality: For given nonzero vectors $z_1, \cdots, z_n \in H$,

$$\sum_i \frac{|(x, z_i)|^2}{\sum_j |(z_i, z_j)|} \leq \|x\|^2 \tag{4}$$

holds for all $x \in H$.

In this note, we first point out that the Selberg inequality (4) is refined as follows: If $(y, z_i) = 0$ for given $\{z_i\}$, then

$$(5) \qquad |(y,x)|^2 + \sum_i \frac{|(x,z_i)|^2}{\sum_j |(z_i,z_j)|}\|y\|^2 \le \|x\|^2\|y\|^2$$

holds for all x. Though the refinement (5) is motivated by Theorem A, it gives us further extensions of Theorem A, precisely a simultaneous extension of Selberg and Heinz-Kato-Furuta inequalities. We moreover consider its generalizations via the Furuta inequality.

This is a joint work with R.Nakamoto.

1. Preliminary. For the sake of convenience, we first give a proof of Theorem A:

Proof of Theorem A. We only use the positivity of the Gram matrix

$$G = G(U|T|^\alpha x, |T^*|^\beta y, U|T|^\alpha z).$$

Noting that

$$(|T^*|^\beta y, U|T|^\alpha z) = (y, |T^*|^\beta U|T|^\alpha z) = (y, T|T|^{\alpha+\beta-1}z) = 0$$

by the assumption, we have

$$G = \begin{pmatrix} \||T|^\alpha x\|^2 & (U|T|^\alpha x, |T^*|^\beta y) & (U|T|^\alpha x, U|T|^\alpha z) \\ (U|T|^\alpha x, |T^*|^\beta y)^* & \||T^*|^\beta y\|^2 & 0 \\ (U|T|^\alpha x, U|T|^\alpha z)^* & 0 & \||T|^\alpha z\|^2 \end{pmatrix}.$$

Since $|T|^\alpha z \neq 0$, we have

$$|(T|T|^{\alpha+\beta-1}x, y)|^2 + \frac{|(|T|^{2\alpha}x, z)|^2(|T^*|^{2\beta}y, y)}{(|T|^{2\alpha}z, z)} \le (|T|^{2\alpha}x, x)(|T^*|^{2\beta}y, y).$$

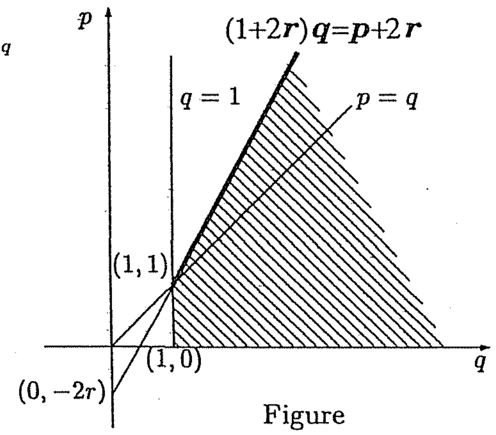
Next we cite the Furuta inequality, [2] and [3] for a one-page proof:

The Furuta inequality. *If $A \ge B \ge 0$, then for each $r \ge 0$,*

$$(B^rA^pB^r)^{1/q} \ge (B^rB^pB^r)^{1/q}$$

holds for $p \ge 0$ and $q \ge 1$ with

$$(*) \qquad (1+2r)q \ge p + 2r.$$



Figure

2. Refinements of Selberg inequality.

We begin with the proof of (5), which is done along with Furuta's way [4]:

Lemma 1. *If* $(y,z_i)=0$ *for given nonzero vectors* $\{z_i; i=1,2,\cdots,n\}$, *then*

$$|(x,y)|^2+\sum_i \frac{|(x,z_i)|^2}{\sum_j |(z_i,z_j)|}\|y\|^2 \le \|x\|^2\|y\|^2 \tag{5}$$

holds for all x.

Proof. We put

$$u=x-\sum_i \frac{(x,z_i)}{\sum_j |(z_j,z_i)|}z_i = x-\sum_i a_i z_i.$$

Then we have

$$\begin{aligned}\|u\|^2 &= \|x-\sum_i a_i z_i\|^2\\ &\le \|x\|^2-2\mathrm{Re}\sum \bar{a}_i(x,z_i)+\sum\{|a_i|^2\sum_j |(z_i,z_j)|\}\\ &= \|x\|^2-\sum_i \frac{|(x,z_i)|^2}{\sum_j |(z_i,z_j)|}.\end{aligned}$$

Hence it follows that

$$\begin{aligned}\|y\|^2\{\|x\|^2-\sum_i \frac{|(x,z_i)|^2}{\sum_j |(z_i,z_j)|}\} &\ge \|y\|^2\|u\|^2 \ge |(y,u)|^2\\ &= |(y,x-\sum_i \frac{(x,z_i)}{\sum_j |(z_j,z_i)|}z_i)|^2 = |(y,x)|^2.\end{aligned}$$

Now Furuta showed the following extension of the Selberg inequality: Let T be an operator on H with the kernel ker (T). For given $\{z_i\} \not\subset \ker(T^*)$,

$$\sum_i \frac{|(Tx,z_i)|^2}{\sum_j |(|T^*|^{2(1-\alpha)}z_i,z_j)|} \le \||T|^\alpha x\|^2 \tag{6}$$

holds for all $x \in H$ and $\alpha \in [0,1]$.

Thus we have a refinement of (6) by Lemma 1.

Theorem 2. *Let* $T=U|T|$ *be the polar decomposition of an operator* T *on* H, $\{z_i; i=1,2,\cdots,n\} \not\subset \ker(T^*)$ *and* $\alpha\in[0,1]$. *If* $(U|T|^{1-\alpha}y,z_i)=0$ *for all* i, *then*

$$|(|T|^\alpha x,y)|^2+\sum_i \frac{|(Tx,z_i)|^2}{\sum_j |(|T^*|^{2(1-\alpha)}z_i,z_j)|}\|y\|^2 \le \||T|^\alpha x\|^2\|y\|^2 \tag{7}$$

holds for all $x\in H$.

Theorem 3. *Suppose that* $\{z_i; i = 1, 2, \cdots, n\} \not\subset \ker(T^*)$ *and* $\alpha, \beta \geq 0$ *with* $\alpha + \beta \geq 1 \geq \alpha - \beta$. *If* $(|T^*|^{\beta+1-\alpha}y, z_i) = 0$ *for all* i, *then*

$$(8) \qquad |(T|T|^{\alpha+\beta-1}x, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 \| |T^*|^{\beta} y\|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|} \leq \| |T|^{\alpha} x\|^2 \| |T^*|^{\beta} y\|^2$$

holds for all $x \in H$. *In particular, if* $(|T^*|^{2(1-\alpha)}y, z_i) = 0$ *for* $\alpha \in [0, 1]$, *then*

$$(9) \qquad |(Tx, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 \| |T^*|^{1-\alpha} y\|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|} \leq \| |T|^{\alpha} x\|^2 \| |T^*|^{1-\alpha} y\|^2$$

holds for all $x \in H$.

By a similar way to Lemma 1, we have an alternative simultaneous extension of Selberg and generalized Schwarz inequalities:

Theorem 4. *Suppose that* $\{z_i; i = 1, 2, \cdots, n\} \not\subset \ker(T)$ *and* $\alpha, \beta \geq 0$ *with* $\alpha + \beta \geq 1$. *If* $(T|T|^{\alpha+\beta-1}z_i, y) = 0$ *for all* i, *then*

$$(10) \qquad |(T|T|^{\alpha+\beta-1}x, y)|^2 + \sum_i \frac{|(|T|^{2\alpha}x, z_i)|^2 \| |T^*|^{\beta} y\|^2}{\sum_j |(|T|^{2\alpha} z_i, z_j)|} \leq \| |T|^{\alpha} x\|^2 \| |T^*|^{\beta} y\|^2$$

holds for all $x \in H$.

3. Refinements of Heinz-Kato-Furuta inequality.

Corollary 5. *Suppose that* $\alpha, \beta \geq 0$ *with* $\alpha + \beta \geq 1 \geq \alpha - \beta$ *and* $z_i \notin \ker(T^*)$ *satisfies* $(|T^*|^{\beta+1-\alpha}y, z_i) = 0$ *for* $i = 1, \cdots, n$. *If* $|T|^2 \leq A^2$ *and* $|T^*|^2 \leq B^2$ *for* $A, B \geq 0$, *then*

$$(11) \qquad |(T|T|^{\alpha+\beta-1}x, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 \| |T^*|^{\beta} y\|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|} \leq \|A^{\alpha} x\|^2 \|B^{\beta} y\|^2$$

holds for all $x \in H$. *In particular, if* $(|T^*|^{2(1-\alpha)}y, z_i) = 0$ *for* $\alpha \in [0, 1]$, *then*

$$(12) \qquad |(Tx, y)|^2 + \sum_i \frac{|(Tx, z_i)|^2 \| |T^*|^{1-\alpha} y\|^2}{\sum_j |(|T^*|^{2(1-\alpha)} z_i, z_j)|} \leq \|A^{\alpha} x\|^2 \|B^{1-\alpha} y\|^2$$

holds for all $x \in H$.

Corollary 6. *Suppose that* $\alpha, \beta \geq 0$ *with* $\alpha + \beta \geq 1 \geq \alpha - \beta$ *and* $z_i \notin \ker(T)$ *satisfies* $(T|T|^{\alpha+\beta-1}z_i, y) = 0$ *for* $i = 1, 2, \cdots, n$. *If* $|T|^2 \leq A^2$ *and* $|T^*|^2 \leq B^2$ *for* $A, B \geq 0$, *then*

$$(13) \qquad |(T|T|^{\alpha+\beta-1}x, y)|^2 + \sum_i \frac{|(|T|^{2\alpha}x, z_i)|^2 \| |T^*|^{\beta} y\|^2}{\sum_j |(|T|^{2\alpha} z_i, z_j)|} \leq \|A^{\alpha} x\|^2 \|B^{\beta} y\|^2$$

holds for all $x \in H$.

To give further extensions of the Heinz-Kato-Furuta inequality, we apply the Furuta inequality [2] and [3].

Theorem 7. *Let A and B be positive operators on H and T an operator such that $T^*T \le A^2$ and $TT^* \le B^2$. Then for each $r,\ s \ge 0$*

$$|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)|^2 + \sum_i \frac{|(Tx,z_i)|^2(|T^*|^{2(1+2s)\beta}y,y)}{\sum_j(|T|^{2(1-\alpha-2r\alpha)}z_i,z_j)} \tag{14}$$

$$\le ((|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x,x)((|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}y,y)$$

for all $p,\ q \ge 1$, $\alpha, \beta \in [0,1]$ with $(1+2r)\alpha+(1+2s)\beta \ge 1 \ge (1+2r)\alpha$ and $x, y, z_1, \cdots, z_n \in H$ such that $z_i \notin$ ker(T^*) and $(|T^*|^{(1+2s)\beta+1-(1+2r)\alpha}y, z_i) = 0$ for $i = 1, \cdots, n$.

Similarly we have the following further extensions by Theorem 4:

Theorem 8. *Let A and B be positive operators on H and T an operator such that $T^*T \le A^2$ and $TT^* \le B^2$. Then for each $r,\ s \ge 0$*

$$|(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}x,y)|^2 + \sum_i \frac{|(|T|^{2(1+2r)\alpha}x,z_i)|^2(|T^*|^{2(1+2s)\beta}y,y)}{\sum_j(|T|^{2(1+2r)\alpha}z_i,z_j)} \tag{15}$$

$$\le ((|T|^{2r}A^{2p}|T|^{2r})^{\frac{(1+2r)\alpha}{p+2r}}x,x)((|T^*|^{2s}B^{2q}|T^*|^{2s})^{\frac{(1+2s)\beta}{q+2s}}y,y)$$

for all $p,\ q \ge 1$, $\alpha, \beta \in [0,1]$ with $(1+2r)\alpha + (1+2s)\beta \ge 1$ and $x, y, z_1, \cdots, z_n \in H$ such that $z_i \notin$ ker(T) and $(T|T|^{(1+2r)\alpha+(1+2s)\beta-1}z_i, y) = 0$ for $i = 1, \cdots, n$.

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Hanner type inequality with random coefficients

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Abstract

Let $1 \leq p < \infty$, and let (S, Σ, μ) be a measure space with $\mu(S) = 1$ and $L^p = L^p(S, \Sigma, \mu)$. The norm of L^p is given by $\|x\| = (\int_S |x(s)|^p d\mu(s))^{\frac{1}{p}}$. Hanner proved the following inequalities. For $x_1, x_2 \in L^p$, it holds that for $1 \leq p \leq 2$

$$\|x_1 + x_2\|^p + \|x_1 - x_2\|^p \geq |\|x_1\| + \|x_2\||^p + |\|x_1\| - \|x_2\||^p$$

and for $2 \leq p < \infty$

$$\|x_1 + x_2\|^p + \|x_1 - x_2\|^p \leq |\|x_1\| + \|x_2\||^p + |\|x_1\| - \|x_2\||^p.$$

In our recent works, we extended Hanner's inequality to the n-element inequality as follows. Let $\delta_1, \delta_2, \ldots, \delta_n$ be the independent symmetric real valued random variables and let $x_1, x_2, \ldots, x_n \in L^p_{\mathrm{C}}$, then it holds that

$$\begin{aligned} \mathbf{E}\left\|\sum_{i=1}^{n} \delta_i x_i\right\|^p &\geq \mathbf{E}\left|\sum_{i=1}^{n} \delta_i \|x_i\|\right|^p \quad for1 \leq p \leq 2, and \\ \mathbf{E}\left\|\sum_{i=1}^{n} \delta_i x_i\right\|^p &\leq \mathbf{E}\left|\sum_{i=1}^{n} \delta_i \|x_i\|\right|^p \quad for2 \leq p < \infty, \end{aligned}$$

where $\mathbf{E}$ means the expectation with respect to the distribution of $\{\delta_i\}$. In this short note, we consider the complex valued random variables $\{\delta_i(w)\}$. We shall give the range condition for $\{\delta_i(w)\}$ which implies the above inequalities.

1 Introduction

Hannerの不等式，Pavlovićの不等式については以下の結果が得られている．

Hannerの不等式([1]) $x_1, x_2 \in L^p_{\mathrm{C}}$，$1 \leq p \leq 2$．とするとき次の不等式が成り立つ．

$$\|x_1 + x_2\|^p + \|x_1 - x_2\|^p \geq |\|x_1\| + \|x_2\||^p + |\|x_1\| - \|x_2\||^p.$$

$2 \leq p < \infty$ なら逆向きの不等式が成り立つ.
特に $p = 1$ の場合はノルムの三角不等式と同等であり任意のノルム空間で成り立つ. $p = 2$ なら等号成立. これを Rademacher 列を使って次のようにあらわす事ができる.

$\varepsilon_1, \varepsilon_2$ を独立な Rademacher 列とする. すなわち確率 $\frac{1}{2}$ で ± 1 をとる.

$$\mathbf{E}\left\|\sum_{i=1}^{2} \varepsilon_i x_i\right\|^p \geq \mathbf{E}\left|\sum_{i=1}^{2} \varepsilon_i \|x_i\|\right|^p \quad (1 \leq p \leq 2) \tag{1}$$

$$x_1, x_2 \in L^p_{\mathrm{C}}$$

Hanner の不等式は次のように自然に n 要素に拡張できる.

n 要素 Hanner の不等式 ([2,3])

$$\mathbf{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\|^p \geq \mathbf{E}\left|\sum_{i=1}^{n} \varepsilon_i \|x_i\|\right|^p \quad (1 \leq p \leq 2) \tag{2}$$

$$x_1, x_2, \ldots, x_n \in L^p_{\mathrm{C}}$$

一方、Hanner の不等式の拡張としては、Pavlović の不等式がある. Hanner の不等式に複素数値の係数をつけたものであるが、この場合 $x_1, x_2 \in L^p_{\mathrm{C}}$ では必ずしも成り立たず、$x_1, x_2 \in L^p_{\mathrm{R}}$ となる.
Pavlović の不等式 ([5])

$$\|w_1 x_1 + w_2 x_2\|^p + \|w_1 x_1 - w_2 x_2\|^p \geq |w_1\|x_1\| + w_2\|x_2\||^p + |w_1\|x_1\| - w_2\|x_2\||^p \tag{3}$$

$$(1 \leq p \leq 2)$$

次に n 要素 Pavlović の不等式を示す. Hanner の不等式と同様に独立な Rademacher 列を使って拡張している.

n 要素 Pavlović([4])

$$\mathbf{E}\left\|\sum_{i=1}^{n} \varepsilon_i w_i x_i\right\|^p \geq \mathbf{E}\left|\sum_{i=1}^{n} \varepsilon_i w_i \|x_i\|\right|^p \quad (1 \leq p \leq 2) \tag{4}$$

$$w_i \in \mathbf{C}, x_i \in L^p_{\mathrm{R}} \quad i = 1, 2, \ldots, n$$

(3)(4) において $x_i \in L^p_{\mathrm{C}}$ では必ずしも成立しない. n 要素 Hanner の不等式と n 要素 Pavlović の不等式の係数を $\{\varepsilon_i w_i\}$ から一般の独立確率変数へ拡張したものが次である.

$\{\delta_i(w)\}_{i=1}^{n}$:独立対称 実数値 確率変数列
$x_1, x_2, \ldots, x_n \in \underline{L^p_{\mathrm{C}}}$
このとき次の不等式が成り立つ.

$$\mathbf{E}\left\|\sum_{i=1}^{n} \delta_i x_i\right\|^p \geq \mathbf{E}\left|\sum_{i=1}^{n} \delta_i \|x_i\|\right|^p \quad (1 \leq p \leq 2) \tag{5}$$

$\{\delta_i(w)\}_{i=1}^{n}$:独立対称 $\underline{\text{複素数値}}$ 確率変数列
$x_1, x_2, \dots, x_n \in \underline{L^p_{\mathbf{R}}}$

$$\Longrightarrow \mathbf{E}\left\|\sum_{i=1}^{n}\delta_i(w)x_i\right\|^p \geq \mathbf{E}\left|\sum_{i=1}^{n}\delta_i(w)\|x_i\|\right|^p \quad (1 \leq p \leq 2) \tag{6}$$

上記 (1) から (6) は全て $2 \leq p < \infty$ では逆向きの不等式となる．Pavlović 不等式は $x_i \in L^p_{\mathbf{C}}$ では成立しない．ここでは成立するための条件について考察する．

2 Results

$p = 2$ では常に等号が成立する．

Theorem 1

$x_i \in L^2_{\mathbf{C}}$, $\{\delta_i(w)\}_i^n = 1$ を独立複素数値確率変数とする．ただし，$\mathbf{E}[\delta_i] = 0, \mathbf{E}[|\delta_i|^2] < \infty$．このとき次が成り立つ．

$$\mathbf{E}\left|\,|\sum_{i=1}^{n}\delta_i x_i\right|\,|^2 = \mathbf{E}\left|\sum_{i=1}^{n}\delta_i\|x_i\|\right|^2$$

Theorem 2

$\{\delta_i(w)\}_{i=1}^{n} \in \mathbf{C}^n$, $x_1, x_2, \dots, x_n \in L^p_{\mathbf{C}}$ とし、集合 V を

$$\begin{aligned} V &= V(\delta_1, \dots, \delta_n) \\ &= \left\{(z_1, \dots, z_n) \in \mathbf{C}^n \,\middle|\, \mathbf{E}\left|\sum_{i=1}^{n}\delta_i z_i\right|^p \leq \mathbf{E}\left|\sum_{i=1}^{n}\delta_i\,|z_i|\right|^p\right\} \end{aligned}$$

が成り立つ領域とする．このとき次の不等式が成り立つ．

$$\mathbf{E}\left\|\sum_{i=1}^{n}\delta_i(w)x_i\right\|^p \geq \mathbf{E}\left|\leq \sum_{i=1}^{n}\delta_i(w)\|x_i\|\right|^p \quad (1 \leq p \leq 2)$$

Theorem 2 を Pavlović 不等式の形に直すと
$w_i \in \mathbf{C}, x_i \in L^p_{\mathbf{C}}; (x_1(t), \dots, x_n(t)) \in V$ a.e.

$$\begin{aligned} V &= V(w_1, \dots, w_n) \\ &= \left\{(z_1, \dots, z_n) \in \mathbf{C}^n \,\middle|\, \mathbf{E}\left|\sum_{i=1}^{n}\varepsilon_i w_i z_i\right|^p \leq \mathbf{E}\left|\sum_{i=1}^{n}\varepsilon_i w_i\,|z_i|\right|^p\right\} \quad (1 \leq p \leq 2) \end{aligned}$$

$$\Longrightarrow \mathbf{E}\left\|\sum_{i=1}^{n}\varepsilon_i w_i x_i\right\|^p \geq \mathbf{E}\left|\leq \sum_{i=1}^{n}\varepsilon_i w_i\|x_i\|\right|^p$$

となる．

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On minimal Banach spaces

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Abstract. We consider a Banach space X which embeds into any one of its infinite-dimensional subspaces. Such a space X is called minimal. It is shown that a non-reflexive Banach space X is minimal if and only if it is isomorphic to a subspace of c_0 or ℓ_1. It is also shown that a minimal Banach space X is isomorphic to a Hilbert space if and only if any quotient space of any subspace of X has the Gordon-Lewis (GL)-property.

Definitions and Preliminaries

Let X be an infinite-dimensional Banach space and X^* be its dual.

(1) X is minimal if it embeds into each of its subspaces. (Unless otherwise stated, a subspace means an infinite-dimensional closed linear subspace; the notion of minimality was introduced by Rosenthal.)

In 1960, Pelczynski [Pe] showed that c_0 and spaces ℓ_p $(1 \leqq p < \infty)$ are minimal. (For the other examples of minimal Banach spaces; see Casazza-Johnson-Tzafriri [CJT] and Schlumprecht [Sc].) It is easy to see that any minimal Banach space is separable; and all subspaces of minimal spaces are also minimal.

(2) X is homogeneous if it is isomorphic to each of its subspaces. Of course, any separable Hilbert space is homogeneous, and homogeneous Banach spaces are minimal.

Homogeneous Banach space (HBS) problem: Is any homogeneous Banach space isomorphic to a Hilbert space ?

This problem has frequently been called the Banach's HBS problem (cf. [Ba], 1932). In 1988, Johnson [Jo] showed that if X and X^* are homogeneous and have the Gordon-Lewis (GL)-property, then X is isomorphic to ℓ_2. Recall that X is called to have the GL-property if any absolutely summing operator from X into L_2 factors through some L_1-space. In 1995, Komorowski and Tomczak-Jaegermann [KT] showed that if X is homogeneous and has a subspace with an unconditional basis, then X is isomorphic to ℓ_2. Recently the HBS problem was solved in the positive by Gowers [G] proving that any homogeneous Banach space contains a subspace with an unconditional basis. It is also true that any minimal Banach space has a subspace with an unconditional basis, see [G].

Main Results

It is easy to see that an infinite-dimensional Banach space X is isomorphic to a Hilbert space if and only if any separable subspace of X is isomorphic to ℓ_2. Hence the positive answer to the HBS problem implies the following result.

Theorem 1. Let X be a Banach space such that all separable subspaces are mutually isomorphic, then X is isomorphic to a Hilbert space. (X is not necessarily separable.)

Remark 1. Even up to this day no direct proof is known that if X is homogeneous, then X is uniformly isomorphic to all of its subspaces, that is, there is a constant C>0 such that the Banach-Mazur distance $d(X,Y) \leqq C$ for all subspaces Y; in this case we say that X and Y are C-isomorphic.

The following result is easily proved.

Theorem 2. Let X be a Banach space such that for each n and for all n-dimensional subspaces Y and Z of X, Y and Z are C-isomorphic, where C>0 is a constant. Then X is isomorphic to a Hilbert space.

Remark 2. If C = 1, then X is isometric to a Hilbert space.

As mentioned above, any minimal Banach space has a subspace with an unconditional basis. Hence, by a rerult of James [Ja], we have

Theorem 3. Let X be a non-reflexive Banach space. Then X is minimal if and only if it is isomorphic to a subspace of c_0 or ℓ_1.

It is well-known that any subspace of ℓ_p $(1 \leqq p \leqq 2)$ has the GL-property and any quotient space of c_0 or ℓ_q $(2 \leqq q \leqq \infty)$ has the GL-property. In general, X has the GL-property if and only if X^* has it, and if X is of cotype 2 and has the GL-property, then any subspace of X and any quotient space of X^* has the GL-property (cf. [Pi1]). In the following, we consider Banach spaces X such that any quotient space of any subspace of X has the GL-property.

Lemma 4 (Johnson [Jo]). If any subspace of X has the GL-property, then X is of weak cotype 2.

Remark 3. It is known that weak cotype 2 implies cotype q for all $q > 2$, but the converse is false. For the details of cotype and weak cotype, see Pisier [Pi2].

Weak Hilbert space (cf. [Pi2]): X is called a weak Hilbert space if X and X^* are of weak cotype 2 and X is of type p for some $p > 1$.

Lemma 5. If any subspace of X and any quotient space of X^{**} have the GL-property, then X is a weak Hilbert space.

Lemma 6 (cf. [Jo]). Let X be a minimal Banach space. Then X is isomorphic to a Hilbert space if and only if it is a weak Hilbert space.

Theorem 7. Let X be a minimal Banach space. Then X is isomorphic to ℓ_2 if and only if any quotient space of any subspace of X has the GL-property.

The proof can be done by Theorem 2, Lemmas 5 and 6.

Remark 4. Without the assumption of minimality, it can be shown that if X is reflexive and any quotient space of any subspace of $\ell_2(X)$ has the GL-property, then X is isomorphic to a Hilbert space. It is known that if any quotient space of any subspace of $\ell_2(X)$ has a basis , then X is isomorphic to ℓ_2 (cf. [MT]). It seems to be unknown that if any quotient space of any subspace of $\ell_2(X)$ has the approximation property, is X isomorphic to a Hilbert space ?

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EXTRAPOLATION THEOREM ON LORENTZ SPACES

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Abstract. Extrapolation theory was constructed by B. Jawerth and M. Milman ([JM]) to treat Zygmund class $L\log^{\alpha} L$ as the "limit case" of the family L^p, $p \searrow 1$. In the present paper, we shall investigate the "limit" $p \searrow p_0 \geq 1$ or, more generally, such "limit case" on Lorentz spaces.

1. Introduction and Results.

関数空間における補外理論は，次の Yano の定理がその出発点である．

Theorem A([Y]). (Ω, μ) を有限測度空間とする．T を 任意の $1 < p < 2$ に対して $L^p(\Omega)$ 上で定義された半線形な有界作用素とし，

$$\left[\int_{\Omega} |Tf(x)|^p d\mu(x)\right]^{1/p} \leq \frac{A}{(p-1)^k} \left[\int_{\Omega} |f(x)|^p d\mu(x)\right]^{1/p}$$

という評価が任意の $f \in L^p(\Omega)$, $1 < \forall p \leq 2$ に対して成り立つものとする．このとき T は

$$\int_{\Omega} |Tf(x)| d\mu(x) \leq B \int_{\Omega} |f(x)|(1 + \log|f(x)|)^k d\mu(x) + C,$$

という評価を満たす．すなわち T は任意の $L\log^k L$ に属する関数を L^1 へ写す作用素となる．

この定理をもとに，B.Jawerth と M.Milman が一般の (quasi-)Banach spaces に対して抽象的な補外空間として Σ_q 空間を構成した．

定義. $\{A_\theta\}_{0<\theta<1}$ は準ノルム可換群の族であり，次の性質を満たすとする：

(1) (strongly compatible) 2つの準ノルム可換群 Δ と Σ があって，

$$\Delta \subset A_\theta \subset \Sigma \qquad \text{for any } 0 < \theta < 1$$

であり，その埋め込み写像は連続でかつ，

(2) (weak Σ_p-condition) $a \in \Sigma$ が $a = \sum_n a_n$ (convergence in Σ, $a_n \in A_{\theta_n}$, $0 < \theta_n < 1$) と分解されるとき，そのような分解のうち少なくとも1つは

$$\sum_{n=1}^{\infty} \left(\frac{\|a_n\|_\Sigma}{\|a_n\|_{A_{\theta_n}}}\right)^{p'} < \infty$$

をみたすようなものが存在する (この条件は [JM] にある条件よりも弱い).

このとき 任意の $q \geq 1$ に対して準ノルム可換群

$$\sum\nolimits_{q} \limits_{0<\theta<1} A_\theta = \sum\nolimits_q \{A_\theta;\, 0<\theta<1\}$$

を，$a = \sum_{i=1}^{\infty} a_i$ (in Σ), $a_i \in A_{\theta_i}$ という分解が少なくとも1通り存在して，

$$\|a\|_{\Sigma_q\{A_\theta:0<\theta<1\}} = \inf \left[\sum_{i=1}^{\infty} \|a_i\|_{A_{\theta_i}}^q \right]^{\frac{1}{q}} < \infty$$

を満たすような $a \in \Sigma$ 全体として定義する．ただし，ここでの inf は $a = \sum_{i=1}^{\infty} a_i$, $a_i \in A_{\theta_i}$ という分解全体をとるものとする．

このとき，次のことが示される．

Main Lemma[1]**.** $\{A_\theta\}_{0<\theta<1}$ および $\{B_\theta\}_{0<\theta<1}$ は上の *Weak* Σ_p 条件を $p=1$ でみたすとする．T が劣加法的な A_θ から B_θ への縮小作用素 $(\theta \in (0,1))$ であれば，T は $\sum_p \limits_{0<\theta<1} \{A_\theta\}$ *to* $\sum_p \limits_{0<\theta<1} \{B_\theta\}$ の有界作用素である $(1 \leq p < \infty)$.

このことを Σ_p が extrapolation method であるともいう．
次のことが示される．

Theorem B[2]**.** (Ω, μ) を有限測度空間とし，$1 \leq p_0 < \infty$, $\alpha \geq 0$ とする．このとき L^p 空間の族に対して次のように Σ 空間が決定される．

$$\sum\nolimits_{p_0} \left\{ \left(\frac{p}{p-p_0} \right)^{\alpha} L^p(\Omega) : p_0 < p < \infty \right\} = L^{p_0} \log^{p_0 \alpha} L(\Omega)$$

$p_0 = 1$ の場合を用いれば，上の Lemma から Yano の定理が導かれることがわかる ([JM]).

一方筆者はこの定理を無限測度空間上の L^p 空間の族に対して拡張することを試み，次の結果を得た ([S1]).

Theorem C. (Ω, μ) を σ有限測度空間とし，$1 \leq p_0 < p < p_1 < \infty$, $\alpha \geq 0$ とする．このとき L^p 空間の Σ_1 空間が次のように求められる．

$$\sum \left\{ \left(\left(\frac{1}{p-p_0} \right)^{\alpha} L^p(\Omega) \right)^{\frac{p}{p_1}} : p_0 < p < p_1 \right\} = (L^{p_0} \log^{p_0 \alpha} L(\Omega))^{\frac{p_0}{p_1}} + (L^{p_1}(\Omega))^{\frac{p_1}{p_1}}$$

ここで定数 $k, \alpha > 0$ および $1 \leq p < \infty$ に対し $(kL^p)^\alpha$ とは

$$|||f||| = \left[k \int_\Omega |f(x)|^p d\mu(x) \right]^{\frac{\alpha}{p}} < \infty$$

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[1] $q = 1$ の場合は [JM], $q > 1$ の場合には [S2].

[2] $p_0 = 1$ の場合は [JM] に証明がある．$p_0 > 1$ の場合にもこの定理が成り立つことは M.Milman が筆者に証明なしに示唆した．証明は [S3] を参照．

を満たす関数全体（集合としては L^p と同一）からなる quasi-Banach 空間である．

この結果から Yano の定理を含む結果，および $p \searrow p_0 > 1$ のときに発散する作用素の有界性について，同様の結果が得られる．

しかしこの方法は L^p 空間そのものを扱っていない．そこで次に考えるのは

$$p_0 \geq 1 \text{ のときに } \sum_{p_0<p<p_1} \{(p-p_0)^{-\alpha} L^p\} \text{ という空間を具体的に特徴づける}$$

ことである．

このことを調べるうちに次の結果が得られた．

Main theorem. (Ω, μ) を一般の測度空間とする．$1 \leq p_0 < p_1 \leq \infty$, $\alpha \geq 0$, $1 \leq q_0 < \infty$ とする．

$p_1 < \infty$ の場合には

$$\sum_{p_0<p<p_1}{}_{q_0} \left\{ \left(\frac{p}{p-p_0} \right)^{\alpha} L^{p,q(p)}(\Omega) \right\} = L^{p_0,q_0} \log^{q_0 \alpha} L + L^{p_1,q_0}(\Omega).$$

$p_1 = \infty$ の場合には

$$\sum_{p_0<p<\infty}{}_{q_0} \left\{ \left(\frac{p}{p-p_0} \right)^{\alpha} L^{p,q(p)}(\Omega) \right\} = L^{p_0,q_0} \log^{q_0 \alpha} L + L^{\infty}(\Omega).$$

ここで $q(\cdot)$ は p に依存して変わる $1 \leq q(p) \leq \infty$ なる任意の数，$L^{p_0,q_0} \log^{q_0 \alpha_0} L + L^{p_1,q_1} \log^{q_1 \alpha_1} L$ はその再配列関数 f^* に対して

$$\left(\int_0^1 [t^{\frac{1}{p_0}} (1+\log t)^{\alpha_0} f^*(t)]^{q_0} \frac{dt}{t} \right)^{\frac{1}{q_0}} + \left(\int_1^{\infty} [t^{\frac{1}{p_1}} (1+\log t)^{\alpha_1} f^*(t)]^{q_1} \frac{dt}{t} \right)^{\frac{1}{q_1}} < \infty.$$

をみたす関数 f 全体の集合である．

2. Our result is ⋯.

2.1. Natural.

Extrapolation theory は interpolation theory と密接な関係がある．実際，良く知られた次の補間定理を考える．

$$((L^{p_0})^{p_0}, (L^{p_1})^{p_1})_{\theta,1,K} = (L^p)^p \qquad p = (1-\theta)p_0 + \theta p_1$$

$$(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,q,K} = L^{p,q} \qquad p = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

Σ_p-space と $K_{\theta,p}$ space は，定義を見ればおよそ空間の族もしくは pair の "ℓ^p-sum" であると見ることができる．ということは，このような Lorentz 型の空間を考えることは自然であると言えよう．

2.2. General.

$q_0 = 1$, $q(p) = p$ とおけば我々の問題の答えが得られる．また考える測度空間が有限測度空間なら $p_0 = q_0 = q(p)$ とおけば Theorem B が得られる．よって，これまでの結果を含む一般的な結果であることがわかる．

2.3. Yields Yano's type estimation.

Corollary. $1 \leq p_0 < p_1 \leq \infty$ $1 \leq r_0 < r_1 \leq \infty$ とする．T が *sub-additive* で $L^{p,q(p)}$ から $L^{r,s(r)}$ への有界作用素，ただし $p_0 < \forall p < p_1$ $r_0 < \forall r < r_1$ は次をみたす指数とする：

$$\left(\frac{1}{p_0} - \frac{1}{p}\right) / \left(\frac{1}{p_0} - \frac{1}{p_1}\right) = \left(\frac{1}{r_0} - \frac{1}{r}\right) / \left(\frac{1}{r_0} - \frac{1}{r_1}\right)$$

また $1 \leq q(p), s(r) \leq \infty$, $\alpha \geq 0$. さらにこの作用素 T の $L^{p,q(p)}$ から $L^{r,s(r)}$ への作用素ノルムが任意の p に対して $(p-p_0)^{-\alpha}$ の定数倍でおさえられるとする．このとき T は

$$L^{p_0,q}\log^{\beta+\alpha} L + L^{p_1,q} \qquad \text{から} \qquad L^{r_0,q}\log^{\beta} L + L^{r_1,q} \qquad \text{への有界作用素である}$$

$(\beta \geq 0,\ 1 \leq q < \infty)$

2.4. Partially improves known results.

この結果は Bennett - Rudnick の interpolation theorem ([BR]) の評価を部分的に改良する．その様子を見るために，１つ具体例を挙げる．

Example. $T = I_\gamma$ を *Riesz potential operator,* すなわち

$$I_\lambda f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n\lambda}} dy \qquad (0 < \lambda < 1).$$

とする．

I_λ は weak type $(1, 1/\lambda)$ かつ strong type (p, r) $(\frac{1}{p} - \frac{1}{r} = 1 - \lambda)$ であることが知られている．

Bennett - Rudnick の定理と Riesz-Thorin の補間定理を用いると

$$I_\lambda : L^{1,1}\log^{\beta+1} L + L^{p_1,1} \longrightarrow L^{\frac{1}{\lambda},1}\log^{\beta} L + L^{r_1,1} \quad \text{bounded}(\beta \geq 0)$$

という評価が得られる．

一方, Marcinkiewicz の補間定理の評価を精密に見て Riesz-Thorin の定理と組み合わせると

$$\|I_\lambda f\|_r \leq A\left(\frac{p}{p-1}\right)^{\lambda} \|f\|_p$$

という評価を得る $(\frac{1}{p} - \frac{1}{r} = 1 - \lambda$. 我々の得た Corollary をここに用いると $\alpha = \lambda$, $p_0 = 1$, $r_0 = \frac{1}{\lambda}$ $\frac{1}{p_1} - \frac{1}{r_1} = 1 - \lambda$ に対して

$$I_\lambda : L^{1,1}\log^{\beta+\lambda} L + L^{p_1,1} \longrightarrow L^{\frac{1}{\lambda},1}\log^{\beta} L + L^{r_1,1} \quad \text{bounded}(\beta \geq 0)$$

という評価を得る．

$0 < \lambda < 1$ である．

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An Introduction and Consideration of the Robbins' Paper on BSE-Banach modules

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Abstract. By using bundle techniques, D. A. Robbins answered affirmatively a question on a BSE-Banach module over a commutative Banach algebra with bounded approximate identity, which raised by the author.

局所可換群上の有界Borel測度のFourier-Stieltjes変換を連続関数のある種の性質で特徴付けたBochner-Shoenberg-Eberleinの定理(cf. [5, p. 32])は一般の可換Banach環の上に完全に焼き直され、そのような定理の成り立つ可換Banach環はBSE-環と定義され、BSE-環の構造や応用が研究されてきた(cf. [1], [2], [7] and [8])。BSE-環の定義とほぼ平行して、可換Banach環上のBanach moduleにもBSE構造が導入され、BSE-環の定義を特別な場合として含むBSE-Banach moduleが定義された(cf. [6])。これは先にLiu-Rooji-Wang [3]によってコンパクト可換群の群環上のBanach moduleに持ち込まれたBochner-Shoenberg-Eberlein型の定理を基本的に是正したものでもあった。実際、彼等の定理は彼等自身も論文で述べているように、もとの定理の完全な一般化ではなかったからである。

筆者はBSE-Banach moduleを定義した際、誘導モジュールがもとの可換Banach環上のBanach moduleとなるのかを問題として提起した。これに対して、最近D. A. Robbins [4]は、Banach space bundleのテクニックを用いて、moduleとなることを示した。また彼はsup-normでは完備とならないBanach moduleの例をあげているが、これは筆者の問題の出し仕方が曖昧だったせいもあり、実際には後で定義するBSE-normで完備かどうかまだopenである。

もう少し詳しく述べよう。今可換Banach環A上のBanach module Xを考える。Aの（正則）極大イデアル空間 Φ_A の各元 $\varphi \in \Phi_A$ 対して M_φ を対応するAの極大正則イデアルとし、

$$X^\varphi = \overline{\mathrm{span}}\{M_\varphi X + (1-e_\varphi)X\}$$

と置く。ここで e_φ は $\varphi(e_\varphi)=1$ を満たすAの元である。今各 $\varphi \in \Phi_A$ 対して、$X_\varphi = X/X^\varphi$ と置き、Φ_A 上の X_φ に関するベクトル場の全体を $\prod X_\varphi = \prod_{\varphi \in \Phi_A} X_\varphi$ で表わす。また各 $\varphi \in \Phi_A$, $x \in X$ に対して、

$$\pi_\varphi(x) \equiv \hat{x}(\varphi) = x + X^\varphi$$

と置く。ベクトル場 $\sigma \in \prod X_\varphi$ は、次の不等式を満たす正数 β が存在するときBSEと呼ぶ：

$$\left| \sum_{i=1}^{n} f_i(\sigma(\varphi_i)) \right| \leq \beta \left\| \sum_{i=1}^{n} f_i \circ \pi_{\varphi_i} \right\|_{X^*}$$

$$(\forall \varphi_1, \ldots, \varphi_n \in \Phi_A, \forall f_1 \in (X_{\varphi_1})^*, \ldots, \forall f_n \in (X_{\varphi_n})^*, \forall n = 1, 2, \ldots)$$

そのような β の下限を $\|\sigma\|_{BSE}$ で表わす。このとき、BSE ベクトル場の全体 $\prod_{BSE} X_\varphi$ はノルム $\|\sigma\|_{BSE}$ のもとで、Banach A-module となることが分かる。

さてAからXへの連続なA-準同型写像を multiplier と呼びその全体を M(A, X) で表わす。このとき各 $T \in M(A, X)$ に対して、

$$(Ta)^\wedge(\varphi) = \varphi(a)\hat{T}(\varphi) \ \ (\forall a \in A, \forall \varphi \in \Phi_A)$$

を満たすベクトル場 $\hat{T}$ が一意に定まる。今

$$\hat{M}(A, X) = \{\hat{T} : T \in M(A, X)\}$$

と置く。更に連続なBSEベクトル場の全体を $\prod^c_{BSE} X_\varphi$ で表わし、これをXの誘導モジュールと呼ぶ。ここでベクトル場 $\sigma \in \prod X_\varphi$ が連続であるとは、次の意味である：積位相を導入した空間 $\Phi_A \times X$ を考え、

$$\pi(\varphi, x) = (\varphi, \hat{x}(\varphi)) \ \ (\varphi \in \Phi_A,\ x \in X)$$

で定義される $\Phi_A \times X$ から $\bigcup_{\varphi \in \Phi_A} \{\varphi\} \times X_\varphi$ への写像 π に関する quotient topology を $\bigcup_{\varphi \in \Phi_A} \{\varphi\} \times X_\varphi$ に導入したとき、Φ_A から $\bigcup_{\varphi \in \Phi_A} \{\varphi\} \times X_\varphi$ への写像: $\varphi \to (\varphi, \sigma(\varphi))$ が連続である。

我々の興味は $\hat{M}(A, X) = \prod^c_{BSE} X_\varphi$ となる場合である。このような Banach module XをBSEと呼んでおり、これは multiplier の Gelfand 変換がいわゆるBSE-不等式：

$$\left| \sum_{i=1}^{n} f_i(\sigma(\varphi_i)) \right| \leq \|\sigma\|_{BSE} \left\| \sum_{i=1}^{n} f_i \circ \pi_{\varphi_i} \right\|_{X^*}$$

で完全に特徴付けられることを意味している。可換 Banach 環はそれ自身の上の Banach module と見ることができるが、それがBSEであるとき、BSE-環と呼ぶ。円板環、ハーディ環、群環、(可換) C*-環などはBSE-環の代表的なものである (cf. [7])。また擬中心的 C*-環はその中心上の BSE Banach module である。更にコンパクト可換群Gに対して、C(G), $L^p(G)$ $(1 \leq p \leq \infty)$, M(G) は皆 BSE-Banach $L^1(G)$-module である ([6])。

さて任意の可換 Banach 環Aをそれ自身の上の Banach module と見たとき、その誘導モジュールは $C_{BSE}(\Phi_A)$ と書かれるが、これは BSE-norm のもとで常に半単純可換 Banach 環である。従って、任意の可換 Banach 環A上の任意の Banach module Xの誘導モジュール $\prod^c_{BSE} X_\varphi$ もまた BSE-norm のもとで Banach A-module となることが期待されて良いだろう。これに対して Robbins は bundle：

$$\bigcup_{\varphi \in \Phi_A} \{\varphi\} \times X_\varphi \to \Phi_A$$

が Banach space bundle となり且つ任意の $\hat{T} \in \hat{M}(A, X)$ が連続となるような bundle topology が一意に存在し、更にAが有界近似単位元を持てば、この bundle topology が quotient topology に一致することを示して、$\prod^c_{BSE} X_\varphi$ がA-module となることを肯定的に解決した。然しながらこれが BSE-norm で完備であるかどうかはまだ open である。彼は単位区間上の有界 Borel 測度の全体のなす Banach $C([0, 1])$ − module を考察し、この誘導モジュールが sup-norm のもとで完備でないことを示しているが、これが BSE-norm で完備であるかどうかは不明である。また彼はその Banach space bundle のテクニックを用いて、筆者の結果のいくつかの

短い証明を与えている。

最後に有界近似単位元を持たない重要な可換 Banach 環は沢山あるが、これらの上の Banach module についての考察がなされたい。

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Cellular Automata in Function Spaces II

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1. Introduction

Cellular automata are discrete dynamical systems with simple construction and some of the limit sets show fractal patterns. Cellular automata are used as models for physical and biological phenomena [6, 10]. The existence of the limit set is studied by some people [5, 7] and its fractal dimension is also examined [7, 8, 9]. For non-linear cellular automata, chaotic phenomena occur [2, 3]. Linear cellular automata (LCA) are defined as

$$La(x) = \sum_{k\in G} \alpha_k a(x+k) \qquad (mod\ p) \qquad a \in \mathcal{P}^d, \tag{1.1}$$

where $\mathcal{P}^d$ is the set of all maps $a : \mathbb{Z}^d \to \mathbb{Z}/p$ with compact support and G is a finite subset of $\mathbb{Z}$ with $\sharp G \geq 2$ and $\alpha_k \in \{1, 2, \cdots, p-1\}$ and the summation $\sum$ is taken as the summation with mod p, in this paper. In case of $p = 2$, S. J. Willson [7] investigated the limit set of LCA. For $n \in \mathbb{N}$ and $a \in \mathcal{P}^d$, he consider the set

$$K(n,a) = \{(x,t) \in \mathbb{Z}^d \times \mathbb{N} \mid 0 \leq t \leq 2^n - 1,\ L^t a(x) = 1\},$$

which is obtained from the nth stage of LCA with the initial state $a \in \mathcal{P}^d$ and showed the existence of the limit set of LCA, where the limit set Y_a is considered as a subset of $\mathbb{R}^d \times [0,1]$ such that $Y_a = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n\geq k} \frac{K(n,a)}{p^n}}$ if the set $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n\geq k} \frac{K(n,a)}{p^n}}$ coincides with the set $\overline{\bigcup_{k=1}^{\infty} \bigcap_{n\geq k} \frac{K(n,a)}{p^n}}$. He also showed that the limit set does not depend on the initial state. F.v.Haeseler, H.-O.Peitgen and G.Skordev [1] studied the existence of a limit set of LCA by using matrix substitution system and hierarchical iterated function systems.

For a prime number $p \geq 2$, S. Takahashi [5] investigated the set

$$K(n,\delta_0) = \{(x,t) \in \mathbb{Z}^d \times \mathbb{N} \mid 0 \leq t \leq p^n - 1,\ L^t \delta_0(x) \neq 0\}$$

for $\delta_0 \in \mathcal{P}^d$ and $n \in \mathbb{N}$, where the value is paid attention on whether zero or nonzero. By using the set $K(n,\delta_0)$, he also defined the limit set as a subset of $\mathbb{R}^d \times [0,1]$ in the same way as $p = 2$ and showed the existence of the limit set Y_{δ_0} of $\frac{K(n,\delta_0)}{p^n}$ for special $\delta_0 \in \mathcal{P}^d$. For $a \in \mathcal{P}^d$ ($a \neq \delta_0$), the limit set of $\frac{K(n,a)}{p^n}$ is not considered. He also considered the existence of the limit set of $\frac{K(n,\delta_0)}{p^n}$ for any $p \in \mathbb{N}$. The set $\frac{K(n,\delta_0)}{p^n}$ has correspondence with $\mathbb{Z}/p$-valued upper semi continuous functions on $\mathbb{R}^d \times [0,1]$. In [4], we have investigated the limit function in case of $p = 2$.

We shall consider the case of a prime number $p \geq 2$. We consider $\mathbb{Z}/p$-valued upper semi continuous functions $T^n(\psi_0(a))$ instead of $\frac{K(n,\delta_0)}{p^n}$. We also consider the topology of convergence. We will prove that the limit set doesn't depend on the initial state for a prime number p (≥ 3). When p is not a prime number, it occurs that the limit function takes more than two values. So if we apply this function theory to the case of non prime number p, we would get more interesting conclusion than treated as the subset of $\mathbb{R}^d \times [0,1]$.

2. The limit function of linear cellular automata

Let p be a prime number and let $\mathcal{P}^d$ be the set of all configurations $a : \mathbb{Z}^d \to \mathbb{Z}/p$ with compact support. We define $\delta_0 \in \mathcal{P}^d$ as

$$\delta_0(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Let USC be the space of all $\mathbb{Z}/p$-valued upper semi continuous functions $g : \mathbb{R}^d \times [0,1] \to \mathbb{Z}/p$ with compact support. The space USC is an order complete lattice, that is, for any $\{f_n\}_n \subset USC$, there exist functions $\bigvee f_n$ and $\bigwedge f_n$ in USC such that

$$\bigvee_{n\geq 1} f_n(x,t) = \inf\{\phi(x,t) | \phi \in USC, \phi(x,t) \geq f_n(x,t) \text{ for any } n \in N\}$$

and

$$\bigwedge_{n\geq 1} f_n(x,t) = \inf\{f_n(x,t) | n \in N\}.$$

We define an operator $T : USC \to USC$ by

$$Tg(x,t) = \sum_{j=0}^{p-1} \sum_{l \in G_j} L^j\delta_0(l) g(S_{l,j}^{-1}(x,t)) \quad \text{for } g \in USC, \tag{2.1}$$

where $G_j = \{l \in \mathbb{Z} \,|L^j\delta_0(l) \neq 0\}$ and $S_{l,j}(x,t) = (\frac{x}{p}, \frac{t}{p}) + (\frac{l}{p}, \frac{j}{p})$.

For $n \in \mathbb{N}$ let $X_n = \{(\frac{x}{p^n}, \frac{t}{p^n}) \in \mathbb{R}^d \times [0,1] | x \in \mathbb{Z}^d, t \in \mathbb{N}, 0 \leq t \leq p^n - 1\}$ and define $\psi_n : \mathcal{P}^d \to USC$ by

$$(\psi_n a)(\frac{x}{p^n}, \frac{t}{p^n}) = \begin{cases} L^t a(x) & \text{if } (\frac{x}{p^n}, \frac{t}{p^n}) \in X_n \\ 0 & \text{if } (\frac{x}{p^n}, \frac{t}{p^n}) \in (\mathbb{R}^d \times [0,1]) \setminus X_n \end{cases}$$

for $a \in \mathcal{P}^d$.

For a configuration $a \in \mathcal{P}^d$, put $l(a) = \min\{i | a(i) \neq 0\}$ and $u(a) = \max\{i | a(i) \neq 0\}$. Then we have the following proposition concerning T.

Proposition 1. *Suppose $a \in \mathcal{P}$ is nonzero and $l(a) = 0$. Then*

(i) *for $(x,t) \in \mathbb{R}^d \times [0,1]$, there exists $\lim_{n\to\infty} T^n(\psi_0(a))(x,t)$.*

(ii) *For $a \in \mathcal{P}^d$, put $g_1(x,t) = \lim_{n\to\infty} T^n(\psi_0(a))(x,t)$. Then $Tg_1 = g_1$, that is, g_1 is an T-invariant function.*

(iii) *For $(x,t) \in \mathbb{R}^d \times [0,1]$, put $g_0(x,t) = \lim_{n\to\infty} T^n(\psi_0(\delta_0))(x,t)$. Then $a(0)g_0 = g_1$.*

We will consider the convergence of $T^n(\psi_0(\delta_0))$ to g_0 using the distances d_f and D_f (which are defined in the next section) and investigate the relation among g_0, g_1 and Y_{δ_0} in section 4.

3. Metrics in the space *USC*

By using the Hausdorff metric $D(A,B)$ of compact sets A and B, we shall introduce two metrics d_f, D_f in USC for considering the convergence to the limit set as follows:

$$\begin{aligned} d_f(g_1,g_2) &= \max_{1\leq j\leq p-1} D(\overline{g_1^{-1}(j)},\overline{g_2^{-1}(j)}), \\ D_f(g_1,g_2) &= \max_{1\leq s\leq p-1} D(g_1^{-1}[s+],g_2^{-1}[s+]), \end{aligned}$$

for $g_1, g_2 \in USC$, where $g^{-1}[s+] = \{(x,t) | g(x,t) \geq s\}$ and $\overline{g_1^{-1}(j)}$ is the closure of the set $g_1^{-1}(j)$. We have the following theorem concerning d_f and D_f.

Theorem 2. *For* $f_n \in USC$, *suppose* $d_f(f_n, f_m) \to 0$ *as* $n, m \to \infty$. *Let* $g = \bigwedge_{k=1}^{\infty} \bigvee_{n\geq k} f_n$. *Then we have*

$$D_f(f_n, g) \to 0 \text{ as } n \to \infty.$$

4. Convergence of $T^n g$

We will consider the case of one dimensional cellular automata ($d = 1$). We first rewrite an operator $T : USC \to USC$ as follows:

Let α_k be defined in (1.1). Put $k_- = \min\{k : \alpha_k \neq 0\}$, $k_+ = \max\{k : \alpha_k \neq 0\}$ and $k_0 = k_+ - k_-$. Let $N = p + \frac{p(p-1)}{2}k_0$. For $1 \leq l \leq N$ there exists a unique integer v_l such that

$$v_l + \frac{v_l(v_l-1)}{2}k_0 < l \leq v_l + 1 + \frac{v_l(v_l+1)}{2}k_0.$$

Let

$$\begin{aligned} u_l &= l - v_l - \frac{v_l(v_l-1)}{2}k_0 - v_l k_+ - 1, \\ c_l &= L^{v_l}\delta_0(u_l), \end{aligned}$$

$$S_l(x,t) = (\frac{x}{p}, \frac{t}{p}) + (\frac{u_l}{p}, \frac{v_l}{p}) \tag{4.1}$$

and

$$Tg(x,t) = \sum_l c_l g(S_l^{-1}(x,t)) \quad \text{for } g \in USC, \tag{4.2}$$

where the summation $\sum_l$ is taken over l satisfying $(x,t) \in S_l(\mathbb{R} \times [0,1])$ with mod p. Then it equals to the definition of T in section 2. So we use this definition of T to prove the following lemmas and propositions.

By Theorem 2, we shall show $\{T^n(\psi_0(\delta_0))\}_n$ is a Cauchy sequence w.r.t. d_f in order to consider the convergence to the limit set of LCA. So we define some kind of distance $M_0^{n,n'}$ of ψ_n and $\psi_{n'}$. In order to define the distance $M_0^{n,n'}$, we prepare some notations. We first define the disjoint partition $\{E_\gamma^n\}_\gamma$ of *supp* $\psi_n(\delta_0)$, where $\{S_l(\bigcup_{\gamma\in\Gamma} E_\gamma^n)\}_{l=1}^N$ are either disjoint or the union of E_γ^{n+1}(Proposition 3).

Put $n_0 = \min\{n | \ k_0 < p^n\}$. For $s \in \{0, \cdots, pk_0\}$, put $t_s^{n_0} = [\frac{sp^{n_0}-1+k_0}{k_0}]$, where [ ] is the Gauss's symbol. For $s \in \{0, \cdots, pk_0\}$, there exist $l_s \in \{0, \cdots, p^2\}$ and $r_s \in \{0, \cdots, k_0 - 1\}$ such that

$$sp = l_s k_0 + r_s. \tag{4.3}$$

For $n \geq n_0 + 1$ and $s \in \{0, \cdots, pk_0\}$, define t_s^n inductively as follows:

$$t_s^n = l_s p^{n-1} + t_{r_s}^{n-1}. \tag{4.4}$$

Divide the region including $supp\ \psi_n(\delta_0)$ into disjoint subsets $\{A^n_{i,j,s}\}_{i,j,s}$ or $\{E^n_\gamma\}_\gamma$, which satisfy Proposition 3 as follows:

For $n \geq n_0$, $1 \leq s \leq k_0$ and $1 \leq j \leq s$, put

$$A^n_{1,j,s} = \bigcup_{i=t^n_{s-1}}^{t^n_s - 1} \{(x + j - 1, t) | \frac{i}{p^n} \leq t < \frac{i+1}{p^n},\ \frac{k_- i}{p^n} \leq x < \frac{k_+ i + 1}{p^n} - (s-1)\}$$

and for $n \geq n_0$, $2 \leq s \leq k_0$ and $1 \leq j \leq s - 1$, put

$$A^n_{2,j,s} = \bigcup_{i=t^n_{s-1}}^{t^n_s - 1} \{(x + j - 1, t) | \frac{i}{p^n} \leq t < \frac{i+1}{p^n},\ \frac{k_+ i + 1}{p^n} - (s-1) \leq x < \frac{k_- i}{p^n} + 1\}.$$

Put

$$\Gamma = \{(1, j, s) \mid 1 \leq s \leq pk_0,\ \ 1 \leq j \leq s\} \bigcup \{(2, j, s) \mid 2 \leq s \leq pk_0,\ \ 1 \leq j \leq s - 1\}.$$

For $n \geq n_0 + 1$ and $\gamma = (1, j, s) \in \Gamma$, put

$$E^n_\gamma = \bigcup_{i=t^{n-1}_{s-1}}^{t^{n-1}_s - 1} \{(x + j - 1, t) | \frac{i}{p^n} \leq t < \frac{i+1}{p^n},\ \frac{k_- i}{p^n} \leq x < \frac{k_+ i + 1}{p^n} - (s-1)\}$$

and for $n \geq n_0 + 1$ and $\gamma = (2, j, s) \in \Gamma$, put

$$E^n_\gamma = \bigcup_{i=t^{n-1}_{s-1}}^{t^{n-1}_s - 1} \{(x + j - 1, t) | \frac{i}{p^n} \leq t < \frac{i+1}{p^n},\ \frac{k_+ i + 1}{p^n} - (s-1) \leq x < \frac{k_- i}{p^n} + 1\}.$$

Then we have

Proposition 3. *Put* $E^n = \bigcup_{\gamma \in \Gamma} E^n_\gamma$ *. Then*

(i) $supp\ \psi_n(\delta_0) \subset E^n$.

(ii) $E^n_\gamma \cap E^n_{\gamma'} = \phi$ *if* $\gamma \neq \gamma'$.

(ii) *If* $S_l(E^n) \cap S_{l'}(E^n) \neq \phi$, *then there exists* $\gamma_1, \cdots, \gamma_k \in \Gamma (k \geq 1)$ *such that*

$$S_l(E^n) \bigcap S_{l'}(E^n) = \bigcup_{i=1}^{k} E^{n+1}_{\gamma_i}.$$

(iv) *Each* $S_l^{-1}(E^n_\gamma)$ *corresponds to some* $A^{n-1}_{i,j,s}$.

Moreover we define the sets I_γ, V and a function h_v. For $\gamma = (b, j, s) \in \Gamma$, put

$$I_\gamma = \{(l, i) \mid 1 \leq e_s \leq k_0 \text{ with } v_l k_0 + e_s = s,\ S_l(A^{n-1}_{b,i,e_s}) = E^n_{b,j,s}\}.$$

Let

$$V = \{v = \{\gamma_1, \cdots, \gamma_m\} \quad | \quad m \in \mathbf{N}, \gamma_k \in \Gamma, I_{\gamma_k} \neq \phi,\ (l_k^o, i_k^o) \in I_{\gamma_k}, E_{\gamma_{k+1}}^n \subset S_{l_k^o}^{-1}(E_{\gamma_k}^{n+1}) \\ \text{for any } k \in \{1, \cdots, m\} \text{ and any } n \in \mathbb{N}\},$$

where $(l_k^o, i_k^o) \in I_{\gamma_k}$ is taken such that $i_k^o \leq i_k$ for any $(l_k, i_k) \in I_{\gamma_k}$. If $E_{\gamma_{k+1}}^n \subset S_{l_k^o}^{-1}(E_{\gamma_k}^{n+1})$ holds for some $n \in \mathbb{N}$, then it holds for all $n \in \mathbb{N}$. For $v = \{\gamma_1, \cdots, \gamma_m\} \in V$, define h_v^n by

$$h_v^n(x,t) = \sum_{(l_1,i_1)\in I_{\gamma_1}} \cdots \sum_{(l_m,i_m)\in I_{\gamma_m}} c_{l_1} \cdots c_{l_m} \psi_n(\delta_0)(x + \sum_{k=1}^{m} p^{m-k}(i_k - i_k^o), t) 1_{S_{l_m^o}^{-1}(E_{\gamma_m}^{n+1})}(x,t).$$

Put

$$M_0^{n,n'} = \max\{d_f(h_v^n, h_v^{n'}) \ |v \in V\}.$$

Then we have the following proposition.

Proposition 4. $d_f(\psi_{n+1}(\delta_0), \psi_{n'+1}(\delta_0)) \leq \frac{1}{p} M_0^{n,n'}$.

Proposition 5. *For a sufficiently large* n_1,

$$M_0^{n_1,n_1+1} < \infty.$$

Proposition 6. *For* $n, n' \geq 2$,

$$M_0^{n+1,n'+1} \leq \frac{1}{p} M_0^{n,n'}.$$

Using these propositions, we have

Theorem 7. *Let* T *be defined as* (2.1). *Then*

(i) $d_f(T^n(\psi_0(\delta_0)), T^m(\psi_0(\delta_0))) \to 0$ *as* $n, m \to \infty$.

(ii) *Put* $f_0 = \bigwedge_{k\geq 1} \bigvee_{n \geq k} T^n(\psi_0(\delta_0))$, *where* $\bigwedge$ *and* $\bigvee$ *are lattice operations in* USC. *Then we have*

$$D_f(T^n(\psi_0(\delta_0)), f_0) \to 0 \qquad as \qquad n \to \infty.$$

We will get the following theorem in a similar way to Theorem 7.

Theorem 8. *Let* T *be defined as* (2.1) *and* $a \in \mathcal{P}$ *be nonzero. Then*

(i) $d_f(T^n(\psi_0(a)), T^m(\psi_0(a))) \to 0$ *as* $n, m \to \infty$.

(ii) *Put* $f_a = \bigwedge_{k\geq 1} \bigvee_{n\geq k} T^n(\psi_0(a)) \in USC$. *Then we have*

$$D_f(T^n(\psi_0(a)), f_a) \to 0 \qquad as \qquad n \to \infty.$$

We will consider the relation among f_0 in Theorem 7, f_a in Theorem 8 and the limit set $Y_a = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n\geq k} \frac{K(n,a)}{p^n}}$. Let $\hat{g}$ be the upper envelope of g, that is,

$$\hat{g}(x,t) = \inf\{\phi(x,t) | \phi \in USC, \phi(x,t) \geq g(x,t)\}.$$

Theorem 9. *For* $a \in \mathcal{P}$, *let* $Y_a = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} \overline{\frac{K(n,a)}{p^n}}$ *and* g_a *be defined by* $g_a(x,t) = \lim_{n \to \infty} T^n(\psi_0(a))(x,t)$.

(i)*For* $\delta_0 \in \mathcal{P}$, *the characteristic function* $1_{Y_{\delta_0}}$ *of the set* Y_{δ_0} *satisfies* $(p-1)1_{Y_{\delta_0}} = \hat{g}_{\delta_0}$ *and* $\hat{g}_{\delta_0} = \bigwedge_{k \geq 1} \bigvee_{n \geq k} T^n(\psi_0(\delta_0))$.

(ii) *For any* $a \in \mathcal{P}$, *we have*

$$\bigwedge_{k \geq 1} \bigvee_{n \geq k} T^n(\psi_0(\delta_0)) = \bigwedge_{k \geq 1} \bigvee_{n \geq k} T^n(\psi_0(a)).$$

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Chaotic semigroups of linear operators

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1 Introduction

The hypercyclic or chaotic operator is consistent with the topologically transitive or chaotic, respectively in a topological space defined by Devaney [2]. The property of hypercyclic and chaotic operators has been studied by some people [1, 3, 4, 6]. In [6], the theory of hypercyclic and chaotic bounded linear operators has been developed in connection with the invariant subspace problem of Hilbert spaces. In [1], it is given a necessary and sufficient condition for the semigroup to be hypercyclic in a separable Banach space. Concerning to the chaotic semigroup, a necessary condition for the semigroup is given in a separable Banach space. In [5], chaotic semigroups are associated with the idea of exactness and applied to partial differential equations. In this paper, we investigate necessary and sufficient conditions for translation semigroups to be chaotic in weigted function spaces and give an example which shows that some solutions of partial differential equations become chaotic semigroups.

2 Preliminaries

Let X be a Banach space and $\{T(t)\}$ be a strongly continuous semigroup in X. The semigroup $\{T(t)\}$ is called *hypercyclic* if there exists $x \in X$ such that $\{T(t)x \mid t > 0\}$ is dense in X. The semigroup $\{T(t)\}$ is called *chaotic* if $\{T(t)\}$ is hypercyclic and the set of periodic points $X_p = \{x \in X \mid \exists t > 0 \ s.t. \ T(t)x = x\}$ is dense in X.

Let I be $(-\infty, \infty)$ or $I = [0, \infty)$. By *an admissible weight function* on I we mean a measurable function $\rho : I \to \mathbb{R}$ satisfying the following conditions:

(i) $\rho(\tau) > 0 \quad$ for all $\tau \in I$;

(ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(\tau) \leq Me^{\omega t}\rho(t+\tau)$ for all $\tau \in I$ and for all $t > 0$.

With an admissible weight function, we construct the following weighted function spaces.

$$L^p_\rho(I, \mathbb{C}) = \left\{ u : I \to \mathbb{C} \mid u \textit{ measurable}, \int_I |u(\tau)|^p \rho(\tau)\ d\tau < \infty \right\}$$

$$\text{with } \|u\| = \left(\int_I |u(\tau)|^p \rho(\tau)\ d\tau \right)^{\frac{1}{p}},$$

$$C_{0,\rho}(I, \mathbb{C}) = \left\{ u : I \to \mathbb{C} \mid u \textit{ continuous}, \lim_{\tau \to \pm\infty} \rho(\tau)u(\tau) = 0 \right\}$$

$$\text{with } \|u\| = \sup_{\tau \in I} |u(\tau)|\rho(\tau).$$

In the above spaces, we consider the family of following operators $\{T(t)\}$ with the parameter $t > 0$ as a translation semigroup:

$$[T(t)u](\tau) = u(\tau + t) \qquad \text{for } u \in C_{0,\rho}(I) \text{ or } L^p_\rho(I).$$

When $\rho(\tau) = 1$, weighted function spaces are equal to L^p or C_0 and the hypercyclicity of the translation semigroup doesn't occur, since the norm of $T(t)$ is equal to 1 for all $t \geq 0$ in L^p or C_0. However, if $\rho(\tau) \neq 1$, hypercyclic or chaotic phenomena may occur. So we shall investigate a hypercyclic or chaotic translation semigroup in L^p_ρ or $C_{0,\rho}$. A condition for the translation semigroup to be hypercyclic in L^p_ρ or $C_{0,\rho}$ is given in [1] as follows.

Theorem A [1]. *Let X be one of the spaces $L^p_\rho(I)$ or $C_{0,\rho}(I)$ with an admissible weight function ρ. Then the following (1) and (2) are equivalent.*

(1) The translation semigroup $\{T(t)\}$ in X is hypercyclic.

(2) (i) If $I = [0, \infty)$, then $\liminf_{t\to\infty} \rho(t) = 0$ holds.
(ii) If $I = (-\infty, \infty)$, then for each $\theta \in \mathbb{R}$ there exists a sequence $\{t_j\}_{j=1}^{\infty}$ of positive real numbers such that

$$\lim_{j\to\infty} \rho(t_j + \theta) = \lim_{j\to\infty} \rho(-t_j + \theta) = 0.$$

3 Main results

We give necessary and sufficient conditions for the translation semigroup to be chaotic in weighted function spaces. We also show that if the set X_p is dense in X, the semigroup $\{T(t)\}$ becomes hypercyclic.

The necessary and sufficient condition in Theorem A for the translation semigroup to be hypercyclic depends on whether $I = [0, \infty)$ or $I = (-\infty, \infty)$, but does not depend on whether X is $L^p_\rho(I)$ or $C_{0,\rho}(I)$. As for the chaotic condition, it depends on whether X is $L^p_\rho(I)$ or $C_{0,\rho}(I)$, but does not depend on whether $I = [0, \infty)$ or $I = (-\infty, \infty)$ as shown below.

Theorem 1. *Let X be $L^p_\rho(I)$, where $I = (-\infty, \infty)$ (resp.$I = [0, \infty)$). The translation semigroup $\{T(t)\}$ is chaotic if and only if for all $\epsilon > 0$ and for all $l > 0$, there exist $P > 0$ such that*

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \rho(l + nP) < \epsilon \quad (resp. \sum_{n=1}^{\infty} \rho(l + nP) < \epsilon).$$

The above condition is stronger than "$\lim_{\tau\to\infty} \rho(\tau) = 0$" from the property of the admissible weight function.

The next theorem is the case of $C_{0,\rho}(I)$.

Theorem 2. *Let X be $C_{0,\rho}(I)$, where $I = (-\infty, \infty)$ (resp. $I = [0, \infty)$). Then the following assertions are equivalent:*

(i) *the translation semigroup $\{T(t)\}$ in X is chaotic;*

(ii) *for all $\epsilon > 0$ and for all $l > 0$, there exists $P > 0$ such that $\rho(l+nP) < \epsilon$ for any $n \in \mathbb{Z} \setminus \{0\}$ (resp. $n \in \mathbb{N}$);*

(iii) *there exists $\{l_i\}_{i=1}^{\infty} \subset \mathbb{R}^+$ whose limit is infinity, such that for all $\epsilon > 0$ and for all $i \in \mathbb{N}$, there exists $P > 0$ such that $\rho(l_i + nP) < \epsilon$, for all $n \in \mathbb{Z} \setminus \{0\}$(resp.$n \in \mathbb{N}$).*

The equivalence between (i) and (ii) is obtained by the property of the admissible weight function. The condition (iii) is weaker than the condition "$\lim_{\tau\to\infty} \rho(\tau) = 0$". In fact, there exist admissible weight functions which satisfy the condition (iii) and whose limit is not 0.

So we give the equivalent condition to $\lim_{\tau\to\infty} \rho(\tau) = 0$.

Theorem 3. *Let I be $(-\infty, \infty)$ (resp.$I = [0, \infty)$), and let X be $C_{0,\rho}(I)$. Then for a translation semigroup $\{T(t)\}$, the following conditions are equivalent:*

(i) $\lim_{\tau\to\pm\infty}\rho(\tau)=0$ *(resp.*$\tau\to\infty$*);*

(ii) $\{T(t)\}$ *is chaotic. In addition, for all* $\epsilon>0$ *and for all* $x\in X$ *there exists* $t_0>0$ *such that, for all* $t\geq t_0$ *there exists* $v_t\in X_p$ *such that*

$$\|x-v_t\|<\epsilon \quad \text{and} \quad T(t)v_t=v_t.$$

The next theorem means that the hypercyclicity comes from the denseness of the set of periodic points when I is a half line.

Theorem 4. *Let* I *be* $[0,\infty)$ *and* X *be* $L^p_\rho(I)$ *or* $C_{0,\rho}(I)$. *Then the set of periodic points* X_p *is dense in* X *if and only if* $\{T(t)\}$ *is chaotic.*

The next example is the application of Theorem 3.

Example. Let $C_0([0,\infty))$ be the set of continuous functions on $[0,\infty)$ which vanish at infinity with sup norm. We shall consider the following partial differential equation on the space $C_0([0,\infty))$, :

$$\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial u}{\partial x}-\omega u & \omega<0 \\ u(0,x)=f(x) & f\in C^1_0([0,\infty)). \end{cases}$$

Then the solution is

$$u(t,x)=e^{-\omega t}f(x+t).$$

So if we define an operator $T(t)$ on $C_0([0,\infty))$ by $T(t)f(x)=u(t,x)$, then $\{T(t)\}$ becomes a chaotic semigroup.

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Two Yorke's conjectures on chaotic bifurcations

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Abstract

We discuss problems of non-monotone bifurcations for two cases of one-parameter families: real quadratic rational maps and real cubic polynomials. We present counter examples by computer experiments to the monotonicity conjecture and the antimonotonicity conjecture.

1 Introduction

System of iterated maps, viewed as real dynamical systems is considered as an important model for the chaotic behavior in certain parameterized systems. Creation and annihilation of periodic orbits is one of the most fundamental bifurcation processes, often illustrated by the pitchforks oriented either one-way or both-ways.

We discuss in this paper some topics from the bifurcation problems for a one parameter real family of quadratic rational maps or of cubic polynomials. J. Milnor and W. Thurston ([11]) proved by using Teichmüller theory that the logistic family $\{\lambda x(1-x)\ ;\ \lambda \in [1,4]\}$, which is a family of simple maps with extremely complicated dynamics, has only orbit-creation parameter values and no orbit-annihilation values as the parameter increases. Unlike monotonicity of the logistic family, however, there exist many one-parameter families exhibiting a non-monotone orbit-bifurcation structure, namely the pitchforks oriented both-ways.

We discuss monotonicity conjecture (M) indicated in several papers, now reformulated by [18] as follows:

> (M) *Let $f_m(x) = mf(x)$ be a one-parameter family of differential maps from closed interval I_m into itself which satisfies the following properties:* (1) *f_m is concave on I_m,* (2) *the set of periodic points of f_1 consists of two fixed points,* (3) *f_m has a negative schwarzian derivative. As the parameter m is increased, this one-parameter family is monotone.*

We consider a family $\{mf(x)\}$, where $f(x) = r + \frac{x}{1+x^2}$. The bifurcation diagram of this family can be monotone, non-monotone, or antimonotone according to the choice of the function f, namely the choice of r (cf. [6]). To the monotonicity conjecture, we will give a counter example using the defining equation of the lower escape locus, obtained in the section 2.1.

Next, we present an example to the antimonotonicity conjecture (A), enounced in the paper ([2]) with their heuristic argument and numerical evidence:

> (A) *A smooth one-dimensional map depending on one parameter has an antimonotone parameter value whenever at least two independent critical points are contained in the interior of a chaotic attractor.*

Hereafter we call the part *"at least two independent critical points are contained in the interior of a chaotic attractor"*, anti-condition:(Anti). To construct a one-parameter family under (Anti), having no antimonotone parameter value, we use an algebraic curve, so-called center curve defined in our papers ([16], [4]), in the moduli space of the cubic maps with the multiplier-coordinates system.

Our method of approach to a bifurcation problem is to analyze an algebraic curve, defined by one-parameter family in the moduli space associated of a family, e.g., we examine "which hyperbolic locus does the curve lie in?" or "which dynamical curves does the curve intersect with?"

2 Moduli space of quadratic rational maps

Let $\overline{\mathbf{C}}$ be the Riemann sphere and $\mathrm{Rat}_2(\mathbf{C})$ the space of all quadratic rational maps from $\overline{\mathbf{C}}$ to itself. The group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations acts on the space $\mathrm{Rat}_2(\mathbf{C})$ by conjugation, $g \circ f \circ g^{-1} \in \mathrm{Rat}_2(\mathbf{C})$ for $g \in \mathrm{PSL}_2(\mathbf{C})$, $f \in \mathrm{Rat}_2(\mathbf{C})$. The quotient space of $\mathrm{Rat}_2(\mathbf{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbf{C})$, and called the moduli space of holomorphic conjugacy classes $\langle f \rangle$ of quadratic rational maps f. The multipliers coordinates are introduced in $\mathcal{M}_2(\mathbf{C})$. For each $f \in \mathrm{Rat}_2(\mathbf{C})$, let z_1, z_2, z_3 be the fixed points of f and μ_i the multipliers of z_i; $\mu_i = f'(z_i)$ $(1 \leq i \leq 3)$. Consider the elementary symmetric functions of the three multipliers, $\sigma_1 = \mu_1 + \mu_2 + \mu_3$, $\sigma_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1$, $\sigma_3 = \mu_1\mu_2\mu_3$, which are subject only to the restriction that $\sigma_3 = \sigma_1 - 2$. Hence the moduli space $\mathcal{M}_2(\mathbf{C})$ is canonically isomorphic to $\mathbf{C}^2$ (Lemma 3.1 in [10]). Let $\mathrm{Rat}_2(\mathbf{R})$ be the set of real quadratic rational maps. We remark that the real moduli space $\mathcal{M}_2(\mathbf{R})$ for $\mathrm{Rat}_2(\mathbf{R})$ is the real cut of $\mathcal{M}_2(\mathbf{C})$ (see [6]).

By an automorphism of a quadratic rational map f, we will mean $g \in \mathrm{PSL}_2(\mathbf{C})$ which commutes with f. The collection $\mathrm{Aut}(f)$ of all automorphisms of f forms a finite group. Since $\mathrm{Aut}(\tilde{f})$ is isomorphic to $\mathrm{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$, the set

$$\mathcal{S} = \{\langle f \rangle\, ;\ \mathrm{Aut}(f) \text{ is non-trivial}\} \subset \mathcal{M}_2(\mathbf{C})$$

is defined and called the symmetry locus.

For each $\mu \in \mathbf{C}$, let $\mathrm{Per}_1(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps f having a fixed point with multiplier μ. Each of $\mathrm{Per}_1(\mu)$ forms a straight line as follows:

$$\mathrm{Per}_1(\mu) = \left\{\langle f \rangle \in \mathcal{M}_2(\mathbf{C}); \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1})\right\}$$

(Lemmas 3.4 and 3.6 in [10]).

Topological partition

For map $f \in \mathrm{Rat}_2(\mathbf{R})$, the two critical points of f are two real numbers or a pair of complex conjugate numbers. If f has a pair of complex conjugate critical points, this map is two-to-one covering map on $S^1 = \mathbf{R} \cup \{\infty\}$. In this case, if $f' > 0$ then f is called the map of degree $+2$, else $f' < 0$ then the map of degree -2.

While a map f with real critical points is called monotone (resp. unimodal, bimodal) if the interval $I = int(f(S^1))$ contains no (resp. one, two) critical points ([10]).

2.1 Real slices of hyperbolic escape locus

A rational map is hyperbolic if and only if the orbit of every critical point converges to some attracting periodic orbit. The hyperbolic maps form an open subset of moduli space, and the connected components of this open set are called hyperbolic components. M. Rees ([19]) shows that the hyperbolic components can be divided into four classes,Type B: Bitransitive, Type C: Capture, Type D: Disjoint attractors, and Type E: Escape. The names are due to J. Milnor ([10]).

Type E: Escape. Both critical orbits converge to the same attracting fixed point. There is just one such hyperbolic component.

In the complex case the escape locus is connected. But the real cut of this component splits into two parts; the upper part and the lower part. The boundary curve of the upper part is given by Milnor (Caption of Figure 16 in [10]).

Now, we specify the lower boundary. Proof is given in [13] and [7]. This boundary curve will play a key role in our later discussions of section 3.

Theorem 1 *Escape loci on the real moduli space is the union of the following sets;*

$$\{\sigma_2 > -2\sigma_1 + 1, \quad \sigma_2 > 2\sigma_1 - 3\}, \ \{\sigma_2 < 2\sigma_1 - 3, \quad \sigma_1 < -1\},$$
$$\{\sigma_2 < \tfrac{-2\sigma_1^2 - 7\sigma_1 - 10}{2+\sigma_1}, \quad \sigma_1 \geq -1\}.$$

3 Bifurcations

Let $\{f_\lambda\}_\Lambda$ be a one-parameter family of discrete dynamical systems on $\mathbf{R}$ where Λ is an interval of $\mathbf{R}$. As the parameter increases, a parameter value λ_0 is called orbit creating if, at λ_0, new periodic orbits are created and no periodic orbits are annihilated; λ_0 is called orbit annihilating if periodic orbits are annihilated and no new periodic orbits are created; λ_0 is called neutral if no periodic orbits are annihilated and no periodic orbits are created.

A family $\{f_\lambda\}_\Lambda$ is said to be monotone increasing (resp. decreasing) if every parameter value in Λ is neutral or orbit creating (resp. annihilating). A family $\{f_\lambda\}_\Lambda$ is called non-monotone if Λ contains both orbit creating and orbit annihilating parameter values. A family $\{f_\lambda\}_\Lambda$ is called antimonotone if any neighborhood of a suitable parameter λ_0 in Λ contains both infinitely many orbit creating and orbit annihilating parameter values.

4 Counter examples

4.1 Counter example to the Monotonicity conjecture

In this section we shall present a counter example, which is a one parameter family of quadratic rational maps, to the monotonicity conjecture enounced in the paper [18].

4.1.1 Monotone and non-monotone bifurcations of quadratic rational families

Now, we investigate the dynamics of a certain real 2-parameter family given by M. Bier and T. C. Bountis [1] and rewritten by H. E. Nusse and J. A. Yorke ([18]):

$$\left\{ f_{m,r}(x) = m\left(r + \frac{x}{1+x^2}\right)\right\}_{(m,r)\in\mathbf{R}^2}.$$

Here the map $f_{0,r}(x)$ should be thought of as an ideal limit map, in the natural compactification of $\mathcal{M}_2(\mathbf{C})$ (cf. [9]), of quadratic rational maps which degenerate towards the constant zero map. Then it makes sense to discuss the bifurcations of this family including the parameter value $m = 0$, though in the real moduli space $\mathcal{M}_2(\mathbf{R})$ the maps diverge to infinity according as $m \to \pm 0$. Since the maps $f_{m,r}$ and $f_{m,-r}$ are conjugate to each other for any r, it suffices to consider the case $r \geq 0$.

Theorem 2 *In $\mathcal{M}_2(\mathbf{R})$, the one-parameter family $\{f_{m,r}(x)\}_m$ for each fixed r ($r \geq 0$) lies exactly on an irreducible algebraic curve $\mathcal{H}_r$:*

For $r \neq \frac{1}{2}, 0$, the curve $\mathcal{H}_r$ is of degree 4 defined by the equation

$$\begin{aligned} H_r(\sigma_1, \sigma_2) \;=\; & -r^2\sigma_1^4 + (8r^2-2)\sigma_1^3 + ((8r^2-1)\sigma_2 - 128r^4 + 8r^2 + 1)\sigma_1^2 + ((-32r^2+8)\sigma_2 \\ & +512r^4 - 96r^2 - 12)\sigma_1 + (-16r^2+4)\sigma_2^2 + (512r^4 - 96r^2 - 12)\sigma_2 \\ & -4096r^6 + 1536r^4 - 144r^2 + 36 = 0. \end{aligned} \tag{1}$$

For $r = \frac{1}{2}$ or $r = 0$, the curve $\mathcal{H}_r$ is of degree 3.

The proof is given in our paper [6].

Example (Antimonotone) Consider the one-parameter family defined on a suitable interval I_m, $F_m(x) = m\frac{x^2+ax+b}{1+x^2}$, where constant a is the positive root of the following equation $49a^2 - 32 = 0$, and b is the unique positive root of the following equation

$$117649b^7 + 684285b^6 + 1721517b^5 + 2358566b^4 + 1670655b^3 + 991301b^2 - 257125b = 0.$$

It is clear that this family satisfies the conditions of monotonicity conjecture (M), namely, (1) each F_m is concave, (2) the set of periodic points in I_1 of F_1 consists of two fixed points, and (3) F_m has a negative schwarzian derivative.

In this moduli space, a defining equation of the algebraic curve defined by $\{F_m\}_m$ is given as follows;

$$\begin{aligned} S_{a,b} = \; & (2\sigma_1^3 + (\sigma_2-1)\sigma_1^2 + (-8\sigma_2+12)\sigma_1 - 4\sigma_2^2 + 12\sigma_2 - 36)a^6 + ((2\sigma_1^3 + (\sigma_2+24)\sigma_1^2 + (12\sigma_2+72)\sigma_1 + \\ & 36\sigma_2)b^2 + (-14\sigma_1^3 + (-6\sigma_2-20)\sigma_1^2 + (32\sigma_2+24)\sigma_1 + 16\sigma_2^2 + 24\sigma_2 + 144)b + \sigma_1^4 + 4\sigma_1^3 + (-3\sigma_2 - \\ & 12)\sigma_1^2 - 12\sigma_2\sigma_1 + 36\sigma_2)a^4 + ((-10\sigma_1^3 + (-4\sigma_2 - 132)\sigma_1^2 + (-48\sigma_2 - 504)\sigma_1 - 144\sigma_2 - 432)b^3 + \\ & (2\sigma_1^4 + 46\sigma_1^3 + (4\sigma_2 + 188)\sigma_1^2 + (-16\sigma_2 - 216)\sigma_1 - 240\sigma_2 - 720)b^2 + (-4\sigma_1^4 - 30\sigma_1^3 + (4\sigma_2 + \\ & 84)\sigma_1^2 + (48\sigma_2 + 152)\sigma_1 - 112\sigma_2 - 336)b + 2\sigma_1^4 - 6\sigma_1^3 + (-4\sigma_2 - 12)\sigma_1^2 + (16\sigma_2 + 56)\sigma_1 - 16\sigma_2 - \\ & 48)a^2 + (\sigma_1^4 + 24\sigma_1^3 + 216\sigma_1^2 + 864\sigma_1 + 1296)b^4 + (-4\sigma_1^4 - 64\sigma_1^3 - 288\sigma_1^2 + 1728)b^3 + (6\sigma_1^4 + 48\sigma_1^3 - \\ & 48\sigma_1^2 - 576\sigma_1 + 864)b^2 + (-4\sigma_1^4 + 96\sigma_1^2 - 256\sigma_1 + 192)b + \sigma_1^4 - 8\sigma_1^3 + 24\sigma_1^2 - 32\sigma_1 + 16 = 0. \end{aligned}$$

We remark that this curve tangent to a boundary curve of the lower locus of escape. Then we see this family is antimonotone at this tangent point [3].

4.2 Antimonotonicity conjecture

In this section we shall present an example, which is a one parameter family of cubic polynomials, to the antimonotonicity conjecture enounced in the paper [2]. The one-parameter family $f_\lambda(x) = -x^3 + 1.2675x - \lambda$, defined in [2], is antimonotone under (Anti). It turns out that this family exactly on a half line $\sigma_1 = -3.8025$ in the moduli space. On the other hand, we can present a set BC1: $\sigma_3 = -\frac{8}{3}(\sigma_1 - 6)^2$, of classes of the maps one of whose two critical points maps to another one (see [16], [17]). The set BC1 corresponds to the one parameter family: $BC1$: $g_a(x) = -x^3 + ax + (1 + \frac{2}{3}a)\sqrt{\frac{a}{3}}$. We can show with computer experiments that this family is monotone (naturally not antimonotone) under (Anti).

Recently we know that J. Milnor and Ch. Tresser also treat of this problem and they said in [12] that

> *The analogue of the Antimonotonicity Conjecture for the stunted sawtooth families is certainly false, since by 5.8, it is very easy to find smooth curves along which there are only orbit creations. Thus, if the conjecture is true for the cubic family, then any complexity preserving correspondence between the stunted sawtooth and cubic parameter triangles must be very wild indeed.*

We remark that the entropy of the family $\{f_\lambda\}_\lambda$ is not monotone but one of our family $\{g_a\}_a$ is monotone.

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Isometries of Nevanlinna-type spaces
（Nevanlinna 型空間における等長写像について）

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Abstract. Let A be a linear isometry of X into X. When $X = H^q$ $(0 < q < \infty, q \neq 2)$, A is characterized by Rudin ([R], Theorem 4.4). The case, where $A(H^q) = H^q$, is described by Forelli ([F]). In the case $X = N_*$, A is completely described by Stephenson ([Ste], Theorem 2.2 and Corollary 2.3). We shall show that, when $X = N^p$, A has the same form as in the case $X = N_*$.

0. 序

Nevanlinna 型空間における等長写像について、H^p については Forelli, Rudin らに、N_* については Stephenson に、N^p については Iida-Mochizuki によって結果が得られている。

ここでは、それらの結果の紹介とともに、未解決である N 上と H^∞ 上の等長写像について触れる。

1. 準備

まず、代表的な空間である Nevanlinna class, Smirnov class, Hardy spaces の定義を与える：

定義 1 (N, N_*, H^p)

$U = \{z \in \mathbf{C} \mid |z| < 1\}, T = \{z \in \mathbf{C} \mid |z| = 1\}$ とする。また、$d\sigma$ を T 上の normalized Lebesgue measure とする。さらに f を U 上の正則関数とする。

1. $\displaystyle\lim_{r \to 1^-} \int_T \log(1 + |f(r\zeta)|)\, d\sigma(\zeta) < +\infty$ を満たすとき、$f \in N$ とする。

（注）　$f \in N$ のとき、$\displaystyle f^*(\zeta) := \lim_{r \to 1^-} f(r\zeta)$ が a.e. $\zeta \in T$ で存在することが知られている。

2. $f \in N$ で、$\displaystyle\lim_{r \to 1^-} \int_T \log(1 + |f(r\zeta)|)\, d\sigma(\zeta) = \int_T \log(1 + |f^*(\zeta)|)\, d\sigma(\zeta)$ を満たすとき、$f \in N_*$ とする。

3. $0 < p < \infty$ に対し　$\displaystyle\lim_{r \to 1^-} \int_T |f(r\zeta)|^p\, d\sigma(\zeta) < +\infty$ を満たすとき、$f \in H^p$ とする。

また、U 上の有界正則関数全体を H^∞ で表す。

N を Nevanlinna class, N_* を Smirnov class, H^p $(0 < p \leqq \infty)$ を Hardy spaces と呼ぶ。

これらの空間のあいだに、以下のような包含関係が成り立つ：

$$H^\infty \subset H^q \subset H^p \subset N_* \subset N \qquad (0 < p < q < \infty)$$

以上のような包含関係は昔からよく知られていたが、1977 年に M. Stoll は N_* と H^p の間に位置する空間 N^p を次のように導入した ([Sto])：

定義2 (N^p)

$p > 1$ とする。U 上の正則関数 f が

$$\lim_{r \to 1^-} \int_T [\log(1 + |f(r\zeta)|)]^p \, d\sigma(\zeta) < +\infty$$

を満たすとき、 $f \in N^p$ とする。

この N^p には、以下の特徴がある：

$$N^p \subset N^q \quad (1 < q < p) \ , \quad \bigcup_{q>0} H^q \subset \bigcap_{p>1} N^p \ , \quad \bigcup_{p>1} N^p \subset N_*$$

以上の空間 N, N_*, N^p, H^p をまとめて Nevanlinna-type spaces と呼ぶ ([CK])。

2. H^p, N_*, N^p 上の線形等長写像について

① H^p-isometry

$f \in H^p \ (0 < p < \infty)$ に対して、

$$\|f\|_p = \left\{ \int_T |f^*(\zeta)|^p \, d\sigma(\zeta) \right\}^{\frac{1}{p}}$$

とおくと、これは $p \geqq 1$ のとき H^p におけるノルムとなる。

一方、 $0 < p < 1$ のとき $\|f\|_p$ は三角不等式を満たさないのでノルムとはならないが、

$$\rho_p(f, g) = \|f - g\|_p^p = \int_T |f^*(\zeta) - g^*(\zeta)|^p \, d\sigma(\zeta)$$

とおくと、ρ_p は H^p における距離を定義する。

そこで、H^p から H^p への線形写像 A が任意の $f \in H^p$ について $\int_T |(Af)^*(\zeta)|^p \, d\sigma(\zeta) = \int_T |f^*(\zeta)|^p \, d\sigma(\zeta)$ を満たすとき、 A を H^p-等長写像と呼ぶことにする。

この H^p-等長写像については、de Leeuw-Rudin-Wermer ([DRW]), Forelli([F]), Rudin([R]) らに研究されているが、ここでは Rudin の結果を記しておく：

定理1 ([R])

$0 < p < \infty, p \neq 2$ とし、$S : H^p \to H^p$ が H^p-等長写像であるとする。このとき $\Psi \in H^p$ と U 上の inner function Φ が存在して、

$$(Sf)(z) = \Psi(z) f(\Phi(z)) \quad , \quad z \in U, f \in H^p$$

が成り立つ。ここに、T 上の任意の有界な Borel function $h(\zeta)$ に対し、

$$\int_T h(\zeta) \, d\sigma(\zeta) = \int_T h(\Phi^*(\zeta)) |\Psi^*(\zeta)|^p \, d\sigma(\zeta)$$

が成り立つ。

② N_*-isometry

次に、N_* について以下のような距離を定義しよう：

$$d(f, g) = \int_T \log(1 + |f^*(\zeta) - g^*(\zeta)|) \, d\sigma(\zeta) \quad (f, g \in N_*)$$

この距離 d に関する N_* 上の線形等長写像については、Stephenson によって次の結果が得られている：

定理 2 ([Ste])

$T: N_* \to N_*$ が線形等長写像であるとする。このとき U 上の inner functions Ψ, Φ（ただし、Φ^* は T 上 measure-preserving）が存在して、

$$(Tf)(z) = \Psi(z)f(\Phi(z)) \quad , \quad z \in U\,,\, f \in N_*$$

となる。

③ N^p-isometry

N^p $(p > 1)$ について以下のような距離を定義しよう：

$$d_p(f\,,\,g) = \left\{\int_T [\log(1 + |f^*(\zeta) - g^*(\zeta)|)]^p\, d\sigma(\zeta)\right\}^{\frac{1}{p}} \quad (f\,,\,g \in N^p)$$

この距離 d_p に関する N^p 上の線形等長写像については、Iida-Mochizuki によってその結果が今年になって得られた。これは N_* の場合とまったく同じ形である ([IM])：

定理 3

$p > 1$ とし、$A: N^p \to N^p$ が線形等長写像であるとする。このとき U 上の inner functions Ψ, Φ（ただし、Φ^* は T 上 measure-preserving）が存在して、

(☆) $$(Af)(z) = \Psi(z)f(\Phi(z)) \quad , \quad z \in U\,,\, f \in N^p$$

となる。

逆に、上記のような Ψ と Φ が与えられたとき、(☆) は N^p から N^p への線形等長写像となる。

系 ([IM])

$p > 1$ とし、$A: N^p \to N^p$ が上への線形等長写像であるとする。このとき $a, b \in T$ が存在して、

$$(Af)(z) = af(bz) \quad , \quad z \in U\,,\, f \in N^p$$

となる。

3. Nevanlinna 型空間上の線形等長写像の未解決問題について

④ H^∞-isometry

H^∞ 上のノルムは $\|f\|_\infty = \sup_{z \in U} |f(z)|$ で与えられる。このノルムに関する線形等長写像については、onto の場合は de Leeuw-Rudin-Wermer によって以下の結果が得られている ([DRW])。

定理 4

$S: H^\infty \to H^\infty$ が上への H^∞-等長写像であるとする。このとき $c \in T$ と $\phi(z) = \frac{z - a}{1 - \bar{a}z}b$ $(a \in U\,,\, b \in T)$ が存在して

$$(Sf)(z) = cf(\phi(z)) \quad , \quad z \in U\,,\, f \in H^\infty$$

となる。

"into" の場合の H^∞-等長写像についてはその形はまだ決定されていないようである。

⑤ N-isometry

N 上には以下のような距離が通常定義されている。

$$d_N(f\,,g)=\lim_{r\to 1^-}\int_T \log(1+|f(r\zeta)-g(r\zeta)|)\,d\sigma(\zeta)\quad (f\,,g\in N)$$

ここでは、これとは異なる形の距離について考えていく。
以下の結果は昔から良く知られている N の因数分解定理である：

定理 5

$f\in N\,,f\not\equiv 0$ とする。このとき f は以下の形に分解される。

$$f(z)=\frac{aB(z)F(z)S_1(z)}{S_2(z)}$$

ここで、

- a は絶対値 1 の複素数
- $B(z)=z^m\prod_{n=1}^{\infty}\frac{|a_n|}{a_n}\frac{a_n-z}{1-\overline{a_n}z}$ は f の零点 $\{a_n\}$ からなる Blaschke product
- $F(z)=\exp\left(\int_T\frac{\zeta+z}{\zeta-z}\log|f^*(\zeta)|\,d\sigma(\zeta)\right)$: N に対する outer function と呼ばれる。
- $S_j(z)=\exp\left(-\int_T\frac{\zeta+z}{\zeta-z}\,d\nu_j(\zeta)\right)$ $(j=1,2.\ \ \nu_1\,,\nu_2$ は違いに特異な測度) : (singular) inner function と呼ばれる。

（注） $f\in N_*$ のときは $S_2(z)\equiv 1$ となる。

このとき、以下の式が成り立つ：

$$||f||=\lim_{r\to 1^-}\int_T\log(1+|f(r\zeta)|)\,d\sigma(\zeta)=\int_T\log(1+|f^*(\zeta)|)\,d\sigma(\zeta)+\nu_2(T)$$

Stephenson は、[Ste] において $\nu_2(T)$ や singular inner function と N-isometry との関わりあいを調べている。ここでは代表的な結果のみを挙げておく。

まず、$f\,,g\in N_*$ が **common inner factor** h を持つとは、$\frac{f}{h}\,,\frac{g}{h}\in N_*$ となる non-constant inner function h が存在することである。

さらに $f\,,g\in N_*$ が common inner factor を持たないとき、$f\wedge g=1$ と書く。

また、U 上の inner functions $\psi\,,\phi$ (ただし $\phi(0)=0$) に対し、$A(\psi\,,\phi):N\to N$ は $(Af)(z)=\psi(z)(f(\phi(z)))$ $(f\in N)$ のことを表すものとする。

$A(\psi\,,\phi):N\to N$ の形のものが N-isometry になる条件は以下のようになる。

定理 6 ([Ste])

$A(\psi\,,\phi):N\to N$ が N-isometry になる必要十分条件は任意の singular inner function S に対して $\psi\wedge(S(\phi))=1$ が成り立つことである。

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On the type of von Neumann algebras generated by partial isometry operators
(部分等距離作用素から生成されるフォン・ノイマン環の型について)

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Abstract *We shall talk the generation of von Neumann algebras in this talk. In particular, we talk the generation ofvon Neumann algebras by partial isometries and give an answer for the probrem of Saito.*

この講演では我々は次のNotationの下で話を進めて行く。

Hは可分なヒルベルトであり，$B(H)$をH上の有界作用素全体からなる C*-環とする。
$B(H)$ の部分集合 $\mathscr{R}$ に対して,

$M(\mathscr{R})$: $\mathscr{R}$ を含む最小のフォン・ノイマン環($\mathscr{R}$ から生成されたフォン・ノイマン環)

特に, $\mathscr{R}$ = {A, B, C, …} であるとき, $M(\mathscr{R})$=M(A, B, C, …) であらわす。

フォン・ノイマン環の生成については，１９６０年代において Behncke, Fillmore, Saito, Stempfli, Suzuki, Topping, Pearcy, Putnum, 等によって進められていた。

（１）可換フォン・ノイマン環は１個の自己共役作用素から生成される。(von Neumann: 1929)

（２）Ｉ型フォン・ノイマン環は１個から生成される。(Pearcy:1952)

(3) A, B : 可換フォン・ノイマン環$\Rightarrow M(A, B)$ は１個から生成される。

(4) (ⅰ) M_n : １個から生成される　(ⅱ) $\{M_n\}$: 互いに可換

⇒

$\{M_n\}$ から生成されるフォン・ノイマン環Mは１個から生成される。

(5) Hyperfinite フォン・ノイマン環は１個から生成される。(Saito:1963)

(6) II_1 型, II_∞型, Ⅲ型のフォン・ノイマン環を生成する作用素がそれぞれ存在する。しかも, それらは partial isometry である。(Pearcy:1962 and Saito:1963)

(7) $M \cong M_2(M)$ であるフォン・ノイマン環Mに対して次は同値である。(Saito:1968)

(ⅰ) Mは１個から生成される。

(ⅱ) Mは１個の partial isometry によって生成される。

特別な partial isometry がどんな型のフォン・ノイマン環を生成するかについて考える。

斎藤は[8]で, T^n $(n=1,2,\cdots)$が partial isometries であるとき, T が power partial isometry であるといい, 次の問題を提起した。

一般の power partial isometry から生成されるフォン・ノイマン環の型については知られていない。

さらに, 斎藤は [8] で次のことを問題として提起している。

任意の Properly infinite なフォン・ノイマン環は１個の power partial isometry によって生成される？

我々はこの問題に対して否定的な完全な回答を与えることができた。すなわち, 次の定理によって, II_∞型, Ⅲ型のフォン・ノイマン環を生成する power partial isometry が存

在しないことがわかる。

定理 A.

Power partial isometry はⅠ型のフォン・ノイマン環を生成する。

これの準備として次のようなことを考える。

定義

ヒルベルト空間K上の作用素 U: truncated sift operator of index n

であるとは，Kが次の性質をみたすように部分空間K_0 の n-fold copy によってあらわされる。

$$K = K_0 \oplus K_0 \oplus \cdots \oplus K_0$$

であり，

$U = 0 \quad \text{if } n = 1$

and

$U\langle f_1, f_2, \cdots, f_n\rangle = \langle 0, f_1, f_2, \cdots, f_{n-1}\rangle \quad \text{if } n \geqq 2$

そのとき，斎藤[7]は次のことを示した。

T : truncated sift operators の finite direct sum,

すなわち

$$T = \Sigma \oplus U_{n(k)} \, (1 \leqq n(1) < n(2) < \cdots < n(r))$$

ここで，$U_{n(k)}$ は truncated sift operator of index n(k) である。

このとき，$\mathcal{M}(T)$ はⅠ型のフォン・ノイマン環である。

我々は上の finite direct sum という条件を落としても同様のことが言えることを示した。そして，その結果が最終的に有用な役割を演ずる。

定理 B.

T : truncated sift operators の finite or infinite direct sum,

すなわち

$$T = \Sigma \oplus U_{n(k)} (1 \leqq n(1) < n(2) < \cdots < n(r) < \cdots < K \leqq +\infty)$$

ここで，$U_{n(k)}$ は truncated sift operator of index n(k) である。

このとき，$\mathcal{M}(T)$ はＩ型のフォン・ノイマン環である。

以上の事柄と次の重要な power partial isometry 分解定理（Halmos and Wallen[3], also see Saito[9]）をあげる。

定理 C.

U: power partial isometry

$\Rightarrow$

$U = U_1 \oplus U_2 \oplus U_3 \oplus U_4$ (central direct sum)

ここで，

U_1; unitary operator　　U_2; unilateral sift operator

U_3; adjoint of unilateral sift operator

U_4; direct sum of truncated shift operators

この定理 C を考えると，$\mathcal{M}(U_1)$ は可換フォン・ノイマン環で，$\mathcal{M}(U_2)$，$\mathcal{M}(U_3)$ はＩ型のフォン・ノイマン環である。したがって，truncated shift operator U_4 に対して，$\mathcal{M}(U_4)$ の型を知る必要がある。これは定理Ｂにより，Ｉ型であるので，

$$\mathcal{M}(U) = \mathcal{M}(U_1) \oplus \mathcal{M}(U_2) \oplus \mathcal{M}(U_3) \oplus \mathcal{M}(U_4)$$

はＩ型フォン・ノイマン環となる。

Truncated shift operators の直和で表された operator によって生成されるフォン・ノイマン環がＩ型であることを示した定理Ｂ はグラフに付随した隣接作用素に対しても適用

される。この結果は藤井等[2]による隣接作用素が partial isometry であることの特性を考えることによって，[11]で次のことを示した。

定理 D.

T: 有向グラフに付随した隣接作用素で partial isometry

⇒

$\mathcal{M}$(T)：I 型フォン・ノイマン環

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A subclass of paranormal including class of log-hyponormal and several related classes

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1 Introduction

ここではヒルベルト空間 H 上の有界線形作用素について考える。以下、単に作用素と呼ぶことにする。ここで作用素 T が positive であるとは positive definite、即ち $(Tx, x) \geq 0$ for all $x \in H$ と定義し、$T \geq 0$ と表す。また、T が positive かつ invertible であるとき、T は strictly positive であるといい、$T > 0$ と表す。

normal ($\overset{\text{def}}{\Longleftrightarrow} T^*T = TT^*$) を含む作用素の class として次の3つが知られている。

Definition A.

T : *hyponormal* $\overset{\text{def}}{\Longleftrightarrow} T^*T \geq TT^*$.

T : *log-hyponormal* $\overset{\text{def}}{\Longleftrightarrow} T$: *invertible and* $\log T^*T \geq \log TT^*$.

T : *paranormal* $\overset{\text{def}}{\Longleftrightarrow} \|T^2x\| \geq \|Tx\|^2$ *for every unit vector* $x \in H$.

hyponormal$\Longrightarrow$paranormal はよく知られた結果であり、invertible hyponormal$\Longrightarrow$log-hyponormal は $\log t$ が作用素単調関数であることよりわかる。また、log-hyponormal$\Longrightarrow$paranormal は Ando[1] によって示されている。なお、log-hyponormal については [1][10] で研究されており、paranormal は 1960 年代後半に多くの研究者によって研究された class であった [1][5][8]。

今回、我々は log-hyponormal と paranormal の間に新しい作用素の class (class A と名づける) を定義することにより、log-hyponormal$\Longrightarrow$paranormal の見通しのよい証明を operator inequality の結果を使って与えることができた。次に、その class A と paranormal のそれぞれを拡張した新しい作用素の class を定義すると、log-hyponormal, class A 及び paranormal の関係を自然に理解できることがわかった。更に、これらの class の関係が全て proper であることを示す具体例を挙げる。

2 A subclass of paranormal including class of log-hyponormal

新しい作用素の class として次のような class を定義する。なお、作用素 T の絶対値 $|T| = (T^*T)^{\frac{1}{2}} \geq 0$ であることに注意しておく。

Definition 1. T : *class A* $\overset{\text{def}}{\Longleftrightarrow} |T^2| \geq |T|^2$. (2.1)

class "A" という名前は "absolute" value にちなんだものである。この class A に関して、次の Theorem 1 が得られた。なお、Theorem 1 は Ando[1] の結果 "log-hyponormal$\Longrightarrow$paranormal" を含んだ定理である。

Theorem 1. *Let $T \in B(H)$. Then the following assertions hold;*
(1) T : *log-hyponormal* $\Longrightarrow T$: *class* A.
(2) T : *class* $A \Longrightarrow T$: *paranormal.*

Theorem 1 の証明の前にいくつかの定理と補題を紹介しておく。これらの定理は以下の証明において重要な役割を果たす。

Theorem A ([3][6]). *Let A and B be positive invertible operators. Then the following properties are mutually equivalent:*

(i) $\log A \geq \log B$.
(ii) $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ *for all* $p \geq 0$.
(iii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all* $p \geq 0$ *and* $r \geq 0$.

なお、Theorem A の (i) と (ii) の同値性については [2] で示されている。

Theorem B (Hölder-McCarthy inequality [9]). *Let A be a positive operator. Then the following inequalities hold for all $x \in H$:*

(i) $(A^r x, x) \leq (Ax, x)^r \|x\|^{2(1-r)}$ *for* $0 < r \leq 1$.
(ii) $(A^r x, x) \geq (Ax, x)^r \|x\|^{2(1-r)}$ *for* $r \geq 1$.

次の Lemma A は [7, Lemma 1] を少し改良したものである。

Lemma A. *Let A and B be invertible operators. Then*

$$(BAA^*B^*)^\lambda = BA(A^*B^*BA)^{\lambda-1}A^*B^* \quad \textit{holds for any real number } \lambda.$$

Proof of Theorem 1.
Proof of (1). T が log-hyponormal ($\overset{\text{def}}{\Longleftrightarrow} \log T^*T \geq \log TT^*$) であることは、

$$\log |T|^2 \geq \log |T^*|^2 \tag{2.2}$$

と同値である。また、Theorem A の (i) と (ii) の同値性より (2.2) は

$$|T|^{2p} \geq (|T|^p |T^*|^{2p} |T|^p)^{\frac{1}{2}} \quad \text{for all } p \geq 0 \tag{2.3}$$

と同値である。そこで、(2.3) に $p = 1$ を代入すると、

$$|T|^2 \geq (|T||T^*|^2|T|)^{\frac{1}{2}}. \tag{2.4}$$

Lemma A と $|T^*|^2 = TT^*$ より、(2.4) は

$$|T|^2 \geq |T|T(T^*|T|^2T)^{\frac{-1}{2}}T^*|T| \tag{2.5}$$

と変形できる。そして、(2.5) は

$$(T^*|T|^2T)^{\frac{1}{2}} \geq T^*T \tag{2.6}$$

と同値である。さらに、$|T|^2 = T^*T$ より (2.6) は

$$|T^2| \geq |T|^2$$

と同値であるので T は class A である。

Proof of (2). T が class A であることの定義は次のようなものであった。

$$T: \text{ class A } \overset{\text{def}}{\Longleftrightarrow} |T^2| \geq |T|^2. \tag{2.1}$$

すると、任意の $\|x\| = 1$ に対して、

$$\begin{aligned} \|T^2x\|^2 &= ((T^2)^*T^2x, x) \\ &= (|T^2|^2x, x) \\ &\geq (|T^2|x, x)^2 \qquad \text{by (ii) of Theorem B} \\ &\geq (|T|^2x, x)^2 \qquad \text{by (2.1)} \\ &= \|Tx\|^4. \end{aligned}$$

よって、

$$\|T^2x\| \geq \|Tx\|^2 \quad \text{for every unit vector } x \in H$$

が成り立つので T は paranormal である。

以上により、Theorem 1 は証明された。 □

3 Several classes related to class A and paranormal

第2章では新しい作用素の class として "class A" を定義したが、この章ではさらに class A, paranormal の拡張である class をそれぞれ次の様に定義する。

Definition 2. *For each* $k > 0$,

$$T: \textit{class } A(k) \overset{\text{def}}{\Longleftrightarrow} (T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2.$$

$$T: \textit{absolute-k-paranormal} \overset{\text{def}}{\Longleftrightarrow} \||T|^kTx\| \geq \|Tx\|^{k+1} \quad \textit{for every unit vector } x \in H.$$

class A, paranormal はそれぞれ class A(k), absolute-k-paranormal において $k = 1$ としたものとなっている。

この class A(k), absolute-k-paranormal に関して、次の結果が得られた。

Theorem 2. *Let* $T \in B(H)$. *Then the following assertions hold;*

(1) T : *invertible and class A* $\Longrightarrow T$: *class A(k) for* $k \geq 1$.

(2) T : *paranormal* $\Longrightarrow T$: *absolute-k-paranormal for* $k \geq 1$.

(3) *For each* $k > 0$, T : *class A(k)* $\Longrightarrow T$: *absolute-k-paranormal.*

更に、Theorem 2 の (1), (2) の拡張として class A(k), absolute-k-paranormal についてそれぞれ次の結果を得ることが出来た。

Theorem 3. *Let T be an invertible class A(k) operator for* $k > 0$. *Then*

$$f(l) = (T^*|T|^{2l}T)^{\frac{1}{l+1}}$$

is increasing for $l \geq k > 0$, *and the following inequality holds;*

$$f(l) \geq |T|^2, \ \textit{i.e., } T \textit{ is class } A(l) \textit{ for } l \geq k > 0.$$

Theorem 4. *Let T be an absolute-k-paranormal operator for $k > 0$. Then for every unit vector $x \in H$,*

$$F(l) = \left\| |T|^l Tx \right\|^{\frac{1}{l+1}}$$

is increasing for $l \geq k > 0$, and the following inequality holds;

$$F(l) \geq \left\| Tx \right\|, \text{ i.e., } T \text{ is absolute-}l\text{-paranormal for } l \geq k > 0.$$

Theorem 3 は、ある $k > 0$ に対して class A(k) であるならば、$l \geq k > 0$ なる l に対して class A(l) であり、さらにそのことは関数 $f(l)$ が l についての単調増加関数であるという形で表現できる、ということを主張している。同様に Theorem 4 は、ある $k > 0$ に対して absolute-k-paranormal であるならば、$l \geq k > 0$ なる l に対して absolute-l-paranormal であり、さらにそのことは関数 $F(l)$ が l についての単調増加関数であるという形で表現できる、ということを主張している。

4 Examples

この章では absolute-k-paranormal の characterization を与え、更にそれらを利用して第2章と第3章で定義した class の関係が proper であることを示すいくつかの具体例を挙げる。

paranormal の characterization として Ando[1] は次の Theorem E を得た。

Theorem E ([1]). *An operator T is paranormal if and only if*

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0 \quad \text{for all } \lambda > 0.$$

この Theorem E の拡張として、次のような absolute-k-paranormal の characterization が得られた。なお、Theorem 5 において $k = 1$ としたものが Theorem E である。

Theorem 5. *For each $k > 0$, an operator T is absolute-k-paranormal if and only if*

$$T^*|T|^{2k}T - (k+1)\lambda^k|T|^2 + k\lambda^{k+1} \geq 0 \quad \text{for all } \lambda > 0.$$

また、log-hyponormal, class A(k) の定義と Theorem 5 より次の Proposition 6 を得ることができる。

Proposition 6. *Let $K = \bigoplus_{n=-\infty}^{\infty} H_n$ where $H_n \cong H$. For given positive operators A, B on H, define the operator $T_{A,B}$ on K as follows:*

$$T_{A,B} = \begin{pmatrix} \ddots & & & & & & \\ & \ddots & 0 & & & & \\ & & B & 0 & & & \\ & & & B & \boxed{0} & & \\ & & & & A & 0 & \\ & & & & & A & 0 \\ & & & & & & \ddots \quad \ddots \end{pmatrix} \tag{4.1}$$

where $\boxed{}$ shows the place of the $(0,0)$ matrix element. Then the following assertions hold:

(i) *$T_{A,B}$ is log-hyponormal if and only if A and B are invertible and*

$$\log A \geq \log B.$$

(ii) *For each* $k > 0$, $T_{A,B}$ *is class A(k) if and only if*

$$(BA^{2k}B)^{\frac{1}{k+1}} \geq B^2.$$

(iii) *For each* $k > 0$, $T_{A,B}$ *is absolute-k-paranormal if and only if*

$$BA^{2k}B - (k+1)\lambda^k B^2 + k\lambda^{k+1} \geq 0 \quad \textit{for all } \lambda > 0.$$

以下、この Proposition 6 を使ってここで議論した class の関係が全て proper であることを示す具体例を挙げる。なお、ここでは行列 X のトレースと行列式をそれぞれ $\operatorname{tr} X$, $\det X$ と表すものとする。

Example 1. *Let* $K = \bigoplus_{n=-\infty}^{\infty} H_n$ *where* $H_n \cong \mathbb{R}^2$. *For given positive matrices* A, B *on* $\mathbb{R}^2$, *define the operator* $T_{A,B}$ *on* K *as* (4.1) *in Proposition 6. Then we have the following examples.*

(1) *An example of non-log-hyponormal, class A operator.*

$$A = \begin{pmatrix} 17 & 7 \\ 7 & 5 \end{pmatrix}^2, \ B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^2$$

とする。Fujii, Furuta, Wang[4] は $\log A \ngeq \log B$ かつ $A^2 \geq (AB^2A)^{\frac{1}{2}}$ であることを示した。一方、$A^2 \geq (AB^2A)^{\frac{1}{2}}$ と $(BA^2B)^{\frac{1}{2}} \geq B^2$ は Lemma A より同値である。よって、Proposition 6 の (i) と (ii) より $T_{A,B}$ は non-log-hyponormal かつ class A である。

(2) *An example of non-class A, class A(2), paranormal operator.*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2\sqrt{23} \end{pmatrix}, \ B = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$$

とする。そのとき、

$$(BA^2B)^{\frac{1}{2}} - B^2 = \begin{pmatrix} 0.17472\ldots & -3.1798\ldots \\ -3.1798\ldots & 11.770\ldots \end{pmatrix}$$

である。$(BA^2B)^{\frac{1}{2}} - B^2$ の固有値は 12.585 ... と-0.64001 ... なので、$(BA^2B)^{\frac{1}{2}} \ngeq B^2$ である。よって、Proposition 6 の (ii) より $T_{A,B}$ は non-class A である。また、同様の計算により $T_{A,B}$ は class A(2) である。

一方、$\lambda > 0$ に対して $X_1(\lambda)$ を次の様に定義する。

$$X_1(\lambda) = BA^2B - 2\lambda B^2 + \lambda^2 = \begin{pmatrix} 404 - 26\lambda + \lambda^2 & -576 + 24\lambda \\ -576 + 24\lambda & 844 - 26\lambda + \lambda^2 \end{pmatrix}.$$

$p_1(\lambda) = \operatorname{tr} X_1(\lambda)$, $q_1(\lambda) = \det X_1(\lambda)$ とすると、

$$p_1(\lambda) = 2\lambda^2 - 52\lambda + 1248 = 2(\lambda - 13)^2 + 910 > 0$$

である。また、

$$q_1(\lambda) = \lambda^4 - 52\lambda^3 + 1348\lambda^2 - 4800\lambda + 9200$$

である。$q_1(\lambda)$ を微分すると、

$$\begin{aligned} q_1'(\lambda) &= 4\lambda^3 - 156\lambda^2 + 2696\lambda - 4800 \\ &= 4(\lambda - 2)\{(\lambda - 18.5)^2 + 257.75\} \end{aligned}$$

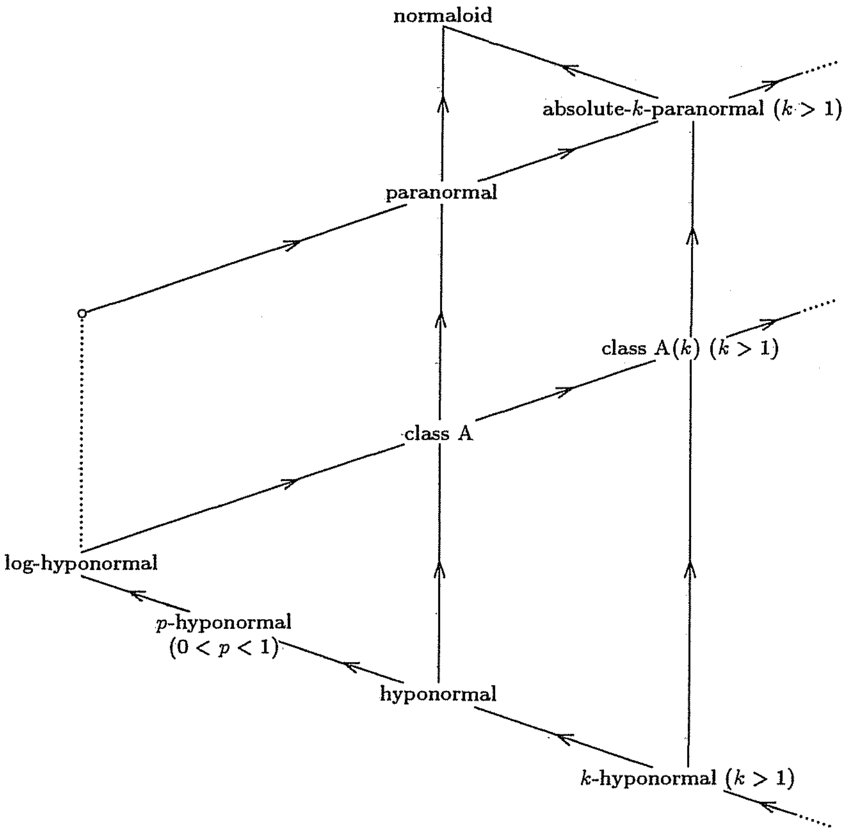
となる。よって、$q_1'(\lambda) = 0$ のとき $\lambda = 2$ であるので、任意の $\lambda > 0$ に対して $q_1(\lambda) \geq q_1(2) = 4592 > 0$ である。ゆえに、任意の $\lambda > 0$ に対して $\operatorname{tr} X_1(\lambda) = p_1(\lambda) > 0$ かつ $\det X_1(\lambda) = q_1(\lambda) > 0$ であるので、$X_1(\lambda) \geq 0$ である。よって、Proposition 6 の (iii) より $T_{A,B}$ は paranormal である。

5 Remarks

Remark 1. paranormal$\Longrightarrow$normaloid ($\overset{\text{def}}{\Longleftrightarrow}$ $\|T^n\| = \|T\|^n$ for all positive integers n) はよく知られた結果である [5][8]。この結果の拡張として次の Theorem 7 が得られた。

Theorem 7. *T : absolute-k-paranormal for some $k > 0 \Longrightarrow T$: normaloid.*

Remark 2. ここで論じた作用素の class の関係を図にまとめると次のようになる。なお、p-hyponormal ($\overset{\text{def}}{\Longleftrightarrow}$ $(T^*T)^p \geq (TT^*)^p$ for a positive number p) については多くの研究者によって研究がなされている。



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Characterizations of chaotic order associated with Kantorovich inequality

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Abstract

By using the order preserving operator inequality shown in [11] which is associated with Kantorovich inequality, we shall give some characterizations of chaotic order.

1 Introduction

ここではヒルベルト空間上の有界線形作用素について考える。作用素 T が positive ($T \geq 0$ と書く) とは $(Tx,x) \geq 0$ for all $x \in H$ のことと定義する。そして、T が strictly positive ($T > 0$ と書く) とは、T が positive かつ invertible と定義する。この positive operator に対して、"$A \geq B \geq 0$ *ensures* $A^p \geq B^p$ *for any* $p \in [0,1]$" というのは有名な Löwner-Heinz theorem であるが、この指数 p が $p > 1$ の時は $A \geq B \geq 0$ であっても必ずしも $A^p \geq B^p$ が成り立つとは限らないことも知られている。これに対して、次の定理が示されている。

Theorem A **([5]).** *If* $A \geq B > 0$ *and* $MI \geq B \geq mI > 0$, *then*

$$\left(\frac{M}{m}\right)^p A^p \geq B^p \quad \text{for } p \geq 1.$$

また、最近 Theorem A よりも精密な結果として、次の結果が示されている。

Theorem B **([11]).** *If* $A \geq B > 0$ *and* $MI \geq B \geq mI > 0$, *then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq K_+(m,M,p)A^p \geq B^p \quad \text{for } p \geq 1, \tag{1.1}$$

where

$$K_+(m,M,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p-m^p)^p}{(M-m)(mM^p-Mm^p)^{p-1}}. \tag{1.2}$$

この Theorem B は Hölder-McCarthy inequality [13] と Kantorovich inequality "*If* A *is an operator on a Hilbert space* H *such that* $MI \geq A \geq mI > 0$, *then* $(A^{-1}x,x)(Ax,x) \leq (m+M)^2/4mM$ *holds for every unit vector* x *in* H" に関連して示されたものである。

また、Löwner-Heinz theorem は指数 p が $p > 1$ では成り立つとは限らないということから、応用上不便であったが、この不便さを解消するものとして次の定理が示された。

Theorem F (Furuta inequality [7]).

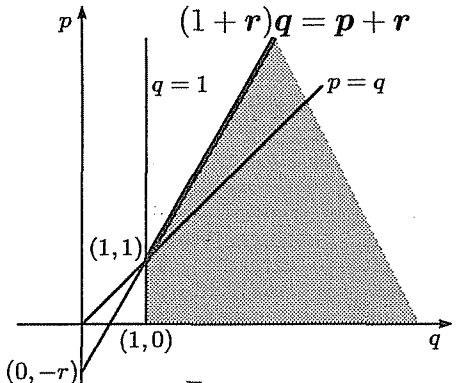
If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.



FIGURE

Theorem F の (i) または (ii) において、$r = 0$ とおくと、Löwner-Heinz theorem が得られる。また、[3] や [12] で Theorem F の別証明が得られており、さらに [8] では Theorem F の簡単な 1 ページの証明が得られている。そして、[17] では上の Figure で書かれた p, q, r の domain が Theorem F の best possible domain であることが示されている。

また、Ando [1] は、$\log A \geq \log B$ (chaotic order と呼ぶ) と $(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}} \geq B^p$ for all $p \geq 0$ が同値であることを示しているが、Theorem F を使うことによって その Ando の結果の拡張が得られる。

Theorem C ([4][6][9]). *Let* A *and* B *be positive and invertible operators on a Hilbert space* H. *Then the following assertions are mutually equivalent*:

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ for all $p \geq 0$ and $r \geq 0$.

我々は、今回 Theorem B と Theorem C を使うことによって、新しい chaotic order の characterization を得ることができた。

2 Results

Theorem 1. *Let* A *and* B *be positive and invertible operators on a Hilbert space* H *satisfying* $\log A \geq \log B$ *and* $MI \geq B \geq mI > 0$. *Then*

$$\left(\frac{M}{m}\right)^p A^p \geq K_+(m, M, p+1)A^p \geq B^p \quad \textit{for } p \geq 0, \tag{2.1}$$

where $K_+(m, M, p)$ *is defined in* (1.2).

Theorem 1 は Theorem A の拡張と見ることができる。また、次の chaotic order の characterization も得た。

Theorem 2. *Let* A *and* B *be positive and invertible operators on a Hilbert space* H *satisfying* $MI \geq B \geq mI > 0$. *Then the following assertions are mutually equivalent*:

(i) $\log A \geq \log B$.

(ii) $\dfrac{(m^p + M^p)^2}{4m^pM^p}A^p \geq B^p$ for all $p \geq 0$.

さらに、Theorem 1 と Theorem 2 の (i) $\Longrightarrow$ (ii) を補間するものとして、次の結果を得た。

Theorem 3. *Let A and B be positive and invertible operators on a Hilbert space H satisfying* $\log A \geq \log B$ *and* $MI \geq B \geq mI > 0$. *Then*

$$K_+\left(m^r, M^r, 1+\frac{p}{r}\right)A^p \geq B^p \qquad \textit{for } p > 0 \textit{ and } r > 0, \tag{2.2}$$

where $K_+(m, M, p)$ *is defined in* (1.2).

Theorem 3 において、$r = 1$ と置くと、 Theorem 1 を得ることができ、Theorem 3 において、$r = p$ と置くと、 Theorem 2 の (i) $\Longrightarrow$ (ii) を得ることができる。また、(1.2) における定数 $K_+(m, M, p)$ に関連して、次の proposition を得た。

Proposition 4. *Let* $K_+(m, M, p)$ *be defined in* (1.2). *Then*

$$F(p, r, m, M) = K_+\left(m^r, M^r, \frac{p+r}{r}\right)$$

is an increasing function of p, r and M, and also a decreasing function of m for $p > 0$, $r > 0$ and $M > m > 0$. And the following inequality holds:

$$\left(\frac{M}{m}\right)^p \geq K_+\left(m^r, M^r, \frac{p+r}{r}\right) \geq 1 \qquad \text{for any } p > 0,\ r > 0 \text{ and } M > m > 0. \tag{2.3}$$

そして、この Proposition 4 を考えることによって、Theorem 2 よりもさらによい評価である chaotic order の characterization を得ることができた。

Theorem 5. *Let A and B be positive and invertible operators on a Hilbert space H satisfying* $MI \geq B \geq mI > 0$. *Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $M_h(p)A^p \geq B^p$ holds for all $p > 0$, where $h = \frac{M}{m} > 1$ and

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e\log(h^{\frac{p}{h^p-1}})}. \tag{2.4}$$

なお、Theorem 5 において、$M_h(1) = \dfrac{(h-1)h^{\frac{1}{h-1}}}{e\log h}$ は Specht's ratio と呼ばれ、[2][16] などで研究されている。

3 Proof of results

以下、Theorem 2 、Proposition 4 と Theorem 5 の証明の概略を紹介する。

Proof of Theorem 2.

(a) *Proof of* (i) $\Longrightarrow$ (ii). Theorem C の (ii) において、$r = p$ と置くことによって、$\log A \geq \log B$ から、次の式を得る。

$$(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}} \geq B^p \quad \text{for } p \geq 0.$$

そこで、$A_1 = (B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}}$、$B_1 = B^p$ と置くと、条件から $A_1 \geq B_1 > 0$、$M^p \geq B_1 \geq m^p > 0$ を満たすことがわかる。そして、A_1 と B_1 に Theorem B を適用すると、次の式が得られる。

$$K_+(m^p, M^p, p_1)(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{p_1}{2}} \geq (B^p)^{p_1} \quad \text{for } p \geq 0 \text{ and } p_1 \geq 1. \tag{3.1}$$

さらに、(3.1) 式において、$p_1 = 2 \geq 1$ と置いて、(3.1) 式を整理すると、次の式が得られる。

$$K_+(m^p, M^p, 2)A^p \geq B^p \quad \text{for } p \geq 0.$$

よって、$K_+(m^p, M^p, 2) = \dfrac{(m^p + M^p)^2}{4m^pM^p}$ であることから、(i) $\Longrightarrow$ (ii) が示せた。

(b) *Proof of* (ii) $\Longrightarrow$ (i). $\log t$ は operator monotone function であるので、(ii) の両辺に log をとることによって、次の式が得られる。

$$\log\left\{\left(\frac{(m^p + M^p)^2}{4m^pM^p}\right)^{\frac{1}{p}} A\right\} \geq \log B \quad \text{for all } p \geq 0. \tag{3.2}$$

そこで、(3.2) 式において $p \to +0$ とすると、

$$\lim_{p \to 0}\left\{\frac{(m^p + M^p)^2}{4m^pM^p}\right\}^{\frac{1}{p}} = 1.$$

であることから $\log A \geq \log B$ を得ることができ、Theorem 2 の証明ができた。 □

Proof of Proposition 4. $h = \frac{M}{m} > 1$ とし、

$$g(p,r,h) = \left(\frac{r}{p+r}\frac{h^{p+r}-1}{h^r-1}\right)^{\frac{1}{p}} \tag{3.3}$$

とすると、

$$\begin{aligned} K_+\left(m^r, M^r, \frac{p+r}{r}\right) &= \frac{(\frac{p}{r})^{\frac{p}{r}}}{(1+\frac{p}{r})^{1+\frac{p}{r}}} \frac{(M^{p+r} - m^{p+r})^{1+\frac{p}{r}}}{(M^r - m^r)(m^rM^{p+r} - M^rm^{p+r})^{\frac{p}{r}}} && \text{by (1.2)} \\ &= \left(\frac{r}{p+r}\right)\left(\frac{p}{p+r}\right)^{\frac{p}{r}} \frac{(h^{p+r}-1)^{1+\frac{p}{r}}}{(h^r-1)(h^{p+r}-h^r)^{\frac{p}{r}}} && \text{by } h = \frac{M}{m} > 1 \\ &= \left\{\frac{1}{h}\left(\frac{r}{p+r}\frac{h^{p+r}-1}{h^r-1}\right)^{\frac{1}{p}}\left(\frac{p}{p+r}\frac{h^{p+r}-1}{h^p-1}\right)^{\frac{1}{r}}\right\}^p \\ &= \left\{\frac{1}{h}\cdot g(p,r,h)\cdot g(r,p,h)\right\}^p && \text{by (3.3).} \end{aligned} \tag{3.4}$$

となる。さらに、$g(p,r,h)$ について次の 2 つのが示せるが証明は省略する。

(I). $g(p,r,h)$ は $p > 0$ と $r > 0$ についての increasing function である。

(II). $p > 0, r > 0$ について $h \geq g(p,r,h) \geq h^{\frac{1}{2}}$.

よって、上の (II) により次の不等式を得ることができる。

$$h \geq \frac{1}{h}\cdot g(p,r,h)\cdot g(r,p,h) \geq 1 \qquad \text{for } p > 0 \text{ and } r > 0. \tag{3.5}$$

また、(3.4) 式と (3.5) 式によって、(2.3) 式を得る。つまり、

$$\left(\frac{M}{m}\right)^p \geq K_+\left(m^r, M^r, \frac{p+r}{r}\right) \geq 1 \qquad \text{for any } p > 0,\ r > 0 \text{ and } M > m > 0. \tag{2.3}$$

さらに、上の (I) と (3.4) 式、(3.5) 式によって、$F(p,r,m,M) = K_+(m^r, M^r, \frac{p+r}{r})$ が $p > 0$ と $r > 0$ についての increasing function であることもわかる。また、同様にして $F(p,r,m,M) = K_+(m^r, M^r, \frac{p+r}{r})$ の M と m についての単調性も示すことができる。 □

Proof of Theorem 5.

(a) *Proof of* (i) $\Longrightarrow$ (ii). Theorem 3 によって, $\log A \geq \log B$ ならば

$$K_+\left(m^r, M^r, 1+\frac{p}{r}\right) A^p \geq B^p \quad \text{for } p > 0 \text{ and } r > 0. \tag{2.2}$$

ここで、$h = \frac{M}{m} > 1$ として、(2.2) 式において $r \to +0$ とすると、

$$\lim_{r \to +0} K_+\left(m^r, M^r, 1+\frac{p}{r}\right) = M_h(p)$$

であることから、$M_h(p)A^p \geq B^p$ for $p > 0$ を得る。

(b) *Proof of* (ii) $\Longrightarrow$ (i). (ii) の両辺に log をとると、次の式を得る。

$$\log(\{M_h(p)\}^{\frac{1}{p}} A) \geq \log B \quad \text{for } p > 0. \tag{3.6}$$

そこで、(3.6) 式において $p \to +0$ とすると、

$$\lim_{p \to +0} \{M_h(p)\}^{\frac{1}{p}} = 1$$

より、$\log A \geq \log B$ を得ることができる。よって、Theorem 5 が証明できた。 □

4 Concluding Remarks

Remark 1. A と B を positive invertible な作用素とする。この時、$A \geq B$ と $\log A \geq \log B$ を補間する order として、$A^\delta \geq B^\delta$ for $\delta \in (0, 1]$ を考える。そして、この order に対して次の結果を得た。

Proposition 6. *Let A and B be positive and invertible operators on a Hilbert space H satisfying $A^\delta \geq B^\delta$ for $\delta \in (0,1]$ and $MI \geq B \geq mI > 0$, then*

$$K_+\left(m^\delta, M^\delta, \frac{p}{\delta}\right) A^p \geq B^p \qquad \text{for } p \geq \delta,$$

where $K_+(m, M, p)$ is defined in (1.2).

証明は $A^\delta \geq B^\delta$ に対して Theorem B を適用するだけである。また、

$$\lim_{\delta \to +0} K_+\left(m^\delta, M^\delta, \frac{p}{\delta}\right) = M_h(p) \tag{4.1}$$

であることが示されるので、Theorem B と Theorem 5 の自然なつながりを次のようにまとめることができる。

Let $A > 0$ and $MI \geq B \geq mI > 0$. Then the following assertions hold:

(i) $A \geq B$ implies $K_+(m, M, p)A^p \geq B^p$ for $p > 1$,

(ii) for each $\delta \in (0, 1]$, $A^\delta \geq B^\delta$ implies $K_+\left(m^\delta, M^\delta, \frac{p}{\delta}\right) A^p \geq B^p$ for $p > \delta$,

(iii) $\log A \geq \log B$ implies $M_h(p)A^p \geq B^p$ for $p > 0$,

where $h = \frac{M}{m} > 1$, and $K_+(m, M, p)$ and $M_h(p)$ are defined in (1.2) and (2.4), respectively.

上の (ii) において、$\delta = 1$ とすると (i) が得られ、(4.1) 式によって、(ii) において $\delta \to +0$ とすると (iii) を得る。

Remark 2. 最近、chaotic order の characterization として、次の定理が示されている。

Theorem D ([6]). *If* $A, B > 0$, *then* $\log A \geq \log B$ *if and only if for any* $\delta \in (0,1]$ *there exists an* $\alpha = \alpha_\delta > 0$ *such that* $(e^\delta A)^\alpha > B^\alpha$.

これに対して、Theorem 2 と Theorem 5 は、次のように書き換えることができる。

Theorem 2'. *If* $A, B > 0$, *then* $\log A \geq \log B$ *if and only if for any* $p \geq 0$ *there exists a* $K_p > 1$ *such that* $K_p \to 1$ *as* $p \to +0$, *and* $(K_pA)^p \geq B^p$.

また、Theorem 2 から Theorem 2' を得たように、Theorem 2 から Theorem D も形式的に得ることができる。

Remark 3. Theorem 2' は 次の Theorem E [10] と非常に似た形をしている。

Theorem E ([10]). *If* $A, B > 0$, *then* $\log A \geq \log B$ *if and only if for any* $p \geq 0$ *there exists the unique unitary operator* U_p *such that* $U_p \to I$ *as* $p \to +0$, *and* $(U_pAU_p^*)^p \geq B^p$.

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Order among Furuta type inequalities

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ABSTRACT. The order between parametrized Furuta inequality and parametrized grand Furuta inequality is determined as follows; if $A \geq B > 0$, then

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta$$

$$B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq B^u \sharp_{\frac{\delta-u}{p-u}} A^p \geq A^\delta$$

for $t \in [0,1]$, $0 \leq t < p \leq \beta$, $u \leq 0$ *and* $\delta \in [0,p]$.
Especially, if $\delta \in [0,1]$ under the above conditions, then

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta$$

$$\leq A^\delta \leq B^u \sharp_{\frac{\delta-u}{p-u}} A^p \leq B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

The case of $\delta = 1$ gives the order between the Furuta inequality and grand Furuta inequality.

1. Introduction. Throughout this note, we use a capital letter as an operator on a Hilbert space H. An operator A is said to be positive (in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$, and also an operator A is strictly positive (in symbol: $A > 0$) if A is positive and invertible.

The original form of the Furuta inequality [5] given by Furuta himself is the following(cf.[6],[17]).

Furuta inequality: *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \quad and \quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for p and q such that $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

The case of $r = 0$ in this inequality is the Löwner-Heinz inequality:

$$(LH) \qquad A^\alpha \geq B^\alpha \quad for \;\; A \geq B \geq 0 \;\; and \;\; 0 \leq \alpha \leq 1.$$

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From the viewpoint of operator mean ([2],[3],[10],[11] etc.), the Furuta inequality is rewritten as follows;

$$A^u \sharp_{\frac{1-u}{p-u}} B^p \leq A \quad and \quad B \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

for $p \geq 1$ and $u \leq 0$. The notations $\sharp_\alpha$ and $\natural_\alpha$ are defined for positive operators A and B by

$$A \natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}, \quad for\ \alpha \in \mathbf{R}$$

and $\sharp_\alpha = \natural_\alpha$ when $\alpha \in [0,1]$. Note that $\sharp_\alpha$ is an operator mean in the sense of Kubo-Ando [16] which corresponds to the operator monotone function x^α in the Löwner theory.

As shown in [11], we had arranged these inequalities in one line by using the operator mean $\sharp_\alpha$ as follows:

Satellite theorem of the Furuta inequality: *If* $A \geq B \geq 0$, *then*

$$A^u \sharp_{\frac{1-u}{p-u}} B^p \leq B \leq A \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

for all $p \geq 1$ and $u \leq 0$.

We can generalize this inequality as the following and called it a paramertrization of the Furuta inequality ([13], [14]).

Theorem A. *If* $A \geq B > 0$, *then for* $0 \leq \delta \leq p$ *and* $u \leq 0$

$$A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \ and\ B^u \sharp_{\frac{\delta-u}{p-u}} A^p \geq A^\delta,$$

and for $u \leq \gamma \leq 0$ *and* $p \geq 0$

$$A^u \sharp_{\frac{\gamma-u}{p-u}} B^p \leq A^\gamma \ and\ B^u \sharp_{\frac{\gamma-u}{p-u}} A^p \geq B^\gamma.$$

As a corollary we have the following, in which the case of $\delta = 1$ is the satellite theorem.

Corollary A. *If* $A \geq B > 0$, *then for* $0 \leq \delta \leq 1$, $\delta \leq p$ *and* $u \leq 0$

$$A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \leq A^\delta \leq B^u \sharp_{\frac{\delta-u}{p-u}} A^p,$$

and for $-1 \leq \gamma \leq 0$, $u \leq \gamma$ *and* $p \geq 0$

$$A^u \sharp_{\frac{\gamma-u}{p-u}} B^p \leq A^\gamma \leq B^\gamma \leq B^u \sharp_{\frac{\gamma-u}{p-u}} A^p.$$

As a generalization of the Furuta inequality, Furuta [7] had given an inequality which we called the grand Furuta inequality. It interpolates the Furuta inequality and the Ando-Hiai inequality [1] equivalent to the main result of log majorization. We cite here it in terms of operator mean ([3]):

The grand Furuta inequality: *If $A \geq B \geq 0$ and A is invertible, then for each $p \geq 1$ and $0 \leq t \leq 1$,*

$$A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A$$

holds for $r \geq t$ and $s \geq 1$.

The best possibility of $\frac{1-t+r}{(p-t)s+r}$ is shown in [18]. This theorem also has satellte form [14] and more generally we have shown the next theorem as a parametrized form of the grand Furuta inequality [15].

Theorem B. *If $A \geq B > 0$, then for $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$ and $0 \leq \delta \leq \beta$*

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}}$$

and

$$B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}}.$$

The satellite theorem of grand Furuta inequality is just the case of $\delta = 1$ and we can align these inequalities as follows;

Corollary B. *If $A \geq B > 0$, then for $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$, $0 \leq \delta \leq 1$ and $\delta \leq \beta$*

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq B^\delta \leq A^\delta \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}} \leq B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

On the complementary domain of the Furuta inequality, that is, $0 \leq t < p \leq 1$, the following inequality holds ([12],[15]).

Theorem C. *If $A \geq B > 0$, then for $0 \leq t < p \leq 1$, $p \leq \delta \leq \min\{1, 2p\}$ and $\beta \geq \delta$*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq A^t \natural_{\frac{\delta-t}{p-t}} B^p \leq B^\delta \leq A^\delta \leq B^t \natural_{\frac{\delta-t}{p-t}} A^p \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{\delta}{\beta}}.$$

If $A \geq B > 0$, then for $0 \leq t \leq 1 \leq p$, $p \neq t$ and $\beta \geq p$

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{1}{\beta}}.$$

2. Results and Proofs. At the begining, we modify Theorem C.

Theorem 1. *If* $A \geq B > 0$, *then for* $0 \leq t \leq 1$ *and* $0 \leq t < p \leq \beta$

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p \quad \textit{and} \quad (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{p}{\beta}} \geq A^p.$$

The following theorems are obtained by the use of Theorem A and Theorem 1 in which we can give an order between the Furuta inequality and the grand Furuta inequality.

Theorem 2. *If* $A \geq B > 0$, *then for* $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$ *and* $0 \leq \delta \leq p$

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta$$

and

$$B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq B^u \sharp_{\frac{\delta-u}{p-u}} A^p \geq A^\delta.$$

Proof. It follows from Theorem B and Theorem 1 that

$$\begin{aligned} & A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \\ = \ & A^u \sharp_{\frac{\delta-u}{p-u}} (A^u \sharp_{\frac{p-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)) \\ \leq \ & A^u \sharp_{\frac{\delta-u}{p-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq A^u \sharp_{\frac{\delta-u}{p-u}} B^p. \end{aligned}$$

The first inequality follows from Theorem B and the second one is Theorem 1.

Under a little stronger conditions of Theorem 2, we can arrange these inequalities in one line, which is completely parallel to Corollary B, too:

Corollary 3. *If* $A \geq B > 0$, *then for* $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$, $0 \leq \delta \leq 1$ *and* $\delta \leq p$

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \leq A^\delta \leq B^u \sharp_{\frac{\delta-u}{p-u}} A^p \leq B^u \sharp_{\frac{\delta-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

The case of $\delta = 1$ shows the order between the Furuta inequality and grand Furuta inequality.

Theorem 4. *If* $A \geq B > 0$, *then for* $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$ *and* $u \leq \gamma \leq 0$

$$A^u \sharp_{\frac{\gamma-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \sharp_{\frac{\gamma-u}{p-u}} B^p \leq A^\gamma$$

and

$$B^u \sharp_{\frac{\gamma-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p) \geq B^u \sharp_{\frac{\gamma-u}{p-u}} A^p \geq B^\gamma.$$

Proof.

$$\begin{aligned} & A^u \sharp_{\frac{\gamma-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \\ = \; & A^u \sharp_{\frac{\gamma-u}{p-u}} (A^u \sharp_{\frac{p-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)) \\ \leq \; & A^u \sharp_{\frac{\gamma-u}{p-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq A^u \sharp_{\frac{\gamma-u}{p-u}} B^p. \end{aligned}$$

We can arrange these inequalities in one line also under a little stronger condition for γ.

Corollary 5. *If* $A \geq B > 0$, *then for* $0 \leq t \leq 1$, $0 \leq t < p \leq \beta$, $u \leq 0$, $-1 \leq \gamma \leq 0$ *and* $u \leq \gamma$

$$A^u \sharp_{\frac{\gamma-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq A^u \sharp_{\frac{\gamma-u}{p-u}} B^p \leq A^\gamma \leq B^\gamma \leq B^u \sharp_{\frac{\gamma-u}{p-u}} A^p \leq B^u \sharp_{\frac{\gamma-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

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OPERATOR MOMENT PROBLEMS ON ABELIAN *-SEMIGROUPS

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ABSTRACT. Let S be an abelian $*$-semigroup with the identity. Our main concerns in this paper are the integral representations for operator-valued, positive definite functions on S and operator-valued functions of positive type on S. In Theorem 2.1, we characterize moment functions whose representing measures have compact support. Furthermore, in section 3, we discuss the relation between (semi)perfectness of S and operator semiperfectness of S.

1. PRELIMINARIES

Let $S = (S, +, *)$ be an abelian $*$-semigroup with the identity 0. A function $\rho : S \to \mathbb{C}$ is called a *semicharacter* if it is a nonzero multiplicative function satisfying $\rho(s^*) = \overline{\rho(s)}$ for all $s \in S$. Note that $\rho(0) = 1$ and every semicharacter is not bounded. The set of semicharacters on S is denoted by S^*. We equip S^* with the topology inherited from $\mathbb{C}^S$, having the topology of pointwise convergence. Then S^* is a topological semigroup under pointwise multiplication with involution $\rho \mapsto \overline{\rho}$ and the identity $\mathbf{1}$. In particular, S^* is a Hausdorff space.

Let $\mathcal{H}$ be a complex Hilbert space, $\langle \cdot, \cdot \rangle$ the inner product on $\mathcal{H}$, $B(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$, and $B(\mathcal{H})_+$ the set of positive operators in $B(\mathcal{H})$. A function $\varphi : S \to B(\mathcal{H})$ is called *of positive type* if

$$\sum_{i,j=1}^{n} \langle \varphi(s_i + s_j^*)\xi_i, \xi_j \rangle \geq 0$$

for all $n \geq 1$, $s_1, s_2, \cdots, s_n \in S$ and $\xi_1, \xi_2, \cdots, \xi_n \in \mathcal{H}$. Moreover, φ is called *positive definite* if

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \langle \varphi(s_i + s_j^*)\xi, \xi \rangle \geq 0$$

for all $n \geq 1$, $s_1, s_2, \cdots, s_n \in S$, $c_1, c_2, \cdots, c_n \in \mathbb{C}$ and $\xi \in \mathcal{H}$. If $\mathcal{H} = \mathbb{C}$, the functions of positive type are the same as positive definite functions. In particular, every semicharacter is a positive definite function on S.

Let X be a subset of S^*. The set of all Borel subsets of X is denoted by $\mathfrak{B}(X)$. Let $M_+(X)$ denote the set of all regular Borel (i.e. Radon) measures defined on $\mathfrak{B}(X)$, and $E_+(X)$ the set of measures $\mu \in M_+(X)$ such that

$$\int_X |\rho(s)| d\mu(\rho) < \infty \quad \text{for } s \in S.$$

The set of signed measures of the form $\mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ with $\mu_j \in E_+(X)$ will be denoted by $E(X)$. Moreover, let $E_+(X, \mathcal{H})$ denote the set of functions $F : \mathfrak{B}(X) \to B(\mathcal{H})_+$ satisfying $\langle F(\cdot)\xi, \eta\rangle \in E(X)$ for all $\xi, \eta \in \mathcal{H}$. A function $\varphi : S \to B(\mathcal{H})$ is called a *moment function* if there exists $F \in E_+(S^*, \mathcal{H})$ such that $\varphi(s) = \int_{S^*} \rho(s) dF(\rho)$, i.e.

$$\langle \varphi(s)\xi, \eta\rangle = \int_{S^*} \rho(s) d\langle F(\rho)\xi, \eta\rangle \quad \text{for } s \in S,\ \xi, \eta \in \mathcal{H}.$$

We have the following.

Proposition 1.1. *Let $\varphi : S \to B(\mathcal{H})$ and consider the following conditions :*

(1) *φ is a moment function ;*
(2) *φ is of positive type ;*
(3) *φ is positive definite.*

Then the implication (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) *holds.*

Proof. (1) $\Longrightarrow$ (2) If φ is a moment function, then φ can be written in the form $\varphi(s) = \int_{S^*} \rho(s) dF(\rho)$, $s \in S$, for some $F \in E_+(S^*, \mathcal{H})$. Let $\xi_1, \xi_2, \cdots, \xi_n \in \mathcal{H}$ and $s_1, s_2, \cdots, s_n \in S$. Define $\mu_{ij} \in E(S^*)$, $i, j = 1, 2, \cdots, n$, and $\mu \in E_+(S^*)$ by

$$\mu_{ij}(B) := \langle F(B)\xi_i, \xi_j\rangle, \quad B \in \mathfrak{B}(S^*),$$
$$\mu(B) := \sum_{i=1}^{n} \mu_{ii}(B), \quad B \in \mathfrak{B}(S^*).$$

Since the matrix

$$\begin{bmatrix} \mu_{ii}(B) & \mu_{ij}(B) \\ \mu_{ji}(B) & \mu_{jj}(B) \end{bmatrix}$$

is positive, we have

$$|\mu_{ij}(B)| \le \mu_{ii}(B)^{1/2} \mu_{jj}(B)^{1/2} \le \mu(B).$$

Therefore each μ_{ij} is absolutely continuous with respect to μ. Let $g_{ij} := \dfrac{d\mu_{ij}}{d\mu}$, $i, j = 1, 2, \cdots, n$, be the Radon-Nikodym derivative. Then for $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$ and $B \in \mathfrak{B}(S^*)$,

$$\int_B \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} g_{ij}(\rho) d\mu(\rho) = \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \langle \mu(B)\xi_i, \xi_j\rangle \ge 0.$$

Since $B \in \mathfrak{B}(S^*)$ is arbitrary, it follows that the matrix $[g_{ij}(\rho)]_{i,j=1}^{n}$ is positive for μ-a.e.ρ. Consequently,

$$\begin{aligned} \sum_{i,j=1}^{n} \langle \varphi(s_i + s_j^*)\xi_i, \xi_j\rangle &= \sum_{i,j=1}^{n} \int_{S^*} \rho(s_i + s_j^*) d\mu_{ij}(\rho) \\ &= \int_{S^*} \sum_{i,j=1}^{n} \rho(s_i)\overline{\rho(s_j)} g_{ij}(\rho) d\mu(\rho) \ge 0, \end{aligned}$$

which shows that φ is of positive type.

The implication (2) $\Longrightarrow$ (3) is clear. □

Remark. (i) In the special case of S being an abelian group G with the involution $s^* = -s$, every positive definite function on G is of positive type (see [8, p.58, Corollary 1]). Furthermore, since positive definite functions on G are bounded, they are moment functions by Theorem 2.1.
(ii) If $\mathcal{H} = \mathbb{C}$, a positive definite function is not necessarily a moment function, and a representing measure is not unique if any (see [1, Chapter 6, §2 & §3]). So an abelian $*$-semigroup S is called *semiperfect* if every positive definite, scalar-valued function on S is a moment function. If, furthermore, a representing measure is unique, then S is called *perfect.*
(iii) Examples are known showing that (3) $\Longrightarrow$ (2) is not true (see [4, Proof of Theorem 1]).

Let $\mathcal{K}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{K}$. By a $*$-*representation* of a $*$-semigroup S with the identity 1 (S is not necessarily abelian) on $\mathcal{D}$, we mean a family $\{\pi(s)\}_{s\in S}$ of densely defined linear operators on $\mathcal{K}$ with domain $\mathcal{D}$ such that

(i) $\pi(1) = I$ (the identity operator),
(ii) $\pi(s)\mathcal{D} \subset \mathcal{D}$ and $\pi(s)\pi(t) = \pi(st)$ for $s, t \in S$,
(iii) $\pi(s)^*|_{\mathcal{D}} = \pi(s^*)$ for $s \in S$.

The following theorem is known (cf. [8, p.27, Theorem 1]).

Theorem 1.2. *Let S be a $*$-semigroup with the identity 1 and let $\varphi : S \to B(\mathcal{H})$ be of positive type, i.e. for any $s_1, s_2, \cdots, s_n \in S$ and $\xi_1, \xi_2, \cdots, \xi_n \in \mathcal{H}$*

$$\sum_{i,j=1}^{n} \langle \varphi(s_j^* s_i)\xi_i, \xi_j\rangle \geq 0.$$

Then there exist a Hilbert space $\mathcal{K}$, a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ and a $*$-representation $\{\pi(s)\}_{s\in S}$ of S on a dense subspace $\mathcal{D}$ of $\mathcal{K}$ such that

(1) *$V\mathcal{H} \subset \mathcal{D}$ and $\{\pi(s)V\xi : s \in S, \xi \in \mathcal{H}\}$ is total in $\mathcal{K}$,*
(2) *$\varphi(s) = V^*\pi(s)V$ for $s \in S$.*

2. MOMENT PROBLEM : CASE OF COMPACT SUPPORT

Every exponentially bounded, positive definite, scalar-valued function on S is a moment function whose representing measure has compact support (see [2, Theorem 2.1]). In this section, we shall prove this theorem for operator-valued function. A function $\alpha : S \to [0, \infty)$ is called an *absolute value* if

(i) $\alpha(0) = 1$,
(ii) $\alpha(s + t) \leq \alpha(s)\alpha(t)$ for $s, t \in S$,
(iii) $\alpha(s^*) = \alpha(s)$ for $s \in S$.

A function $\varphi : S \to B(\mathcal{H})$ is called *α-bounded* if there exists a constant $C > 0$ such that

$$||\varphi(s)|| \leq C\alpha(s) \quad \text{for } s \in S,$$

and φ is called *exponentially bounded* if there exists an absolute value α with respect to which φ is α-bounded.

Theorem 2.1. *Let $\varphi : S \to B(\mathcal{H})$. Then the following conditions are mutually equivalent :*

(1) *There exist a compact subset X of S^* and $F \in E_+(X, \mathcal{H})$ such that*

$$\varphi(s) = \int_X \rho(s) dF(\rho) \quad \textit{for } s \in S;$$

(2) *φ is of positive type and exponentially bounded ;*

(3) *φ is positive definite and exponentially bounded.*

The representing function $F \in E_+(X, \mathcal{H})$ is uniquely determined by φ.

Proof. (1) $\Longrightarrow$ (2) Suppose that (1) holds. Then φ is of positive type by Proposition 1.1. We define an absolute value α on S by

$$\alpha(s) := \sup\{|\rho(s)| : \rho \in X\} \quad \text{for } s \in S.$$

Then for $\xi \in \mathcal{H}$ with $||\xi|| = 1$, we have

$$\begin{aligned} |\langle \varphi(s)\xi, \xi\rangle| &= \left| \int_X \rho(s) d\langle F(\rho)\xi, \xi\rangle \right| \\ &\le \int_X |\rho(s)| d\langle F(\rho)\xi, \xi\rangle \\ &\le \alpha(s)\langle F(X)\xi, \xi\rangle \\ &\le \alpha(s)||F(X)||, \end{aligned}$$

so that $|\langle \varphi(s)\xi, \xi\rangle| \le \alpha(s)||F(X)||$. By polarization, we get

$$|\langle \varphi(s)\xi, \eta\rangle| \le 2\alpha(s)||F(X)||$$

for ξ, $\eta \in \mathcal{H}$ with $||\xi|| = ||\eta|| = 1$. Therefore $||\varphi(s)|| \le 2||F(X)||\alpha(s)$ and φ is α-bounded.

(2) $\Longrightarrow$ (3) It follows from Proposition 1.1.

(3) $\Longrightarrow$ (1) Suppose that there exist an absolute value α on S and a constant $C > 0$ such that

$$||\varphi(s)|| \le C\alpha(s) \quad \text{for } s \in S.$$

Then for each $\xi \in \mathcal{H}$, the function on S defined by $s \mapsto \langle \varphi(s)\xi, \xi\rangle$ is positive definite and α-bounded since

$$|\langle \varphi(s)\xi, \xi\rangle| \le ||\varphi(s)||\,||\xi||^2 \le C||\xi||^2\alpha(s).$$

By [2, Theorem 2.1], there exists a unique measure $\mu_\xi \in E_+(S^*)$ satisfying

$$\langle \varphi(s)\xi, \xi\rangle = \int_{S^*} \rho(s) d\mu_\xi(\rho) \quad \text{for } s \in S.$$

Moreover, the support of μ_ξ is a subset of the compact set X of α-bounded semicharacters. For $\xi, \eta \in \mathcal{H}$, define

$$\mu_{\xi,\eta} := \frac{1}{4}\{\mu_{\xi+\eta} - \mu_{\xi-\eta} + i\mu_{\xi+i\eta} - i\mu_{\xi-i\eta}\} \in E(X).$$

Then, by polarization,

$$\langle \varphi(s)\xi, \eta\rangle = \int_X \rho(s) d\mu_{\xi,\eta}(\rho) \quad \text{for } s \in S. \tag{2.1}$$

Fix $B \in \mathfrak{B}(X)$ and put $\Phi(\xi,\eta) := \mu_{\xi,\eta}(B)$ for $\xi,\eta \in \mathcal{H}$. Then Φ defines a sesquilinear form on $\mathcal{H} \times \mathcal{H}$ since the transformation $\mu \in E(X) \mapsto \widehat{\mu} \in \mathbb{C}^S$ given by

$$\widehat{\mu}(s) = \int_X \rho(s)\,d\mu(\rho), \quad s \in S,$$

is linear and injective (see [1, 4.2.10]). Moreover, Φ is positive and bounded since

$$\begin{aligned} 0 \le \Phi(\xi,\xi) &= \mu_{\xi,\xi}(B) \\ &\le \mu_{\xi,\xi}(X) \\ &= \langle \varphi(0)\xi,\xi\rangle \\ &\le \|\varphi(0)\|\,\|\xi\|^2 \end{aligned}$$

(cf. [7, §18]). Therefore, by [7, p.38, Theorem 1], there exists a unique operator $F(B) \in B(\mathcal{H})_+$ satisfying

$$\mu_{\xi,\eta}(B) = \Phi(\xi,\eta) = \langle F(B)\xi,\eta\rangle, \quad \xi,\eta \in \mathcal{H}.$$

Since $B \in \mathfrak{B}(X)$ is arbitrary, it follows that $\langle F(\cdot)\xi,\eta\rangle = \mu_{\xi,\eta} \in E(X)$, i.e. $F \in E_+(X,\mathcal{H})$. Consequently, we have

$$\langle \varphi(s)\xi,\eta\rangle = \int_X \rho(s)\,d\langle F(\rho)\xi,\eta\rangle \quad \text{for } s \in S.$$

Thus the proof is complete. □

Remark. The equivalence (1) $\Longleftrightarrow$ (3) of Theorem 2.1 is found in [2, Theorem 2.6] (without proof).

3. MOMENT PROBLEM : RELATION TO (SEMI)PERFECTNESS

As mentioned in Remark (ii) of section 1, even for scalar-valued functions on S, the implication (3) $\Longrightarrow$ (1) of Proposition 1.1 does not necessarily hold. In this section, firstly we shall prove that the implication (3) $\Longrightarrow$ (1) of Proposition 1.1 holds for perfect $*$-semigroups. Furthermore, for semiperfect $*$-semigroups, we discuss whether the implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) hold or not.

Theorem 3.1. *Let S be a perfect $*$-semigroup and $\varphi : S \to B(\mathcal{H})$. If φ is positive definite, then it is a moment function. Furthermore, a representing function $F \in E_+(S^*,\mathcal{H})$ is uniquely determined by φ.*

Proof. In the proof of Theorem 2.1 (3) $\Longrightarrow$ (1), use the perfectness of S instead of [2, Theorem 2.1] and [1, Proposition 6.5.2] instead of [1, 4.2.10]. □

By Theorem 3.1 and Proposition 1.1 the following conditions are mutually equivalent:

(i) S is perfect;
(ii) Every function of positive type on S is a moment function whose representing function is uniquely determined;
(iii) Every positive definite function on S is a moment function whose representing function is uniquely determined.

Accordingly, when S is perfect, we may assume that not only positive definite, scalar-valued functions but also positive definite functions or functions of positive type have the unique integral representation. For semiperfect $*$-semigroups, is this equivalent relation true ? Unfortunately we do not know the explicit answer to this question. So as long as this problem remains unsettled, let us agree to define the following.

Definition. An abelian $*$-semigroup S is called *operator semiperfect* (resp. *strongly operator semiperfect*) if every function of positive type (resp. positive definite function) on S is a moment function.

The following Theorem 3.2 and Theorem 3.3 are obtained if we consider operator-valued functions instead of scalar-valued functions in the proof of [6, Proposition 1] and [3, Proposition 1], respectively.

Theorem 3.2. *Let S_1 and S_2 be abelian $*$-semigroups with the identity and let $h : S_1 \to S_2$ be a surjective $*$-homomorphism.*

1. *If S_1 is operator semiperfect, then S_2 is operator semiperfect.*
2. *If S_1 is strongly operator semiperfect, then S_2 is strongly operator semiperfect.*

Theorem 3.3. *Let S_1 be an abelian $*$-semigroup with the identity and S_2 a finitely generated abelian $*$-semigroup with the identity.*

1. *If S_1 is perfect and S_2 is operator semiperfect, then the product $S_1 \times S_2$ is operator semiperfect.*
2. *If S_1 is perfect and S_2 is strongly operator semiperfect, then the product $S_1 \times S_2$ is strongly operator semiperfect.*

By Theorem 3.2 and 3.3, we note that operator semiperfect $*$-semigroups and strongly operator semiperfect $*$-semigroups have the properties that semiperfect $*$-semigroups have. We know only three essential examples, which are semiperfect and not perfect:

(i) $(\mathbb{N}_0, +, s^* = s)$, where $\mathbb{N}_0 := \{0, 1, 2, \cdots\}$;
(ii) $(\mathbb{Z}, +, s^* = s)$;
(iii) $(\mathbb{Z}^2, +, (s,t)^* = (t,s))$.

We shall prove that they are operator semiperfect.

Theorem 3.4. *The semigroup $(\mathbb{N}_0, +, s^* = s)$ is operator semiperfect.*

Proof. Let $\varphi : \mathbb{N}_0 \to B(\mathcal{H})$ be of positive type. By Theorem 1.2, there exist a Hilbert space $\mathcal{K}$, a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ and a $*$-representation $\{\pi(n)\}_{n\in\mathbb{N}_0}$ of $\mathbb{N}_0$ on a dense subspace of $\mathcal{K}$ such that

$$\varphi(n) = V^*\pi(n)V \quad \text{for } n \in \mathbb{N}_0.$$

Since $\pi(1) = \pi(1^*) \subset \pi(1)^*$, the operator $\pi(1)$ is symmetric and closable. Let T denote its closure and define the operator $\widetilde{T}$ on the direct sum $\mathcal{K} \oplus \mathcal{K}$ by

$$\widetilde{T} := T \oplus (-T).$$

The operator $\widetilde{T}$ is symmetric and its deficiency indices are equal. Therefore $\widetilde{T}$ has a self-adjoint extension T' on $\mathcal{K} \oplus \mathcal{K}$ (see [9, p.341]). Let F' be the spectral measure of T' and P the orthogonal projection of $\mathcal{H}\oplus\mathcal{H}$ onto $\mathcal{H}\oplus\{0\}$ and define $V':\mathcal{H}\oplus\mathcal{H}\to\mathcal{K}\oplus\mathcal{K}$ by $V':=V\oplus 0$. Then, identifying $\mathcal{H}\oplus\{0\}$ with $\mathcal{H}$, we have

$$\begin{aligned}\varphi(n) &= V^*T^nV\\ &= PV'^*T'^nV'|_{\mathcal{H}}\\ &= \int_{\mathbb{R}}\lambda^n\,dF(\lambda),\end{aligned}$$

where $F(\cdot)=PV'^*F'(\cdot)V'|_{\mathcal{H}}$. Since

$$\mathbb{N}_0^*=\{n\mapsto\lambda^n:\lambda\in\mathbb{R}\}\cong\mathbb{R},$$

it follows that φ is a moment function. Hence $(\mathbb{N}_0,+,s^*=s)$ is operator semiperfect. □

Theorem 3.5. *The semigroup* $(\mathbb{Z},+,s^*=s)$ *is operator semiperfect.*

Proof. We use the same notation as in the proof of Theorem 3.4. Let $\varphi:\mathbb{Z}\to B(\mathcal{H})$ be of positive type and let $(\pi,V,\mathcal{K})$ be the triple as in Theorem 3.4. Since $\pi(1)\pi(-1)=\pi(0)=I$ and the set $\{\pi(s)V\xi: s\in S,\ \xi\in\mathcal{H}\}$ is total in $\mathcal{K}$, the range of T' is dense in $\mathcal{K}\oplus\mathcal{K}$. Hence $T'=T'^*$ is one-to-one. Let F' be the spectral measure of T' and put $F(\cdot)=PV'^*F'(\cdot)V'|_{\mathcal{H}}$. Then $F(\{0\})=0$ and

$$\varphi(n)=\int_{\mathbb{R}\setminus\{0\}}\lambda^n dF(\lambda)\quad\text{for } n\in\mathbb{Z}.$$

This shows that φ is a moment function since

$$\mathbb{Z}^*=\{n\mapsto\lambda^n:\lambda\in\mathbb{R}\setminus\{0\}\}\cong\mathbb{R}\setminus\{0\}.$$

Therefore $(\mathbb{Z},+,s^*=s)$ is operator semiperfect. □

Theorem 3.6. *The $*$-semigroup* $(\mathbb{Z}^2,+,(s,t)^*=(t,s))$ *is operator semiperfect.*

Proof. Let $S:=(\mathbb{Z},+,s^*=-s)$ and $T:=(\mathbb{Z},+,s^*=s)$. Since S is perfect by [1, p.203] and T is operator semiperfect, the group

$$G:=(S\times T,+,(s,t)^*=(-s,t))$$

with the product involution is operator semiperfect by Theorem 3.3. Define the function $h:(\mathbb{Z}^2,+,(s,t)^*=(t,s))\to G$ by

$$h(n,m):=(n-m,n+m)\quad\text{for } (n,m)\in\mathbb{Z}^2.$$

Then h is a $*$-isomorphism onto the $*$-stable subgroup

$$G_0:=\{(s,t)\in G: s+t\in 2\mathbb{Z}\},$$

of G. In view of Theorem 3.2, it suffices to show that G_0 is operator semiperfect. However, considering the operator-valued function in [3, Theorem 1], we see that G_0 is operator semiperfect. □

Remark. In special case of φ in Theorem 3.4 being a matrix-valued function of positive type, K. Schmüdgen has proved that φ is a moment function (see [10, Proposition 2.3]). In [5], T. M. Bisgaard discusses operator semiperfectness, but our proofs and methods are different and independent.

As can be seen from the definitions, we have the following implications:

$$\begin{aligned} &\text{(i) } S \text{ is perfect}\\ \Longrightarrow &\text{(ii) } S \text{ is strongly operator semiperfect}\\ \Longrightarrow &\text{(iii) } S \text{ is operator semiperfect}\\ \Longrightarrow &\text{(iv) } S \text{ is semiperfect.}\end{aligned}$$

We do not know whether the implications (ii) $\Longrightarrow$ (i) and (iv) $\Longrightarrow$ (iii) are true or not. On the contrary, the implication (iii) $\Longrightarrow$ (ii) is not true. In fact, $\mathbb{N}_0$ is not strongly operator semiperfect (see [4, Proof of Theorem 1]). We shall prove that $(\mathbb{Z}, +, x^* = x)$ is not strongly operator semiperfect, too.

Theorem 3.7. *There exists a function $\varphi : \mathbb{Z} \to M_2(\mathbb{C})$ which is positive definite and not of positive type.*

Proof. Let $a_n := 2^{(n+2)!}$, $n \geq 0$. Define $\varphi : \mathbb{Z} \to M_2(\mathbb{C})$ as follows:

$$\varphi(0) := \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad \varphi(1) := \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \quad \varphi(2) := \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad \varphi(-1) := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \varphi(n) &:= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \ (|n| \geq 3, n : \text{odd}),\\ \varphi(n) &:= \begin{bmatrix} a_n & 0 \\ 0 & a_n \end{bmatrix} \ (n \geq 4, n : \text{even}),\\ \varphi(n) &:= \begin{bmatrix} a_{|n|+1} & 0 \\ 0 & a_{|n|+1} \end{bmatrix} \ (n \leq -2, n : \text{even}). \end{aligned}$$

Put $\xi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\xi_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Then

$$\sum_{i,j=0}^{1} \langle \varphi(i+j)\xi_i, \xi_j \rangle = -2 < 0,$$

which implies that φ is not of positive type. To see that φ is positive definite, we shall prove that

$$D_n(\xi) := \begin{vmatrix} \langle \varphi(-2n)\xi, \xi \rangle & \cdots & \langle \varphi(0)\xi, \xi \rangle \\ \vdots & \ddots & \vdots \\ \langle \varphi(0)\xi, \xi \rangle & \cdots & \langle \varphi(2n)\xi, \xi \rangle \end{vmatrix}, \quad n \geq 0,$$

and

$$D'_n(\xi) := \begin{vmatrix} \langle \varphi(-2n+2)\xi, \xi \rangle & \cdots & \langle \varphi(1)\xi, \xi \rangle \\ \vdots & \ddots & \vdots \\ \langle \varphi(1)\xi, \xi \rangle & \cdots & \langle \varphi(2n)\xi, \xi \rangle \end{vmatrix}, \quad n \geq 1,$$

are positive for $\xi = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \in \mathbb{C}^2$ with $|\alpha_1|^2 + |\alpha_2|^2 = 1$ (cf. [1, Remark 6.4.2]). We have

$$\begin{aligned} D_0(\xi) &= 4|\alpha_1|^2 + |\alpha_2|^2 \geq 1, \\ D_1'(\xi) &= 4|\alpha_1|^4 + 4|\alpha_2|^4 + 17|\alpha_1\overline{\alpha_2}|^2 - 16(\mathrm{Re}(\alpha_1\overline{\alpha_2}))^2 \\ &\geq 2(|\alpha_1|^2 + |\alpha_2|^2)^2 \geq 1, \\ D_1(\xi) &= a_3 D_1'(\xi) - (4|\alpha_1|^2 + |\alpha_2|^2)^3 \\ &\geq 2^{5!} - 2^6 \geq 1. \end{aligned}$$

Let $n \geq 2$ and suppose that $D_{n-1}'(\xi) \geq 1$ and $D_{n-1}(\xi) \geq 1$. We show that $D_n'(\xi) \geq 1$ and $D_n(\xi) \geq 1$. For $k = -2n+2, -2n+3, \cdots, 2n-1$, we have

$$\begin{aligned} |\langle \varphi(k)\xi, \xi\rangle| &\leq ||\varphi(k)|| \\ &\leq \max\{||\varphi(-2n+2)||,\ ||\varphi(2n-2)||\} \\ &\leq a_{2n-1}, \end{aligned}$$

so that

$$|\langle \varphi(k)\xi, \xi\rangle| \leq a_{2n-1}. \tag{3.1}$$

Similarly for $k = -2n+1, -2n+2, \cdots, 2n$,

$$|\langle \varphi(k)\xi, \xi\rangle| \leq a_{2n}. \tag{3.2}$$

Then the estimate (3.1) yields

$$\begin{aligned} D_n'(\xi) &\geq a_{2n} D_{n-1}(\xi) - (2n-1)a_{2n-1}^{2n}(2n-1)! \\ &\geq 2^{(2n+2)!} - (2n)!2^{(2n+1)!2n} \\ &= 2^{2n(2n+1)!}(2^{2(2n+1)!} - (2n)!) \\ &\geq 2^{(2n)!} - (2n)! \geq 1. \end{aligned}$$

Accordingly, the estimate (3.2) yields

$$\begin{aligned} D_n(\xi) &\geq a_{2n+1} D_n'(\xi) - (2n)a_{2n}^{2n+1}(2n)! \\ &\geq 2^{(2n+3)!} - (2n+1)!2^{(2n+2)!(2n+1)} \\ &= 2^{(2n+2)!(2n+1)}(2^{2(2n+2)!} - (2n+1)!) \\ &\geq 2^{(2n+1)!} - (2n+1)! \geq 1. \end{aligned}$$

Thus, by induction, we get $D_n'(\xi) \geq 1$, $n \geq 1$, and $D_n(\xi) \geq 1$, $n \geq 0$. □

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On the product of Riesz sets in dual objects of compact groups

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Abstract. Let E_i be a Riesz set in the dual object of a compact group $K_i (i = 1, 2)$. We show that the product set $E_1 \times E_2$ is a Riesz set in the dual object of $K_1 \oplus K_2$. We also give a result on compact groups related to a result of Glicksberg and Graham concerned with "small p set".

§1　　$\mathbf{T}$ 上の F.and M. Riesz の定理の $\mathbf{T}^2$ への拡張として次のBochner の結果がある。

定理 1.1.　　$\mu \in M(\mathbf{T}^2)$, $\hat{\mu}(n, m) = 0$ for $(n, m) \notin \mathbf{Z}^+ \times \mathbf{Z}^+ \Longrightarrow \mu \ll m_{\mathbf{T}^2}$.

定理1.1 は $\hat{\mathbf{T}} \cong \mathbf{Z}$ の Riesz 集合 $\mathbf{Z}^+$ の積 $\mathbf{Z}^+ \times \mathbf{Z}^+$ は $\hat{\mathbf{T}^2} \cong \mathbf{Z} \oplus \mathbf{Z}$ の Riesz 集合になることを示している。この種のことについては、局所コンパクト可換群についても成り立つ。

定義 1.1.　　G を局所コンパクト可換群とし、p を自然数とする。$\hat{G}$ の閉集合 E が次を満たすとき、small p set と呼ばれる。

(1. 1)　　$\forall \mu \in M_E(G) \Longrightarrow \mu^p = \overbrace{\mu * \cdots * \mu}^{p} \in L^1(G)$.

但し、$M_E(G) = \{\mu \in M(G) : \hat{\mu} = 0 \text{ on } E^c\}$. 特に、small 1 set は Riesz 集合と呼ばれる。

定理 1.2 (cf.[6]).　　G_1, G_2 を局所コンパクト可換群とし、p を自然数とする。E_1, E_2 をそれぞれ $\hat{G}_1, \hat{G}_2$ の small p set とする。すると、$E_1 \times E_2$ は $G_1 \oplus G_2$ の small p set である。

small 2 set になるための条件としては、Graham によって与えられた次の結果がある。

定理 1.3 (cf.[3]).　　G を局所コンパクト可換群とし、S を次を満たす $\hat{G}$ の Borel 集合とする。

(1. 2)　　$\{\gamma \in \hat{G} : m_{\hat{G}}(S \cap (\gamma - S)) < \infty\}$ は $\hat{G}$ で稠密。

すると、$\mu, \nu \in M_S(G) \Longrightarrow |\mu| * |\nu| \in L^1(G)$.

ここでは、定理 1.1、定理 1.3 に関連したことを（非可換）コンパクト群にたいして考えてみる。

§2　K をコンパクト群とし、Σ_K を K の dual object とする。$M(K)$ を K 上の bounded regular measures の空間とし、m_K を K 上の Haar measure とする。$\sigma \in \Sigma_K$ に対して、$U^{(\sigma)}$ を σ に属し、H_σ を表現空間として持つ K の continuous irreducible unitary representation とする。$\mu \in M(K)$ に対して、μ の Fourier 変換 $\hat{\mu}$ を次のように定義する：$\sigma \in \Sigma_K; \xi, \eta \in H_\sigma$ に対して、

$$< \hat{\mu}(\sigma)\xi, \eta >= \int_K < \bar{U}_x^{(\sigma)}\xi, \eta > d\mu(x).$$

但し、$\bar{U}_x^{(\sigma)} = D_\sigma U_x^{(\sigma)} D_\sigma$。又、$D_\sigma$ は H_σ 上の conjugation。そして、$spec(\mu) = \{\sigma \in \Sigma_K : \hat{\mu}(\sigma) \neq 0\}$ とおく。Σ_K には、conjugation ”$^-$” と積 ”×” の 2 つの operations が定義される。

定義 2.1.　p を自然数とする。Σ_K の部分集合 E が次を満たすとき s-small p set と言うことにする。
(2. 1)　$\forall \mu_1, \cdots, \mu_p \in M_E(K) \Longrightarrow \mu_1 * \cdots * \mu_p \in L^1(K)$.
但し、$M_E(K) = \{\mu \in M(K) : spec(\mu) \subset E\}$.　特に、s-small 1 set は Riesz 集合と呼ばれる。

注意.　K が compact abelian group の時は、”s-small p set” と ”small p set” は同じ概念である。

定理 2.1.　p を自然数とし、K_1, K_2 を compact groups とする。E_1, E_2 をそれぞれ $\Sigma_{K_1}, \Sigma_{K_2}$ の s-small p sets とする。すると、$E_1 \times E_2$ は　$\Sigma_{K_1 \oplus K_2} \cong \Sigma_{K_1} \times \Sigma_{K_2}$ における s-small p set である。

系.　E_1, E_2 をそれぞれ $\Sigma_{K_1}, \Sigma_{K_2}$ の Riesz 集合とする。すると、$E_1 \times E_2$ は $\Sigma_{K_1 \oplus K_2} \cong \Sigma_{K_1} \times \Sigma_{K_2}$ における Riesz 集合である。

次に、定理 1.3 に対応したことを compact group の場合について考えてみる。G が compact abelian group の場合は、定理 1.3 の条件 (1. 2) は次の条件 (1. 2)' となる。

(1. 2)'　$\forall \gamma \in \hat{G}$ にたいして、$S \cap (\gamma - S)$ は有限集合。

又、条件 (1. 2)' は次の条件 (1. 2)" と同値である。

(1. 2)"　$\forall \gamma_1, \gamma_2 \in \hat{G}$ に対して、$(\gamma_1 + S) \cap (\gamma_2 - S)$ は有限集合。

定理 2.2.　K を compact group とし、Δ を次を満たす Σ_K の部分集合とする。
(2. 2)　$\forall \sigma, \tau \in \Sigma_K$ に対して、$(\sigma \times \Delta) \cap (\tau \times \bar{\Delta})$ は有限集合。
すると、$\forall \mu, \nu \in M_\Delta(K) \Longrightarrow |\mu| * |\nu| \in L^1(K)$。
但し、$\bar{\Delta} = \{\bar{\omega} : \omega \in \Delta\}, \sigma \times \Delta = \{\sigma \times \eta : \eta \in \Delta\}$。特に、$\Delta$ は s-small 2 set である。

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CONTROLS OF THE OUTPUTS BY MEANS OF INPUTS

(abstract)

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Abstract: Let $\mathbf{f}_j$ be a member of a Hilbert space $\mathcal{H}_j$, S_j be a linear system of $\mathcal{H}_j$ and f_j be the output of $\mathbf{f}_j$ in the system. We assume that the outputs f_j are functions on a same set E. Then we consider the problems :

How to find the sum $f_1 + f_2$, the product $f_1 f_2$, and etc by means of their inputs $\mathbf{f}_j$?

The theory of reproducing kernels will give natural answers in natural situations for these problems.

Surprising enough, for very general nonlinear system S_j, we will be able to discuss the similar problems.

AMS Subj. Classification: 30C40, 46E32, 44A10, 35A22.

Key Words: Hilbert space, reproducing kernel, linear transform, nonlinear transform, convolution, norm inequality, integral equation, nonlinear differential equation, algebraic structure in Hilbert spaces.

1. A General Concept

Following Saitoh [1], we shall introduce a general theory for linear transforms in the framework of Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space. Let E be an abstract set and $\mathbf{h}$ be a Hilbert $\mathcal{H}$-valued function on E. Then we shall consider the linear transform

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \mathbf{f} \in \mathcal{H} \tag{1.1}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued function on E. In order to investigate the linear transform (1.1), we form a positive matrix $K(p,q)$ on E defined by

$$K(p,q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}} \text{ on } E \times E. \tag{1.2}$$

Then, we obtain the following :

(I) The range in the linear transform (1.1) by $\mathcal{H}$ is characterized as the reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(p,q)$.

(II) In general, we have the inequality

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}}.$$

Here, for a member f of $H_K(E)$ there exists a uniquely determined $\mathbf{f}^* \in \mathcal{H}$ satisfying

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}} \text{ on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}.$$

(III) In general, we have the inversion formula in (1.1) in the form

$$f \to \mathbf{f}^* \tag{1.3}$$

in (II) by using the reproducing kernel Hilbert space $H_K(E)$. However, this formula is, in general, involved and delicate. We need, case by case, arguments. In this paper, we assume that the inversion formula (1.3) is established.

(IV) Conversely, we assume that an isometrical mapping $\widetilde{L}$ from a reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(p,q)$ on E onto a Hilbert space $\mathcal{H}$. Then we have the representation (1.1) by

$$\mathbf{h}(p) := \widetilde{L}K(\cdot,p).$$

Furthermore, $\{\mathbf{h}(p); p \in E\}$ is complete in $\mathcal{H}$.

Now we shall consider two systems

$$f_j(p) = (\mathbf{f}_j, \mathbf{h}_j(p))_{\mathcal{H}_j}, \quad \mathbf{f}_j \in \mathcal{H}_j \tag{1.4}$$

in the above way by using $\{\mathcal{H}_j, E, \mathbf{h}_j\}_{j=1}^2$. Here, we assume that E is a same set for the two systems in order to have the output functions $f_1(p)$ and $f_2(p)$ on the same set E.

For example, we can consider the operators

$$f_1(p) + f_2(p)$$

and

$$f_1(p)\ f_2(p)$$

in $\mathcal{F}(E)$. Then, we can consider the following problems : How to represent the sum $f_1(p) + f_2(p)$ and the product $f_1(p)\ f_2(p)$ on E in terms of their inputs $\mathbf{f}_1$ and $\mathbf{f}_2$?

In this abstract we shall show that by using the theory of reproducing kernels we can give natural answers for these problems.

2. Sum

By (I), $f_1 \in H_{K_1}(E)$ and $f_2 \in H_{K_2}(E)$, and we note that for the reproducing kernel Hilbert space $H_{K_1+K_2}(E)$ admitting the reproducing kernel

$$K_1(p,q) + K_2(p,q) \quad \text{on } E,$$

$H_{K_1+K_2}(E)$ is composed of all functions

$$f(p) = f_1(p) + f_2(p); \quad f_j \in H_{K_j}(E) \tag{2.1}$$

and its norm $\|f\|_{H_{K_1+K_2}(E)}$ is given by

$$\|f\|^2_{H_{K_1+K_2}(E)} = \min\{\|f_1\|^2_{H_{K_1}(E)} + \|f_2\|^2_{H_{K_2}(E)}\} \tag{2.2}$$

where the minimum is taken over $f_j \in H_{K_j}(E)$ satisfying (2.1) for f. Hence, in general, we have the inequality

$$\|f_1 + f_2\|^2_{H_{K_1+K_2}(E)} \leq \|f_1\|^2_{H_{K_1}(E)} + \|f_2\|^2_{H_{K_2}(E)}. \tag{2.3}$$

For the positive matrix $K_1 + K_2$ on E, we assume the expression in the form

$$K_1(p,q) + K_2(p,q) = (\mathbf{h}_S(q), \mathbf{h}_S(p))_{\mathcal{H}_S} \quad \text{on } E \times E \tag{2.4}$$

with a Hilbert space $\mathcal{H}_S$-valued function on E and further we assume that

$$\{\mathbf{h}_S(p); p \in E\} \text{ is complete in } \mathcal{H}_S. \tag{2.5}$$

Such a representation is, in general, possible (Saitoh [1], page 36 and see chapter 1, §5). Then, we can consider the linear mapping from $\mathcal{H}_S$ onto $H_{K_1+K_2}(E)$

$$f_S(p) = (\mathbf{f}_S, \mathbf{h}_S(p))_{\mathcal{H}_S}, \quad \mathbf{f}_S \in \mathcal{H}_S \tag{2.6}$$

and we obtain the isometrical identity

$$\|f_S\|_{H_{K_1+K_2}(E)} = \|\mathbf{f}_S\|_{\mathcal{H}_S}. \tag{2.7}$$

Hence, for such representations (2.4) with (2.5), we obtain the isometrical mappings among the Hilbert space $\mathcal{H}_S$.

Now, for the sum $f_1(p) + f_2(p)$ there exists a uniquely determined $\mathbf{f}_S \in \mathcal{H}_S$ satisfying

$$f_1(p) + f_2(p) = (\mathbf{f}_S, \mathbf{h}_S(p))_{\mathcal{H}_S} \text{ on } E. \tag{2.8}$$

Then, $\mathbf{f}_S$ will be considered as a sum of $\mathbf{f}_1$ and $\mathbf{f}_2$ through these transforms and so, we shall introduce the notation

$$\mathbf{f}_S = \mathbf{f}_1[+]\mathbf{f}_2. \tag{2.9}$$

This sum for the members $\mathbf{f}_1 \in \mathcal{H}_1$ and $\mathbf{f}_2 \in \mathcal{H}_2$ is introduced through the three transforms induced by $\{\mathcal{H}_j,\ E,\ \mathbf{h}_j\}$ $(j = 1,2)$ and $\{\mathcal{H}_s,\ E,\ \mathbf{h}_S\}$.

The operator $\mathbf{f}_1[+]\mathbf{f}_2$ is expressible in terms of $\mathbf{f}_1$ and $\mathbf{f}_2$ by the inversion formula

$$(\mathbf{f}_1, \mathbf{h}_1(p))_{\mathcal{H}_1} + (\mathbf{f}_2, \mathbf{h}_2(p))_{\mathcal{H}_2} \longrightarrow \mathbf{f}_1[+]\mathbf{f}_2 \tag{2.10}$$

in the sense (II) from $H_{K_1+K_2}(E)$ onto $\mathcal{H}_S$. Then, from (II) and (2.5) we have

Theorem 2.1. *We have a triangle inequality*

$$\|\mathbf{f}_1[+]\mathbf{f}_2\|^2_{\mathcal{H}_S} \leq \|\mathbf{f}_1\|^2_{\mathcal{H}_1} + \|\mathbf{f}_2\|^2_{\mathcal{H}_2}. \tag{2.11}$$

If $\{\mathbf{h}_j(p);\ p \in E\}$ are complete in $\mathcal{H}_j$ $(j = 1,2)$, then $\mathcal{H}_j$ and H_{K_j} are isometrical. By using the isometrical mappings induced by Hilbert space valued function $\mathbf{h}_j$ $(j = 1,2)$ and $\mathbf{h}_S$, we can introduce the sum space of $\mathcal{H}_1$ and $\mathcal{H}_2$ in the form

$$\mathcal{H}_1[+]\mathcal{H}_2 \tag{2.12}$$

through the transforms.

For example, if for some positive number γ

$$K_1 \ll \gamma^2 K_2 \text{ on } E \tag{2.13}$$

that is, $\gamma^2 K_2 - K_1$ is a positive matrix on E, we have

$$H_{K_1}(E) \subset H_{K_2}(E) \tag{2.14}$$

and

$$\|f_1\|_{H_{K_2}(E)} \leq \gamma \|f_1\|_{H_{K_1}(E)} \text{ for } f_1 \in H_{K_1}(E) \tag{2.15}$$

(Saitoh [1], page 37). Hence, in this case, we need not to introduce a Hilbert space $\mathcal{H}_S$ and the linear mapping (2.6) in Theorem (2.1) and we can use the linear mapping

$$(\mathbf{f}_2, \mathbf{h}_2(p))_{\mathcal{H}_2}, \quad \mathbf{f}_2 \in \mathcal{H}_2$$

instead of (2.6) in Theorem 2.1.

3. Product

The product $K_1(p,q)\ K_2(p,q)$ is a positive matrix on E and the reproducing kernel Hilbert space $H_{K_1 K_2}(E)$ admitting the reproducing kernel $K_1(p,q)\ K_2(p,q)$ is composed of all functions

$$f(p) = \sum_{n=1}^{\infty} f_{1,n}(p) f_{2,n}(p) \text{ on } E; \tag{3.1}$$

$$f_{j,n}(p) \in H_{K_j}(E) \quad (j = 1,2)$$

and the norm in $H_{K_1K_2}(E)$ is given by

$$\|f\|^2_{H_{K_1K_2}(E)} = \min \sum_{n=1}^{\infty} \|f_{1,n}\|^2_{H_{K_1}(E)} \|f_{2,n}\|^2_{H_{K_2}(E)} \tag{3.2}$$

where the minimum is taken over all functions satisfying (3.1) for f. In particular, (3.1) converges absolutely on E. Especially we obtain the inequality

$$\|f_1 f_2\|_{H_{K_1K_2}(E)} \leq \|f_1\|_{H_{K_1}(E)} \|f_2\|_{H_{K_2}(E)}. \tag{3.3}$$

As in the sum, we assume that the representation

$$K_1(p,q)K_1(p,q) = (\mathbf{h}_P(q), \mathbf{h}_P(p))_{\mathcal{H}_P} \text{ on } E \times E \tag{3.4}$$

with a Hilbert space $\mathcal{H}_P$-valued function on E, and we assume that

$$\{\mathbf{h}_P(p); p \in E\} \text{ is complete in } \mathcal{H}_P. \tag{3.5}$$

Then we consider the linear mapping

$$f_P(p) = (\mathbf{f}_P, \mathbf{h}_P(p))_{\mathcal{H}_P}, \ \mathbf{f}_P \in \mathcal{H}_P \tag{3.6}$$

and we obtain the isometrical identity

$$\|f_P\|_{H_{K_1K_2}}(E) = \|\mathbf{f}_P\|_{H_P}. \tag{3.7}$$

Hence, for any product $f_1(p)f_2(p)$ there exists a uniquely determined $\mathbf{f}_P \in \mathcal{H}_P$ satisfying

$$f_1(p)f_2(p) = (\mathbf{f}_P, \mathbf{h}_P(p))_{\mathcal{H}_P} \text{ on } E. \tag{3.8}$$

Then, $\mathbf{f}_P$ will be considered as a product of $\mathbf{f}_1$ and $\mathbf{f}_2$ through these transforms and so, we shall introduce the notation

$$\mathbf{f}_P = \mathbf{f}_1[\times]\mathbf{f}_2. \tag{3.9}$$

This product for the members $\mathbf{f}_j \in \mathcal{H}_j$ $(j = 1,2)$ is introduced through the three transforms induced by $\{\mathcal{H}_j,\ E,\ \mathbf{h}_j\}$ $(j = 1,2)$ and $\{\mathcal{H}_P,\ E,\ \mathbf{h}_P\}$. The operator $\mathbf{f}_1[\times]\mathbf{f}_2$ is expressible in terms of $\mathbf{f}_1$ and $\mathbf{f}_2$ by the inversion formula

$$(\mathbf{f}_1, \mathbf{h}_1(p))_{\mathcal{H}_1} (\mathbf{f}_2, \mathbf{h}_2(p))_{\mathcal{H}_2} \longrightarrow \mathbf{f}_1[\times]\mathbf{f}_2 \tag{3.10}$$

in the sense (III) from $H_{K_1K_2}(E)$ onto $\mathcal{H}_P$. Then, we obtain

Theorem 3.1. *We have a Schwarz type inequality*

$$\|\mathbf{f}_1[\times]\mathbf{f}_2\|_{\mathcal{H}_P} \leq \|\mathbf{f}_1\|_{\mathcal{H}_1} \|\mathbf{f}_2\|_{\mathcal{H}_2}. \tag{3.11}$$

As in the sum space $\mathcal{H}_1[+]\mathcal{H}_2$ we can introduce the product space

$$\mathcal{H}_1[\times]\mathcal{H}_2 \tag{3.12}$$

through the three transforms under the completeness assumptions of $\mathbf{h}_j$ in $\mathcal{H}_j$ $(j = 1, 2)$.

For example, if for a positive γ

$$K_1 K_2 \ll \gamma^2 K_1 \text{ on } E, \tag{3.13}$$

as in the sum, we can consider the linear transform

$$(\mathbf{f}_1, \mathbf{h}_1(p))_{\mathcal{H}_1}, \quad \mathbf{f}_1 \in \mathcal{H}_1$$

instead of (3.6).

In particular, in the setting in Section 1, we obtain

Corollary 3.1. *If* $K^2 \ll \gamma^2 K$ *on* E *for a positive constant* γ *and* $\{\mathbf{h}(p); p \in E\}$ *is complete in* $\mathcal{H}$*, then* $\mathcal{H}$ *is a commutative ring with the product* $\mathrm{f}[\times]\mathrm{g}$ *through the same three transforms* $\{\mathcal{H},\ E,\ \mathbf{h}\}$*. Furthermore if* $\gamma = 1$*,* $\mathcal{H}$ *is a Banach ring with the product.*

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Genus of the boundary generating curve of the numerical range of a matrix

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Abstract

In this talk, I discuss the boundary generating curve of the numerical range of a matrix. The main theorem determines the numerical range of a certain type of nilpotent Toeplitz matrices.

1　行列の数域の境界を形成する代数曲線

$n \times n$ 複素行列 Tに対し、その数域 $W(T)$ が、

$$W(T) = \{(T\xi, \xi) : \xi \in \mathbf{C}^n,\ ||\xi|| = 1\}$$

によって定義される。数域 $W(T)$ は、Toeplitz-Hausdorff の定理よりコンパクト凸集合となる。実代数幾何学における Tarski=Seidenberg 定理より、 $W(T)$ の境界 $\partial W(T)$ は、有限個の代数曲線弧の和集合となっていることがわかる (cf. [N-N-T])。

A を $n \times n$ 複素行列とし、次のようなエルミット行列の固有値$\lambda_j(\theta)$ を各$\theta \in \mathbf{R}$ に対して考える：

$$(\cos\theta)(A + A^*)/2 - (\sin\theta)(A - A^*)/(2\,i) = (\exp(i\,\theta)\,A)_h,$$

$$\det(t\,I_n - (\cos\theta)\,(A + A^*)/2 + (\sin\theta)\,(A - A^*)/(2\,i)) = \prod_{j=1}^{n}(t - \lambda_j(\theta)).$$

このとき、 点 $[(\lambda_j(\theta), -\cos\theta, \sin\theta)]$ は、次のような代数曲線 C:

$$\{[(t, x, y)] \in \mathbf{RP}^2 : \det(t\,I_n + (x/2)(A + A^*) - i\,(y/2)(A - A^*)) = 0\}$$

上にある。逆に曲線 C は、次のような弧の有限個の和集合である:

$$\{[(\lambda_j(\theta), -\cos\theta, \sin\theta)] : \alpha_j \leq \theta \leq \beta_j\},$$

ここで、各　$\lambda_j(\theta)$ は変数θについての実解析関数であり、C の双対曲線 $C^\wedge$は、次のような弧の和集合となる $\{x_j(\theta) + i\,y_j(\theta) : \alpha_j \leq \theta \leq \beta_j\}$ ここで、関数 $x_j(\theta)$, $x_j(\theta)$ は、次のようにして与えられる：

$$x_j(\theta) = \cos\theta\lambda_j(\theta) - \sin\theta\lambda_j^{'}(\theta),$$

$$y_j(\theta) = -\sin\theta\lambda_j(\theta) - \cos\theta\lambda_j^{'}(\theta).$$

ここで、行列 A の数域 $W(A)$ は、複素数平面 $\mathbf{C}$ における曲線 $\{x_j(\theta) + i\,y_j(\theta) : \alpha_j \leq \theta \leq \beta_j\}$ の凸包と一致する。この意味で、曲線 $x_j(\theta) + i\,y_j(\theta)$ を $W(A)$ の境界を形成する代数曲線 *boundary generating curve* of $W(A)$ と言うことにする。

2 Toelitz 型の巾零行列で数域の境界が有理曲線となるもの

複素アフィン 曲線 $C = \{(z, w) : f(z, w) = 0\}$ は、その有限個の点を除いて 2 つの 1 変数有理関数 $\phi(t)$, $\psi(t)$ を用いて

$$\{(\phi(t), \psi(t)) : t \in \mathbf{C} \backslash E\}$$

と媒介変数表示されるとき、有理曲線であると言われる。ここで、Eは、ϕ, ψの分母の零点からなる有限集合である。ここで複素射影代数曲線Cの非特異モデルと呼ばれるCと双有理同値な特異点を持たない空間"曲線" [複素曲線、リーマン面] $C^\sharp$を取れば、Cが有理曲線であることと、リーマン面 $C^\sharp$の種数 (genus) が 0 であることとは、同値である (cf. [Wal])。

$n = 2m$ を 2 以上の偶数とし、$\beta_1, \ldots, \beta_{(n-2)/2}$ を任意の複素数とし、 $\beta_{n/2} \in \mathbf{R}$ とするとき、次のような $n \times n$ 巾零行列 $B = A(\beta_1, \ldots, \beta_{(n-2)/}, \beta_{n/2})$ を

$$B = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \cdots & \beta_{n/2-1} & \beta_{n/2} & \overline{\beta_{n/2-1}} & \cdots & \overline{\beta_2} & \overline{\beta_1} \\ 0 & 0 & \beta_1 & \cdots & \beta_{n/2-2} & \beta_{n/2-1} & \beta_{n/2} & \cdots & \overline{\beta_3} & \overline{\beta_2} \\ 0 & 0 & 0 & \cdots & \beta_{n/2-3} & \beta_{n/2-2} & \beta_{n/2-1} & \cdots & \overline{\beta_4} & \overline{\beta_3} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \beta_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.1}$$

により定める。また $n = 2m - 1$ を 3 以上の奇数とし$\beta_1, \ldots, \beta_{(n-1)/2}$ を任意の複素数とするとき、次のような $n \times n$ 巾零行列 $B = A(\beta_1, \ldots, \beta_{(n-2)/}, \beta_{(n-1)/2})$ を次のように定める。

$$B = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \cdots & \beta_{(n-1)/2} & \overline{\beta_{(n-1)/2}} & \cdots & \overline{\beta_2} & \overline{\beta_1} \\ 0 & 0 & \beta_1 & \cdots & \beta_{(n-3)/2} & \beta_{(n-1)/2} & \cdots & \overline{\beta_3} & \overline{\beta_2} \\ 0 & 0 & 0 & \cdots & \beta_{(n-5)/2} & \beta_{(n-3)/2} & \cdots & \overline{\beta_4} & \overline{\beta_3} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \beta_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.2}$$

Theorem [M. T. Chien（簡　茂丁)-Nakazato [C-N]] Bは、 Toeplitz 行列であるような巾零行列 (2.1) または (2.2) とする。. このとき $W(B)$　の境界を形成する代数曲線 は、有理曲線である。数域 $W(B)$ は次のような曲線の凸包である C:

$$x(\theta)+i\,y(\theta)=\exp(-i\,\theta)\lambda(\theta)-i\,\exp(-i\,\theta)\lambda'(\theta),$$

ここで、$0\le\theta\le n\pi$とする。ただし、n が偶数 $n=2m$ ならば、

$$\lambda(\theta)=(1/2)\beta_m+\Re(\sum_{j=1}^{m-1}\beta_{m-j}\exp(i\,[\frac{2j}{n}\theta])),$$

であり、n が奇数 $n=2m-1$ ならば、次のようになる。

$$\lambda(\theta)=\Re(\sum_{j=1}^{m-1}\beta_{m-j}\exp(i\,[\frac{2j-1}{n}\theta])).$$

[定理が適用できる具体例]

$$B_1=\begin{pmatrix}0 & 1+i & 2+3i & 4 & 2-3i & 1-i\\ 0 & 0 & 1+i & 2+3i & 4 & 2-3i\\ 0 & 0 & 0 & 1+i & 2+3i & 4\\ 0 & 0 & 0 & 0 & 1+i & 2+3i\\ 0 & 0 & 0 & 0 & 0 & 1+i\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

3　一般化された循環行列の固有値

前節で述べた定理を証明するためのアイデアについて述べよう。第 1 節で述べたような理由で、行列 A の数域 $W(A)$ は、各実数 θに対し、エルミット行列 $(\exp(i\,\theta)\,A)_h$ を考え、その固有値が与えられれば、決定される。例えば、

$$A=\begin{pmatrix}0 & r_1\exp(i\eta) & r_2\exp(i\zeta) & r_2\exp(-i\zeta) & r_1\exp(-i\eta)\\ 0 & 0 & r_1\exp(i\eta) & r_2\exp(i\zeta) & r_2\exp(-i\zeta)\\ 0 & 0 & 0 & r_1\exp(i\eta) & r_2\exp(i\zeta)\\ 0 & 0 & 0 & 0 & r_1\exp(i\eta)\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

($r_1,r_2\in\mathbf{R}$, $\eta,\zeta\in\mathbf{R}$) ならば、

$$2(\exp(i\,\theta)\,A)_h=$$

$$\begin{pmatrix}0 & r_1\exp(i[\theta+\eta]) & r_2\exp(i[\theta+\zeta]) & r_2\exp(i[\theta-\zeta]) & r_1\exp(i[\theta-\eta])\\ r_1\exp(-i[\theta+\eta]) & 0 & r_1\exp(i[\theta+\eta]) & r_2\exp(i[\theta+\zeta]) & r_2\exp(i[\theta-\zeta])\\ r_2\exp(-i[\theta+\zeta]) & r_1\exp(-i[\theta+\eta]) & 0 & r_1\exp(i[\theta+\eta]) & r_2\exp(i[\theta+\zeta])\\ r_2\exp(-i[\theta-\zeta]) & r_2\exp(-i[\theta+\zeta]) & r_1\exp(-i[\theta+\eta]) & 0 & r_1\exp(i[\theta+\eta])\\ r_1\exp(-i[\theta-\eta]) & r_2\exp(-i[\theta-\zeta]) & r_2\exp(-i[\theta+\zeta]) & r_1\exp(-i[\theta+\eta]) & 0\end{pmatrix}$$

となるが、この $2(\exp(i\,\theta)\,A)_h$ =は、$\alpha = \exp(-2i\theta)$ に対し、

$$\begin{pmatrix} 0 & c_1 & c_2 & c_3 & c_4 \\ \alpha c_4 & 0 & c_1 & c_2 & c_3 \\ \alpha c_3 & \alpha c_4 & 0 & c_1 & c_2 \\ \alpha c_2 & \alpha c_3 & \alpha c_4 & 0 & c_1 \\ \alpha c_1 & \alpha c_2 & \alpha c_3 & \alpha c_4 & 0 \end{pmatrix}$$

という形をしている。一般に、複素数 $\alpha \neq 0$ および、複素数 $c_1, c_2, \ldots, c_{n-1}$ に対し、$n \times n$ 行列を、

$$\begin{pmatrix} 0 & c_1 & c_2 & c_3 & \ldots & c_{n-1} \\ \alpha c_{n-1} & 0 & c_1 & c_2 & \ldots & c_{n-2} \\ \alpha c_{n-2} & \alpha c_{n-1} & 0 & c_1 & \ldots & c_{n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha c_1 & \alpha c_2 & \alpha c_3 & \alpha c_4 & \ldots & 0 \end{pmatrix}$$

で定義し、これを一般化された循環行列と言う。この行列は、次のような weighted shift S を考えるとき、

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ \alpha & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

これを用いて、 $c_1 S + c_2 S^2 + \ldots + c_{n-1} S^{n-1}$ と表わすことができる。行列 S は、多項式 $\phi(x) = x^n - \alpha$ に対する companion matrix であり、Sの固有多項式は、$\phi(x)$ であり、αの n 乗根が Sの固有値となる。これより、$c_1 S + \ldots + c_{n-1} S^{n-1}$ の固有値も求まる。このような原理で、(2.1), (2.2) で登場した巾零行列の数域が決定される。

4　通常多重点と代数曲線の種数

A が、$n \times n$ 複素行列とし、複素代数曲線 $C^{\mathbf{C}}$を次のような集合として定める:

$$\{[(t,x,y)] \in \mathbf{CP}^2 : \det(t\,I_n + (x/2)(A + A^*) - i\,(y/2)(A - A^*)) = 0\}.$$

ここで、複素同次多項式 $F(t,x,y) = F_A(t,x,y)$ を次の関係式により定める：

$$F(t,x,y) = \det(t\,I_n + (x/2)(A + A^*) - i\,(y/2)(A - A^*)).$$

曲線 $C^{\mathbf{C}}$の一点 $P_0 = [(t_0, x_0, y_0)]$ を取る 。ここで、必要ならば $F(t,x,y) = t^p F_1(t,x,y)$ と因数分解する [$p \geq 0$] ことにより $t_0 \neq 0$ 従って $t_0 = 1$ と仮定できる。さて、 $f(x,y) = F(1,x,y)$ と置く。さて、

$$f(x_0 + x, y_0 + y) = \sum_{j=0}^{m} a_{j,m-j} x^j\, y^{m-j} + \sum_{j+k>m} a_{j,k} x^j\, y^k,$$

と表わすことができる。ここで、 m は自然数であって $a_{j,m-j} \neq 0$ が或る $0 \leq j \leq m$ に対して成り立つものとする。ここで、次のような 2 変数同次多項式を考える

$$T_{P_0}(x,y)=\sum_{j=0}^{m} a_{j,m-j}x^j y^{m-j}=\prod_{k=1}^{m}(\alpha_k x+\beta_k y),$$

ここで各 k に対し $(\alpha_k,\beta_k)\in \mathbf{C}\backslash\{(0,0)\}$ とする。このとき、各直線

$$\{[(1,x,y)]\in \mathbf{CP}^2 : \alpha_k(x-x_0)+\beta_k(y-y_0)=0\}\cup\{[(0,\beta_k,-\alpha_k)]\}$$

は点 P_0 における曲線 $C^{\mathbf{C}}$の接線 tangent (line) と言われる。$m\geq 2$ ならば, 点 P_0 は曲線 $C^{\mathbf{C}}$の特異点 *singular* point とか多重点 *multiple* point と呼ばれる。$m=1$ ならば, 点 P_0 は非特異点 a non-singular point であると言う。$m=1$ ならば曲線 $C^{\mathbf{C}}$ は点 P_0で唯一つの接線を持つ。多項式 F が重複因子を含まないならば、$C^{\mathbf{C}}$ の特異点の個数は有限個である。$m\geq 2$ であって $C^{\mathbf{C}}$ の点 P_0における接線 $\alpha_k(x-x_0)+\beta_k(y-y_0)=0$ が各 kごとに異なるとき、すなわち $\alpha_j\beta_k-\alpha_k\beta_j\neq 0$ が、任意の $1\leq j\neq k\leq m$ に対して成り立つとき、点 P_0は、通常 m 重点 *ordinary* m-ple point であると言われる (cf. [Wal])。

$C^{\mathbf{C}}$が既約な n 次曲線であってその特異点がすべて通常多重点であり、その多重度が r_j ($j=1,2,\ldots,p$) であるとき、曲線 $C^{\mathbf{C}}$の種数、示性数 *genus* g が次の式で与えられる（定義される)。

$$g=\frac{1}{2}(n-1)(n-2)-\frac{1}{2}\sum_{j=1}^{p} r_j(r_j-1).$$

5　双有理変換による特異点の reduction

代数曲線で尖点 (cusp) や接触結節点という "複雑な" 特異点を持つものを、もっと単純な特異点である通常多重点しか、特異点として持たない代数曲線に変換する方法が知られている。

複素射影平面 $\mathbf{CP}^2$ においてアフィン座標系 (x,y) を採用し、変換 $(x,y)\mapsto(X,Y)$:

$$X=\phi(x,y),\ Y=\psi(x,y),$$

で以下に述べるような性質を持つものを考える。ここで、ϕ,ψ は有理関数であって、上記の写像の逆写像 $(X,Y)\mapsto(x,y)$ が

$$x=\Phi(X,Y),\ y=\Psi(X,Y),$$

という形で与えられる。ここでΦ,Ψ もまた有理関数である。このような変換を双有理変換 *birational transformation* と言う。任意の既約な複素代数曲線 Cに対し、双有理変換 T で 曲線 $C'=T(C)$ が特異点として、通常多重点以外のものを含まないものが存在する。このとき、このような C'の種数でもって、曲線 C の種数が与えられる。このような変換により、Cの種数を計算することが原理的には可能である ([Wal], [Shii])。

6 種数 1 の boundary generating curve の例

次のような 4×4 行列 A, A' および、8×8 行列 B の 数域の境界を形成する代数曲線の種数はいずれも 1 である。

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A' = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$B = \begin{pmatrix} O_4 & I_4 \\ O_4 & N_4 \end{pmatrix}, N_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

これらの行列 A, A', B の数域の境界を、楕円関数を使って径数表示できる。上記の B との対比で不思議である事実としては、つぎのような行列 K

$$K = \begin{pmatrix} 0_3 & I_3 \\ 0_3 & N_3 \end{pmatrix}, N_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

の数域の boundary generating curve が有理曲線であることが挙げられる。

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Finite Rank Intermediate Hankel Operators On The Bergman Space

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Abstract. Let $L^2 = L^2(D, rdrd\theta/\pi)$ be the Lebesgue space on the open unit disc and $L_a^2 = L^2 \cap Hol(D)$ be the Bergman space. Let P be the orthogonal projection of L^2 onto L_a^2 and let Q be the orthgonal projection onto $\overline{L}_{a,0}^2 = \{g \in L^2 : \overline{g} \in L_a^2, g(0) = 0\}$. Then $I - P \ge Q$. The big Hankel operator and the small Hankel operator on L_a^2 are defined as the following: For ϕ in L^∞, $H_\phi^{big}(f) = (I-P)(\phi f)$ and $H_\phi^{small}(f) = Q(\phi f)$ $(f \in L_a^2)$. In this paper, the finite rank intermediate Hankel operators between H_ϕ^{big} and H_ϕ^{small} are studied. We give two theorems which describe general finite rank intermediate Hankel operators. Using them we study some special intermediate Hankel operators more exactly.

§ 1. Introduction

D は *open unit disc*、$dA = rdrd\theta/\pi$ は D 上の *normalized area mesure* を示す。$L^2 = L^2(D, dA)$ とする。$Hol(D)$ は D 上の *holomorphic function* から成る集合とする。$L_a^2 = L_a^2(D, dA) = L^2 \cap Hol(D)$ とし、L_a^2 を *Bergman space* と呼ぶ。M が L^2 の *invariant subspace* であるとは、M が L^2 の *closed subspace* であり、かつ、$zM \subset M$ になるときをいう。

M を L^2 の *closed subspace*、P^M を L^2 から M への *orthogonal projection* とする。$\phi \in L^\infty$ に対して $H_\phi^M f = (I-P)(\phi f)$　$(f \in L_a^2)$ と定義される H_ϕ^M を $M-$ *Hankel operator* と呼ぶ。

$M = L_a^2$ のときは、H_ϕ^{big} と書き *big Hankel operator*, $M = \left(\overline{\bar{z}L_a^2}\right)^\perp$ のときは、H_ϕ^{small} と書き *small Hankel operator* と呼ばれる。そして、$L_a^2 \subseteq M \subseteq \left(\overline{\bar{z}L_a^2}\right)^\perp$ のとき、H_ϕ^M は *intermediate Hankel operator* と呼ばれる。この *operator* については、Peng - Rochberg - Wu [8], Janson - Rochberg [2], Wang - Wu [11], Strouse [10] らが、M の特別な場合に研究した。それらの例は、すべて M が L^2 の *invariant subspace* となっている。そのうちの１つを次に挙げる。

<u>例</u>　$0 < \beta < 1$ のとき、$T_\beta = \overline{span}\{z^m z^n : \beta n \ge m \ge 0\}$ とする。このとき、$M = T_\beta$ とすると、T_β は *invariant subspace* となる。(Janson - Rochberg [2])

H_ϕ^M が compact や schatten class になるときは調べられている([8], [2], [11])が、いつ finite rank となるかは、[10] を除いては調べられたことはないようである。ただ、finite rank についても、ひじょうに特殊な M の場合のみに研究されているにすぎない。我々は、これまで研究されたことのあるすべての例が、満足している次のひじょうに弱い仮定のもとに、次の問題を研究する。

仮定　$L_a^2 \subseteq M$ かつ M は L^2 の invariant subspace とする。

問題　仮定のもとに、H_ϕ^M が finite rank となる symbol ϕ を決定せよ。

§ 2. Kronecker type の定理

$M^\infty = M \cap L^\infty$ かつ $M^{\infty,l} = \left\{ f \in L^\infty : f(z) = g(z)\prod_{j=1}^{t}(z-a_j)^{-1},\ a.e. on\ D\ ,\ g \in M^\infty,\ a_1,\dots,a_t \in D\ ,\ t \le l \right\}$ と定義すると、$M^{\infty,l}$ は l について単調増加となる。次の定理は、Bergman space における Kronecker type の定理である。

定理1　M を L^2 の invariant subspace とする。$\phi \in L^\infty$ に対して、H_ϕ^M の rank が l 以下であるならば、$\phi \in M^{\infty,l}$ となる。

この定理は、Hardy space におけるよく知られた Kronecker の定理と全く同じ形をしているが、必ずしも finite rank $M-$Hnakel operator を決定しているとは言えない。なぜなら、M が Hardy space のとき、$M^{\infty,l}\,(l=0,1,2,\dots)$ はすべて異なることが知られているが、M が Bergman space のとき、$M^{\infty,l}\,(l=0,1,2,\dots)$ がすべて異なるかどうかは知られていないからである。よって、我々は、$M^{\infty,l}$ をもっと研究することが必要であると思われる。次の定理は、finite rank $M-$Hankel operator は zero operator 以外に存在しないのはいつであるかを決定している。

定理2　M は L^2 の invariant subspace とするとき、次の (1) ~ (3) は同値である。

(1) $\phi \in L^\infty$ に対して H_ϕ^M の rank が l 以下であるならば、$H_\phi^M = 0$ となる。

(2) 任意の l に対して、$M^\infty = M^{\infty,l}$ となる。

(3) 任意の $g \in M^\infty$ に対して、$a \in D$ かつ $g(z)-g(a)/z-a \in L^\infty$ であるならば、$g(z)-g(a)/z-a \in M^\infty$ となる。

この定理から、次の系が簡単に得られる。

系１　$M=L_a^2$ のとき、H_ϕ^{big} の *rank* が有限であるならば、$H_\phi^{big}=0$ となる。

系２　D_0 を D の *open subset* とする。$M=\left\{f\in L^2 : f : \textit{holomorphic function on } D_0\right\}$ のとき、H_ϕ^M の *rank* が有限であるならば、$H_\phi^M=0$ となる。

§ 3.Invariant subspace の Fourier coefficient

$\mathscr{L}^2\equiv L^2\left(\left[0,1\right), dr/\pi\right)$ とすると、$L^2=\sum_{j=-\infty}^{\infty}\oplus\mathscr{L}^2 e^{ij\theta}, L_a^2=\sum_{j=0}^{\infty}\oplus\left\langle cr^j\right\rangle e^{ij\theta}$ と分解される。M は L^2 の *invariant subspace* とする。$M_j\equiv\left\{f_j\in\mathscr{L}^2 : f\in M, f(z)=\sum_{j=-\infty}^{\infty}f_j(r)e^{ij\theta}\right\}$ と定義するとき、$\left\{M_j\right\}_{j=-\infty}^{\infty}$ を M の *Fourier coefficient* と呼ぶ。さらに $M_j e^{ij\theta}\subset M$ のとき、$M=\sum_{j=-\infty}^{\infty}\oplus M_j e^{ij\theta}$ となるが、これを M の *Fourier decomposition* と呼ぶ。次の定理は、M の *Fourier decomposition* を用いて、*finite rank* M − *Hankel operator* を決定している。

定理３　M は L^2 の *invariant subspace* とする。$\phi\in L^\infty$ に対して、$\phi=\sum_{j=-\infty}^{\infty}\phi_j(r)e^{ij\theta}$ のとき、H_ϕ^M の *rank* が l 以下である必要十分条件は、$b_l=1$ かつ 任意の n に対して、$\sum_{j=0}^{l}b_j r^j\phi_{n-j}(r)\in M_n$ を満たすような、$b_0,\ldots,b_l\in\mathbb{C}$ が存在することである。

この定理は、定理１を用いて証明できる。定理３は、すべての n について調べたが、次の系が示すように、特別な *symbol* ϕ では必ずしもすべてを調べる必要はない。

系３　$\phi=\phi_t(r)e^{it\theta}$ のとき H_ϕ^M の *rank* が l 以下である必要十分条件は、$b_l=1$ かつ $t\leq n\leq l+t$ を満たす任意の n に対して、$b_{n-t}r^{n-t}\phi_t(r)\in M_n$ となるような、$b_0,\ldots,b_l\in\mathbb{C}$ が存在することである。

系４　$\phi=\sum_{j=1}^{\infty}a_j z^j+\sum_{j=0}^{\infty}a_{-j}\overline{z}^{j}$ のとき H_ϕ^M の *rank* が l 以下である必要十分条件は、$b_l=1$ かつ $n\leq 0$ を満たす任意の n に対しては、$\sum_{j=0}^{l}b_j a_{n-j}r^{2j-n}\in M_n$ であり、$0<n<l$ を満たす任意の n に対しては、$\sum_{j=n}^{l}b_j a_{n-j}r^{2j-n}\in M_n$ となるような、$b_0,\ldots,b_l\in\mathbb{C}$ が存在することである。

§4. M が特別の場合

$\mathbb{H}^2 \equiv \sum_{j=0}^{\infty} \oplus \mathscr{L}^2 e^{ij\theta}$ と定義すると、$L_a^2 \subset \mathbb{H}^2 \subset \left(\overline{z}\,\overline{L_a^2}\right)^{\perp}, L^2 = \mathbb{H}^2 \oplus e^{-i\theta}\overline{\mathbb{H}^2}$ となる。次の定理は、*big Hankel operator* の場合を含み、それに近い場合を調べている。

<u>定理４</u>　$zL_a^2 \subset M \subset \mathbb{H}^2$ で、M は L^2 の *invariant subspace* とする。$\phi = \sum_{j=1}^{\infty} \phi_{-j}(r)e^{ij\theta} \in L^{\infty}$ のとき、次の (1), (2) が示せる。

(1) $j \geq 0$ を満たす任意の j に対して、$M_j \cap r^{j+1}\mathscr{L}^2 = \{0\}$ であるならば、$H_{\phi}^{M} = 0$ を除く *finite rank* H_{ϕ}^{M} は存在しない。

(2) $H_{\phi}^{M} = 0$ を除く *finite rank* H_{ϕ}^{M} は存在しないならば、$j \geq 0$ を満たす任意の j に対して、$M_j \cap r^{j+1}\mathscr{L}^{\infty} = \{0\}$ となる。

(1) の仮定については、$M = L_a^2$ のときは明らかに満たしている条件である。次の定理は、*small Hankel operator* を含み、それに近い場合を調べている。

<u>定理５</u>　$M \supseteq \mathbb{H}^2$ で、M は L^2 の *invariant subspace* とする。このとき、L^2 ではないどんな M に対しても、$H_{\phi}^{M} \neq 0$ かつ H_{ϕ}^{M} が *finite rank* となる *symbol* ϕ が存在する。

次の定理は、M がちょうど真ん中の $\mathbb{H}^2$ のときは、簡単に *symbol* ϕ が決定されることを示している。

<u>定理６</u>　$M = \mathbb{H}^2$ で、M は L^2 の *invariant subspace* とする。$\phi \in L^{\infty}$ に対して、H_{ϕ}^{M} の *rank* が *1* である必要十分条件は、$\phi = \sum_{j=-1}^{\infty} \phi_j(r)e^{ij\theta}$ となることである。但し、$\phi_{-1}(r) \neq 0$ とする。

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Carleson inequalities in classes of derivatives of harmonic Bergman functions with $0 < p \leq 1$

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ABSTRACT. We give a necessary and sufficient condition for a positive measure μ on the upper half-space of $\mathbb{R}^n$ to satisfy the inequalities

$$\left(\int |D^\alpha u|^q d\mu\right)^{1/q} \leq C\left(\int |D_y^m u|^p y^r dV\right)^{1/p}$$

for all u in a subclass of a harmonic Bergman space when $0 < p \leq 1$ and $p \leq q$, where D_y denotes the partial differentiation operator with respect to the last coordinate y. We also show that the Bergman norm is comparable to derivative norms and harmonic conjugation is bounded on the harmonic Bergman space b^p when $0 < p \leq 1$.

1. Introduction

Let H be the upper half-space of the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, that is, $H = \{z = (x, y) \in \mathbb{R}^n \ ; \ y > 0\}$, where we have written a point $z \in \mathbb{R}^n$ as $z = (x, y)$ with $x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. For $0 < p < \infty$, let $b^p = b^p(H, dV)$ be the class of all harmonic functions u on H such that

$$\| u \|_p = \left(\int_H |u|^p dV\right)^{1/p} < \infty$$

where dV denotes the Lebesgue volume measure on H. The class b^p is called the harmonic Bergman space. Recently, properties of functions in the harmonic Bergman space b^p for $1 \leq p < \infty$ have been studied by Ramey and Yi [9], and several important results have been given. Our aim is to investigate properties in the harmonic Bergman space b^p when $p \leq 1$.

In this paper, we study conditions on a σ-finite positive Borel measure μ on H for which there is a constant C satisfying $\int |u|^p d\mu \leq C \int |D_y u|^p y^r dV$ for all u in a subclass of b^p when $p \leq 1$, where D_y denotes the partial derivative with respect to y and $r > -1$. (Our consideration is more general.) Such inequalities on the unit disk in the complex plane were studied by Stegenga [10]. It was proved that when $r \geq 1$ a finite positive Borel measure ν on the unit disk satisfies the inequality $\int |f|^2 d\nu \leq C \int |f'|^2 (1 - |\zeta|)^r dA$ for all

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holomorphic functions f if and only if there is a constant K such that $\nu(S(I)) \leq K|I|^r$ for any interval I in the unit circle, where dA denotes the Lebesgue area measure, $|I|$ denotes the normalized arc length of I, and $S(I) = \{\zeta : \zeta/|\zeta| \in I, 1-|I| < |\zeta| < 1\}$. It was also proved that when $0 \leq r < 1$ such measures are those satisfying $\nu(\cup S(I_j)) \leq K\mathrm{Cap}(\cup I_j)$ for all finite disjoint collections of intervals $\{I_j\}$, where Cap is an appropriate Bessel capacity (if $r < 0$ any finite Borel measure satisfies this inequality). It is known that these characterizations can be generalized to the case of $p > 1$ (see also [10]). When $p \leq 1$, the characterization in Ahern and Jevtić [1] is simpler. Indeed, ν satisfies the inequality $\int |f|^p d\nu \leq C \int |f'|^p(1-|\zeta|)^r dA$ if and only if $\nu(S(I)) \leq K|I|^{2-p+r}$ when $p \leq 1$. In the proof of the case $p \leq 1$, a Hausdorff capacity was used in stead of the Bessel capacity. When $p > 1$ investigations for several variables are given in [3]. In these investigations, necessary and sufficient conditions were not obtained completely, and it was also shown that, in general, the above condition is not necessary, in contrast to the result on the unit disk. In case $p \leq 1$, no necessary and sufficient conditions are known.

In §3, we give a necessary and sufficient condition for a measure μ on the upper half-space H to satisfy the inequality $\int |u|^p d\mu \leq C \int |D_y u|^p y^r dV$ for all u in a subclass of b^p when $p \leq 1$ (see Theorem 1). §2 is devoted to some preliminary lemmas for this investigation in §3. In the proofs of characterizations of measures on the unit disk to satisfy such inequalities in [10] and [1], capacity estimates are used. However, in the proof of Theorem 1 in §3, we use integral representations for harmonic functions.

In §4, we study properties of functions in the harmonic Bergman space b^p when $p \leq 1$. All results described in §4 were proved in [9] when $p \geq 1$. In [9], it was shown that if $p \geq 1$ and $u \in b^p$ then there exist unique harmonic conjugates $u_1, \cdots, u_{n-1}$ of u that belong to b^p. Using the ideas used in the proof of Theorem 1, we show that these conjugation results are also valid in the case of $p \leq 1$. Therefore, harmonic conjugation is bounded on the harmonic Bergman space b^p for all $0 < p < \infty$ and all dimensions n. It is well known that such conjugation result is not valid in the theory of Hardy spaces (see [5, pp.102–123] and [4, pp.167–172]). Moreover, we show that when $p \leq 1$ the Bergman norm is comparable to several "derivative norms" as in [9]. These results are consequences of Theorem 1 and the boundedness of harmonic conjugation.

2. Preliminary lemmas

Recall that a point $z \in H$ will be written as $z = (x, y)$ with $x \in \mathbb{R}^{n-1}$ and $y > 0$. We use the absolute value symbol $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$. For $z = (x, y)$, let $\bar{z} = (x, -y)$. The pseudohyperbolic metric ρ in H is defined by $\rho(z, w) = |z-w|/|z-\bar{w}|$. It is clear that ρ is invariant under horizontal translations and dilations. Let $D_\varepsilon(w) = \{z \in H\ ;\ \rho(z, w) < \varepsilon\}$ when $w = (s, t) \in H$ and $0 < \varepsilon < 1$. $D_\varepsilon(w)$ is a Euclidean ball whose center and radius are $(s, \frac{1+\varepsilon^2}{1-\varepsilon^2}t)$ and $\frac{2\varepsilon t}{1-\varepsilon^2}$ respectively. It follows that there is a constant $C = C_\varepsilon > 0$ such that $C^{-1}t^n \leq V(D_\varepsilon(w)) \leq Ct^n$ for all $w \in H$. The proof of (3) of Lemma 1 is parallel to that of Lemma 4.3.6 in [12].

LEMMA 1. *Let $0 < \varepsilon < 1$. Then, the following are true.*

(1) *If z, w, ζ are in H and $\rho(z,w) < \varepsilon$, then $C^{-1}|z-\bar{\zeta}| \leq |w-\bar{\zeta}| \leq C|z-\bar{\zeta}|$ with a positive constant C depending only on ε.*

(2) *If $z = (x,y), w = (s,t)$ are in H and $\rho(z,w) < \varepsilon$, then $C^{-1}y \leq t \leq Cy$ with a positive constant C depending only on ε.*

(3) *If $0 < \varepsilon < 1/2$ then there exist a positive integer N and a sequence $\{\zeta_j\}$ in H satisfying the following conditions :* (a) *$H = \cup D_\varepsilon(\zeta_j)$,* (b) *$D_{\varepsilon/4}(\zeta_i) \cap D_{\varepsilon/4}(\zeta_j) = \emptyset$ if $i \neq j$,* (c) *any point in H belongs to at most N of the sets $D_{2\varepsilon}(\zeta_j)$.*

For a function u on H and $\delta > 0$, let $\tau_\delta u$ denote the function on H defined by $\tau_\delta u(x,y) = u(x, y+\delta)$, and let $\mathcal{T}^p = \{\tau_\delta u \ ; \ u \in b^p, \delta > 0\}$. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index of nonnegative integers with order ℓ, then D^α denotes the partial differentiation operator $\partial^\ell / \partial x_1^{\alpha_1} \cdots \partial x_{n-1}^{\alpha_{n-1}} \partial y^{\alpha_n}$. The following lemma is stated in [2, Corollary 8.2] when $p \geq 1$.

LEMMA 2. *Let $0 < p \leq 1$. Then, the following are true.*

(1) *For any $u \in b^p$, there is a constant $C > 0$ such that $|D^\alpha u(s,t)| \leq C/t^{n/p+|\alpha|}$ for all $(s,t) \in H$.*

(2) *For any $u \in b^p$, there is a constant $C > 0$ such that $|(D^\alpha \tau_\delta u)(s,t)| \leq C/(t+\delta)^{n/p+|\alpha|}$ for all $(s,t) \in H$.*

Let $w = (s,t) \in H$. The Poisson kernel P_w is the function on $\mathbb{R}^{n-1}$ given by $P_w(x) = P(s-x,t) = \gamma_n t/(|s-x|^2+t^2)^{n/2}$ (γ_n is the positive constant $\gamma_n = 2/(nV(\mathbb{B}_n))$, where $\mathbb{B}_n$ denotes the unit ball in $\mathbb{R}^n$). The harmonic extension of this function to H is $P(s-x, t+y)$. If $z = (x,y) \in H$, then we may write $P_w(z)$. We note that $P_w(z) = \gamma_n(t+y)/|w-\bar{z}|^n$, $|D_z^\alpha P_w(z)| \leq C/|w-\bar{z}|^{n+|\alpha|-1}$, and $D_z^\alpha P_w(z) = (-1)^{\alpha_1+\cdots+\alpha_{n-1}} D_w^\alpha P_w(z)$. The following lemma is useful and stated in [9, Lemma 3.1]

LEMMA 3. *Let $0 < c < 1$. Then, there is a constant $C > 0$ depending on c and n such that*

$$\int_H \frac{y^{-c}}{|w-\bar{z}|^n} dV(z) = Ct^{-c}$$

for all $w = (s,t) \in H$.

Let m be a nonnegative integer and let $c_m = (-2)^m/m!$. The following Lemma 4 is given in [2, Chaper 8] and [9], when $u \in b^p$ and $p \geq 1$. The proofs of (1) and (2) of Lemma 4 are parallel to the proofs of Theorem 8.22 in [2, Chapter 8] and Lemma 4.1 in [9] respectively, except only minor changes.

LEMMA 4. *Let $0 < p \leq 1$. If $u \in \mathcal{T}^p$, then the following equalities hold.*

(1) *$u(w) = -2 \int_H u(z) D_y P_w(z) dV(z)$ for all $w \in H$.*

(2) $u(w) = -2c_m \int_H y^m (D_y^{m+1}u)(z) P_w(z) dV(z)$ *for all* $w \in H$, $m = 0, 1, 2, \cdots$.

The following Lemma 5 is a consequence of Lemma 4.

LEMMA 5. *Let* $0 < p \leq 1$. *If* $u \in \mathcal{T}^p$, *then*

$$u(w) = -2c_{m+k} \int_H y^{m+k} (D_y^m u)(z) D_y^{k+1} P_w(z) dV(z)$$

for all $m, k \geq 0$ *and* $w \in H$.

3. Carleson inequalities

Let $B_t(s)$ denote the ball in $\mathbb{R}^{n-1}$ with center $s \in \mathbb{R}^{n-1}$ and radius $t > 0$. When no confusion arises we may write B_t in stead of $B_t(s)$. For each ball B_t in $\mathbb{R}^{n-1}$ set $S(B_t) = \{(x, y)\ ;\ x \in B_t,\ y < 2t\}$. We now state our main result in this section.

THEOREM 1. *Suppose that* $0 < p \leq 1$, $p \leq q$ *and* $r > -1$. *Let* μ *be a* σ*-finite positive Borel measure on* H, *and let* ℓ *and* m *be nonnegative integers. Then, the following* $(1) \sim (3)$ *are equivalent.*

(1) *There is a constant* $C > 0$ *such that*

$$\left(\int_H |D^\alpha u|^q d\mu \right)^{1/q} \leq C \left(\int_H |D_y^m u|^p y^r dV \right)^{1/p}$$

for all $u \in \mathcal{T}^p$ *and for all multi-indices* α *of order* ℓ.

(2) *There is a constant* $C > 0$ *such that*

$$\left(\int_H |D_y^\ell u|^q d\mu \right)^{1/q} \leq C \left(\int_H |D_y^m u|^p y^r dV \right)^{1/p}$$

for all $u \in \mathcal{T}^p$.

(3) *There is a constant* $K > 0$ *such that* $\mu(S(B_t)) \leq K t^{(n+r)q/p + (\ell - m)q}$ *for all balls* $B_t \subset \mathbb{R}^{n-1}$.

We note that in case $(n+r)q/p + (\ell - m)q = 0$ (or equivalently, $n + r = p(m - \ell)$), μ satisfies the above inequalities if and only if μ is a finite measure. In fact, in this case, condition (3) of Theorem 1 is reduced to $\mu(S(B)) \leq K$ for all balls B. For each compact set $E \subset H$, we can choose a ball B satisfying $E \subset S(B)$. Therefore, we have $\mu(E) \leq K$ for all compact sets $E \subset H$, and thus μ is finite. Similarly, we can see that in case $(n+r)q/p + (\ell - m)q < 0$, μ satisfies the above inequalities if and only if $\mu = 0$. In the inequality in (2) of Theorem 1, if $m \geq \ell$, then, of course, we can replace $D_y^\ell u$ and $D_y^m u$ by u and $D_y^{m-\ell} u$ respectively. Similarly, if $m < \ell$, then we can replace $D_y^\ell u$ and $D_y^m u$ by $D_y^{\ell-m} u$ and u respectively.

We give a sufficient condition for a measure μ to satisfy the inequality.

PROPOSITION 2. *Under the assumptions on p, q, r, ℓ and m in Theorem 1, let k be a nonnegative integer such that $p(n+k)-2n>0$. Suppose that there is a constant $K>0$ such that*

$$\int_H \frac{1}{|w-\bar{z}|^{q(n+\ell+k)}}d\mu(z) \leq Kt^{(n+r)q/p-q(n+m+k)}$$

for all $w=(s,t)\in H$. Then, there is a constant $C>0$ such that

$$\left(\int_H |D^\alpha u|^q d\mu\right)^{1/q} \leq C\left(\int_H |D_y^m u|^p y^r dV\right)^{1/p}$$

for all $u\in\mathcal{T}^p$ and for all multi-indices α of order ℓ.

In order to give a necessary condition for a measure μ to satisfy the inequality in (2) of Theorem 1, we need the following lemma.

LEMMA 6. *Let k be a nonnegative integer. Then, there exist constants $0<\sigma\leq 1$ and $C>0$ such that $|D_y^k P_w(z)|\geq C/t^{n+k-1}$ for all $w=(s,t)\in H$ and $z\in S(B_{\sigma t}(s))$.*

PROPOSITION 3. *Under the assumptions on p, q, r, ℓ and m in Theorem 1, suppose that there is a constant $C>0$ such that*

$$\left(\int_H |D_y^\ell u|^q d\mu\right)^{1/q} \leq C\left(\int_H |D_y^m u|^p y^r dV\right)^{1/p}$$

for all $u\in\mathcal{T}^p$. Then, there is a constant $K>0$ such that $\mu(S(B_t))\leq Kt^{(n+r)q/p+(\ell-m)q}$ for all balls $B_t\subset\mathbb{R}^{n-1}$.

PROOF OF THEOREM 1. (1) ⇒ (2) is trivial. (2) ⇒ (3) was already shown in Proposition 3. We will show (3) ⇒ (1). Let $c=(n+r)q/p+(\ell-m)q$ and suppose that $\mu(S(B_\eta))\leq K\eta^c$ for all balls $B_\eta\subset\mathbb{R}^{n-1}$. By Proposition 2, it is sufficient to prove that there exists a nonnegative integer k such that $p(n+k)-2n>0$ and $\int_H 1/|w-\bar{z}|^\gamma d\mu(z)\leq Ct^{c-\gamma}$ for all $w=(s,t)\in H$, where $\gamma=q(n+\ell+k)$. Let $w\in H$. Without loss of generality we may assume that $w=(0,t)$, and k will be determined later. Let $S_j=S(B_{2^jt}(0))$ $(j\geq 0)$. Clearly, if $z\notin S_{j-1}$, then $|w-\bar{z}|\geq 2^{j-1}t$ $(j\geq 1)$. Therefore,

$$\int_H \frac{1}{|w-\bar{z}|^\gamma}d\mu(z)\leq t^{-\gamma}\int_{S_0}d\mu+t^{-\gamma}\sum_{j=1}^{\infty}\frac{1}{2^{\gamma(j-1)}}\int_{S_j\setminus S_{j-1}}d\mu\leq Ct^{c-\gamma}+C't^{c-\gamma}\sum_{j=1}^{\infty}\frac{1}{(2^{\gamma-c})^j}.$$

Since $\gamma-c=q(n+m+k)-(n+r)q/p$, we can choose an integer k such that $\gamma-c>0$ and $p(n+k)-2n>0$. It follows that $\int_H 1/|w-\bar{z}|^\gamma d\mu(z)\leq Ct^{c-\gamma}$.

4. Derivative norms and harmonic conjugates of b^p-functions

When $p \geq 1$, properties of the harmonic Bergman space b^p have been studied by Ramey and Yi [9]. We show that some of these properties are also valid for $0 < p < 1$. For each $\delta > 0$, set $\Omega_\delta = \{z \in H \ ; \ y > \delta\}$ and denote by χ_δ the characteristic function of Ω_δ. We use the expression $A \approx B$ meaning that there is a constant $C > 0$ such that $C^{-1}A \leq B \leq CA$. We show that the Bergman norm is comparable to "derivative norms". The following theorem is a consequence of Theorem 1.

THEOREM 4. *Let* $0 < p \leq 1$ *and* ℓ *be a nonnegative integer. Then*

$$\| u \|_p \approx \sum_{|\alpha|=\ell} \| y^\ell D^\alpha u \|_p \approx \| y^\ell D_y^\ell u \|_p$$

for all $u \in b^p$.

Given a harmonic function u on H, recall that functions $u_1, \cdots, u_{n-1}$ are called harmonic conjugates of $u = u_n$ if

$$\sum_{j=1}^{n} D_j u_j = 0 \quad \text{and} \quad D_i u_j = D_j u_i \quad (1 \leq i, j \leq n),$$

where $D_j = \partial/\partial x_j \ \ (1 \leq j \leq n-1)$ and $D_n = D_y = \partial/\partial y$.

In [9], it was shown that harmonic conjugation is bounded on the harmonic Bergman space b^p when $p \geq 1$. We show that this conjugation result is also valid in the case of $p \leq 1$. That conjugation is bounded on the Bergman space on the unit disk for $p \leq 1$ was observed in [6]. An analogous result holds for the upper half-space in all dimensions.

THEOREM 5. *Let* $0 < p \leq 1$ *and* $u \in b^p$. *Then, there exist harmonic conjugates* $u_1, \cdots, u_{n-1}$ *of* u *such that* $u_j \in b^p$. *Moreover, they are uniquely determined and*

$$\| u \|_p \approx \sum_{j=1}^{n-1} \| u_j \|_p \, .$$

By Theorems 4 and 5, we see that Bergman norms are also comparable to tangential derivative norms. In the proof of Theorem 6.2 in [9], if we replace Theorems 4.4 and 6.1 in [9] by Theorems 4 and 5 respectively, then the following Theorem 6 is obtained.

THEOREM 6. *Let* $0 < p \leq 1$ *and* ℓ *be a nonnegative integer. Then,*

$$\| u \|_p \approx \sum_{|\alpha|=\ell, \alpha_n=0} \| y^\ell D^\alpha u \|_p$$

for all $u \in b^p$.

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Ordered topological linear space

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ABSTRACT

In partially ordered linear space, we define a generalized supremum as the set of all minimal elements of the set of upper bound. Also we consider the facial structure of positive cone in order to investigate the properties of the generalized supremum, especially the condition $U(A) = (\operatorname{Sup} A) + P$.

Let E be a linear space over $\mathbb{R}$, and P be a convex cone in E satisfying

(P1) $E = P - P$,
(P2) $P \cap (-P) = \{0\}$.

The order relation in E is defined by $x \leq y \Longleftrightarrow y - x \in P$. It can easily be seen that

(1) $x \leq y$ and $y \leq x \Longrightarrow x = y$,
(2) $x \leq y$ and $y \leq z \Longrightarrow x \leq z$,
(3) $x \leq y \Longrightarrow x + z \leq y + z$ for all $z \in E$,
(4) $0 \leq x$ and $0 \leq \lambda \in \mathbb{R} \Longrightarrow 0 \leq \lambda x$,
(5) For every $x \in E$, there exists $x_1, x_2 \in E$ such that $x = x_1 - x_2$, and $0 \leq x_1, x_2$.

Conversely, if an order in E satisfies (1) $\sim$ (5), then $P = \{x \in E \mid 0 \leq x\}$ is a convex cone satisfying (P1) and (P2). A linear space E equipped with such a positive cone P is called a partially ordered linear space, and is sometimes denoted by (E, P).

Definition. *For a subset A of E, the generalized supremum* $\operatorname{Sup} A$ *is defined to be the set of all minimal elements of $U(A)$, where $U(A)$ is the set of all upper bound of A.*

The generalized infimum $\operatorname{Inf} A$ can be defined similarly. In order to distinguish this notion from the least upper bound and the greatest lower bound, we denote the latter ones by $\sup A$ and $\inf A$ respectively. If E is order complete, then $\operatorname{Sup} A = \{\sup A\}$ holds whenever the subset A is upper bounded (i.e.,$U(A) \neq \emptyset$). When $E = \mathbb{R}^n$ and P is closed and not a lattice cone, $\operatorname{Sup} A$ becomes a infinite set in most cases. However, it is possibly empty, even when A is upper bounded.

Proposition 1. *For $a \in E$ and $\lambda > 0$, we have*

(1) $\operatorname{Sup}(A + a) = \operatorname{Sup} A + a$,
(2) $\operatorname{Sup} \lambda A = \lambda \operatorname{Sup} A$,
(3) $\operatorname{Sup} A = -\operatorname{Inf}(-A)$.

Proposition 2. *For an arbitrary set $A \subset E$ with $U(A) \neq \emptyset$,* $\operatorname{Sup} A = \operatorname{Sup}(coA)$ *holds where coA is the convex hull of A.*

$a \vee b$ and $a \wedge b$ are defined to be $\operatorname{Sup}\{a, b\}$ and $\operatorname{Inf}\{a, b\}$ respectively.

Proposition 3. *For every $a, b, c \in E$ and $\lambda \in \mathbb{R}$,*

(1) $(a + c) \vee (b + c) = (a \vee b) + c$,
(2) $\lambda a \vee \lambda b = \lambda(a \vee b)$.

Theorem 1.

(1) *For* $a, b \in E$, $a \vee b \neq \emptyset \Longleftrightarrow a \wedge b \neq \emptyset$,

(2) *For* $a, b \in E$, $a + b - (a \vee b) = a \wedge b$,

(3) $a \in a_+ + a_-$ *where* $a_+ = a \vee 0$ *and* $a_- = a \wedge 0$.

A partially ordered linear space (E, P) is said to be monotone order complete (m.o.c. for short) if every upper bounded totally ordered subset of E has the least upper bound in E. If E is finite dimensional, (E, P) is m.o.c. if and only if P is closed. Moreover, if E is reflexive Banach space and $E^* = P^* - P^*$ holds where $P^* = \{x^* \mid x^*(x) \geq 0, \ \forall x \in P\}$, then (E, P) is m.o.c.([3]). In [1], some conditions which guarantee $E^* = P^* - P^*$ are shown.

Theorem 2. *Suppose that a partially ordered linear space* (E, P)*is monotone order complete, then for every subset* A *of* E,

$$U(A) = (\operatorname{Sup} A) + P \tag{F.1}$$

holds. In particular, $a \vee b \neq \emptyset, a \wedge b \neq \emptyset$ *for every* $a, b \in E$,*and* $U(a, b) = (a \vee b) + P$.

Corollary 1. *Suppose that* (E, P) *satisfies the hypotheses in Theorem 2 and let* A *be a subset of* E. *If* $\operatorname{Sup} A$ *consists of a single element* a, *then* a *is the least upper bound of* A.

Corollary 2. *For every subset* A *of* E, $U(L(U(A))) = U(A)$ *holds where* $L(U(A))$ *denotes the lower bound of* $U(A)$. *Moreover, if* (E, P) *satisfies the hypotheses in Theorem 2, then we have* $\operatorname{Sup} \operatorname{Inf} \operatorname{Sup} A = \operatorname{Sup} A$.

Let (E, P) be a partially ordered linear space, and suppose that P is algebraically closed, that is, every straight line of E meets P by a closed interval. A point x of a convex subset $A \subset E$ is called an algebraic interior point of A if for every $z \in E$, there exists $\lambda > 0$ such that $x + \lambda z \in A$. We denote the algebraic interior and the algebraic boundary of A by $\operatorname{int} A$ and ∂A respectively. A convex subset C of P is called an exposed face of P if there exists a supporting hyperplane H of P such that $C = P \cap H$. By $\mathfrak{F}(P)$, we denote the set of all exposed faces of P. For $C \in \mathfrak{F}(P)$, $\dim C$ is defined as the dimension of aff$C$ where aff$C$ denotes the affine hull of C. The following theorem is fundamental and is useful when we intend to determine the set $a \vee b$ explicitly.

Theorem 3. *Suppose that* P *is algebraically closed and* $\operatorname{int} P \neq \emptyset$. *If* $\dim C \leq 1$ *for every* $C \in \mathfrak{F}(P)$, *then* $a \vee b = \partial U(a) \cap \partial U(b)$ *holds for every incomparable pair* $a, b \in E$.

Lemma 1. *If* $0 \leq x \leq y$ *and* $y \in \partial P$, *then* $x \in \partial P$.

Proof. Suppose that $x \in \operatorname{int} P$ and put $z = 2y - x$, then $z = y + (y - x) \in P + P = P$. Since P is convex and $x \in \operatorname{int} P$, $y = \frac{1}{2}(x + z) \in \operatorname{int} P$. This contradicts the assumption.

Proof of Theorem 3. Let x_0 be an element of $a \vee b$, and suppose that $x_0 \in \operatorname{int} U(a)$. Then there exists $\lambda > 0$ such that $c \underset{\text{def}}{=} (1 - \lambda) x_0 + \lambda b \in U(a)$. It is easy to see that $c \in U(a) \cap U(b) = U(a, b)$ and $c \lneqq x_0$. This contradicts the fact that x_0 is a minimal element of $U(a, b)$, and hence $a \vee b \subset \partial U(a) \cap \partial U(b)$.

Conversely, take $x_0 \in \partial U(a) \cap \partial U(b)$ arbitrarily and suppose that $y_0 \leq x_0$, $y_0 \in U(a, b)$. Since $a \leq y_0 \leq x_0$, it follows by Lemma 1 that

$$y_0 \in [a, x_0] \subset \partial U(a),$$

where $[a, x_0] = \{x \in E | \ a \leq x \leq x_0\}$ is an order interval. Obviously every order interval is a convex set. Similarly we have

$$y_0 \in [b, x_0] \subset \partial U(b),$$

and hence

$$[a, x_0] \cap \operatorname{int} U(a) = \emptyset, \quad [b, x_0] \cap \operatorname{int} U(b) = \emptyset,$$

while $\operatorname{int} U(a)$ and $\operatorname{int} U(b)$ are both assumed to be nonempty . Applying the separation theorem, we can find hyperplanes H_1, H_2 of E such that

(1) H_1 separates $[a, x_0]$ and $U(a)$ and,

(2) H_2 separates $[b, x_0]$ and $U(b)$.

Since $[a, x_0] \subset U(a)$ and $[b, x_0] \subset U(b)$, we can see that $[a, x_0] \subset U(a) \cap H_1$ and $[b, x_0] \subset U(b) \cap H_2$. By the condition of $\mathfrak{F}(P)$, these two faces are actually half lines. On the other hand, a, b, and x_0 cannot be in any single straight line because a and b are not comparable. Hence $[a, x_0]$ and $[b, x_0]$ are respectively included in two different lines, and in particular, both x_0 and y_0 belong to the intersection of those two lines. This means $x_0 = y_0$ and so $x_0 \in a \vee b$. Q.E.D.

In the case when a linear topology is given in E, the assertion of Theorem 3 can be translated into the terms of topology and is still valid.

Theorem 4. *Suppose that P is algebraically closed and* $\operatorname{int} P \neq \emptyset$. *If* $\dim C \leq \infty$ *for every $C \in \mathfrak{F}(P)$,* (F.1) *holds and $a \vee b \neq \emptyset$ for every $a, b \in E$ in particular.*

Lemma 2. *If $x \in \partial U(A)$ for a subset A of E, then $U(A)_x \subset \partial U(A)$ where $U(A)_x = \{y \in U(A) \mid y \leq x\}$.*

Proof. Let y be an arbitrary point in $U(A)_x$. Since $x \in \partial U(A)$ there exists a point $z \in E$ such that $\{x + tz \mid t > 0\} \cap U(A) = \emptyset$. By the definition of $U(A)$, $U(A) + P = U(A)$, and this yields $\{y + tz \mid t > 0\} \cap U(A) = \emptyset$. This means that $y \in \partial U(A)$.

Proof of Theorem 4. Let x_0 be an arbitrary point in $U(A)$. Since P is algebraically closed, P can not include any straight line. Indeed if $\{x + ty \mid t \in \mathbb{R}\} \subset P$ for some $y \neq 0$, then $\{ty \mid t \in \mathbb{R}\} \subset P \cup \partial P = P$ and this contradicts (P2). Hence for a positive element $x \neq 0$, there exists $t_1 = \max\{t \geq 0 \mid x_0 - tx \in U(A)\}$. If we put $x_1 = x_0 - t_1 x$, then $x_1 \in \partial U(A)$ and it follows from Lemma 2 that $U(A)_{x_1} \subset \partial U(A)$. Since $U(A)_{x_1}$ is a convex set and $\operatorname{int} U(A) \neq \emptyset$, we can apply the separation theorem and there exists a hyper plane H which separates $U(A)_{x_1}$ and $U(A)$. Clearly, $U(A)_{x_1} \subset (x_1 - P) \cap H$ and $\dim((x_1 - P) \cap H) < \infty$ by the assumption. Put $H' = H - x_1, P' = H' \cap P$, and $X = P' - P'$, then $\dim X < \infty$, and P' is a closed convex cone with nonempty interior relative to X. Thus the partially ordered linear space (X, P') is m.o.c. Since the recession cone of $U(A)$ is P, the recession cone of $U(A)_{x_1} = U(A) \cap (x_1 - P)$ is $\{ 0 \}$. Hence by the convexity of $U(A)_{x_1}$ it is bounded in X and also bouded below in (X, P). By an analogy of the proof of Theorem 2, we can conclude that there exists a minimal element z of $U(A)_{x_1}$ which is also a minimal element of $U(A)$. Q.E.D.

Example 1. Let E be the space of all symmetric matices of $M_2(\mathbb{R})$, and let P be the set of all positive semi definite matrices in E. Then (E, P) is m.o.c., but it is not a lattice. E and P can be identified with $\mathbb{R}^3$ and $P = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 \leq xy, \ 0 \leq x, \ 0 \leq y\}$ respectively. It is easy to see that every exposed face of the positive cone P is 1-dimensional except the trivial face $\{0\}$, and P satisfies the condition in Theorem 3.

Hence, by some simple culculations, we can determine the set $a \vee b$ for incomparable pair $a, b \in E$.

Example 2. Next we investigate the relation between the conditions of Theorem 2 and that of Theorem 3. We say that the positive cone P satisfies condition ($\mathfrak{F}$) when $\dim C \leq 1$ for every $C \in \mathfrak{F}(P)$. The monotone order completeness does not imply the condition ($\mathfrak{F}$). Now we show an example in order to see the converse implication is also not true.

Let E be the linear space consisting of all sequences $x = (x_1, x_2, \cdots)$ $(x_i \in \mathbb{R})$ such that $x_i = 0$ except for finite number of $i = 1, 2, \cdots$. We define

$$P = \{\ x = (x_1, x_2, \cdots) \mid x_1 \geq (\textstyle\sum_{i=2}^{\infty} x_i^2)^{\frac{1}{2}}\ \}.$$

Then P is algebraically closed and $\operatorname{int} P \neq \emptyset$. Moreover we can prove (E, P) is not m.o.c. while P satisfies the condition ($\mathfrak{F}$). We will show that (E, P) is not m.o.c. We define a sequence $\{a_n\} \subset E$ by

$$a_n = (\tfrac{1}{2^n}, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \cdots, \tfrac{1}{2^n}, 0, 0, \cdots) \qquad (n = 1, 2, \cdots).$$

Then we have $a_1 \geq a_2 \geq a_3 \geq \cdots$. Moreover, since $(\frac{1}{2})^2 + (\frac{1}{4})^2 + (\frac{1}{8})^2 + \cdots = \frac{1}{3}$, we can see that $(-\sqrt{\frac{1}{3}}, 0, 0, \cdots)$ is a lower bound of $\{a_n\}$. Let $b = (b_1, b_2, \cdots, b_i, 0, 0, \cdots)$ be an arbitrary lower bound of $\{a_n\}$. Then an element of the form $c = (b_1 + \lambda, b_2, b_3, \cdots, b_i, \mu, 0, 0, \cdots)$ always satisfies $b \not\geq c$ when $\lambda > 0$. It is easy to see that we can choose λ and μ such that c is also a lower bound of $\{a_n\}$. This means that the greatest lower bound of $\{a_n\}$ does not exist, and (E, P) is not m.o.c..

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Q-Algebras And Dilations

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Abstract. Let $\mathcal{B}$ be a commutative Banach algebra with a unit and let A/J be a Q-algebra where A is a uniform algebra and J is a closed ideal. In this paper we show the following two results. (1) If $\mathcal{B}$ is of two dimension and an operator algebra on a Hilbert space then $\mathcal{B}$ is a Q-algebra. (2) A two dimensional Q-algebra is isometrically isomorphic to $\mathcal{A}/\mathcal{J}$ where $\mathcal{A}$ is the disc algebra and $\mathcal{J}$ is a closed ideal of $\mathcal{A}$. As a result of (2) we can show a dilation theorem. This is a survey article on the author's two papers [5] and [6].

定義

A を X 上の uniform algebra かつ J を A の closed ideal とするとき、A/J と isometrically isomorphic な Banach algebra は Q-algebra と呼ばれる。これは Varopoulous の定義である。$M(A)$ は A の maximal ideal space を示す。$x \in M(A)$ に対して δ は x における point derivation を示す。すなわち δ は $\delta(fg) = f(x)\delta(g) + \delta(f)g(x)$ となる continuous linear functional を示す。$x, y \in M(A)$ かつ $x \neq y$ に対して

$$\begin{aligned} \sigma &= \sigma(x,y) = \sup\{|f(y)| \ ; \ f \in A, \ f(x)=0, \ \|f\|_\infty \leq 1\}、\\ \omega &= \omega(x,\delta) = \sup\{|\delta(f)| \ ; \ f \in A, \ f(x)=0, \ \|f\|_\infty \leq 1\}、\end{aligned}$$

とする。

Part I. Dilation

$C(X)$ を X 上の continuous function の全体とすると uniform algebra A は $C(X)$ の closed subalgebra である。H は Hilbert space 、$L(H)$ は H 上の bounded linear operator の全体とする。$\|\cdot\|_\infty$ は $C(X)$ の norm 、$\|\cdot\|$ は H または $L(H)$ の norm を示す。

Φ が A から $L(H)$ への unital bounded homomorphism とは、Φ (1) は H 上の identity operator 、Φ は A 上で linear かつ multiplicative, かつ

$$\|\Phi(f)\| \leq \gamma \|f\|_\infty \quad (f \in A)$$

となる γ が存在するとする。Φ_n を $(f_{ij}) \in M_{n\times n}(A)$ から $(\Phi(f_{ij})) \in M_{n\times n}(L(H))$ への写像とすると、$\|\Phi_n\| \leq \|\Phi_{n+1}\|$ である。

$$\|\Phi\|_{cb} = \sup_n \|\Phi_n\|$$

と書く。一般に $\|\Phi\| \leq \|\Phi\|_{cb}$ である。$\|\Phi\|_{cb} < \infty$ のとき Φ は completely bounded といわれる。$\|\Phi\| = 1$ のとき、$\tilde{\Phi}$ が Φ の dilation とは、$\tilde{\Phi}$ が $C(X)$ から $L(K)$ への unital bounded homomorphism かつ $\|\tilde{\Phi}\| = 1$ で、

$$\Phi(f) = P\tilde{\Phi}(f)|H \quad (f \in A)$$

となることである。ここで K は H を含む Hilbert space で、P は K から H への orthogonal projection である。このとき $\|\Phi\|_{cb} = \|\tilde{\Phi}\|_{cb} = 1$ である。$\|\Phi\| = 1$ とは限らないとき、もし $\|\Phi\|_{cb} < \infty$ ならば Φ は dilation をもつ unital bounded homomorphism Ψ (だから $\|\Psi\| = 1$) に similar であることが知られている。逆も成立することが知られている。

$\|\Phi\| < \infty$ ならば $\|\Phi\|_{cb} < \infty$ となるかどうかは大変興味ある問題であるが、$\dim H = \infty$ のとき A が disc algebra であってもこれが成立しないことは Pisier [12] によって最近示された。$\dim H < \infty$ のとき、任意の uniform algebra A についてこれは成立することが知られている [18, Exercises 3.11]。ただし $\|\Phi\| = \|\Phi\|_{cb}$ とは限らない。

我々は $\dim H < \infty$ のとき、$\|\Phi\| = \|\Phi\|_{cb}$ となるのはいつかを問題とする。$4 \leq \dim H < \infty$ のとき、たとえ $\|\Phi\| = 1$ でも $\|\Phi\| = \|\Phi\|_{cb}$ とは限らない uniform algebra が存在することは Parrot [10] によって示された。$\dim H = 3$ のとき全く知られていない様であるが、$\dim H = 2$ のときは任意の uniform algebra と任意の Φ について $\|\Phi\| = \|\Phi\|_{cb}$ となることは Chu [5] によって示された。

問題 1　A を任意の uniform algebra かつ Φ を任意の unital bounded homomorphism とする。もし $\dim A/\ker\Phi = 2$ ならば $\|\Phi\| = \|\Phi\|_{cb}$ が成立するか？

問題 1 は Nakazi-Takahashi [4] によって $\|\Phi\| = 1$ のとき正しいことが示された。次の定理は、もし A が disc algebra のとき問題 1 に答えれば良いことを示している。定理 1 の証明は [8, Theorem 9] の証明から導ける。

定理 1　A を任意の uniform algebra、Φ を A から $L(H)$ への unital bounded homomorphism とする。もし $\dim A/\ker\Phi = 2$ ならば、disc algebra $\mathcal{A}$ のある closed ideal $\mathcal{J}$ があって、$A/\ker\Phi$ から $\mathcal{A}/\mathcal{J}$ への unital isometrically isomorphism ϕ が存在して、$\Phi\circ\phi$ は $\mathcal{A}$ から $L(H)$ への unital bounded homomorphism となる。

定理 1 は、任意の uniform algebra A の Q-algebra A/J が 2 次元のとき、A/J は disc algebra $\mathcal{A}$ の Q-algebra $\mathcal{A}/\mathcal{J}$ と isometrically isomorphic であることを示している。

Part II. Commutative Banach Algebra

$\mathcal{B}$ を identity I をもつ 2 次元 commutative Banach algebra とする。このとき $\mathcal{B} = \{aI + bB : a, b \in \mathbf{C}\}$ と書ける。ここで $\mathcal{B}$ が semi-simple のとき $B^2 = I$, そうでないとき $B^2 = 0$ である。

問題 2　$\mathcal{B}$ が 2 次元 commutative Banach algebra が Q-algebra である必要十分条件は $\mathcal{B}$ は Hilbert space H 上の operator algebra であることであるか？

Q-algebra は常に Hilbert space H 上の operator algebra であることは Cole [4] によって示された。$\mathcal{B}$ が 2 次元 commutative Banach algebra かつ $\mathcal{B} \subset L(H)$ とする。もし 仮定 $\dim H = 2$ かつ $\mathcal{B}$ が semi-simple のとき Drury [4] によって示された ([1] を参照)。よって上の問題 2 は 仮定 なしに Q-algebra を示すことであるが、次の定理 2 は問題 2 に答えている。(参照 [5])。

定理 2　$\mathcal{B}$ が 2 次元 commutative Banach algebra が Hilbert space 上の operator algebra ならば Q-algebra である。

2 次元ということより、$\mathcal{B} = \{aI + bB : a, b \in \mathbf{C}\}$ と書けるが、$\mathcal{B} \cong A/J$ (isometrically isomorphic) となる uniform algebra A とその closed ideal J が存在する。

$B^2 = I$ のとき、$J_1 = \{g \in A : g(x) = g(y) = 0\}$ とする。ここで $x, y \in M(A)$ かつ $x \neq y$ である。$F_1 \in A$ を $F_1(x) = 1$ かつ $F_1(y) = -1$ とすると $aI + bB \to a1 + bF_1$ は $\mathcal{B}$ から A/J_1 への isomorphism を与えている。Cole の定理 [4] より、A/J_1 は Hilbert space 上の operator algebra となるので、Feldman-Krupnik-Markus の定理 [3] より、$^\forall f \in A$ に対して

$$\begin{aligned}
&\|f + J_1\| \\
&= \sqrt{\left|\frac{f(x) - f(y)}{2}\right|^2 \left(\frac{1}{\sigma^2} - 1\right) + \left(\frac{|f(x)| + |f(y)|}{2}\right)^2} \\
&+ \sqrt{\left|\frac{f(x) - f(y)}{2}\right|^2 \left(\frac{1}{\sigma^2} - 1\right) + \left(\frac{|f(x)| - |f(y)|}{2}\right)^2}。
\end{aligned}$$

ここで $\sigma = \sigma(x, y)$ である。

$B^2 = 0$ のとき、$J_2 = \{g \in A : f(x) = \delta(f) = 0\}$ とする。ここで $x \in M(A)$ かつ δ は x における point derivation である。$F_2 \in A$ を $F_2(x) = 0$ かつ $\delta(F_2) = 1$ とすると、$aI + bB \to \alpha 1 + \beta F_2$ は $\mathcal{B}$ から A/J_2 への isomorphism を与えている。Cole の定理 [4] より、A/J_2 は Hilbert space 上の operator algebra となることを用いて (今度は書かれた定理はない様であるが)、$^\forall f \in A$ に対して

$$\begin{aligned}
&\|f + J_2\| \\
&= \sqrt{\left|\frac{\delta(f)}{2}\right|^2 \frac{1}{\omega^2} + |f(x)|^2} + \left|\frac{\delta(f)}{2}\right| \frac{1}{\omega}
\end{aligned}$$

を示すことができる。ここで $\omega = \omega(x, \delta)$ である。

$\mathcal{B}$ が Hilbert space 上の operator algebra を仮定せずに定理 2 は成立するかは興味あるが、これはたとえ $\mathcal{B}$ が semi-simple であっても成立しない (参照 [9])。

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Norm of a linear combination of two projections on a Banach space

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Abstract. Let $L(B)$ denote the algebra of all bounded linear operators on a Banach space B. Let $P \in L(B)$ satisfy $P^2 = P$, $P \neq 0, I$, and let $Q = I - P$. For $1 \leq p \leq \infty$ let B_1, B_2 be nonzero closed subspaces of B such that $B_1 \cap B_2 = \{0\}$, $B_1 + B_2 = B$ and

$$\|F_1 + F_2\|_B = (\|F_1\|_B^p + \|F_2\|_B^p)^{1/p}, \quad (F_1 \in B_1, F_2 \in B_2).$$

Then we write $B = B_1 \oplus_p B_2$. Let α, β be complex numbers. If $\dim B = 2$, then we calculate the norm of $\alpha P + \beta Q$ on B. For any $F \in B$, there is a unique decomposition $F = F_1 + F_2$. In Part I, we calculate $\|\alpha P + \beta Q\|_{L(B)}$ when $\dim B = 2$, $B = B_1 \oplus_p B_2$ and $\mathrm{ran} P = B_1$. In Part II, we calculate $\|\alpha P + \beta Q\|_{L(B)}$ when $\dim B = 2$ and $B = B_1 \oplus_p B_2$. If B is an infinite dimensional Hilbert space H, then we can reduce it to two dimensional case. In Part III, we give the elementary proof of the Feldman-Krupnik-Markus theorem using the cosine of the angle between P and Q and the results of Part I and Part II when $\dim B = 2$, $p = 2$. Furthermore we calculate the norm of $(\alpha A_1 + \beta A_2)(\gamma A_1 + \delta A_2)^{-1}$ for complex numbers $\alpha, \beta, \gamma, \delta$ $(\gamma\delta \neq 0)$ and operators A_1, A_2 which are not necessarily linear and satisfy $A_2 \gamma A_1 = A_1 \delta A_2$.

Theorem (Feldman-Krupnik-Markus) Let P be a bounded linear operator on a Hilbert space H satisfying $P^2 = P$, $P \neq 0, I$, and let $Q = I - P$. Let α, β be complex numbers. Then

$$\begin{aligned} \|\alpha P + \beta Q\|_{L(H)} &= \sqrt{\left|\frac{\alpha - \beta}{2}\right|^2 \left(\|P\|_{L(H)}^2 - 1\right) + \left(\frac{|\alpha| + |\beta|}{2}\right)^2} \\ &+ \sqrt{\left|\frac{\alpha - \beta}{2}\right|^2 \left(\|P\|_{L(H)}^2 - 1\right) + \left(\frac{|\alpha| - |\beta|}{2}\right)^2}. \end{aligned}$$

次の Lemma 1 の (3) は Feldman-Krupnik-Markus(cf. [1]) によりもっと一般的に知られていたが、この場合はもっと初等的に証明でき、Part I, II, III において本質的である。

Lemma 1. $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ と複素数 a, b, c, d について不等式：

$$\begin{aligned} &\max\left((|a|^p + |c|^p)^{1/p}, \ (|b|^p + |d|^p)^{1/p}\right) \leq \sup_{\|(x,y)^T\|_p = 1} \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_p \\ &= \sup_{x,y \in C} \frac{(|ax + by|^p + |cx + dy|^p)^{1/p}}{(|x|^p + |y|^p)^{1/p}} \leq \left((|a|^q + |b|^q)^{p/q} + (|c|^q + |d|^q)^{p/q}\right)^{1/p}. \end{aligned}$$

が成り立つ。よって、$\|\alpha P+\beta Q\|_{L(B)} := \sup_{\|F\|_B=1} \|(\alpha P+\beta Q)F\|_B$

$$= \sup_{\|F_1+F_2\|_B=1} \|(\alpha F_1+(\alpha-\beta)CF_2)+\beta F_2\|_B .$$

とくに、$1 \le p \le \infty, \quad \|F\|_B = (\|F_1\|_B^p + \|F_2\|_B^p)^{1/p}$ のとき、

$$\|\alpha P+\beta Q\|_{L(B)} = \sup_{(\|F_1\|_B^p+\|F_2\|_B^p)^{1/p}=1} (\|\alpha F_1+(\alpha-\beta)CF_2\|_B^p + \|\beta F_2\|_B^p)^{1/p} .$$

とくに　$C=0$ のとき、$\|\alpha P+\beta Q\|_{L(B)} = \max(|\alpha|,\ |\beta|)$.

Theorem 1.　$1 \le p \le \infty,\ \ \dim B=2,\ \ B=B_1 \oplus_p B_2,\ \ \mathrm{ran}P=B_1$ のとき、$\|e^1\|_B = \|e^2\|_B = 1$ を満たすような B_1 の基底 e^1 と B_2 の基底 e^2 が存在し、任意の $F_1 \in B_1, F_2 \in B_2$ は複素数 x, y により $F_1 = xe^1,\ \ F_2 = ye^2$ とかけ、先に定義した作用素 C に対し、$Ce^2 = ae^1$ を満たす複素数 a が存在して等式：

$$\|\alpha P+\beta Q\|_{L(B)} = \sup_{(|x|^p+|y|^p)^{1/p}=1} (|\alpha x+(\alpha-\beta)ay|^p + |\beta y|^p)^{1/p}$$

が成り立ち、$\frac{1}{p}+\frac{1}{q}=1$ のとき不等式：

$$\begin{aligned}\max\Big(|\alpha|,\quad (|\alpha-\beta|^p|a|^p+|\beta|^p)^{1/p}\Big) &\le \|\alpha P+\beta Q\|_{L(B)}\\ &\le \Big((|\alpha|^q+|\alpha-\beta|^q|a|^q)^{p/q}+|\beta|^p\Big)^{1/p}\end{aligned}$$

が成り立ち、等式：$\|P\|_{L(B)} = (|a|^q+1)^{1/q}$ が成り立つ。
とくに、$p=\infty,1,2$ のとき、それぞれ次の (1), (2), (3) が成り立つ。
(1)　$p=\infty$ のとき、　$\|\alpha P+\beta Q\|_{L(B)} = \sup_{\max(|x|,|y|)=1} \max(|\alpha x+(\alpha-\beta)ay|,\quad |\beta y|)$

$$= \max(|\alpha|+|\alpha-\beta|\cdot|a|,\quad |\beta|) .$$

(2)　$p=1$ のとき、　$\|\alpha P+\beta Q\|_{L(B)} = \sup_{|x|+|y|=1} (|\alpha x+(\alpha-\beta)ay|+|\beta y|)$

$$= \max(|\alpha|,\quad |\alpha-\beta|\cdot|a|+|\beta|) .$$

(3)　$p=2$ のとき、　$\|\alpha P+\beta Q\|_{L(B)} = \sup_{(|x|^2+|y|^2)^{1/2}=1} \left(|\alpha x+(\alpha-\beta)ay|^2+|\beta y|^2\right)^{1/2}$

$$= \sqrt{\left|\frac{\alpha-\beta}{2}\right|^2 |a|^2 + \left(\frac{|\alpha|+|\beta|}{2}\right)} + \sqrt{\left|\frac{\alpha-\beta}{2}\right|^2 |a|^2 + \left(\frac{|\alpha|-|\beta|}{2}\right)}.$$

が成り立つ。とくに$ad-bc=0$のとき、2つめの不等号は等号になる。とくに$p=\infty,1,2$のとき、それぞれ次の(1), (2), (3)が成り立つ。

(1) $p=\infty$のとき、 $\sup_{\|(x,y)^T\|_\infty=1}\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\|_\infty = \sup_{x,y\in\mathbf{C}}\frac{\max(|ax+by|,|cx+dy|)}{\max(|x|,|y|)}$

$$= \max(|a|+|b|,\ |c|+|d|).$$

(2) $p=1$のとき、 $\sup_{\|(x,y)^T\|_1=1}\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\|_1 = \sup_{x,y\in\mathbf{C}}\frac{|ax+by|+|cx+dy|}{|x|+|y|}$

$$= \max(|a|+|c|,\ |b|+|d|).$$

(3) $p=2$のとき、 $\sup_{\|(x,y)^T\|_2=1}\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\|_2 = \sup_{x,y\in\mathbf{C}}\frac{\sqrt{|ax+by|^2+|cx+dy|^2}}{\sqrt{|x|^2+|y|^2}}$

$$= \sqrt{\frac{|a|^2+|b|^2+|c|^2+|d|^2}{4}+\frac{|ad-bc|}{2}} + \sqrt{\frac{|a|^2+|b|^2+|c|^2+|d|^2}{4}-\frac{|ad-bc|}{2}}.$$

とくに(3)の証明のアイデアは任意の正の数kに対して次の(i) ～ (vi)の同値性による。

(i) $k \geq \sup_{x,y\in\mathbf{C}}\frac{|ax+by|^2+|cx+dy|^2}{|x|^2+|y|^2}$

(ii) すべての複素数x,yについて、

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} k-|a|^2-|c|^2 & a\bar{b}+c\bar{d} \\ \bar{a}b+\bar{c}d & k-|b|^2-|d|^2 \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \geq 0$$

(iii) $k \geq \max(|a|^2+|c|^2,\ |b|^2+|d|^2)$ かつ

$$\begin{vmatrix} k-|a|^2-|c|^2 & a\bar{b}+c\bar{d} \\ \bar{a}b+\bar{c}d & k-|b|^2-|d|^2 \end{vmatrix} \geq 0$$

(iv) $k \geq \max(|a|^2+|c|^2,\ |b|^2+|d|^2)$ かつ $k^2-(|a|^2+|b|^2+|c|^2+|d|^2)\,k+|ad-bc|^2 \geq 0$

(v) $k \geq \frac{|a|^2+|b|^2+|c|^2+|d|^2+\sqrt{(|a|^2+|b|^2+|c|^2+|d|^2)^2-4|ad-bc|^2}}{2}$

(vi) $\sqrt{k} \geq \sqrt{\frac{|a|^2+|b|^2+|c|^2+|d|^2}{4}+\frac{|ad-bc|}{2}}+\sqrt{\frac{|a|^2+|b|^2+|c|^2+|d|^2}{4}-\frac{|ad-bc|}{2}}$

Part I ran$P=B_1$のとき、有界線形作用素$C: B_2\to B_1$が存在して、$B=B_1\oplus_p B_2$という分解について、

$$P=\begin{pmatrix} I_1 & C \\ 0 & 0 \end{pmatrix},\quad Q=\begin{pmatrix} 0 & -C \\ 0 & I_2 \end{pmatrix},\quad \alpha P+\beta Q=\begin{pmatrix} \alpha I_1 & (\alpha-\beta)C \\ 0 & \beta I_2 \end{pmatrix}$$

Part II $\mathrm{ran}P = B_1$ を仮定しないときを考える。$Pk \neq 0, Qk \neq 0$ を満たす $k \in B$ を用いて、2 次元複素数ベクトル空間に、

$$\|(s,t)\|_{B_k} = \|sPk + tQk\|_B, \quad (s, t \in C)$$

というノルムを定めてできる 2 次元 Banach 空間を B_k で表す。B_k 上の作用素 P_0, Q_0 を、

$$P_0(s,t) = (s,0), \quad Q_0(s,t) = (0,t), \quad (s,t \in C)$$

と定める。このとき、$\|\alpha P + \beta Q\|_{L(B)}$ は

$$\|\alpha P_0 + \beta Q_0\|_{L(B_k)} := \sup_{\|(s,t)\|_{B_k}=1} \|(\alpha P_0 + \beta Q_0)(s,t)\|_{B_k}$$

を用いて、

$$\|\alpha P + \beta Q\|_{L(B)} = \sup_{k\in B, Pk\neq 0, Qk\neq 0} \|\alpha P_0 + \beta Q_0\|_{L(B_k)}$$

とかくことができる。$\|\alpha P_0 + \beta Q_0\|_{L(B_k)}$ について次の Theorem 2 が成り立つ。

Theorem 2. $1 \leq p \leq \infty$, $\dim B = 2$, $B = B_1 \oplus_p B_2$ のとき、$\|e^1\|_B = \|e^2\|_B = 1$ を満たすような B_1 の基底 e^1 と B_2 の基底 e^2 が存在し、$Pk \neq 0, Qk \neq 0$ を満たす $k \in B$ に対し、複素数 f_1, f_2, g_1, g_2 が存在して

$$Pk = f_1 e^1 + f_2 e^2, \quad Qk = g_1 e^1 + g_2 e^2$$

とかける。このとき、$f_1 g_2 - f_2 g_1 \neq 0$ が成り立ち、

$$a := \frac{\alpha f_1 g_2 - \beta f_2 g_1}{f_1 g_2 - f_2 g_1}, \quad b := \frac{(\beta - \alpha) f_1 g_1}{f_1 g_2 - f_2 g_1}, \quad c := \frac{(\alpha - \beta) f_2 g_2}{f_1 g_2 - f_2 g_1}, \quad d := \frac{\beta f_1 g_2 - \alpha f_2 g_1}{f_1 g_2 - f_2 g_1}$$

と定めると、等式： $\|\alpha P_0 + \beta Q_0\|_{L(B_k)} = \sup_{\|xe^1 + ye^2\|_B = 1} \|(ax+by)e^1 + (cx+dy)e^2\|_B$

$$= \sup_{(|x|^p + |y|^p)^{1/p} = 1} (|ax+by|^p + |cx+dy|^p)^{1/p} = \sup_{\|(x,y)^T\|_p = 1} \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_p$$

が成り立ち、$\frac{1}{p} + \frac{1}{q} = 1$ のとき不等式： $\max\left((|a|^p + |c|^p)^{1/p}, \quad (|b|^p + |d|^p)^{1/p}\right)$

$$\leq \|\alpha P_0 + \beta Q_0\|_{L(B_k)} \leq \left((|a|^q + |b|^q)^{p/q} + (|c|^q + |d|^q)^{p/q}\right)^{1/p}$$

が成り立ち、等式：

$$\|P_0\|_{L(B_k)} = \frac{(|f_1|^p + |f_2|^p)^{1/p} (|g_1|^q + |g_2|^q)^{1/q}}{|f_1 g_2 - f_2 g_1|}$$

が成り立つ。とくに、$p=\infty,1,2$ のとき、それぞれ次の (1), (2), (3) が成り立つ。

(1) $p=\infty$ のとき、$\|\alpha P_0+\beta Q_0\|_{L(B_k)}=\max(|a|+|b|,\quad |c|+|d|)$.

(2) $p=1$ のとき、$\|\alpha P_0+\beta Q_0\|_{L(B_k)}=\max(|a|+|c|,\quad |b|+|d|)$.

(3) $p=2$ のとき、$\|\alpha P_0+\beta Q_0\|_{L(B_k)}$

$$= \sqrt{\frac{|a|^2+|b|^2+|c|^2+|d|^2}{4}+\frac{|ad-bc|}{2}}+\sqrt{\frac{|a|^2+|b|^2+|c|^2+|d|^2}{4}-\frac{|ad-bc|}{2}}$$

$$= \sqrt{\left|\frac{\alpha-\beta}{2}\right|^2\cdot\left|\frac{f_1\bar{g_1}+f_2\bar{g_2}}{f_1g_2-f_2g_1}\right|^2+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}+\sqrt{\left|\frac{\alpha-\beta}{2}\right|^2\cdot\left|\frac{f_1\bar{g_1}+f_2\bar{g_2}}{f_1g_2-f_2g_1}\right|^2+\left(\frac{|\alpha|-|\beta|}{2}\right)^2}.$$

Example. $Pk=e^1+e^2$, $Qk=e^1-e^2$ のとき、$a=d=\frac{\alpha+\beta}{2}$, $b=c=\frac{\alpha-\beta}{2}$ より、

$$\frac{(|\alpha+\beta|^p+|\alpha-\beta|^p)^{1/p}}{2}\le\|\alpha P_0+\beta Q_0\|_{L(B_k)}\le\left(\frac{|\alpha+\beta|^q+|\alpha-\beta|^q}{2}\right)^{1/q}.$$

とくに $p=2$ のとき、$\|\alpha P_0+\beta Q_0\|_{L(B_k)}=\max(|\alpha|,|\beta|)$. とくに $p=1,\infty$ のとき、

$$\|\alpha P_0+\beta Q_0\|_{L(B_k)}=\frac{|\alpha+\beta|+|\alpha-\beta|}{2}.$$

これは、Q-algebra でない例を与えている (cf. [2], [3])。

Part III Feldman-Krupnik-Markus の定理の別証明：$Pk\neq 0, Qk\neq 0$ を満たす $k\in H$ に対し、正規直交系 e^1, e^2 が存在して $\mathrm{span}(Pk,Qk)=\mathrm{span}(e^1,e^2)$ が成り立ち、これは 2 次元 Hilbert 空間になる。このとき、

$$\|xe^1+ye^2\|_H=(|x|^2+|y|^2)^{1/2}$$

が成り立つから、Theorem 2 (3) より、

$$\|\alpha P_0+\beta Q_0\|_{L(B_k)}=\sqrt{\gamma+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}+\sqrt{\gamma+\left(\frac{|\alpha|-|\beta|}{2}\right)^2},$$

$$\text{ここで、}\quad \gamma=\left|\frac{\alpha-\beta}{2}\right|^2\frac{\rho(Pk,Qk)^2}{1-\rho(Pk,Qk)^2},\quad \rho(Pk,Qk)=\frac{|(Pk,Qk)_H|}{\|Pk\|_H\|Qk\|_H}.$$

よって、$0\le\rho(Pk,Qk)<1$. γ は $\rho=\rho(Pk,Qk)$ について単調増加であるから、

$$\begin{aligned}\|\alpha P+\beta Q\|_{L(H)} &= \sup_{k\in B, Pk\neq 0, Qk\neq 0}\|\alpha P_0+\beta Q_0\|_{L(B_k)}\\ &= \sqrt{\gamma+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}+\sqrt{\gamma+\left(\frac{|\alpha|-|\beta|}{2}\right)^2},\end{aligned}$$

$$\text{ここで、}\quad \gamma = \left|\frac{\alpha-\beta}{2}\right|^2 \frac{\rho(P,Q)^2}{1-\rho(P,Q)^2}, \quad \rho(P,Q) = \sup_{k\in H, Pk\neq 0, Qk\neq 0} \rho(Pk,Qk).$$

よって、$\|\alpha P+\beta Q\|_{L(H)}$ が $\rho(P,Q)$ を用いて表された。とくに、$\alpha=1,\beta=0$ の場合より $\rho(P,Q)=\sup_{f\in H}\frac{|(Pf,Qf)_H|}{\|Pf\|_H\cdot\|Qf\|_H}=\sqrt{1-\|P\|_{L(H)}^{-2}}$ が成り立ち、$\|\alpha P+\beta Q\|_{L(H)}$ が $\|P\|_{L(H)}$ を用いて表された。証明終。

2つの作用素の角度の cosine に着目し、つぎの新しい定理を証明できる。とくに、$A_1=P,\ A_2=Q,\ \gamma=\delta=1$ のとき、(2) は Feldman-Krupnik-Markus の定理になる。

Theorem 3. Hilbert 空間 H 全体で定義された作用素 A_1,A_2 （線形とは限らない）と複素数 $\alpha,\beta,\gamma,\delta$ （ただし $\gamma\delta\neq 0$ ）について、(1), (2) が成り立つ。

(1)
$$\frac{\|\alpha A_1+\beta A_2\|}{\|\gamma A_1+\delta A_2\|} \leq \sqrt{\gamma_\rho+\left(\frac{|\alpha\delta|+|\beta\gamma|}{2|\gamma\delta|}\right)^2}+\sqrt{\gamma_\rho+\left(\frac{|\alpha\delta|-|\beta\gamma|}{2|\gamma\delta|}\right)^2}$$

$$\text{ここで、}\quad \gamma_\rho = \left|\frac{\alpha\delta-\beta\gamma}{2\gamma\delta}\right|^2 \frac{\rho(A_1,A_2)^2}{1-\rho(A_1,A_2)^2}, \quad \rho(A_1,A_2) = \sup_{f\in H} \frac{|(A_1f,A_2f)_H|}{\|A_1f\|_H\cdot\|A_2f\|_H}.$$

(2) とくに $A_2\gamma A_1 = A_1\delta A_2 = 0$ かつ $\gamma A_1+\delta A_2$ が逆作用素をもつとき、

$$\begin{aligned}\|(\alpha A_1+\beta A_2)(\gamma A_1+\delta A_2)^{-1}\| &= \sqrt{\left|\frac{\alpha\delta-\beta\gamma}{2\gamma\delta}\right|^2\frac{\rho(A_1,A_2)^2}{1-\rho(A_1,A_2)^2}+\left(\frac{|\alpha\delta|+|\beta\gamma|}{2|\gamma\delta|}\right)^2}\\ &+ \sqrt{\left|\frac{\alpha\delta-\beta\gamma}{2\gamma\delta}\right|^2\frac{\rho(A_1,A_2)^2}{1-\rho(A_1,A_2)^2}+\left(\frac{|\alpha\delta|-|\beta\gamma|}{2|\gamma\delta|}\right)^2}.\end{aligned}$$

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Toeplitz type operators on half-spaces of $L^2(R)$ defined by orthogonal wavelets

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§1. Introduction Let ψ be an orthonormal wavelet of $L^2(R)$. As usual we define

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \quad (j, k \in Z). \tag{1}$$

$\{\psi_{j,k} : j, k \in Z\}$ forms an othonormal basis of $L^2(R)$. It is easy to see that the inverse Fourier transform of $\psi_{j,k}(\xi)$ is given by $\check{\psi_{j,k}}(\xi) = 2^{-j/2}\check{\psi}(2^{-j}\xi)e^{2^{-j}k\xi i}$. By the Plancherel theorem, the set $\{\check{\psi_{j,k}} : j, k \in Z\}$ forms an orthonormal basis of $L^2(R)$.

Definition 1 Let ψ be an orthonormal wavelet of $L^2(R)$. We define $H^2_\psi(R)$ as the closed subspace of $L^2(R)$ generated by the set $\{\psi_{j,k} : j \in Z, k \in Z_+(= \{0, 1, 2, ...\})\ \}$, and define $H_{\check{\psi}}$ as the image of the inverse Fourier transform of the space H^2_ψ. $H^2_{\check{\psi}}$ is equal to the space generated by $\{\check{\psi_{j,k}} : j \in Z, k \in Z_+(= \{0, 1, 2, ...\})\ \}$.

H^p $(0 < p \le \infty)$ denotes the classical Hardy space on the real line. As we will see in Example 1 below, H^2 is equal to $H^2_{\check{\psi}}$ for the Haar wavelet ψ.

Definition 2 For $g \in L^\infty(R)$, we define a Toeplitz type operator T_g on $H^2_{\check{\psi}}$ by

$$T_g(f) := P_{\check{\psi}}(M_g(f)) \quad (f \in H^2_{\check{\psi}}),$$

where $P_{\check{\psi}}$ stand for the orthogonal projection of $L^2(R)$ onto $H^2_{\check{\psi}}$.

In this paper, we study some fundamental properties of T_g for $g \in L^\infty(R)$. For the definitions and the fundamental properties of Toeplitz operators and orthonormal wavelets, we refer to [1,2,3].

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§2. Some fundamental properties of Toeplitz type operators on $H^2_{\check{\psi}}$

以下これまで得られた結果について述べる。証明は Lemma 3, 定理４、定理１１についてのみ述べる。

Theorem 1 ([5]) *If $g \in L^\infty(R)$ and $T_g = 0$, we have $g = 0$.*

Lemma 2 ([5]) *For each $g \in L^\infty(R)$, we have $T_{\bar{g}} = T_g^*$.*

Lemma 3 *Let V be a finite dimensional subspace of $H^2_{\check{\psi}}$ and and let $0 < \varepsilon < 1$. For each $f \in L^2(R)$ we can choose positive integers N and h such that $e^{2^{N+1}h\xi i}f$ has a decomposition of the form*

$$e^{2^{N+1}h\xi i}f = f_1 + f_2$$

with the properties (i) $f_1 \in H^2_{\check{\psi}} \ominus V$, (ii) $\|f_1\|_2 \geq (1-\varepsilon)^2\|f\|_2$ and (iii) $\|f_2\| \leq 2\varepsilon\|f\|_2$.

Proof. Let $e_1, ..., e_n$ be an orthonormal basis of V. For positive integers N and h, we define closed subspaces $H_j : j = 1, 2, 3$ as follows. H_1 is the closed subspace of $L^2(R)$ generated by the set of vectors $\{\check{\psi}_{j,k} :\mid j \mid \leq N, k \geq -2^{N+j}h\}, H_2 = L^2(R) \ominus H_1$ and H_3 is the closed subspace generated by $\{\check{\psi}_{j,k} :\mid j \mid > N$ or $\mid j \mid \leq N, k \geq 2^{N+j}h\}$. We can choose N and h so that the following (2) and (3) are valid.

$$f = f' + f'', f' \in H_1,\ f'' \in H_2,\ \|f''\|_2 \leq \varepsilon\|f\|_2 \tag{2}$$

$$e_k = {e_k}' + {e_k}'',\ \ {e_k}' \in H^2_{\check{\psi}} \ominus H_3,\ {e_k}'' \in H_3,\ \|{e_k}''\|_2 \leq \frac{\varepsilon}{\sqrt{n}} \quad k = 1, ..., n \tag{3}$$

From (2) we can deduce at once

$$\|f'\|_2 \geq \|f\|_2 - \|f''\|_2 \geq (1-\varepsilon)\|f\|_2. \tag{4}$$

Here we put $\tilde{f} := f'e^{2^{N+1}h\xi i}$. Then we have

$$\begin{aligned} \tilde{f} &= \Bigl(\sum_{|j|\leq N, k\geq -2^{N+j}h} < f', \check{\psi}_{j,k} > \check{\psi}_{j,k}\Bigr)e^{2^{N+1}h\xi i} \\ &= \Bigl(\sum_{|j|\leq N, k\geq 2^{N+j}h} < f', \check{\psi}_{j,k-2^{N+j+1}} > \check{\psi}_{j,k}\Bigr) \in H_3. \end{aligned}$$

Hence if we decompose $\tilde{f} = \tilde{f}' + \tilde{f}''$ with $\tilde{f}' \in H^2_{\check{\psi}} \ominus V$ and $\tilde{f}'' \in V$, we get

$$\begin{aligned} \|\tilde{f}''\|_2^2 &= \sum_{k=1}^{n} |< \tilde{f}, e_k >|^2 = \sum_{k=1}^{n} |< \tilde{f}, e_k'' >|^2 \\ &\leq \Bigl(\sum_{k=1}^{n} \|e_k''\|_2^2\Bigr)\|\tilde{f}\|_2^2 \leq \varepsilon^2\|\tilde{f}\|_2^2. \end{aligned} \tag{5}$$

$$\|\tilde{f}'\| \geq \|\tilde{f}\|_2 - \|\tilde{f}''\|_2^2 \geq \|\tilde{f}\|_2 - \varepsilon\|\tilde{f}\|_2 = (1-\varepsilon)\|\tilde{f}\|_2. \tag{6}$$

Therefore we get

$$\begin{aligned} f e^{2^{N+1}h\xi i} &= f' e^{2^{N+1}h\xi i} + f'' e^{2^{N+1}h\xi i} \\ &= \tilde{f}' + \tilde{f}'' + f'' e^{2^{N+1}h\xi i} = f_1 + f_2, \end{aligned} \tag{7}$$

where $f_1 = \tilde{f}'$ and $f_2 = \tilde{f}'' + f'' e^{-2^{N+1}h\xi i}$. For f_1 and f_2 we get the following norm estimation from (2), (4), (6) and (7) as required.

$$\|f_1\|_2 = \|\tilde{f}'\|_2 \geq (1-\varepsilon)\|\tilde{f}\|_2 = (1-\varepsilon)\|f'\|_2 \geq (1-\varepsilon)^2\|f\|_2.$$
$$\|f_2\|_2 \leq \|\tilde{f}''\|_2 + \|f''\| \leq \varepsilon\|\tilde{f}\|_2 + \varepsilon\|f\|_2 \leq 2\varepsilon\|f\|_2.$$

Q.E.D.

Theorem 4 *If* $g \in L^\infty(R)$ *and* T_g *is a Fredholm operator, then* g *is invertible in* $L^\infty(R)$.

Proof. Since T_g is a Fredholm operator, $V := \ker T_g$ is finite dimensional and $W := \mathrm{Im} T_g$ is a closed subspace of $H^2_{\check{\psi}}$. Therefore T_g induces a one to one bounded linear operator from $H^2_{\check{\psi}} \ominus V$ onto W, and there exists a $\delta > 0$ which satisfies

$$\|T_g f\|_2 \geq \delta\|f\|_2 \quad (f \in H^2_{\check{\psi}} \ominus V) \tag{8}$$

Let f be an arbitrary element of $L^2(R)$, and put $\varepsilon := \min\{\frac{1}{4}, \frac{\delta}{8\|g\|_\infty}\}$. For these f and ε we choose, by Lemma 3, positive integers N and h such that

$$f e^{2^{N+1}h\xi i} = f_1 + f_2, \qquad f_1 \in H^2_{\check{\psi}} \ominus V,$$
$$\|f_1\|_2 \geq (1-\varepsilon)^2\|f\|_2 > \frac{\|f\|_2}{2}, \tag{9}$$
$$\|f_2\|_2 \leq 2\varepsilon\|f\|_2 \leq \frac{\delta\|f\|_2}{4\|g\|_\infty}. \tag{10}$$

Thus, by (8), (9) and (10), we can calculate as follows:

$$\begin{aligned} \|M_g f\|_2 &= \|gf\|_2 = \|gfe^{2^{N+1}h\xi i}\|_2 \\ &= \|g(f_1+f_2)\|_2 \geq \|gf_1\|_2 - \|gf_2\|_2 \\ &\geq \|T_g f_1\|_2 - \|g\|_\infty \frac{\delta\|f\|_2}{4\|g\|_\infty} \\ &\geq \delta\frac{\|f\|_2}{2} - \frac{\delta}{4}\|f\|_2 = \frac{\delta}{4}\|f\|_2. \end{aligned}$$

That is , we get

$$\|M_g f\|_2 \geq \frac{\delta}{4}\|f\|_2 \quad (f \in L^2(R)). \tag{11}$$

In the same way, since $T_g^* = T_{\overline{g}}$ is a Fredholm operator, we get for some $\delta' > 0$:

$$\|M_g^* f\|_2 \geq \frac{\delta'}{4}\|f\|_2 \quad (f \in L^2(R)). \tag{12}$$

By (11) and (12), we have that M_g is invertible in $\mathcal{B}(L^2(R))$ and hence g is invertible in $L^\infty(R)$. Q.E.D.

Corollary 5 *For each non-zero element $g \in L^\infty(R)$, T_g is not a compact operator.*

Corollary 6 *If g is in $L^\infty(R)$, then $\mathcal{R}(g) = \sigma(M_g) \subseteq \sigma(T_g)$.*

Corollary 7 *If we put $\xi(g) = T_g$ $(g \in L^\infty(R))$, ξ is a *-linear isometry from $L^\infty(R)$ into $\mathcal{B}(H^2_{\tilde{\psi}})$.*

Example 1 Let ψ be the Haar wavelete:

$$\psi(x) = \begin{cases} 1 & (x \in [0, 1/2)) \\ -1 & (x \in [1/2, 1)) \\ 0 & \text{otherwise} \end{cases}$$

Then we have $H^2_\psi = L^2(R_+)$, and hence $H^2_{\tilde{\psi}} = H^2$, the classical Hardy space. Further for $g \in L^\infty(R)$, $W_g(f) := \mathcal{F}^{-1}T_g\mathcal{F}(f)$ $(f \in L^2(R_+))$ is a Wiener-Hopf operator with a symbol function g. Therefore, if we apply Theorem 4 for the Haar wavelet ψ, we get at once the follwing well-known result.

Corollary 8 If $g \in L^\infty(R)$ and if the Wiener-Hopf operator W_g is a Fredholm operator, then g is invertible in $L^\infty(R)$.

§3. $e^{i\lambda\xi}H^2_{\check{\psi}} \subseteq H^2_{\check{\psi}}$ $(\lambda > 0)$ を満たす $H^2_{\check{\psi}}$ の特徴付け

In this section, we characterize the class of $H^2_{\check{\psi}}$ which satisfies $e^{i\lambda\xi}H^2_{\check{\psi}} \subseteq H^2_{\check{\psi}}$ $(\lambda > 0)$.

Lemma 9(cf. [4]) *If W is a closed subspace of $H^2_{\check{\psi}}$ and $e^{i\xi}W \subseteq W$ for each $\lambda > 0$, then the following (i) or (ii) hold;*

(i) there exists a measurable set F of R such that $W = \mathcal{X}_F \cdot L^2(R)$,

(ii) there exists a unimodular function u on R which satisfies

$$H^2_{\check{\psi}} = u \cdot H^2.$$

Lemma 10 *Let g be a function in $L^\infty(R)$. Then g is an element of $H^\infty(R)$ if and oly if g satisfies $gH^2 \subseteq H^2$.*

Theorem 11 *Let ψ be an orthonormal wavelet for $L^2(G)$ such that*

$$e^{i\lambda\xi}H^2_{\check{\psi}} \subseteq H^2_{\check{\psi}} \text{ for each } \lambda > 0. \tag{13}$$

Then there exists a unimodular function on R such that $H^2_{\check{\psi}} = u \cdot H^2$, where u satisfies

$$u(2^{-1}\xi) = e^{ia}u(\xi) \ \ (a.e. \xi \in R) \ \ \text{for some } a \in R. \tag{14}$$

Conversely, if u is a unimodular function on R satisfying (14), there exists an orthonormal wavelet ψ which satisfies $H^2_{\check{\psi}} = u \cdot H^2$.

Proof. Suppose that ψ is an orthonormal wavelet satisfying (13). Then by Lemma 9, (i) or (ii) of Lemma 9 holds with $W = H^2_{\check{\psi}}$.

Suppose that (i) of Lemma 9 holds, that is ther exists a measurable function F of R such that $H^2_{\check{\psi}} = \mathcal{X}_F \cdot L^2(R)$. Then $e^{-i\xi}\check{\psi} = \check{\psi}_{0,-1} \in H^{2\perp}_{\check{\psi}} = \mathcal{X}_{R\setminus F} \cdot L^2(R)$. This implies $|\check{\psi}| \in \mathcal{X}_F \cdot L^2(R) \cap \mathcal{X}_{R\setminus F} \cdot L^2(R) = \{0\}$, a contradiction. Therefore we nay assume that (i) holds, that is, there exists a measurable unimodular function u on R such that

$$H^2_{\check{\psi}} = u \cdot H^2. \tag{15}$$

Let $f_1 \in H^2$ be arbitrary. Then by (15), there exists $F_1 \in H^2_{\psi}$ such that $u \cdot f_1 = F_1$. Since both spaces H^2_{ψ} and H^2 are 2^j-dilation invariant for each $j \in Z$, we have

$$u(2\xi) \cdot f_1(2\xi) = F_1(2\xi), \ \ f_1(2\xi) \in H^2, \ \ F_1(2\xi) \in H^2_{\psi} \tag{16}$$

By (15) and (16) we have a representation of the form;

$$u(2\xi) \cdot f_1(2\xi) = u(\xi) f_2(\xi), \ \text{ where } \ f_2 \in H^2. \tag{17}$$

If we put $\zeta = 2\xi \in$ in (17), we get the relation : $u(\zeta)u(2^{-1}\zeta)^{-1}f_1(\zeta) = f_2(2^{-1}\zeta) \in H^2$. Then by Lemma 10, we have

$$u(\zeta)u(2^{-1}\zeta)^{-1} \in H^{\infty}. \tag{18}$$

In the same way, if we start with 2^{-1}-dilation, we get

$$u(2^{-1}\zeta)u(\zeta)^{-1} \in H^{\infty}. \tag{19}$$

By (18) and (19), we can conclude that $u(2^{-1}\zeta)u(\zeta)^{-1}$ is a constant of absolute value 1, that is

$$u(2^{-1}\zeta)u(\zeta)^{-1} = e^{ia} \ \ \text{for some } a \in R.$$

Conversely, suppose that u is a unimodular function on R stisfying (14). Then if we choose $\psi \in L^2(R)$ such that $\check{\psi} = u \cdot \check{\psi}_1$ for the Haar wavelet ψ_1, it is easy to see that ψ is an orthonormal wavelet for $L^2(R)$ which satisfies $H^2_{\psi} = u \cdot H^2$. Q.E.D.

* In the conferece, we reduce the case (ii) directly from (13), and Prof. Nakazi pointed out that from (13) the case (i) also can occure. In this report we added the consideration on the case (i), by which we can fill a gap in the proof.

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