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Confluent hypergeometric systems
associated with principal nilpotent p -tuples
(主冪零 p 組に付随する合流型超幾何系)

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CHAPTER 1

Introduction

In this paper, we define and study a hypergeometric system associated with a principal nilpotent p -tuple. Its parameter space is the dual space of the centralizer of the principal nilpotent p -tuple.

Aomoto [1] and Gel'fand [3] independently defined a hypergeometric system of Gauss type, which is now called an Aomoto-Gel'fand system. The parameter space of an Aomoto-Gel'fand system is the dual space of a Cartan subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. In [5], Gel'fand-Retahk-Serganova defined and studied a hypergeometric system of Airy type, which is a confluent version of an Aomoto-Gel'fand system. The parameter space of this system is the dual space of the Lie algebra of a Jordan Lie group (called a Jordan Lie subalgebra in [12]).

Inspired by the papers above, Kimura-Haraoka-Takano [7] introduced hypergeometric systems associated with regular elements of $\mathfrak{gl}(n, \mathbb{C})$. Their parameter spaces are the dual spaces of the centralizers $\mathfrak{z}_{\mathfrak{gl}(n, \mathbb{C})}(x)$ of the regular elements. (Recall that $x \in \mathfrak{gl}(n, \mathbb{C})$ is regular if $\dim \mathfrak{z}_{\mathfrak{gl}(n, \mathbb{C})}(x) = n$.) These systems coincide with Aomoto-Gel'fand systems in the case where the regular elements are semisimple, and Gel'fand-Retahk-Serganova's systems in the case where the regular elements are nilpotent. For the other regular elements, we obtain systems in-between. For example, we obtain Gauss, Kummer, Bessel, Hermite-Weber, and Airy for $n = 4$. Furthermore, in [10], Kimura-Takano explained the process of confluence between their hypergeometric systems by taking limits of regular elements.

There exist many Lie subalgebras which are limits of Cartan subalgebras but not the centralizers of regular elements (cf. [12]). It is natural to ask if the systems of differential equations associated with these Lie subalgebras are obtained as the limits of systems of Gauss type.

In this paper, we introduce principal nilpotent p -tuples. Principal nilpotent 1-tuples are regular nilpotent elements, and principal nilpotent 2-tuples (pairs) were introduced by Ginzburg [6]. He also classified principal nilpotent pairs of type A_n by using Young diagrams. Principal nilpotent pairs of the other classical types were classified by Elashvili-Panyushev [2] and R. Yu [14]. The centralizer of a principal nilpotent p -tuple is a limit of Cartan subalgebras as follows:

LEMMA 1.1 (Lemma 6.1). *Let $(e_1, \dots, e_p)$ be a principal nilpotent p -tuple, and let $\mathfrak{h}$ be the Cartan subalgebra corresponding to $(e_1, \dots, e_p)$. Set $\mathbf{e} := e_1 + \dots + e_p$ and $\mathbf{e}(\tau) := \exp\left(\frac{\tau-1}{\tau}\mathbf{e}\right) \in GL(n)$. Then we have*

$$\lim_{\tau \rightarrow 0} \text{Ad}(\mathbf{e}(\tau)) \mathfrak{h} = \mathfrak{z}_{\mathfrak{gl}(n, \mathbb{C})}(e_1, \dots, e_p) := \bigcap_{i=1}^p \mathfrak{z}_{\mathfrak{gl}}(e_i).$$

In this paper, we take the limit of the action of $\mathbf{e}(\tau)$ to deform hypergeometric systems of Gauss type into the system associated with $(e_1, \dots, e_p)$, along with the corresponding integrands of integral representations of solutions.

This paper is organized as follows. In Chapter 2 we construct hypergeometric systems whose parameter spaces are n -dimension abelian Lie subalgebra of $\mathfrak{gl}(n)$ inspired by the definition of hypergeometric systems defined by Gel'fand et al. In Chapter 3 we review the result of [10]. In Chapter 4 we introduce principal nilpotent p -tuples. In Chapter 5 we exhibit an integral representation of a solution to a hypergeometric system associated with a principal nilpotent p -tuple. In Chapter 6 we consider the deformation of integrands of integral representations of solutions. In Chapter 7 we study the action of the normalizer of the abelian subalgebra generated by a principal nilpotent p -tuple, to specialize the parameters of the corresponding hypergeometric systems, as in [9]. In Chapters 8-10, as one we concentrate on considering the hypergeometric system associated with a principal nilpotent pair (i.e. $p = 2$). In Chapter 8 we show that the system corresponding to the Young diagram $(n - 1, 1)$ is generically holonomic, and the solutions coincide with generalized Airy functions. In Chapter 9 we consider the normalizer of $\mathfrak{a}$. In Chapter 10 we consider the hypergeometric systems in $2n$ variables corresponding to a Young diagram λ with $|\lambda| \leq 6$, and describe their classical counterparts. Chapters 2, 4-8 are taken from [13].

CHAPTER 2

Hypergeometric systems à la Gel'fand

Throughout this paper, set $G = GL(n, \mathbb{C})$ (or $GL(V)$) and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ (or $\mathfrak{gl}(V)$), where V is complex n -dimensional vector space.

In this chapter, we recall the definitions of hypergeometric systems of Gauss type [3] and of Airy type [5] defined by Gel'fand et al.

Let $G = GL(n, \mathbb{C})$, and

$$N = \left\{ \begin{bmatrix} I_m & B \\ O & D \end{bmatrix} \in G \mid D \in GL(n-m) \right\}.$$

Then we have the morphism

$$G \ni g \mapsto g \begin{bmatrix} I_m \\ O \end{bmatrix} \in G/N = \{z \in M_{n,m}(\mathbb{C}) \mid \text{rank}(z) = m\} =: Z.$$

Let Θ_Z denote the sheaf of vector fields on Z . Naturally we have a Lie algebra homomorphism $\partial : \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \ni a \mapsto \partial_a \in \Theta_Z(Z)$,

$$(\partial_a(f))(z) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(-ta)z) \quad (a \in \mathfrak{g}, f \in \mathcal{O}_Z, z \in Z).$$

Let $\mathfrak{a}$ be an n -dimensional abelian subalgebra of $\mathfrak{g}$. For a given $\alpha \in \mathfrak{a}^*$, we consider a hypergeometric system $M_{\mathfrak{a}, \alpha}$ defined on Z :

$$(1) \quad \mathcal{D}_Z \Big/ \left(\sum_{p,q=1}^n \sum_{j_1, j_2=1}^m \mathcal{D}_Z (\partial_{p,j_1} \partial_{q,j_2} - \partial_{p,j_2} \partial_{q,j_1}) + \sum_{i,j=1}^m \mathcal{D}_Z \left(\sum_{k=1}^n z_{k,i} \partial_{k,j} + \delta_{i,j} \right) + \sum_{a \in \mathfrak{a}} \mathcal{D}_Z (\partial_a - \alpha(a)) \right).$$

When we take the Cartan subalgebra consisting of diagonal matrices as $\mathfrak{a}$, the system $M_{\mathfrak{a}, \alpha}$ coincides with the Gel'fand's hypergeometric system [3]. When we take the Jordan Lie subalgebra as $\mathfrak{a}$:

$$(2) \quad \mathfrak{a} = \langle \Lambda_n^i \mid i = 0, 1, \dots, n-1 \rangle,$$

where $\Lambda = \sum_{i=1}^{n-1} E_{i+1,i}$, and $E_{i,j}$ is the matrix with (i, j) -entry 1 and with 0 for the other entries, the system $M_{\mathfrak{a}, \alpha}$ is the generalized Airy system [5].

LEMMA 2.1. For $a = [a_{ij}] \in \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$,

$$\partial_a = - \sum_{k,l,s} a_{k,s} z_{s,l} \partial_{k,l}.$$

In particular, for $i, j \leq n$,

$$\partial_{E_{i,j}} = - \sum_{l=1}^m z_{j,l} \partial_{i,l}.$$

PROOF.

$$\begin{aligned}
(\partial_a f)(Z) &= \left. \frac{d}{dt} \right|_{t=0} f(\exp(-ta)Z) \\
&= \sum_{k,l} \left(\left. \frac{d}{dt} \right|_{t=0} [\exp(-ta)Z]_{k,l} \right) \partial_{k,l} f(Z) \\
&= \sum_{k,l} \left(\left. \frac{d}{dt} \right|_{t=0} [Z - taZ + O(t^2)]_{k,l} \right) \partial_{k,l} f(Z) \\
&= - \sum_{k,l} [aZ]_{k,l} \partial_{k,l} f(Z) \\
&= - \sum_{k,l,s} a_{k,s} z_{s,l} \partial_{k,l} f(Z).
\end{aligned}$$

□

CHAPTER 3

Hypergeometric systems of Kimura-Takano

Kimura-Takano [10] studied the confluence of general hypergeometric systems, by focusing on taking limits of regular elements. In this chapter, we summarize the part of [10] concerning the limit of the centralizers of regular elements.

Let $b \in \mathfrak{g}$ be a regular element. For a Young diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$, put

$$b_\lambda = \bigoplus_{i=1}^l (b^{(i)} I_{\lambda_i} + \Lambda_i).$$

Then a regular element b is expressed by

$$b = \text{Ad}(g_b) b_\lambda$$

for a Young diagram λ and $g_b \in G$.

For simplicity, we consider the case of $b = b_\lambda$ for some Young diagram λ . Let $\mathfrak{h}_\lambda = \mathfrak{z}_{\mathfrak{g}}(b_\lambda)$. Then we have

$$\mathfrak{h}_\lambda = \langle e_i^j \mid 1 \leq i \leq s, 0 \leq j \leq \lambda_i - 1 \rangle$$

which

$$e_i = O_{\lambda_1} \oplus O_{\lambda_2} \oplus \dots \oplus \Lambda_{\lambda_i} \oplus \dots \oplus O_{\lambda_s}.$$

Let $\alpha \in (\mathfrak{h}_\lambda)^*$ satisfy $\alpha(I_n) = -m$. Then $M_{\mathfrak{h}_\lambda, \alpha}$ coincides with the general hypergeometric system defined by Kimura-Takano [10] (cf. Lemma 2.1).

Let λ, μ be Young diagrams with $|\lambda| = |\mu| = n$. We denote by $\lambda \rightarrow \mu$ if

- (1) $l(\mu) = l(\lambda) - 1$, where $l(\cdot)$ denotes the length of Young diagram,
- (2) there exist $0 \leq j < l(\mu)$, $0 \leq j_2 < j_1 < l(\lambda)$ with $\mu_j = \lambda_{j_1} + \lambda_{j_2}$ such that $\{\mu_k\}_{k \neq j} = \{\lambda_k\}_{k \neq j_1, j_2}$.

For example, the Young diagrams with $|\lambda| = 4$ hold

$$(1, 1, 1, 1) \rightarrow (2, 1, 1) \begin{array}{c} \nearrow (2, 2) \\ \searrow (3, 1) \end{array} \begin{array}{c} \searrow (4) \\ \nearrow (4) \end{array}$$

For any Young diagram λ, μ satisfying $\lambda \rightarrow \mu$, we review the limit of Lie subalgebras from $\mathfrak{h}_\lambda$ to $\mathfrak{h}_\mu$. First we consider the case $\lambda = (p, q) \rightarrow \mu = (n)$. Let $X(\varepsilon) \in \mathfrak{h}_\lambda$ be the regular element

$$X(\varepsilon) = \Lambda_p \oplus (\varepsilon I_q + \Lambda_q),$$

and let $g(\varepsilon) \in G$ satisfy

$$\begin{aligned} & (I_n, X(\varepsilon), X(\varepsilon)^2, \dots, X(\varepsilon)^{n-1}) \\ = & (I_p \oplus O_q, \Lambda_p \oplus O_q, \dots, \Lambda_p^{p-1} \oplus O_q, O_p \oplus I_q, O_p \oplus \Lambda_q, \dots, O_p \oplus \Lambda_q^{q-1}) g(\varepsilon). \end{aligned}$$

Then $g(\varepsilon)$ is expressed by

$$g(\varepsilon) = {}^t \begin{bmatrix} I_p & G_{12} \\ 0 & G_{22} \end{bmatrix},$$

where

$$\begin{bmatrix} G_{12} \\ G_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon^2 & \cdots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & 0 & \varepsilon^{n-1} \end{bmatrix} \begin{bmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{q-1} \\ \binom{1}{0} & \binom{1}{1} & \cdots & \binom{1}{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n-1}{0} & \binom{n-1}{1} & \cdots & \binom{n-1}{q-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon^{-2} & \cdots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & 0 & \varepsilon^{-(q-1)} \end{bmatrix}$$

($\binom{i}{j} = 0$ for $i \leq j$). Then we have

$$\lim_{\varepsilon \rightarrow 0} \text{Ad}(g(\varepsilon)^{-1})X(\varepsilon) = \Lambda_n$$

(cf. [10, Lemma 4.1]) and the centralizer version:

$$\lim_{\varepsilon \rightarrow 0} \text{Ad}(g(\varepsilon)^{-1})\mathfrak{h}_\lambda = \mathfrak{h}_\mu$$

(cf. [10, Lemma 5.1]).

In general, let λ, μ be $\lambda \rightarrow \mu$. In the same way as above, there exists $g(\varepsilon) \in G$ such that

$$\lim_{\varepsilon \rightarrow 0} \text{Ad}(g(\varepsilon))\mathfrak{h}_\lambda = \mathfrak{h}_\mu$$

(cf. [10, Lemma 4.2]).

CHAPTER 4

Principal nilpotent p -tuples

In this chapter, we introduce principal nilpotent p -tuples, which were defined by Ginzburg [6] for $p = 2$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. From now on, we suppose that there exist $h_1, \dots, h_p \in \mathfrak{h}$ and $e_1, \dots, e_p \in \mathfrak{g}$ satisfying

- (1) $[h_i, e_j] = \delta_{ij}e_j \quad (1 \leq i, j \leq p),$
- (2) $[e_i, e_j] = 0 \quad (1 \leq i, j \leq p),$
- (3) Let $\mathfrak{a} := \langle e^{\mathbf{l}} \mid \mathbf{l} \in \mathbb{N}^p \rangle$, where $e^{\mathbf{l}} = \prod_{i=1}^p e_i^{l_i}$ for $\mathbf{l} = (l_1, \dots, l_p) \in \mathbb{N}^p$. Then the centralizer $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ equals $\mathfrak{a}$, and $\dim \mathfrak{a} = n$.

We say such p -tuple $(e_1, \dots, e_p)$ to be *principal* (cf. [6] ($p = 2$ case)).

Clearly each e_i is nilpotent, and

$$(3) \quad [h_i, e^{\mathbf{l}}] = l_i e^{\mathbf{l}}.$$

Put

$$L := \{\mathbf{l} \in \mathbb{N}^p \mid e^{\mathbf{l}} \neq 0\}.$$

Then $\{e^{\mathbf{l}} \mid \mathbf{l} \in L\}$ forms a basis of $\mathfrak{a}$.

The following lemma can be proved in the same way as in the proof of [6, Theorem 1.2 (ii)].

LEMMA 4.1. $\mathfrak{z}_{\mathfrak{g}}(h_1, \dots, h_p) := \cap_{i=1}^p \mathfrak{z}_{\mathfrak{g}}(h_i) = \mathfrak{h}$.

EXAMPLE 4.2. Let $p = 1$. Let e be a regular nilpotent element. Since there exists a semisimple element H with $[H, e] = 2e$, putting $h = \frac{1}{2}H$, we see that h and e satisfy the above conditions. For example, we can take $h = \sum_{i=1}^n iE_{i,i}$, $e = \sum_{i=1}^{n-1} E_{i+1,i}$. Then $\mathfrak{a}$ is the Jordan Lie subalgebra (2).

Since $h_1, \dots, h_p$ are commuting semisimple elements, they are simultaneously diagonalizable. Let $A = \{\mathbf{a} = (a_1, \dots, a_p)\}$ be the set of joint eigenvalues, and let $\{v_{\mathbf{a}} \mid \mathbf{a} \in A\}$ be a basis of V consisting of joint eigenvectors:

$$h_i v_{\mathbf{a}} = a_i v_{\mathbf{a}} \quad (i = 1, \dots, p; \mathbf{a} \in A).$$

Multiplying $v_{\mathbf{a}}$ by some constants if necessary, we may suppose that

$$e^{\mathbf{l}} v_{\mathbf{a}} = v_{\mathbf{a}+\mathbf{l}} \quad (\mathbf{a}, \mathbf{a}+\mathbf{l} \in A, \mathbf{l} \in L)$$

by (3). Throughout this paper, we suppose that there exists $\mathbf{a}(0) \in A$ such that

$$A = \mathbf{a}(0) + L.$$

With respect to the basis $\{v_{\mathbf{a}}\}$, h_i and e_i are represented by

$$(4) \quad \sum_{\mathbf{a} \in A} a_i E_{\mathbf{a}, \mathbf{a}}, \quad \sum_{\mathbf{a}+\mathbf{1}_i, \mathbf{a} \in A} E_{\mathbf{a}+\mathbf{1}_i, \mathbf{a}},$$

respectively. More generally $h^{\mathbf{l}}$ and $e^{\mathbf{l}}$ are respectively represented by

$$(5) \quad \sum_{\mathbf{a} \in A} \mathbf{a}^{\mathbf{l}} E_{\mathbf{a}, \mathbf{a}}, \quad \sum_{\mathbf{a} + \mathbf{l}, \mathbf{a} \in A} E_{\mathbf{a} + \mathbf{l}, \mathbf{a}},$$

where $h^{\mathbf{l}} = h_1^{l_1} \cdots h_p^{l_p}$ and $\mathbf{a}^{\mathbf{l}} = a_1^{l_1} \cdots a_p^{l_p}$ for $\mathbf{l} = (l_1, \dots, l_p)$.

EXAMPLE 4.3. Let $n = 4$, $p = 2$, and let

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$h_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then (e_1, e_2) is a principal nilpotent pair,

$$A = L = \{(0,0), (1,0), (2,0), (0,1)\} = \begin{array}{|c|c|c|} \hline (0,0) & (1,0) & (2,0) \\ \hline (0,1) & & \\ \hline \end{array},$$

and

$$\mathbf{a} = \langle I_4, e_1, e_1^2, e_2 \rangle.$$

REMARK 4.4. As in Example 4.3, we can associate L with a p -dimensional Young diagram of size n . For example, L in Example 4.2 is associated with $\square \square \cdots \square$.

In [7], for a Young diagram λ , Kimura, Haraoka and Takano considered a confluent hypergeometric system of type λ . Note that the subalgebra associated with Young diagram λ in this paper is different from the one in [7]. For example, with Young diagram $\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \end{array}$, as in Example 4.3 we associate the subalgebra

$$\left\{ \begin{bmatrix} c_0 & 0 & 0 & 0 \\ c_1 & c_0 & 0 & 0 \\ c_2 & c_1 & c_0 & 0 \\ c_3 & 0 & 0 & c_0 \end{bmatrix} \mid c_0, \dots, c_3 \in \mathbb{C} \right\},$$

but the subalgebra

$$\left\{ \begin{bmatrix} c_0 & 0 & 0 & 0 \\ c_1 & c_0 & 0 & 0 \\ c_2 & c_1 & c_0 & 0 \\ 0 & 0 & 0 & c_3 \end{bmatrix} \mid c_0, \dots, c_3 \in \mathbb{C} \right\}$$

is associated in [7].

LEMMA 4.5. *The matrix $[\mathbf{a}^{\mathbf{l}}]_{\mathbf{l} \in L, \mathbf{a} \in A}$ is non-singular.*

PROOF. The set of rows is L , and the set of columns is $A = \mathbf{a}(0) + L$. Put

$$D(\mathbf{a}(0) + L) := \det([\mathbf{a}^{\mathbf{l}}]_{\mathbf{l} \in L; \mathbf{a} \in \mathbf{a}(0) + L}).$$

When $p = 1$,

$$D(\mathbf{a}(0) + L) = \prod_{k < l} ((\mathbf{a}(0) + l) - (\mathbf{a}(0) + k)) = \prod_{k < l} \pm(l - k) \neq 0$$

by Vandermonde.

Now we consider the general case. In the matrix $[\mathbf{a}^{\mathbf{l}}]$, we add $-\mathbf{a}(0)_p$ -times of the $\mathbf{l}$ -th row to the $\mathbf{l} + 1_p$ -th row. Then the $(\mathbf{l}, \mathbf{a})$ -component becomes $\mathbf{a}^{\mathbf{l}} - \mathbf{a}(0)_p \mathbf{a}^{\mathbf{l}-1_p} = \mathbf{a}^{\mathbf{l}-1_p}(a_p - a(0)_p)$ for $l_p > 0$; remains $\mathbf{a}^{\mathbf{l}}$ for $l_p = 0$. In particular, the columns $\mathbf{a}$ with $a_p = a(0)_p$ have components 0 in the rows $\mathbf{l}$ with $l_p > 0$. Hence

$$D(\mathbf{a}(0) + L) = D(\mathbf{a}(0) + L')D(\mathbf{a}(0) + 1_p + L'') \prod_{\mathbf{a}=\mathbf{a}(0)+\mathbf{l}; l_p>0} (a_p - a(0)_p),$$

where

$$\begin{aligned} L' &= \{\mathbf{l} \in L \mid l_p = 0\}, \\ L'' &= \{\mathbf{l} - 1_p \mid \mathbf{l} \in L, l_p > 0\}. \end{aligned}$$

Then by repeating the similar discussion for $D(\mathbf{a}(0) + 1_p + L'')$, we have

$$D(\mathbf{a}(0) + L) = \prod_i D(\mathbf{a}(0) + i1_p + L_i) \prod_{\mathbf{a}=\mathbf{a}(0)+\mathbf{l}; l_p>0} (a_p - a(0)_p)^{l_p}$$

where $L_i = \{\mathbf{l} - i1_p \mid \mathbf{l} \in L, l_p = i\}$. Hence by the induction we have proved $D(\mathbf{a}(0) + L) \neq 0$. $\square$

COROLLARY 4.6. $\langle h^{\mathbf{l}} \mid \mathbf{l} \in L \rangle = \mathfrak{h}$.

PROOF. Since $h^{\mathbf{l}}$ acts as $\text{diag}(\mathbf{a}^{\mathbf{l}} : \mathbf{a} \in A)$, the n elements $h^{\mathbf{l}}$ ($\mathbf{l} \in L$) are linearly independent by Lemma 4.5. Hence the assertion holds. $\square$

Let $(e_1, \dots, e_p)$ be a principal nilpotent p -tuple. Take $B \subseteq A$ with $|B| = m$, and let $V_B = \oplus_{\mathbf{a} \in B} \mathbb{C}v_{\mathbf{a}}$. Set

$$\begin{aligned} G &= GL(V), \quad \mathfrak{g} = \mathfrak{gl}(V) = \text{End}(V), \\ N &= N_B = \{g \in G \mid g|_{V_B} = \text{id}_{V_B}\}, \quad Z = G/N_B. \end{aligned}$$

Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_m\}$, and we write j for $\mathbf{b}_j$. Then, by Lemma 2.1, for $\mathbf{a} = \langle e^{\mathbf{l}} \mid \mathbf{l} \in L \rangle$ and $\alpha \in \mathfrak{a}^*$ the system $M_{\mathbf{a}, \alpha}$ (1) is rewritten as follows:

$$(6) \quad \left(\frac{\partial}{\partial z_{\mathbf{a}, j_1}} \frac{\partial}{\partial z_{\mathbf{b}, j_2}} - \frac{\partial}{\partial z_{\mathbf{a}, j_2}} \frac{\partial}{\partial z_{\mathbf{b}, j_1}} \right) \Phi = 0 \quad (\mathbf{a}, \mathbf{b} \in A; 1 \leq j_1, j_2 \leq m),$$

$$(7) \quad \left(\sum_{\mathbf{a} \in A} z_{\mathbf{a}, j_1} \frac{\partial}{\partial z_{\mathbf{a}, j_2}} + \delta_{j_1 j_2} \right) \Phi = 0 \quad (1 \leq j_1, j_2 \leq m),$$

$$(8) \quad \left(\sum_{\mathbf{a}, \mathbf{a}+\mathbf{l} \in A} \sum_{j=1}^m z_{\mathbf{a}, j} \frac{\partial}{\partial z_{\mathbf{a}+\mathbf{l}, j}} + \alpha(e^{\mathbf{l}}) \right) \Phi = 0 \quad (\mathbf{l} \in L).$$

REMARK 4.7. A different L defines a different subalgebra $\mathfrak{a}$, hence a different system. A system in [5] corresponds to the Jordan Lie subalgebra (2) (cf. Example 4.2). Systems $M_{\mathbf{a}, \alpha}$ which are also systems in [7] are only the systems in [5], since the semisimple part of our subalgebra $\mathfrak{a} = \langle e^{\mathbf{l}} \mid \mathbf{l} \in L \rangle$ is 1-dimensional as in Remark 4.4.

CHAPTER 5

Integral Representations

In this chapter, after recalling an integral representation of a solution to the system for $\mathfrak{h}$, we exhibit an integral representation of a solution to the system associated with a principal nilpotent p -tuple.

Set

$$dt := \sum_{i=1}^m (-1)^{i-1} t_i dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_m.$$

PROPOSITION 5.1 ([3]).

$$\psi(\alpha, z) := \int \prod_{\mathbf{a} \in A} \left(\sum_{j=1}^m z_{\mathbf{a},j} t_j \right)^{\alpha_{\mathbf{a}}} dt$$

is a solution to the system $M_{\mathfrak{h}, -\alpha}$, where $\alpha \in \mathfrak{h}^*$ is defined by $\alpha(E_{\mathbf{a}, \mathbf{a}}) = \alpha_{\mathbf{a}}$ ($\mathbf{a} \in A$).

PROOF. Let $g \in GL(m)$. Then

$$\begin{aligned} \psi(\alpha, zg) &= \int \prod_{\mathbf{a} \in A} \left(\sum_{j=1}^m \sum_{k=1}^m z_{\mathbf{a},k} g_{k,j} t_j \right)^{\alpha_{\mathbf{a}}} dt \\ &= \int \prod_{\mathbf{a} \in A} \left(\sum_{k=1}^m z_{\mathbf{a},k} \left(\sum_{j=1}^m g_{k,j} t_j \right) \right)^{\alpha_{\mathbf{a}}} dt. \end{aligned}$$

Put

$$T_k := \sum_{j=1}^m g_{k,j} t_j.$$

Then $t_j = \sum_{k=1}^m (g^{-1})_{j,k} T_k$ and $dt = \det(g)^{-1} dT$. Hence

$$\psi(\alpha, zg) = \det(g)^{-1} \int \prod_{\mathbf{a} \in A} \left(\sum_{k=1}^m z_{\mathbf{a},k} T_k \right)^{\alpha_{\mathbf{a}}} dT = \det(g)^{-1} \psi(\alpha, z),$$

and ψ satisfies (7).

We have

$$\begin{aligned} \frac{\partial}{\partial z_{\mathbf{b},k}} \left(\prod_{\mathbf{a} \in A} \left(\sum_{j=1}^m z_{\mathbf{a},j} t_j \right)^{\alpha_{\mathbf{a}}} \right) &= \prod_{\mathbf{a} \neq \mathbf{b}} \left(\sum_{j=1}^m z_{\mathbf{a},j} t_j \right)^{\alpha_{\mathbf{a}}} \frac{\partial}{\partial z_{\mathbf{b},k}} \left(\sum_{j=1}^m z_{\mathbf{b},j} t_j \right)^{\alpha_{\mathbf{b}}} \\ &= \alpha_{\mathbf{b}} t_k \prod_{\mathbf{a} \neq \mathbf{b}} \left(\sum_{j=1}^m z_{\mathbf{a},j} t_j \right)^{\alpha_{\mathbf{a}}} \left(\sum_{j=1}^m z_{\mathbf{b},j} t_j \right)^{\alpha_{\mathbf{b}}-1}. \end{aligned}$$

Hence

$$\left(\frac{\partial}{\partial z_{\mathbf{b},k}} \frac{\partial}{\partial z_{\mathbf{c},l}} - \frac{\partial}{\partial z_{\mathbf{b},l}} \frac{\partial}{\partial z_{\mathbf{c},k}} \right) \left(\prod_{\mathbf{a} \in A} \left(\sum_{j=1}^m z_{\mathbf{a},j} t_j \right)^{\alpha_{\mathbf{a}}} \right) = 0,$$

and

$$\sum_{k=1}^m z_{\mathbf{b},k} \frac{\partial}{\partial z_{\mathbf{b},k}} \left(\prod_{\mathbf{a} \in A} \left(\sum_{j=1}^m z_{\mathbf{a},j} t_j \right)^{\alpha_{\mathbf{a}}} \right) = \alpha_{\mathbf{b}} \prod_{\mathbf{a} \in A} \left(\sum_{j=1}^m z_{\mathbf{a},j} t_j \right)^{\alpha_{\mathbf{a}}}.$$

Hence ψ is a solution to the system $M_{\mathfrak{h}, -\alpha}$. □

As in [5], we introduce the polynomials $\theta_{\mathbf{k}}$; let

$$(9) \quad \sum_{\mathbf{l} \in \mathbb{N}^p} b_{\mathbf{l}} T^{\mathbf{l}} = \exp\left(\sum_{\mathbf{k} \in \mathbb{N}^p \setminus \{\mathbf{0}\}} \theta_{\mathbf{k}}(b) T^{\mathbf{k}}\right),$$

where $b_{\mathbf{0}} = 1$.

LEMMA 5.2.

$$\sum_{j \geq l} b_{j-l} \frac{\partial \theta_{\mathbf{k}}}{\partial b_j} = \begin{cases} 1 & (\mathbf{k} = \mathbf{l}) \\ 0 & (\mathbf{k} \neq \mathbf{l}). \end{cases}$$

PROOF. By definition, in $\mathbb{C}[b_{\mathbf{l}} (\mathbf{l} \in \mathbb{N}^p \setminus \{\mathbf{0}\})][[T_1, \dots, T_p]]$,

$$\log\left(\sum_{\mathbf{i} \in \mathbb{N}^p} b_{\mathbf{i}} T^{\mathbf{i}}\right) = \sum_{\mathbf{k} \in \mathbb{N}^p \setminus \{\mathbf{0}\}} \theta_{\mathbf{k}} T^{\mathbf{k}}.$$

By executing $\frac{\partial}{\partial b_j}$, we have

$$\frac{T^j}{(\sum_{i \geq 0} b_i T^i)} = \sum_{\mathbf{k} \geq \mathbf{0}} \frac{\partial \theta_{\mathbf{k}}}{\partial b_j} T^{\mathbf{k}}.$$

Hence

$$\frac{T^{j-l}}{(\sum_{i \geq 0} b_i T^i)} = \sum_{\mathbf{k} \geq \mathbf{0}} \frac{\partial \theta_{\mathbf{k}}}{\partial b_j} T^{\mathbf{k}-\mathbf{l}},$$

and

$$1 = \frac{\sum_{j \geq l} b_{j-l} T^{j-l}}{(\sum_{i \geq 0} b_i T^i)} = \sum_{\mathbf{k} \geq \mathbf{0}} \sum_{j \geq l} b_{j-l} \frac{\partial \theta_{\mathbf{k}}}{\partial b_j} T^{\mathbf{k}-\mathbf{l}},$$

Therefore we have the assertion. □

PROPOSITION 5.3. For $\mathbf{l} \in L$, let $b_{\mathbf{l}} = \sum_{j=1}^m z_{\mathbf{a}(0)+\mathbf{l},j} t_j$, $b'_{\mathbf{l}} := b_{\mathbf{l}}/b_{\mathbf{0}}$, and

$$(10) \quad \phi(\alpha, z, t) := b_{\mathbf{0}}^{\alpha_0} \exp\left(\sum_{\mathbf{k} \in L_+} \alpha_{\mathbf{k}} \theta_{\mathbf{k}}(b')\right),$$

where $L_+ = L \setminus \{\mathbf{0}\}$. Then

$$\sum_{i \geq l} \sum_{j=1}^m z_{\mathbf{a}(0)+\mathbf{i}-\mathbf{l},j} \frac{\partial}{\partial z_{\mathbf{a}(0)+\mathbf{i},j}} \phi = \alpha_{\mathbf{l}} \phi.$$

PROOF. Let $\mathbf{l} \neq \mathbf{0}$. Then

$$\begin{aligned} \sum_{i \geq l} \sum_{j=1}^m z_{\mathbf{a}(0)+\mathbf{i}-\mathbf{l},j} \frac{\partial}{\partial z_{\mathbf{a}(0)+\mathbf{i},j}} \phi &= \sum_{i \geq l} \sum_{j=1}^m z_{\mathbf{a}(0)+\mathbf{i}-\mathbf{l},j} \phi \sum_{\mathbf{k} \in L_+} \alpha_{\mathbf{k}} \frac{\partial}{\partial z_{\mathbf{a}(0)+\mathbf{i},j}} \theta_{\mathbf{k}}(b') \\ &= \sum_{i \geq l} \sum_{j=1}^m z_{\mathbf{a}(0)+\mathbf{i}-\mathbf{l},j} \phi \sum_{\mathbf{k} \in L_+} \alpha_{\mathbf{k}} \frac{\partial \theta_{\mathbf{k}}}{\partial b_i} t_j \\ &= \phi \sum_{\mathbf{k} \in L_+} \alpha_{\mathbf{k}} \sum_{i \geq l} b_{i-l} \frac{\partial \theta_{\mathbf{k}}}{\partial b_i} = \alpha_{\mathbf{l}} \phi \end{aligned}$$

by Lemma 5.2. Clearly $\phi(\alpha, e^{\tau} z, t) = e^{\alpha_0 \tau} \phi(\alpha, z, t)$. Hence we have the equation for $\mathbf{l} = \mathbf{0}$. □

PROPOSITION 5.4 ([5] for $p = 1$).

$$\psi(\alpha, z) := \int \phi(\alpha, z, t) dt$$

is a solution to the system $M_{\mathbf{a}, -\alpha}$, where $\mathbf{a} = \langle e^{\mathbf{l}} \mid \mathbf{l} \in L \rangle$, and $\alpha \in \mathfrak{a}^*$ is defined by $\alpha(e^{\mathbf{l}}) = \alpha_{\mathbf{l}}$ ($\mathbf{l} \in L$).

PROOF. By Proposition 5.3, ψ satisfies (8). As in the proof of Proposition 5.1, it is easy to see

$$\psi(\alpha, zg) = \det(g)^{-1} \psi(\alpha, z),$$

and ψ satisfies (7).

We have, for $\mathbf{i} \neq \mathbf{0}$,

$$\begin{aligned} \frac{\partial}{\partial z_{\mathbf{a}(0)+\mathbf{i},j}} \phi &= \phi \cdot \frac{\partial}{\partial z_{\mathbf{a}(0)+\mathbf{i},j}} \left(\sum_{\mathbf{k}} \alpha_{\mathbf{k}} \theta_{\mathbf{k}} \right) \\ &= \phi \cdot \sum_{\mathbf{k}} \alpha_{\mathbf{k}} \frac{\partial \theta_{\mathbf{k}}}{\partial z_{\mathbf{a}(0)+\mathbf{i},j}} \\ &= \phi \cdot \sum_{\mathbf{k}} \alpha_{\mathbf{k}} \frac{\partial \theta_{\mathbf{k}}}{\partial b_{\mathbf{i}}} t_j. \end{aligned}$$

We also have

$$\begin{aligned} \frac{\partial}{\partial z_{\mathbf{a}(0),j}} \phi &= \alpha_{\mathbf{0}} b_{\mathbf{0}}^{-1} t_j \phi + \phi \cdot \sum_{\mathbf{k}} \alpha_{\mathbf{k}} \frac{\partial \theta_{\mathbf{k}}}{\partial z_{\mathbf{a}(0),j}} \\ &= \alpha_{\mathbf{0}} b_{\mathbf{0}}^{-1} t_j \phi - \phi \cdot \sum_{\mathbf{k}, \mathbf{i}} \alpha_{\mathbf{k}} \frac{\partial \theta_{\mathbf{k}}}{\partial b'_{\mathbf{i}}} \frac{b_{\mathbf{i}}}{b_{\mathbf{0}}^2} t_j \\ &= b_{\mathbf{0}}^{-1} t_j \phi \left(\alpha_{\mathbf{0}} - \sum_{\mathbf{k}, \mathbf{i}} \alpha_{\mathbf{k}} b_{\mathbf{i}} \frac{\partial \theta_{\mathbf{k}}}{\partial b_{\mathbf{i}}} \right). \end{aligned}$$

From the above equations,

$$\begin{aligned} &\frac{\partial^2}{\partial z_{\mathbf{a}(0)+\mathbf{i}_1, j_1} \partial z_{\mathbf{a}(0)+\mathbf{i}_2, j_2}} \phi \\ &= \frac{\partial}{\partial z_{\mathbf{a}(0)+\mathbf{i}_1, j_1}} \left(\phi \cdot \sum_{\mathbf{k}} \alpha_{\mathbf{k}} \frac{\partial \theta_{\mathbf{k}}}{\partial b_{\mathbf{i}_2}} t_{j_2} \right) \\ &= \phi \left(\left(\sum_{\mathbf{k}'} \alpha_{\mathbf{k}'} \frac{\partial \theta_{\mathbf{k}'}}{\partial b_{\mathbf{i}_1}} \right) \left(\sum_{\mathbf{k}} \alpha_{\mathbf{k}} \frac{\partial \theta_{\mathbf{k}}}{\partial b_{\mathbf{i}_2}} \right) + \sum_{\mathbf{k}} \alpha_{\mathbf{k}} \frac{\partial^2 \theta_{\mathbf{k}}}{\partial b_{\mathbf{i}_1} \partial b_{\mathbf{i}_2}} \right) t_{j_1} t_{j_2} \end{aligned}$$

for $\mathbf{i}_1, \mathbf{i}_2 \neq \mathbf{0}$, $\mathbf{i}_1 \neq \mathbf{i}_2$, $j_1 \neq j_2$, and

$$\begin{aligned} &\frac{\partial^2}{\partial z_{\mathbf{a}(0), j_1} \partial z_{\mathbf{a}(0)+\mathbf{i}_2, j_2}} \phi \\ &= \frac{\partial}{\partial z_{\mathbf{a}(0), j_1}} \left(\phi \cdot \sum_{\mathbf{k}} \alpha_{\mathbf{k}} \frac{\partial \theta_{\mathbf{k}}}{\partial b_{\mathbf{i}_2}} t_{j_2} \right) \\ &= b_{\mathbf{0}}^{-1} \phi \left(\left(\alpha_{\mathbf{0}} - \sum_{\mathbf{k}', \mathbf{i}} \alpha_{\mathbf{k}'} b_{\mathbf{i}} \frac{\partial \theta_{\mathbf{k}'}}{\partial b_{\mathbf{i}}} \right) \left(\sum_{\mathbf{k}} \alpha_{\mathbf{k}} \frac{\partial \theta_{\mathbf{k}}}{\partial b_{\mathbf{i}_2}} \right) - \sum_{\mathbf{k}, \mathbf{i}} \alpha_{\mathbf{k}} b_{\mathbf{i}} \frac{\partial^2 \theta_{\mathbf{k}}}{\partial b_{\mathbf{i}} \partial b_{\mathbf{i}_2}} \right) t_{j_1} t_{j_2} \end{aligned}$$

for $\mathbf{i}_2 \neq 0$, $j_1 \neq j_2$. Then we see that

$$\frac{\partial^2}{\partial z_{\mathbf{a}(0)+\mathbf{i}_1,j_1} \partial z_{\mathbf{a}(0)+\mathbf{i}_2,j_2}} \phi$$

is symmetric in j_1 and j_2 . Hence

$$\left(\frac{\partial^2}{\partial z_{\mathbf{a}(0)+\mathbf{i}_1,j_1} \partial z_{\mathbf{a}(0)+\mathbf{i}_2,j_2}} - \frac{\partial^2}{\partial z_{\mathbf{a}(0)+\mathbf{i}_1,j_2} \partial z_{\mathbf{a}(0)+\mathbf{i}_2,j_1}} \right) \phi = 0.$$

□

CHAPTER 6

Deformation

In the previous chapter, we saw an integral representation of a solution to $M_{\mathfrak{h}, -\alpha}$ (Proposition 5.1) and that to $M_{\mathfrak{a}, -\alpha}$ (Proposition 5.4). In this chapter, we naturally deform the integrand of the former to that of the latter (Theorem 6.6), which is the main theorem of this paper.

Recall that e_i is represented by

$$\sum_{\mathbf{a}+1_i, \mathbf{a} \in A} E_{\mathbf{a}+1_i, \mathbf{a}},$$

and e^l is represented by

$$\sum_{\mathbf{a}+l, \mathbf{a} \in A} E_{\mathbf{a}+l, \mathbf{a}}.$$

Recall that for $\mathfrak{h} = \langle h^l \rangle$ the system $M_{\mathfrak{h}, -\alpha}$, where $\alpha \in \mathfrak{h}^*$ is defined by $\alpha(E_{\mathbf{a}, \mathbf{a}}) = \alpha_{\mathbf{a}}$ ($\mathbf{a} \in A$), is the following:

$$(11) \quad \left(\frac{\partial}{\partial z_{\mathbf{a}, i}} \frac{\partial}{\partial z_{\mathbf{b}, j}} - \frac{\partial}{\partial z_{\mathbf{a}, j}} \frac{\partial}{\partial z_{\mathbf{b}, i}} \right) \Phi = 0 \quad (\mathbf{a}, \mathbf{b} \in A; 1 \leq i, j \leq m)$$

$$(12) \quad \left(\sum_{\mathbf{a} \in A} z_{\mathbf{a}, i} \frac{\partial}{\partial z_{\mathbf{a}, j}} + \delta_{ij} \right) \Phi = 0 \quad (1 \leq i, j \leq m)$$

$$(13) \quad \left(\sum_{k=1}^m z_{\mathbf{a}, k} \frac{\partial}{\partial z_{\mathbf{a}, k}} - \alpha_{\mathbf{a}} \right) \Phi = 0 \quad (\mathbf{a} \in A).$$

Let

$$\alpha_l = \sum_{\mathbf{a} \in A} \mathbf{a}^l \alpha_{\mathbf{a}}.$$

The equations (13) are equivalent to

$$(14) \quad \left(\sum_{\mathbf{a} \in A} \mathbf{a}^l \sum_{k=1}^m z_{\mathbf{a}, k} \frac{\partial}{\partial z_{\mathbf{a}, k}} - \alpha_l \right) \Phi = 0 \quad (l \in L),$$

since h^l corresponds to

$$\sum_{\mathbf{a} \in A} \mathbf{a}^l E_{\mathbf{a}, \mathbf{a}}.$$

The equation (14) can be rewritten as

$$(15) \quad (\partial_{h^l} + \alpha_l) \Phi = 0 \quad (l \in L).$$

Set $\mathbf{e} := e_1 + \cdots + e_p$, and consider $\mathbf{e}(\tau) := \exp(\frac{\tau-1}{\tau} \mathbf{e}) \in G = GL(n)$. Then $\mathbf{e}(1) = 1$.

LEMMA 6.1. (1) $\lim_{\tau \rightarrow 0} \text{Ad}(\mathbf{e}(\tau))(\tau h)^M = e^M$ for all $M \in \mathbb{N}^p$.

(2) $\lim_{\tau \rightarrow 0} \text{Ad}(\mathbf{e}(\tau)) \mathfrak{h} = \mathfrak{a}$.

PROOF. Since $[\mathbf{e}, h_i] = -e_i$, we have

$$\text{Ad}(\mathbf{e}(\tau))(h_i) = \exp \text{ad}\left(\frac{\tau-1}{\tau} \mathbf{e}\right)(h_i) = h_i + \frac{1-\tau}{\tau} e_i.$$

Since $\text{Ad}(\mathbf{e}(\tau))$ is an automorphism of an associative algebra,

$$\text{Ad}(\mathbf{e}(\tau))((\tau h)^{\mathbf{l}}) = \prod_{i=1}^p (\tau h_i + (1 - \tau)e_i)^{l_i}.$$

Hence we see the assertion (1), and hence (2). $\square$

We have the left multiplication map $L_{\mathbf{e}(\tau)}$ on $Z = G/N$. In general,

$$L_g \circ \partial_a \circ L_g^{-1} = \partial_{\text{Ad}(g)a} \quad (g \in G, a \in \mathfrak{g}).$$

LEMMA 6.2.

$$L_{\mathbf{e}(\tau)} \cdot z_{\mathbf{a},j} = \sum_{\mathbf{k} \geq 0} \frac{1}{\mathbf{k}!} \left(\frac{1 - \tau}{\tau} \right)^{|\mathbf{k}|} z_{\mathbf{a} - \mathbf{k},j}.$$

PROOF. By an abuse of notation, we denote a general matrix in Z by Z again. By definition,

$$(L_{\mathbf{e}(\tau)} \cdot z_{\mathbf{a},j})(Z) = z_{\mathbf{a},j}(\mathbf{e}(\tau)^{-1}Z).$$

We have

$$\mathbf{e}(\tau)^{-1}Z = \sum_{\mathbf{k} \geq 0} \frac{1}{\mathbf{k}!} \left(\frac{1 - \tau}{\tau} \right)^{|\mathbf{k}|} \mathbf{e}^{\mathbf{k}} Z.$$

Hence

$$(\mathbf{e}(\tau)^{-1}Z)_{\mathbf{a},j} = \sum_{\mathbf{k} \geq 0} \frac{1}{\mathbf{k}!} \left(\frac{1 - \tau}{\tau} \right)^{|\mathbf{k}|} Z_{\mathbf{a} - \mathbf{k},j}.$$

$\square$

COROLLARY 6.3.

$$L_{\mathbf{e}(\tau)} \cdot \partial_{\mathbf{a},j} = \sum_{\mathbf{b} \geq \mathbf{a}} \frac{1}{(\mathbf{b} - \mathbf{a})!} \left(\frac{\tau - 1}{\tau} \right)^{|\mathbf{b} - \mathbf{a}|} \partial_{\mathbf{b},j}.$$

PROOF. We have

$$\begin{aligned} ((L_{\mathbf{e}(\tau)} \cdot \partial_{\mathbf{a},j})f)(Z) &= ((L_{\mathbf{e}(\tau)} \circ \partial_{\mathbf{a},j} \circ L_{\mathbf{e}(\tau)}^{-1})f)(Z) \\ &= L_{\mathbf{e}(\tau)} \cdot (\partial_{\mathbf{a},j}(f(\mathbf{e}(\tau)Z))). \end{aligned}$$

By Lemma 6.2,

$$\frac{\partial(\mathbf{e}(\tau)Z)_{\mathbf{b},i}}{\partial z_{\mathbf{a},j}} = \begin{cases} \delta_{i,j} \frac{1}{(\mathbf{b} - \mathbf{a})!} \left(\frac{\tau - 1}{\tau} \right)^{|\mathbf{b} - \mathbf{a}|} & (\mathbf{b} - \mathbf{a} \geq 0) \\ 0 & (\text{otherwise}). \end{cases}$$

Hence

$$\begin{aligned} \partial_{\mathbf{a},j}(f(\mathbf{e}(\tau)Z)) &= \sum_{(\mathbf{b},i)} (\partial_{\mathbf{b},i}f)(\mathbf{e}(\tau)Z) \frac{\partial(\mathbf{e}(\tau)Z)_{\mathbf{b},i}}{\partial z_{\mathbf{a},j}} \\ &= \sum_{\mathbf{b} \geq \mathbf{a}} (\partial_{\mathbf{b},j}f)(\mathbf{e}(\tau)Z) \frac{1}{(\mathbf{b} - \mathbf{a})!} \left(\frac{\tau - 1}{\tau} \right)^{|\mathbf{b} - \mathbf{a}|}, \end{aligned}$$

and the assertion holds. $\square$

COROLLARY 6.4.

$$L_{\mathbf{e}(\tau)} \cdot \left(\sum_{\mathbf{k} \geq 0} \frac{1}{\mathbf{k}!} \left(\frac{1 - \tau}{\tau} \right)^{|\mathbf{k}|} \partial_{\mathbf{a} + \mathbf{k},j} \right) = \partial_{\mathbf{a},j}.$$

PROOF. Clear from Corollary 6.3. $\square$

We consider the system $M_{\text{Ad}(\mathbf{e}(\tau))\mathfrak{h}, -\alpha(\tau)}$, where $\alpha(\tau) \in (\text{Ad}(\mathbf{e}(\tau))\mathfrak{h})^*$ is defined by

$$(\alpha(\tau))(\text{Ad}(\mathbf{e}(\tau))h^l) = \frac{\alpha_l}{\tau^{|l|}}.$$

Hence the equations (15) become

$$(16) \quad (\partial_{\text{Ad}(\mathbf{e}(\tau))h^l} + \frac{\alpha_l}{\tau^{|l|}})\Phi = 0 \quad (l \in L).$$

Put

$$\phi_\tau(\alpha, z) = \prod_{\mathbf{a} \in A} (L_{\mathbf{e}(\tau)} \cdot b_{\mathbf{a}-\mathbf{a}(0)})^{\alpha_{\mathbf{a}}}.$$

Then $\phi_1(\alpha, z)$ coincides with the integrand in Proposition 5.1. We have

$$\begin{aligned} \partial_{\text{Ad}(\mathbf{e}(\tau))h^l} \phi_\tau(\alpha, z) &= L_{\mathbf{e}(\tau)} \circ \partial_{h^l} \circ L_{\mathbf{e}(\tau)}^{-1} \phi_\tau(\alpha, z) \\ &= L_{\mathbf{e}(\tau)} \circ \partial_{h^l} \phi(\alpha, z) \\ &= -L_{\mathbf{e}(\tau)} \sum_{\mathbf{a} \in A} \mathbf{a}^l \alpha_{\mathbf{a}} \phi(\alpha, z) \\ &= -\sum_{\mathbf{a} \in A} \mathbf{a}^l \alpha_{\mathbf{a}} \phi_\tau(\alpha, z). \end{aligned}$$

Let $[\beta_{\mathbf{a}}^l]$ be the inverse of $[\mathbf{a}^l]$. Then for $\alpha_{\mathbf{a}} = \sum_l \beta_{\mathbf{a}}^l \alpha_l / \tau^{|l|}$

$$\partial_{\text{Ad}(\mathbf{e}(\tau))h^l} \phi_\tau(\alpha, z) = -\frac{\alpha_l}{\tau^{|l|}} \phi_\tau(\alpha, z).$$

Hence we have the following proposition:

PROPOSITION 6.5. *Let $[\beta_{\mathbf{a}}^l]$ be the inverse of $[\mathbf{a}^l]$, and let*

$$\phi_\tau(\alpha, z) = \prod_{l \in L} \left(\prod_{\mathbf{a} \in A} (L_{\mathbf{e}(\tau)} \cdot b_{\mathbf{a}-\mathbf{a}(0)})^{\beta_{\mathbf{a}}^l} \right)^{\alpha_l / \tau^{|l|}}.$$

Then

$$\int \phi_\tau(\alpha, z) dt$$

satisfies the system $M_{\text{Ad}(\mathbf{e}(\tau))\mathfrak{h}, -\alpha(\tau)}$:

$$\begin{aligned} & \left(\frac{\partial}{\partial z_{\mathbf{a},i}} \frac{\partial}{\partial z_{\mathbf{b},j}} - \frac{\partial}{\partial z_{\mathbf{a},j}} \frac{\partial}{\partial z_{\mathbf{b},i}} \right) \Phi = 0 \quad (\mathbf{a}, \mathbf{b} \in A; i, j \in B) \\ & \left(\sum_{\mathbf{a} \in A} z_{\mathbf{a},i} \frac{\partial}{\partial z_{\mathbf{a},j}} + \delta_{ij} \right) \Phi = 0 \quad (i, j \in B) \\ & (\partial_{\text{Ad}(\mathbf{e}(\tau))h^l} + \frac{\alpha_l}{\tau^{|l|}}) \Phi = 0 \quad (l \in L), \end{aligned}$$

where the compatibility condition is $\alpha_0 = -m$.

PROOF. The compatibility condition is

$$\sum_{\alpha \in A} \alpha_{\alpha} = -m.$$

We have

$$\begin{aligned} \sum_{\mathbf{a} \in A} \alpha_{\mathbf{a}} &= \sum_{\mathbf{a} \in A} \sum_{l \in L} \beta_{\mathbf{a}}^l \alpha_l / \tau^{|l|} \\ (17) \quad &= \sum_{l \in L} \sum_{\mathbf{a} \in A} \mathbf{a}^0 \beta_{\mathbf{a}}^l \alpha_l / \tau^{|l|} \\ &= \alpha_0 / \tau^0 = \alpha_0. \end{aligned}$$

Since $\sum_{\mathbf{a} \in A} \mathbf{a}^0 \beta_{\mathbf{a}}^l \alpha_l / \tau^l = \delta_{l,0} \alpha_0$, we have

$$\left(\prod_{\mathbf{a} \in A} (L_{e(\tau)} \cdot b_{\mathbf{a}-\mathbf{a}(0)})^{\beta_{\mathbf{a}}^l} \right)^{\alpha_l / \tau^{|\mathbf{l}|}} = \begin{cases} (\prod_{\mathbf{a} \in A} (L_{e(\tau)} \cdot b'_{\mathbf{a}-\mathbf{a}(0)})^{\beta_{\mathbf{a}}^l})^{\alpha_l / \tau^{|\mathbf{l}|}} & (\mathbf{l} \neq \mathbf{0}) \\ b_0^{-m} (\prod_{\mathbf{a} \in A} (L_{e(\tau)} \cdot b'_{\mathbf{a}-\mathbf{a}(0)})^{\beta_{\mathbf{a}}^0})^{\alpha_0} & (\mathbf{l} = \mathbf{0}). \end{cases}$$

Hence

$$\phi_{\tau}(\alpha, z) = b_0^{-m} \prod_{l \in L} \left(\prod_{\mathbf{a} \in A} (L_{e(\tau)} \cdot b'_{\mathbf{a}-\mathbf{a}(0)})^{\beta_{\mathbf{a}}^l} \right)^{\alpha_l / \tau^{|\mathbf{l}|}}.$$

Note that $\phi_{\tau} = L_{e(\tau)} \cdot \phi_1$. Put

$$T_A := \left\langle \frac{\partial}{\partial z_{\mathbf{a},i}} \frac{\partial}{\partial z_{\mathbf{b},j}} - \frac{\partial}{\partial z_{\mathbf{a},j}} \frac{\partial}{\partial z_{\mathbf{b},i}} \middle| \mathbf{a}, \mathbf{b} \in A, 1 \leq i < j \leq m \right\rangle.$$

Since $T_A \phi_1 = 0$, we have

$$\begin{aligned} (L_{e(\tau)} \cdot T_A) \phi_{\tau} &= (L_{e(\tau)} \circ T_A \circ L_{e(\tau)}^{-1})(L_{e(\tau)} \phi_1) \\ &= (L_{e(\tau)} \circ T_A)(\phi_1) = 0. \end{aligned}$$

By Corollary 6.4, we have $L_{e(\tau)} \cdot T_A \supset T_A$, and hence ϕ_{τ} satisfies (17). $\square$

Put

$$\varphi_{\tau}(\alpha, z) := \left(\frac{\tau}{1-\tau} \right)^{\sum_{\mathbf{a} \in A} \alpha_{\mathbf{a}} |\mathbf{a}-\mathbf{a}(0)|} \prod_{\mathbf{a} \in A} ((\mathbf{a} - \mathbf{a}(0))!)^{\alpha_{\mathbf{a}}} \phi_{\tau}(\alpha, z).$$

Then $\int \varphi_{\tau}(\alpha, z)$ is also a solution of the system $M_{\text{Ad}(e(\tau))\mathfrak{h}, -\alpha(\tau)}$. The following theorem corresponds to [10, Theorem 6.3].

THEOREM 6.6.

$$\varphi_{\tau}(\alpha, z) \rightarrow b_0^{-m} \exp\left(\sum_{l \in L} \alpha_l \theta_l(b')\right) = \phi(\alpha, z, t) \quad (\tau \rightarrow 0).$$

PROOF. By definition,

$$\begin{aligned} \varphi_{\tau}(\alpha, z) &= b_0^{-m} \prod_{l \in L} \left(\prod_{\mathbf{a} \in A} \left(\sum_{\mathbf{k} \geq 0} \frac{(\mathbf{a} - \mathbf{a}(0))!}{\mathbf{k}!} \left(\frac{\tau}{1-\tau} \right)^{|\mathbf{a}-\mathbf{a}(0)-\mathbf{k}|} b'_{\mathbf{a}-\mathbf{a}(0)-\mathbf{k}} \right)^{\beta_{\mathbf{a}}^l} \right)^{\alpha_l / \tau^{|\mathbf{l}|}} \\ &= b_0^{-m} \prod_{l \in L} \left(\prod_{\mathbf{a} \in A} \left(\sum_{\mathbf{a}(0) \leq \mathbf{d} \leq \mathbf{a}} \frac{(\mathbf{a} - \mathbf{a}(0))!}{(\mathbf{a} - \mathbf{d})!} \left(\frac{\tau}{1-\tau} \right)^{|\mathbf{d}-\mathbf{a}(0)|} b'_{\mathbf{d}-\mathbf{a}(0)} \right)^{\beta_{\mathbf{a}}^l} \right)^{\alpha_l / \tau^{|\mathbf{l}|}}. \end{aligned}$$

For $\mathbf{k}' \leq \mathbf{k}$, define $c_{\mathbf{k}', \mathbf{k}} \in \mathbb{C}$ by

$$\prod_{i=1}^p (X_i - \mathbf{a}(0)_i)(X_i - \mathbf{a}(0)_i - 1) \cdots (X_i - \mathbf{a}(0)_i - k_i + 1) = \sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k}} c_{\mathbf{k}', \mathbf{k}} X^{\mathbf{k}'}.$$

Note that $c_{\mathbf{k}, \mathbf{k}} = 1$. Then for $\mathbf{a}(0) \leq \mathbf{d} \leq \mathbf{a}$

$$\frac{(\mathbf{a} - \mathbf{a}(0))!}{(\mathbf{a} - \mathbf{d})!} = \sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{d}-\mathbf{a}(0)} c_{\mathbf{k}', \mathbf{d}-\mathbf{a}(0)} \mathbf{a}^{\mathbf{k}'}.$$

Hence

$$\begin{aligned} &\prod_{\mathbf{a} \in A} \left(\sum_{\mathbf{a}(0) \leq \mathbf{d} \leq \mathbf{a}} \frac{(\mathbf{a} - \mathbf{a}(0))!}{(\mathbf{a} - \mathbf{d})!} \left(\frac{\tau}{1-\tau} \right)^{|\mathbf{d}-\mathbf{a}(0)|} b'_{\mathbf{d}-\mathbf{a}(0)} \right)^{\beta_{\mathbf{a}}^l} \\ &= \prod_{\mathbf{a} \in A} \left(\sum_{\mathbf{a}(0) \leq \mathbf{d} \leq \mathbf{a}} \sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{d}-\mathbf{a}(0)} c_{\mathbf{k}', \mathbf{d}-\mathbf{a}(0)} \mathbf{a}^{\mathbf{k}'} \left(\frac{\tau}{1-\tau} \right)^{|\mathbf{d}-\mathbf{a}(0)|} b'_{\mathbf{d}-\mathbf{a}(0)} \right)^{\beta_{\mathbf{a}}^l}. \end{aligned}$$

We have by Lemma 6.7 below

$$\begin{aligned} & \sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k} \leq \mathbf{a} - \mathbf{a}(0)} c_{\mathbf{k}', \mathbf{k}} \mathbf{a}^{\mathbf{k}'} \left(\frac{\tau}{1 - \tau} \right)^{|\mathbf{k}|} b'_{\mathbf{k}} \\ &= \exp \left(\sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k}} \theta_{\mathbf{k}', \mathbf{k}}(\{c_{s', s} b'_s\}) \mathbf{a}^{\mathbf{k}'} \left(\frac{\tau}{1 - \tau} \right)^{|\mathbf{k}|} \right), \end{aligned}$$

where in the statement of Lemma 6.7 we put $S_i = \frac{\tau}{1 - \tau}$ for $i = 1, 2, \dots, p$. Hence

$$\begin{aligned} & \prod_{\mathbf{a} \in A} \left(\sum_{\mathbf{a}(0) \leq \mathbf{d} \leq \mathbf{a}} \frac{(\mathbf{a} - \mathbf{a}(0))!}{(\mathbf{a} - \mathbf{d})!} \left(\frac{\tau}{1 - \tau} \right)^{|\mathbf{d} - \mathbf{a}(0)|} b'_{\mathbf{d} - \mathbf{a}(0)} \right)^{\beta_{\mathbf{a}}^l} \\ &= \prod_{\mathbf{a} \in A} \exp \left(\sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k}} \theta_{\mathbf{k}', \mathbf{k}}(\{c_{s', s} b'_s\}) \mathbf{a}^{\mathbf{k}'} \left(\frac{\tau}{1 - \tau} \right)^{|\mathbf{k}|} \right)^{\beta_{\mathbf{a}}^l} \\ &= \exp \left(\sum_{\mathbf{a} \in A} \sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k}} \beta_{\mathbf{a}}^l \theta_{\mathbf{k}', \mathbf{k}}(\{c_{s', s} b'_s\}) \mathbf{a}^{\mathbf{k}'} \left(\frac{\tau}{1 - \tau} \right)^{|\mathbf{k}|} \right) \\ &= \exp \left(\sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k}} \sum_{\mathbf{a} \in A} \beta_{\mathbf{a}}^l \mathbf{a}^{\mathbf{k}'} \theta_{\mathbf{k}', \mathbf{k}}(\{c_{s', s} b'_s\}) \left(\frac{\tau}{1 - \tau} \right)^{|\mathbf{k}|} \right) \\ &= \exp \left(\sum_{l \leq \mathbf{k}} \theta_{l, \mathbf{k}}(\{c_{s', s} b'_s\}) \left(\frac{\tau}{1 - \tau} \right)^{|\mathbf{k}|} \right). \end{aligned}$$

Hence

$$\begin{aligned} \varphi_{\tau}(\alpha, z) &= b_0^{-m} \prod_{l \in L} \left(\exp \left(\sum_{l \leq \mathbf{k}} \theta_{l, \mathbf{k}}(\{c_{s', s} b'_s\}) \left(\frac{\tau}{1 - \tau} \right)^{|\mathbf{k}|} \right) \right)^{\alpha_l / \tau^{|\mathbf{l}|}} \\ &= b_0^{-m} \prod_{l \in L} \exp \left(\frac{\alpha_l}{\tau^{|\mathbf{l}|}} \sum_{l \leq \mathbf{k}} \theta_{l, \mathbf{k}}(\{c_{s', s} b'_s\}) \left(\frac{\tau}{1 - \tau} \right)^{|\mathbf{k}|} \right). \end{aligned}$$

Hence as $\tau \rightarrow 0$

$$\begin{aligned} \varphi_{\tau}(\alpha, z) &\rightarrow b_0^{-m} \prod_{l \in L} \exp(\alpha_l \theta_{l, l}(\{c_{s', s} b'_s\})) \\ &= b_0^{-m} \prod_{l \in L} \exp(\alpha_l \theta_l(\{b'_s\})), \end{aligned}$$

where the last equation follows from Lemma 6.7. Therefore

$$\varphi_{\tau}(\alpha, z) \rightarrow b_0^{-m} \exp \left(\sum_{l \in L} \alpha_l \theta_l(b') \right) \quad (\tau \rightarrow 0).$$

□

LEMMA 6.7. *Let*

$$\sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k}} c_{\mathbf{k}', \mathbf{k}} b'_{\mathbf{k}} T^{\mathbf{k}'} S^{\mathbf{k}} = \exp \left(\sum_{\mathbf{0} \leq \mathbf{k}', \mathbf{k}} \theta_{\mathbf{k}', \mathbf{k}}(\{c_{s', s} b'_s\}) T^{\mathbf{k}'} S^{\mathbf{k}} \right),$$

where $c_{\mathbf{0}, \mathbf{0}} b'_0 = 1$. Then $\theta_{\mathbf{k}', \mathbf{k}} = 0$ unless $\mathbf{k}' \leq \mathbf{k}$. Moreover, if $c_{\mathbf{k}, \mathbf{k}} = 1$ for all $\mathbf{k}$, then

$$\theta_{\mathbf{k}, \mathbf{k}}(\{c_{s', s} b'_s\}) = \theta_{\mathbf{k}}(\{b'_s\}).$$

PROOF. The first statement is clear from the Taylor expansion of $\log(1 + U)$:

$$\log(1 + U) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} U^n.$$

We have

$$\begin{aligned} \log\left(\sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k}} c_{\mathbf{k}', \mathbf{k}} b'_{\mathbf{k}} T^{\mathbf{k}'} S^{\mathbf{k}}\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k}} c_{\mathbf{k}', \mathbf{k}} b'_{\mathbf{k}} T^{\mathbf{k}'} S^{\mathbf{k}}\right)^n \\ &= \sum_{\mathbf{0} \leq \mathbf{k}', \mathbf{k}} \theta_{\mathbf{k}', \mathbf{k}}(\{c_{s', s} b'_s\}) T^{\mathbf{k}'} S^{\mathbf{k}}. \end{aligned}$$

Put $U := TS$. Then

$$\begin{aligned} \log\left(\sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k}} c_{\mathbf{k}', \mathbf{k}} b'_{\mathbf{k}} U^{\mathbf{k}'} S^{\mathbf{k}-\mathbf{k}'}\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\sum_{\mathbf{0} \leq \mathbf{k}' \leq \mathbf{k}} c_{\mathbf{k}', \mathbf{k}} b'_{\mathbf{k}} U^{\mathbf{k}'} S^{\mathbf{k}-\mathbf{k}'}\right)^n \\ &= \sum_{\mathbf{0} \leq \mathbf{k}', \mathbf{k}} \theta_{\mathbf{k}', \mathbf{k}}(\{c_{s', s} b'_s\}) U^{\mathbf{k}'} S^{\mathbf{k}-\mathbf{k}'}. \end{aligned}$$

Let $S = \mathbf{0}$. Then

$$\begin{aligned} \log\left(\sum_{\mathbf{0} \leq \mathbf{k}'} c_{\mathbf{k}', \mathbf{k}'} b'_{\mathbf{k}'} U^{\mathbf{k}'}\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\sum_{\mathbf{0} \leq \mathbf{k}'} c_{\mathbf{k}', \mathbf{k}'} b'_{\mathbf{k}'} U^{\mathbf{k}'}\right)^n \\ &= \sum_{\mathbf{0} \leq \mathbf{k}'} \theta_{\mathbf{k}', \mathbf{k}'}(\{c_{s', s} b'_s\}) U^{\mathbf{k}'}. \end{aligned}$$

Since $c_{\mathbf{k}', \mathbf{k}'} = 1$ for all $\mathbf{k}'$,

$$\begin{aligned} \log\left(\sum_{\mathbf{0} \leq \mathbf{k}'} b'_{\mathbf{k}'} U^{\mathbf{k}'}\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\sum_{\mathbf{0} \leq \mathbf{k}'} b'_{\mathbf{k}'} U^{\mathbf{k}'}\right)^n \\ &= \sum_{\mathbf{0} \leq \mathbf{k}'} \theta_{\mathbf{k}', \mathbf{k}'}(\{c_{s', s} b'_s\}) U^{\mathbf{k}'}, \end{aligned}$$

which means

$$\theta_{\mathbf{k}', \mathbf{k}'}(\{c_{s', s} b'_s\}) = \theta_{\mathbf{k}'}(\{b'_s\}) \quad (\forall \mathbf{k}').$$

□

CHAPTER 7

The normalizer of $\mathfrak{a}$ and its action

In this chapter, we consider the normalizer $N_G(\mathfrak{a})$ of $\mathfrak{a}$ and its action following [9]. Using the results in this chapter, we specialize parameters of hypergeometric systems associated with principal nilpotent p -tuples in the next chapter.

Put $L_+ := L \setminus \{\mathbf{0}\}$ and $\mathfrak{a}_+ := \langle e^l \mid l \in L_+ \rangle$. Note that $N_G(\exp \mathfrak{a}) = N_G(\exp \mathfrak{a}_+) \supseteq N_G(\mathfrak{a}_+) = N_G(\mathfrak{a})$. The action of $N_G(\mathfrak{a})$ on $Z/GL(m)$ induces that of $N_G(\mathfrak{a})$ on $\exp(\mathfrak{a}) \backslash Z/GL(m)$.

Let χ_α denote the (multivalued) character of $\exp \mathfrak{a}$ whose differential is equal to $\alpha = \sum_l \alpha_l (e^l)^* \in \mathfrak{a}^*$. Then we have the following commutative diagram:

$$\begin{array}{ccc} b \in \exp(\mathfrak{a}) & \xrightarrow{\chi_\alpha} & \mathbb{C}^\times \\ \exp \uparrow & & \uparrow \exp \\ \theta(b) = \sum_l \theta_l(b) e^l \in \mathfrak{a} & \xrightarrow{\alpha} & \mathbb{C}. \end{array}$$

Namely we have

$$\chi_\alpha(b) = \exp(\alpha(\theta(b))).$$

LEMMA 7.1. $\theta(gbg^{-1}) = \text{Ad}(g)(\theta(b))$ for $g \in N_G(\mathfrak{a}), b \in \exp \mathfrak{a}$.

PROOF. We have

$$gbg^{-1} = g \exp(\theta(b)) g^{-1} = \exp(g\theta(b)g^{-1}) = \exp(\text{Ad}(g)(\theta(b))).$$

Hence $\theta(gbg^{-1}) = \text{Ad}(g)(\theta(b))$. □

LEMMA 7.2. $\chi_\alpha(gbg^{-1}) = \chi_{\text{Ad}^*(g)\alpha}(b)$ for $g \in N_G(\mathfrak{a}), b \in \exp \mathfrak{a}$, where Ad^* is the coadjoint action.

PROOF. By Lemma 7.1,

$$\begin{aligned} \chi_\alpha(gbg^{-1}) &= \exp(\alpha(\theta(gbg^{-1}))) = \exp(\alpha(\text{Ad}(g)(\theta(b)))) \\ &= \exp((\text{Ad}^*(g)(\alpha))(\theta(b))) = \chi_{\text{Ad}^*(g)\alpha}(b). \end{aligned}$$
□

LEMMA 7.3. Let $b = \sum_l b_l e^l \in \exp \mathfrak{a}$. Then

$$b = b_0 \exp\left(\sum_{k \geq 0} \theta_k(b') e^k\right),$$

where $b'_l = b_l/b_0$. Hence

$$\chi_\alpha(b) = b_0^{\alpha_0} \exp\left(\sum_{k \geq 0} \alpha_k \theta_k(b')\right).$$

PROOF. This is clear, since $b = b_0 \sum_l b'_l e^l$. □

We denote by ι the isomorphism $\mathfrak{a} \ni \sum_l c_l e^l \mapsto [c_l]_l \in \mathbb{C}^L$, and by ι^* its dual isomorphism: $\mathbb{C}^L \rightarrow \mathfrak{a}^*$. For $g \in N_G(\mathfrak{a})$, let $M(g)$ denote the matrix representation of $\text{Ad}(g)$ on $\mathfrak{a}$ with respect to the basis $\{e^l \mid l \in L\}$.

Then by the definitions

$$(18) \quad \text{Ad}(g)(\iota^{-1}(h)) = \iota^{-1}(M(g)h), \quad \text{Ad}^*(g)(\iota^*(h)) = \iota^*({}^tM(g)h)$$

for $h \in \mathbb{C}^L$.

PROPOSITION 7.4. $\phi(\alpha, M(g)z, t) = \phi(\text{Ad}^*(g)\alpha, z, t)$ for $g \in N_G(\mathfrak{a})$.

PROOF. Recall the definition of $\phi(\alpha, z, t)$ in Proposition 5.3;

$$\phi(\alpha, z, t) = \chi_\alpha\left(\sum_{\mathbf{l}} b_{\mathbf{l}} e^{\mathbf{l}}\right) = \chi_\alpha(\iota^{-1}([b_{\mathbf{l}}]_{\mathbf{l}}))$$

by Lemma 7.3, where $b_{\mathbf{l}} = (zt)_{\mathbf{l}}$. Hence by (18) we have

$$\begin{aligned} \phi(\alpha, M(g)z, t) &= \chi_\alpha(\iota^{-1}([(M(g)zt)_{\mathbf{l}}]_{\mathbf{l}})) = \chi_\alpha(\text{Ad}(g)\iota^{-1}([(zt)_{\mathbf{l}}]_{\mathbf{l}})) \\ &= \chi_{\text{Ad}^*(g)\alpha}(\iota^{-1}([(zt)_{\mathbf{l}}]_{\mathbf{l}})) = \phi(\text{Ad}^*(g)\alpha, z, t). \end{aligned}$$

□

COROLLARY 7.5. $\psi(\alpha, M(g)z) = \psi(\text{Ad}^*(g)\alpha, z)$ for $g \in N_G(\mathfrak{a})$.

To describe the matrix representation of $\text{Ad}(g)$ ($g \in N_G(\mathfrak{a})$), we introduce some notation.

Put

$$\begin{aligned} f_i(x, T) &:= \sum_{\mathbf{a} \in \mathbb{N}^p} x_{\mathbf{a}}^{(i)} T^{\mathbf{a}} \quad (i = 1, 2, \dots, p) \\ f(x, T)^{\mathbf{l}} &:= \prod_{i=1}^p (f_i(x, T))^{l_i} = \sum_{\mathbf{l}' \in \mathbb{N}^p} \phi_{\mathbf{l}', \mathbf{l}}(x) T^{\mathbf{l}'} \end{aligned}$$

for $\mathbf{l} \in \mathbb{N}^p$.

LEMMA 7.6. (1) $\phi_{\mathbf{l}', \mathbf{0}}(x) = \begin{cases} 1 & (\mathbf{l}' = \mathbf{0}) \\ 0 & (\text{otherwise}). \end{cases}$

(2) $\phi_{\mathbf{l}', \mathbf{l}}(x) = \sum_{\sum_{i=1}^p \sum_{j=1}^{l_i} \mathbf{a}_{ij} = \mathbf{l}'} \prod_{i=1}^p x_{\mathbf{a}_{i1}}^{(i)} \cdots x_{\mathbf{a}_{il_i}}^{(i)}$.

(3) $\phi_{\mathbf{l}', \mathbf{l}}(x) = 0$ for $|\mathbf{l}| > |\mathbf{l}'|$.

(4) $\phi_{\mathbf{k}, \mathbf{l} + \mathbf{l}'}(x) = \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} \phi_{\mathbf{k}_1, \mathbf{l}}(x) \phi_{\mathbf{k}_2, \mathbf{l}'}(x)$.

PROOF. (1) follows from $f(x, T)^{\mathbf{0}} = 1$.

Since we have

$$\begin{aligned} f(x, T)^{\mathbf{l}} &= \prod_{i=1}^p \left(\sum_{\mathbf{a} \in \mathbb{N}^p} x_{\mathbf{a}}^{(i)} T^{\mathbf{a}} \right)^{l_i} \\ &= \prod_{i=1}^p \left(\sum_{\mathbf{a}_{i1}, \dots, \mathbf{a}_{il_i} \in \mathbb{N}^p} x_{\mathbf{a}_{i1}}^{(i)} \cdots x_{\mathbf{a}_{il_i}}^{(i)} T^{\sum_{j=1}^{l_i} \mathbf{a}_{ij}} \right), \end{aligned}$$

we obtain (2).

Since $|\sum_{i=1}^p \sum_{j=1}^{l_i} \mathbf{a}_{ij}| \geq |\mathbf{l}|$, (3) follows from (2).

The equation $f(x, T)^{\mathbf{l} + \mathbf{l}'} = f(x, T)^{\mathbf{l}} f(x, T)^{\mathbf{l}'}$ leads to (4). □

COROLLARY 7.7. Suppose that $|\mathbf{l}| = |\mathbf{l}'|$. Then any term of $\phi_{\mathbf{l}', \mathbf{l}}$ is a product of $x_{1_j}^{(i)}$ ($i, j = 1, 2, \dots, p$). □

PROOF. This is clear from Lemma 7.6 (2). □

COROLLARY 7.8. Let L_k be a subset of $\mathbb{N}_k^p := \{\mathbf{l} \in \mathbb{N}^p \mid |\mathbf{l}| = k\}$ ($k > 0$). Suppose that $C = [c_{1j}^{(i)}]_{1 \leq i, j \leq p} \in GL(p)$ satisfies $\phi_{\mathbf{l}', \mathbf{l}}(C) = 0$ for all $\mathbf{l}' \in L_k$, $\mathbf{l} \in \mathbb{N}_k^p \setminus L_k$.

Then $\det([\phi_{\mathbf{l}', \mathbf{l}}]_{\mathbf{l}', \mathbf{l} \in L_k}) \neq 0$.

PROOF. First note that for $\mathbf{l}', \mathbf{l} \in \mathbb{N}_k^p$ the notation $\phi_{\mathbf{l}', \mathbf{l}}(C)$ is justified by Corollary 7.7.

Let $U = \bigoplus_{i=1}^p \mathbb{C}\mathbf{u}_i$, and let $S_k(U)$ be the space of symmetric product of degree k . Then there exists a natural representation

$$\rho_k : GL(U) \rightarrow GL(S_k(U)).$$

With respect to the basis $\{\mathbf{u}^{\mathbf{l}} \mid \mathbf{l} \in \mathbb{N}_k^p\}$, $\rho_k(C)$ is represented by $[\phi_{\mathbf{l}', \mathbf{l}}(C)]_{\mathbf{l}', \mathbf{l} \in \mathbb{N}_k^p}$. Hence

$$\begin{aligned} 0 &\neq \det([\phi_{\mathbf{l}', \mathbf{l}}(C)]_{\mathbf{l}', \mathbf{l} \in \mathbb{N}_k^p}) \\ &= \det([\phi_{\mathbf{l}', \mathbf{l}}(C)]_{\mathbf{l}', \mathbf{l} \in L_k}) \det([\phi_{\mathbf{l}', \mathbf{l}}(C)]_{\mathbf{l}', \mathbf{l} \in \mathbb{N}_k^p \setminus L_k}) \end{aligned}$$

by the assumption. Hence we have $\det([\phi_{\mathbf{l}', \mathbf{l}}]_{\mathbf{l}', \mathbf{l} \in L_k}) \neq 0$. $\square$

PROPOSITION 7.9. Let $g \in N_G(\mathfrak{a})$, and let

$$\text{Ad}(g)e_i = \sum_{\mathbf{l} \in L_+} c_{\mathbf{l}}^{(i)} e^{\mathbf{l}} = f_i(\{c_{\mathbf{l}}^{(i)}\}, e).$$

Then

$$\text{Ad}(g)e^{\mathbf{l}} = \sum_{\mathbf{l}' \in L_+} \phi_{\mathbf{l}', \mathbf{l}}(\{c_{\mathbf{k}}^{(i)}\}) e^{\mathbf{l}'} = f(\{c_{\mathbf{k}}^{(i)}\}, e)^{\mathbf{l}}.$$

PROOF. This is immediate from $\text{Ad}(g)e^{\mathbf{l}} = \prod_{i=1}^p (\text{Ad}(g)e_i)^{l_i}$. $\square$

For $g \in N_G(\mathfrak{a})$, let $M(g)$ denote the matrix representation of $\text{Ad}(g)$ on $\mathfrak{a}$ with respect to the basis $\{e^{\mathbf{l}} \mid \mathbf{l} \in L\}$.

COROLLARY 7.10. Put $\phi_{\mathbf{0}, \mathbf{l}} = \delta_{\mathbf{0}, \mathbf{l}}$. Then

$$M(g) = [m(g)_{\mathbf{l}', \mathbf{l}}]_{\mathbf{l}', \mathbf{l} \in L} = [\phi_{\mathbf{l}', \mathbf{l}}(\{c_{\mathbf{k}}^{(i)}\})]_{\mathbf{l}', \mathbf{l} \in L},$$

where $c_{\mathbf{l}}^{(i)}$ are those in Proposition 7.9.

PROOF. Since $\text{Ad}(g)e^{\mathbf{l}}$ is nilpotent, we have $m(g)_{\mathbf{0}, \mathbf{l}} = \delta_{\mathbf{0}, \mathbf{l}}$. $\square$

DEFINITION 7.11. (1) $\mathbf{a} \in L$ is called an *inner corner* of L , if $\mathbf{a} + \mathbf{1}_i \notin L$ for any $i = 1, 2, \dots, p$.

(2) $\mathbf{a} \in \mathbb{N}^p$ is called an *outer corner* of L , if $\mathbf{a} \notin L$, and for every $i = 1, 2, \dots, p$ we have $\mathbf{a} - \mathbf{1}_i \notin \mathbb{N}^p$ or $\mathbf{a} - \mathbf{1}_i \in L$.

EXAMPLE 7.12. Let $p = 2$. Then L can be given by a Young diagram (see Remark 4.4). Suppose that L corresponds to the Young diagram $(5, 5, 3, 2, 1)$.

(19)

Then in (19) $\cdot$ indicates an inner corner, and $\circ$ an outer corner.

LEMMA 7.13. Let $g \in N_G(\mathfrak{a})$, and let $\text{Ad}(g)e_i = \sum_{\mathbf{l} \in L_+} c_{\mathbf{l}}^{(i)} e^{\mathbf{l}}$.

(1) Let $\mathbf{a}$ be an inner corner. Then $\prod_{i=1}^p (\sum_{\mathbf{l} \in L_+} c_{\mathbf{l}}^{(i)} e^{\mathbf{l}})^{a_i} \neq 0$.

(2) Let $\mathbf{a}$ be an outer corner. Then $\prod_{i=1}^p (\sum_{\mathbf{l} \in L_+} c_{\mathbf{l}}^{(i)} e^{\mathbf{l}})^{a_i} = 0$.

PROOF. Note that

$$(20) \quad f(c, e)^{\mathfrak{a}} = \text{Ad}(g)e^{\mathfrak{a}} = \prod_{i=1}^p \left(\sum_{\mathbf{l} \in L_+} c_{\mathbf{l}}^{(i)} e^{\mathbf{l}} \right)^{a_i}.$$

Hence the assertions are immediate, since an inner corner belongs to L and an outer corner does not. $\square$

REMARK 7.14. Let $g \in N_G(\mathfrak{a})$, and let $\text{Ad}(g)e_i = \sum_{\mathbf{l} \in L_+} c_{\mathbf{l}}^{(i)} e^{\mathbf{l}}$. Suppose that $\mathbf{l}$ is an outer corner of L . Then, by Lemma 7.6 (4),

$$\begin{aligned} 0 &= \phi_{\mathbf{k}, \mathbf{l}}(c) = \sum_{\mathbf{k}_1 + \dots + \mathbf{k}_p = \mathbf{k}} \prod_{i=1}^p \phi_{\mathbf{k}_i, \mathbf{l}_i 1_i}(c) \\ &= \sum_{\sum_{i=1}^p \sum_{j=1}^{l_i} \mathbf{k}_{ij} = \mathbf{k}} \prod_{i=1}^p \prod_{j=1}^{l_i} \phi_{\mathbf{k}_{ij}, 1_i}(c) \\ &= \sum_{\sum_{i=1}^p \sum_{j=1}^{l_i} \mathbf{k}_{ij} = \mathbf{k}} \prod_{i=1}^p \prod_{j=1}^{l_i} c_{\mathbf{k}_{ij}}^{(i)} \end{aligned}$$

for $\mathbf{k} \in L_+$. These equations are equivalent to Lemma 7.13.

EXAMPLE 7.15. Let $p = 1$, and $L = \{0, 1, \dots, n-1\}$. Then $n-1$ is the inner corner and n is the outer corner. We have

$$\left(\sum_{l=1}^{n-1} c_l e^l \right)^{n-1} \neq 0, \quad \left(\sum_{l=1}^{n-1} c_l e^l \right)^n = 0.$$

We obtain $c_1 \neq 0$ from the first equation.

EXAMPLE 7.16. Let $p = 2$, and $L = \{(i, 0) \mid 0 \leq i \leq n-2\} \cup \{(1, 0)\}$. Then the inner corners are $(n-2, 0)$ and $(0, 1)$. The outer corners are $(n-1, 0)$, $(1, 1)$, and $(0, 2)$.

Then we have

$$\begin{aligned} &\left(\sum_{l=1}^{n-2} c_{(l,0)}^{(1)} e_1^l + c_{(0,1)}^{(1)} e_2 \right)^{n-2} \neq 0, \\ &\left(\sum_{l=1}^{n-2} c_{(l,0)}^{(2)} e_1^l + c_{(0,1)}^{(2)} e_2 \right) \neq 0, \\ &\left(\sum_{l=1}^{n-2} c_{(l,0)}^{(1)} e_1^l + c_{(0,1)}^{(1)} e_2 \right)^{n-1} = 0, \\ &\left(\sum_{l=1}^{n-2} c_{(l,0)}^{(1)} e_1^l + c_{(0,1)}^{(1)} e_2 \right) \left(\sum_{l=1}^{n-2} c_{(l,0)}^{(2)} e_1^l + c_{(0,1)}^{(2)} e_2 \right) = 0, \\ &\left(\sum_{l=1}^{n-2} c_{(l,0)}^{(2)} e_1^l + c_{(0,1)}^{(2)} e_2 \right)^2 = 0. \end{aligned}$$

From these equations, we obtain $c_{(1,0)}^{(1)} \neq 0$, $c_{(l,0)}^{(2)} = 0$ for $l \leq n-3$.

Put

$$(21) \quad L_k := \{\mathbf{l} \in L_+ \mid |\mathbf{l}| = k\}.$$

LEMMA 7.17. Let $g \in N_G(\mathfrak{a})$, and let $\text{Ad}(g)e_i = \sum_{\mathbf{l} \in L_+} c_{\mathbf{l}}^{(i)} e^{\mathbf{l}}$. For any k ,

$$\det[\phi_{\mathbf{l}', \mathbf{l}}(\{c_{\mathbf{k}}^{(i)}\})]_{\mathbf{l}, \mathbf{l}' \in L_k} \neq 0.$$

PROOF. Since $\text{Ad}(g)\mathfrak{a}^k = \mathfrak{a}^k$ for any k , we have $\text{Ad}(g)(\mathfrak{a}^k/\mathfrak{a}^{k+1}) = \mathfrak{a}^k/\mathfrak{a}^{k+1}$. Hence the assertion holds. $\square$

EXAMPLE 7.18. Let $p = 2$, and $L = \{(i, 0) \mid 0 \leq i \leq n-2\} \cup \{(1, 0)\}$. Then $L_1 = \{(1, 0), (0, 1)\}$. Hence

$$\det \begin{bmatrix} c_{(1,0)}^{(1)} & c_{(1,0)}^{(2)} \\ c_{(0,1)}^{(1)} & c_{(0,1)}^{(2)} \end{bmatrix} \neq 0.$$

For $2 \leq k \leq n-2$, $L_k = \{(k, 0)\}$. Hence

$$0 \neq \phi_{(k,0),(k,0)}(c) = \phi_{(1,0),(1,0)}(c)^k = (c_{(1,0)}^{(1)})^k.$$

Hence $c_{(1,0)}^{(1)} \neq 0$, which is the inner corner condition.

PROPOSITION 7.19. Let $g \in N_G(\mathfrak{a})$, and let $\text{Ad}(g)e_i = \sum_{\mathbf{l} \in L_+} c_{\mathbf{l}}^{(i)} e^{\mathbf{l}}$. Suppose that $A = \alpha(0) + L$. Put $\mathbf{u}_{\mathbf{l}} := v_{\alpha(0)+\mathbf{l}} = e^{\mathbf{l}} v_{\alpha(0)}$. Define $\tilde{g} \in GL(V)$ by

$$\tilde{g}\mathbf{u}_{\mathbf{l}} = \sum_{\mathbf{l}'} \phi_{\mathbf{l}', \mathbf{l}}(c) \mathbf{u}_{\mathbf{l}}.$$

Then $\tilde{g} \in N_G(\mathfrak{a})$ and $M(\tilde{g}) = \tilde{g} = M(g)$.

PROOF. We know that $M(g) = [\phi_{\mathbf{l}', \mathbf{l}}(c)]$ is invertible. Let $[\psi_{\mathbf{l}', \mathbf{l}}]$ be its inverse. Then

$$\tilde{g}^{-1}\mathbf{u}_{\mathbf{l}} = \sum_{\mathbf{l}'} \psi_{\mathbf{l}', \mathbf{l}} \mathbf{u}_{\mathbf{l}'}.$$

Hence by Lemma 7.6 (4)

$$\begin{aligned} \tilde{g}e^{\mathbf{k}}\tilde{g}^{-1}\mathbf{u}_{\mathbf{l}} &= \sum_{\mathbf{l}'} \psi_{\mathbf{l}', \mathbf{l}} \tilde{g}\mathbf{u}_{\mathbf{l}'+\mathbf{k}} \\ &= \sum_{\mathbf{l}', \mathbf{l}''} \psi_{\mathbf{l}', \mathbf{l}} \phi_{\mathbf{l}'', \mathbf{l}'+\mathbf{k}} \mathbf{u}_{\mathbf{l}''} \\ &= \sum_{\mathbf{l}', \mathbf{l}_1, \mathbf{l}_2} \psi_{\mathbf{l}', \mathbf{l}} \phi_{\mathbf{l}_1, \mathbf{l}'} \phi_{\mathbf{l}_2, \mathbf{k}} \mathbf{u}_{\mathbf{l}_1+\mathbf{l}_2} \\ &= \sum_{\mathbf{l}_1, \mathbf{l}_2} \delta_{\mathbf{l}_1, \mathbf{l}} \phi_{\mathbf{l}_2, \mathbf{k}} \mathbf{u}_{\mathbf{l}_1+\mathbf{l}_2} \\ &= \sum_{\mathbf{l}_2} \phi_{\mathbf{l}_2, \mathbf{k}} \mathbf{u}_{\mathbf{l}+\mathbf{l}_2} = \sum_{\mathbf{l}_2} \phi_{\mathbf{l}_2, \mathbf{k}} e^{\mathbf{l}_2} \mathbf{u}_{\mathbf{l}}. \end{aligned}$$

Hence

$$\tilde{g}e^{\mathbf{k}}\tilde{g}^{-1} = \sum_{\mathbf{l}_2} \phi_{\mathbf{l}_2, \mathbf{k}} e^{\mathbf{l}_2}.$$

We have thus proved $\tilde{g} \in N_G(\mathfrak{a})$ and $M(\tilde{g}) = \tilde{g} = M(g)$. $\square$

COROLLARY 7.20. Let $c_{\mathbf{l}}^{(i)}$ ($1 \leq i \leq p$, $\mathbf{l} \in L_+$) satisfy the conditions

- (1) $\det([\phi_{\mathbf{l}', \mathbf{l}}(c)]_{\mathbf{l}, \mathbf{l}' \in L_+}) \neq 0$.
- (2) $\phi_{\mathbf{l}', \mathbf{l}}(c) = 0$ for every $\mathbf{l} \notin L_+$, $\mathbf{l}' \in L_+$.

Then $g = [\phi_{\mathbf{l}', \mathbf{l}}(c)]_{\mathbf{l}, \mathbf{l}' \in L} \in N_G(\mathfrak{a})$ and $M(g) = g$.

PROOF. This is clear from the proof of Proposition 7.19. $\square$

LEMMA 7.21. *The following two conditions are equivalent:*

- (1) $\prod_{i=1}^p (\sum_{\mathbf{l}' \in L_+} c_{\mathbf{l}}^{(i)} e^{\mathbf{l}'})^{l_i} = 0$ for all outer corners $\mathbf{l}$.
- (2) $\phi_{\mathbf{l}', \mathbf{l}}(c) = 0$ for every $\mathbf{l} \notin L_+, \mathbf{l}' \in L_+$.

PROOF. The condition (2) is equivalent to the condition

$$f(c, e)^{\mathbf{l}} = 0 \text{ for all } \mathbf{l} \notin L_+,$$

which is equivalent to the condition

$$f(c, e)^{\mathbf{l}} = 0 \text{ for all outer corners } \mathbf{l},$$

which is exactly the condition (1). □

PROPOSITION 7.22. *Each element of $N_G(\mathfrak{a}_+)/Z_G(\mathfrak{a}_+)$ is uniquely written as $[\phi_{\mathbf{l}', \mathbf{l}}(c)]_{\mathbf{l}', \mathbf{l} \in L_+}$ with $c = (c_{\mathbf{l}}^{(i)} : 1 \leq i \leq p; \mathbf{l} \in L_+)$ satisfying*

- (1) $\det[c_{1_j}^{(i)}]_{1 \leq i, j \leq p} \neq 0$.
- (2) $f(c, e)^{\mathbf{l}} = 0$ for all outer corners $\mathbf{l}$.

PROOF. The condition (1) in Corollary 7.20 is equivalent to the condition

$$(22) \quad \det([\phi_{\mathbf{l}', \mathbf{l}}(c)]_{\mathbf{l}, \mathbf{l}' \in L_k}) \neq 0 \quad (\forall k > 0)$$

by Lemma 7.6 (3). Then by Corollary 7.8 we see that the conditions

$$\det([\phi_{\mathbf{l}', \mathbf{l}}(c)]_{\mathbf{l}, \mathbf{l}' \in L_k}) \neq 0 \quad (k > 1)$$

are derived from the other conditions. □

CHAPTER 8

Type $(n-1, 1)$

As stated in the paragraph just above Lemma 2.1, when $\mathbf{a}$ is associated with $L = \{0, 1, 2, \dots, n-1\} = \{(l, 0) | 0 \leq l \leq n-1\}$, which we may call type (n) , the system $M_{\mathbf{a}, \alpha}$ is the generalized Airy system [5].

In this chapter, we consider the system $M_{\mathbf{a}, \alpha}$ of type $(n-1, 1)$, which we also denote by $M_{(n-1, 1), \alpha}$. Namely let $L = \{(l, 0) | 0 \leq l \leq n-2\} \cup \{(0, 1)\}$ and $\mathbf{a} = \langle \mathbf{e}^l | l \in L \rangle$;

$$L = \begin{array}{|c|c|} \hline & \\ \hline \end{array} \cdots \begin{array}{|c|} \hline \\ \hline \end{array}.$$

Then the system $M_{\mathbf{a}, \alpha}$ with a parameter vector $\alpha \in \mathfrak{a}_+^* \simeq \mathbb{C}^{L+}$ is $\mathcal{D}_Z / J_{\mathbf{a}, \alpha}$, where the left ideal $J_{\mathbf{a}, \alpha}$ is generated by the following:

$$(23) \quad \frac{\partial}{\partial z_{\mathbf{a}, j_1}} \frac{\partial}{\partial z_{\mathbf{b}, j_2}} - \frac{\partial}{\partial z_{\mathbf{a}, j_2}} \frac{\partial}{\partial z_{\mathbf{b}, j_1}} \quad (\mathbf{a}, \mathbf{b} \in L; 1 \leq j_1, j_2 \leq m),$$

$$(24) \quad \sum_{\mathbf{a} \in L} z_{\mathbf{a}, j_1} \frac{\partial}{\partial z_{\mathbf{a}, j_2}} + \delta_{j_1 j_2} \quad (1 \leq j_1, j_2 \leq m),$$

$$(25) \quad \sum_{i=0}^{n-k-2} \sum_{j=1}^m z_{(i, 0), j} \frac{\partial}{\partial z_{(i+k, 0), j}} + \alpha_{(k, 0)} \quad (k = 1, \dots, n-2),$$

$$(26) \quad \sum_{j=1}^m z_{(0, 0), j} \frac{\partial}{\partial z_{(0, 1), j}} + \alpha_{(0, 1)}.$$

Note that by (24) we have

$$\sum_{i=0}^{n-2} \sum_{j=1}^m z_{(i, 0), j} \frac{\partial}{\partial z_{(i, 0), j}} + \sum_{j=1}^m z_{(0, 1), j} \frac{\partial}{\partial z_{(0, 1), j}} + m = 0,$$

and thus $\alpha_{(0, 0)}$ is always equal to m ; hence we consider $\alpha \in \mathfrak{a}_+^* \simeq \mathbb{C}^{L+}$.

In this chapter, we prove that $M_{(n-1, 1), \alpha}$ is generically holonomic (Theorem 8.2), and is reduced to $M_{(n-1), \beta}$, when $\alpha_{(n-2, 0)} \neq 0$ (Proposition 8.5, Theorem 8.6).

LEMMA 8.1. *Let $\alpha = (\alpha_l)_{l \in L+} \in \mathbb{C}^{L+}$ be a parameter vector. Then*

$$\alpha_{(n-2, 0)} \frac{\partial}{\partial z_{(0, 1), j}} - \alpha_{(0, 1)} \frac{\partial}{\partial z_{(n-2, 0), j}} \in J_{\mathbf{a}, -\alpha}$$

for all $j = 1, \dots, m$.

PROOF. Since

$$\sum_{l=1}^m z_{(0, 0), l} \frac{\partial}{\partial z_{(n-2, 0), l}} - \alpha_{(n-2, 0)}, \quad \sum_{l=1}^m z_{(0, 0), l} \frac{\partial}{\partial z_{(0, 1), l}} - \alpha_{(0, 1)}$$

belong to the ideal $J_{\mathbf{a}, -\alpha}$, we see by (23) that

$$\begin{aligned} & \alpha_{(n-2,0)} \frac{\partial}{\partial z_{(0,1),j}} - \alpha_{(0,1)} \frac{\partial}{\partial z_{(n-2,0),j}} \\ \equiv & \frac{\partial}{\partial z_{(0,1),j}} \left(\sum_{l=1}^m z_{(0,0),l} \frac{\partial}{\partial z_{(n-2,0),l}} - \alpha_{(n-2,0)} \right) \\ & - \frac{\partial}{\partial z_{(n-2,0),j}} \left(\sum_{l=1}^m z_{(0,0),l} \frac{\partial}{\partial z_{(0,1),l}} - \alpha_{(0,1)} \right) \end{aligned}$$

belongs to $J_{\mathbf{a}, -\alpha}$. □

THEOREM 8.2. *Let $\alpha = (\alpha_l)_{l \in L_+} \in \mathbb{C}^{L_+}$ satisfy $\alpha_{(n-2,0)} \neq 0$. Then $M_{\mathbf{a}, -\alpha}$ is generically holonomic.*

PROOF. Let (z, ξ) belong to the characteristic variety of $M_{\mathbf{a}, -\alpha}$. For a generic z , we show $\xi = 0$. We follow the proof of [4, Lemma 6].

By (23), we may put $\xi_{lj} = a_l b_j$ for all l, j .

By (24),

$$0 = \sum_{l \in L} z_{l,j_1} a_l b_{j_2} = b_{j_2} \sum_{l \in L} z_{l,j_1} a_l$$

for all $j_1, j_2 \leq m$.

Suppose that $\xi \neq 0$. Then $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$. Hence

$$\sum_{l \in L} z_{l,k} a_l = 0 \quad (k = 1, \dots, m).$$

Put $I_L := \{l \in L \mid a_l \neq 0\}$. Then

$$(27) \quad \sum_{l \in I_L} z_{l,k} a_l = 0 \quad (k = 1, \dots, m).$$

Since z is generic, we see

$$(28) \quad |I_L| > m.$$

By (25), we have for $1 \leq k \leq n-2$

$$(29) \quad 0 = \sum_{i=0}^{n-k-2} \sum_{j=1}^m z_{(i,0),j} a_{(i+k,0)} b_j = \sum_{i=0}^{n-k-2} a_{(i+k,0)} c_{(i,0)},$$

where we put

$$(30) \quad c_{(i,0)} = \sum_{j=1}^m z_{(i,0),j} b_j.$$

Put

$$I'_L := \{(k, 0) \mid k = 0, \dots, n-2, a_{(k,0)} \neq 0\} = I_L \setminus \{(0, 1)\}$$

and let $k_0 := \max\{k \mid (k, 0) \in I'_L\}$.

Then we claim

CLAIM 8.3.

$$c_{(k,0)} = 0 \quad \text{for all } k < k_0.$$

PROOF. If $k_0 = 0$, then there exists nothing to prove.

Let $k_0 > 0$. We prove the assertion by induction on k . Plugging $k = k_0$ in (29), we have $a_{(k_0,0)}c_{(0,0)} = 0$. Hence $c_{(0,0)} = 0$.

Now we suppose that $c_{(k,0)} = 0$ for all $k < l < k_0$. Plugging $k = k_0 - l$ in (29), we have $a_{(k_0,0)}c_{(l,0)} = 0$. Hence $c_{(l,0)} = 0$. We have thus proved the claim. $\square$

By Lemma 8.1, (z, ξ) satisfies

$$\alpha_{(n-2,0)}a_{(0,1)}b_j = \alpha_{(0,1)}a_{(n-2,0)}b_j$$

for all $1 \leq j \leq m$, and hence we have

$$(31) \quad \alpha_{(n-2,0)}a_{(0,1)} = \alpha_{(0,1)}a_{(n-2,0)}.$$

We first treat the case when $\alpha_{(0,1)} \neq 0$.

Suppose that $k_0 = n - 2$. Then $a_{(n-2,0)} \neq 0$, and hence $a_{(0,1)} \neq 0$, i. e. $(0, 1) \in I_L$, by (31). Then by (27) and (31) we have $|I'_L| + 1 > m + 1$, otherwise we have $\mathbf{a} = \mathbf{0}$, which is a contradiction. Hence $n > m + 1$ or $n - 2 > m - 1$.

By Claim 8.3, we have

$$c_{(0,0)} = \cdots = c_{(n-3,0)} = 0.$$

By (30), we have $n - 2 < m$, otherwise we have $\mathbf{b} = \mathbf{0}$, which is a contradiction. Hence $m - 1 < n - 2 < m$, which is impossible.

Next suppose that $k_0 < n - 2$. Then $a_{(n-2,0)} = a_{(0,1)} = 0$ by (31). By (28), we have $|I'_L| > m$, and hence $k_0 + 1 > m$.

By Claim 8.3, we have

$$c_{(0,0)} = \cdots = c_{(k_0-1,0)} = 0.$$

By (30), we have $k_0 < m$. Hence $m - 1 < k_0 < m$, which is impossible.

We next treat the case when $\alpha_{(0,1)} = 0$. Then by (31) we have $a_{(0,1)} = 0$. By (28) and the definition of I'_L , we have $|I'_L| > m$, and hence $k_0 + 1 > m$.

By Claim 8.3, we have

$$c_{(0,0)} = \cdots = c_{(k_0-1,0)} = 0.$$

By (30), we have $k_0 < m$. Hence $m - 1 < k_0 < m$, which is impossible. $\square$

Proposition 7.22 implies the form of $g \in N_G(\mathbf{a})$.

PROPOSITION 8.4.

$$N_G(\mathbf{a})/Z_G(\mathbf{a}) = \left\{ [\phi_{\nu, \mathbf{l}}(c)]_{\nu, \mathbf{l} \in L_+} \mid \begin{array}{l} c_{(1,0)}^{(1)}c_{(0,1)}^{(2)} \neq 0, \\ c_{(k,0)}^{(2)} = 0 \end{array} \quad (1 \leq k \leq n-3) \right\}.$$

PROOF. Let $g \in N_G(\mathbf{a})/Z_G(\mathbf{a})$. Then by Proposition 7.22 we have

$$g = [\phi_{\nu, \mathbf{l}}(c)]_{\nu, \mathbf{l} \in L_+}$$

where $c = (c_l^{(i)} : i = 1, 2; \mathbf{l} \in L_+)$.

We have already seen $c_{(1,0)}^{(1)} \neq 0$ and $c_{(k,0)}^{(2)} = 0$ ($1 \leq k \leq n-3$) in Example 7.16, and $c_{(1,0)}^{(1)}c_{(0,1)}^{(2)} \neq 0$ in Example 7.18. These are the necessary and sufficient conditions by Proposition 7.22. $\square$

PROPOSITION 8.5. Let $\Phi(z; \alpha)$ be a solution to $M_{\mathbf{a}, \alpha}$. We assume that a parameter vector

$$\alpha = {}^t(\alpha_{(1,0)}, \dots, \alpha_{(n-2,0)}, \alpha_{(0,1)}) \in \mathbb{C}^{L_+}$$

of $\Phi(z; \alpha)$ satisfies $\alpha_{(n-2,0)} \neq 0$. Then there exists $g \in N_G(\mathfrak{a})$ such that the change of coordinates $z \mapsto z' = gz$ transforms $\Phi(z; \alpha)$ into $\Phi(z'; \alpha')$ with the parameter vector

$$\alpha' = {}^t(0, \dots, 0, 1, 1).$$

PROOF. By Lemma 7.6, we have

$$\begin{aligned} \phi_{(i,0),(0,1)}(c) &= c_{(i,0)}^{(2)} \quad (1 \leq i \leq n-2), \\ \phi_{(0,1),(j,0)}(c) &= \begin{cases} 0 & (j \geq 2) \\ c_{(0,1)}^{(1)} & (j = 1), \end{cases} \\ \phi_{(0,1),(0,1)}(c) &= c_{(0,1)}^{(2)}. \end{aligned}$$

Then by Proposition 8.4, $g \in N_G(\mathfrak{a})$ has the form

$$g = \begin{bmatrix} & & & 0 \\ & [\phi_{(i,0),(j,0)}(c)]_{1 \leq i,j \leq n-2} & & \vdots \\ & & & 0 \\ & & & c_{(n-2,0)}^{(2)} \\ c_{(0,1)}^{(1)} & 0 & \dots & 0 & c_{(0,1)}^{(2)} \end{bmatrix}$$

with respect to the order $(1,0), \dots, (n-2,0), (0,1)$, where $c_{(1,0)}^{(1)} \neq 0$, $c_{(0,1)}^{(2)} \neq 0$ and $c = (c_{(1,0)}^{(1)}, \dots, c_{(n-2,0)}^{(1)}) \in \mathbb{C}^{n-2}$.

By (18), we have

$$(32) \quad \alpha'_{(i,0)} = ({}^t g \alpha)_{(i,0)} = \sum_{j=1}^{n-2} \alpha_{(j,0)} \phi_{ij}(c) + \delta_{i1} \alpha_{(0,1)} c_{(0,1)}^{(1)},$$

$$(33) \quad \alpha'_{(0,1)} = ({}^t g \alpha)_{(0,1)} = \alpha_{(n-2,0)} c_{(1,0)}^{(2)} + \alpha_{(0,1)} c_{(0,1)}^{(2)}.$$

Firstly we consider (32) for $2 \leq i \leq n-2$. Similarly to [9, Corollary 5.4], noting that $\alpha_{(n-2,0)} \neq 0$, we can choose $c_{(1,0)}^{(1)}, \dots, c_{(n-3,0)}^{(1)}$ so that $\alpha'_{(2,0)}, \dots, \alpha'_{(n-3,0)}$ become all zero and $\alpha'_{(n-2,0)} = 1$.

Secondly we consider (32) for $i = 1$, i.e.,

$$\alpha_{(1,0)} c_{(1,0)}^{(1)} + \dots + \alpha_{(n-2,0)} c_{(n-2,0)}^{(1)} + \alpha_{(0,1)} c_{(0,1)}^{(1)}.$$

Using the condition $\alpha_{(n-2,0)} \neq 0$, we can determine $c_{(n-2,0)}^{(1)}$ and $c_{(0,1)}^{(1)}$ so that $\alpha'_{(1,0)} = 0$.

Lastly we consider (33). Noting that $\alpha_{(n-2,0)} \neq 0$, we can determine $c_{(n-2,0)}^{(2)}$ and $c_{(0,1)}^{(2)}$, so that $\alpha'_{(0,1)} = 1$. $\square$

$$(34) \quad \text{Let } Y = \begin{bmatrix} y_{01} & \dots & y_{0m} \\ \vdots & & \vdots \\ y_{n-2,1} & \dots & y_{n-2,m} \end{bmatrix}, \text{ and define } f: Z \rightarrow Y \text{ by}$$

$$f(Z) = \begin{bmatrix} z_{(0,0),1} & \dots & z_{(0,0),m} \\ \vdots & & \vdots \\ z_{(n-3,0),1} & \dots & z_{(n-3,0),m} \\ z_{(n-2,0),1} + z_{(0,1),1} & \dots & z_{(n-2,0),m} + z_{(0,1),m} \end{bmatrix}.$$

THEOREM 8.6. Let $\alpha = {}^t(0, \dots, 0, 1, 1) \in \mathbb{C}^{n-1}$ and $\beta = {}^t(0, \dots, 0, 1) \in \mathbb{C}^{n-2}$. Then

$$f^* M_{(n-1),\beta} = M_{(n-1,1),\alpha}.$$

PROOF. Let

$$Z' = \begin{bmatrix} z_{(0,0),1} & \cdots & z_{(0,0),m} \\ \vdots & & \vdots \\ z_{(n-3,0),1} & \cdots & z_{(n-3,0),m} \\ z_{(n-2,0),1} + z_{(0,1),1} & \cdots & z_{(n-2,0),m} + z_{(0,1),m} \\ z_{(0,1),1} & \cdots & z_{(0,1),m} \end{bmatrix}.$$

Then $f = p_{n-1} \circ f'$, where $f'(Z) = Z'$ and p_{n-1} is the projection on the first $n-1$ rows.

Since p_{n-1} is the projection, we easily see $p_{n-1}^* M_{(n-1),\beta} = D_{Z'}/J'$, where J' is generated by

$$\begin{aligned} & \frac{\partial}{\partial z'_{(k_1,0),j_1}} \frac{\partial}{\partial z'_{(k_2,0),j_2}} - \frac{\partial}{\partial z'_{(k_1,0),j_2}} \frac{\partial}{\partial z'_{(k_2,0),j_1}} \\ & \quad (0 \leq k_1, k_2 \leq n-2; 1 \leq j_1, j_2 \leq m), \\ & \sum_{i=0}^{n-2} z'_{(i,0),j_1} \frac{\partial}{\partial z'_{(i,0),j_2}} + \delta_{j_1 j_2} \quad (1 \leq j_1, j_2 \leq m), \\ & \sum_{i=0}^{n-k-2} \sum_{j=1}^m z'_{(i,0),j} \frac{\partial}{\partial z'_{(i+k,0),j}} + \beta_{(k,0)} \quad (k = 1, \dots, n-2), \\ & \frac{\partial}{\partial z'_{(0,1),j}} \quad (j = 1, \dots, m). \end{aligned}$$

By the change f' of coordinates

$$\begin{aligned} z'_{(i,0),j} &= z_{(i,0),j} \quad (i = 0, 1, \dots, n-2) \\ z'_{(n-2,0),j} &= z_{(n-2,0),j} + z_{(0,1),j} \\ z'_{(0,1),j} &= z_{(0,1),j}, \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial}{\partial z'_{(i,0),j}} &= \frac{\partial}{\partial z_{(i,0),j}} \quad (i = 0, 1, \dots, n-2) \\ \frac{\partial}{\partial z'_{(0,1),j}} &= \frac{\partial}{\partial z_{(0,1),j}} - \frac{\partial}{\partial z_{(n-2,0),j}}. \end{aligned}$$

Taking into account that we have

$$\frac{\partial}{\partial z_{(n-2,0),j}} - \frac{\partial}{\partial z_{(0,1),j}} \in J_{(n-1,1),\alpha}$$

by Lemma 8.1, we see $f^* M_{(n-1),\beta} = f'^* p_{n-1}^* M_{(n-1),\beta} = M_{(n-1,1),\alpha}$. $\square$

COROLLARY 8.7. *Let $\alpha = {}^t(0, \dots, 0, 1, 1) \in \mathbb{C}^{n-1}$ and $\beta = {}^t(0, \dots, 0, 1) \in \mathbb{C}^{n-2}$. Then we have a bijection between the space of solutions to $M_{(n-1,1),\alpha}$ and that to $M_{(n-1),\beta}$.*

PROOF. Let f be (34). By Theorem 8.6, we have an isomorphism

$$\begin{aligned} f^{-1} \text{Hom}_{\mathcal{D}_Y}(M_{(n-1),\beta}, \mathcal{O}_Y) &\simeq \text{Hom}_{\mathcal{D}_Y}(f^* M_{(n-1),\beta}, f^* \mathcal{O}_Y) \\ &= \text{Hom}_{\mathcal{D}_Y}(M_{(n-1,1),\alpha}, \mathcal{O}_Z). \end{aligned}$$

More explicitly, $\Phi(f(Z))$ is a solution to $M_{(n-1,1),\alpha}$ for a solution Φ to $M_{(n-1),\beta}$. Conversely any solution to $M_{(n-1,1),\alpha}$ is of this form. $\square$

CHAPTER 9

$p = 2$ case; the normalizer of $\mathfrak{a}$

In this chapter, we consider the normalizer of $\mathfrak{a}$ in the case of $p = 2$.

Let (e_1, e_2) with $e_i \in \mathfrak{g}$ be a principal nilpotent pair, and set $L = \{\mathbf{l} = (l_1, l_2) \in \mathbb{N}^2 \mid e_1^{l_1} e_2^{l_2} \neq 0\}$ and $\mathfrak{a} = \langle e_1^{l_1} e_2^{l_2} \mid (l_1, l_2) \in L \rangle$. We say $L \subseteq \mathbb{N}^2$ to be asymmetric if there exists $(l_1, l_2) \in L$ such that $(l_2, l_1) \notin L$.

Let $g \in N_G(\mathfrak{a}_+)/Z_G(\mathfrak{a}_+)$. Then there exists $c = (c_{\mathbf{l}}^{(i)} : 1 \leq i \leq p; \mathbf{l} \in L_+)$ satisfying the conditions

- (a) $\det[c_{1_j}^{(i)}]_{1 \leq i, j \leq p} \neq 0$,
- (b) $\phi_{\mathbf{l}', \mathbf{l}}(c) = 0$ for every $\mathbf{l} \notin L_+, \mathbf{l}' \in L_+$

such that

$$g = [\phi_{\mathbf{l}', \mathbf{l}}(c)]_{\mathbf{l}', \mathbf{l} \in L}$$

by Proposition 7.22.

LEMMA 9.1. *Let $c = (c_{\mathbf{l}}^{(i)} : 1 \leq i \leq p; \mathbf{l} \in L_+)$ satisfy the conditions (a) and (b).*

- (1) *If $L \neq \{(l_1, l_2) \in \mathbb{N}^2 \mid l_1 + l_2 \leq m\}$ for any $m \in \mathbb{N}$, then we have $c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} = 0$ or $c_{(1,0)}^{(2)} c_{(0,1)}^{(1)} = 0$.*
- (2) *If L is asymmetric, then we have $c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} \neq 0$.*

PROOF. (1) Let $m = \max\{|\mathbf{l}| \mid \mathbf{l} \in L\}$. By the assumption, there exist $\mathbf{l}, \mathbf{l}' \in \mathbb{N}^2$ with $|\mathbf{l}| = |\mathbf{l}'| = m$ such that $\mathbf{l} \in L, \mathbf{l}' \notin L$. If $(m, 0) \in L$, then $\phi_{(m,0), \mathbf{l}'}(c) = (c_{(1,0)}^{(1)})^{l'_1} (c_{(1,0)}^{(2)})^{l'_2} = 0$. If $(m, 0) \notin L$, then $\phi_{\mathbf{l}, (m,0)}(c) = (c_{(1,0)}^{(1)})^{l_1} (c_{(0,1)}^{(1)})^{l_2} = 0$. Thus if $c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} \neq 0$, we have $c_{(1,0)}^{(2)} = 0$ or $c_{(0,1)}^{(1)} = 0$.

(2) By the assumption, there exists $(l_1, l_2) \in L$ such that $(l_2, l_1) \notin L$. Then we have $\phi_{(l_1, l_2), (l_2, l_1)}(c) = 0$. By (1), we have $c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} = 0$ or $c_{(1,0)}^{(2)} c_{(0,1)}^{(1)} = 0$. If $c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} = 0$, we have $\phi_{(l_1, l_2), (l_2, l_1)}(c) = (c_{(1,0)}^{(2)})^{l_1} (c_{(0,1)}^{(1)})^{l_2} = 0$. This contradicts the fact that $\det[c_{1_j}^{(i)}]_{1 \leq i, j \leq 2} \neq 0$. □

We denote by $\text{Inn}(L)$ the set of inner corners of L and by $\text{Out}(L)$ the set of outer corners of L . Define $D_1(L), D_2(L)$ by

$$\begin{aligned} D_1(L) &= \{l_1 - l'_1 \mid (l_1, l_2) \in \text{Inn}(L), (l'_1, l'_2) \in \text{Out}(L), l_2 - l'_2 = -1\}, \\ D_2(L) &= \{l_2 - l'_2 \mid (l_1, l_2) \in \text{Inn}(L), (l'_1, l'_2) \in \text{Out}(L), l_1 - l'_1 = -1\}. \end{aligned}$$

Put $d_i = \max D_i(L)$ for $i = 1, 2$.

PROPOSITION 9.2. *Let $c = (c_{\mathbf{l}}^{(i)} : 1 \leq i \leq p; \mathbf{l} \in L_+)$ satisfy $c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} \neq 0$ and $\phi_{\mathbf{l}', \mathbf{l}}(c) = 0$ for every $\mathbf{l} \notin L_+, \mathbf{l}' \in L_+$. Then $(c_{\mathbf{l}}^{(i)} : 1 \leq i \leq p; \mathbf{l} \in L_+)$ satisfies the following:*

(1) If $d_1 > 0$, then we have

$$c_{(1,0)}^{(2)} = c_{(2,0)}^{(2)} = \cdots = c_{(d_1,0)}^{(2)} = 0.$$

(2) If $d_2 > 0$, then we have

$$c_{(0,1)}^{(1)} = c_{(0,2)}^{(1)} = \cdots = c_{(0,d_2)}^{(1)} = 0.$$

PROOF. We assume $d_1 > 0$. Since $d_1 > 0$, there exist $\mathbf{l} \in \text{Inn}(L)$, $\mathbf{l}' \in \text{Out}(L)$ which satisfy $l_2 - l'_2 = -1$ and $l_1 - l'_1 > 0$. Let $s = l_1 - l'_1$ and $\mathbf{m} = (l'_1 + 1, l_2)$. Since $\mathbf{l}' \in \text{Out}(L)$, we have $\mathbf{m} + (i, 0) \in L$ for $i = 0, 1, \dots, s-1$. We show $c_{(1,0)}^{(2)} = c_{(2,0)}^{(2)} = \cdots = c_{(s,0)}^{(2)} = 0$ by induction.

First, we show $c_{(1,0)}^{(2)} = 0$. Since $c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} \neq 0$, we have $c_{(1,0)}^{(2)} c_{(0,1)}^{(1)} = 0$ by Lemma 9.1 (1). Then $\phi_{\mathbf{m}, \mathbf{l}'}(c) = (c_{(1,0)}^{(1)})^{l'_1} c_{(1,0)}^{(2)} (c_{(0,1)}^{(2)})^{l_2}$. Thus $c_{(1,0)}^{(2)} = 0$ by $\phi_{\mathbf{m}, \mathbf{l}'}(c) = 0$.

Next, we assume $c_{(1,0)}^{(2)} = c_{(2,0)}^{(2)} = \cdots = c_{(i,0)}^{(2)} = 0$. By Lemma 7.6 (4) we have

$$\phi_{\mathbf{m}+(i,0), \mathbf{l}'}(c) = \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{m}+(i,0)} \phi_{\mathbf{k}_1, (\mathbf{l}'_1, 0)}(c) \phi_{\mathbf{k}_2, (0, l'_2)}(c).$$

Since $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{m} + (i, 0)$ and Lemma 7.6 (3), $\mathbf{k}_2$ satisfies that $l'_2 \leq |\mathbf{k}_2| \leq l'_2 + i$ and $k_{2,2} \leq l_2$.

Let $\mathbf{u}_1, \dots, \mathbf{u}_{l'_2} \in \mathbb{N}^2$ satisfy $\mathbf{u}_1 + \cdots + \mathbf{u}_{l'_2} = \mathbf{k}_2$. Since $k_{2,2} \leq l_2 = l'_2 - 1$, there exists $1 \leq i \leq l'_2$ such that $u_i = (j, 0)$ for some $j \in \mathbb{N}$. Let $a = |\mathbf{k}_2| - l'_2$. Then we have $1 \leq j \leq 1 + a$. If $j = 1 + a$, then there exists a unique i satisfies $\mathbf{u}_i = (1 + a, 0)$, and then $\mathbf{u}_{i'} = \mathbf{1}_2$ for all $i' \neq i$.

Then, since

$$\phi_{\mathbf{k}_2, (0, l'_2)}(c) = \sum_{\mathbf{u}_1 + \cdots + \mathbf{u}_{l'_2} = \mathbf{k}_2} c_{\mathbf{u}_1}^{(2)} \cdots c_{\mathbf{u}_{l'_2}}^{(2)},$$

we have

$$\phi_{\mathbf{k}_2, (0, l'_2)}(c) = 0$$

for all $\mathbf{k}_2 \neq (i+1, l_2)$ and

$$\phi_{(i+1, l_2), (0, l'_2)}(c) = c_{(i+1,0)}^{(2)} (c_{(0,1)}^{(2)})^{l_2}$$

by the assumption. Thus

$$\begin{aligned} \phi_{\mathbf{m}+(i,0), \mathbf{l}'}(c) &= \phi_{(\mathbf{l}'_1, 0), (\mathbf{l}'_1, 0)}(c) \phi_{(i+1, l_2), (0, l'_2)}(c) \\ &= (c_{(1,0)}^{(1)})^{l'_1} c_{(i+1,0)}^{(2)} (c_{(0,1)}^{(2)})^{l_2} = 0, \end{aligned}$$

and we have $c_{(i+1,0)}^{(2)} = 0$.

Similarly, if $d_2 > 0$, then we have $c_{(0,1)}^{(1)} = c_{(0,2)}^{(1)} = \cdots = c_{(0,d_2)}^{(1)} = 0$. □

THEOREM 9.3. Assume $d_i = \max D_i(L) > 0$ for $i = 1, 2$. Furthermore, assume that $(c_{\mathbf{l}}^{(i)} : 1 \leq i \leq 2; \mathbf{l} \in L_+)$ satisfies $c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} \neq 0$,

$$c_{(1,0)}^{(2)} = c_{(2,0)}^{(2)} = \cdots = c_{(d_1,0)}^{(2)} = 0,$$

and

$$c_{(0,1)}^{(1)} = c_{(0,2)}^{(1)} = \cdots = c_{(0,d_2)}^{(1)} = 0.$$

Then we have $[\phi_{\mathbf{l}', \mathbf{l}}(c)]_{\mathbf{l}', \mathbf{l} \in L_+} \in N_G(\mathfrak{a}_+)/Z_G(\mathfrak{a}_+)$.

In order to prove Theorem 9.3, we show two lemmas.

LEMMA 9.4. *For any $n, m \in \mathbb{N}$, we have*

$$\max_{\substack{(\lambda_1, \dots, \lambda_m), \\ \lambda_1 + \dots + \lambda_m = n}} (\min_i \lambda_i) = \left\lfloor \frac{n}{m} \right\rfloor.$$

PROOF. Put $k = \left\lfloor \frac{n}{m} \right\rfloor$. We have

$$n = mk + r \quad (0 \leq r < m).$$

Let $(\lambda_1, \dots, \lambda_m)$ be

$$\lambda_i = \begin{cases} k+1 & (1 \leq i \leq r) \\ k & (r+1 \leq i \leq m). \end{cases}$$

Then we have $\min_i \lambda_i = k$. On the other hand, if $\min_i \lambda_i > k$, we have

$$\lambda_1 + \dots + \lambda_m \geq m(k+1) > n.$$

This is a contradiction. □

LEMMA 9.5. *Let $\mathbf{l} \in \text{Inn}(L)$, $\mathbf{l}' \in \text{Out}(L)$.*

(1) *If $l_2 < l'_2$, then*

$$\left\lfloor \frac{l_1 - l'_1}{l'_2 - l_2} \right\rfloor \leq \max D_1(L).$$

(2) *If $l_1 < l'_1$, then*

$$\left\lfloor \frac{l_2 - l'_2}{l'_1 - l_1} \right\rfloor \leq \max D_2(L).$$

PROOF. We show (1). By the definitions of inner corners and outer corners, there exist $\mathbf{k}_1, \dots, \mathbf{k}_m \in \text{Inn}(L)$ and $\mathbf{k}'_1, \dots, \mathbf{k}'_m \in \text{Out}(L)$ such that $\mathbf{k}_1 = \mathbf{l}$, $\mathbf{k}'_m = \mathbf{l}'$ and $k_{i,2} - k'_{i,2} = -1$ for any i . Since $m \leq l'_2 - l_2$, we have

$$\begin{aligned} \frac{l_1 - l'_1}{l'_2 - l_2} &\leq \frac{l_1 - l'_1}{m} \\ &= \frac{1}{m} \sum_{i=1}^m (k_{i,1} - k'_{i,1}) \\ &\leq \frac{1}{m} m \max D_1(L) = \max D_1(L). \end{aligned}$$

□

PROOF OF THEOREM 9.3. We show $\phi_{\mathbf{l}, \mathbf{l}'}(c) = 0$ for all $\mathbf{l} \in L$, $\mathbf{l}' \in \text{Out}(L)$.

Fix $\mathbf{l}' \in \text{Out}(L)$. By Lemma 7.6 (3), it is sufficient to show $\phi_{\mathbf{l}, \mathbf{l}'}(c) = 0$ for all $\mathbf{l} \in L$ satisfying $|\mathbf{l}| \geq |\mathbf{l}'|$. If $l_2 = l'_2$, we have $l_1 < l'_1$. Then we have $|\mathbf{l}| \leq |\mathbf{l}'|$. Thus we consider the case of $l_2 < l'_2$ and that of $l_2 > l'_2$. By Lemma 7.6 (4),

$$(35) \quad \phi_{\mathbf{l}, \mathbf{l}'}(c) = \sum_{\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{l}} \phi_{\mathbf{k}_1, (l'_1, 0)}(c) \phi_{\mathbf{k}_2, (0, l'_2)}(c).$$

First, we assume $l_2 < l'_2$. Since $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{l}$, we have

$$l'_2 \leq |\mathbf{k}_2| \leq l'_2 + |\mathbf{l}| - |\mathbf{l}'|, \quad k_{2,2} \leq l_2.$$

Let $\mathbf{u}_1, \dots, \mathbf{u}_{l'_2} \in \mathbb{N}^2$ satisfy $\mathbf{u}_1 + \dots + \mathbf{u}_{l'_2} = \mathbf{k}_2$. Since $k_{2,2} \leq l_2 < l'_2$, for any $j = 1, \dots, l'_2 - k_{2,2}$, there exist $\mathbf{u}_{i_1}, \dots, \mathbf{u}_{i_{l'_2 - k_{2,2}}}$ such that $\mathbf{u}_{i_j} = (s_j, 0)$ for some $s_j \in \mathbb{N}$. We have

$$s_1 + \dots + s_{l'_2 - k_{2,2}} \leq k_{2,1},$$

and then there exists j such that

$$s_j \leq \left\lfloor \frac{k_{2,1}}{l'_2 - k_{2,2}} \right\rfloor, \quad 1 \leq j \leq l'_2 - k_{2,2}$$

by Lemma 9.4.

Since $l'_2 \leq |\mathbf{k}_2| \leq l'_2 + |\mathbf{l}| - |\mathbf{l}'|$ and $k_{2,2} \leq l_2$, we have

$$\begin{aligned} \frac{k_{2,1}}{l'_2 - k_{2,2}} &= \frac{|\mathbf{k}_2| - k_{2,2}}{l'_2 - k_{2,2}} \\ &\leq \frac{l'_2 + |\mathbf{l}| - |\mathbf{l}'| - k_{2,2}}{l'_2 - k_{2,2}} \\ &= 1 + \frac{|\mathbf{l}| - |\mathbf{l}'|}{l'_2 - k_{2,2}} \\ &\leq 1 + \frac{|\mathbf{l}| - |\mathbf{l}'|}{l'_2 - l_2} \\ &= \frac{l_1 - l'_1}{l'_2 - l_2}. \end{aligned}$$

Since $\mathbf{l} \in L$, there exists $\mathbf{l}'' \in \text{Inn}(L)$ such that $l_i \leq l''_i$ for $i = 1, 2$. Then we have

$$s_j \leq \left\lfloor \frac{l_1 - l'_1}{l'_2 - l_2} \right\rfloor \leq \left\lfloor \frac{l''_1 - l'_1}{l'_2 - l''_2} \right\rfloor \leq d_1$$

by Lemma 9.5.

By the assumption, we have $\phi_{\mathbf{k}_2, (0, l'_2)}(c) = 0$ for any $l'_2 \leq |\mathbf{k}_2| \leq l'_2 + |\mathbf{l}| - |\mathbf{l}'|$, $k_{2,2} \leq l_2$. Hence we have proved $\phi_{\mathbf{l}, \mathbf{l}'}(c) = 0$ for any $\mathbf{l} \in L$, $\mathbf{l}' \in \text{Out}(L)$ satisfying $l_2 < l'_2$.

Next, we assume $l_2 > l'_2$. By the definition of outer corner, we have $l_1 < l'_1$. Then we similarly show $\phi_{\mathbf{l}, \mathbf{l}'}(c) = 0$ by Lemma 9.4 and Lemma 9.5. $\square$

PROPOSITION 9.6. *Let λ be a Young diagram and put $d_i = \max D_i(L)$. Then we have the following:*

(1) *If L_λ is asymmetric, then we have*

$$N_G(\mathfrak{a}_\lambda)/Z_G(\mathfrak{a}_\lambda) = \left\{ [\phi_{\mathbf{l}, \mathbf{l}'}(c)]_{\mathbf{l}, \mathbf{l}' \in L_\lambda} \mid \begin{array}{l} c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} \neq 0, \quad c_{(i,0)}^{(2)} = 0 \quad (1 \leq i \leq d_1) \\ c_{(0,j)}^{(1)} = 0 \quad (1 \leq j \leq d_2) \end{array} \right\}.$$

(2) *If L_λ is not asymmetric and $L_\lambda \neq \{\mathbf{l} \in \mathbb{N}^2 \mid |\mathbf{l}| \leq q\}$ for any $q \in \mathbb{N}$, then we have*

$$N_G(\mathfrak{a}_\lambda)/Z_G(\mathfrak{a}_\lambda) = \left\{ [\phi_{\mathbf{l}, \mathbf{l}'}(c)]_{\mathbf{l}, \mathbf{l}' \in L_\lambda}, [g\phi_{\mathbf{l}, \mathbf{l}'}(c)]_{\mathbf{l}, \mathbf{l}' \in L_\lambda} \mid \begin{array}{l} c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} \neq 0, \quad c_{(i,0)}^{(2)} = 0 \quad (1 \leq i \leq d) \\ c_{(0,j)}^{(1)} = 0 \quad (1 \leq j \leq d) \end{array} \right\},$$

where $d = d_1$ and $g \in GL(V)$ satisfies

$$\text{Ad}(g)e_1 = e_2, \quad \text{Ad}(g)e_2 = e_1.$$

PROOF. (1) Let L_λ be asymmetric. We assume $[\phi_{\mathbf{l}, \mathbf{l}'}(c)]_{\mathbf{l}, \mathbf{l}' \in L_\lambda} \in N_G(\mathfrak{a}_\lambda)/Z_G(\mathfrak{a}_\lambda)$.

Then we have $c_{(1,0)}^{(1)}, c_{(0,1)}^{(2)} \neq 0$ by Lemma 9.1(2) Thus

$$[\phi_{\mathbf{l}, \mathbf{l}'}(c)]_{\mathbf{l}, \mathbf{l}' \in L_\lambda} \in N_G(\mathfrak{a}_\lambda)/Z_G(\mathfrak{a}_\lambda) \Leftrightarrow \begin{array}{l} c_{(i,0)}^{(2)} = 0 \quad (1 \leq i \leq d_1), \\ c_{(0,j)}^{(1)} = 0 \quad (1 \leq j \leq d_2) \end{array}$$

by Proposition 9.2 and Theorem 9.3.

- (2) By the assumption, we have $d_1 = d_2 > 0$. Let $d = d_1$. Moreover we have $c_{(1,0)}^{(1)}c_{(0,1)}^{(2)} = 0$ or $c_{(1,0)}^{(2)}c_{(0,1)}^{(1)} = 0$ by Lemma 9.1(1). If $c_{(1,0)}^{(2)}c_{(0,1)}^{(1)} = 0$, then we have $c_{(1,0)}^{(1)}c_{(0,1)}^{(2)} \neq 0$ and thus

$$\begin{aligned} c_{(i,0)}^{(2)} &= 0 \quad (1 \leq i \leq d), \\ c_{(0,j)}^{(1)} &= 0 \quad (1 \leq j \leq d) \end{aligned}$$

by Proposition 9.2.

If $c_{(1,0)}^{(1)}c_{(0,1)}^{(2)} = 0$, then we have $c_{(1,0)}^{(2)}c_{(0,1)}^{(1)} \neq 0$. Let $g \in G$ satisfy $\text{Ad}(g)e_1 = e_2$, $\text{Ad}(g)e_2 = e_1$. Then we have

$$\begin{aligned} (36) \quad \text{Ad}(g[\phi_{\mathbf{l},\mathbf{l}'}(c)])e_1^{l_1}e_2^{l_2} &= \text{Ad}(g)\text{Ad}([\phi_{\mathbf{l},\mathbf{l}'}(c)])e_1^{l_1}e_2^{l_2} \\ &= \text{Ad}(g) \sum_{\mathbf{k} \in L_\lambda^+} \phi_{\mathbf{k},(l_1,l_2)} e_1^{l_1}e_2^{l_2} \\ &= \sum_{\mathbf{k} \in L_\lambda^+} \phi_{\mathbf{k},(l_1,l_2)} e_1^{l_2}e_2^{l_1}. \end{aligned}$$

Thus $g[\phi_{\mathbf{l},\mathbf{l}'}(c)] \in N_G(\mathfrak{a})/Z_G(\mathfrak{a})$, and there exists $c' = (c_{\mathbf{l}}'^{(i)})_{i=1,2, \mathbf{l} \in L_\lambda}$ such that $g[\phi_{\mathbf{l},\mathbf{l}'}(c)] = [\phi_{\mathbf{l},\mathbf{l}'}(c')]$. By (36), $c_{\mathbf{l}}'^{(1)} = c_{\mathbf{l}}^{(2)}$, $c_{\mathbf{l}}'^{(2)} = c_{\mathbf{l}}^{(1)}$ for any $\mathbf{l} \in L_\lambda$. Then c' satisfies $c_{(1,0)}'^{(1)}c_{(0,1)}'^{(2)} \neq 0$, and we have $c_{(i,0)}'^{(2)} = 0$ ($1 \leq i \leq d$), $c_{(0,j)}'^{(1)} = 0$ ($1 \leq j \leq d$) by Proposition 9.2. Hence we have $[\phi_{\mathbf{l},\mathbf{l}'}(c)] = g[\phi_{\mathbf{l},\mathbf{l}'}(c')]$ which satisfies $c_{(1,0)}'^{(1)}c_{(0,1)}'^{(2)} \neq 0$, $c_{(i,0)}'^{(2)} = 0$ ($1 \leq i \leq d$) and $c_{(0,j)}'^{(1)} = 0$ ($1 \leq j \leq d$).

□

CHAPTER 10

$p = 2$ and $m = 2$

In this chapter, we consider confluent hypergeometric systems associated with principal nilpotent pairs with $m = 2$. For any Young diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$, there exists a principal nilpotent pair (e_1, e_2) which corresponds to λ . Put $\mathfrak{a}_\lambda = \langle e_1^{l_1} e_2^{l_2} | \mathbf{l} \in \mathbb{N}^2 \rangle$ and $L_\lambda = \{\mathbf{l} | e^\mathbf{l} \neq 0\}$. Then we have

$$L_\lambda = \{(l_1, l_2) \in \mathbb{N}^2 | 0 \leq l_2 \leq s-1, 0 \leq l_1 \leq \lambda_{l_2} - 1\}.$$

We denote $M_{\mathfrak{a}_\lambda, \alpha}$ simply by $M_{\lambda, \alpha}$ as in Chapter 8.

We have already considered the case of $\lambda = (n-1, 1)$ in Chapter 8. This classical counterpart coincides with the system of generalized Airy functions by Theorem 8.6. Here we describe the classical counterpart of $M_{\lambda, \alpha}$ by Risa/Asir [11]. First, we prove the next proposition.

PROPOSITION 10.1. *Suppose $L \neq \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Then we have*

$$[z_{i,j}]_{i=1,2,\dots,n, j=1,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & x_1 \\ \vdots & \vdots \\ 0 & x_{n-3} \end{bmatrix}$$

in $\exp(\mathfrak{a}) \backslash Z/GL(m)$.

PROOF. First, we have

$$[z_{i,j}]_{i=1,\dots,n, j=1,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ W_{n-2} \end{bmatrix} \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix},$$

where

$$W_{n-2} = \begin{bmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \\ \vdots & \vdots \\ w_{n-2,1} & w_{n-2,2} \end{bmatrix} = \begin{bmatrix} z_{3,1} & z_{3,2} \\ z_{4,1} & z_{4,2} \\ \vdots & \vdots \\ z_{n,1} & z_{n,2} \end{bmatrix} \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix}^{-1}.$$

Since

$$\exp(c_{(1,0)}e_1) = \begin{bmatrix} 1 & & & \\ c_{(1,0)} & 1 & & \\ \frac{1}{2}c_{(1,0)}^2 & c_{(1,0)} & 1 & \\ & & & \ddots \end{bmatrix}$$

with respect to the order $(0, 0), (1, 0), (2, 0), \dots$, we have

$$\exp(-w_{3,2}e_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ w_{1,1} & w_{1,2} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -w_{1,2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ w'_{1,1} & 0 \\ \vdots & \vdots \end{bmatrix},$$

where $c_{(1,0)} = -w_{3,2}$ and $w'_{1,1} = w_{1,1} - \frac{1}{2}w_{1,2}^2$.

Next, since

$$\exp(c_{(2,0)}e_1^2) = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ c_{(2,0)} & 0 & 1 & \\ & & & \ddots \end{bmatrix}$$

with respect to the order $(0, 0), (1, 0), (2, 0), \dots$, we have

$$\exp(c_{(2,0)}e_1^2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ w'_{1,1} & 0 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix},$$

where $c_{(2,0)} = -w'_{1,1}$.

Similarly, we can determine c_l for any $l \in L$ such that

$$\prod_{l \in L} \exp(c_l e^l) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ W_{n-2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -w_{1,2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & x_1 \\ \vdots & \vdots \\ 0 & x_3 \end{bmatrix}.$$

□

1. $n = 4$ case

First, we consider $\lambda = (3, 1)$ case. By Proposition 8.5 and Proposition 10.1, it is sufficient to consider the restriction of $M_{(3,1),\alpha}$ with $\alpha = (-2, 0, 1, 1)$ on

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & x \end{bmatrix}.$$

Then the restriction of $M_{(3,1),\alpha}$ is

$$(\partial + x)u = 0$$

and a solution of this equation is $u = \exp(-\frac{x}{2})$.

Next, we consider the system $M_{(2,2),\alpha}$. Then $L_\lambda = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$.

Since

$$\exp(c_1 e_1) \exp(-w_4 e_2) \exp(-w_3 e_1 e_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ w_1 & w_2 \\ w_3 & w_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x \\ 0 & 0 \end{bmatrix},$$

where $c_1 = \frac{w_4 - w_1}{w_2}$, $x = w_2$, we have

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x \\ 0 & 0 \end{bmatrix}$$

in $\exp(\mathfrak{a}_{(2,2)}) \backslash Z/GL(2)$.

PROPOSITION 10.2. *Let $\Phi(z; \alpha)$ be a solution to $M_{(2,2), \alpha}$. We assume that a parameter vector*

$$\alpha = {}^t(\alpha_{(1,0)}, \alpha_{(0,1)}, \alpha_{(1,1)}) \in \mathbb{C}^{L+}$$

of $\Phi(z; \alpha)$ satisfies $\alpha_{(1,1)} \neq 0$. Then there exists $g \in N_G(\mathfrak{a})$ such that the change of coordinates $z \mapsto z' = gz$ transforms $\Phi(z; \alpha)$ into $\Phi(z'; \alpha')$ with the parameter vector

$$\alpha' = {}^t(0, 0, 1).$$

PROOF. By Proposition 9.6, there exists $g \in N_G(\mathfrak{a})$ such that

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{(1,0)}^{(1)} & 0 & c_{(1,1)}^{(1)} \\ 0 & 0 & c_{(0,1)}^{(2)} & c_{(1,1)}^{(2)} \\ 0 & 0 & 0 & c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} \end{bmatrix}$$

with respect to the order $(0,0), (1,0), (0,1), (1,1)$, where $c_{(1,0)}^{(1)} \neq 0$, $c_{(0,1)}^{(2)} \neq 0$ and $c = (c_{(1,0)}^{(1)}, c_{(1,1)}^{(1)}, c_{(0,1)}^{(2)}, c_{(1,1)}^{(2)}) \in \mathbb{C}^4$.

We have

$$(37) \quad \alpha'_{(1,0)} = ({}^t g \alpha)_{(1,0)} = c_{(1,0)}^{(1)} \alpha_{(1,0)} + c_{(1,1)}^{(1)} \alpha_{(1,1)},$$

$$(38) \quad \alpha'_{(0,1)} = ({}^t g \alpha)_{(0,1)} = c_{(0,1)}^{(2)} \alpha_{(0,1)} + c_{(1,1)}^{(2)} \alpha_{(1,1)},$$

$$(39) \quad \alpha'_{(1,1)} = ({}^t g \alpha)_{(1,1)} = c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} \alpha_{(1,1)}.$$

Since $\alpha_{(1,1)} \neq 0$, by (39), we can determine $c_{(1,0)}^{(1)}$ and $c_{(0,1)}^{(2)}$, so that $\alpha'_{(1,1)} = 1$.

By (37) and (38), we can determine $c_{(1,1)}^{(1)}$ and $c_{(1,1)}^{(2)}$ so that $\alpha'_{(1,0)} = \alpha'_{(0,1)} = 0$. $\square$

The restriction of $M_{(2,2), \alpha}$ with $\alpha = (-2, 0, 0, 1)$ on

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & x \\ 0 & 0 \end{bmatrix}$$

is

$$\left(2x \frac{d}{dx} + 1\right) u = 0,$$

and a solution of this equation is $u = \frac{1}{\sqrt{x}}$.

REMARK 10.3. In the case of $n = 4$, the classical counterparts were also calculated in [8].

2. $n = 5$ case

We consider the systems $M_{\lambda,\alpha}$, where $\lambda = (3, 2)$ and $\lambda = (3, 1, 1)$. By Proposition 10.1, we consider the restriction of $M_{\mathbf{a},\alpha}$ on

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & x_1 \\ 0 & x_2 \end{bmatrix}.$$

First we treat the case of $\lambda = (3, 2)$.

PROPOSITION 10.4. *Let $\Phi(z; \alpha)$ be a solution to $M_{(3,2),\alpha}$. We assume that a parameter vector*

$$\alpha = {}^t(\alpha_{(1,0)}, \alpha_{(0,1)}, \alpha_{(2,0)}, \alpha_{(1,1)}) \in \mathbb{C}^{L+}$$

of $\Phi(z; \alpha)$ satisfies $\alpha_{(1,1)} \neq 0$. Then there exists $g \in N_G(\mathbf{a})$ such that the change of coordinates $z \mapsto z' = gz$ transforms $\Phi(z; \alpha)$ into $\Phi(z'; \alpha')$ with the parameter vector

$$\alpha' = {}^t(0, 0, 0, 1).$$

PROOF. By Proposition 9.6, there exists $g \in N_G(\mathbf{a})$ such that

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & c_{(1,0)}^{(1)} & 0 & 0 & 0 \\ 0 & c_{(0,1)}^{(1)} & c_{(0,1)}^{(2)} & 0 & 0 \\ 0 & c_{(2,0)}^{(1)} & c_{(2,0)}^{(2)} & (c_{(1,0)}^{(1)})^2 & 0 \\ 0 & c_{(1,1)}^{(1)} & c_{(1,1)}^{(2)} & 2c_{(1,0)}^{(1)}c_{(0,1)}^{(1)} & c_{(1,0)}^{(1)}c_{(0,1)}^{(2)} \end{bmatrix}$$

with respect to the order $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1)$, where $c_{(1,0)}^{(1)} \neq 0, c_{(0,1)}^{(2)} \neq 0$.

We have

$$(40) \quad \alpha'_{(1,0)} = ({}^t g \alpha)_{(1,0)} = c_{(1,0)}^{(1)} \alpha_{(1,0)} + c_{(0,1)}^{(1)} \alpha_{(0,1)} + c_{(2,0)}^{(1)} \alpha_{(2,0)} + c_{(1,1)}^{(1)} \alpha_{(1,1)},$$

$$(41) \quad \alpha'_{(0,1)} = ({}^t g \alpha)_{(0,1)} = c_{(0,1)}^{(2)} \alpha_{(0,1)} + c_{(2,0)}^{(2)} \alpha_{(2,0)} + c_{(1,1)}^{(2)} \alpha_{(1,1)},$$

$$(42) \quad \alpha'_{(2,0)} = ({}^t g \alpha)_{(2,0)} = (c_{(1,0)}^{(1)})^2 \alpha_{(2,0)} + 2c_{(1,0)}^{(1)} c_{(0,1)}^{(1)} \alpha_{(0,1)} \alpha_{(1,1)},$$

$$(43) \quad \alpha'_{(1,1)} = ({}^t g \alpha)_{(1,1)} = c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} \alpha_{(1,1)}.$$

Since $\alpha_{(1,1)} \neq 0$, by (43) we can determine $c_{(1,0)}^{(1)}$ and $c_{(0,1)}^{(2)}$ so that $\alpha'_{(1,1)} = 1$.

By (40), (41), (42), we can determine $c_{(0,1)}^{(1)}, c_{(2,0)}^{(1)}, c_{(1,1)}^{(1)}, \dots, c_{(1,1)}^{(2)}$ so that $\alpha'_{(1,0)} = \alpha'_{(0,1)} = \alpha'_{(2,0)} = 0$. $\square$

The restriction of $M_{(3,2),\alpha}$ with $\alpha = {}^t(0, 0, 0, 1)$ on

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & x_1 \\ 0 & x_2 \end{bmatrix}$$

is

$$\partial_1 - \partial_2^2, \quad 2x_1 \partial_2 + x_2, \quad 2x_1 \partial_2^2 + x_2 \partial_2 + 1,$$

and a solution of this system is $f(x_1, x_2) = \frac{1}{\sqrt{x_1}} \exp\left(-\frac{x_2^2}{4x_1}\right)$.

Next, we treat the case of $\lambda = (3, 1, 1)$.

PROPOSITION 10.5. *Let $\Phi(z; \alpha)$ be a solution to $M_{(3,1,1),\alpha}$. We assume that a parameter vector*

$$\alpha = {}^t(\alpha_{(1,0)}, \alpha_{(0,1)}, \alpha_{(2,0)}, \alpha_{(0,2)}) \in \mathbb{C}^{L+}$$

of $\Phi(z; \alpha)$ satisfies $\alpha_{(2,0)} \neq 0$ and $\alpha_{(0,2)} \neq 0$. Then there exists $g \in N_G(\mathfrak{a})$ such that the change of coordinates $z \mapsto z' = gz$ transforms $\Phi(z; \alpha)$ into $\Phi(z'; \alpha')$ with the parameter vector

$$\alpha' = {}^t(0, 0, 1, 1).$$

PROOF. By Proposition 9.6, there exists $g \in N_G(\mathfrak{a})$ such that

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & c_{(1,0)}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & c_{(0,1)}^{(2)} & 0 & 0 \\ 0 & c_{(2,0)}^{(1)} & c_{(2,0)}^{(2)} & (c_{(1,0)}^{(1)})^2 & 0 \\ 0 & c_{(0,2)}^{(1)} & c_{(0,2)}^{(2)} & 0 & (c_{(0,1)}^{(2)})^2 \end{bmatrix}$$

with respect to the order $(0, 0), (1, 0), (0, 1), (2, 0), (0, 2)$, where $c_{(1,0)}^{(1)} \neq 0, c_{(0,1)}^{(2)} \neq 0$.

We have

$$(44) \quad \alpha'_{(1,0)} = ({}^t g \alpha)_{(1,0)} = c_{(1,0)}^{(1)} \alpha_{(1,0)} + c_{(2,0)}^{(1)} \alpha_{(2,0)} + c_{(0,2)}^{(1)} \alpha_{(0,2)},$$

$$(45) \quad \alpha'_{(0,1)} = ({}^t g \alpha)_{(0,1)} = c_{(0,1)}^{(2)} \alpha_{(0,1)} + c_{(2,0)}^{(2)} \alpha_{(2,0)} + c_{(0,2)}^{(2)} \alpha_{(0,2)},$$

$$(46) \quad \alpha'_{(2,0)} = ({}^t g \alpha)_{(2,0)} = (c_{(1,0)}^{(1)})^2 \alpha_{(2,0)},$$

$$(47) \quad \alpha'_{(0,2)} = ({}^t g \alpha)_{(0,2)} = (c_{(0,1)}^{(2)})^2 \alpha_{(0,2)}.$$

Since $\alpha_{(2,0)} \neq 0$ and $\alpha_{(0,2)} \neq 0$, by (46) and (47) we can determine $c_{(1,0)}^{(1)}$ and $c_{(0,1)}^{(2)}$, so that $\alpha'_{(2,0)} = \alpha'_{(0,2)} = 1$.

By (44), (45), we can determine $c_{(2,0)}^{(1)}, c_{(0,2)}^{(1)}, \dots, c_{(0,2)}^{(2)}$ so that $\alpha'_{(1,0)} = \alpha'_{(0,1)} = 0$. $\square$

The restriction of $M_{(3,1,1),\alpha}$ with $\alpha = {}^t(0, 0, 1, 1)$ on

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & x_1 \\ 0 & x_2 \end{bmatrix}$$

is

$$\partial_1 - x_1 \partial_2^2, (x_1^2 + 1) \partial_2 + x_2,$$

and a solution of this system is $f(x_1, x_2) = \frac{1}{\sqrt{x_1^2 + 1}} \exp\left(-\frac{x_2^2}{2(x_1^2 + 1)}\right)$.

3. $n = 6$ case

We consider the hypergeometric systems associated with Young diagrams with $|\lambda| = 6$ without $(5, 1)$.

(1) Let $\lambda = (4, 2)$. Since

$$N_G(\mathfrak{a}_\lambda)/Z_G(\mathfrak{a}_\lambda) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{(1,0)}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & c_{(0,1)}^{(1)} & c_{(0,1)}^{(2)} & 0 & 0 & 0 \\ 0 & c_{(2,0)}^{(1)} & c_{(2,0)}^{(2)} & (c_{(1,0)}^{(1)})^2 & 0 & 0 \\ 0 & c_{(1,1)}^{(1)} & c_{(1,1)}^{(2)} & 0 & c_{(1,0)}^{(1)} c_{(0,1)}^{(2)} & 0 \\ 0 & c_{(3,0)}^{(1)} & c_{(3,0)}^{(2)} & 2c_{(1,0)}^{(1)} c_{(2,0)}^{(1)} & c_{(1,0)}^{(1)} c_{(2,0)}^{(2)} & (c_{(1,0)}^{(1)})^3 \end{bmatrix} \in GL_6(\mathbb{C}) \mid c_{(1,0)}^{(1)} c_{(2,0)}^{(2)} \neq 0 \right\},$$

a solution $\Phi(z; \alpha)$ with $\alpha_{(3,0)} \neq 0$ is transformed into $\Phi(z'; \alpha')$ with the parameter vector

$$\alpha' = {}^t(0, 0, 0, 0, 1).$$

The restriction of systems $M_{(4,2),\alpha}$ with $\alpha = {}^t(0, 0, 0, 0, 1)$ on

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & x_1 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix}$$

is

$$\partial_1^2 + x_1, \partial_2, \partial_3.$$

(2) Let $\lambda = (4, 1, 1)$. Since

$$N_G(\mathfrak{a}_\lambda)/Z_G(\mathfrak{a}_\lambda) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{(1,0)}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{(0,1)}^{(2)} & 0 & 0 & 0 \\ 0 & c_{(2,0)}^{(1)} & 0 & (c_{(1,0)}^{(1)})^2 & 0 & 0 \\ 0 & c_{(0,2)}^{(1)} & c_{(0,2)}^{(2)} & 0 & (c_{(0,1)}^{(2)})^2 & 0 \\ 0 & c_{(3,0)}^{(1)} & c_{(3,0)}^{(2)} & 2c_{(1,0)}^{(1)} c_{(2,0)}^{(1)} & 0 & (c_{(1,0)}^{(1)})^3 \end{bmatrix} \in GL_6(\mathbb{C}) \mid c_{(1,0)}^{(1)} c_{(2,0)}^{(2)} \neq 0 \right\},$$

a solution $\Phi(z; \alpha)$ with $\alpha_{(3,0)} \neq 0$ and $\alpha_{(0,2)} \neq 0$ is transformed into $\Phi(z'; \alpha')$ with the parameter vector

$$\alpha' = {}^t(0, 0, 0, 1, 1).$$

The restriction of systems $M_{(4,1,1),\alpha}$ with $\alpha = {}^t(0, 0, 0, 1, 1)$ on

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & x_1 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix}$$

is

$$\partial_1 - \partial_3, \partial_2 - x_2 \partial_3^2, \partial_3^2 + x_2 \partial_3 + (x_1 + x_3).$$

(3) Let $\lambda = (3, 3)$. Since

$$= \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{(1,0)}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{(0,1)}^{(2)} & 0 & 0 & 0 \\ 0 & c_{(2,0)}^{(1)} & 0 & (c_{(1,0)}^{(1)})^2 & 0 & 0 \\ 0 & c_{(1,1)}^{(1)} & c_{(1,1)}^{(2)} & 0 & c_{(1,0)}^{(1)}c_{(0,1)}^{(2)} & 0 \\ 0 & c_{(2,1)}^{(1)} & c_{(2,1)}^{(2)} & 2c_{(1,0)}^{(1)}c_{(1,1)}^{(1)} & c_{(2,0)}^{(1)}c_{(0,1)}^{(2)} + c_{(1,0)}^{(1)}c_{(1,1)}^{(2)} & 3(c_{(1,0)}^{(1)})^2c_{(0,1)}^{(2)} \end{bmatrix} \in GL_6(\mathbb{C}) \mid c_{(1,0)}^{(1)}c_{(2,0)}^{(2)} \neq 0 \right\},$$

a solution $\Phi(z; \alpha)$ with $\alpha_{(2,1)} \neq 0$ is transformed into $\Phi(z'; \alpha')$ with the parameter vector

$$\alpha' = {}^t(0, 0, 0, 0, 1).$$

The restriction of systems $M_{(3,3),\alpha}$ with $\alpha' = {}^t(0, 0, 0, 0, 1)$ on

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & x_1 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix}$$

is

$$\partial_2 - \partial_3^2, \partial_1 - \partial_2\partial_3, 3x_1\partial_3^2 + 2x_2\partial_3 + x_3.$$

(4) Let $\lambda = (3, 2, 1)$. Since

$$N_G(\mathbf{a}_\lambda)/Z_G(\mathbf{a}_\lambda) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{(1,0)}^{(1)} & c_{(1,0)}^{(2)} & 0 & 0 & 0 \\ 0 & c_{(0,1)}^{(1)} & c_{(0,1)}^{(2)} & 0 & 0 & 0 \\ 0 & c_{(2,0)}^{(1)} & c_{(2,0)}^{(2)} & & & \\ 0 & c_{(1,1)}^{(1)} & c_{(1,1)}^{(2)} & [\phi_{\mathbf{l},\mathbf{k}}(c)]_{\mathbf{l},\mathbf{k} \in L_2} & & \\ 0 & c_{(0,2)}^{(1)} & c_{(0,2)}^{(2)} & & & \end{bmatrix} \in GL_6(\mathbb{C}) \right\},$$

a solution $\Phi(z; \alpha)$ that satisfies $\alpha_{(2,0)} \neq 0$, $\alpha_{(1,1)} \neq 0$ or $\alpha_{(0,2)} \neq 0$ is transformed into $\Phi(z'; \alpha')$ with the parameter vector

$$\alpha' = {}^t(0, 0, 0, 0, 1).$$

The restriction of systems $M_{(3,2,1),\alpha}$ with $\alpha' = {}^t(0, 0, 0, 0, 1)$ on

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & x_1 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix}$$

is

$$\partial_2, \partial_1 - x_1\partial_3^2, x_1^2\partial_3 + x_3.$$

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