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On a class of explicit Cauchy–Stieltjes transforms related to monotone stable and free Poisson laws

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We consider a class of probability measures $\mu_{s,r}^\alpha$ which have explicit Cauchy–Stieltjes transforms. This class includes a symmetric beta distribution, a free Poisson law and some beta distributions as special cases. Also, we identify $\mu_{s,2}^\alpha$ as a free compound Poisson law with Lévy measure a monotone α -stable law. This implies the free infinite divisibility of $\mu_{s,2}^\alpha$. Moreover, when symmetric or positive, $\mu_{s,2}^\alpha$ has a representation as the free multiplication of a free Poisson law and a monotone α -stable law. We also investigate the free infinite divisibility of $\mu_{s,r}^\alpha$ for $r \neq 2$. Special cases include the beta distributions $B(1 - \frac{1}{r}, 1 + \frac{1}{r})$ which are freely infinitely divisible if and only if $1 \leq r \leq 2$.

Keywords: beta distribution; free infinite divisibility; free Poisson law; monotone stable law

1. Introduction

In random matrix theory, a Marchenko–Pastur law describes the asymptotic behavior of the spectrum of the so-called Wishart matrices [11]. In free probability, a Marchenko–Pastur (or free Poisson) law plays the role that a Poisson distribution does in probability theory: it is the limiting distribution of $((1 - \frac{\lambda}{N})\delta_0 + \frac{\lambda}{N}\delta_1)^{\boxplus N}$ when $N \rightarrow \infty$. For this reason, it is called a free Poisson law in the context of free probability. On the other hand, an arcsine law appears in probability theory as the law of the proportion of the time during which a Wiener process is nonnegative. In monotone probability, an arcsine law plays the role of a Gaussian law [13]. In particular, an arcsine law is a monotone stable law with stability index $\alpha = 2$ [9].

Arizmendi *et al.* [3] found an interplay between Marchenko–Pastur and arcsine laws. They introduced a class *FTA* of freely infinitely divisible distributions whose Lévy measures are mixtures of a symmetric arcsine law. The building block of this class is a symmetric beta distribution

$$b_s(dx) = \frac{1}{\pi\sqrt{s}}|x|^{-1/2}(\sqrt{s} - |x|)^{1/2} dx, \quad -\sqrt{s} \leq x \leq \sqrt{s}.$$

The free Lévy measure of b_s coincides with an arcsine law. Moreover, b_s is equal to the free multiplicative convolution of an arcsine law with a Marchenko–Pastur law, and hence, is freely infinitely divisible. Moreover, its Cauchy–Stieltjes transform (or Cauchy transform for short) can

be calculated explicitly as

$$G_{b_s}(z) = -\sqrt{\frac{2}{s}} \sqrt{1 - \sqrt{1 - sz^{-2}}}, \quad s > 0. \quad (1.1)$$

This paper studies a class of Cauchy–Stieltjes (or Cauchy for short) transforms related to Marchenko–Pastur laws and monotone stable laws. We deform the above Cauchy transform (1.1) to introduce a family of probability measures which include the symmetric beta distribution b_s , Marchenko–Pastur and some other beta distributions as special cases. More explicitly, for $0 < \alpha \leq 2$, we define

$$G_{s,r}^\alpha(z) = -r^{1/\alpha} \left(\frac{1 - (1 - s(-1/z)^\alpha)^{1/r}}{s} \right)^{1/\alpha}, \quad r > 0, s \in \mathbb{C} \setminus \{0\}. \quad (1.2)$$

The branches of powers have to be defined carefully and the precise definition is presented in Section 3. It can be shown that the function (1.2) defines the Cauchy transform of a probability measure $\mu_{s,r}^\alpha$ for $1 \leq r < \infty$ and (α, s) satisfying what we call an admissible condition. This condition is related to stable distributions.

The reciprocal Cauchy transforms $F_{s,r}^\alpha = \frac{1}{G_{s,r}^\alpha}$ satisfy

$$F_{s,r}^\alpha \circ F_{us,u}^\alpha = F_{us,ur}^\alpha.$$

We note that the same relation appears for probability measures introduced by Młotkowski [12]. This relation enables us to calculate the inverse map explicitly:

$$(F_{s,r}^\alpha)^{-1} = F_{s/r,1/r}^\alpha. \quad (1.3)$$

The inverse map of the reciprocal Cauchy transform, which is hard to calculate in general, is crucial to investigate free infinite divisibility. Therefore, the explicit form of $(F_{s,r}^\alpha)^{-1}$ is quite useful and we can prove the free infinite divisibility of $\mu_{s,r}^\alpha$ for some parameters.

The probability measure $\mu_{s,2}^\alpha$ turns out to be a free compound Poisson distribution with Lévy measure a monotone α -stable law $a_{s/4}^\alpha$. From Proposition 4 of [14], if symmetric or positive, $\mu_{s,2}^\alpha$ coincides with the free multiplicative convolution of a Marchenko–Pastur law m and the monotone α -stable distribution $a_{s/4}^\alpha$:

$$\mu_{s,2}^\alpha = m \boxtimes a_{s/4}^\alpha.$$

Moreover, $\mu_{s,r}^\alpha$ is freely infinitely divisible for other parameters, not only for $r = 2$. An interesting case of $\mu_{s,r}^\alpha$ is $\mu_{-1,r}^1$ which is a beta distribution with the density $\frac{r \sin(\pi/r)}{\pi} x^{-1/r} (1-x)^{1/r}$ on $(0, 1)$. We prove that this is freely infinitely divisible if and only if $1 \leq r \leq 2$. We also mention that, while an arcsine law is not freely infinitely divisible, some monotone stable laws are. This fact was implicitly proved by Biane in a different context; see Corollary 4.5 of [8].

2. Preliminary results

2.1. The Voiculescu transform and the R -transform

In this paper, $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, denote the upper half-plane and the lower half-plane of $\mathbb{C}$.

An additive free convolution $\mu \boxplus \nu$ of compactly supported probability measures μ and ν on $\mathbb{R}$ is the probability distribution of $X + Y$, where X and Y are self-adjoint free independent random variables with distributions μ and ν , respectively, [17]. This convolution was extended to all Borel probability measures in [7]. A probability measure μ on $\mathbb{R}$ is said to be $\boxplus$ -infinitely divisible if for any $n \in \mathbb{N}$, there is μ_n such that $\mu = \mu_n^{\boxplus n}$.

For a probability measure μ on $\mathbb{R}$, let us denote by G_μ the Cauchy transform and by F_μ its reciprocal: $G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z-x}$ and $F_\mu(z) = \frac{1}{G_\mu(z)}$. Bercovici and Voiculescu [7] proved the existence of $\eta, \eta' > 0$ and $M, M' > 0$ such that F_μ is univalent in $\Gamma_{\eta, M} := \{z \in \mathbb{C}_+ : \operatorname{Im} z > M, |\operatorname{Im} z| > \eta |\operatorname{Re} z|\}$ and $\Gamma_{\eta', M'} \subset F_\mu(\Gamma_{\eta, M})$. The Voiculescu transform ϕ_μ is defined in $\Gamma_{\eta', M'}$ to be $F_\mu^{-1}(z) - z$. The free convolution $\mu \boxplus \nu$ is characterized by

$$\phi_{\mu \boxplus \nu}(z) = \phi_\mu(z) + \phi_\nu(z)$$

in $\Gamma_{\eta'', M''}$ for some $\eta'', M'' > 0$. $R_\mu(z) := z\phi_\mu(\frac{1}{z})$ is called an R -transform. A probability measure μ is $\boxplus$ -infinitely divisible if and only if ϕ_μ is the restriction of an analytic map from $\mathbb{C}_+$ into $\mathbb{C}_- \cup \mathbb{R}$ [7]. This is also equivalent to the Lévy–Khintchine type representation suggested in [4]

$$R_\mu(z) = cz + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1-xz} - 1 - xz\mathbf{1}_{\{|x| \leq 1\}}(x) \right) \nu(dx), \quad (2.1)$$

for some $c \in \mathbb{R}, a \geq 0$ and a nonnegative measure ν satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{1, x^2\} \times \nu(dx) < \infty$. We call ν the Lévy measure of μ .

The following is useful to calculate the Lévy measure. For a $\boxplus$ -infinitely divisible measure μ , its Voiculescu transform can be written as

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \left(\frac{1}{z-x} - \frac{x}{1+x^2} \right) (1+x^2)\tau(dx)$$

for some $\gamma \in \mathbb{R}$ and a nonnegative finite measure τ [7]. The measure τ can be calculated, by using the Stieltjes inversion formula [1, 16], as

$$\int_u^v (1+x^2)\tau(dx) = -\frac{1}{\pi} \lim_{y \searrow 0} \int_u^v \operatorname{Im} \phi_\mu(x+iy) dx \quad (2.2)$$

for all continuity points u, v of τ . Considering the relation $R_\mu(z) = z\phi_\mu(\frac{1}{z})$ and (2.1), we obtain $\frac{1+x^2}{x^2} \tau|_{\mathbb{R} \setminus \{0\}} = \nu|_{\mathbb{R} \setminus \{0\}}$ and $\tau(\{0\}) = a$, where a is the real number of (2.1). In particular, if the functions $f_\mu^y(x) := -\frac{1}{\pi} \operatorname{Im} \phi_\mu(x+iy)$ converges uniformly to a continuous function $f_\mu(x)$ ($y \searrow$

0) on an interval $[u, v]$, then τ is absolutely continuous in $[u, v]$ with density $f_\mu(x)$. Hence, ν is also absolutely continuous in $[u, v]$ with density

$$\frac{1+x^2}{x^2} f_\mu(x). \quad (2.3)$$

Regarding atoms, the following formula holds: $\tau(\{x\}) = \frac{1}{1+x^2} \lim_{y \searrow 0} \text{Im} \phi_\mu(x+iy)$ for any $x \in \mathbb{R}$.

2.2. The S -transform

Multiplicative free convolution $\boxtimes$ for probability measures on $[0, \infty)$ was investigated in [7, 18]. This convolution corresponds to the probability distribution of $X^{1/2} Y X^{1/2}$, or equivalently $Y^{1/2} X Y^{1/2}$, where X and Y are positive free independent random variables. This convolution is characterized by S -transforms defined as follows. For a probability measure μ on $\mathbb{R}$, we let $\psi_\mu(z) := \int_{\mathbb{R}} \frac{zx}{1-zx} \mu(dx)$. ψ_μ coincides with a moment generating function if μ has finite moments of all orders. In [7], ψ_μ was proved to be univalent in the left half-plane $i\mathbb{C}_+$ for a probability measure μ on $[0, \infty)$ with $\mu(\{0\}) < 1$. Moreover, $\psi_\mu(i\mathbb{C}_+)$ contains the interval $(1 - \mu(\{0\}), 0)$. Then a map $\chi_\mu: \psi_\mu(i\mathbb{C}_+) \rightarrow i\mathbb{C}_+$ is defined by the inverse of ψ_μ . The S -transform is defined as

$$S_\mu(z) := \frac{1+z}{z} \chi_\mu(z), \quad z \in \psi_\mu(i\mathbb{C}_+). \quad (2.4)$$

Using the S -transform, $\mu \boxtimes \nu$ is characterized as

$$S_{\mu \boxtimes \nu}(z) = S_\mu(z) S_\nu(z) \quad (2.5)$$

in a common domain including an interval of the form $(-\varepsilon, 0)$.

More generally, a multiplicative convolution $\mu \boxtimes \nu$ can be defined if μ or ν is supported on $[0, \infty)$. While (2.5) is expected to hold also in this case, it is not known whether an S -transform can be defined for every probability measure. It was shown in [18] to hold for measures with bounded support and nonvanishing mean, while the bounded case when μ has vanishing mean was solved in [15]. For the unbounded case, as a partial solution, Arizmendi and Pérez-Abreu [2] defined an S -transform of a symmetric probability measure as follows. For a symmetric distribution $\mu \neq \delta_0$, there is a unique probability distribution $\mu^2 \neq \delta_0$ on $[0, \infty)$ such that $\psi_\mu(z) = \psi_{\mu^2}(z^2)$ for $z \in \mathbb{C}_+$. Using a property of ψ_{μ^2} , we can conclude that ψ_μ is univalent in $\mathbb{H} := \{z \in \mathbb{C}_+ : \text{Im } z > |\text{Re } z|\}$. Moreover, $\psi_\mu(\mathbb{H})$ contains the interval $(1 - \mu(\{0\}), 0)$. Therefore, we can define $\chi_\mu = \psi_\mu^{-1}: \psi_\mu(\mathbb{H}) \rightarrow \mathbb{H}$ and $S_\mu(z) := \frac{1+z}{z} \chi_\mu(z)$. Then (2.5) still holds if μ or ν is symmetric and the other is supported on $[0, \infty)$.

Finally, we recall the analogues of compound Poisson distributions, which will be important in this paper.

Definition 2.1. A probability measure μ is said to be free compound Poisson if $R_\mu(z) = \lambda \psi_\nu(z)$ for a probability measure ν with $\nu(\{0\}) = 0$ and a $\lambda \geq 0$. In this case, $\lambda \nu$ coincides with the Lévy measure of μ .

The Marchenko–Pastur law m with mean one belongs to the class of free compound Poisson measures; the pair (λ, ν) is given by $(1, \delta_1)$. m is also characterized by $S_m(z) = \frac{1}{z+1}$ in terms of the S -transform.

3. Probability measures $\mu_{s,r}^\alpha$

Let $r > 0$, $2 \geq \alpha > 0$ and $s \in \mathbb{C} \setminus \{0\}$. For any $\eta > 0$, we will find an $M > 0$ such that the function

$$G_{s,r}^\alpha(z) = -r^{1/\alpha} \left(\frac{1 - (1 - s(-1/z)^\alpha)^{1/r}}{s} \right)^{1/\alpha} \quad (3.1)$$

is defined as an analytic map in $\Gamma_{\eta,M}$. To make the definition precise, we take branches of powers $z^{1/\alpha}$, $z^{1/r}$ and z^α as follows:

(1) $z^{1/\alpha}$ and z^α are, respectively, defined as $e^{1/\alpha \log_{(1)} z}$ and $e^{\alpha \log_{(1)} z}$ in $\mathbb{C} \setminus [0, \infty)$, where $\log_{(1)}$ denotes a logarithm satisfying $\text{Im}(\log_{(1)} z) \in (0, 2\pi)$;

(2) $z^{1/r}$ is defined to be $e^{1/r \log_{(2)} z}$ in $\mathbb{C} \setminus (-\infty, 0]$, where $\log_{(2)}$ is a logarithm satisfying $\text{Im}(\log_{(2)} z) \in (-\pi, \pi)$.

We show that these branches enable us to define $G_{s,r}^\alpha$ as an analytic function in $\Gamma_{\eta,M}$ for an $M > 0$ depending on $\eta > 0$, $s \in \mathbb{C} \setminus \{0\}$, $r > 0$. Under the definition (2), the function $(1+w)^{1/r}$ is equal to the generalized binomial expansion $\sum_{n=0}^{\infty} {}_{1/r}C_n w^n$ for $|w| < 1$, where ${}_{1/r}C_n$ is the generalized binomial coefficient $\frac{1/r(1/r-1)\cdots(1/r-n+1)}{n!}$. Therefore, for $z \in \mathbb{C}_+$ with large $|z|$, the function $\frac{1 - (1 - s(-1/z)^\alpha)^{1/r}}{s}$ can be written as

$$\frac{1 - (1 - s(-1/z)^\alpha)^{1/r}}{s} = \left(-\frac{1}{z} \right)^\alpha \sum_{n=1}^{\infty} {}_{1/r}C_n (-s)^{n-1} \left(-\frac{1}{z} \right)^{(n-1)\alpha}, \quad (3.2)$$

where $(-\frac{1}{z})^{n\alpha}$ is defined by $((-\frac{1}{z})^\alpha)^n$. For any $\eta > 0$, $s \in \mathbb{C} \setminus \{0\}$, $r > 0$, there is an $M > 0$, independent of $\alpha \in (0, 2]$, such that the image of the map $\Gamma_{\eta,M} \ni z \mapsto (-\frac{1}{z})^\alpha \sum_{n=1}^{\infty} {}_{1/r}C_n (-s)^{n-1} \times (-\frac{1}{z})^{(n-1)\alpha}$ is contained in the sector $\{z \in \mathbb{C} \setminus \{0\} : \arg z \in (0, \alpha\pi)\}$. Therefore, we can take the power of (3.2) by $1/\alpha$ and $G_{s,r}^\alpha$ is well-defined as an analytic map in $\Gamma_{\eta,M}$.

We note that $G_{s,r}^\alpha(z)$ can be expanded in a series regarding $(-\frac{1}{z})^\alpha$:

$$\begin{aligned} G_{s,r}^\alpha(z) &= -r^{1/\alpha} \left(\left(-\frac{1}{z} \right)^\alpha \sum_{n=1}^{\infty} {}_{1/r}C_n (-s)^{n-1} \left(-\frac{1}{z} \right)^{(n-1)\alpha} \right)^{1/\alpha} \\ &= \frac{1}{z} \left(1 + r \sum_{n=1}^{\infty} {}_{1/r}C_{n+1} (-s)^n \left(-\frac{1}{z} \right)^{n\alpha} \right)^{1/\alpha} \\ &= \frac{1}{z} \sum_{n=0}^{\infty} c_n(\alpha, s, r) \left(-\frac{1}{z} \right)^{n\alpha}, \quad z \in \Gamma_{\eta,M} \end{aligned} \quad (3.3)$$

for some complex coefficients $c_n(\alpha, s, r)$ with $c_0 = 1$. In the second line, we used the formula $((-\frac{1}{z})^\alpha(1 + o(1/z)))^{1/\alpha} = -\frac{1}{z}(1 + o(1/z))^{1/\alpha}$. This formula is valid in $\Gamma_{\eta, M}$ if the function $(1 + o(1/z))^{1/\alpha}$ is understood to be the generalized binomial expansion.

Let us define $F_{s,r}^\alpha(z) := \frac{1}{G_{s,r}^\alpha(z)}$ for $z \in \Gamma_{\eta, M}$, where $M > 0$ is large enough depending on (η, s, r) . Then we have the following.

Theorem 3.1. *Let $r, u > 0$, $2 \geq \alpha > 0$, $\eta > 0$ and $s \in \mathbb{C} \setminus \{0\}$. Then*

$$F_{s,r}^\alpha \circ F_{us,u}^\alpha = F_{us,ur}^\alpha$$

holds in $\Gamma_{\eta, M}$ for some $M > 0$.

Proof. We note that $(-G_{us,u}^\alpha(z))^\alpha$ is equal to $\frac{1-(1-us(-1/z)^\alpha)^{1/u}}{s}$ in $\Gamma_{\eta, M}$ with large $M > 0$. Also, we note that $((1+w)^{1/r})^{1/u} = (1+w)^{1/(ru)}$ for small $|w|$. Then

$$-r^{1/\alpha} \left(\frac{1 - (1 - s(-G_{us,u}^\alpha(z))^\alpha)^{1/r}}{s} \right)^{1/\alpha} = -(ur)^{1/\alpha} \left(\frac{1 - (1 - us(-1/z)^\alpha)^{1/(ru)}}{us} \right)^{1/\alpha}$$

for $z \in \Gamma_{\eta, M}$. $\square$

Under further conditions on (r, α, s) , the function $G_{s,r}^\alpha$ is well-defined in $\mathbb{C}_+$ with values in $\mathbb{C}_-$, and therefore defines a probability measure.

Theorem 3.2. *Suppose $1 \leq r < \infty$, $0 < \alpha \leq 2$ and $s \in \mathbb{C} \setminus \{0\}$. Assume that either of the following conditions is satisfied:*

- (1) $0 < \alpha \leq 1$ and $(1 - \alpha)\pi \leq \arg s \leq \pi$;
- (2) $1 < \alpha \leq 2$ and $0 \leq \arg s \leq (2 - \alpha)\pi$.

Then $G_{s,r}^\alpha$ is the Cauchy transform of a probability measure, which we denote by $\mu_{s,r}^\alpha$. Moreover, $G_{s,r}^\alpha$ is univalent in $\mathbb{C}_+$. If (α, s) satisfies (1) or (2), it is said to be admissible.

Proof. Let $r \geq 1$. We can immediately check that $zG_{s,r}^\alpha(z) \rightarrow 1$ as $z \rightarrow \infty$, $z \in \mathbb{C}_+$, nontangentially. Therefore, what needs to be proved is that $G_{s,r}^\alpha$ analytically maps the upper half-plane to the lower half-plane.

We first focus on the case $0 < \alpha \leq 1$ and $\theta := \arg s \in [\pi(1 - \alpha), \pi]$. Then the image of the map $\frac{1-(1-s(-1/z)^\alpha)^{1/r}}{s}$ in $\mathbb{C}_+$ can be described as in Figure 4 after some steps described in Figures 1–3.

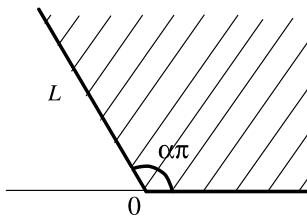


Figure 1. The image of $\mathbb{C}_+$ under the map $z \mapsto (-\frac{1}{z})^\alpha$.

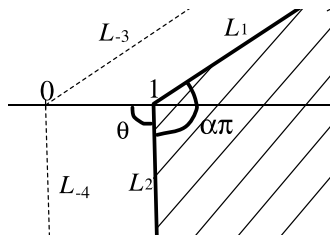


Figure 2. The image of $\mathbb{C}_+$ under the map $z \mapsto 1 - s(-\frac{1}{z})^\alpha$. L_1 and L_2 are half lines contained in the upper half-plane and the lower half-plane, respectively. L_{-3} and L_{-4} are preimages of L_3 and L_4 of Figure 3 for the map $z \mapsto z^{1/r}$, respectively.

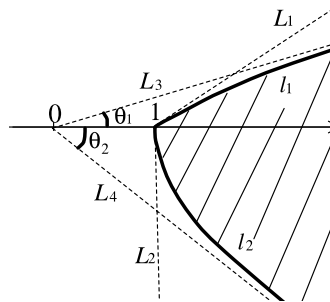


Figure 3. The image of $\mathbb{C}_+$ under the map $z \mapsto (1 - s(-\frac{1}{z})^\alpha)^{1/r}$. θ_1 and θ_2 are defined by $\theta_1 = \frac{\theta - (1-\alpha)\pi}{r}$ and $\theta_2 = \frac{\pi - \theta}{r}$. L_1 and L_2 are the same half lines as in Figure 2. L_3 and L_4 are starting at 0. l_1 is tangent to L_1 at 1 since $z^{1/r}$ is a conformal mapping. Moreover, it approaches L_3 asymptotically. l_2 is tangent to L_2 at 1 from the same reason and approaches L_4 asymptotically.

We can see that the image of the map $\frac{1 - (1 - s(-1/z)^\alpha)^{1/r}}{s}$ is contained in the sector $\{z \in \mathbb{C} : 0 < \arg z < \alpha\pi\}$. This implies the desired conclusion.

In the case $1 < \alpha \leq 2$, we draw similar pictures; see Figure 5–8. In Figure 8, the image of $\frac{1 - (1 - s(-1/z)^\alpha)^{1/r}}{s}$ is contained in the sector $\{z \in \mathbb{C} : 0 < \arg z < \alpha\pi\}$. Therefore, the image of the map $(\frac{1 - (1 - s(-1/z)^\alpha)^{1/r}}{s})^{1/\alpha}$ is contained in $\mathbb{C}_+$.

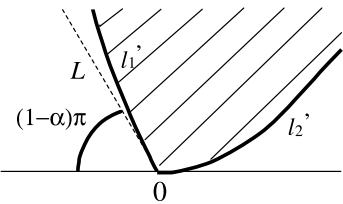


Figure 4. The image of $\mathbb{C}_+$ under the map $z \mapsto \frac{1 - (1 - s(-1/z)^\alpha)^{1/r}}{s}$ which can be obtained from the rotation and the translation of Figure 3. l_1' is tangent to L and l_2' is tangent to the x axis at 0.

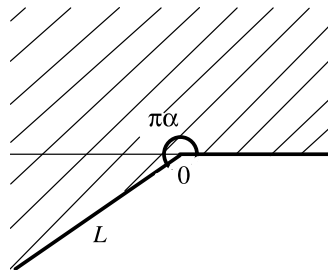


Figure 5. The image of $\mathbb{C}_+$ under the map $z \mapsto (-\frac{1}{z})^\alpha$.

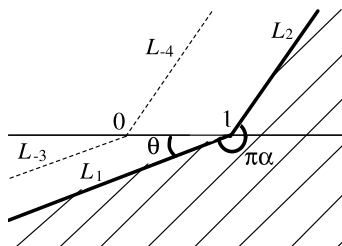


Figure 6. The image of $\mathbb{C}_+$ under the map $z \mapsto 1 - s(-\frac{1}{z})^\alpha$. L_1 and L_2 are half lines starting at 1. L_{-3} and L_{-4} are preimages of L_3 and L_4 of Figure 7 for the map $z \mapsto z^{1/r}$, respectively.

In each step described in the figures, a new univalent map is added, so that after all the steps, the map $G_{s,r}^\alpha$ is also univalent in $\mathbb{C}_+$. $\square$

Remark 3.3. (i) The admissible condition is related to monotone stable distributions as mentioned in the next section.

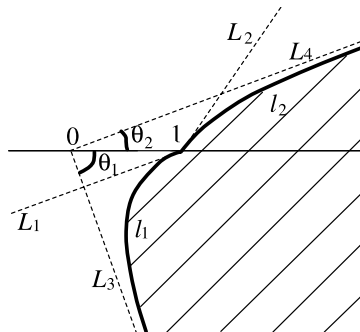


Figure 7. The image of $\mathbb{C}_+$ under the map $z \mapsto (1 - s(-\frac{1}{z})^\alpha)^{1/r}$. θ_1 and θ_2 are defined by $\theta_1 = \frac{\pi - \theta}{r}$ and $\theta_2 = \frac{\pi(\alpha - 1) + \theta}{r}$. L_1 and L_2 are the same half lines as in Figure 6. L_3 and L_4 are starting at 0. l_1 is tangent to L_1 at 1 and approaches L_3 asymptotically. l_2 is tangent to L_2 at 1 and approaches L_4 asymptotically.

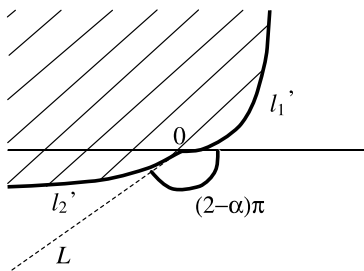


Figure 8. The image of $\mathbb{C}_+$ under the map $z \mapsto \frac{1-(1-s(-1/z)^\alpha)^{1/r}}{s}$ which can be obtained from the rotation and the translation of Figure 7. l_1' is tangent to the x axis and l_2' is tangent to L at 0.

(ii) We have $\mu_{s,1}^\alpha = \delta_0$ for any admissible (α, s) . Therefore, the right inverse of $F_{s,r}^\alpha$ can be calculated as $(F_{s,r}^\alpha)^{-1} = F_{s/r,1/r}^\alpha$ from Theorem 3.1.

(iii) From the relation $(F_{s,r}^\alpha)^{-1} = F_{s/r,1/r}^\alpha$, we can conclude that $G_{s,r}^\alpha$ does not define a probability measure for $0 < r < 1$ and admissible (α, s) . The reason is as follows. If μ is a probability measure and not a point mass, then $\text{Im } F_\mu(z) > \text{Im } z$ for any $z \in \mathbb{C}_+$; see Corollary 5.3 of [7]. Hence, $\text{Im } F_\mu^{-1}(z) < \text{Im } z$ if $z = F_\mu(w)$ and F_μ is univalent around w . Therefore, F_μ^{-1} cannot be written as F_ν for a probability measure ν on $\mathbb{R}$.

(iv) The measure $\mu_{s,r}^\alpha$ satisfies self-similarity with respect to s as follows. If μ is a probability distribution of a random variable X , then let $D_c\mu$ denote the distribution of cX . For $c > 0$, we have

$$\mu_{cs,r}^\alpha = D_{c^{1/\alpha}}\mu_{s,r}^\alpha.$$

4. A relation to monotone stable and free Poisson laws

Let a_s^α be a monotone (strictly) α -stable distribution [9] characterized by

$$F_{a_s^\alpha}(z) = (z^\alpha + (-1)^{\alpha-1}s)^{1/\alpha}, \quad z \in \mathbb{C}_+,$$

where (α, s) satisfies the admissible condition. a_s^2 is the centered arcsine law with variance $s/2$ and a_s^1 is a Cauchy distribution or a delta measure. The following properties are valuable to note here.

- (1) a_s^α is supported on $[0, \infty)$ if and only if $0 < \alpha \leq 1$ and $\arg s = \pi$.
- (2) a_s^α is symmetric if and only if $\arg s = (1 - \frac{\alpha}{2})\pi$.
- (3) Both $a_{Ri}^{1/2}$ and $a_{-R}^{1/2}$ are free $\frac{1}{2}$ -stable distributions, but not strictly stable.

The proofs are as follows. Let $s := re^{i\theta}$, $r > 0$. From the Stieltjes inversion formula, the density $p_s^\alpha(x)$ of a_s^α is given by

$$p_s^\alpha(x) = \begin{cases} \frac{\sin[(1/\alpha) \arg(|x|^\alpha + re^{i(\alpha\pi - \pi + \theta)})]}{\pi(|x|^{2\alpha} - 2r|x|^\alpha \cos(\alpha\pi + \theta) + r^2)^{1/(2\alpha)}}, & x > 0, \\ \frac{\sin[(1/\alpha) \arg(|x|^\alpha + re^{i(\pi - \theta)})]}{\pi(|x|^{2\alpha} - 2r|x|^\alpha \cos \theta + r^2)^{1/(2\alpha)}}, & x < 0, \end{cases}$$

where $\arg z$ is defined in $(\mathbb{C}_+ \cup \mathbb{R}) \setminus \{0\}$ so that it takes values in $[0, \pi]$. Now the properties (1) and (2) can be proved easily.

(3) It was proved in [7] that free $\frac{1}{2}$ -stable distributions are characterized in terms of the Voiculescu transform $\phi_\mu(z) = bz^{1/2} + c$, where $c \in \mathbb{R}$ and $\arg b \in [\pi, 3\pi/2]$. Moreover, strictly stable laws correspond to the case $c = 0$. Since $F_{a_s^{1/2}}(z) = (z^{1/2} - is)^2$, we have $\phi_{a_s^{1/2}}(z) = F_{a_s^{1/2}}^{-1}(z) - z = 2isz^{1/2} - s^2$, which for $s = -R$ or $s = iR$ means that $a_s^{1/2}$ is free $\frac{1}{2}$ -stable, but not strictly stable.

The main theorem of this section is the following.

Theorem 4.1. $\mu_{s,2}^\alpha$ is a free compound Poisson distribution for any admissible (α, s) . Moreover, its Lévy measure $\nu_{s,2}^\alpha$ is given by the monotone stable distribution $a_{s/4}^\alpha$.

Proof. Thanks to Proposition 4 of [14], it suffices to prove that $R_{s,2}^\alpha(z) = \psi_{a_{s/4}^\alpha}(z)$, or equivalently, $\phi_{s,2}^\alpha(z) = z^2 G_{a_{s/4}^\alpha}(z) - z$, in an open set of the form $\Gamma_{\eta,M}$.

As in (3.3), a naive relation $(zw)^\alpha = z^\alpha w^\alpha$ may not be valid. To avoid this problem, we understand that $(1 - \frac{s}{4}(-\frac{1}{z})^\alpha)^{1/\alpha}$, appearing below, is defined by using the generalized binomial expansion $(1+w)^{1/\alpha} = \sum_{n=0}^{\infty} \frac{1}{\alpha} C_n w^n$ for $|w| < 1$. Then for any $\eta > 0$, the following calculation is correct in $\Gamma_{\eta,M}$ with large $M > 0$:

$$\begin{aligned} F_{a_{s/4}^\alpha}(z) &= \left(z^\alpha + \frac{s}{4}(-1)^{\alpha-1} \right)^{1/\alpha} \\ &= \left(z^\alpha - \frac{s}{4}(-1)^\alpha \right)^{1/\alpha} \\ &= \left(z^\alpha \left(1 - \frac{s}{4z^\alpha}(-1)^\alpha \right) \right)^{1/\alpha} \\ &= \left(z^\alpha \left(1 - \frac{s}{4} \left(-\frac{1}{z} \right)^\alpha \right) \right)^{1/\alpha} \\ &= z \left(1 - \frac{s}{4} \left(-\frac{1}{z} \right)^\alpha \right)^{1/\alpha}. \end{aligned}$$

Therefore,

$$z^2 G_{a_{s/4}^\alpha}(z) - z = \frac{z}{(1 - (s/4)(-1/z)^\alpha)^{1/\alpha}} - z.$$

On the other hand, the Voiculescu transform of $\mu_{s,2}^\alpha$ is given as

$$\begin{aligned} \phi_{s,2}^\alpha(z) &= F_{s/2,1/2}^\alpha(z) - z \\ &= -\frac{1}{((1 - (1 - (s/2)(-1/z)^\alpha)^2)/s)^{1/\alpha}} - z \\ &= -\frac{1}{((s(-1/z)^\alpha - (s^2/4)(-1/z)^{2\alpha})/s)^{1/\alpha}} - z \\ &= -\frac{1}{((-1/z)^\alpha(1 - (s/4)(-1/z)^\alpha))^{1/\alpha}} - z \\ &= \frac{z}{(1 - (s/4)(-1/z)^\alpha)^{1/\alpha}} - z \end{aligned}$$

in $\Gamma_{\eta,M}$. Therefore, we have proved $\phi_{s,2}^\alpha(z) = z^2 G_{a_{s/4}^\alpha}(z) - z$. $\square$

With Proposition 4 of [14], the above result implies $\mu_{s,2}^\alpha = m \boxtimes a_{s/4}^\alpha$ if $\mu_{s,2}^\alpha$ and $a_{s/4}^\alpha$ are symmetric or supported on $[0, \infty)$. We do not know if this holds for any admissible pair (α, s) since S -transforms are not defined for probability measures which are not symmetric or supported on $[0, \infty)$.

Theorem 4.2. *Let (α, s) satisfy either of the following conditions: $0 < \alpha \leq 1$ and $\arg s \in \{(1 - \alpha/2)\pi, \pi\}$; $1 < \alpha \leq 2$ and $\arg s = (1 - \alpha/2)\pi$. Then $\mu_{s,2}^\alpha = m \boxtimes a_{s/4}^\alpha$.*

Example 4.3. In general, the density of $\mu_{s,2}^\alpha$ is difficult to calculate. In some cases, however, the density is explicit as we show below.

(1) Let us consider $(\alpha, s, r) = (1, i, 2)$. Then $\mu_{i,2}^1$ is the free multiplicative convolution of the Marchenko–Pastur law and a symmetric Cauchy distribution. This is absolutely continuous with a strictly positive density on $\mathbb{R}$ written as

$$\frac{\sqrt{2}}{\pi} \left(\sqrt{1 + \sqrt{1 + \frac{1}{x^2}}} - \sqrt{2} \right).$$

We mention that this probability measure belongs to a class proposed in [10].

(2) Let $(\alpha, s, r) = (\frac{1}{2}, -1, 2)$. Then the corresponding probability measure is supported on $[0, \infty)$ with a density

$$\frac{4\sqrt{2}}{\pi} \left(\frac{1}{\sqrt{2x}} - \sqrt{-1 + \sqrt{1 + \frac{1}{x}}} \right).$$

(3) As shown in [3], $\mu_{s,2}^2$ for $s > 0$ is a symmetric beta distribution:

$$\mu_{s,2}^2(dx) = \frac{1}{\pi\sqrt{s}} |x|^{-1/2} (\sqrt{s} - |x|)^{1/2} dx, \quad -\sqrt{s} \leq x \leq \sqrt{s}.$$

In addition to $\mu_{s,2}^\alpha$, some monotone stable distributions are also $\boxplus$ -infinitely divisible. This property was essentially proved by Biane [8].

Proposition 4.4. a_s^α is $\boxplus$ -infinitely divisible if and only if (α, s) satisfies either of the following conditions:

- (1) $\frac{1}{2} \leq \alpha < 1$ and $\arg s \in \{(1 - \alpha)\pi, \pi\}$;
- (2) $\alpha = 1$.

In fact, Biane considered only special values for $\arg s$, but the same proof can be applied to the above result.

Finally, we note the S -transforms of $\mu_{s,2}^\alpha$ and a_s^α .

Proposition 4.5. Let (α, s) satisfy either of the following conditions: $0 < \alpha \leq 1$ and $\arg s \in \{(1 - \alpha/2)\pi, \pi\}$; $1 < \alpha \leq 2$ and $\arg s = (1 - \alpha/2)\pi$. Then

- (1) $S_{a_s^\alpha}(z) = -\frac{1}{z} \left(\frac{(1+z)^\alpha - 1}{s} \right)^{1/\alpha}$, $z \in (-1, 0)$,
- (2) $S_{\mu_{s,2}^\alpha}(z) = -\frac{4^{1/\alpha}}{z(z+1)} \left(\frac{(1+z)^\alpha - 1}{s} \right)^{1/\alpha} = S_m(z) S_{a_{s/4}^\alpha}(z)$, $z \in (-1, 0)$.

Proof. The Voiculescu transform $\phi_{a_s^\alpha}$ can be calculated as $\phi_{a_s^\alpha}(w) = F_{a_s^\alpha}^{-1}(w) - w = (w^\alpha + (-1)^\alpha s)^{1/\alpha} - w$. Let us define $z := R_{a_s^\alpha}(w) = w \phi_{a_s^\alpha}^{-1}(\frac{1}{w})$. Then $(1+z)^\alpha = 1 + s(-w)^\alpha$. Since $R_{a_s^\alpha}(z S_{a_s^\alpha}(z)) = z$ holds, the desired formula follows. A similar calculation is possible for $\mu_{s,2}^\alpha$. $\square$

5. More on free infinite divisibility of $\mu_{s,r}^\alpha$

In the previous section, we proved that $\mu_{s,r}^\alpha$ is $\boxplus$ -infinitely divisible whenever $r = 2$. In this section we will determine infinite divisibility for $r \neq 2$. We found the general case is too difficult to treat, so that we only consider the problem for some parameters. The main results of this section are the following.

- (1) If $0 < \alpha \leq 1$ and $1 \leq r \leq 2$, then $\mu_{s,r}^\alpha$ is $\boxplus$ -infinitely divisible.
- (2) If $1 \leq \alpha \leq 2$ and $1 \leq r \leq \frac{2}{\alpha}$, then $\mu_{s,r}^\alpha$ is $\boxplus$ -infinitely divisible.
- (3) $\mu_{s,3}^1$ is $\boxplus$ -infinitely divisible if and only if $\arg s = \frac{\pi}{2}$.
- (4) If $\alpha > 1$, there exists an $r_0 = r_0(\alpha, s) > 1$ such that $\mu_{s,r}^\alpha$ is not $\boxplus$ -infinitely divisible for $r > r_0$.

We also show that some beta distributions are $\boxplus$ -infinitely divisible, and some are not.

5.1. The case $1 \leq r \leq 2$

To prove the free infinite divisibility of $\mu_{s,r}^\alpha$, we introduce a subclass of $\boxplus$ -infinitely divisible distributions.

Definition 5.1. A probability measure μ is said to be in class $\mathcal{UI}^1$ if F_μ is univalent in $\mathbb{C}_+$ and, moreover, F_μ^{-1} has an analytic continuation from $F_\mu(\mathbb{C}_+)$ to $\mathbb{C}_+$ as a univalent function.

The following property was implicitly used in [6].

Proposition 5.2. $\mu \in \mathcal{UI}$ implies that μ is $\boxplus$ -infinitely divisible.

Proof. The Voiculescu transform ϕ_μ has an analytic continuation to $\mathbb{C}_+$ defined by $F_\mu^{-1}(z) - z$. If there existed a point $z_0 \in \mathbb{C}_+$ such that $\text{Im } \phi_\mu(z_0) > 0$, then $\text{Im } F_\mu^{-1}(z_0) = \text{Im}(z_0 + \phi_\mu(z_0)) > \text{Im } z_0 > 0$. Since $\text{Im } F_\mu(w) \geq \text{Im } w$ for $w \in \mathbb{C}_+$, $\text{Im } F_\mu^{-1}(z) \leq \text{Im } z$ for $z \in F_\mu(\mathbb{C}_+)$. Therefore, z_0 never belongs to $F_\mu(\mathbb{C}_+)$. However, since F_μ^{-1} is univalent in $\mathbb{C}_+$ and $F_\mu^{-1}(F_\mu(\mathbb{C}_+)) = \mathbb{C}_+$, z_0 must satisfy $\text{Im } F_\mu^{-1}(z_0) \leq 0$, which contradicts the inequality $\text{Im } F_\mu^{-1}(z_0) > 0$. Therefore, ϕ_μ maps $\mathbb{C}_+$ into $\mathbb{C}_- \cup \mathbb{R}$. $\square$

Remark 5.3. If μ is $\boxplus$ -infinitely divisible, then F_μ is always univalent in $\mathbb{C}_+$. This can be proved for instance by using the so-called subordination functions. Let μ be $\boxplus$ -infinitely divisible and $\mu_t = \mu^{\boxplus t}$ be the probability measure corresponding to the Voiculescu transform $t\phi_\mu$. For $s \leq t$, an analytic function $\omega_{s,t}: \mathbb{C}_+ \rightarrow \mathbb{C}_+$ exists so that it satisfies $F_{\mu_s} \circ \omega_{s,t} = F_{\mu_t}$. $\omega_{s,t}$ is called a subordination function. The reader is referred also to equation (5.4) of [5], where the following replacements are required: μ by $\mu^{\boxplus s}$ and t by t/s . The relation $F_{\mu_s} \circ \omega_{s,t} = F_{\mu_t}$ is equivalent to

$$F_{\mu_t}(z) = \frac{t/s}{t/s - 1} \omega_{s,t}(z) - \frac{z}{t/s - 1}. \quad (5.1)$$

Moreover, it is proved in Theorem 4.6 of [5] that

$$|\omega_{s,t}(z_1) - \omega_{s,t}(z_2)| \geq \frac{1}{2}|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{C}_+.$$

Taking the limit $s \rightarrow 0$ in (5.1), we get

$$|F_{\mu_t}(z_1) - F_{\mu_t}(z_2)| \geq \frac{1}{2}|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{C}_+,$$

so that F_{μ_t} is univalent in $\mathbb{C}_+$.

For instance, the normal law $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$ is in $\mathcal{UI}$ from the result of [6]. Moreover, we can easily prove that Wigner's semicircle law, the Marchenko–Pastur law and the Cauchy distribution belong to $\mathcal{UI}$.

¹The symbol $\mathcal{UI}$ stands for *univalent inverse reciprocal Cauchy transforms*.

$\mathcal{UI}$ is closed under the weak topology. This is proved as follows. The convergence of μ_n implies the local uniform convergence of the Voiculescu transforms ϕ_{μ_n} [7]. Since $F_{\mu_n}^{-1}(z) = z + \phi_{\mu_n}(z)$ converges locally uniformly, the limit function is univalent. Also F_{μ_n} itself converges to a univalent function. Therefore, the limit measure belongs to the class $\mathcal{UI}$.

We note that $\mathcal{UI}$ is a proper subset of all $\boxplus$ -infinitely divisible distributions. For instance, let μ be a probability measure characterized by the Voiculescu transform $\phi_\mu(z) = \frac{1}{z-1} + \frac{1}{z+1}$. Then $F_\mu^{-1}(z) = z + \phi_\mu(z) = z + \frac{1}{z-1} + \frac{1}{z+1}$. We can find two distinct points z_1, z_2 such that $F_\mu^{-1}(z_1) = F_\mu^{-1}(z_2)$ with $z_1 = iy$ for small $y > 0$ and z_2 near to i . This example also proves that $\mathcal{UI}$ is not closed under the free convolution, since the measures ν and λ , respectively, defined by $\phi_\nu(z) := \frac{1}{z-1}$ and $\phi_\lambda(z) := \frac{1}{z+1}$, both belong to $\mathcal{UI}$.

From Theorem 3.2, the map $F_{s,r}^\alpha$ is univalent in $\mathbb{C}_+$ for any admissible (α, s) and $r \geq 1$, so that we only have to prove the inverse $(F_{s,r}^\alpha)^{-1}$ is univalent in $\mathbb{C}_+$.

Theorem 5.4. *Let (α, s) be an admissible pair. Then $\mu_{s,r}^\alpha \in \mathcal{UI}$ if either of the following conditions holds:*

- (1) $0 < \alpha \leq 1$ and $1 \leq r \leq 2$;
- (2) $1 \leq \alpha \leq 2$ and $1 \leq r \leq \frac{2}{\alpha}$.

Proof. By Remark 3.3, the explicit formula for $(F_{s,r}^\alpha)^{-1}(z)$ is

$$(F_{s,r}^\alpha)^{-1}(z) = -\frac{1}{((1 - (1 - (s/r)(-1/z)^\alpha)r)/s)^{1/\alpha}}.$$

Let us define $\theta := \arg s$ and $E_{s,r}^\alpha(z) := \frac{1 - (1 - (s/r)(-1/z)^\alpha)r}{s}$. First, we consider $1 \leq \alpha \leq 2$. Since the image of the function $1 - \frac{s}{r}(-\frac{1}{z})^\alpha$ for $z \in \mathbb{C}_+$ is contained in the sector $\{z \in \mathbb{C}: z \neq 0, -(\pi - \theta) < \arg z < -(\pi - \theta) + \alpha\pi\}$, one can easily see that $E_{s,r}^\alpha(\mathbb{C}_+)^c$ contains a half line starting from 0. In particular, $E_{s,r}^\alpha$ is univalent in $\mathbb{C}_+$. Therefore, we can take that line as a slit for the function $z \mapsto z^{1/\alpha}$, which then becomes univalent outside the slit.

Let us focus on the case $0 < \alpha \leq 1$. If $1 \leq r \leq 2$, one can prove that $E_{s,r}^\alpha(\mathbb{C}_+)$ is contained in a sector with central angle $r\alpha\pi$ and therefore $E_{s,r}^\alpha$ is univalent in $\mathbb{C}_+$. Since $r\alpha\pi \leq 2\alpha\pi$, the map $z \mapsto z^{1/\alpha}$ can be defined as a univalent map in that sector. $\square$

We take $\alpha = 1$ and $s = -1$ as a special case. Then $\mu_{-1,r}^1$ is the beta distribution $B(1 - \frac{1}{r}, 1 + \frac{1}{r})$ for $1 < r < \infty$:

$$\mu_{-1,r}^1(dx) = \frac{r \sin(\pi/r)}{\pi} x^{-1/r} (1-x)^{1/r} dx, \quad 0 < x < 1.$$

Indeed, now we have $G_{-1,r}^1(z) = r(1 - (1 - \frac{1}{z})^{1/r})$. It holds that

$$\lim_{y \searrow 0} \operatorname{Im} G_{-1,r}^1(x + iy) = 0$$

if $x > 1$ or $x < 0$ and

$$\lim_{y \searrow 0} \operatorname{Im} G_{-1,r}^1(x + iy) = -r \operatorname{Im} \left(e^{i\pi/r} \left(\frac{1-x}{x} \right)^{1/r} \right) = -r \sin(\pi/r) \left(\frac{1-x}{x} \right)^{1/r}$$

if $x \in (0, 1)$. The Stieltjes inversion formula [1] $\mu(dx) = -\frac{1}{\pi} \lim_{y \searrow 0} \operatorname{Im} G_{-1,r}^1(x + iy) dx$ implies the conclusion.

A consequence of Theorem 5.4 is that the beta distribution $B(1 - \frac{1}{r}, 1 + \frac{1}{r})$ is $\boxplus$ -infinitely divisible for $1 < r \leq 2$. More strongly, we can prove the following.

Theorem 5.5. *The beta distribution $B(1 - \frac{1}{r}, 1 + \frac{1}{r})$ ($1 < r < \infty$) is $\boxplus$ -infinitely divisible if and only if $1 < r \leq 2$. The Lévy measure $\nu_{-1,r}^1$ for $1 < r < 2$ can be calculated as*

$$\nu_{-1,r}^1(dx) = \frac{|\sin(r\pi)|}{\pi} \frac{x^{r-2}(1/r-x)^r}{(1/r-x)^{2r} - 2x^r(1/r-x)^r \cos(r\pi) + x^{2r}} dx, \quad 0 < x < \frac{1}{r}.$$

Proof. By Remark 3.3(ii), $(F_{-1,r}^1)^{-1}$ is calculated as

$$(F_{-1,r}^1)^{-1}(z) = \left(1 - \left(1 - \frac{1}{rz} \right)^r \right)^{-1}.$$

If $r > 2$, the function $1 - (1 - \frac{1}{rz})^r$ has a zero point in the upper half-plane, so that $(F_{-1,r}^1)^{-1}$ never be defined as an analytic function. If $r \leq 2$, $(F_{-1,r}^1)^{-1}$ is analytic and univalent in the upper half-plane.

For the Lévy measure, the Voiculescu transform is $\phi_{-1,r}^1(z) = (1 - (1 - \frac{1}{rz})^r)^{-1} - z$. It holds that $\operatorname{Im}(1 - (1 - \frac{1}{r(x+iy)})^r) \rightarrow 0$ as $y \searrow 0$ if $x > 1/r$ or $x < 0$ and that

$$1 - \left(1 - \frac{1}{r(x+iy)} \right)^r \rightarrow 1 - e^{ir\pi} \left(\frac{x-1/r}{x} \right)^r, \quad x \in (0, 1/r)$$

as $y \searrow 0$. After some more calculations, one can see

$$\begin{aligned} \tau(dx) &= -\frac{1}{\pi(1+x^2)} \lim_{y \searrow 0} \operatorname{Im} \phi_{-1,r}^1(x + iy) dx \\ &= \begin{cases} \frac{1}{\pi(1+x^2)} \frac{|\sin(r\pi)| x^r (1/r-x)^r}{(1/r-x)^{2r} - 2x^r(1/r-x)^r \cos(r\pi) + x^{2r}} dx, & x \in (0, 1/r), \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where τ is the measure in (2.2). τ does not have an atom since $\lim_{y \searrow 0} iy \phi_{-1,r}^1(x + iy) = 0$ for any $x \in \mathbb{R}$. The Lévy measure $\nu_{-1,r}^1$ is equal to $\frac{1+x^2}{x^2} \tau$ as explained in Section 2. $\square$

If $s = Re^{i\theta}$ is not real, the support of $\mu_{s,r}^1$ is unbounded. The density for large $|x|$ can be calculated as

$$\mu_{Re^{i\theta},r}^1|_{|x|>R}(dx) = -\frac{r}{\pi} \sum_{n=1}^{\infty} \binom{1/r}{n+1} \frac{R^n \sin(n\theta)}{x^{n+1}} dx.$$

In particular, $\mu_{s,r}^1$ belongs to a class introduced in [10].

5.2. The case $\alpha = 1, r = 3$

In Section 5.1, the free infinite divisibility of $\mu_{s,r}^\alpha$ was proved for some parameters in terms of the class $\mathcal{UI}$. In Section 3, we succeeded in proving the free infinite divisibility of $\mu_{s,2}^\alpha$ since the Voiculescu transform had a quite explicit form. For other parameters, it is difficult to investigate the free infinite divisibility. A possible case is for $\alpha = 1$ and $r = 3$. In this case, the Voiculescu transform has a quite explicit form as in the case $r = 2$ and $\boxplus$ -infinite divisibility can be determined completely. Indeed, the Voiculescu transform is

$$\phi_{3s,3}^1(z) = \frac{-3s}{1 - (1 + s/z)^3} - z = \frac{-3sz^2 - s^2z}{3z^2 + 3zs + s^2}.$$

In contrast to the case $r = 2$, infinite divisibility depends on the parameter s if $r = 3$.

Theorem 5.6. *Let $0 \leq \arg s \leq \pi$. Then $\mu_{s,3}^1$ is $\boxplus$ -infinitely divisible if and only if $\arg s = \frac{\pi}{2}$. The Lévy measure $\nu_{3i,3}^1$ for $\mu_{3i,3}^1$ can be calculated as*

$$\nu_{3i,3}^1(dx) = \frac{9x^2}{\pi(9x^4 + 3x^2 + 1)} dx, \quad x \in \mathbb{R}.$$

Proof. Because of Remark 3.3(iv), let us consider $s = e^{i\theta}$ for simplicity. After some calculations, we get

$$\operatorname{Im} \phi_{3s,3}^1(x + i0) = -\frac{9x^4 \sin \theta + 3x^3 \sin 2\theta}{|3x^2 + 3xs + s^2|^2}.$$

Therefore, if $\theta \neq 0, \pi, \frac{\pi}{2}$, we can find a point $x_0 \in \mathbb{R}$ such that $\operatorname{Im} \phi_{3s,3}(x_0 + i0) > 0$. If $\theta = \pi$, we can calculate

$$\operatorname{Im} \phi_{-3,3}^1(x + iy) = -\frac{y[6y^2 + 6(x - 1/2)^2 - 1/2]}{|3x^2 + 3xs + s^2|^2},$$

and therefore $\phi_{3s,3}$ takes a positive value at a point. By symmetry, also $\phi_{3,3}$ can take a positive value. Therefore, $\mu_{3s,3}$ is not $\boxplus$ -infinitely divisible for $\theta \neq \frac{\pi}{2}$.

For $\theta = \frac{\pi}{2}$, after some calculations, it holds that

$$\operatorname{Im} \phi_{3i,3}^1(x + iy) = -\frac{9x^4 + 18x^2y^2 + 9y^4 + 12x^2y + 12y^3 + 6y^2 + y}{|3x^2 + 3xi - 1|^2} < 0,$$

so that $\mu_{3i,3}^1$ is $\boxplus$ -infinitely divisible. The measure τ in (2.2) is absolutely continuous with respect to the Lebesgue measure since

$$-\frac{1}{\pi} \operatorname{Im} \phi_{3i,3}^1(x+iy) \rightarrow \frac{9x^4}{\pi|3x^2+3ix-1|^2} = \frac{9x^4}{\pi(9x^4+3x^2+1)}$$

locally uniformly in $\mathbb{R}$ as $y \searrow 0$. The Lévy measure is given by $\frac{1+x^2}{x^2}\tau$, where τ is defined in (2.2). $\square$

5.3. Noninfinite divisibility for $1 < \alpha \leq 2$ and large r

We prove the following.

Proposition 5.7. *For $\alpha > 1$ and $\arg s \in [0, (2-\alpha)\pi]$, there exists an $r_0 = r_0(\alpha, s) > 1$ such that $\mu_{s,r}^\alpha$ is not $\boxplus$ -infinitely divisible for $r > r_0$.*

Proof. Let $\theta := \arg s$. It is sufficient to find a zero point of the function $E_{s,r}^\alpha(z) := \frac{1-(1-(s/r)(-1/z)^\alpha)^r}{s}$. The function $1 - \frac{s}{r}(-\frac{1}{z})^\alpha$ maps $\mathbb{C}_+$ to a shifted sector $\Omega := \{z \in \mathbb{C}: z \neq 0, -(\pi-\theta) < \arg(z-1) < -(\pi-\theta) + \alpha\pi\}$. If $\alpha > 1$, Ω and the unit circle $\{z \in \mathbb{C}: |z|=1\}$ have intersection which is an arc with an end point 1. Let us denote by $\varphi \in (-\pi, \pi) \setminus \{0\}$ the angle of the other end point of that arc. We can take $r_0(\alpha, s)$ to be $\frac{2\pi}{|\varphi|}$. $\square$

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