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Doctoral Dissertation

On the Falk invariant of an arrangement

(超平面配置のFalk不変量)

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Preface

A hyperplane H is an affine subspace with codimension 1 in $\mathbb{C}^\ell$, and an arrangement $\mathcal{A}$ is a finite set of hyperplanes. Let $M = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$ be the complement of the arrangement of $\mathcal{A}$. There are many interesting topological objects related to the complement M . One of them is the fundamental group $G = \pi_1(M)$. And from the fundamental group, we can get its lower central series

$$G = G_1 \supset G_2 \supset G_3 \supset \cdots,$$

where $G_i = [G_{i-1}, G_1]$ for $i \geq 2$.

In Section 1.3, we show that some important information can be got from the series. To study such problems, Falk introduced in [1] a multiplicative invariant that is the rank of the abelian group G_3/G_4 and posed as an open question to give a combinatorial interpretation of the rank for some arrangements. We call it **Falk invariant**.

Several authors already studied this invariant. In [2], Schenck and Suciu studied the lower central series of arrangements and described a formula for the Falk invariant in the case of graphic arrangements. In [3], we gave a formula for ϕ_3 in the case of simple signed graphic arrangements. In the paper [4], we extended the previous result for signed graphic arrangements coming from graphs without loops. In [5], we described a combinatorial formula for the Falk invariant of a signed graphic arrangement that do not have a B_2 as sub-arrangement. In [6], we gave a formula for the Falk invariant ϕ_3 of the arrangements that are canonical linear gain representations of gain graphs that do not have a subgraph isomorphic to B_2 , or loops adjacent to a θ -graph with only three edges and with at most triple parallel edges.

The doctoral thesis is mainly based on [5], [6] and another joint work with Michele Torielli on the Falk invariant of Shi arrangements.

The organization of this thesis is as follows: in the first chapter we give some definitions and examples of hyperplane arrangements, we also give some definitions about Orlik-Solomon algebra and lower central series with some definitions and theorems. In the second chapter, we recall definitions and theorems of graphic arrangements. In the third chapter, we focus on computing the Falk invariant for a signed graphic arrangement. This chapter is mainly based on [5]. In the fourth chapter, we extend to the Falk invariant

for multiplicative gain graphic arrangements and give the matroidal interpretation. This chapter is mainly based on [6]. In the final chapter, after calculating the Falk invariant for an additive gain graphic arrangement, we apply the formula for the formula to calculate the cone of Shi, Linial and semiorder arrangements.

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Chapter 1

Preliminaries

1.1 Hyperplane arrangements

In this section we show some basic definitions of the theory of hyperplane arrangements.

Let $\mathbb{K}$ be a field and V an ℓ -dimensional vector space over $\mathbb{K}$.

Definition 1.1.1. A *hyperplane* H in V is an $(\ell - 1)$ -dimensional affine subspace of V . A *hyperplane arrangement* $\mathcal{A}$ is a finite set of hyperplanes in V . We call $\mathcal{A}$ an ℓ -*arrangement* when we want to emphasize the dimension of V . The *empty* ℓ -arrangement is denoted by Φ_ℓ . If each hyperplane H in $\mathcal{A}$ passes through the origin O , that is $O \in \cap_{H \in \mathcal{A}} H$, we call $\mathcal{A}$ *central*.

Let $S = S(V^*)$ be the symmetric algebra of V^* , where V^* is the dual space of V . Choose a basis $\{v_1, v_2, \dots, v_\ell\}$ in V and let $\{x_1, x_2, \dots, x_\ell\}$ be the dual basis in V^* so $x_i(v_j) = \delta_{i,j}$. We may identify $S(V^*)$ with the polynomial algebra $S = \mathbb{K}[x_1, x_2, \dots, x_\ell]$. Let α_H be a polynomial of degree 1, such that $H = \{\alpha_H = 0\}$.

Definition 1.1.2. The *defining polynomial* $Q(\mathcal{A})$ of an arrangement $\mathcal{A}$ is defined by the product

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H.$$

We agree that $Q(\Phi_\ell) = 1$ is the defining polynomial of the empty arrangement.

Definition 1.1.3. An affine ℓ -arrangement $\mathcal{A}$ defined by $Q(\mathcal{A}) \in S$ gives rise to a central $(\ell + 1)$ -arrangement $c\mathcal{A}$, called the *cone* over $\mathcal{A}$. Let $Q' \in \mathbb{K}[x_0, x_1, \dots, x_\ell]$ be the polynomial $Q(\mathcal{A})$ homogenized and define $Q(c\mathcal{A}) = x_0 Q'$. We call $K_0 = \ker(x_0)$ the *additional hyperplane*.

Definition 1.1.4. For an ℓ -arrangement $\mathcal{A}$ in V , define the **intersection poset** by

$$L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}, \bigcap_{H \in \mathcal{B}} H \neq \emptyset \right\},$$

where we agree that $V = \bigcap_{H \in \emptyset} H$ when $\mathcal{B}$ is the empty set. Define a **partial order** on $L(\mathcal{A})$ by reverse inclusion. In other words, define $x \leq y$ in $L(\mathcal{A})$ if $x \supseteq y$ (as subsets of V).

Example 1.1.5. The **braid arrangement** $\mathcal{B}_\ell$ in $\mathbb{K}^\ell$ is defined as:

$$\mathcal{B}_\ell = \{H_{ij}\}_{1 \leq i < j \leq \ell},$$

where:

$$H_{ij} = \ker(x_i - x_j).$$

The defining polynomial is

$$Q(\mathcal{A}) = \prod_{1 \leq i < j \leq \ell} (x_i - x_j).$$

We can find the number of hyperplanes in $\mathcal{B}_\ell$ is $\binom{\ell}{2}$, and the intersection lattice $L(\mathcal{B}_\ell)$ is isomorphic to the partition lattice of the set $\{1, 2, \dots, \ell\}$, with the maximal element $\bigcap_{1 \leq i < j \leq \ell} H_{ij} = \{x_1 = x_2 = \dots = x_\ell\}$, for the proof we refer the reader to [7, Proposition 2.9].

1.2 Orlik-Solomon algebra

In this section we introduce the Orlik-Solomon algebras. These algebras was given firstly by Arnold, Brieskorn, and Orlik-Solomon [8] when they proved theorems on the cohomology algebras of the complements to complex hyperplane arrangements. For more details we refer the reader to [9].

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement of hyperplanes in $\mathbb{K}^\ell$. Let $E^1 := \bigoplus_{j=1}^n \mathbb{K}e_j$ be the free module generated by $e_1, e_2, \dots, e_n$, where e_i is a symbol corresponding to the hyperplane H_i . Let $E := \bigwedge E^1$ be the exterior algebra over $\mathbb{K}$. The algebra E is graded via $E = \bigoplus_{p=0}^n E^p$, where $E^p := \bigwedge^p E^1$. The $\mathbb{K}$ -module E^p is free and has the distinguished basis consisting of monomials $e_S := e_{i_1} \wedge \dots \wedge e_{i_p}$, where $S = \{i_1, \dots, i_p\}$ is running through all the subsets of $\{1, \dots, n\}$ of cardinality p with $i_1 < i_2 < \dots < i_p$. The graded algebra E is a commutative differential graded algebra with respect to the differential ∂ of degree -1 uniquely defined by the conditions $\partial e_i = 1$ for all

$i = 1, \dots, n$ and the graded Leibniz formula. Then for every $S \subseteq \{1, \dots, n\}$ of cardinality p , we have

$$\partial e_S = \sum_{j=1}^p (-1)^{j-1} e_{S_j},$$

where S_j is the complement in S to its j -th element.

For $S \subseteq \{1, \dots, n\}$, put $\bigcap S := \bigcap_{i \in S} H_i$. Since $\mathcal{A}$ is central, the intersection lattice of $\mathcal{A}$ can be written as $L(\mathcal{A}) := \{\bigcap S \mid S \subseteq \{1, \dots, n\}\}$. A subset $S \subseteq \{1, \dots, n\}$ is called **dependent** if the set of linear polynomials $\{\alpha_i \mid i \in S\}$, with $H_i = \alpha_i^{-1}(0)$, are linearly dependent.

Definition 1.2.1. *The **Orlik–Solomon ideal** of $\mathcal{A}$ is the ideal $I = I(\mathcal{A})$ of E generated by $\{\partial e_S \mid S \text{ dependent}\}$. The algebra $A := A^\bullet(\mathcal{A}) = E/I(\mathcal{A})$ is called the **Orlik–Solomon algebra** of $\mathcal{A}$.*

Clearly I is a homogeneous ideal of E and $I^p = I \cap E^p$ whence A is a graded algebra and we can write $A = \bigoplus_{p \geq 0} A^p$, where $A^p = E^p/I^p$. The map ∂ induces a well-defined differential $\partial: A^p(\mathcal{A}) \longrightarrow A^{p-1}(\mathcal{A})$, for any $p > 0$.

Let I_k be the ideal of E generated by $\sum_{j \leq k} I^j$. We call I_k the **k -adic Orlik–Solomon ideal** of $\mathcal{A}$. It is clear that I_k is a graded ideal and $(I_k)^p = E^p \cap I_k$. Write $A_k := E/I_k$ and $A_k^p := E^p/(I_k)^p$ which is called **k -adic Orlik–Solomon algebra** by Falk [10].

According to the book [7], we decompose the algebra A into a direct sum indexed by elements of $L(\mathcal{A})$.

Definition 1.2.2. *For $Y \in L(\mathcal{A})$ let $S_Y = \{S' \subseteq S \mid \bigcap S' = Y\}$ and let*

$$E_Y := \sum_{S' \in S_Y} \mathbb{K} e_{S'}.$$

Lemma 1.2.3. *(Lemma 3.17, [7]) Since $S = \bigcup_{Y \in L(\mathcal{A})} S_Y$ is a disjoint union, $E = \bigoplus_{Y \in L(\mathcal{A})} E_Y$.*

Let $\delta: E \rightarrow A$ be the natural homomorphism and let $A_Y = \delta(E_Y)$, for $Y \in L(\mathcal{A})$.

Theorem 1.2.4. *(Theorem 3.26, [7]) Let $\mathcal{A}$ be a central arrangement. Then*

$$A = \bigoplus_{Y \in L(\mathcal{A})} A_Y.$$

Corollary 1.2.5. (Corollary 3.27, [7]) Let $\mathcal{A}$ be a central arrangement and $L_p(\mathcal{A}) = \{Y \in L(\mathcal{A}) \mid \text{codim}(Y) = p\}$. Then

$$A^p = \bigoplus_{Y \in L_p(\mathcal{A})} A_Y.$$

Next we construct a basis of the Orlik-Solomon algebra A as described in the book [7].

We introduce an arbitrary linear order $\prec$ in the arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$. Call a p -tuple $S' = (H_1, \dots, H_p)$ **standard** if $H_1 \prec \dots \prec H_p$.

Definition 1.2.6. A p -tuple $S' = (H_1, \dots, H_p)$ is a **circuit** if it is minimally dependent. Thus $(H_1, \dots, H_p)$ is dependent, but for $1 \leq k \leq p$ the $(p-1)$ -tuple $(H_1, \dots, \widehat{H}_k, \dots, H_p)$ is independent, where $\widehat{H}_k$ is the omitted element.

Definition 1.2.7. Given $S' = (H_1, \dots, H_p)$, let $\max S'$ be the maximal element of S' in the linear order $\prec$ in $\mathcal{A}$.

Definition 1.2.8. A standard p -tuple $S' \in S$ is a **broken circuit** if there exists $H \in \mathcal{A}$ such that $\max S' \prec H$ and (S', H) is a circuit.

Definition 1.2.9. A standard p -tuple S' is called **χ -independent** if it does not contain any broken circuit.

Theorem 1.2.10. (Theorem 3.43, [7]) In a central arrangement $\mathcal{A}$ with a linear order $\prec$, the set

$$\{e_{S'} + I \in A(\mathcal{A}) \mid S' \text{ is standard and } \chi\text{-independent}\}$$

is a basis for $A(\mathcal{A})$ as a graded $\mathbb{K}$ -module.

1.3 Lower central series

For a hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$, the complement

$$M = M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$$

is of significant interest in the study of hyperplane arrangements. The fundamental group $G = \pi_1(M)$ is arguably the most important group associated to an arrangement. The lower central series of G is

$$G = G_1 \supset G_2 \supset G_3 \supset \dots,$$

where $G_i = [G_{i-1}, G_1]$ for $i \geq 2$. Then by [11, Theorem 5.4], G_i/G_{i+1} is a finitely generated abelian group. The graded vector space $\text{gr}(G; \mathbb{Q}) := (\bigoplus G_i/G_{i+1}) \otimes \mathbb{Q}$ is in fact a Lie algebra, with bracket induced by group commutators, hence called the *associated graded Lie algebra* of G . The dimension of each graded piece $\phi_i(\mathcal{A}) = \phi_i(G) = \text{rank}(G_i/G_{i+1})$ is then an invariant of the arrangement $\mathcal{A}$. In the literature,

$$U_{\mathcal{A}}(t) := \prod_{i=1}^{\infty} (1 - t^i)^{\phi_i(\mathcal{A})} \in \mathbb{Z}[[t]]$$

is used frequently to encapsulate all information about $\phi_i(\mathcal{A})$. An explicit equality relating $U_{\mathcal{A}}(t)$ with some polynomial is usually called a *LCS (lower central series) formula*. Kohno [12] found one formula for braid arrangement:

$$U_{\mathcal{A}}(t) = P_M(-t),$$

where $P_M(t)$ is the Poincaré polynomial of M . The same formula has been proved for *fiber-type arrangements* by Falk and Randell [13], Shelton and Yuzvinsky [14] also gave an interpretation using Koszul duality. The LCS formula for M a *formal rational $K(\pi, 1)$ space* has been given by Papadima and Yuzvinsky [15]. An analogue formula for *hypersolvable arrangements* has been proved by Jambu and Papadima [16]. For *decomposable arrangements*, Papadima and Suciu [17] gave a LCS formula. Lima-Filho and Schenck [18] proved a LCS formula for graphic arrangements (subarrangements of braid arrangements), which was conjectured in [19].

In Kohno's paper [20], he gave a description of the holonomy Lie algebra for M the complement of a hyperplane arrangement $\mathcal{A}$ in terms of the intersection lattice of $\mathcal{A}$ up to rank 2. To state Kohno's result, we will show some definitions and notations.

At first, we recall the basics of free Lie algebras (over $\mathbb{Q}$). For more details, see [21].

Let X be a finite set (of letters) and $M(X)$ the free magma on X . Pictorially $M(X)$ is the set of full rooted binary trees whose leaves are labeled by letters in X . For each tree $t \in M(X)$, let the length $\ell(t)$ be the number of leaves of t . Write $M(X)_n := \{t \in M(X) \mid \ell(t) = n\}$ and $M(X)_0 = \{\emptyset\}$. The noncommutative polynomial algebra $\mathbb{Q}\langle X \rangle$ is the $\mathbb{Q}$ -algebra on $M(X)$. Then $\mathbb{Q}\langle X \rangle$ has a natural grading $\mathbb{Q}\langle X \rangle = \bigoplus_{n \geq 0} \mathbb{Q}\langle X \rangle_n$, where the homogeneous component $\mathbb{Q}\langle X \rangle_n$ is the $\mathbb{Q}$ -vector space on $M(X)_n$. Define a Lie bracket on $\mathbb{Q}\langle X \rangle$

$$[P, Q] = PQ - QP$$

for $P, Q \in \mathbb{Q}\langle X \rangle$. Therefore $\mathbb{Q}\langle X \rangle$ becomes a Lie algebra. Let $\mathbb{L}(X)$ be the smallest Lie subalgebra of $\mathbb{Q}\langle X \rangle$ containing X , it is called the *free Lie*

algebra on X . Indeed, $\mathbb{L}(X)$ has the universal property: for an arbitrary Lie algebra K and an arbitrary mapping $f : X \rightarrow K$, there is a unique Lie algebra homomorphism $F : \mathbb{L}(X) \rightarrow K$ such that $f = F \circ i$, where $i : X \rightarrow \mathbb{L}(X)$ is the inclusion. As a Lie subalgebra of $\mathbb{Q}\langle X \rangle$, $\mathbb{L}(X)$ inherits a grading $\mathbb{L}(X) = \bigoplus_{n \geq 0} \mathbb{L}(X)_n$.

Note that there is a canonical mapping $\Psi : M(X) \rightarrow \mathbb{L}(X)$ taking Lie brackets according to the tree. For instance $\Psi(x(yz)) = [x, [y, z]]$.

For a vector space V with basis X , we also let $\mathbb{L}(V) = \mathbb{L}(X)$.

Let $\mathcal{A}$ be a central arrangement and $\mathbb{L}(\mathcal{A})$ the free Lie algebra on $\mathcal{A}$. For a rank 2 flat $L \in L_2(\mathcal{A})$, let $\mathcal{A}_L = \{H \in \mathcal{A} \mid H < L\}$ and write $T_L = \sum_{H \in \mathcal{A}_L} H \in \mathbb{L}(\mathcal{A})_1$. Then define $\mathcal{I}(\mathcal{A})$ to be the ideal of $\mathbb{L}(\mathcal{A})$ generated by

$$\{[H, T_L] \mid L \in L_2(\mathcal{A}), H < L\}.$$

Note that $\mathcal{I}(\mathcal{A})$ is a graded ideal $\mathcal{I}(\mathcal{A}) = \bigoplus_n \mathcal{I}(\mathcal{A})_n$, where $\mathcal{I}(\mathcal{A})_n = \mathcal{I}(\mathcal{A}) \cap \mathbb{L}(\mathcal{A})_n$ and $\mathcal{I}(\mathcal{A})_0 = \mathcal{I}(\mathcal{A})_1 = 0$. Let $\mathfrak{h}(\mathcal{A}) := \mathbb{L}(\mathcal{A})/\mathcal{I}(\mathcal{A})$ be the quotient Lie algebra with the natural grading.

Theorem 1.3.1 ([20]). *Let M be the complement of a complex hypersurface in $\mathbb{C}^\ell$, put $G = \pi_1(M)$, and $G = G_1 \supset G_2 \supset G_3 \supset \dots$ the lower central series of G . Then there is an isomorphism of graded Lie algebras*

$$\mathfrak{h}(M) \cong \left(\bigoplus G_i/G_{i+1} \right) \otimes \mathbb{Q}.$$

If M is the complement of a hyperplane arrangement $\mathcal{A}$, we have

$$\mathfrak{h}(M) \cong \mathfrak{h}(\mathcal{A}).$$

We take $\mathfrak{h}(\mathcal{A})$ as the definition of **holonomy Lie algebra** of the arrangement $\mathcal{A}$.

We call the set $\{t_1, t_2, \dots, t_k\} \subseteq E$ the **minimal generators** of the Orlik-Solomon ideal I if $I = \langle t_1, \dots, t_k \rangle$, but $\langle t_1, \dots, \widehat{t_i}, \dots, t_k \rangle \subsetneq I$, where $\widehat{t_i}$ ($1 \leq i \leq k$) denotes the omitted element.

Proposition 1.3.2. *Let $\mathcal{A}$ be a central arrangement with n hyperplanes in $\mathbb{C}^\ell$, and let a_j denote the number of minimal generators of degree j in the Orlik-Solomon ideal I . Then $\phi_1(\mathcal{A}) = |\mathcal{A}|$ and $\phi_2(\mathcal{A}) = a_2$.*

Proof. By Theorem 1.3.1 we can compute the invariant $\phi_i(G) = \text{rank}(G_i/G_{i+1})$ of arrangement $\mathcal{A}$ through calculating the dimension of $\mathfrak{h}(M)_i$. Since $\mathfrak{h}(M)_i = (\mathbb{L}(\mathcal{A})/\mathcal{I}(\mathcal{A}))_i$, then $\phi_i = \dim(\mathbb{L}(\mathcal{A})_i) - \dim(\mathcal{I}(\mathcal{A})_i)$. From the fact $\mathcal{I}(\mathcal{A})_0 = \mathcal{I}(\mathcal{A})_1 = 0$, we can get $\phi_1 = \dim(\mathbb{L}(\mathcal{A})_1) = |\mathcal{A}|$.

To compute ϕ_2 , we will consider the relation between the ideal $\mathcal{I}(\mathcal{A})_2$ and the Orlik-Solomon algebra A_2 .

At first we give a linear order $\prec$ for the arrangement $\mathcal{A}$ such that $H_1 \prec \dots \prec H_n$. From the definitions we know $\mathcal{I}(\mathcal{A})_2 = \text{span}_{\mathbb{Q}}\{[H, T_L] \mid L \in L_2(\mathcal{A}), H < L\}$. For each rank 2 flat Y , write $\mathcal{I}_Y = \text{span}_{\mathbb{Q}}\{[H, T_Y] \mid H < Y\}$, so $\mathcal{I}(\mathcal{A})_2 = \sum_{Y \in L_2(\mathcal{A})} \mathcal{I}_Y$. Since any pair of hyperplanes can only appear under one rank 2 flat, so for all $Y_1, \dots, Y_m \in L_2(\mathcal{A})$, the subspaces $\mathcal{I}_{Y_1}, \dots, \mathcal{I}_{Y_m}$ are linearly independent, then $\mathcal{I}(\mathcal{A})_2 = \bigoplus_{Y \in L_2(\mathcal{A})} \mathcal{I}_Y$.

From Corollary 1.2.5, we can get $A_2 = \bigoplus_{Y \in L_2(\mathcal{A})} A_Y$. Next, we will consider the relation between $\mathcal{I}_Y$ and A_Y for each rank 2 flat Y .

Assume that under the rank 2 flat Y , there are k hyperplanes $H_{i_1} \prec \dots \prec H_{i_k}$. Then $\mathcal{I}_Y = \text{span}_{\mathbb{Q}}\{[H_{i_j}, H_{i_1} + \dots + \widehat{H}_{i_j} + \dots + H_{i_k}] \mid 1 \leq j \leq k\}$. Since $\sum_{1 \leq j \leq k} [H_{i_j}, H_{i_1} + \dots + \widehat{H}_{i_j} + \dots + H_{i_k}] = 0$ and the set $\{[H_{i_j}, H_{i_1} + \dots + \widehat{H}_{i_j} + \dots + H_{i_k}] \mid 1 \leq j \leq k-1\}$ is linearly independent, so this set can be a basis of $\mathcal{I}_Y$ and $\dim(\mathcal{I}_Y) = k-1$.

In the subarrangement $\mathcal{A}_Y = \{H_{i_1}, \dots, H_{i_k}\}$, if $1 \leq t < j < k$, the tuple $(H_{i_t}, H_{i_j}, H_{i_k})$ is a circuit, then the standard 2-tuple (H_{i_t}, H_{i_j}) is not χ -independent. Then we can find that all of 2-tuples which are χ -independent are (H_{i_t}, H_{i_k}) for $1 \leq t \leq k-1$. From Theorem 1.2.10, we can get that the set $\{e_{i_t i_k} + I_2 \mid 1 \leq t \leq k-1\}$ is a basis of A_Y and $\dim(A_Y) = \dim(\mathcal{I}_Y) = k-1$.

Therefore, $\dim(\mathcal{I}(\mathcal{A})_2) = \sum_{Y \in L_2(\mathcal{A})} \dim(\mathcal{I}_Y) = \sum_{Y \in L_2(\mathcal{A})} \dim(A_Y) = \dim(A_2)$. Since $\dim(\mathbb{L}(\mathcal{A})_2) = \binom{n}{2} = \dim(E_2)$, So

$$\begin{aligned} \phi_2 &= \dim(\mathbb{L}(\mathcal{A})_2) - \dim(\mathcal{I}(\mathcal{A})_2) \\ &= \dim(E_2) - \dim(A_2) \\ &= \dim(I_2) = a_2. \end{aligned}$$

□

In [1], Falk firstly gave a formula for ϕ_3 of an arbitrary arrangement, but now we do not have a general formula for ϕ_k when $k \geq 4$. Next, we will introduce the Falk invariant from the lower central series.

Definition 1.3.3. *The **Falk invariant** is defined as the third rank of successive quotients in the lower central series, that is,*

$$\phi_3 = \text{rank}(G_3/G_4).$$

Remark 1.3.4. *Consider the map $d: E^1 \otimes I^2 \rightarrow E^3$. According to Falk's paper [10], the invariant ϕ_3 can also be considered as the dimension of the kernel of the map d .*

In [10] and [22], Falk gave a beautiful formula to compute such invariant.

Theorem 1.3.5. [22, Theorem 4.7] *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement of hyperplanes in $\mathbb{K}^\ell$. Then*

$$\phi_3 = 2 \binom{n+1}{3} - n \dim(A^2) + \dim(A_2^3). \quad (1.1)$$

Remark 1.3.6. *Since $\dim(A_2^3) = \dim((E/I_2)^3) = \dim(E^3) - \dim((I_2)^3)$ and $\dim(E^3) = \binom{n}{3}$, then we obtain*

$$\phi_3 = 2 \binom{n+1}{3} - n \dim(A^2) + \binom{n}{3} - \dim((I_2)^3). \quad (1.2)$$

Chapter 2

Gain graphic arrangements

In this chapter we recall the basic notions of graphs, and we describe the connection between hyperplane arrangements and these graphs. See [23], [24], [25] and [26] for a thorough treatment of the subject. See also [27], for generalities on graph theory.

2.1 Gain graphs

Definition 2.1.1. A **graph** G is an ordered triple $(\mathcal{V}_G, \mathcal{E}_G, \psi_G)$ consisting of a nonempty set $\mathcal{V}_G$ of **vertices**, a set $\mathcal{E}_G$, disjoint from $\mathcal{V}_G$, of **edges**, and an **incidence function** ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G . If $\mathbf{e}$ is an edge and u and v are vertices such that $\psi_G(\mathbf{e}) = uv$, then $\mathbf{e}$ is said to **join** u and v ; the vertices u and v are called the **endpoints** of $\mathbf{e}$. An edge is a **link** if it has two distinct endpoints, a **loop** if two coincident endpoints.

In the following, we will give an example to clarify the definitions.

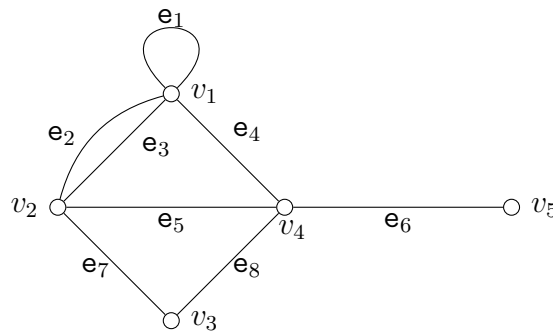


Figure 2.1: Example of a graph.

Example 2.1.2. In Figure 2.1, we have a graph $G = (\mathcal{V}_G, \mathcal{E}_G, \psi_G)$, where

$$\begin{aligned}\mathcal{V}_G &= \{v_1, v_2, v_3, v_4, v_5\}, \\ \mathcal{E}_G &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\},\end{aligned}\tag{2.1}$$

and ψ_G is defined by

$$\begin{aligned}\psi_G(e_1) &= v_1v_1, \psi_G(e_2) = v_1v_2, \psi_G(e_3) = v_1v_2, \psi_G(e_4) = v_1v_4, \\ \psi_G(e_5) &= v_2v_4, \psi_G(e_6) = v_4v_5, \psi_G(e_7) = v_2v_3, \psi_G(e_8) = v_3v_4.\end{aligned}\tag{2.2}$$

Definition 2.1.3. A graph G is *isomorphic* to a graph H if there is a bijection between the vertex sets of G and H

$$f: \mathcal{V}(G) \rightarrow \mathcal{V}(H),$$

such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H , and the number of edges joining u and v is same as the number of edges joining $f(u)$ and $f(v)$.

Definition 2.1.4. Let $\mathfrak{G}$ be a group. A *gain graph* $\mathcal{G} = (G, \varphi)$ consists of an underlying graph $|\mathcal{G}| = G = (\mathcal{V}_G, \mathcal{E}_G)$ and a *gain map* $\varphi: \mathcal{E}_G \rightarrow \mathfrak{G}$ from the edges of G into the *gain group* $\mathfrak{G}$ such that when $v_1v_2 \in \mathcal{E}_G$, $\varphi(v_1v_2) = \varphi(v_2v_1)^{-1}$. (This applies to loops as well as links.) To be precise we may call $\mathcal{G}$ an *additive gain graph* if $\mathfrak{G}$ is an additive group. While call $\mathcal{G}$ a *multiplicative gain graph* if $\mathfrak{G}$ is a multiplicative group.

In this paper, we consider the vertex set $\mathcal{V}_G = \{v_1, v_2, \dots, v_\ell\}$ is a finite set, and the group $\mathfrak{G}$ is commutative. Since $\varphi(e^{-1}) = \varphi(e)^{-1}$, then $\varphi(e)$ depends on the orientation of e but neither orientation is preferred.

Definition 2.1.5. A *signed graph* is a multiplicative gain graph with gain group $\mathfrak{G}$ equal to $\{1, -1\}$.

For more details, we refer the reader to [23] and [26].

A **subgraph** of $\mathcal{G}$ is a subgraph of the underlying graph $|\mathcal{G}|$ with the same gain map, restricted to the subgraph's edges.

A **walk** is a chain of vertices and edges,

$$P = \{v_0, e_1, v_1, e_2, \dots, e_k, v_k\},$$

where $v_i \in \mathcal{V}_G$, $e_i \in \mathcal{E}_G$, and $\psi_G(e_i) = v_{i-1}v_i$, $i \in \{1, 2, \dots, k\}$. To indicate its endpoints we may write $P: v_0 \rightarrow v_k$. To indicate the edge sequence of P , it may be written as a word

$$P = e_1e_2 \cdots e_k$$

in the free group $\mathfrak{F}(\mathcal{E}_G)$ generated by $\mathcal{E}_G$. A walk is a **path** if it has no repeated vertices except possibly for $v_k = v_0$ if $k > 0$ (then it is closed, otherwise open). A **circle** is a closed path. We consider some circles are **equal** when they contain the same vertices and edges. For example, in Figure 2.1, the circles $\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5$, $\mathbf{e}_4\mathbf{e}_3\mathbf{e}_5$ and $\mathbf{e}_5\mathbf{e}_4\mathbf{e}_3$ are equal.

If the gain graph is multiplicative, a path $P = \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_k$ has gain value $\varphi(P) = \varphi(\mathbf{e}_1)\varphi(\mathbf{e}_2)\cdots\varphi(\mathbf{e}_k)$ under φ . If the gain graph is additive, $\varphi(P) = \varphi(\mathbf{e}_1) + \varphi(\mathbf{e}_2) + \cdots + \varphi(\mathbf{e}_k)$. If P is a circle, its gain depends on the starting point and direction, but whether or not the gain equals the identity element is independent of the starting point and direction. For a multiplicative gain graph, a circle whose gain value is 1 is called **balanced**. While for an additive gain graph, a circle P is a **balanced** circle if and only if $\varphi(P) = 0$. In a multiplicative or additive gain graph, a circle is **unbalanced** if it is not balanced. The class of balanced circles is denoted by $\mathcal{B}(\mathcal{G})$. We write $\langle \mathcal{G} \rangle = (G, \mathcal{B}(\mathcal{G}))$. We call $\mathcal{G}$ **balanced** if all its circles are balanced, and **contrabalanced** if it contains no balanced circles at all.

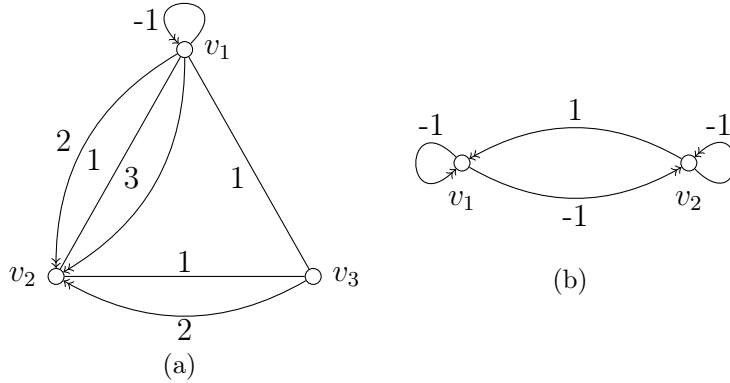


Figure 2.2: Example of two multiplicative gain graphs.

Example 2.1.6. In Figure 2.2(a), we have a multiplicative gain graph $\mathcal{G}$ with gains in $\mathbb{Q}^*$, the multiplicative group of rational numbers. We adopt the simplified notation $\mathbf{e}_{ij}(g)$ for an edge $\{v_i, v_j\}$ with gain $\varphi(\mathbf{e}_{ij}(g)) = g$. (Then for instance $\mathbf{e}_{12}(2) = \mathbf{e}_{21}(2^{-1})$.) The balanced circles are $C_1 := \{\mathbf{e}_{12}(1), \mathbf{e}_{23}(1), \mathbf{e}_{13}(1)\}$ and $C_2 := \{\mathbf{e}_{12}(2), \mathbf{e}_{32}(2), \mathbf{e}_{13}(1)\}$. In fact their gains are $\varphi(C_1) = 1 \cdot 1 \cdot 1 = 1$ and $\varphi(C_2) = 2 \cdot 2^{-1} \cdot 1 = 1$. Therefore $\langle \mathcal{G} \rangle = (G, \{C_1, C_2\})$.

While in Figure 2.2(b), there is no balanced circle, we call it B_2 .

Example 2.1.7. Next, we illustrate a signed graph as follows:

$$G = (\mathcal{V}_G, \mathcal{E}_G^+, \mathcal{E}_G^-, \mathcal{L}_G) = \left(\begin{array}{cc} 1 & 4 \\ \circ & \bullet \\ | & \diagup \\ \circ & \bullet \\ 2 & 3 \end{array}, \begin{cases} \mathcal{V}_G = \{1, 2, 3, 4\}, \\ \mathcal{E}_G^+ = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}, \\ \mathcal{E}_G^- = \{\{1, 3\}, \{2, 3\}, \{2, 4\}\}, \\ \mathcal{L}_G = \{3, 4\}. \end{cases} \right),$$

Where $\mathcal{V}_G$ is the vertices set, $\mathcal{E}_G^+$ (respectively $\mathcal{E}_G^-$) is the set of positive (respectively negative) edges with the gain 1 (respectively -1), shown in the figure with solid lines (respectively dashed lines) and $\mathcal{L}_G$ is the set of loops shown in the figure with black points. The double edges joining v_1 and v_3 are shown with double lines, one is positive and the other is negative. There is two balanced circles $C_1 = \{\mathbf{e}_{12}(1), \mathbf{e}_{23}(-1), \mathbf{e}_{31}(-1)\}$ and $C_2 = \{\mathbf{e}_{14}(1), \mathbf{e}_{42}(-1), \mathbf{e}_{23}(-1), \mathbf{e}_{31}(1)\}$. Therefore $\langle \mathcal{G} \rangle = (G, \{C_1, C_2\})$.

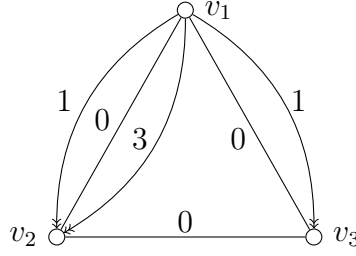


Figure 2.3: Examples of an additive gain graph.

Example 2.1.8. In Figure 2.3, we see a gain graph $\mathcal{G}$ containing 3 vertices with gains in $\mathbb{Q}$, the additive group of rational numbers. Here, $\mathbf{e}_{13}(1) = \mathbf{e}_{31}(-1)$. The balanced circles are $C_1 := \{\mathbf{e}_{12}(0), \mathbf{e}_{23}(0), \mathbf{e}_{13}(0)\}$ and $C_2 := \{\mathbf{e}_{12}(1), \mathbf{e}_{23}(0), \mathbf{e}_{31}(-1)\}$. In fact their gains are $\varphi(C_1) = 0 + 0 + 0 = 0$ and $\varphi(C_2) = 1 + 0 - 1 = 0$. Therefore $\langle \mathcal{G} \rangle = (G, \{C_1, C_2\})$.

Definition 2.1.9. A **theta graph** is a subdivision of a triple link, that is, three open paths meeting only at their endpoints. A **handcuff** consists of a pair of edge sets, C_1 and C_2 , each of which is a circle or a half-edge singleton set, and the edge set of a connecting open path $P : u_1 \rightarrow u_2$ such that P meets C_i at u_i and nowhere else and C_1 meets C_2 only at $\{u_1\} \cap \{u_2\}$. If P has positive length the handcuff is **loose**. Otherwise it is **tight**.

For example, in the following figures there are one theta graph, one loose handcuff and one tight handcuff.

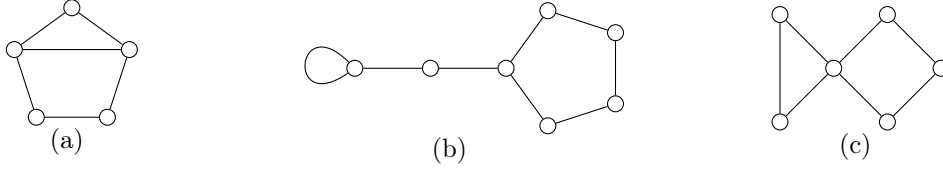


Figure 2.4: Examples of (a) theta graph, (b) loose handcuff, (c) tight handcuff.

Definition 2.1.10. A **biased graph** $\mathcal{H}$ consists of an underlying graph H and a set $\mathcal{D}(\mathcal{H})$ of circles of H called **distinguished circles**. We will always let $\mathcal{H} = (H, \mathcal{D}(\mathcal{H})) = (\mathcal{V}_H, \mathcal{E}_H, \mathcal{D})$ denote a biased graph with underlying graph $H = (\mathcal{V}_H, \mathcal{E}_H)$ and distinguished circle class $\mathcal{D}(\mathcal{H})$.

Let $\mathcal{G}$ be a gain graph, the graph $\langle \mathcal{G} \rangle = (G, \mathcal{B}(\mathcal{G}))$ is a biased graph, where the class of balanced circles $\mathcal{B}(\mathcal{G})$ is the set of distinguished circles.

Definition 2.1.11. Two biased graphs $\langle \mathcal{G}_1 \rangle = (G_1, \mathcal{D}_1)$ and $\langle \mathcal{G}_2 \rangle = (G_2, \mathcal{D}_2)$ are **isomorphic**, written $\langle \mathcal{G}_1 \rangle \cong \langle \mathcal{G}_2 \rangle$, if the two underlying graphs are isomorphic, and a circle is in $\mathcal{D}_1$ if and only if its image is in $\mathcal{D}_2$.

Theorem 2.1.12. [24, Theorem 2.1] Let $\mathcal{G}$ be a multiplicative gain graph. Then there is a matroid $M(\mathcal{G})$, whose points are the edges of $\mathcal{G}$ and whose circuits are the edge sets of balanced circles, contrabalanced theta graphs, contrabalanced loose handcuffs and contrabalanced tight handcuffs.

[24, Theorem 3.1] Similarly, let $\mathcal{G}$ be an additive gain graph. Then there is a matroid $L_0(\mathcal{G})$, whose points are the edges of $\mathcal{G}$ together with an extra point $\mathbf{e}_0$ and whose circuits consists of the edge sets of all balanced circles along with all contrabalanced theta graphs, all contrabalanced loose handcuffs, and all the unions of $\mathbf{e}_0$ and an unbalanced circle.

Definition 2.1.13. Let $\mathcal{G}$ be a multiplicative gain graph, then the matroid $M(\mathcal{G})$ is called the **frame matroid** associated to $\mathcal{G}$. While let $\mathcal{G}$ be an additive gain graph, the matroid $L_0(\mathcal{G})$ is called **complete lift matroid** associated to $\mathcal{G}$.

Definition 2.1.14. A set of edges of a gain multiplicative graph $\mathcal{G}$ (respectively an additive graph $\mathcal{G}$) is called a **circuit** if the corresponding points form a circuit in $M(\mathcal{G})$ (respectively $L_0(\mathcal{G})$).

Consider a multiplicative (respectively additive) gain graph $\mathcal{G} = (G, \varphi)$ with an underlying graph $G = (\mathcal{V}_G, \mathcal{E}_G)$. Let $\mathfrak{G}$ be a multiplicative group (respectively additive group) and $\lambda: \mathcal{V}_G \rightarrow \mathfrak{G}$ be any function. Switching the

gain graph $\mathcal{G}$ by λ means replacing $\varphi(\mathbf{e})$ by $\varphi^\lambda(\mathbf{e}) := \lambda(v)^{-1}\varphi(\mathbf{e})\lambda(w)$ (respectively $\varphi^\lambda(\mathbf{e}) := -\lambda(v) + \varphi(\mathbf{e}) + \lambda(w)$), where $\mathbf{e}$ is oriented from v to w . The switched graph, $\mathcal{G}^\lambda = (G, \varphi^\lambda)$, is called **switching equivalent** to $\mathcal{G}$. In general, we will denote by $[\mathcal{G}]$ any gain graph that is switching equivalent to $\mathcal{G}$ for some λ .

Lemma 2.1.15. [23, Lemma 5.2] $\langle [\mathcal{G}] \rangle = \langle \mathcal{G} \rangle$.

Directly from Theorem 2.1.12, we have the following result.

Proposition 2.1.16. *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two multiplicative gain graphs (respectively additive gain graphs) such that $\langle \mathcal{G}_1 \rangle = \langle \mathcal{G}_2 \rangle$, then $M(\mathcal{G}_1) = M(\mathcal{G}_2)$ (respectively $L_0(\mathcal{G}_1) = L_0(\mathcal{G}_2)$).*

By Proposition 2.1.16 and Lemma 2.1.15, we have the following

Corollary 2.1.17. $M([\mathcal{G}]) = M(\mathcal{G})$ (or $L_0([\mathcal{G}]) = L_0(\mathcal{G})$).

2.2 Hyperplane arrangement realizations of gain graphs.

In this section we consider a gain graph $\mathcal{G} = (G, \varphi)$ with an underlying graph $G = (\mathcal{V}_G, \mathcal{E}_G, \psi_G)$, a gain map φ , a gain group $\mathfrak{G}$ contained in the field $\mathbb{Q}$ (either multiplicative or additive), the edge set $\mathcal{E}_G$, and the vertices set $\mathcal{V}_G = \{v_1, v_2, \dots, v_\ell\}$. If $\psi_G(\mathbf{e}) = v_i v_j$, we will write the edge $\mathbf{e}$ as $\mathbf{e}_{ij}$.

For a multiplicative gain graph $\mathcal{G}$, we assume that all 2-circles and loops in $\mathcal{G}$ are unbalanced, then if $\mathbf{e}_{ii}$ is a loop, we have $\varphi(\mathbf{e}_{ii}) \neq 1$, and we attach to it the hyperplane $\{x_i = 0\}$.

Definition 2.2.1. *For a multiplicative group $\mathfrak{G}$ in $\mathbb{Q}^*$, let $\mathcal{A}(\mathcal{G})$ be the hyperplane arrangement in $\mathbb{C}^\ell$ consisting of the following hyperplanes*

$$\{x_i = \varphi(\mathbf{e}_{ij})x_j\} \text{ for } \mathbf{e}_{ij} \in \mathcal{E}_G.$$

*We will call $\mathcal{A}(\mathcal{G})$ the **canonical linear hyperplane representation** of $\mathcal{G}$.*

Example 2.2.2. *Consider the gain graph described in Figure 2.2(a). Then we obtain the hyperplane arrangement $\mathcal{A}(\mathcal{G}) \subseteq \mathbb{R}^3$ with defining polynomial $Q(\mathcal{A}(\mathcal{G})) = x(x-y)(x-2y)(x-3y)(y-z)(2y-z)(x-z)$.*

Example 2.2.3. *Consider the signed graph described in Example 2.1.7. Then we obtain the hyperplane arrangement $\mathcal{A}(\mathcal{G}) \subseteq \mathbb{R}^4$ with defining polynomial $Q(\mathcal{A}(\mathcal{G})) = x_3 x_4 (x_1 - x_2)(x_1 - x_3)(x_1 + x_3)(x_1 - x_4)(x_2 + x_4)(x_2 + x_3)$.*

For an arrangement $\mathcal{A}(\mathcal{G})$, we can find the matroid $M(\mathcal{A}(\mathcal{G}))$ associated to the intersection lattice of $\mathcal{A}(\mathcal{G})$, the independent sets in $M(\mathcal{A}(\mathcal{G}))$ are determined by the linear independent subsets of $\mathcal{A}(\mathcal{G})$ and the matroid $M(\mathcal{A}(\mathcal{G}))$ contains the same information as the intersection lattice $L(\mathcal{A}(\mathcal{G}))$. For more details, see [28].

Given a multiplicative gain graph $\mathcal{G}$, we can now associate to it two matroids: the frame matroid $M(\mathcal{G})$ and the matroid $M(\mathcal{A}(\mathcal{G}))$ associated to the intersection lattice of $\mathcal{A}(\mathcal{G})$. In [25], Zaslavsky proved that these two matroids coincide. In particular, he proved the following.

Theorem 2.2.4. [25, Corollary 2.2] $M(\mathcal{G}) \cong M(\mathcal{A}(\mathcal{G}))$.

For an additive gain graph $\mathcal{G}$, to study its corresponding arrangements we have the definition as following.

Definition 2.2.5. For an additive group $\mathfrak{G}$ in $\mathbb{Q}$, let $\mathcal{A}(\mathcal{G})$ be the hyperplane arrangement in $\mathbb{C}^{\ell+1}$ consisting of the following hyperplanes

$$\{x_0 = 0\} \cup \{x_i - x_j + \varphi(\mathbf{e}_{ij})x_0 = 0\} \text{ for } \mathbf{e}_{ij} \in \mathcal{E}_{\mathcal{G}},$$

where the coordinate x_i is corresponding to the vertex v_i for $1 \leq i \leq \ell$, and x_0 is an extra coordinate we put in $\mathbb{C}^{\ell+1}$. We will call $\mathcal{A}(\mathcal{G})$ the **canonical linear complete lift representation** of $\mathcal{G}$.

Example 2.2.6. Consider the additive gain graph $\mathcal{G}$ described in Example 2.1.8. Then we obtain the hyperplane arrangement $\mathcal{A}(\mathcal{G}) \subseteq \mathbb{R}^4$ with defining polynomial $Q(\mathcal{A}(\mathcal{G})) = x_0(x_1 - x_2 + x_0)(x_1 - x_2)(x_1 - x_2 + 3x_0)(x_2 - x_3)(x_3 - x_1)(x_1 - x_3 + x_0)$.

Similar to a multiplicative gain graph, given an additive gain graph $\mathcal{G}$, we can now associate to it two matroids: the complete lift matroid $L_0(\mathcal{G})$ and the matroid of intersections of $\mathcal{A}(\mathcal{G})$. In [25], Zaslavsky proved that these two matroids coincide. In particular, he proved the following

Theorem 2.2.7. $L_0(\mathcal{G}) \cong M(\mathcal{A}(\mathcal{G}))$, where $M(\mathcal{A}(\mathcal{G}))$ is the matroid associated with $\mathcal{A}(\mathcal{G})$.

Proposition 2.2.8. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two gain graphs (multiplicative or additive) such that $\langle \mathcal{G}_1 \rangle = \langle \mathcal{G}_2 \rangle$. Then $\phi_3(\mathcal{A}(\mathcal{G}_1)) = \phi_3(\mathcal{A}(\mathcal{G}_2))$.

Proof. By Proposition 2.1.16 and Theorem 2.2.4, $M(\mathcal{A}(\mathcal{G}_1)) \cong M(\mathcal{A}(\mathcal{G}_2))$. This implies that $\mathcal{A}(\mathcal{G}_1)$ and $\mathcal{A}(\mathcal{G}_2)$ have isomorphic Orlik–Solomon algebra, and hence they have the same Falk invariant ϕ_3 . $\square$

Corollary 2.2.9. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two switching equivalent gain graphs. Then $\phi_3(\mathcal{A}(\mathcal{G}_1)) = \phi_3(\mathcal{A}(\mathcal{G}_2))$.

2.3 Falk invariant of a graphic arrangement

In this section we give our idea of computing the Falk invariant ϕ_3 associated to a graphic arrangement, this idea can be applied to both multiplicative graphs and additive graphs that we assume. Here we assume the multiplicative gain graph has no subgraph isomorphic to $\langle B_2 \rangle$ (see Figure 2.2(b)) or to a contrabalanced triple parallel edge with an adjacent loop and has at most triple parallel edges. And we assume the additive gain graph contains no loops and at most double parallel edges.

To fix the notation, we will suppose $\mathcal{G}$ is a gain graph whose underlying graph $|\mathcal{G}|$ is on ℓ vertices. Since our result will only depend on $\langle \mathcal{G} \rangle$ and not on the specific gain value of the edges (see 2.2.8), we will label the corresponding hyperplanes of $\mathcal{A}(\mathcal{G})$ as elements of $[n] := \{1, 2, \dots, n\}$.

In order to compute ϕ_3 , we will use Theorem 1.3.5, hence we need firstly to identify the triples S in $[n]$ that are dependent. In fact a dependent triple S corresponds to a circuit of size 3 in the matroid associated to the gain graphic arrangement. We call such S a **3-circuit**. Moreover, we will write

$$\mathcal{C}_3 := \text{span}\{e_S \in E \mid S \text{ is a 3-circuit}\}$$

which is a subset of E as a vector space over $\mathbb{C}$. Let $C'_3 = \{e_S \mid S \text{ is a 3-circuit}\}$ be a basis of $\mathcal{C}_3$. By construction $I^2 = \text{span}\{\partial e_{ijk} \mid e_{ijk} \in C'_3\}$. Notice that if e_{ijk}, e_{pqr} are distinct elements of C'_3 then the corresponding subgraphs of $\langle \mathcal{G} \rangle$ share at most one edge and hence ∂e_{ijk} and ∂e_{pqr} are linearly independent. This implies that $\dim(I^2) = \dim(\mathcal{C}_3) = \#C'_3$. Since by definition $A = E/I$, hence

$$\dim(A^2) = \dim(E^2) - \dim(I^2) = \binom{n}{2} - \#C'_3.$$

Using Theorem 1.3.5 and Remark 1.3.6, to calculate ϕ_3 , we just need to describe $\dim((I_2)^3)$. To do so, consider

$$C_3 := \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in \{i, j, k\}\},$$

and

$$F_3 := \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}\}.$$

By construction $(I_2)^3 = I^2 \cdot E^1 = \text{span}\{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n]\}$, and hence

$$(I_2)^3 = \text{span}(C_3) + \text{span}(F_3).$$

In the remainder, we will show that for some gain graphic arrangements, $(I_2)^3$ can be written as the direct sum of $\text{span}(C_3)$ and $\text{span}(F_3)$. Through calculating their dimensions, we will get the dimension of $(I_2)^3$. Finally, by Theorem 1.3.5 we can get the value of ϕ_3 .

Chapter 3

Signed graphic arrangements

In this chapter we describe how to compute the Falk invariant ϕ_3 for $\mathcal{A}(\mathcal{G})$, a signed graphic arrangement associated to a signed graph $\mathcal{G} = (G, \varphi)$ that does not contain a subgraph which is isomorphic to a sign complete graph with 2 vertices and a full set of loops.

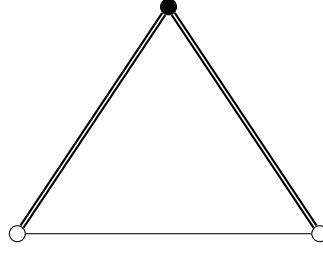
As introduced in the second chapter, let $\mathcal{G} = (G, \varphi)$ be a signed graph with an underlying graph $G = (\mathcal{V}_G, \mathcal{E}_G)$ and a gain map $\varphi: \mathcal{E}_G \rightarrow \mathfrak{G}$ where $\mathfrak{G} = \{1, -1\}$ is a multiplicative group with two elements. To indicate the vertex set and edge set, we write $\mathcal{G} = (\mathcal{V}_G, \mathcal{E}_G^+, \mathcal{E}_G^-, \mathcal{L}_G)$. We consider the vertices set $\mathcal{V}_G$ a finite set with ℓ elements and edge set $\mathcal{E}_G$ a finite set with n edges, ℓ and n are positive integers. Next we will label only the edges as elements of $[n] := \{1, 2, \dots, n\}$ and the corresponding hyperplanes in $\mathcal{A}(\mathcal{G})$ in $\mathbb{C}^\ell$ as $\{H_1, H_2, \dots, H_n\}$.

3.1 List of distinguished signed graphs

In this section, taking inspiration from graph theory and the study of hyperplane arrangements, we will describe all the signed graphs that we need to express our main theorem. Write $[\ell] = \{1, 2, \dots, \ell\}$ and $[\ell]_{<}^2 = \{(i, j) \mid i, j \in [\ell], i < j\}$.

- (1) The signed graph K_ℓ is the complete graph with ℓ vertices and all edges being positive, i.e. $K_\ell = ([\ell], [\ell]_{<}^2, \emptyset, \emptyset)$.
- (2) The signed graph D_ℓ is the complete signed graph with ℓ vertices and no loops, i.e. $D_\ell = ([\ell], [\ell]_{<}^2, [\ell]_{<}^2, \emptyset)$.
- (3) The signed graph B_ℓ is the complete signed graph with ℓ vertices and a full set of loops, i.e. $B_\ell = ([\ell], [\ell]_{<}^2, [\ell]_{<}^2, [\ell])$.

- (4) The signed graph K_ℓ^ℓ is the complete graph with ℓ vertices, all edges being positive and a full set of loops, i.e. $K_\ell^\ell = ([\ell], [\ell]_{<}^2, \emptyset, [\ell])$.
- (5) The signed graph D_ℓ^1 is the complete signed graph with ℓ vertices and one loop, i.e. $D_\ell^1 = ([\ell], [\ell]_{<}^2, [\ell]_{<}^2, \{1\})$.
- (6) The signed graph $G_\circ$ is the signed graph in Figure 3.1.

Figure 3.1: The signed graph $G_\circ$.

The figure is following the rules of Example 2.1.6.

To clarify these signed graphs, we will show the following graphs with $\ell = 3$ vertices.

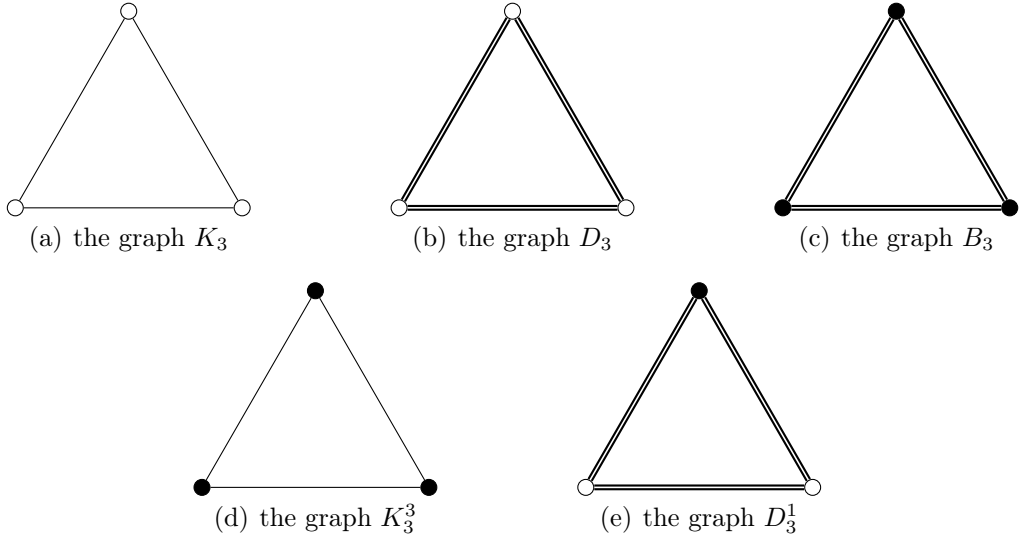


Figure 3.2: The signed graphs with 3 vertices

Remark 3.1.1. *In the figure 3.2, we can find that the graph K_3 is isomorphic to the subgraph of the other four graphs, while B_3 contains the subgraphs isomorphic to K_3, D_3, K_3^3 and D_3^1 . In the same way, K_3 is isomorphic to the subgraph of K_3^3 , and D_3 is isomorphic to the subgraph of D_3^1 .*

3.2 Falk invariant of a signed graph

The goal of this section is to prove the following theorem.

Theorem 3.2.1. *For a signed graphic arrangement associated to a signed graph $\mathcal{G}$ such that $\langle \mathcal{G} \rangle$ does not contain a subgraph isomorphic to $\langle B_2 \rangle$, we have*

$$\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_o) + 5d_{3,1}, \quad (3.1)$$

where k_l denotes the number of subgraph of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_l \rangle$, d_l denotes the number of subgraph of $\langle \mathcal{G} \rangle$ isomorphic to $\langle D_l \rangle$ but not contained in $\langle D_l^1 \rangle$, $d_{l,1}$ denotes the number of subgraph of $\langle \mathcal{G} \rangle$ isomorphic to $\langle D_l^1 \rangle$, $k_{l,\ell}$ denotes the number of subgraph of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_\ell^\ell \rangle$ and g_o denotes the number of subgraph of $\langle \mathcal{G} \rangle$ isomorphic to the $\langle G_o \rangle$ but not contained in $\langle D_\ell^1 \rangle$.

In order to compute ϕ_3 , we will apply the idea introduced in section 2.3. Firstly, we need to identify the 3-circuits S in $[n]$. Clearly, we have the following

Lemma 3.2.2. *$S = (i_1, i_2, i_3)$ is a 3-circuit if and only if i_1, i_2, i_3 correspond to the edges of a subgraph of $\langle \mathcal{G} \rangle$ that is isomorphic to a $\langle K_3 \rangle$, or a $\langle D_2^1 \rangle$ or a $\langle K_2^2 \rangle$.*

Remark 3.2.3. *Notice that the 3-circuits are exactly the balanced 3-cycles together with the subgraphs isomorphic to $\langle K_2^2 \rangle$. In particular, If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two signed graphs with the same underlying graph such that $\mathcal{G}_1$ is switching equivalent to $\mathcal{G}_2$, then $\mathcal{C}_3(\mathcal{G}_1) = \mathcal{C}_3(\mathcal{G}_2)$, where $\mathcal{C}_3 = \text{span}\{e_S \in E \mid S \text{ is a 3-circuit}\}$.*

Since $e_i e_j e_k = -e_j e_i e_k$, it is clear that the dimension of the vector space $\mathcal{C}_3$ is $k_3 + d_{2,1} + k_{2,2}$. Moreover, we can consider C'_3 a basis of $\mathcal{C}_3$. Then each element of C'_3 is in a one-to-one correspondence of the subgraph of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_3 \rangle$, or a $\langle D_2^1 \rangle$ or a $\langle K_2^2 \rangle$.

Lemma 3.2.4. $\dim(A^2) = \binom{n}{2} - k_3 - d_{2,1} - k_{2,2}$.

Proof. By definition $A = E/I$, hence

$$\dim(A^2) = \dim(E^2) - \dim(I^2) = \binom{n}{2} - \dim(I^2).$$

Since $I^2 = \text{span}\{\partial e_{ijk} \mid e_{ijk} \in \mathcal{C}_3\}$, then $\dim(I^2) = k_3 + d_{2,1} + k_{2,2}$, and the formula follows. $\square$

Lemma 3.2.5. *For a signed graphic arrangement associated to a signed graph $\mathcal{G}$ such that $\langle \mathcal{G} \rangle$ does not contain a subgraph isomorphic to $\langle B_2 \rangle$, we have*

$$I_2^3 = \text{span}(C_3) \oplus \text{span}(F_3).$$

Proof. Since $\langle \mathcal{G} \rangle$ does not contain a subgraph isomorphic to $\langle B_2 \rangle$, any two triangles shares at most one element. This then gives us that $\text{span}(C_3) \cap \text{span}(F_3) = \{0\}$. $\square$

Remark 3.2.6. *Notice that if we allow $\langle \mathcal{G} \rangle$ to have subgraphs isomorphic to $\langle B_2 \rangle$, then the previous lemma is not true anymore.*

By the previous lemma, we can write

$$\dim(I_2^3) = \dim(\text{span}(C_3)) + \dim(\text{span}(F_3)) = k_3 + d_{2,1} + k_{2,2} + \dim(\text{span}(F_3)).$$

To prove Theorem 3.2.1 we need to be able to compute $\dim(\text{span}(F_3))$. To do so, consider the following sets

$$\begin{aligned} F_3^1 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are not in the same } \langle K_4 \rangle, \langle D_3 \rangle, \\ &\quad \langle G_\circ \rangle, \langle D_3^1 \rangle, \langle K_3^3 \rangle\}, \\ F_3^2 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are in the same } \langle K_4 \rangle\}, \\ F_3^3 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are in the same } \langle D_3 \rangle \text{ but not} \\ &\quad \text{same } \langle D_3^1 \rangle\}, \\ F_3^4 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are in the same } \langle G_\circ \rangle \text{ but} \\ &\quad \text{not same } \langle D_3^1 \rangle\}, \\ F_3^5 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are in the same } \langle D_3^1 \rangle\}, \\ F_3^6 &:= \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \text{ are in the same } \langle K_3^3 \rangle\} \end{aligned}$$

Lemma 3.2.7. *For a signed graphic arrangement associated to a signed graph $\mathcal{G}$ such that $\langle \mathcal{G} \rangle$ does not contain a subgraph isomorphic to $\langle B_2 \rangle$, we have*

$$\text{span}(F_3) = \bigoplus_{i=1}^6 \text{span}(F_3^i).$$

Proof. There is an evident direct summand decomposition

$$\text{span}(F_3) = \bigoplus_{X \in L_3(\mathcal{A})} \text{span}\{e_t \partial e_{ijk} \mid H_t \cap H_i \cap H_j \cap H_k = X\},$$

where $\{i, j, k\}$ is a 3-circuit and $L_3(\mathcal{A})$ is the set of rank three flats of the lattice of intersections.

Then the result follows from recognizing $F_3^2, F_3^3, F_3^4, F_3^5, F_3^6$ as particular groups of summands of the above direct sum, corresponding to particular types of rank three flats. Specifically, F_3^2 corresponds to the $X = H_t \cap H_i \cap H_j \cap H_k$ where t, i, j, k are in the same $\langle K_4 \rangle$, F_3^3 corresponds to the X where t, i, j, k are in the same $\langle D_3 \rangle$ but not same $\langle D_3^1 \rangle$, F_3^4 corresponds to the X where t, i, j, k are in the same $\langle G_\circ \rangle$ but not same $\langle D_3^1 \rangle$, F_3^5 corresponds to the X where t, i, j, k are in the same $\langle D_3^1 \rangle$, and F_3^6 corresponds to the X where t, i, j, k are in the same $\langle K_3^3 \rangle$. Finally, F_3^1 consists of the rest of the summands. $\square$

We now proceed to computing the dimensions of F_3^i for $i = 1, \dots, 6$, beginning with two examples which illustrate the general idea.

Example 3.2.8. We consider the dimension of $\text{span}(F_3)$ for the sign graphic arrangement A_3 associated to the graph $G_\circ$ (see Figure 3.3).

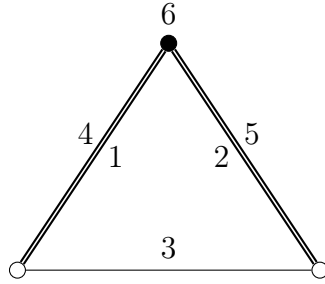
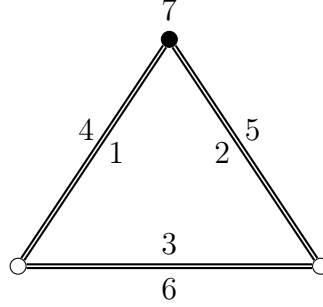


Figure 3.3: The sign graph $G_\circ$

In this situation we have $\mathcal{E}^+ = \{1, 2, 3\}$, $\mathcal{E}^- = \{4, 5\}$ and $\mathcal{L} = \{6\}$. Then the number of the elements in F_3 is 12, listed as follows.

$$\begin{aligned}
e_4 \partial e_{123} &= e_{234} - e_{134} + e_{124}, & e_5 \partial e_{123} &= e_{235} - e_{135} + e_{125}, \\
e_6 \partial e_{123} &= e_{236} - e_{136} + e_{126}, & e_1 \partial e_{345} &= e_{145} - e_{135} + e_{134}, \\
e_2 \partial e_{345} &= e_{245} - e_{235} + e_{234}, & e_6 \partial e_{345} &= e_{456} - e_{356} + e_{346}, \\
e_2 \partial e_{146} &= e_{246} + e_{126} - e_{124}, & e_3 \partial e_{146} &= e_{346} + e_{136} - e_{136}, \\
e_5 \partial e_{146} &= -e_{456} + e_{156} + e_{145}, & e_1 \partial e_{256} &= e_{156} - e_{126} + e_{125}, \\
e_3 \partial e_{256} &= e_{356} + e_{236} - e_{235}, & e_4 \partial e_{256} &= e_{456} + e_{246} - e_{245}.
\end{aligned}$$

Then an easy computation shows that in this case $\dim(\text{span}(F_3)) = 10$.

Figure 3.4: The sign graph D_3^1

Example 3.2.9. We consider the dimension of $\text{span}(F_3)$ for the sign graphic arrangement associated to the graph D_3^1 (see Figure 3.4).

In this situation we have $\mathcal{E}^+ = \{1, 2, 3\}$, $\mathcal{E}^- = \{4, 5, 6\}$ and $\mathcal{L} = \{7\}$. Then the number of the elements in F_3 is 24, listed as follows.

$$\begin{aligned}
e_4 \partial e_{123} &= e_{124} - e_{134} + e_{234}, & e_5 \partial e_{123} &= e_{125} - e_{135} + e_{235}, \\
e_6 \partial e_{123} &= e_{126} - e_{136} + e_{236}, & e_7 \partial e_{123} &= e_{127} - e_{137} + e_{237}, \\
e_2 \partial e_{156} &= -e_{125} + e_{126} + e_{256}, & e_3 \partial e_{156} &= -e_{135} + e_{136} + e_{356}, \\
e_4 \partial e_{156} &= -e_{145} + e_{146} + e_{456}, & e_7 \partial e_{156} &= e_{157} - e_{167} + e_{567}, \\
e_1 \partial e_{246} &= e_{124} - e_{126} + e_{146}, & e_3 \partial e_{246} &= -e_{234} + e_{236} + e_{346}, \\
e_5 \partial e_{246} &= e_{245} + e_{256} - e_{456}, & e_7 \partial e_{246} &= e_{247} - e_{267} + e_{467}, \\
e_1 \partial e_{345} &= e_{134} - e_{135} + e_{145}, & e_2 \partial e_{345} &= e_{234} - e_{235} + e_{245}, \\
e_6 \partial e_{345} &= e_{346} - e_{356} + e_{456}, & e_7 \partial e_{345} &= e_{347} - e_{357} + e_{457}, \\
e_2 \partial e_{147} &= -e_{124} + e_{127} + e_{247}, & e_3 \partial e_{147} &= -e_{134} + e_{137} + e_{347}, \\
e_5 \partial e_{147} &= e_{145} + e_{157} - e_{457}, & e_6 \partial e_{147} &= e_{146} + e_{167} - e_{467}, \\
e_1 \partial e_{257} &= e_{125} - e_{127} + e_{157}, & e_3 \partial e_{257} &= -e_{235} + e_{237} + e_{357}, \\
e_4 \partial e_{257} &= -e_{245} + e_{247} + e_{457}, & e_6 \partial e_{257} &= e_{256} + e_{267} - e_{567}.
\end{aligned}$$

Then an easy computation shows that in this case $\dim(\text{span}(F_3)) = 19$.

Remark 3.2.10. Similarly to the previous examples, we can directly compute $\dim(\text{span}(F_3))$ for several sign graph. In particular, if we consider D_3 , K_4 and K_3^3 , then $\dim(\text{span}(F_3)) = 10$.

Lemma 3.2.11. $\dim(\text{span}(F_3^2)) = 10k_4$, $\dim(\text{span}(F_3^3)) = 10d_3$, $\dim(\text{span}(F_3^4)) = 10g_o$, $\dim(\text{span}(F_3^5)) = 19d_{3,1}$ and $\dim(\text{span}(F_3^6)) = 10k_{3,3}$.

Proof. Assume that in the signed graph $\langle \mathcal{G} \rangle$ there are exactly $g_o = p$ distinct subgraphs isomorphic to a $\langle G_o \rangle, \langle G_1 \rangle, \dots, \langle G_p \rangle$, none of which is a subgraph of a graph isomorphic to $\langle D_3^1 \rangle$. Consider

$$F_{3,r}^4 := \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \in \langle G_r \rangle\}.$$

Since four edges in the graph $\langle \mathcal{G} \rangle$ can not appear in two distinct $\langle G_o \rangle$ at the same time, then none of the terms of the element $e_t \partial e_{ijk} \in F_{3,r}^2$ appear in the elements of $F_3^4 \setminus F_{3,r}^4$. This shows that

$$\text{span}(F_3^4) = \bigoplus_{r=1}^p \text{span}(F_{3,r}^4).$$

By Example 3.2.8, we have that $\dim(\text{span}(F_{3,r}^4)) = 10$ for all $r = 1, \dots, p$. This then implies that

$$\dim(\text{span}(F_3^4)) = \sum_{r=1}^p \dim(\text{span}(F_{3,r}^4)) = 10g_o.$$

Using Remark 3.2.10 and Example 3.2.9, the same exact argument used in this case will prove the other equalities. $\square$

Lemma 3.2.12. *For a signed graphic arrangement associated to a signed graph $\mathcal{G}$ such that $\langle \mathcal{G} \rangle$ does not contain a subgraph isomorphic to $\langle B_2 \rangle$, we have*

$$\dim(I_2^3) = (n-2)(k_3 + d_{2,1} + k_{3,3}) - 2k_4 - 2d_3 - 2g_o - 2k_{3,3} - 5d_{3,1}.$$

Proof. By the previous lemmas

$$\begin{aligned} \dim(\text{span}(F_3)) &= \sum_{i=1}^6 \dim(\text{span}(F_3^i)) = \\ &= [(n-3)(k_3 + d_{2,1} + k_{3,3}) - 12k_4 - 12d_3 - 12g_o - 12k_{3,3} - 24d_{3,1}] + \\ &\quad + 10k_4 + 10d_3 + 10g_o + 10k_{3,3} + 19d_{3,1} = \\ &= (n-3)(k_3 + d_{2,1} + k_{3,3}) - 2k_4 - 2d_3 - 2g_o - 2k_{3,3} - 5d_{3,1}. \end{aligned}$$

The thesis follows from the equality

$$\dim(I_2^3) = k_3 + d_{2,1} + k_{2,2} + \dim(\text{span}(F_3)).$$

$\square$

Proof of Theorem 3.2.1. By Remark 1.3.6 and Lemma 3.2.4 we have

$$\phi_3 = 2 \binom{n+1}{3} - n \binom{n}{2} - k_3 - d_{2,1} - k_{2,2} + \binom{n}{3} - \dim(I_2^3).$$

Because $2 \binom{n+1}{3} - n \binom{n}{2} + \binom{n}{3} = 0$, then from Lemma 3.2.12 we obtain

$$\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_o) + 5d_{3,1}.$$

□

Let us see how our formula works on a non-trivial example.

Example 3.2.13. We want to compute ϕ_3 for the arrangement associated to the graph $\mathcal{G}$ of Figure 3.5.

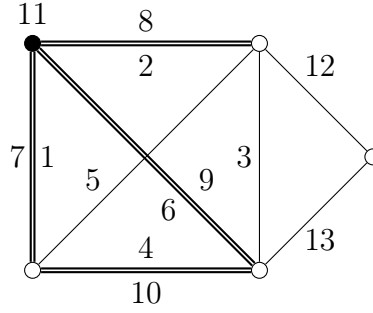


Figure 3.5: The sign graph G

In this situation we have $\mathcal{E}^+ = \{1, 2, 3, 4, 5, 6, 12, 13\}$, $\mathcal{E}^- = \{7, 8, 9, 10\}$ and $\mathcal{L} = \{11\}$. In order to compute ϕ_3 with the formula (3.1), we need to compute the following:

- $k_3 = |\{\{1, 2, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{3, 4, 5\}, \{1, 9, 10\}, \{6, 7, 10\}, \{4, 7, 9\}, \{3, 8, 9\}, \{5, 7, 8\}, \{3, 12, 13\}\}| = 10$;
- $k_4 = |\{\{1, 2, 3, 4, 5, 6\}, \{3, 4, 5, 7, 8, 9\}\}| = 2$;
- $d_3 = 0$;
- $d_{2,1} = |\{\{1, 7, 11\}, \{6, 9, 11\}, \{2, 8, 11\}\}| = 3$;
- $k_{2,2} = 0$;
- $k_{3,3} = 0$;
- $g_o = |\{\{1, 2, 5, 7, 8, 11\}, \{2, 3, 6, 8, 9, 11\}\}| = 2$;

- $d_{3,1} = |\{\{1, 4, 6, 7, 9, 10, 11\}\}| = 1.$

From formula (3.1), we obtain

$$\phi_3 = 2(10 + 2 + 0 + 3 + 0 + 0 + 2) + 5 = 39.$$

Notice that if we would try to compute the dimension of F_3 directly, we would have to write 130 equations in the e_{ijk} .

Chapter 4

Multiplicative gain graphic arrangements

In this chapter, we will show how to compute the Falk invariant ϕ_3 for a multiplicative gain graphic arrangement $\mathcal{A}(\mathcal{G})$, associated to a multiplicative gain graph $\mathcal{G} = (G, \varphi)$ where $G = (\mathcal{V}_G, \mathcal{E}_G)$ is the underlying graph and φ is the map from the edges of G to the group $\mathbb{Q}^*$. And the vertices set $\mathcal{V}_G = \{1, 2, \dots, \ell\}$ is a finite set with ℓ elements, the edge set $\mathcal{E}_G = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is also a finite set, ℓ and n are positive integers.

4.1 List of distinguished multiplicative gain graphs

In this section, we will describe all the gain graphs that we need to express our main theorem. Since we will consider the gain map as $\mathbb{Q}^*$, we will describe them by their balanced circles.

- (1) The gain graph K_n has as underlying graph the complete graph on n vertices, and all edges have gain 1, hence it is a balanced gain graph.
- (2) The gain graph K_ℓ^ℓ is the gain graph K_n with an unbalanced loop at every vertex.
- (3) The gain graph D_2^1 is the one depicted in Figure 4.1(a) and it is contrabalanced.
- (4) The gain graph B_2 is the one depicted in Figure 4.1(b) and it is contrabalanced.
- (5) The gain graph D_3 is the one depicted in Figure 4.1(c). The list of its balanced circles is $\{\mathbf{e}_{12}(1)\mathbf{e}_{23}(1)\mathbf{e}_{31}(1), \mathbf{e}_{21}(-1)\mathbf{e}_{13}(-1)\mathbf{e}_{32}(1), \mathbf{e}_{21}(-1)\mathbf{e}_{31}(1)\mathbf{e}_{32}(-1), \mathbf{e}_{12}(1)\mathbf{e}_{13}(-1)\mathbf{e}_{32}(-1)\}$.

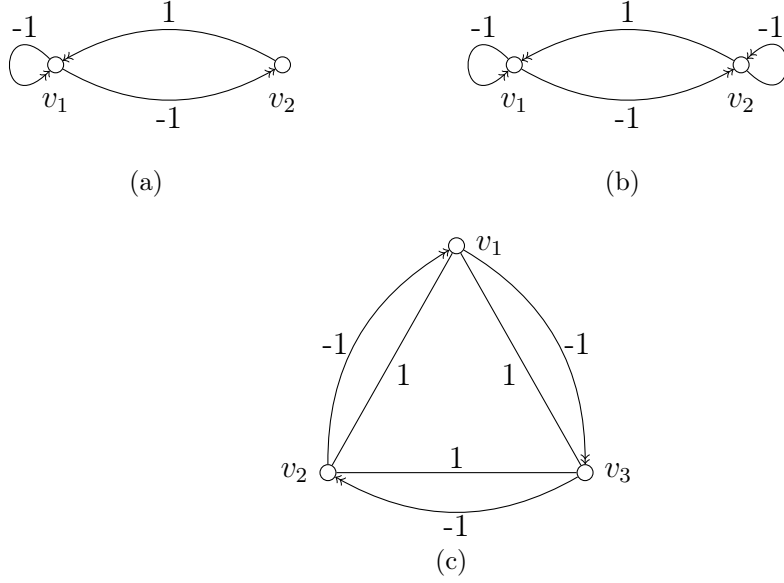


Figure 4.1: List of underlying graphs.

- (6) The gain graph $G_\circ$ is the one depicted in Figure 4.2(a). Its balanced circles are $\{\mathbf{e}_{21}(-1)\mathbf{e}_{13}(-1)\mathbf{e}_{23}(1), \mathbf{e}_{12}(1)\mathbf{e}_{23}(1)\mathbf{e}_{31}(1)\}$.
- (7) The gain graph D_3^1 is the one depicted in Figure 4.2(b). Its balanced circles are $\{\mathbf{e}_{21}(-1)\mathbf{e}_{13}(-1)\mathbf{e}_{23}(1), \mathbf{e}_{12}(1)\mathbf{e}_{23}(1)\mathbf{e}_{31}(1), \mathbf{e}_{21}(-1)\mathbf{e}_{31}(1)\mathbf{e}_{32}(-1), \mathbf{e}_{12}(1)\mathbf{e}_{13}(-1)\mathbf{e}_{32}(-1)\}$.
- (8) The gain graph $\mathbb{Z}_3K_3 \setminus \mathbf{e}$ is the one depicted in Figure 4.2(c). Its balanced circles are $\{\mathbf{e}_{12}(1)\mathbf{e}_{23}(1)\mathbf{e}_{31}(1), \mathbf{e}_{12}(2)\mathbf{e}_{23}(1)\mathbf{e}_{13}(2), \mathbf{e}_{12}(2)\mathbf{e}_{32}(2)\mathbf{e}_{31}(1), \mathbf{e}_{12}(1)\mathbf{e}_{32}(2)\mathbf{e}_{31}(2), \mathbf{e}_{21}(2)\mathbf{e}_{31}(2)\mathbf{e}_{23}(1)\}$.
- (9) The gain graph $\mathbb{Z}_3K_3$ is the one depicted in Figure 4.2(d). Its balanced circles are $\{\mathbf{e}_{12}(1)\mathbf{e}_{23}(1)\mathbf{e}_{31}(1), \mathbf{e}_{21}(2)\mathbf{e}_{31}(2)\mathbf{e}_{23}(1), \mathbf{e}_{21}(2)\mathbf{e}_{31}(1)\mathbf{e}_{23}(2), \mathbf{e}_{12}(1)\mathbf{e}_{13}(2)\mathbf{e}_{23}(2), \mathbf{e}_{12}(2)\mathbf{e}_{13}(2)\mathbf{e}_{23}(1), \mathbf{e}_{21}(2)\mathbf{e}_{13}(2)\mathbf{e}_{23}(4)\}$.

Notice that by construction we have the following

Lemma 4.1.1. *The graphs $\langle \mathbb{Z}_3K_3 \setminus \mathbf{e} \rangle$ and $\langle \mathbb{Z}_3K_3 \rangle$ do not have any subgraphs isomorphic to $\langle D_3 \rangle$.*

4.2 The computation of Falk invariant

In this section we will describe how to compute the Falk invariant ϕ_3 for a multiplicative gain graphic arrangement $\mathcal{A}(\mathcal{G})$, an arrangement associated

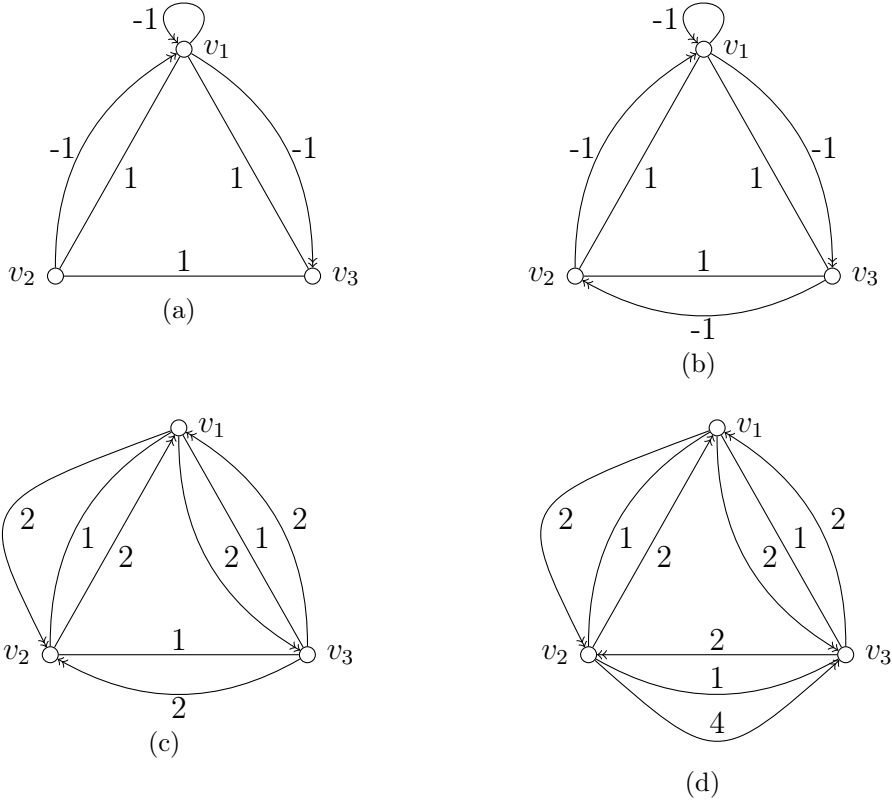


Figure 4.2: List of underlying graphs.

to a gain graph $\mathcal{G}$ such that $\langle \mathcal{G} \rangle$ does not have a subgraph isomorphic to $\langle B_2 \rangle$, it has at most triple parallel edges and it has no loops adjacent to a theta graph. Moreover, we will assume that all 2-circles and loops of $\mathcal{G}$ are unbalanced.

We define the numbers of some subgraphs of a graph $\langle \mathcal{G} \rangle$ as the followings:

- k_l denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_l \rangle$,
- d_3 denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to $\langle D_3 \rangle$ but not contained in $\langle D_3^1 \rangle$,
- $d_{\ell,1}$ denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to $\langle D_\ell^1 \rangle$,
- $k_{\ell,\ell}$ denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_\ell^\ell \rangle$,
- g_0 denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle G_\circ \rangle$ but not contained in $\langle D_3^1 \rangle$,

- g_1 denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle \mathbb{Z}_3 K_3 \setminus e \rangle$ but not contained in $\langle \mathbb{Z}_3 K_3 \rangle$,
- g_2 denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle \mathbb{Z}_3 K_3 \rangle$,
- Θ counts contrabalanced triple parallel edges.

We still apply the idea in Section 2.3 and we get the following theorem.

Theorem 4.2.1. *For an arrangement associated to a multiplicative gain graph $\mathcal{G}$ such that $\langle \mathcal{G} \rangle$ has no subgraph isomorphic to $\langle B_2 \rangle$ or to a contrabalanced triple parallel edge with an adjacent loop and has at most triple parallel edges, we have*

$$\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_0 + g_2 + \Theta) + 5d_{3,1} + g_1. \quad (4.1)$$

In order to describe the proof of this theorem, we will adopt the same notation as Chapter 3. We prefer to label the edge set $\mathcal{E}_G$ as elements of $[n] = \{1, 2, \dots, n\}$ and the corresponding hyperplanes in $\mathcal{A}(\mathcal{G})$ in $\mathbb{C}^\ell$ as $\{H_1, H_2, \dots, H_n\}$. Firstly, we should identify the triples S in $[n]$ that are dependent (that is 3-circuit) for a multiplicative gain graphic arrangement. Then we get the following

Lemma 4.2.2. *$S = (i_1, i_2, i_3)$ is dependent if and only if i_1, i_2, i_3 correspond to the edges of a subgraph of $\langle \mathcal{G} \rangle$ that is isomorphic to $\langle K_3 \rangle$, or $\langle D_2^1 \rangle$ or $\langle K_2^2 \rangle$ or a contrabalanced theta graph with only three edge.*

Remark 4.2.3. *Notice that the 3-circuits are exactly the balanced 3-circles, the contrabalanced theta graphs, loose handcuffs and tight handcuffs.*

Since $e_i e_j e_k = -e_j e_i e_k$, it is clear that the dimension of the vector space $\mathcal{C}_3 = \text{span}\{e_S \in E \mid S \text{ is a 3-circuit}\}$ is $k_3 + d_{2,1} + k_{2,2} + \Theta$. Let C'_3 be a basis of $\mathcal{C}_3$ consisting of monomials in one-to-one correspondence with the subgraph of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_3 \rangle$, or a $\langle D_2^1 \rangle$ or a $\langle K_2^2 \rangle$ or a contrabalanced theta graph with only three edges.

Lemma 4.2.4. $\dim(A^2) = \binom{n}{2} - k_3 - d_{2,1} - k_{2,2} - \Theta$.

Proof. By definition $A = E/I$, hence

$$\dim(A^2) = \dim(E^2) - \dim(I^2) = \binom{n}{2} - \dim(I^2).$$

By construction $I^2 = \text{span}\{\partial e_{ijk} \mid e_{ijk} \in \mathcal{C}_3\}$. Notice that if e_{ijk}, e_{pqr} are distinct elements of C'_3 then the corresponding subgraphs of $\langle \mathcal{G} \rangle$ share at most one edge and hence ∂e_{ijk} and ∂e_{pqr} are linearly independent. This implies that $\dim(I^2) = \dim(\mathcal{C}_3) = k_3 + d_{2,1} + k_{2,2} + \Theta$, and the thesis follows.

□

Lemma 4.2.5. *For an arrangement associated to a multiplicative gain graph $\mathcal{G}$ without loops adjacent to a theta graph with only three edges, with at most triple parallel edges, and such that $\langle \mathcal{G} \rangle$ does not contain a subgraph isomorphic to $\langle B_2 \rangle$, we have*

$$(I_2)^3 = \text{span}(C_3) \oplus \text{span}(F_3).$$

Proof. Since $\langle \mathcal{G} \rangle$ does not contain a subgraph isomorphic to $\langle B_2 \rangle$ or loops adjacent to a theta graph or quadruple parallel edges, any two 3-circuits share at most one element. This then gives us that $\text{span}(C_3) \cap \text{span}(F_3) = \{0\}$. $\square$

Remark 4.2.6. *Notice that if we allow $\langle \mathcal{G} \rangle$ to have subgraphs isomorphic to $\langle B_2 \rangle$ or a loop adjacent to a theta graph or quadruple parallel edges, then the previous lemma is not true any more.*

By the previous lemma, we can write

$$\begin{aligned} \dim((I_2)^3) &= \dim(\text{span}(C_3)) + \dim(\text{span}(F_3)) \\ &= k_3 + d_{2,1} + k_{2,2} + \Theta + \dim(\text{span}(F_3)). \end{aligned}$$

To prove our main result we need to be able to compute $\dim(\text{span}(F_3))$. To do so, consider the following sets

$$\begin{aligned} F_3^1 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are not in the same } \langle K_4 \rangle, \langle D_3 \rangle, \langle G_o \rangle, \langle D_3^1 \rangle, \\ &\quad \langle K_3^3 \rangle, \langle \mathbb{Z}_3 K_3 \setminus \mathbf{e} \rangle, \langle \mathbb{Z}_3 K_3 \rangle\}, \\ F_3^2 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are in the same } \langle K_4 \rangle\}, \\ F_3^3 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are in the same } \langle D_3 \rangle \text{ but not the} \\ &\quad \text{same } \langle D_3^1 \rangle\}, \\ F_3^4 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are in the same } \langle G_o \rangle \text{ but} \\ &\quad \text{not the same } \langle D_3^1 \rangle\}, \\ F_3^5 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are in the same } \langle D_3^1 \rangle\}, \\ F_3^6 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are in the same } \langle K_3^3 \rangle\}, \\ F_3^7 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are in the same } \langle \mathbb{Z}_3 K_3 \setminus \mathbf{e} \rangle \text{ but not in} \\ &\quad \text{the same } \langle \mathbb{Z}_3 K_3 \rangle\}, \\ F_3^8 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are in the same } \langle \mathbb{Z}_3 K_3 \rangle\}. \end{aligned}$$

Lemma 4.2.7. *For an arrangement associated to a multiplicative gain graph $\mathcal{G}$ without loops adjacent to a theta graph with only three edges, with at most triple parallel edges, and such that $\langle \mathcal{G} \rangle$ does not contain a subgraph switching equivalent to $\langle B_2 \rangle$, we have*

$$\text{span}(F_3) = \bigoplus_{i=1}^8 \text{span}(F_3^i).$$

Proof. By hypothesis, $\langle \mathcal{G} \rangle$ does not contain a subgraph isomorphic to $\langle B_2 \rangle$ or loops adjacent to a theta graph with only three edges or quadruple parallel edges. This fact together with Lemma 4.1.1 implies that $\text{span}(F_3^p) \cap \text{span}(F_3^q) = \{0\}$ for all $p, q = 2, \dots, 8$ such that $p \neq q$.

For any element $e_t \partial e_{ijk}$ of F_3^1 , we assert that at least one of the terms $e_{tjk}, e_{tik}, e_{tij}$ appears only in the expression of $e_t \partial e_{ijk}$. So $e_t \partial e_{ijk}$ can not be expressed linearly by the elements of $F_3^2, \dots, F_3^8$.

Since the edges t, i, j, k are not in the same $\langle K_4 \rangle, \langle D_3 \rangle, \langle G_o \rangle, \langle D_3^1 \rangle, \langle K_3^3 \rangle, \langle \mathbb{Z}_3 K_3 \setminus \mathbf{e} \rangle, \langle \mathbb{Z}_3 K_3 \rangle$, and we do not consider the graphs having subgraphs isomorphic to $\langle B_2 \rangle$ or loops adjacent to a theta graph with only three edges or quadruple parallel edges, we should only consider three cases about the edge t : it can be adjacent to none of the edges i, j, k , to two of them, or to all of them.

Assume that the edge t is adjacent to none of the edges i, j, k . This implies that t and none of i, j, k can appear in the same 3-circuit. Hence any element $e_t \partial e_{ijk}$ of F_3^1 will not appear in any of $F_3^2, \dots, F_3^8$.

Assume now that the edge t is adjacent to two of the edges i, j, k , then we should consider several possibilities. Suppose that in the set $\{t, i, j, k\}$ there is no loop. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in $F_3^2, \dots, F_3^8$, then t, i, j, k have to appear in the same $\langle K_4 \rangle$, but this is impossible by construction. Suppose that t is a loop and there is no loop in the set $\{i, j, k\}$. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in $F_3^2, \dots, F_3^8$, then t, i, j, k have to appear in the same $\langle G_o \rangle$ or in the same $\langle D_3^1 \rangle$, but this is impossible by construction. Suppose that t is not a loop and there is one loop in the set $\{i, j, k\}$. In this case i, j, k are the edges of a $\langle D_2^1 \rangle$. Hence, by assumption, the edges t is not adjacent to the loop. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in $F_3^2, \dots, F_3^8$, then, also in this case, t, i, j, k have to appear in the same $\langle G_o \rangle$ or in the same $\langle D_3^1 \rangle$, but this is impossible by construction. Suppose that t is not a loop and there are two loops in the set $\{i, j, k\}$. In this case i, j, k are the edges of a $\langle K_2^2 \rangle$. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in $F_3^2, \dots, F_3^8$, then t, i, j, k have to appear in the same $\langle K_3^3 \rangle$, but this is impossible by construction.

Finally, assume that the edge t is adjacent to all the edges i, j, k . Since the underlying graph has at most triple edges and no loops adjacent to a theta graph with only three edges, then in this situation, there are just two cases we should consider. Suppose that in the set $\{t, i, j, k\}$ there is no loop. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in $F_3^2, \dots, F_3^8$, then t, i, j, k have to appear in the same $\langle D_3 \rangle$ or in the same $\langle \mathbb{Z}_3 K_3 \setminus \mathbf{e} \rangle$ or in the same $\langle \mathbb{Z}_3 K_3 \rangle$, but this is impossible by construction. Suppose that t is not a loop and there is one loop in the set $\{i, j, k\}$. In this case i, j, k are the edges of a $\langle D_2^1 \rangle$. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in $F_3^2, \dots, F_3^8$,

then t, i, j, k have to appear in the same $\langle G_\circ \rangle$ or in the same $\langle D_3^1 \rangle$, but this is impossible by construction.

Therefore, for any element $e_t \partial e_{ijk} \in F_3^1$, at least one of the terms $e_{tjk}, e_{tik}, e_{tij}$ appears only in the expression of $e_t \partial e_{ijk}$. This shows that $\text{span}(F_3^p) \cap \text{span}(F_3^q) = \{0\}$ for all $p \neq q$. And the subspaces $\text{span}(F_3^i)$ are linearly independent for all $i = 1, 2, \dots, 8$. Since clearly

$$\text{span}(F_3) = \sum_{i=1}^8 \text{span}(F_3^i)$$

this concludes the proof. $\square$

Remark 4.2.8. *Similarly to the examples 3.2.8 and 3.2.9, we can directly compute $\dim(\text{span}(F_3))$ for all the distinguished gain graphs of Section 4.1. In particular, if we consider the gain graphs D_3, K_4, K_3^3 and $G_\circ$, then $\dim(\text{span}(F_3)) = 10$. If we consider D_3^1 , then $\dim(\text{span}(F_3)) = 19$. If we consider $\mathbb{Z}_3 K_3 \setminus \mathbf{e}$, then $\dim(\text{span}(F_3)) = 34$. Finally, if we consider $\mathbb{Z}_3 K_3$, then $\dim(\text{span}(F_3)) = 52$.*

Lemma 4.2.9. *We have the following equalities*

- $\dim(\text{span}(F_3^2)) = 10k_4,$
- $\dim(\text{span}(F_3^3)) = 10d_3,$
- $\dim(\text{span}(F_3^4)) = 10g_0,$
- $\dim(\text{span}(F_3^5)) = 19d_{3,1},$
- $\dim(\text{span}(F_3^6)) = 10k_{3,3},$
- $\dim(\text{span}(F_3^7)) = 34g_1,$
- $\dim(\text{span}(F_3^8)) = 52g_2.$

Proof. Assume that in the graph $\langle \mathcal{G} \rangle$ there are exactly $g_1 = p$ distinct subgraphs isomorphic to a $\langle \mathbb{Z}_3 K_3 \setminus \mathbf{e} \rangle$, $\langle \mathcal{G}_1 \rangle, \dots, \langle \mathcal{G}_p \rangle$, none of which is a subgraph of a graph isomorphic to $\langle \mathbb{Z}_3 K_3 \rangle$. Consider

$$F_{3,r}^7 := \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n] \setminus \{i, j, k\}, i, j, k \in \mathcal{G}_r\}.$$

Since four edges in the underlying graph of $\langle \mathcal{G} \rangle$ can not appear in two distinct $\langle \mathbb{Z}_3 K_3 \setminus \mathbf{e} \rangle$ at the same time, then none of the terms of the element $e_t \partial e_{ijk} \in F_{3,r}^7$ appear in the elements of $F_3^7 \setminus F_{3,r}^7$. This shows that

$$\text{span}(F_3^7) = \bigoplus_{r=1}^p \text{span}(F_{3,r}^7).$$

By Proposition 2.2.8, we have that $\dim(\text{span}(F_{3,r}^7)) = 34$ for all $r = 1, \dots, p$. This then implies that

$$\dim(\text{span}(F_3^7)) = \sum_{r=1}^p \dim(\text{span}(F_{3,r}^7)) = 34g_1.$$

Using Remark 4.2.8, the same exact argument used in this case will prove the other equalities. $\square$

Lemma 4.2.10. *For an arrangement associated to a multiplicative gain graph $\mathcal{G}$ without loops adjacent to a theta graph with only three edges, with at most triple parallel edges, and such that $\langle \mathcal{G} \rangle$ does not contain a subgraph switching equivalent to $\langle B_2 \rangle$, we have*

$$\dim(I_2^3) = (n-2)(k_3 + d_{2,1} + k_{3,3} + \Theta) - 2k_4 - 2d_3 - 2g_o - 2k_{3,3} - 5d_{3,1} - g_1 - 2g_2.$$

Proof. By the previous lemmas

$$\begin{aligned} \dim(\text{span}(F_3)) &= \sum_{i=1}^8 \dim(\text{span}(F_3^i)) = \\ &= [(n-3)(k_3 + d_{2,1} + k_{3,3} + \Theta) - 12k_4 - 12d_3 - 12g_o - 12k_{3,3} - 24d_{3,1} - 35g_1 - 54g_2] + \\ &\quad + 10k_4 + 10d_3 + 10g_o + 10k_{3,3} + 19d_{3,1} + 34g_1 + 52g_2 = \\ &= (n-3)(k_3 + d_{2,1} + k_{3,3} + \Theta) - 2k_4 - 2d_3 - 2g_o - 2k_{3,3} - 5d_{3,1} - g_1 - 2g_2. \end{aligned}$$

The thesis follows from the equality

$$\dim(I_2^3) = k_3 + d_{2,1} + k_{2,2} + \Theta + \dim(\text{span}(F_3)).$$

$\square$

Proof of Theorem 4.2.1. By Remark 1.3.6 and Lemma 4.2.4 we have

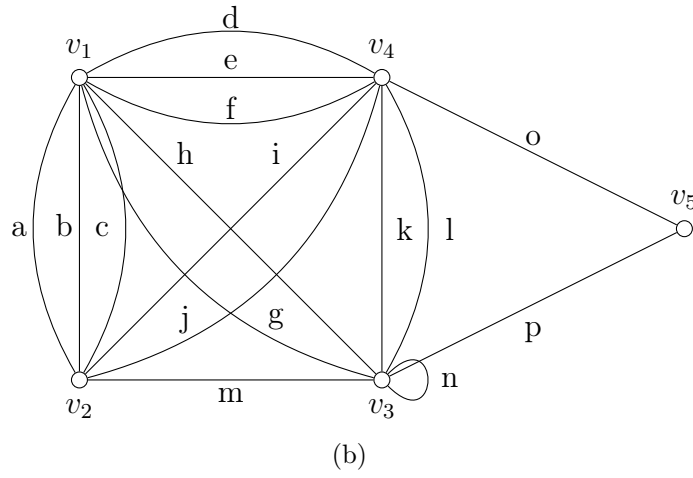
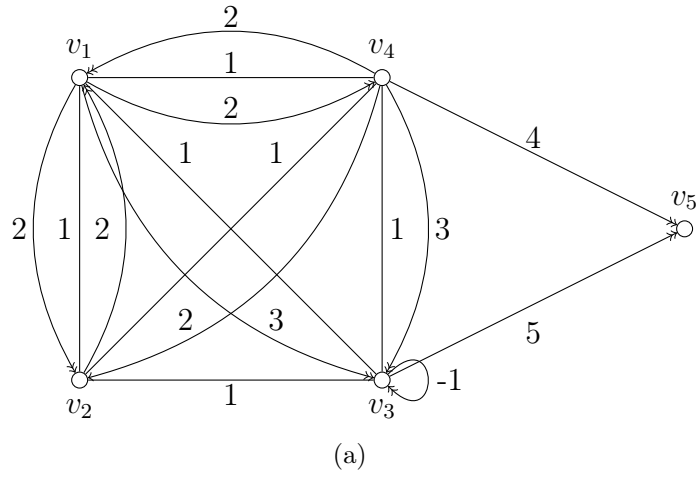
$$\phi_3 = 2 \binom{n+1}{3} - n \binom{n}{2} - k_3 - d_{2,1} - k_{2,2} - \Theta + \binom{n}{3} - \dim(I_2^3).$$

Because $2 \binom{n+1}{3} - n \binom{n}{2} + \binom{n}{3} = 0$, then from Lemma 5.2.10 we obtain

$$\phi_3 = 2(k_3 + k_4 + d_3 + d_{2,1} + k_{2,2} + k_{3,3} + g_o + g_2 + \Theta) + 5d_{3,1} + g_1.$$

$\square$

Let us see how our formula works on a non-trivial example.

Figure 4.3: The gain graph $\mathcal{G}$ and its underlying graph.

Example 4.2.11. We want to compute ϕ_3 for the arrangement associated to the gain graph $\mathcal{G}$ of Figure 4.3(a). In order to not create any confusion, in Figure 4.3(b) we depicted the graph $|\mathcal{G}|$ labeling each edge with a letter

In order to compute ϕ_3 with the formula (4.1), we need to compute the following:

- $k_3 = |\{\{b, e, i\}, \{b, h, m\}, \{e, h, k\}, \{i, k, m\}, \{a, f, i\}, \{a, e, j\}, \{b, d, j\}, \{c, d, i\}, \{e, g, l\}\}| = 9;$
- $k_4 = |\{\{b, e, h, i, k, m\}\}| = 1;$
- $d_3 = 0;$
- $d_{2,1} = |\{\{g, h, n\}, \{k, l, m\}\}| = 2;$
- $k_{2,2} = 0;$
- $k_{3,3} = 0;$
- $g_0 = |\{\{e, g, h, k, l, n\}\}| = 1;$
- $g_2 = 0;$
- $\Theta = |\{\{a, b, c\}, \{d, e, f\}\}| = 2;$
- $d_{3,1} = 0;$
- $g_1 = |\{\{a, b, c, d, e, f, i, j\}\}| = 1.$

From formula (4.1), we obtain

$$\phi_3 = 2(9 + 1 + 0 + 2 + 0 + 0 + 1 + 0 + 2) + 0 + 1 = 31.$$

Notice that if we would try to compute the dimension of F_3 directly, we would have to write 169 equations in the e_{ijk} .

4.3 Matroidal interpretation

In this section, we will give a matroidal interpretation of our main theorem. In Theorem 4.2.1, the formula (4.1) of ϕ_3 is expressed in terms of the numbers of subgraphs of the biased graph associated to the given gain graph. Since some of the subgraphs appearing in the formula (4.1) describe different realizations of the same matroid, we are able to give a new formula just in terms of the numbers of the submatroids of the given gain graph. In this way, we obtain a simpler and more compact formula.

In Theorem 4.2.1, we consider the class of arrangements associated to gain graphs $\mathcal{G}$ such that $\langle \mathcal{G} \rangle$ does not have a subgraph isomorphic to $\langle B_2 \rangle$, it has no loops adjacent to a theta graph with only three edges and it has at most triple parallel edges. This class coincides with the class of arrangements associated to gain graphs such that the underlying matroid has no rank-two flats of size greater than three.

From the list of gain graphs in Section 4.1, it is immediate to prove the following.

Lemma 4.3.1. *The graphs $\langle K_3 \rangle$, $\langle K_2^2 \rangle$, $\langle D_2^1 \rangle$ and the contrabanced theta graph with three edges have isomorphic underlying matroids. Similarly, the graphs $\langle K_4 \rangle$, $\langle D_3 \rangle$, $\langle K_3^3 \rangle$ and $\langle G_\circ \rangle$ have isomorphic underlying matroids.*

Definition 4.3.2. *We denote by $M(K_3)$ the frame matroid associated to $\langle K_3 \rangle$, and by $M(K_4)$ the one associated to $\langle K_4 \rangle$.*

Let $\mathcal{A}(\mathcal{G})$ be an arrangement associated to the gain graph $\mathcal{G}$. Denote by k_l the number of submatroid of $M(\mathcal{G})$ isomorphic to $M(K_l)$, by g_1 the number of submatroid of $M(\mathcal{G})$ isomorphic to $M(\mathbb{Z}_3 K_3 \setminus \mathbf{e})$, by g_2 the number of submatroid of $M(\mathcal{G})$ isomorphic to $M(\mathbb{Z}_3 K_3)$, and by $d_{3,1}$ the number of submatroid of $M(\mathcal{G})$ isomorphic to $M(D_3^1)$.

Remark 4.3.3. *Consider the gain graph $\mathcal{G} = D_3^1$ described in Figure 4.2(b). $M(\mathcal{G})$ is the well-known non-Fano matroid. $\langle \mathcal{G} \rangle$ has two subgraphs isomorphic to $\langle D_2^1 \rangle$ and four isomorphic to $\langle K_3 \rangle$, and hence, $M(\mathcal{G})$ has six submatroid isomorphic to $M(K_3)$. Similarly, $\langle \mathcal{G} \rangle$ has one subgraph isomorphic to $\langle \pm K_3 \rangle$ and two to $\langle G_\circ \rangle$, and hence, $M(\mathcal{G})$ has three submatroid isomorphic to $M(K_4)$. By Theorem 3.2.1, $\phi_3(\mathcal{A}(\mathcal{G})) = 17$, that it coincides with $2(k_3 + k_4) - 1$.*

We are now able to rewrite formula (4.1) of Theorem 4.2.1 just in terms of submatroid.

Theorem 4.3.4. *For an arrangement associated to a gain graph $\mathcal{G}$ such that the underlying matroid has no rank-two flats of size greater than three, we have*

$$\phi_3 = 2(k_3 + k_4 + g_2) - d_{3,1} + g_1, \quad (4.2)$$

Proof. From the formula (4.1), we can get the new formula (4.2), by the use of Lemma 4.3.1 and Remark 4.3.3. $\square$

In the following example we will use the new formula (4.2) to compute ϕ_3 for the gain graph $\mathcal{G}$ of figure 4.3 in the example 4.2.11.

Example 4.3.5. *There are 13 submatroids of $M(\mathcal{G})$ isomorphic to $M(K_3)$ (9 submatroids come from the subgraphs isomorphic to $\langle K_3 \rangle$ and 2 from the subgraphs isomorphic to $\langle D_2^1 \rangle$, and 2 from contrabalanced theta graph with only three edges). There are also 2 submatroids of $M(\mathcal{G})$ isomorphic to $M(K_4)$ (one is isomorphic to $\langle K_4 \rangle$ and the other one to $\langle G_\circ \rangle$). Besides these submatroids, there is 1 submatroid isomorphic to $M(\mathbb{Z}_3 K_3 \setminus \mathbf{e})$.*

Therefore, we obtain

$$\phi_3 = 2(13 + 2 + 0) - 0 + 1 = 31.$$

This coincides with the result of the computation in Example 4.2.11.

Chapter 5

Additive gain graphic arrangements

In this chapter, we will show how to compute the Falk invariant in the case of hyperplane arrangements that are complete lift representation of certain additive gain graphs. As a corollary, we compute the Falk invariant for the cone of Shi, Linial and semiorder arrangements.

Let $\mathcal{G} = (G, \varphi)$ be an additive gain graph with the underlying graph $G = (\mathcal{V}_G, \mathcal{E}_G)$ and the gain map from the edge set $\mathcal{E}_G$ to the additive group $\mathbb{Q}$. And we also consider the vertices set $\mathcal{V}_G$ and edge set $\mathcal{E}_G$ are finite set with ℓ and n elements respectively. Next we will label only the edges as elements of $[n] := \{1, \dots, n\}$. Then we can get the corresponding hyperplane arrangements $\mathcal{A}(\mathcal{G}) = \{H_0, H_1, \dots, H_n\}$ in $\mathbb{C}^{\ell+1}$, where the hyperplane H_0 is $x_0 = 0$ and H_i is corresponding to the edge $\mathbf{e}_i$, for $1 \leq i \leq n$. We label these hyperplanes as elements of $[n]^+ := \{0, 1, \dots, n\}$.

5.1 List of distinguished additive gain graphs

In this section, we will describe all the additive gain graphs that we need to describe our main result. Since we will consider $\mathfrak{G} = \mathbb{Q}$, we will describe them as biased graphs. Hence, we will describe the underlying graph, labeling only the edges, together with the list of balanced circles. We call a balanced circle or a contrabalanced θ -graph, or a contrabalanced (loose or tight) handcuff or an unbalanced circle with two edges jointed by two vertices a **distinguished circle**.

- The biased graph K_3 has as underlying graph the full simple graph on three vertices having the only 3-circle as distinguished circle.

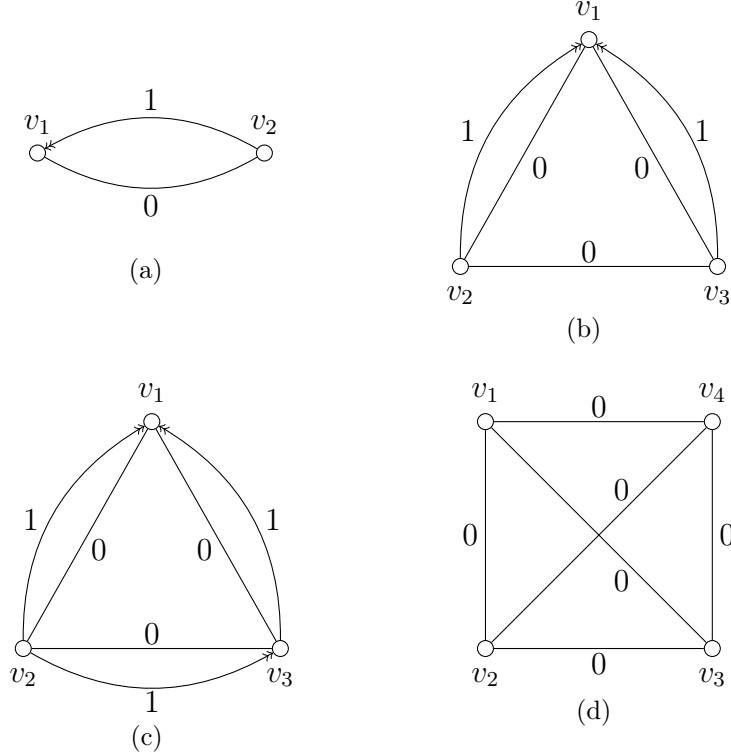


Figure 5.1: List of underlying graphs.

- The biased graph D_2 has as underlying graph the one depicted in Figure 5.1(a) and as distinguished circle $\mathcal{D} := \{\mathbf{e}_{12}(0)\mathbf{e}_{21}(1)\}$.
- The biased graph $G_\circ$ has as underlying graph the one depicted in Figure 5.1(b) and as distinguished circles $\mathcal{D} := \{\mathbf{e}_{12}(0)\mathbf{e}_{21}(1), \mathbf{e}_{13}(0)\mathbf{e}_{31}(1), \mathbf{e}_{21}(1)\mathbf{e}_{13}(-1)\mathbf{e}_{32}(0), \mathbf{e}_{12}(0)\mathbf{e}_{23}(0)\mathbf{e}_{31}(0)\}$.
- The biased graph S_3 has as underlying graph the one depicted in Figure 5.1(c) and as distinguished circles $\mathcal{D} := \{\mathbf{e}_{12}(0)\mathbf{e}_{21}(1), \mathbf{e}_{13}(0)\mathbf{e}_{31}(1), \mathbf{e}_{21}(1)\mathbf{e}_{13}(-1)\mathbf{e}_{32}(0), \mathbf{e}_{12}(0)\mathbf{e}_{23}(0)\mathbf{e}_{31}(0), \mathbf{e}_{32}(0)\mathbf{e}_{23}(1), \mathbf{e}_{21}(1)\mathbf{e}_{13}(0)\mathbf{e}_{32}(-1)\}$.
- The biased graph K_4 has as underlying graph the one depicted in Figure 5.1(d) and as distinguished circles $\mathcal{D} := \{\mathbf{e}_{12}(0)\mathbf{e}_{24}(0)\mathbf{e}_{41}(0), \mathbf{e}_{12}(0)\mathbf{e}_{23}(0)\mathbf{e}_{31}(0), \mathbf{e}_{13}(0)\mathbf{e}_{34}(0)\mathbf{e}_{41}(0), \mathbf{e}_{23}(0)\mathbf{e}_{34}(0)\mathbf{e}_{42}(0)\}$.

5.2 The computation of Falk invariant

In this section, we will describe the computation of Falk invariant for the canonical linear complete lift representation $\mathcal{A}(\mathcal{G})$ of a gain graph $\mathcal{G}$ in which there are no loops and there are at most double parallel edges.

The goal of this section is to prove the following theorem.

Theorem 5.2.1. *For an arrangement associated to a gain graph $\mathcal{G}$ via its complete lift representation in which there are no loops and there are at most double parallel edges, we have*

$$\phi_3 = 2(k_3 + k_4 + d_2 + g_o) + 5s_3, \quad (5.1)$$

where k_l denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_l \rangle$, d_2 denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle D_2 \rangle$, g_o denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle G_o \rangle$ but not contained in $\langle S_3 \rangle$, and s_3 denotes the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to $\langle S_3 \rangle$.

To prove this theorem, we need also to identify the triples S in $[n]^+$ that are dependent (say 3-circuit) for the arrangement $\mathcal{A}(\mathcal{G})$. Then we have the following

Lemma 5.2.2. *$S = (i_1, i_2, i_3)$ is dependent if and only if i_1, i_2, i_3 correspond to the edges of a subgraph of $\langle \mathcal{G} \rangle$ that is isomorphic to a $\langle K_3 \rangle$, or a triple where two of the three corresponding edges form an unbalanced circles and the third one is correspond to H_0 .*

We get that the dimension of the vector space $\mathcal{C}_3$ is $k_3 + d_2$. And each element of $\mathcal{C}_3'$ (the basis of $\mathcal{C}_3$) is in a one-to-one correspondence with the subgraph of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_3 \rangle$, or a $\langle D_2 \rangle$.

Similar to the Section 4.2, we can get the following lemmas.

Lemma 5.2.3. $\dim(A^2) = \binom{n+1}{2} - k_3 - d_2$.

Lemma 5.2.4. *For an arrangement associated to an additive gain graph $\mathcal{G}$ via its complete lift representation in which there are no loops and there are at most double parallel edges, we have*

$$I_2^3 = \text{span}(C_3) \oplus \text{span}(F_3).$$

Proof. Since $\mathcal{G}$ do not contain any loops and two distinct vertices are connected by at most two edges, any two 3-circuit share at most one element. This then gives us that $\text{span}(C_3) \cap \text{span}(F_3) = \{0\}$. $\square$

Next we have

$$\begin{aligned}\dim(I_2^3) &= \dim(\text{span}(C_3)) + \dim(\text{span}(F_3)) \\ &= k_3 + d_2 + \dim(\text{span}(F_3)).\end{aligned}$$

Hence, to prove our main result we need to be able to compute $\dim(\text{span}(F_3))$. To do so, consider the following sets

$$\begin{aligned}F_3^1 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are not in the same } \langle K_4 \rangle, \langle S_3 \rangle, \langle G_o \rangle\}, \\ F_3^2 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are in the same } \langle K_4 \rangle\}, \\ F_3^3 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are in the same } \langle G_o \rangle \text{ but not same } \langle S_3 \rangle\}, \\ F_3^4 &:= \{e_t \partial e_{ijk} \in F_3 \mid t, i, j, k \text{ are in the same } \langle S_3 \rangle\}.\end{aligned}$$

For a circle (i, j) joining by two same vertices, we will consider the 3-circuit as $(0, i, j)$. So e_0 will also appear in F_3^3 and F_3^4 .

Lemma 5.2.5. *For an arrangement associated to a gain graph $\mathcal{G}$ via its complete lift representation in which there are no loops and there are at most double parallel edges, we have*

$$\text{span}(F_3) = \text{span}(F_3^1) \oplus \left(\text{span}\left(\bigcup_{i=2}^4 F_3^i\right) \right).$$

Moreover, we have that $\text{span}(F_3^3) \cap \text{span}(F_3^4) = \{0\}$.

Proof. For any element $e_t \partial e_{ijk}$ of F_3^1 , we assert that at least one of the terms $e_{tijk}, e_{tik}, e_{tij}$ appears only in the expression of $e_t \partial e_{ijk}$. So $e_t \partial e_{ijk}$ can not be expressed linearly by the elements of F_3^2, F_3^3, F_3^4 .

Since the edges t, i, j, k are not in the same $\langle K_4 \rangle, \langle G_o \rangle, \langle S_3 \rangle$, and we do not consider the graphs in which there are loops or triple parallel edges, we should only consider three cases about the edge t : it can be adjacent to none of the edges i, j, k , to two of them, or to all of them.

Assume that the edge t is adjacent to none of the edges i, j, k . This implies that t and none of i, j, k can appear in the same triangle. Hence any element $e_t \partial e_{ijk}$ of F_3^1 will not appear in any of F_3^2, F_3^3, F_3^4 .

Assume now that the edge t is adjacent to two of the edges i, j, k , then we should consider several possibilities. Suppose that $0 \notin \{t, i, j, k\}$. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in F_3^2, F_3^3 , or F_3^4 , then t, i, j, k have to appear in the same $\langle K_4 \rangle$, but this is impossible by construction. Suppose that $t = 0$. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in

F_3^2, F_3^3, F_3^4 , then t, i, j, k have to appear in the same $\langle G_o \rangle$ or in the same $\langle S_3 \rangle$, but this is impossible by construction. Suppose that $t \neq 0$ and $0 \in \{i, j, k\}$. In this case two edges in i, j, k are the edges of a $\langle D_2 \rangle$. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in F_3^2, F_3^3 , or F_3^4 , then, in this case, t, i, j, k have to appear in the same $\langle G_o \rangle$ or in the same $\langle S_3 \rangle$, but this is impossible by construction.

Finally, assume that the edge t is adjacent to all the edges i, j, k . Since the underlying graph has at most double parallel edges and no loops among its edges, then $t \neq 0$ and we should consider only two possibilities. Suppose that $0 \notin \{t, i, j, k\}$. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in F_3^2, F_3^3 , or F_3^4 , then t, i, j, k have to appear in the same $\langle G_o \rangle$ or in the same $\langle S_3 \rangle$, but this is impossible by construction. Suppose that $0 \in \{i, j, k\}$. In this case two edges in $\{i, j, k\}$ are the edges of a $\langle D_2 \rangle$. If all the terms of the element $e_t \partial e_{ijk} \in F_3^1$ appear in F_3^2, F_3^3 , or F_3^4 , then, in this case, t, i, j, k have to appear in the same $\langle G_o \rangle$ or in the same $\langle S_3 \rangle$, but this is impossible by construction.

Therefore, for any element $e_t \partial e_{ijk} \in F_3^1$, at least one of the terms $e_{tjk}, e_{tik}, e_{tij}$ appears only in the expression of $e_t \partial e_{ijk}$. This shows that

$$\text{span}(F_3^1) \cap \left(\text{span} \left(\bigcup_{i=2}^4 F_3^i \right) \right) = \{0\}.$$

Since clearly $\text{span}(F_3) = \sum_{i=1}^4 \text{span}(F_3^i)$, this concludes the proof of the first part of the statement.

The second part of the statement follows directly from the definition of F_3^3 and F_3^4 . $\square$

Differently from the situation discussed in Chapter 4, in this setting it might happen that $\text{span}(F_3^2) \cap \text{span}(F_3^3) \neq \{0\}$ or $\text{span}(F_3^2) \cap \text{span}(F_3^4) \neq \{0\}$. In particular, we have the following straightforward result.

Lemma 5.2.6. *In the graph $\langle \mathcal{G} \rangle$ there is a subgraph isomorphic to a $\langle K_3 \rangle$ with edges i, j, k contained in a subgraph isomorphic to a $\langle G_o \rangle$ and one isomorphic to a $\langle K_4 \rangle$ at the same time if and only if $\{0\} \neq \text{span}(F_3^2) \cap \text{span}(F_3^3) \supseteq \text{span}(e_0 \partial e_{ijk})$. Moreover, the elements of the type $e_0 \partial e_{ijk}$ generate the intersection. Similarly, in the graph $\langle \mathcal{G} \rangle$ there is a subgraph isomorphic to a $\langle K_3 \rangle$ with edges i, j, k contained in a subgraph isomorphic to a $\langle S_3 \rangle$ and one isomorphic to a $\langle K_4 \rangle$ at the same time if and only if $\{0\} \neq \text{span}(F_3^2) \cap \text{span}(F_3^4) \supseteq \text{span}(e_0 \partial e_{ijk})$. Moreover, the elements of the type $e_0 \partial e_{ijk}$ generate the intersection. However, in both cases, $e_{0jk}, e_{0ik}, e_{0ij}$ only appear one time in F_3^2 .*

Remark 5.2.7. Similarly to the examples 3.2.8 and 3.2.9, we can directly compute $\dim(\text{span}(F_3))$ for some distinguished gain graphs in the figure 5.1. In particular, if we consider the gain graphs $G_\circ$, then $\dim(\text{span}(F_3)) = 10$. If we consider S_3 , then $\dim(\text{span}(F_3)) = 19$. If we consider K_4 , then $\dim(\text{span}(F_3)) = 14$.

Similar to the Lemma 4.2.9, we have

Lemma 5.2.8. With the previous notations, we have that $\dim(\text{span}(F_3^3)) = 10g_\circ$, $\dim(\text{span}(F_3^4)) = 19s_3$.

Proof. Assume that in the graph $\langle \mathcal{G} \rangle$ there are exactly $g_1 = p$ distinct subgraphs isomorphic to a $\langle G_\circ \rangle$, $\langle \mathcal{G}_1 \rangle, \dots, \langle \mathcal{G}_p \rangle$, none of which is a subgraph of a graph isomorphic to $\langle S_3 \rangle$. Consider

$$F_{3,r}^3 := \{e_t \partial e_{ijk} \mid e_{ijk} \in C'_3, t \in [n]^+ \setminus \{i, j, k\}, i, j, k \in \langle G_r \rangle\}.$$

Since three edges in the underlying graph of $\mathcal{G}$ can not appear in two distinct $\langle G_\circ \rangle$ at the same time, then none of the terms of the element $e_t \partial e_{ijk} \in F_{3,r}^3$ appear in the elements of $F_3^3 \setminus F_{3,r}^3$. This shows that

$$\text{span}(F_3^3) = \bigoplus_{r=1}^p \text{span}(F_{3,r}^3).$$

By Proposition 2.2.8, we have that $\dim(\text{span}(F_{3,r}^3)) = 10$ for all $r = 1, \dots, p$. This then implies that

$$\dim(\text{span}(F_3^3)) = \sum_{r=1}^p \dim(\text{span}(F_{3,r}^3)) = 10g_\circ.$$

Using Remark 5.2.7, the same exact argument used in this case will prove that $\dim(\text{span}(F_3^4)) = 19s_3$. $\square$

Notice that the argument of the previous lemma cannot be utilized to compute $\dim(\text{span}(F_3^2))$. This is because if in the gain graph $\mathcal{G}$ there is a subgraph isomorphic to a $\langle K_3 \rangle$ with edges i, j, k contained in two distinct subgraphs isomorphic to a $\langle K_4 \rangle$ at the same time, $\mathcal{G}_1, \mathcal{G}_2$, then $\{0\} \neq \text{span}(F_{3,1}^2) \cap \text{span}(F_{3,2}^2) \supseteq \text{span}(e_0 \partial e_{ijk})$. Moreover, the elements of the type $e_0 \partial e_{ijk}$ generate the intersection. This fact together with a similar argument to the one in the proof of Lemma 4.2.9 gives us the following result.

Lemma 5.2.9. With the previous notations, we have that

$$\dim(\text{span}(F_3^2)) = 14k_4 - \sum_{i \geq 2} (i-1)\lambda_i, \quad (5.2)$$

where λ_i is the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_3 \rangle$ contained in exactly i distinct subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_4 \rangle$ at the same time.

Notice that, since we are dealing with finite graphs, the sum in the formula (5.2) is a finite sum.

Lemma 5.2.10. *For an arrangement associated to a gain graph $\mathcal{G}$ via its complete lift representation in which there are no loops and there are at most double parallel edges, we have*

$$\dim(I_2^3) = (n-1)(k_3 + d_2) - 2k_4 - 2g_o - 5s_3.$$

Proof. To prove the statement, we need to compute $\dim(\text{span}(F_3))$. From Lemma 5.2.6, let γ be the number of subgraphs of $\langle \mathcal{G} \rangle$ isomorphic to a $\langle K_3 \rangle$ contained in a subgraph isomorphic to a $\langle G_o \rangle$ and one isomorphic to a $\langle K_4 \rangle$ or in a subgraph isomorphic to a $\langle S_3 \rangle$ and one isomorphic to a $\langle K_4 \rangle$ at the same time. From Lemma 5.2.9, let $\lambda := \sum_{i \geq 2} (i-1)\lambda_i$. By the previous lemmas

$$\begin{aligned} \dim(\text{span}(F_3)) &= \dim(\text{span}(F_3^1)) + \dim(\text{span}(\bigcup_{i=2}^4 F_3^i)) = \\ &= [(n-2)(k_3 + d_2) - 16k_4 + \lambda - 12g_o - 24s_3 + \gamma] + \dim(\text{span}(\bigcup_{i=2}^4 F_3^i)). \\ &= [(n-2)(k_3 + d_2) - 16k_4 + \lambda - 12g_o - 24s_3 + \gamma] + \\ &\quad 14k_4 - \lambda + 10g_o + 19s_3 - \gamma \\ &= (n-2)(k_3 + d_2) - 2k_4 - 2g_o - 5s_3. \end{aligned}$$

The thesis follows from the equality

$$\dim(I_2^3) = k_3 + d_2 + \dim(\text{span}(F_3)).$$

□

Proof of Theorem 5.2.1. By Remark 1.3.6 and Lemma 5.2.3 we have

$$\phi_3 = 2 \binom{n+2}{3} - (n+1) \left(\binom{n+1}{2} - k_3 - d_2 \right) + \binom{n+1}{3} - \dim(I_2^3).$$

Because $2 \binom{n+2}{3} - (n+1) \binom{n+1}{2} + \binom{n+1}{3} = 0$, then from Lemma 5.2.10 we obtain

$$\phi_3 = 2(k_3 + k_4 + d_2 + g_o) + 5s_3.$$

□

Let us see how our formula works on a non-trivial example.

Example 5.2.11. *We want to compute ϕ_3 for the arrangement associated to the gain graph $\mathcal{G}$ of Figure 5.2.*

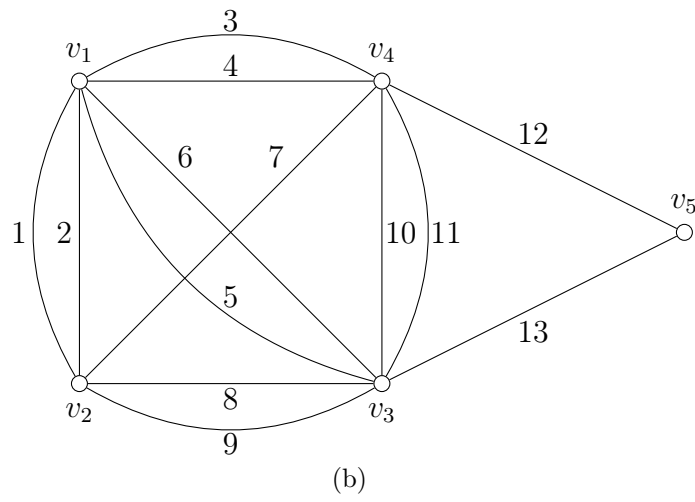
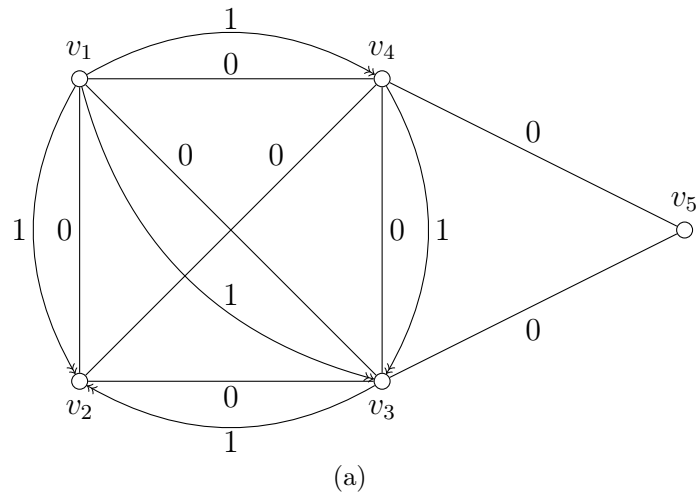
In order to compute ϕ_3 with the formula (5.1), we need to compute the following:

- $k_3 = |\{\{1, 3, 7\}, \{1, 5, 8\}, \{1, 6, 9\}, \{2, 6, 8\}, \{2, 4, 7\}, \{4, 6, 10\}, \{7, 8, 10\}, \{3, 5, 10\}, \{4, 5, 11\}, \{10, 12, 13\}\}| = 10;$
- $d_2 = |\{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}, \{0, 8, 9\}, \{0, 10, 11\}\}| = 5;$
- $k_4 = |\{\{2, 4, 6, 7, 8, 10\}, \{1, 3, 5, 7, 8, 10\}\}| = 2;$
- $g_o = |\{\{1, 2, 3, 4, 7\}\}| = 1.$
- $s_3 = |\{\{1, 2, 5, 6, 8, 9\}, \{3, 4, 5, 6, 10, 11\}\}| = 2.$

From formula (5.1), we obtain

$$\phi_3 = 2(10 + 5 + 2 + 1) + 5 \cdot 2 = 46.$$

Notice that if we would try to compute the dimension of F_3 directly, we would have to write 150 equations in the e_{ijk} .

Figure 5.2: The gain graph $\mathcal{G}$ and its underlying graph.

5.3 The cone of Shi, Linial and semiorder arrangements

The braid arrangement has a number of "deformations" of considerable interests, see [7] and [28] for more details. In this section, we will just define three of those: the Shi arrangement, the Linial arrangement and the semiorder arrangement. Using Theorem 5.2.1, we will compute the Falk invariant for each of these classes.

Definition 5.3.1. *The **Shi arrangement** Shi_ℓ in $\mathbb{C}^\ell$ is the arrangement consisting of the hyperplanes*

$$\{x_i - x_j = 0\} \cup \{x_i - x_j - 1 = 0\} \text{ for } 1 \leq i < j \leq \ell.$$

Notice that the cone of the Shi arrangement $c(Shi_\ell)$ is the arrangement in $\mathbb{C}^{\ell+1}$ consisting of the hyperplanes

$$\{x_0 = 0\} \cup \{x_i - x_j = 0\} \cup \{x_i - x_j - x_0 = 0\} \text{ for } 1 \leq i < j \leq \ell.$$

This implies that $c(Shi_\ell)$ is the canonical linear complete lift representation of the gain graph $\mathcal{G}_{Shi}$, where $\mathcal{G}_{Shi}$ is the gain graph with underlying graph G on $V_G = [\ell]$ such that for any two distinct vertices $i, j \in [\ell]$ there are exactly two parallel edges, with gains respectively 0 and -1 .

Theorem 5.3.2. *The Falk invariant of the cone of the Shi arrangement $c(Shi_\ell)$ is given by*

$$\phi_3(c(Shi_\ell)) = \frac{\ell(\ell-1)(2\ell^2 + \ell - 4)}{6}.$$

Proof. In the graph $\langle \mathcal{G}_{Shi} \rangle$ any two vertices can form a $\langle D_2 \rangle$, so the number of subgraph isomorphic to a $\langle D_2 \rangle$ is $\binom{\ell}{2}$. Any three vertices can form a $\langle S_3 \rangle$ with 3 subgraphs isomorphic to a $\langle K_3 \rangle$, so the number of subgraphs isomorphic to a $\langle S_3 \rangle$ is $\binom{\ell}{3}$, while the number of the subgraphs isomorphic to a $\langle K_3 \rangle$ is $3\binom{\ell}{3}$. Moreover, any four vertices gives us 4 subgraphs isomorphic to a $\langle K_4 \rangle$, so the number of subgraphs isomorphic to a $\langle K_4 \rangle$ is $4\binom{\ell}{4}$. Finally, there is no subgraph isomorphic to a $\langle G_o \rangle$.

From formula (5.1), we obtain

$$\begin{aligned} \phi_3(c(Shi_\ell)) &= 2\left(3\binom{\ell}{3} + \binom{\ell}{2} + 4\binom{\ell}{4} + 0\right) + 5\binom{\ell}{3} \\ &= \frac{\ell(\ell-1)(2\ell^2 + \ell - 4)}{6}. \end{aligned}$$

□

Definition 5.3.3. The *Linial arrangement* $\mathcal{L}_\ell$ in $\mathbb{C}^\ell$ is the arrangement consisting of the hyperplanes

$$\{x_i - x_j - 1 = 0\} \text{ for } 1 \leq i < j \leq \ell.$$

Notice that the cone of the Linial arrangement $c(\mathcal{L}_\ell)$ is the arrangement in $\mathbb{C}^{\ell+1}$ consisting of the hyperplanes

$$\{x_0 = 0\} \cup \{x_i - x_j - x_0 = 0\} \text{ for } 1 \leq i < j \leq \ell.$$

This implies that $c(\mathcal{L}_\ell)$ is the canonical linear complete lift representation of the gain graph $\mathcal{G}_{\text{Linial}}$, where $\mathcal{G}_{\text{Linial}}$ is the gain graph with underlying graph $G = K_\ell$, the complete graph on ℓ vertices, such that each edge has gain equal to -1 .

Theorem 5.3.4. The Falk invariant of the cone of the Linial arrangement $c(\mathcal{L}_\ell)$ is zero for any ℓ .

Proof. In the graph $\langle \mathcal{G}_{\text{Linial}} \rangle$ there are no subgraphs isomorphic to a $\langle D_2 \rangle$, or a $\langle K_3 \rangle$, or a $\langle K_4 \rangle$, or a $\langle G_\circ \rangle$, or a $\langle S_3 \rangle$. This implies by the formula (5.1) that $\phi_3(c(\mathcal{L}_\ell)) = 0$. $\square$

Definition 5.3.5. The *semiorder arrangement* Semi_ℓ in $\mathbb{C}^\ell$ is the arrangement consisting of the hyperplanes

$$\{x_i - x_j + 1 = 0\} \cup \{x_i - x_j - 1 = 0\} \text{ for } 1 \leq i < j \leq \ell.$$

Notice that the cone of the semiorder arrangement $c(\text{Semi}_\ell)$ is the arrangement in $\mathbb{C}^{\ell+1}$ consisting of the hyperplanes

$$\{x_0 = 0\} \cup \{x_i - x_j + x_0 = 0\} \cup \{x_i - x_j - x_0 = 0\} \text{ for } 1 \leq i < j \leq \ell.$$

This implies that $c(\text{Semi}_\ell)$ is the canonical linear complete lift representation of the gain graph $\mathcal{G}_{\text{Semi}}$, where $\mathcal{G}_{\text{Semi}}$ is the gain graph with underlying graph G on $V_G = [\ell]$ such that for any two distinct vertices $i, j \in [\ell]$ there are exactly two parallel edges, with gains respectively 1 and -1 .

Theorem 5.3.6. The Falk invariant of the cone of the semiorder arrangement $c(\text{Semi}_\ell)$ is given by

$$\phi_3(c(\text{Semi}_\ell)) = \ell(\ell - 1).$$

Proof. In the graph $\langle \mathcal{G}_{\text{Semi}} \rangle$ any two vertices can form a $\langle D_2 \rangle$, so the number of subgraph isomorphic to a $\langle D_2 \rangle$ is $\binom{\ell}{2}$. Moreover, there are no subgraphs isomorphic to a $\langle K_3 \rangle$, or a $\langle K_4 \rangle$, or a $\langle G_\circ \rangle$, or a $\langle S_3 \rangle$.

From formula (5.1), we obtain

$$\phi_3(c(\text{Semi}_\ell)) = 2 \binom{\ell}{2} = \ell(\ell - 1).$$

$\square$

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