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Author(s)	Sawasaki, Takahiro
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A Sequent Calculus for K-restricted Common Sense Modal Predicate Logic

Takahiro Sawasaki

Graduate School of Letters, Hokkaido University

Sapporo, Hokkaido, Japan

taka.sawasaki562@gmail.com

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Abstract

In recent years, Common sense Modal Predicate Calculus (CMPC) has been proposed by J. van Benthem in [4, pp. 120–121] and further developed by J. Seligman in [1, 3, 2]. It allows us to ‘take $\exists$ to mean just “exists” while denying the Constant Domain thesis’ [1, p. 8].¹ This is done in terms of *talking about only things in each world in which they exist*. From a proof-theoretical view, the Hilbert-style system for CMPC given by Seligman is a system for modal predicate logic S5 *which has the following axiom K_{inv} instead of axiom K*:

$$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi) \quad \text{provided that all free variables in } \varphi \text{ are free variables in } \psi.$$

It is quite interesting because it might make a clean sweep of all philosophical discussions on possible world semantics between actualists and possibilists. However, neither van Benthem nor Seligman have developed K-restricted CMPC and expansions of the logic with some well known axioms. Moreover, proof-theoretic studies for such logics have not been done yet.

In this talk, I shall propose a sequent calculus for K-restricted CMPC. The main mathematical contributions of this talk are the completeness result (Theorem 1) and cut elimination theorem (Theorem 2) for the calculus. If time allows I shall also introduce sequent calculi for K-restricted CMPC with T axiom and D-like axioms. In what follows, I will outline the contents of this talk.

The language $\mathcal{L}$ of K-restricted Common sense Modal Predicate Calculus **CK** consists of a countably infinite set $\text{Var} = \{x, y, \dots\}$ of variables, a countably infinite set $\text{Pred} = \{P, Q, \dots\}$ of predicate symbols each of which has a fixed finite

¹The Constant Domain thesis is a thesis that ‘[e]very possible world has exactly the same objects as every other possible world.’ [1, p. 5]

arity, and logical symbols, $\perp, \supset, \Box, \forall$. The set Form of formulas of $\mathcal{L}$ is defined recursively as follows:

$$\text{Form} \ni \varphi ::= Px_1 \dots x_n \mid \perp \mid (\varphi \supset \varphi) \mid \forall x\varphi \mid \Box\varphi$$

where P is a predicate symbol with arity n and $x, x_1, \dots, x_n$ are variables. The other connectives are defined as usual. We also define the sets $\text{FV}(\varphi)$ and $\text{FV}(\Gamma)$ of free variables in a formula φ and a set Γ of formulas, respectively, as usual.

Semantics for **CK** is given as follows. A frame is a tuple (W, R, D) , where W is a nonempty set; R is a binary relation on W ; D is a W -indexed family $\{D_w\}_{w \in W}$ of nonempty sets. Thus R does not need to satisfy the *inclusion requirement*: if wRv then $D_w \subseteq D_v$. A model is a tuple (F, V) , where F is a frame and V is a valuation that maps each world w and each predicate P to a subset $V_w(P)$ of D_w . An *assignment* α is a partial function from variables to entities and $\alpha(x|d)$ stands for the same assignment as α except for assigning d to x . In addition to these notions, we follow [1, p. 15] and say that a formula φ is an α_w -formula if $\alpha(x) \in D_w$ for any variable $x \in \text{FV}(\varphi)$. Then, similarly as in [1, pp. 15–16], the satisfaction relation and validity are defined as follows.

Definition 1 (Satisfaction relation). Let M be a model, α be an assignment, and w be a world in W . The *satisfaction relation* $M, \alpha, w \models \varphi$ between M, α, w and an α_w -formula φ is defined as follows:

$$M, \alpha, w \models Px_1 \dots x_n \quad \text{iff} \quad (\alpha(x_1), \dots, \alpha(x_n)) \in V_w(P)$$

$$M, \alpha, w \not\models \perp$$

$$M, \alpha, w \models \psi \supset \gamma \quad \text{iff} \quad M, \alpha, w \models \psi \text{ implies } M, \alpha, w \models \gamma$$

$$M, \alpha, w \models \forall x\psi \quad \text{iff} \quad M, \alpha(x|d), w \models \psi \quad \text{for any } d \in D_w$$

$$M, \alpha, w \models \Box\psi \quad \text{iff} \quad M, \alpha, v \models \psi$$

for any v such that wRv and ψ is an α_v -formula

Definition 2 (Validity). Let $\Gamma \cup \{\varphi\}$ be a set of formulas. We say that φ is *valid in a frame* if for any model M based on the frame, assignment α and world w such that φ is an α_w -formula, $M, \alpha, w \models \varphi$. We also say that φ is *valid in a class of frames* if φ is valid in all frames in the class.

The following propositions that Seligman proves in [1, pp. 16–17] are noteworthy².

Proposition 3 (Converse Barcan formula). A formula $\Box\forall x\varphi \supset \forall x\Box\varphi$ is valid in the class of all frames.

Proof. Fix any model M , assignment α , world w such that $\Box\forall x\varphi \supset \forall x\Box\varphi$ is an α_w -formula. Suppose $M, \alpha, w \models \Box\forall x\varphi$ and fix any element $d \in D_w$, any world v such that wRv and φ is an $\alpha(x|d)_v$ -formula. We show $M, \alpha(x|d), v \models \varphi$. Since

²Strictly speaking, he considers the dual formulas of those in Proposition 3.4.

$FV(\forall x\varphi) \subseteq FV(\varphi)$ and φ is an $\alpha(x|d)_v$ -formula, we have that $\forall x\varphi$ is an $\alpha(x|d)_v$ -formula and thus that $\forall x\varphi$ is an α_v -formula. Hence we get $M, \alpha, v \models \forall x\varphi$ so $M, \alpha(x|d), v \models \varphi$. ■

Proposition 4. A formula $\forall x\Box\varphi \supset \Box\forall x\varphi$ is not valid in the class $\mathbb{F}$ of all frames $F = (W, R, D)$ such that R is an equivalence relation.

Proof. Consider a model $M = (W, R, D, V)$, where $W = \{0, 1\}$; $R = W \times W$; $D_0 = \{a\}$ and $D_1 = \{b\}$; $V_0(P) = \{a\}$ and $V_1(P) = \emptyset$ for some predicate symbol P with arity 1, and $V_i(Q) = \emptyset$ for the other predicate symbols Q with arity n . Then, we can establish $M, \alpha, 0 \models \forall x\Box Px$ but $M, \alpha, 0 \not\models \Box\forall xPx$. Therefore, $\forall x\Box\varphi \supset \Box\forall x\varphi$ is not valid in $\mathbb{F}$. ■

Given finite multisets Γ, Δ of formulas, we call an expression $\Gamma \Rightarrow \Delta$ a sequent. Then a sequent calculus $G(\text{CK})$ for CK is given in Table 1. The rule $\Box_{inv}$ in it plays roles of axiom K_{inv} and the necessitation rule in the Hilbert-style system for CMPC given by Seligman. The notion of a derivation in $G(\text{CK})$ is defined as usual.

Table 1: A Sequent Calculus $G(\text{CK})$ for CK

Initial Sequents	
$\varphi \Rightarrow \varphi$	$\perp \Rightarrow$
Structural Rules	
$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \Rightarrow w$	$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} w \Rightarrow$
$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \Rightarrow c$	$\frac{\varphi, \varphi \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} c \Rightarrow$
$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Theta \Rightarrow \Sigma}{\Gamma, \Theta \Rightarrow \Delta, \Sigma} \text{Cut}$	
Logical Rules	
$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi} \Rightarrow \supset$	$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Theta \Rightarrow \Sigma}{\varphi \supset \psi, \Gamma, \Theta \Rightarrow \Delta, \Sigma} \supset \Rightarrow$
$\frac{\Gamma \Rightarrow \Delta, \varphi(y/x)}{\Gamma \Rightarrow \Delta, \forall x\varphi} \Rightarrow \forall^\dagger$	$\frac{\varphi(t/x), \Gamma \Rightarrow \Delta}{\forall x\varphi, \Gamma \Rightarrow \Delta} \forall \Rightarrow$
$\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \Box_{inv}^\ddagger$	
$\dagger: y \text{ does not occur in } \Gamma, \Delta, \forall x\varphi. \quad \ddagger: FV(\Gamma) \subseteq FV(\varphi).$	

We also say that a sequent $\Gamma \Rightarrow \Delta$ is valid if $(\gamma_1 \wedge \dots \wedge \gamma_m) \supset (\delta_1 \vee \dots \vee \delta_n)$ is valid, where $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ and $\Delta = \{\delta_1, \dots, \delta_n\}$. Then, the following theorems hold under the settings above.

Theorem 1 (Completeness). Let $\Gamma \cup \{ \varphi \}$ be a set of formulas. If $\Gamma \Rightarrow \varphi$ is valid in the class of all frames, then $\Gamma \Rightarrow \varphi$ is derivable in $G(\mathbf{CK})$.

Theorem 2 (Cut elimination). Let Γ, Δ be finite multisets of formulas. If $\Gamma \Rightarrow \Delta$ is derivable in $G(\mathbf{CK})$, then it is also derivable in $G(\mathbf{CK})$ without any application of *Cut*.

References

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