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Proceedings of the 22nd Sapporo Symposium on Partial Differential Equations

Edited by T. Ozawa

Sapporo, 1997

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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on July 30 through August 1 in 1997 at Department of Mathematics, Hokkaido University.

This is the 22nd time of the symposium and also commemorates the 60th birthday of Professor Rentaro Agemi, who made a large contribution to its organization for years.

We wish to dedicate this volume to Professor Agemi in celebration of his 60th birthday.

T. Ozawa

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Programme

第２２回偏微分方程式論札幌シンポジウム

下記の要領でシンポジウムを行ないますのでご案内申し上げます。

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代表者　　小澤　徹

</div>

記

1. 日時　　1997 年 7 月 30 日（水）〜8 月 1 日（金）
2. 場所　　北海道大学大学院理学研究科　数学教室
3. 講演

7 月 30 日

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	いくつかの凸な物体による散乱
15:30-16:30	奈良女・理　　静田　靖（Y. SHIZUTA）
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16:30-17:00	＊

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11:00-12:00	早大・理工　　柴田良弘（Y. SHIBATA）
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	Blow-up Boundaries and Rates for Nonlinear Wave Equations
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11:00-12:00	金沢大・理　　一瀬　孝（T. ICHINOSE） Estimate on the Kac Transfer Operator with Applications
13:30-14:00	＊
14:00-14:30	東京電機大・理工　　高橋秀慈（S. TAKAHASHI） A Weighted Equation Approach to Decay Rate Estimates for the Navier-Stokes Equations
15:00-15:30	東大・数理　　中西賢次（K. NAKANISHI） Global Solutions for Wave Equations with Non-Coercive Critical Nonlinearity
16:00-16:30	北大・理　　伊藤一男（K. ITO） Self-Intersection of Curves Driven by Surface Diffusion Law
16:30-17:00	＊

＊この時間は講演者を囲んで自由な質問の時間とする予定です。

7月31日 18:00 からの懇親会は上見練太郎教授還暦記念パーティーを兼ねて行われます。多数のご参加お待ち申し上げております。

　　　場所　北海道大学　ファカルティハウス　エンレイソウ
　　　電話　（011)726-7601

On scattring by several convex bodies

Mitsuru IKAWA

Let $\mathcal{O}_j$, $j = 1, 2, \ldots, J$ be open bounded sets in $\mathbf{R}^3$ with smooth boundary Γ_j. We set

$$\mathcal{O} = \cup_{j=1}^{J} \mathcal{O}_j$$

and assume the following:

(H.1) Each $\mathcal{O}_j$ is strictly convex.

(H.2) For each $\{j_1, j_2, j_3\} \in \{1, 2, \ldots, J\}^3$ such that $j_l \neq j_{l'}$ for $l \neq l'$

$$(\text{convex hull of } \overline{\mathcal{O}_{j_1}} \text{ and } \overline{\mathcal{O}_{j_2}}) \cap \overline{\mathcal{O}_{j_3}} = \emptyset.$$

Here we consider the case of

$$J \geq 3.$$

We set

$$\Omega = \mathbf{R}^3 - \overline{\mathcal{O}},$$

and consider two dynamics in Ω. The one us the classical dynamics in Ω and the another is the quantum dynamics in Ω, and we are interested in relationships between these two dynamics. As the first step of study of relationships of two dynamics, we take up the zeta function as the subject of the classical dynamics, and the scattering matrix as that of the quantum dynamics. Our interest as to these subjects is to know how the singularities of the zeta function relate to the poles of the scattering matrix, and vice versa.

The problem we shall consider in the talk is the following:

How is the distribution of the poles of the zeta function near the line $\{s \in \mathbf{C};\ \Re s = \alpha_0\}$, where α_0 denotes the abscissa of the absolute convergence of the zeta function?

Recent Topics in the Initial Boundary Value Problem for Linear Symmetric Hyperbolic Systems with Characteristic Boundary

Y. Shizuta

Department of Mathematics

Nara Women's University

July, 1997

First, we define a class of anisotropic Sobolev spaces. Let $\alpha \in \mathbf{Z}_+^n$ be a multi-index and let $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We write $\partial_x = (\partial_1, \cdots, \partial_n)$, $\partial_i = \partial_{x_i} = \partial/\partial x_i$, $1 \leq i \leq n$, and $\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$. A vector field $\Lambda \in C^\infty(\bar{\Omega} ; \mathbf{C}^n)$ is said to be tangential if $\langle \Lambda(x), \nu(x) \rangle = 0$ for all $x \in \Gamma$. Here $\Omega \subseteq \mathbf{R}^n$, $n \geq 2$, is a bounded open set lying on one side of its smooth boundary Γ. Let m and ν be nonnegative integers. Then the function space $H_*^{m,\nu}(\Omega)$ is defined as the set of functions having the following properties:

i) $u \in L^2(\Omega)$.

ii) Let $\Lambda_1, \Lambda_2, \cdots, \Lambda_j$ be tangential vector fields and let $\Lambda_1', \Lambda_2', \cdots, \Lambda_k'$ be any vector fields. Multiply u by the product of these $j + k$ vector fields in an arbitrary order. Then the resulting function belongs to $L^2(\Omega)$, if $j + \varphi(k; \nu) \leq m$.

Here, the auxiliary function $\varphi(k; \nu)$ is defined by $\varphi(k; \nu) = k$ for $0 \leq k \leq \nu$ and $\varphi(k; \nu) = 2k - \nu$ for $\nu \geq k$. $H_*^{m,\nu}(\Omega)$ is normed as follows. We choose an appropriate covering of Γ, diffeomorphisms, and cut off functions, say θ_i, τ_i, χ_i, $1 \leq i \leq N$. Then $u^{(i)} = (\chi_i u) \circ \tau_i^{-1}$ has as its domain $B_+ = \{x \mid |x| < 1, x_n > 0\}$ with Γ corresponding to $x_n = 0$. Let Ω_δ be the set $\{x \in \Omega \mid dist(x, \Gamma) > \delta\}$ and let χ_0 be a cut off function such that $\chi_0 = 0$

on a neighborhood of Γ. Let furthermore $\chi_0 = 1$ on Ω_δ where δ is suitably chosen. We may assume that $\sum_{i=0}^N \chi_i^2 = 1$ on $\bar{\Omega}$. Then the norm in $H_*^{m,\nu}(\Omega)$ is

$$(1) \qquad \|u\|_{m,\nu,*}^2 = \|\chi_0 u\|_m^2 + \sum_{i=1}^N \|\chi_i u\|_{m,\nu,*}^2,$$

$$(2) \qquad \|\chi_i u\|_{m,\nu,*}^2 = \sum_{|\alpha|+\varphi(k;\nu)\leq m} \|\partial_{tan}^\alpha \partial_n^k u^{(i)}\|_{L^2(B_+)}^2,$$

where $\partial_{tan}^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots (x_n \partial_n)^{\alpha_n}$. Note that ∂_{tan}^α can be replaced by $\partial_*^\alpha = x_n^{\alpha_n} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$, because the corresponding norms are equivalent to each other. The norms arising from different choices of θ_i, τ_i, χ_i are also equivalent.

By definition, $H_*^{m,\nu}(\Omega) = H^m(\Omega)$ for $0 \leq m \leq \nu$, where $H^m(\Omega)$ denotes the usual Sobolev space. $H_*^{m,\nu}(\Omega)$ for $0 \leq \nu < m$ is an anisotropic Sobolev space. We are mainly interested in this function space. The function spaces we used in the preceding works are $H_*^m(\Omega)$ and $H_{**}^m(\Omega)$. See [1], [2], [5]. We observe that $H_*^{m,0}(\Omega) = H_*^m(\Omega)$ and that $H_*^{m,1}(\Omega) = H_{**}^m(\Omega)$.

Recently we studied on $H_*^{m,\nu}(\Omega)$ with $0 \leq \nu < m$ and obtained a trace theorem and estimates for the product of functions belonging to these function spaces. Now we want to consider the initial boundary value problem for linear symmetric hyperbolic systems with characterisitic boundary. We shall treat defferential operators of the form

$$L = A_0(x,t)\partial_t + \sum_{j=1}^n A_j(x,t)\partial_j + B(x,t),$$

where the coefficient matrices are real $\ell \times \ell$ matrices depending smoothly (in a certain sense) on their arguments. A_0 and A_j, $1 \leq j \leq n$, are symmetric matrices and A_0 is positive definite.

The problem we are going to study is

$$(3) \qquad Lu = F \qquad \text{in } [0,T] \times \Omega,$$

$$(4) \qquad Mu = 0 \qquad \text{on } [0,T] \times \Gamma,$$

$$(5) \qquad u(0,x) = f(x) \qquad \text{for } x \in \Omega,$$

where the unknown function $u = u(t,x)$ is a vector-valued function with ℓ components. $M(x)$ is an $\ell \times \ell$ real matrix depending smoothly on $x \in \Gamma$. We assume that M is of constant rank everywhere on Γ. The inhomogeneous term F and the initial data f have a certain smoothness on $[0,T] \times \Omega$ and Ω, respectively. The compatibility condition of order $m - 1$ is assumed to hold. The maximal nonnegativity of the boundary matrix is also assumed. We shall discuss on the possibility of solving this problem by using a pair of function spaces, $H_*^{m,\nu}(\Omega)$ and $H_*^{m,\nu+1}(\Omega)$, where ν is fixed and $m > \nu$. Note that a general theory is described for this problem by using a pair of function spaces, $H_*^m(\Omega) = H_*^{m,0}(\Omega)$ and $H_{**}^m(\Omega) = H_*^{m,1}(\Omega)$. See [3],[4],[6].

References

[1] M. Ohno and Y. Shizuta and T. Yanagisawa. *The trace theorem on anisotropic Sobolev spaces.* Tôhoku Math. J. **46**,pp. 393–401 (1994).

[2] M. Ohno and Y. Shizuta and T. Yanagisawa. *The initial boundary value problem for linear symmetric hyperbolic systems with boundary chracteristic of constant multiplicity.* J. Math. Kyoto. Univ. **35**,pp. 143–210 (1995).

[3] M. Ohno and T. Shirota. *On the initial boundary value problem for the linearized MHD equations.* preprint.

[4] P. Secchi. *Linear symmetric hyperbolic systems with characteristic boundary.* Math. Mech. Appl. Sci. **18**,pp. 855–870 (1995).

[5] Y. Shizuta and K. Yabuta. *The trace theorem in anisotropic Sobolev spaces and their applications to the chracteristic initial boundary value problem for symmetric hyperbolic systems.* Math. Mod. Meth. Appl. Sci. **5**,pp. 1079–1092 (1995).

[6] Y. Yamamoto. *Regularity of solutions of initial value problems for symmetric hyperbolic systems with boundary characteristic of constant multiplicity.* to appear in the Proceedings of the Conference on Nonlinear PDE held in Nara on 21 and 22 September 1996.

L^1-solutions of the Navier-Stokes equations in exterior domains

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Introduction

Let Ω be an *exterior* domain in $R^n (n \geq 2)$ with the smooth boundary $\partial\Omega$. Consider the initial-boundary value problem of the Navier-Stokes equations in $\Omega \times (0, T)$:

$$
\text{(N-S)} \quad
\begin{cases}
\dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } x \in \Omega, \, 0 < t < T, \\
\operatorname{div} u = 0 & \text{in } x \in \Omega, \, 0 < t < T, \\
u = 0 \quad \text{on } \partial\Omega, \qquad u(x) \to 0 \quad \text{as } |x| \to \infty, \\
u|_{t=0} = a,
\end{cases}
$$

where $u = u(x, t) = (u_1(x, t), \cdots, u_n(x, t))$ and $p = p(x, t)$ denote the unknown velocity vector and pressure of the fluid at the point $(x, t) \in \Omega \times (0, T)$, while $a = a(x) = (a_1(x), \cdots, a_n(x))$ is the given initial velocity vector field.

The purpose of this talk is to show a necessary and sufficient condition on L^1-summability over Ω of strong solutions u of (N-S). It is proved by Kato [9] and Giga-Miyakawa [7] that if $a \in L^r(\Omega) \cap L^n(\Omega)$ for $1 < r < n$, then there exist $T > 0$ and a solution u of (N-S) in the class $C([0, T); L^r(\Omega) \cap L^n(\Omega))$. Such a solution is actually regular in $\bar{\Omega} \times (0, T)$. We investigate the marginal case when $r = 1$; it is not obvious whether or not for every $a \in L^1(\Omega) \cap L^n(\Omega)$ we can construct a solution u in $C([0, T); L^1(\Omega))$. To solve (N-S), applying the projection operator P onto the solenoidal vector fields to both sides of the equation, we erase the pressure gradient ∇p, which leads us to the integral equation for u itself

$$
\text{(I.E)} \qquad u(t) = e^{-tA} a - \int_0^t e^{-(t-\tau)A} P(u \cdot \nabla u)(\tau) d\tau, \quad 0 < t < T,
$$

where $A \equiv -P\Delta$ denotes the Stokes operator. In case $\Omega = R^n$, the projection operator P is expressed by $P = \{P_{ij}\}_{i,j=1,\cdots,n}$ with $P_{ij} = \delta_{ij} - R_i R_j$, where $R_j (j = 1, \cdots, n)$ are the Riesz transforms. In this case, since P commutes with the Laplacian Δ, A is essentially equal to $-\Delta$ and e^{-tA} is essentially the heat operator. Hence we have $e^{-tA} a \in C([0, \infty); L^1(R^n))$ for all $a \in L^1(R^n)$ with $\operatorname{div} a = 0$. Unfortunately, that is not the case

for $e^{-tA}P$ because $R_j(j = 1, \cdots, n)$ are *not* bounded operators on $L^1(R^n)$. This causes a lot of difficulties to treat the nonlinear term $P(u \cdot \nabla u)$ in (I.E). Recently, Coifman-Lions-Meyer-Semmes [4] proved that for very $u \in W^{1,2}(R^n)$ with div $u = 0$, there holds $u \cdot \nabla u \in H^1(R^n)$, where $H^1(R^n)$ denotes the Hardy space. Making use of their result together with the fact that $R_j(j = 1, \cdots, n)$ are bounded transformations on $H^1(R^n)$, Miyakawa [12] constructed a global weak solution u of (N-S) with $u \in C([0, \infty); L^1(R^n))$ for every $a \in L^1(R^n) \cap L^2(R^n)$. Although his result is on weak solutions, by using his method, one can also show the existence of strong solutions u as Kato [9] and Giga-Miyakawa [7] with $u \in C([0, T); L^1(R^n) \cap L^n(R^n))$ for some $T > 0$ provided $a \in L^1(R^n) \cap L^n(R^n)$.

In our case when Ω is an *exterior* domain, however, P does not have such an explicit representation and furthermore we cannot use the theory of Hardy spaces. In the present talk, we shall show that the L^1-solution on Ω does exist only in a special situation. Actually we shall prove that every strong solution u of (N-S) belongs to $C^1((0, T); L^1(\Omega))$ if and only if the net force exerted by the fluid on $\partial\Omega$ is equal to zero:

$$\text{(N.F)} \qquad \int_{\partial\Omega} T(u(t), p(t)) \cdot \nu dS = 0 \quad \text{for all } 0 < t < T,$$

where $T(u, p)$ and ν denote the stress tensor and the unit outernormal to $\partial\Omega$, respectively.

1 Results

Before stating our results, we first introduce some function spaces. Let $C_{0,\sigma}^\infty(\Omega)$ denote the set of all C^∞ vector functions ϕ with compact support in Ω, such that div $\phi = 0$. $L_\sigma^r(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to the L^r-norm $\|\cdot\|_r \equiv \|\cdot\|_{L^r(\Omega)}$. Recall the Helmholtz decomposition

$$L^r(\Omega) = L_\sigma^r(\Omega) \oplus G^r(\Omega) \quad \text{(direct sum)}, 1 < r < \infty,$$

where $G^r(\Omega) = \{\nabla p \in L^r(\Omega); p \in L_{loc}^r(\bar{\Omega})\}$. We denote by P_r the projection operator from $L^r(\Omega)$ onto $L_\sigma^r(\Omega)$ along $G^r(\Omega)$. Then the Stokes operator A_r is defined by $A_r = -P_r\Delta$ with the domain $D(A_r) = \{u \in W^{2,r}(\Omega) \cap L_\sigma^r(\Omega); u|_{\partial\Omega} = 0\}$.

The solution u of (N-S) which we treat is as follows.

Definition. Let $1 < r \le n$ and let $a \in L_\sigma^r(\Omega) \cap L_\sigma^n(\Omega)$. A measurable function u on $\Omega \times (0, T)$ is called a *strong solution* of (N-S) in the class $S_r(0, T)$ if

(i) $u \in C([0, T); L_\sigma^r(\Omega) \cap L_\sigma^n(\Omega))$;

(ii) $Au, \partial u/\partial t \in C((0, T); L_\sigma^n(\Omega))$;

(iii)

$$\text{(N-S')} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + Au + P(u \cdot \nabla u) = 0 & \text{in } L_\sigma^n(\Omega), 0 < t < T, \\ u(0) = a, \end{cases}$$

Remarks. 1. It is shown by Kato [9] and Giga-Miyakawa [7] that for every $a \in L_\sigma^r(\Omega) \cap L_\sigma^n(\Omega)$ with $1 < r \le n$, there exist $T > 0$ and a unique strong solution u of (N-S) in the class $S_r(0,T)$. For the uniqueness, see Brezis [2] and Cannone [3].

2. Every strong solution u in the class $S_r(0,T)$ satisfies (N-S') also in $L_\sigma^r(\Omega)$ and there holds

$$\frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad \frac{\partial u}{\partial t} \in C(\bar{\Omega} \times (0,T))$$

for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Moreover, there exists a unique(up to an additive function of t) scalar function $p \in C(\Omega \times (0,T))$ with $\nabla p \in C(\Omega \times (0,T))$ and with

$$(1.1) \qquad \nabla p \in C((0,T); L^r(\Omega) \cap L^n(\Omega))$$

such that the pair $\{u,p\}$ satisfies (N-S) in the classical sense. We call such p the pressure *associated* with u.

3. If $1 < r < n$, by (1.1) and the Sobolev embedding([8, Corollary 2.2]), we may take p as $p \in C((0,T); L^{nr/(n-r)}(\Omega))$. It is not known whether or not p can be taken as $p \in C((0,T); L^{n/(n-1)}(\Omega))$.

Our first result on the L^1-solution now reads:

Theorem 1. *Let* $a \in L_\sigma^n(\Omega)$. *Suppose that* u *is the strong solution of (N-S) in the class* $S_n(0,T)$. *If* u *and its associated pressure* p *satisfy*

$$(1.2) \qquad u \in C((0,T); L^1(\Omega)),$$

$$(1.3) \qquad p \in C((0,T); L^{n/(n-1)}(\Omega)),$$

then the net force exerted by the fluid on $\partial\Omega$ *is equal to zero:*

$$(1.4) \qquad \int_{\partial\Omega} T(u(t), p(t)) \cdot \nu \, dS = 0 \quad \text{for all } 0 < t < T,$$

where $T(u,p) = \{\partial u_i/\partial x_j + \partial u_j/\partial x_i - \delta_{ij}p\}_{i,j=1,\cdots,n}$ *denotes the stress tensor and* $\nu = (\nu_1, \cdots, \nu_n)$ *and* dS *denote the unit outernormal and the surface element of* $\partial\Omega$, *respectively.*

As for converse assertion of the above theorem we have

Theorem 2. *Let* $a \in L^1(\Omega) \cap L_\sigma^n(\Omega)$. *Suppose that* u *satisfies the following conditions Case 1 and Case 2 according to the dimension* n.

(1) Case 1. In case $n \ge 3$, u *is the strong solution of (N-S) in the class* $S_{\frac{2n}{n+2}}(0,T)$ *satisfying*

$$(1.5) \qquad \|Au(t)\|_2 = O(t^{\gamma-1}) \quad \text{for some } 1/2 < \gamma < 1 \text{ as } t \to +0.$$

(2) Case 2. In case $n = 2$, u *is the strong solution of (N-S) in the class* $S_r(0,T)$ *for some* $1 < r < 2$ *satisfying*

$$(1.6) \qquad \|Au(t)\|_{\frac{2r}{2-r}} = O(t^{\gamma-1}) \quad \text{for some } 1/2 < \gamma < 1 \text{ as } t \to +0.$$

If u and its associated pressure p satisfy (1.4) on the net force, then there holds

(1.7) $\qquad\qquad u \in C([0,T]; L^1(\Omega)),$

(1.8) $\qquad\qquad p \in C((0,T); L^q(\Omega)) \quad \text{for all } 1 < q < \infty.$

The assumptions (1.5) and (1.6) on the behaviour of $u(t)$ as $t \to +0$ can be removed provided the initial data a has a certain regularity.

Theorem 2'. *(1) Case 1. In case $n \geq 3$, let a be as*

(1.9) $\qquad\qquad a \in L^1(\Omega) \cap L^n_\sigma(\Omega) \cap D(A_2^\gamma) \quad \text{for some } 1/2 < \gamma < 1$

and suppose that u is the strong solution of (N-S) in the class $S_{\frac{2n}{n+2}}(0,T)$.

(2) Case 2. In case $n = 2$, let a be as

(1.10) $\qquad a \in L^1(\Omega) \cap L^2_\sigma(\Omega) \cap D(A_{\frac{2r}{2-r}}^\gamma) \quad \text{for some } 1 < r < 2 \text{ and } 1/2 < \gamma < 1$

and suppose that u is the strong solution of (N-S) in the class $S_r(0,T)$.

Under the above assumption, if u and its associated pressure p satisfy (1.4) on the net force, then we have (1.7) and (1.8).

Remarks. 1. In case $\Omega = R^n$, Miyakawa [12] showed that for every $a \in L^1(R^n) \cap L^2_\sigma(R^n)$ there exists a weak solution u in $C([0,\infty); L^1(R^n))$ with $\|u(t)\|_{L^1(R^n)} \to 0$ as $t \to \infty$. His method is available to construct a local strong solution u in $C([0,T); L^1(R^n) \cap L^n_\sigma(R^n))$ for some $T > 0$ provided $a \in L^1(R^n) \cap L^n_\sigma(R^n)$. Recently, in the case Ω is the half-space R^n_+, Giga-Matsui-Shimizu [6] obtained an L^1-estimate for ∇e^{-tA}.

2. L^r-summability implies the decay of functions at infinity; the lower r, the more rapid decay as $|x| \to \infty$. Our theorems show that in *exterior* domains Ω, existence of solutions decaying rapidly at infinity is governed by the net force exerted by the fluid on $\partial\Omega$. Similar investigation can be found in the following *stationary* problem in 3-dimensional exterior domains Ω:

(S) $\qquad \begin{cases} -\Delta w + w \cdot \nabla w + \nabla \pi = \text{div } F, \quad \text{div } w = 0 \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega, \qquad w(x) \to w^\infty \quad \text{as } |x| \to \infty, \end{cases}$

where $F = F(x) = \{F_{ij}(x)\}_{i,j=1,2,3}$ is the given 3×3-tensor and w^∞ is the prescribed constant vector in R^3.

When $F \equiv 0$, Finn [5] introduced a notion of physically reasonable (PR) solutions to (S) and showed that such PR-solutions w decay like $|w(x) - w^\infty| = o(|x|^{-1})$ as $|x| \to \infty$ if and only if there holds

$$\int_{\partial\Omega} T(w,\pi) \cdot \nu dS = 0.$$

For $F \neq 0$ and $w^\infty = 0$, Borchers-Miyakawa [1] and Kozono-Sohr-Yamazaki [10] proved that the solution w of (S) with $\nabla w \in L^2(\Omega)$ belongs to $L^3(\Omega)$ if and only if there holds

$$\int_{\partial\Omega} \{T(w,\pi) + F\} \cdot \nu dS = 0.$$

These results state that the L^3-solutions exist only in a special situation. Our theorems make it clear that the corresponding phenomena to the *nonstationary* problems occurs in L^1-solutions.

2 Preliminaries

In this section we shall prepare some lemmas.

Lemma 2.1 *Let* $1 \leq q \leq n/(n-1)$ *and let* $f \in C([0,T];L^q(\Omega))$. *Then there exists a sequence* $\{R_m\}_{m=1}^\infty$ *with* $2^m \leq R_m \leq 2^{m+1}, m = 1, 2, \cdots$ *such that*

$$(2.1) \qquad \sup_{0 \leq t \leq T} \left(\int_{|x|=R_m} |f(x,t)|dS \right) \to 0 \quad as\ m \to \infty.$$

We next investigate the mean value of the solenoidal vector fields in $L^1(\Omega)$.

Lemma 2.2 *Let* $u = (u_1, \cdots, u_n)$ *be in* $L^1(\Omega) \cap L^r_\sigma(\Omega)$ *for some* $1 < r < \infty$. *Then we have*

$$(2.2) \qquad \int_\Omega u_j(x)dx = 0 \quad for\ all\ j = 1, \cdots, n.$$

Finally in this section, we consider the following Stokes equations in R^n with the perturbed convective term:

$$(P.S) \qquad \begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + U \cdot \nabla v + \nabla \pi = \operatorname{div} F & \text{in } R^n \times (0,T), \\ \operatorname{div} v = 0 & \text{in } R^n \times (0,T), \\ v|_{t=0} = b, \end{cases}$$

where $\{v, \pi\}$ are the unknown functions, while U is the prescribed coefficient, $F = \{F_{ij}(x,t)\}_{i,j=1,\cdots,n}$ denotes the given $n \times n$-tensor and b is the given initial data.

We impose the following assumption on b, F and U.

Assumption 2.1. (i) $b \in L^1(R^n) \cap L^2_\sigma(R^n)$;
(ii) $F \in C((0,T);L^1(R^n)) \cap L^2(0,T;L^2(R^n))$ with

$$(2.3) \qquad \|F(t)\|_{L^1(R^n)} = O(t^{\delta-1}) \quad \text{for some } \delta > 1/2 \text{ as } t \to +0;$$

(iii)

$$U \in BC([0,T); L^2_\sigma(\mathbf{R}^n)) \cap C([0,T); L^n_\sigma(\mathbf{R}^n)) \cap L^2(0,T; H^1_{0,\sigma}(\mathbf{R}^n)),$$

where BC denotes the set of all bounded continuous functions and $H^1_{0,\sigma}(\mathbf{R}^n)$ is the closure of $C^\infty_{0,\sigma}(\mathbf{R}^n)$ in $W^{1,2}(\mathbf{R}^n)$.

The notion of the weak solution can be defined in an obvious way as the Leray-Hopf weak solution to (N-S). Then we have

Lemma 2.3 *Let b, F and U be as in Assumption 2.1. There is a weak solution v of (P.S) with the following properties.*

(i) (energy identity)

$$\int_{\mathbf{R}^n} |v(x,t)|^2 dx + 2 \int_0^t \int_{\mathbf{R}^n} |\nabla v(x,\tau)|^2 dx d\tau$$

(2.4)
$$= \int_{\mathbf{R}^n} |b(x)|^2 dx - 2 \int_0^t \int_{\mathbf{R}^n} F(x,\tau) \cdot \nabla v(x,\tau) dx d\tau$$

for all $0 \le t \le T$.

(ii)

(2.5)
$$v \in C([0,T); L^1(\mathbf{R}^n)).$$

Moreover, the associated pressure π can be taken as

(2.6)
$$\pi \in C((0,T); L^q(\mathbf{R}^n)) \quad \text{for all } 1 < q \le 2.$$

For the proof of Lemma 2.3 the following proposition plays an important role.

Proposition. *There holds the estimate*

$$\|\nabla e^{t\Delta} a\|_{L^\infty(\mathbf{R}^n)} \le C t^{-\frac{1}{2}} \|a\|_{BMO} \quad \text{for all } a \in BMO \text{ and all } t > 0$$

with $C = C(n)$.

3 Proof of the theorems

3.1 Proof of Theorem 1

Let u be the strong solution of (N-S) in the class $S_n(0,T)$ with (1.2) and let p be its associated pressure with (1.3). Since $u \in C((0,T); W^{2,n}(\Omega) \cap L^1(\Omega))$, the Sobolev embedding states

$$u \in C((0,T); L^1(\Omega) \cap L^\infty(\Omega)) \quad \text{with } \nabla u \in C((0,T); L^1(\Omega) \cap L^q(\Omega)) \text{ for all } 1 < q < \infty.$$

Hence we have by (1.3)

$$T(u, p) - u \otimes u \in C((0, T); L^{n/(n-1)}(\Omega)).$$

Let $0 < \varepsilon < T' < T$. Then it follows from Lemma 2.1 that there exists a sequence $\{R_m\}_{m=1}^{\infty}$ with $2^m \leq R_m \leq 2^{m+1} (m = 1, \cdots)$ such that

$$(3.1) \qquad \sup_{\varepsilon \leq t \leq T'} \left(\int_{|x|=R_m} (T(u(t), p(t)) - u \otimes u(t)) \cdot \nu dS \right) \to 0 \quad \text{as } m \to \infty.$$

Let us consider (N-S) in the domain $\Omega_{R_m} = \Omega \cap B_{R_m}$, where $B_R \equiv \{x \in R^n; |x| < R\}$. Since the pair $\{u, p\}$ satisfies (N-S) in the classical sense, we have by integration by parts

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega_{R_m}} u(x, t) dx &= \int_{\Omega_{R_m}} \frac{\partial}{\partial t} u(x, t) dx \\
&= \int_{\Omega_{R_m}} \text{div } (T(u(t), p(t)) - u \otimes u(t)) \, dx \\
&= \int_{\partial \Omega} T(u(t), p(t)) \cdot \nu dS + \int_{|x|=R_m} (T(u(t), p(t)) - u \otimes u(t)) \cdot \nu dS.
\end{aligned}$$

By (3.1) and the above identity yield

$$(3.2) \qquad \frac{d}{dt} \int_{\Omega_{R_m}} u(x, t) dx \to \int_{\partial \Omega} T(u(t), p(t)) \cdot \nu dS \quad \text{uniformly in } t \text{ for } \varepsilon \leq t \leq T'.$$

On the other hand, since $u \in C((0, T); L^1(\Omega) \cap L^n_\sigma(\Omega))$, it follows from Lemma 2.2 that for each fixed $t \in [\varepsilon, T']$

$$(3.3) \qquad \int_{\Omega_{R_m}} u(x, t) dx \to \int_{\Omega} u(x, t) dx = 0 \quad \text{as } m \to \infty.$$

From (3.2) and (3.3) we obtain

$$\int_{\partial \Omega} T(u(t), p(t)) \cdot \nu dS = 0 \quad \text{for all } \varepsilon \leq t \leq T'.$$

Since ε and T' can be taken arbitrarily, we get the desired result (1.4). This proves Theorem 1.

3.2 Proof of Theorem 2

Let us prove the converse assertion of Theorem 1. The condition (1.4) on the net force enables us to reduce our problem in Ω to that in the whole R^n. Recall first the generalized Stokes formula. Let $D \equiv R^n \setminus \bar{\Omega}$. Note that the boundary of D coincides with $\partial \Omega$. For $1 < r < \infty$, we define a Banach space $E_r(D)$ by

$$E_r(D) \equiv \{u \in L^r(D); \text{div } u \in L^r(D)\}.$$

with the norm $\|u\|_{E_r(D)} = \|u\|_{L^r(D)} + \|\text{div } u\|_{L^r(D)}$. For $u \in E_r(D)$, $u \cdot \nu$ is well-defined as an element of $W^{1-1/r',r'}(\partial D)^*(X^*$; the dual space of $X)$, where $r' = r/(r-1)$. There holds the generalized Stokes formula

$$(3.4) \qquad \int_D u \cdot \nabla p\, dx + \int_D \text{div } u \cdot p\, dx = <u \cdot \nu, p>|_{\partial\Omega}, \quad \text{for all } p \in W^{1,r'}(D),$$

where $< \cdot, \cdot >|_{\partial\Omega}$ denotes the duality between $W^{1-1/r',r'}(\partial\Omega)^*$ and $W^{1-1/r',r'}(\partial\Omega)$. Moreover, we have

$$(3.5) \qquad \|u \cdot \nu\|_{W^{1-1/r',r'}(\partial\Omega)^*} \le C\|u\|_{E_r(D)} \quad \text{for all } u \in E_r(D)$$

with $C = C(D,r)$.

Lemma 3.1 *Let u be the strong solution with the hypotheses of Theorem 2. If u and its associated pressure p satisfy (1.4), then there exists a solution of the boundary-value problem:*

$$(3.6) \qquad \begin{cases} \text{div} F = 0 & \text{in } D \times (0,T), \\ F \cdot \nu = T(u,p) \cdot \nu & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where $D = R^n \setminus \bar{\Omega}$. Such F can be taken in the following class:
(1) Case 1. In case $n \ge 3$, $F \in C((0,T); E_2(D))$ with

$$(3.7) \qquad \|F(t)\|_{L^2(D)} = O(t^{\gamma-1}) \quad \text{as } t \to +0;$$

(2) Case 2. In case $n = 2$, $F \in C((0,T); E_{\frac{2r}{2-r}}(D))$ with

$$(3.8) \qquad \|F(t)\|_{L^{\frac{2r}{2-r}}(D)} = O(t^{\gamma-1}) \quad \text{as } t \to +0,$$

where γ is the same exponent as in (1.5) and (1.6).

Proof of Theorem 2:
Let us define $\tilde{u}$, $\tilde{p}$ and $\tilde{F}$ as

$$\tilde{u}(x,t), \quad \tilde{p}(x,t) = \begin{cases} u(x,t), \quad p(x,t) & \text{for } (x,t) \in \Omega \times (0,T), \\ 0 & \text{for } (x,t) \in D \times (0,T), \end{cases}$$

$$\tilde{F}(x,t) = \begin{cases} 0 & \text{for } (x,t) \in \Omega \times (0,T), \\ F(x,t) & \text{for } (x,t) \in D \times (0,T), \end{cases}$$

where F is the tensor-valued function given by Lemma 3.1. Let $\tilde{a}(x) = a(x)$ for $x \in \Omega$, $= 0$ for $x \in D$. Obviously, there holds

$$(3.9) \qquad \tilde{u} \in C([0,T); L^2_\sigma(R^n) \cap L^n_\sigma(R^n)) \cap L^2(0,T; H^1_{0,\sigma}(R^n)),$$

$$(3.10) \qquad \tilde{a} \in L^1(R^n) \cap L^q_\sigma(R^n) \quad \text{for all } 1 < q \le n.$$

Since $\tilde{F}$ has the compact support in $\bar{D} \times [0, T]$, we have by Lemma 3.1

(3.11) $$\tilde{F} \in C((0, T); L^1(\boldsymbol{R}^n) \cap L^2(\boldsymbol{R}^n)) \quad \text{with}$$

(3.12) $\quad \|F(t)\|_{L^1(\boldsymbol{R}^n)} = O(t^{\gamma - 1}) \quad \text{as } t \to +0 \text{ and } \int_0^T \|F(\tau)\|_{L^2(\boldsymbol{R}^n)}^2 d\tau < \infty.$

Take a test function $\Psi(x, t) = \lambda(t)\phi(x)$ so that $\lambda \in C^1([0, T])$ with $\lambda(T) = 0$ and so that $\phi \in C_0^\infty(\boldsymbol{R}^n)$ with $\text{supp}\phi \in B_R$. By (3.6) and the generalized Stokes formula, we have

$$\int_\varepsilon^T \int_{\boldsymbol{R}^n} \left(T(\tilde{u}, \tilde{p}) - \tilde{u} \otimes \tilde{u} + \tilde{F} \right) \cdot \nabla\Psi \, dx dt$$

$$= \int_\varepsilon^T \int_{\Omega_R} (T(u, p) - u \otimes u) \cdot \nabla\Psi \, dx dt + \int_\varepsilon^T \int_D F \cdot \nabla\Psi \, dx dt$$

$$= \int_\varepsilon^T \int_{\Omega_R} \text{div} \ (T(u, p) - u \otimes u) \cdot \Psi \, dx dt$$

$$+ \int_\varepsilon^T \int_{\partial\Omega} (T(u, p) \cdot \nu) \, \Psi \, dS dt - \int_\varepsilon^T \int_{\partial\Omega} (F \cdot \nu)\Psi \, dS dt$$

$$= - \int_\varepsilon^T \int_{\Omega_R} \partial_t u \cdot \Psi \, dx dt$$

$$= \int_\varepsilon^T \int_{\Omega_R} u \cdot \partial_t \Psi \, dx dt + \int_{\Omega_R} u(x, \varepsilon) \cdot \Psi(x, \varepsilon) dx.$$

Since $u \in C([0, T]; L_\sigma^2(\Omega))$, letting $\varepsilon \to +0$ in the above identity, we obtain

$$\int_0^T \int_{\boldsymbol{R}^n} \left(T(\tilde{u}, \tilde{p}) - \tilde{u} \otimes \tilde{u} + \tilde{F} \right) \cdot \nabla\Psi \, dx dt$$

$$= \int_0^T \int_{\boldsymbol{R}^n} \partial_t \tilde{u} \cdot \partial_t \Psi \, dx dt + \int_{\boldsymbol{R}^n} \tilde{a}(x) \cdot \Psi(x, \varepsilon) dx.$$

This implies that $\tilde{u}$ is weak solution of the following equation for v:

(P.S') $$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + \tilde{u} \cdot \nabla v + \nabla\pi = \text{div} \ \tilde{F} \quad \text{in } \boldsymbol{R}^n \times (0, T), \\ \text{div} \ v = 0 \quad \text{in } \boldsymbol{R}^n \times (0, T), \\ v|_{t=0} = \tilde{a}, \end{cases}$$

By (3.9)–(3.12) we see that $\tilde{a}$, $\tilde{F}$ and $\tilde{u}$ fulfill the hypotheses on b, F and U in Assumption 2.1, respectively. Hence it follows from Lemma 2.3 that there is a weak solution v of (P.S') possessing (2.5) and the energy identity (2.4) with b, F replaced by $\tilde{a}, \tilde{F}$. Since $\tilde{u} \in C([0, T]; L_\sigma^n(\boldsymbol{R}^n))$, the uniqueness criterion of weak solutions as (N-S)(Masuda [11, Theorem 3] states

$$\tilde{u} \equiv v \quad \text{on } \boldsymbol{R}^n \times [0, T),$$

which yields

$$\nabla\pi \equiv \nabla p \quad \text{on } \boldsymbol{R}^n \times [0, T).$$

Since $u = \tilde{u}$, $p = \tilde{p}$ on $\Omega \times (0, T)$, (2.5), (2.6) and the above identities yield (1.7) and (1.8). This proves Theorem 2.

3.3 Proof of Theorem 2'

On account of Theorem 2, it suffices to show that (1.9) and (1.10) imply (1.5) and (1.6), respectively. To this end, we may prove the following lemma.

Lemma 3.2 *Let* $1 < r < \infty$, $1/2 < \gamma < 1$ *and let* $a \in L_\sigma^n(\Omega) \cap D(A_r^\gamma)$. *Suppose that* u *is the strong solution of (N-S) in the class* $S_n(0,T)$. *Then we have*

$$(3.13) \qquad \|Au(t)\|_r = O(t^{\gamma-1}) \quad as \ t \to +0.$$

References

[1] Borchers, W., Miyakawa, T., *On stability of exterior stationary Navier-Stokes flow.* Acta Math. **174**, 311–382 (1995).

[2] Brezis, H., *Remarks on preceding paper by M. Ben-Arzi, "Global solutions of two-dimensional Navier-Stokes and Euler equations".* Arch. Rational Mech. Anal. **128** , 359–360 (1994).

[3] Cannone, M., *Ondelettes, Paraproduits et Navier-Stokes.* Diderot Editeur, Arts et Sciences Paris-New York-Amsterdam 1995

[4] Coifman, R., Lions., P.L., Meyer, Y., Semmes, S., *Compensated compactness and Hardy spaces* J. Math. Pures. Appl. **72**, 247–286 (1993).

[5] Finn, R., *On exterior stationary problem for the Navier-Stokes equations and associated perturbation problems.* Arch. Rational Mech Anal. **19**, 363–404 (1964).

[6] Giga, Y., Matsui, S., Shimizu, Y., L^1 *estimate for Stokes flows i a half space.* preprint

[7] Giga, Y., Miyakawa, T., *Solution in* L_r *of the Navier-Stokes initial value problem.* Arch. Rational Mech. Anal. **89**, 267–281 (1985).

[8] Giga, Y., Sohr, H., *On the Stokes operator in exterior domains.* J. Fac. Sci. Univ. Tokyo, Sec IA **36**, 103–130 (1989).

[9] Kato, T., *Strong* L^p*-solution of the Navier-Stokes equation in* R^m, *with applications to weak solutions.* Math. Z. **187**, 471–480 (1984).

[10] Kozono. H., Sohr, H., Yamazaki, M., *Representation formula, net force and energy relation to the stationary Navier-Stokes equations in 3-dimensional exterior domains.* Kyushu J. Math. **51**, 239–260 (1997).

[11] Masuda, K., *Weak solutions of the Navier-Stokes equations.* Tohoku Math. J. **36**, 623–646 (1984).

[12] Miyakawa, T., *Hardy spaces of solenoidal vector fields, with applications to the Navier-Stokes equations.* Kyushu J. Math. **50**, 1–64 (1996).

[13] Stein, E. M., *Harmonic Analysis.* Princeton University Press 1993.

On the L_p–L_q estimate of Stokes semigroup

in a 2 dimensional exterior domains

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The main part of my talk in this conference is a joint work with Wakako Dan (Tsukuba Univ.).

The motion of the moving body in an incompressible viscous fluid is described by the exterior initial boundary value problem of Navier-Stokes equations if the coordinate system is fixed on the body.

In the three dimensional case, it is well-known that without smallness assumptions, present day analysis yields only a locally in time unique existence of solutions and a globally in time unique existence of small and smooth solutions, while Leray and Hopf proved the existence of square-integrable weak solutions for arbitrary square-integrable initial velocity, whose uniqueness is still open.

Leray also proved the existence of smooth solutions with a finite Dirichlet integral of the stationary problem of the Navier-Stokes equation. But, the solutions obtained by Leray did not provide much qualitative information about the solutions. In particular, nothing was proven about the asymptotic structure of the solutions - especially the wake behind the body. Finn has studied the stationary problem within the class of solutions, termed by him physically reasonable, which tend to a limit at infinity like $|x|^{-\frac{1}{2}-\epsilon}$ for some $\epsilon > 0$. In the three dimensional case, for small data he proved both existence and uniqueness within this class. Furthermore, his solutions exhibit paraboloidal wake region behind the body.

Finn has conjectured that for sufficiently small data physically reasonable solutions are stable. First, Heywood, Masuda, Maremonti and Galdi studied this problem in the L^2 framework. But, Finn also mentioned that the L^2 framework was not so suitable in the three dimensional case, because in general the difference between the stationary solution and its prescribed constant limit at infinity has an infinite L^2-norm. That is, the steady state solution has an infinite wake energy. This situation becomes clearer if we consider the starting problem which was also proposed by Finn and recently solved by Heywood, Galdi and myself. On the other hand, Kato and Fujita proposed to study the three dimensional Navier-Stokes equation in L^3 framework long ago. In fact, Kato and Iwashita proved the stability of the trivial solution of the stationary problem. Recently, Borchers & Miyakawa, and Kozono & Yamazaki proved the stability of the small Finn's physically reasonable solutions with respect to L^3 weak small initial perturbation when the prescribed velocity at infinity is zero. When the prescribed velocity at infinity is small but non-zero, I also proved the stability of the small Finn's physically reasonable solution with respect to L^3 small initial perturbation. Therefore, the stability problem was settled with respect to the small initial perturbation in the L^3 framework.

Recently, I tried to extend the three dimensional stability results to the two dimensional case. But, the situation is completely different from the three dimensional case. The uniqueness of Leray and Hopf solution holds, which was the Lions and Prodi clas-

sical result. Therefore, in the L^2 framework we know the unique existence of solutions for any L^2 initial data. Moreover, Masuda proved that the L^2 norm of nonstationary solutions with zero external force and zero prescribed velocity at infinity tends to zero as time tends to infinity. And also, Kozono and Ogawa proved the decay estimate of nonstationary solutions with zero external force and zero prescribed velocity at infinity in the several different norms.

More precisely, let us consider the solution $u(t, x)$, $p(t, x)$ of the following Navier-Stokes equation :

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \nabla \cdot u = 0 \quad \text{in } \Omega, \ t > 0$$

$$u\big|_{\partial\Omega} = 0, \quad u(0, x) = a(x) \in L^2(\Omega)$$

$$\lim_{|x| \to \infty} u(t, x) = 0$$

where Ω is an exterior domain in $\mathbb{R}^2$ with smooth boundary. Then, Kozono and Ogawa proved the decay estimate :

$$\|u(t, \cdot)\|_{L^p(\Omega)} = o(t^{-\left(\frac{1}{2} - \frac{1}{p}\right)}) \quad 2 \leqq p < \infty$$

(D) $$\|u(t, \cdot)\|_{L^\infty(\Omega)} = o(t^{-\frac{1}{2}} \sqrt{\log t})$$

$$\|\nabla u(t, \cdot)\|_{L^2(\Omega)} = o(t^{-\frac{1}{2}}).$$

as $t \to \infty$. Their method heavily depends on the L^2 decay estimate of solutions, and the theory of the fractional powers in the L^2 framework and some very sharp inequalities of Gagliardo-Nirenberg type. But, they did not investigate any rate of decay of Stokes semigroup $\{e^{tA}\}$, which plays a crucial role according to Kato's argument. Recently, Dan and myself proved the L^p-L^q decay estimate of the Stokes semigroup. Namely, we have

$$\|e^{tA}a\|_{L^p(\Omega)} \leqq C_{p,q} t^{-\left(\frac{1}{q} - \frac{1}{p}\right)} \|a\|_{L^q(\Omega)}, \quad 1 < q \leqq p \leqq \infty$$

$$\|\nabla e^{tA}a\|_{L^p(\Omega)} \leqq C_{p,q} t^{-\left(\frac{1}{q} - \frac{1}{p}\right) - \frac{1}{2}} \|a\|_{L^q(\Omega)}, \quad 1 < q \leqq p \leqq 2.$$

as $t \to \infty$. Using this, Lions and Prodi uniqueness theorem and Masuda's decay result of the L^2-norm of the solution to Navier-Stokes equation and employing the Kato's argument we can improve the decay estimate (D) due to Kozono and Ogawa. Namely, the solution u of the Navier-Stokes equation satisfies the estimate :

$$\|u(t, \cdot)\|_{L^\infty(\Omega)} = o(t^{-\frac{1}{2}}) \quad t \to \infty.$$

Namely, the stability problem of trivial solutions of the stationary problem was completely settled in the two dimensional case.

But, the stability of non-zero solution is the completely open problem. For example, we do not know the existence of non trivial stationary solutions with prescribed zero velocity at infinity even when the external force is very small in the similar sense to the three dimensional case (cf. Borchers and Miyakawa). When the prescribed speed at

infinity is nonzero, Finn and Smith and also Galdi constructed the stationary solutions when the Reynolds number is small enough, which provided a great deal of qualitative asymptotic information, especially about the wake behind the body. But, since we have not yet proved any results concerning the uniform decay estimates of L^p - L^q type of the Oseen semigroup, we can not tell anything about the stability of the stationary solutions constructed by Finn and Smith, and Galdi. From my experience in the treatment of three dimensional stability theory, I feel that the asymptotic behaviour of the stationary solutions seems to be too bad to obtain its stability. Therefore, maybe we must start to study the stability of stationary solutions possessing better asymptotic behaviour at infinity like self-propelled solutions.

In my talk, I would like to talk at least about the L^p-L^q decay estimate of 2 dimensional Stokes semigroup and its application. The proof heavily depends on the sharp representation formula of the coefficients of the asymptotic expansions of the Stokes resolvent near the origin, which seems to have very new aspects in the study of Navier-Stokes equation in this direction. Therefore, if I have time, I would like to mention some essential part of the proof.

Heat convection with a stress free boundary condition and bifurcation problems

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We consider the simple Rayleigh-Benard problem for the heat convection using the Boussinesq equations for the velocity, pressure and temperature :

$$\frac{1}{\mathcal{P}}(u_t + u \cdot \nabla u) + \nabla p = \Delta u - \rho(T)\nabla z , \quad \nabla \cdot u = 0 , \quad T_t + u \cdot \nabla T = \Delta T$$

in the strip $\{ -\infty < x < \infty, \ 0 < z < \pi \}$, where $\rho(T) = G - \mathcal{R}T$ is assumed for the density of the fluid, $\mathcal{P}$ is the Prandtl number and $\mathcal{R}$ is the Rayleigh number.

We assume the stress free boundary condition for the velocity on the both boundaries $\{ -\infty < x < \infty, \ z = 0, \ z = \pi \}$, and the Dirichlet boundary condition for the temperature $T = 1$ on the lower boundary and $T = 0$ on the upper boundary . These equations have the equilibrium solution for any Rayleigh number

$$u = 0 , \quad T = \tilde{T}(z) \equiv 1 - \frac{z}{\pi} , \quad p = \tilde{p}(z) \equiv -\frac{\mathcal{R}}{2\pi}(z - \pi)^2 - G(z - \pi) + p_{\text{air}} ,$$

which represents the purely heat conducting state.

We will consider the bifurcation problems of this equilibrium state under the assumption that all perturbations are periodic in x .

The perturbation (Ψ, Θ) satisfies a nonlinear system, which can be written using the stream function Ψ for the perturbed flow.

$$\Delta \Psi_t + \mathcal{P}\mathcal{R}\Theta_x = \mathcal{P}\Delta^2 \Psi + \Psi_z\Delta \Psi_x - \Psi_x\Delta \Psi_z , \tag{1}$$

$$\Theta_t + \Psi_x = \Delta \Theta + \Psi_z \Theta_x - \Psi_x \Theta_z , \quad \text{in} \quad 0 < z < \pi . \tag{2}$$

$$\Psi = 0 , \quad \Psi_{zz} = 0 , \quad \Theta = 0 \quad \text{on} \quad z = 0 , \pi . \tag{3}$$

We can consider Ψ and Θ of the following form because of the boundary condition :

$$\Psi = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \psi_{mn}(t) \ \sin(a\,mx) \ \sin(nz) , \tag{4}$$

$$\Theta = \sum_{m=0}^{+\infty} \sum_{n=1}^{+\infty} \theta_{mn}(t) \, \cos(\, a \, mx \,) \, \sin(\, nz \,) \quad . \tag{5}$$

The system becomes the infinite dimensional ODE's :

$$\frac{d\psi_{mn}}{dt} = a_{mn} \, \psi_{mn} + b_{mn} \, \theta_{mn} + F_{mn} \quad , \tag{6}$$

$$\frac{d\theta_{mn}}{dt} = c_{mn} \, \psi_{mn} + d_{mn} \, \theta_{mn} + G_{mn} \quad , \tag{7}$$

where a_{mn} , b_{mn} , c_{mn} , d_{mn} are constants and F_{mn} , G_{mn} are the nonlinear terms.

In 1916 Rayleigh considered the linearized stability and found the critical Rayleigh number as follows:

$$\begin{aligned}
\mathcal{R}_c &= \inf_{m,n,a} \frac{(\, a^2 m^2 + n^2 \,)^3}{a^2 \, m^2} \tag{8} \\
&= 6.75 \quad (\, m = 1, \, n = 1, \, a = 1/\sqrt{2} \,) \, .
\end{aligned}$$

The usual bifurcation theory applies to the system and the stationary bifurcation occurs from the above critical point. Veronis (1966), Moore and Weiss (1973) obtained the bifurcated stationary solutions numerically, and Busse and Bolton (1984,1985) examined the stability of the bifurcated solution numerically when a is varied . The stability analysis of the bifurcated solution has been considered by Kagei and Wahl recently in a small neighbourhood of the bifurcation points.

On the other hand Lorenz (1963) treated the truncated system such that only three modes $\{ \, m + n = 2 \, \}$ are included in the system. He obtained not only the bifurcated stationary solution but also a strange attractor for large Rayleigh number. The bifurcation from the stationary solution is the reverse Hopf bifurcation and so the bifurcated periodic solution is not stable. After that Curry (1978) considered a model of 14 components $\{ \, m \leq 3, \, n \leq 4, \, m + n = \text{even} \, \}$ and obtained the stable periodic solution after the Hopf bifurcation and also chaotic solutions numerically.

Here we will obtain bifurcated solutions numerically to have a better bifurcation diagram for the full system.

BLOW-UP BOUNDARIES AND RATES FOR NONLINEAR WAVE EQUATIONS [1]

Dedicated to Professor Rentaro Agemi on his sixtieth birthday

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1 Introduction

In this talk, we shall discuss the blow-up boundaries and the blow-up rates of solutions for the following nonlinear wave equation with a null form:

$$\Box u = u_t^2 - u_x^2 \quad \text{in} \quad \mathbf{R} \times \mathbf{R}, \tag{1.1}$$

where $u = u(t, x)$ is a real valued unknown function of $(t, x) \in \mathbf{R} \times \mathbf{R}$, and $\Box = \partial_t^2 - \partial_x^2$ is the d'Alembertian of one space dimension. For a solution u of (1.1), the blow-up boundary is defined by semilinear nature, i.e.,

$$\partial\{(t, x) \in \mathbf{R} \times \mathbf{R} : |u(t, x)| < +\infty\}.$$

The aim of this talk is to show that there are solutions of (1.1) with an arbitrary blow-up rate and that there are various shapes of blow-up boundaries depending on the initial data.

Before stating our main results, we recall some known results for the blow-up boundaries and the blow-up rates of solutions to other semilinear wave equations of the form:

$$\Box u = F(u) \quad \text{in} \quad \mathbf{R} \times \mathbf{R} \tag{1.2}$$

with some positive nonlinearity $F(u)$, which does not depend on the derivatives of u. For the power nonlinearity $F(u) \sim |u|^p$, Caffarelli and Friedman [1] proved, under some assumptions on the initial data, that the blow-up boundary is a space-like smooth curve and has a monotonicity in the spatial variable x. They also calculated the blow-up rate of the solutions which is the same as that for the o.d.e. $u_{tt} = F(u)$. The analogous result is obtained in their previous paper [2] for two or three dimensional case, in which stronger assumptions on the data are required. The positivity of the fundamental solution of the d'Alembertian plays

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[1]Joint work with Masahito Ohta. (Department of Applied Mathematics, Faculty of Engineering, Shizuoka University, Hamamatsu 432, Japan. e-mail : tsmoota@eng.shizuoka.ac.jp)

a crucial role in [1] and [2]. This kind of problem is studied by Kichenassamy [3] for the exponential nonlinearity $F(u) \sim e^u$ in any number of space dimensions. In [4] Kichenassamy and Littman give examples of solutions with a time-like singular set (see Remark 1 at p.1892 in [4]). These papers [3] and [4] contain general prescriptions to construct solutions with prescribed blow-up set.

On the other hand, we will see in Section 3 that the blow-up nature for (1.1) is completely different from that for (1.2). That is, in Section 3, we show that the blow-up rate for (1.1) is not necessarily the same as that for the o.d.e. $u_{tt} = u_t^2$ and there are infinitely many different blow-up rates for (1.1). Furthermore, it is shown that the blow-up boundary for (1.1) need not be a space-like curve. That is, we give a sufficient condition for which the blow-up boundary is a compact set of the space-time, and also give a sufficient condition on the initial data for which the blow-up boundary is a space-like smooth curve as well as the case for (1.2). We note that the method of [1] and [2] is inapplicable to (1.1) even though the problem is in one space dimension.

This talk is organized as follows. In Section 2, we give an explicit representation of solutions to (1.1) through a linearization. In Section 3, we state and prove our main results using the representation.

2 Preliminary

In this section, we give a concrete representation of solutions to (1.1). It is well known that the nonlinear wave equation (1.1) is transformed into the wave equation $v_{tt} - v_{xx} = 0$ by

$$v(t, x) = \exp\{-u(t, x)\}, \quad u(t, x) = -\log\{v(t, x)\}. \tag{2.1}$$

The transformation (2.1) was first introduced by Hopf [5] for the Burgers equation, and an application of (2.1) to (1.1), which was suggested by L. Nirenberg, appears in Klainerman [6]. However, there does not seem to be a detailed study of the blow-up problem for (1.1) in the literature. By the linearization of (1.1), it is easily seen that for any smooth functions F and G,

$$u(t, x) = -\log\{F(x - t) + G(x + t)\} \tag{2.2}$$

satisfies (1.1) in the domain

$$\Omega = \{(t, x) \in \mathbf{R} \times \mathbf{R} : F(x - t) + G(x + t) > 0\}. \tag{2.3}$$

In fact, from (2.2) we have

$$u_t(t, x) = \frac{F'(x - t) - G'(x + t)}{F(x - t) + G(x + t)}, \tag{2.4}$$

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$$u_x(t, x) = -\frac{F'(x-t) + G'(x+t)}{F(x-t) + G(x+t)}, \tag{2.5}$$

$$u_{tt}(t, x) = -\frac{F''(x-t) + G''(x+t)}{F(x-t) + G(x+t)} + \left(\frac{F'(x-t) - G'(x+t)}{F(x-t) + G(x+t)}\right)^2, \tag{2.6}$$

$$u_{xx}(t, x) = -\frac{F''(x-t) + G''(x+t)}{F(x-t) + G(x+t)} + \left(\frac{F'(x-t) + G'(x+t)}{F(x-t) + G(x+t)}\right)^2. \tag{2.7}$$

Therefore, we obtain the following lemma.

Lemma 1 *Let F and G be smooth functions satisfying $F(x) + G(x) > 0$ for all $x \in \mathbf{R}$. Then $u(t, x)$ given by (2.2) is a smooth solution of (1.1) in the domain Ω defined by (2.3) with the initial data*

$$u(0, x) = -\log\{F(x) + G(x)\}, \quad u_t(0, x) = \frac{F'(x) - G'(x)}{F(x) + G(x)}.$$

Remark It is possible to give the representation in terms of the solution of the wave equation for more general equations. In fact, it is easily seen that

$$u_{tt} - \Delta u + f(u)(|u_t|^2 - |\nabla u|^2) = 0, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^n$$

is transformed into the wave equation $v_{tt} - \Delta v = 0$ by $v(t, x) = \int_0^{u(t,x)} \exp(\int_0^s f(r)dr)ds$.

3 Main Results

In this section, we prove our main results stated in Section 1. We begin with the following theorem, which shows that there are solutions of (1.1) with an arbitrary blow-up rate and that the blow-up may occur at only one point in the space-time.

Theorem 1 *For any positive constant $p > 0$, there exists a solution $u(t, x)$ of (1.1) in $\Omega_0 = \{(t, x) \in \mathbf{R} \times \mathbf{R}; (t, x) \neq (0, 0)\}$ such that $u(t, x) \in C^\infty(\Omega_0)$ and satisfies*

$$\sum_{j,k \in \mathbf{Z}_+, j+k=l} |\partial_t^j \partial_x^k u(t, x)| = O((|t| + |x|)^{-p-l}) \quad \text{for all} \quad l \in \mathbf{Z}_+ \tag{3.1}$$

near $(t, x) = (0, 0)$. Here, $\mathbf{Z}_+$ denotes the set of all non-negative integers.

Proof. Let $H \in C^\infty(\mathbf{R})$ be a function such that

$$H(x) = \begin{cases} 0 & (x = 0) \\ \exp(-|x|^{-p}) & (0 < |x| \leq 1) \\ \text{smooth} & (|x| > 1), \end{cases} \tag{3.2}$$

and $H(x) = H(-x)$, $H'(x) > 0$ for all $x > 0$. Put

$$u(t, x) = -\log\{H(x - t) + H(x + t)\}. \tag{3.3}$$

Then, from Lemma 1, $u(t, x)$ is a smooth solution of (1.1) in Ω_0. Near the singular point $(t, x) = (0, 0)$, it follows from (3.2) and (3.3) that

$$u(t, x) = -\log\{\exp(-|x - t|^{-p}) + \exp(-|x + t|^{-p})\} \tag{3.4}$$

if $0 < |x - t| < 1$ and $0 < |x + t| < 1$. From the expression (3.4), we have the estimate (3.1) near $(t, x) = (0, 0)$. $\qquad\square$

Remark 1 The positivity of $\square u(t, x)$ in the whole existence domain plays an important role in [1]. For the solution $u(t, x)$ of (1.1) given by (3.3), the sign of $\square u(t, x)$ changes around the singular point $(0, 0)$. In fact, from (2.6), (2.7) and (3.3), we have

$$\square u(t, x) = -\frac{4H'(x - t)H'(x + t)}{\{H(x - t) + H(x + t)\}^2},$$

from which together with $xH'(x) > 0$ for $x \neq 0$, it follows that $\square u(t, x) > 0$ if and only if $|x| < |t|$.

Next, we give a sufficient condition for which the blow-up boundary for (1.1) is compact. Therefore, under the assumptions in the following theorem, the singularity of the solution to (1.1) is confined in a
bounded domain of the space-time and does not propagate (for another example of a singularity which does not propagate, see at pp. 433–434 in [4]).

Theorem 2 *Let F and G be non-negative smooth functions, and put $N_F = \{x \in \mathbf{R} : F(x) = 0\}$ and $N_G = \{x \in \mathbf{R} : G(x) = 0\}$. Assume that $-\infty < \inf N_F \leq \sup N_F < \inf N_G \leq \sup N_G < +\infty$. Then the singular set S of the solution $u(t, x)$ to (1.1) given by (2.2) is compact, that is, there exists a bounded closed set S in the space-time $\mathbf{R} \times \mathbf{R}$ such that $u(t, x)$ satisfies (1.1) in $\mathbf{R} \times \mathbf{R} \setminus S$ and $\lim_{(t,x) \to \partial S} u(t, x) = +\infty$.*

Proof. Put $S = \{(t, x) \in \mathbf{R} \times \mathbf{R} : F(x - t) = 0 \text{ and } G(x + t) = 0\}$. Since both F and G are non-negative, we have $\{(t, x) \in \mathbf{R} \times \mathbf{R} : F(x - t) + G(x + t) > 0\} = \mathbf{R} \times \mathbf{R} \setminus S$. Moreover, by the assumption that $\sup N_F < \inf N_G$, we have $F(x) + G(x) > $ for all $x \in \mathbf{R}$. Thus, it follows from Lemma 1 that $u(t, x)$ satisfies (1.1) in $\mathbf{R} \times \mathbf{R} \setminus S$ and $\lim_{(t,x) \to \partial S} u(t, x) = +\infty$. The closedness of S follows from the continuity of the functions F and G. Moreover, S is contained in the rectangle with the vertexes $(t, x) = ((\alpha_i - \beta_j)/2, (\alpha_i + \beta_j)/2)$, $(i, j = 1, 2)$, where $\alpha_1 = \sup N_G$, $\alpha_2 = \inf N_G$, $\beta_1 = \sup N_F$, $\beta_2 = \inf N_F$. Therefore, the boundedness of S follows from the assumption that $-\infty < \beta_2 \leq \beta_1 < \alpha_2 \leq \alpha_1 < +\infty$. $\qquad\square$

Remark 2 In Theorem 2, assuming further that $F - 1/2$, $G - 1/2 \in C_0^\infty(\mathbf{R})$, the initial data $u(0, x)$, $u_t(0, x)$ belong to $C_0^\infty(\mathbf{R})$ by Lemma 1. Moreover, if N_F (or N_G) is not connected, then the singular set S is also not connected.

Remark 3 As a simple example of blow-up solution for (1.1) whose blow-up boundary is not bounded, we can give $u(t, x) = -\log(1 - t)$, which is a smooth solution of (1.1) for $t < 1$ and satisfies $\lim_{t\to 1-0} u(t, x) = +\infty$. That is, the blow-up boundary is a line $\{(t, x) \in \mathbf{R} \times \mathbf{R} : t = 1\}$. On the other hand, $u(t, x) = -\log\{\text{Arctan}(x - t + 1) - \text{Arctan}(x + t - 1)\}$ gives another example of blow-up solution for (1.1) whose blow-up boundary is the same line $\{(t, x) \in \mathbf{R} \times \mathbf{R} : t = 1\}$. We note that both two solutions have the positivity $\Box u(t, x) > 0$ in $(-\infty, 1) \times \mathbf{R}$. Therefore, different initial data may lead to the same blow-up boundary. This point has been made in [3] and [4]. These papers contain general prescriptions to construct solutions with prescribed blow-up set.

Finally, we give a sufficient condition on the initial data for which the blow-up boundary is a space-like smooth convex curve.

Theorem 3 *Assume that $g \in C^\infty(\mathbf{R})$ satisfies*

$$\int_{-\infty}^{+\infty} g(x)dx > 2, \quad g(x) > 0, \quad g(x) = g(-x), \quad \left(\frac{1}{g(x)}\right)'' > 0 \quad for \quad x \in \mathbf{R}, \qquad (3.5)$$

and let $u(t, x)$ be the solution of (1.1) with $u(0, x) = 0$ and $u_t(0, x) = g(x)$ for $x \in \mathbf{R}$. Then, there exist a function $\varphi \in C^\infty(\mathbf{R})$ such that $0 < \varphi(0) = \min_{x \in \mathbf{R}} \varphi(x)$,

$$\varphi(x) = \varphi(-x), \quad |\varphi'(x)| < 1, \quad \varphi''(x) > 0 \quad for \quad x \in \mathbf{R}, \qquad (3.6)$$

$$u \in C^\infty(\{(t, x) \in \mathbf{R} \times \mathbf{R} : t < \varphi(x)\}), \qquad \lim_{t \to \varphi(x) - 0} u(t, x) = +\infty. \qquad (3.7)$$

Proof. Let $v(t, x) = 2 - \int_{x-t}^{x+t} g(y)dy$. Then, from Lemma 1, the solution of (1.1) with $u(0, x) = 0$ and $u_t(0, x) = g(x)$ is given by $u(t, x) = -\log\{(1/2)v(t, x)\}$. Moreover, we have $v_t(t, x) = -g(x + t) - g(x - t) < 0$ for all $(t, x) \in \mathbf{R} \times \mathbf{R}$ by the assumption that $g(x) > 0$ for $x \in \mathbf{R}$, and have $v(0, x) = 2$ and $\lim_{t\to+\infty} v(t, x) < 0$ for all $x \in \mathbf{R}$ by the assumpiton that $\int_{-\infty}^{+\infty} g(x)dx > 2$. Thus, for any $x \in \mathbf{R}$ there exists a unique value $\varphi(x) > 0$ such that $v(\varphi(x), x) = 0$, $v(t, x) > 0$ for $t < \varphi(x)$ and $v(t, x) < 0$ for $t > \varphi(x)$. Therefore, we have (3.7). By the assumption, g is an even function, so is $v(t, \cdot)$ for any $t \in \mathbf{R}$, from which together with the relation $v(\varphi(x), x) = 0$ we have $\varphi(x) = \varphi(-x)$ for $x \in \mathbf{R}$. The smoothness of φ follows from that of g and the relation $v(\varphi(x), x) = 0$. Again from the relation $v(\varphi(x), x) = 0$, we have

$$\varphi'(x) = \frac{g(x - \varphi(x)) - g(x + \varphi(x))}{g(x - \varphi(x)) + g(x + \varphi(x))}. \qquad (3.8)$$

From the assumption that $g(x) > 0$ for $x \in \mathbf{R}$ and (3.8), we have $\varphi'(0) = 0$ and $|\varphi'(x)| < 1$ for all $x \in \mathbf{R}$. From (3.8) we have

$$\varphi''(x) = 2\frac{g(x + \varphi(x))g'(x - \varphi(x))(1 - \varphi'(x)) - g(x - \varphi(x))g'(x + \varphi(x))(1 + \varphi'(x))}{\{g(x - \varphi(x)) + g(x + \varphi(x))\}^2}.$$

Inserting (3.8) into the last equation, we have

$$\varphi''(x) = \frac{4\{G'(x + \varphi(x)) - G'(x - \varphi(x))\}}{\{g(x - \varphi(x))\}^2\{g(x + \varphi(x))\}^2\{g(x - \varphi(x)) + g(x + \varphi(x))\}^3}.$$

Here we have put $G(y) = 1/g(y)$. From the fact that $x - \varphi(x) < x + \varphi(x)$ for $x \in \mathbf{R}$ and the assumption that $G''(y) > 0$ for $y \in \mathbf{R}$, we have $G'(x - \varphi(x)) < G'(x + \varphi(x))$ for all $x \in \mathbf{R}$. Hence we have $\varphi''(x) > 0$ for all $x \in \mathbf{R}$. Finally, from the facts that $\varphi'(0) = 0$ and $\varphi''(x) > 0$ for $x \in \mathbf{R}$, we have $x\varphi'(x) > 0$ for $x \neq 0$ and $\varphi(0) = \min_{x \in \mathbf{R}} \varphi(x)$. □

Remark 4 For example, the function $g(x) = (6/\pi)(1 + x^2)^{-1}$ satisfies the assumption (3.5). In this case, the solution of (1.1) with $u(0, x) = 0$ and $u_t(0, x) = g(x)$ is given by $u(t, x) = -\log\{1 - (3/\pi)(\text{Arctan}(t + x) + \text{Arctan}(t - x))\}$. Furthermore, the explicit form of the blow-up boundary $\varphi(x) = \sqrt{x^2 + 4/3} - \sqrt{1/3}$ is obtained by using the addition theorem for the tangent function: $\tan(x + y) = (\tan x + \tan y)/(1 - \tan x \tan y)$.

Remark 5 For the solution $u(t, x)$ in Theorem 3, the positivity $\Box u(t, x) = 4g(x + t)g(x - t)(2 - \int_{x-t}^{x+t} g(y)dy)^{-2} > 0$ holds in the domain $\{(t, x) \in \mathbf{R} \times \mathbf{R} : t < \varphi(x)\}$.

References

[1] Caffarelli, L. A. and Friedman, A., Differentiability of the blow-up curve for one-dimensional wave equations. *Arch. Rational Mech. Anal.*, 1985, **91**, 83–98.

[2] Caffarelli, L. A. and Friedman, A., The blow-up boundary for nonlinear wave equations. *Trans. Amer. Math. Soc.*, 1986, **297**, 223–241.

[3] Kichenassamy, S., The blow-up problem for exponential nonlinearities. *Comm. Partial Differential Equations*, 1996, **21**, 125–162.

[4] Kichenassamy, S. and Littman, W., Blow-up surfaces for nonlinear wave equations, Part I, II. *Comm. Partial Differential Equations*, 1993, **18**, 431–452, 1869–1899.

[5] Hopf, E., The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.*, 1950, **3**, 201–230.

[6] Klainerman, S., Global existence for nonlinear wave equations. *Comm. Pure Appl. Math.*, 1980, **33**, 43–101.

Global small amplitude solutions of nonlinear hyperbolic systems with a critical exponent under the null condition

Akira Hoshiga [*] and Hideo Kubo [†]

We consider the initial value problem for

$$\Box_i u^i \equiv \partial_t^2 u^i - c_i^2 \Delta u^i = F^i(\partial u, \partial^2 u) \quad \text{in } \mathbf{R}^n \times (0, \infty), \tag{1}$$

$$u^i(x,0) = \varepsilon f^i(x), \quad \partial_t u^i(x,0) = \varepsilon g^i(x) \quad \text{in } \mathbf{R}^n, \tag{2}$$

where $i = 1, \cdots, m$, $n = 2, 3$, c_i are positive constants and $\varepsilon > 0$ is a small parameter. Besides, $F^i \in C^\infty(\mathbf{R}^{(n+1)m} \times \mathbf{R}^{((n+1)m)^2})$ and $f^i, g^i \in C_0^\infty(\mathbf{R}^n)$. We also denoted $u = (u^1, \cdots, u^m)$ and $\partial = (\partial_0, \partial_1, \cdots, \partial_n)$ with $\partial_0 = \partial_t = \partial/\partial t$ and $\partial_j = \partial/\partial x_j$, $j = 1, \cdots, n$. To begin with, we shall make a brief review of known results concerning a special case where $m = 1$, because this case has been studied extensively. (See also [2-7]). Let F^1 satisfy

$$F^1(\partial u, \partial^2 u) = O(|\partial u|^{p-1}|\partial^2 u|) \quad \text{near } \partial u = 0, \tag{3}$$

where $p = (n+1)/(n-1)$. Generally, we do not expect the existence of global solutions, even if the initial data small enough. However, if we suppose the nonlinearity has some special form, there is a unique global solution of the problem for sufficiently small initial data. The additional condition is called Klainerman's null condition.

A role of the null condition is closely connected to the following vector fields $\mathcal{A} = (\Gamma_i)$ which generate a Lie algebra with respect to the usual commutator of linear operators:

$$\partial_t, \partial_1, \cdots, \partial_n, \quad S = t\partial_t + r\partial_r, \quad \Omega_{i,j} = x_i\partial_j - x_j\partial_i \ (1 \le i < j \le n), \tag{4}$$

and

$$L_i = x_i\partial_t + t\partial_i \ (1 \le i \le n),$$

where $r = |x|$. In fact, we may write

$$\partial_i = -\omega_i\partial_t + \frac{1}{t}L_i + \frac{\omega_i}{t+r}S - \sum_{j=1}^n \frac{r\omega_i\omega_j}{t(t+r)}L_j, \quad \partial_t = \frac{1}{t^2-r^2}(tS - \sum_{i=1}^n x_iL_i), \tag{5}$$

where $\omega_i = x_i/|x|$. (See F. John [6]). Therefore, if F^1 satisfies (3) and the null condition, after a direct computation, we arrive at

$$|\Gamma^a F^i(\partial u, \partial^2 u)| \le \frac{C}{t} \sum_{|b+c|\le|a|+2} |\Gamma^b u||\Gamma^c \partial u|^{p-1} \quad \text{for } t > 0, \tag{6}$$

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which gives us an additional decay factor t^{-1}. This is a crucial point to treat the critical nonlinearity.

We now turn our attention to our problem with $m \geq 2$. When the propagation speeds are same, say, $c_i = 1$ ($i = 1, \cdots, m$), the global existence theorem has already been proved by S. Klainerman [9] for $n = 3$ and by S. Katayama [8] for $n = 2$ for the critical nonlinearity satisfying the null condition.

On the other hand, when $n = 2$ and the propagation speeds are different from each other, M. Kovalyov [10] obtained the global existence theorem for special nonlinearities with the critical power. Recently, R. Agemi and K. Yokoyama [1] derived the null condition for the system (1) with different speeds by considering a plane wave solution, which includes the nonlinearity studied in [10]. To state the condition more precisely, we write F^i as

$$F^i(\partial u, \partial^2 u) = \sum_{j,k,l=1}^{m} \sum_{\gamma,\delta=0}^{2} D_{ijkl}^{\alpha\beta\gamma\delta} \partial_\alpha u^j \partial_\beta u^k \partial_\gamma \partial_\delta u^l,$$

where $D_{ijkl}^{\alpha\beta\gamma\delta}$ are constants. Then the condition derived by them is

$$C_i(X) \equiv \sum_{\alpha,\beta,\gamma,\delta=0}^{2} D_{iiii}^{\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma X_\delta = 0 \quad (i = 1, \cdots, m). \tag{7}$$

for any real vector $X = (X_0, X_1, X_2)$ satisfying

$$(X_0)^2 - c_i^2 \{(X_1)^2 + (X_2)^2\} = 0.$$

(See also Proposition 2.1 in [1]). Based on this, we shall rewrite F^i in the following form:

$$F^i(\partial u, \partial^2 u) = N^i(\partial u^i, \partial^2 u^i) + R^i(\partial u, \partial^2 u),$$

where

$$N^i(\partial u^i, \partial^2 u^i) = \sum_{\alpha,\beta,\gamma,\delta=0}^{2} D_{iiii}^{\alpha\beta\gamma\delta} \partial_\alpha u^i \partial_\beta u^i \partial_\gamma \partial_\delta u^i,$$

$$R^i(\partial u, \partial^2 u) = \sum_{(j,k,l)\neq(i,i,i)} \sum_{\alpha,\beta,\gamma,\delta=0}^{2} D_{ijkl}^{\alpha\beta\gamma\delta} \partial_\alpha u^j \partial_\beta u^k \partial_\gamma \partial_\delta u^l.$$

By (7), one can write N^i as linear combinations of the followings:

$$\begin{aligned}
N_1^i &= ((\partial_0 u^i)^2 - c_i^2 |\nabla u^i|^2) \partial_\alpha \partial_\beta u^i, \\
N_2^i &= \partial_\alpha u^i \partial_\beta ((\partial_0 u^i)^2 - c_i^2 |\nabla u^i|^2), \\
N_3^i &= \partial_\alpha u^i \partial_\beta u^i \square_i u^i, \\
N_4^i &= \partial_\alpha u^i (\partial_\beta u^i \partial_\gamma \partial_\delta u^i - \partial_\gamma u^i \partial_\beta \partial_\delta u^i)
\end{aligned}$$

for $\alpha, \beta, \gamma, \delta = 0, 1, 2$. We shall call N^i the Null-form, while R^i the Resonance-form.

Our next interest is then to show the global existence thorem for such nonlinearity. In [1], R. Agemi and K. Yokoyama treated the Resonance-form and proved the existence

theorem. Thus, the aim of this paper is to deal with not only the Resonance-form but also the Null-form. To this end, we rewrite the saptial and time derivative as follows,

$$\nabla = \frac{x}{r}\partial_r - \frac{x^\perp}{r^2}\Omega \quad \text{for} \quad r = |x| > 0$$

and

$$\partial_t = -c_i\partial_r - \frac{\delta(r,t)}{\sqrt{t}}\partial_r + \frac{1}{t}S \quad \text{for} \quad |c_it - r| \le \sqrt{t},$$

where $x^\perp = (x_2, -x_1)$, $\Omega = x_1\partial_2 - x_2\partial_1$, and $-1 \le \delta(r,t) \le 1$. To obtain a variant of (6), we make use of these identities in a region

$$\Lambda_i = \{ (r,t) \in (0,\infty) \times (0,\infty) \mid |c_it - r| \le \sqrt{t} \}, \quad i = 1, \cdots, m. \tag{8}$$

Actually, this is the key of our argument. Furthermore, our proof revises previous one concerning the scalar equations in the sense that we do not need to use the operators L_i. Now we state our main theorem.

Theorem 1 *Let $n = 2$. Suppose (7) holds. Then there exists a positive constant ε_0 such that the initial value problem (1) and (2) has a unique C^∞-solution in $\mathbf{R}^2 \times [0,\infty)$ for $0 < \varepsilon \le \varepsilon_0$.*

References

[1] R. Agemi and K. Yokoyama, *The null condition and global existence of solutions to systems of wave equations with different speeds,* to appear in Series on Adv. in Math. for Appli. Sci., World Scientific.

[2] S. Alinhac, *Temps de vie et comportement explosif des solutions d'équations d'ondes quasi-linéaires en dimension deux II,* Duke Math. J. **73** (1994), pp. 543-560.

[3] P. Godin, *Lifespan of solutions of semilinear wave equations in two space dimensions,* Comm. in P. D. E. **Vol. 18, No. 5&6** (1993), pp. 895-916.

[4] L. Hörmander, *The lifespan of classical solutions of nonlinear hyperbolic equations,* Lecture Note in Math. **1256** (1987), pp. 214-280.

[5] A. Hoshiga, *The initial value problems for quasi-linear wave equations in two space dimensions with small data,* Advances in Math. Sci. Appli. **Vol. 5, No. 1** (1995), pp. 67-89.

[6] F. John, *Existence for large times of strict solutions of nonlinear wave equations in three space dimensions for small initial data,* Comm. Pure Appli. Math. **40** (1987), pp. 79-109.

[7] F. John and S. Klainerman, *Almost global existence to nonlinear wave equations in the three space dimensions,* Comm. Pure Appli. Math. **37** (1984), pp. 443-455.

[8] S. Katayama, *Global existence for systems of nonlinear wave equations in two space dimensions, II*, Publ. RIMS, Kyoto Univ. **31** (1995), pp. 645–665.

[9] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Lectures in Appli. Math. **23** (1986), pp. 293-326.

[10] M. Kovalyov, *Resonance-type behaviour in a system of nonlinear wave equations*, Jour. Diff. Eq. **77** (1989), pp. 73–83.

On the uniqueness of strong solutions to the Navier-Stokes equations

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Abstract

We derive various estimates for strong solutions to the Navier-Stokes equations in $C([0,T), L^3(\mathbb{R}^3))$ that allow us to prove some regularity results on the kinematic bilinear term. We relate these estimates to the proof of uniqueness of strong solutions in $C([0,T), L^3(\mathbb{R}^3))$ recently given by Furioli-Lemarié-Terraneo .

Introduction and definitions

The Cauchy problem for the Navier-Stokes equations governing the time evolution of the velocity $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and the pressure $p(x,t)$ of an incompressible fluid filling all of $\mathbb{R}^3$ is described by the system

(1)
$$\begin{cases} \dfrac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\ \nabla \cdot u &= 0, \\ u(x,0) &= u_0(x),\ x \in \mathbb{R}^3,\ t \geq 0. \end{cases}$$

The existence of solutions to this system which are strongly continuous in time and take value in Lebesgue spaces $L^p(\mathbb{R}^3)$ has been well-known for a long time for $p \geq 3$ (see [2]). In the critical case, $p = 3$, for which solutions of (1) are invariant by rescaling, one can construct strong solutions in a subclass of $C_t(L^3) = C([0,T), L^3(\mathbb{R}^3))$ (see [3, 12, 5, 6]), but their uniqueness within the natural class was proved only recently ([4]). The key tool in obtaining this uniqueness result was the use of the Besov spaces $\dot{B}_q^{-(1-\frac{3}{q}),\infty}$, for $q < 3$.

These spaces have been used previously, but mainly with $q \geq 3$, in obtaining global existence results (see [8],[2],[10]). In addition, it was already noticed in various contexts (see [2],[10]) how the bilinear term, which is the difference between the solution and the solution to the linear heat equation (with same initial data), behaves better than the solution itself. We improve these results in the present paper, and show how this gain in regularity is related to the uniqueness problem, the main estimates involved being of the same kind.

In order to to simplify our study let us introduce the projection operator $\mathbb{P}$ on the divergence free vector fields. We remark that $\mathbb{P}$ is a pseudo-differential operator of order 0 which will be continuous on all spaces subsequently used (primarily because it is continuous on all Lebesgue spaces L^p, for $1 < p < \infty$).

A common method solving (1) is to reduce the system to an integral equation,

$$(2) \qquad u(x,t) = S(t)u_0(x) - \int_0^t \mathbb{P}\, S(t-s)\, \nabla \cdot (u \otimes u)(x,s)\, ds,$$

where $S(t) = e^{t\Delta}$ is the heat kernel, and then to solve it via a fixed point argument in a suitable Banach space (see [2],[6],[7]). Following [2], we remark that the bilinear term in (2) can be reduced to a scalar operator,

$$(3) \qquad B(f,g) = \int_0^t \frac{1}{(t-s)^2} G\left(\frac{\cdot}{\sqrt{t-s}}\right) * (fg)\, ds,$$

where G is analytic, such that

$$(4) \qquad |G(x)| \;\leq\; \frac{C}{1+|x|^4}$$

$$(5) \qquad |\nabla G(x)| \;\leq\; \frac{C}{1+|x|^4}.$$

This can be derived easily from the study of the operator under the integral sum, $\mathbb{P}S(t-s)\nabla\cdot$, since its symbol consists of terms like

$$(6) \qquad -\frac{\xi_j \xi_k \xi_l}{|\xi|^2} e^{-(t-s)|\xi|^2}$$

outside the diagonal, with another term $\xi_j e^{-(t-s)|\xi|^2}$ on it. For the sake of simplicity, we will take G as the inverse Fourier transform of $|\xi|e^{-|\xi|^2}$.

As we mentioned previously, Besov spaces are a useful tool in studying the bilinear operator B. In what follows we will use spaces of functions on $\mathbb{R}^3$, so henceforth the reference to the domain space will be omitted. Let us recall the following definition. The reader will find equivalent definitions of Besov spaces in [9],[11].

DEFINITION 1
Let $\theta(x) \in C^\infty$ be such that

$$\hat{\theta}(\xi) = |\xi|e^{-|\xi|^2}.$$

Let $p,q \in (1,+\infty)$, $s \in \mathbb{R}$, $s < 1$. Then, $f \in \dot{B}_p^{s,q}$ if and only if

$$(7) \qquad \left(\int_0^\infty \|t^{-s}\theta_t * f\|_{L^p}^q\, \frac{dt}{t}\right)^{\frac{1}{q}} < +\infty,$$

where θ_t is the rescaled function $\frac{1}{t^3}\theta(\frac{\cdot}{t})$, and this norm is equivalent to the usual dyadic norm.

We will also make use of the Triebel-Lizorkin spaces, which are very close to the Besov spaces, in the sense that one just exchanges the L^p and L^q norms in the definition :

DEFINITION 2

Let $p,q \in (1, +\infty)$, $s \in \mathbb{R}$, $s < 1$. Then, $f \in \dot{F}_p^{s,q}$ if and only if

$$(8) \qquad \left\| \left(\int_0^\infty (t^{-s} \theta_t * f)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{L^p} < +\infty.$$

The reader familiar with both scales of spaces will note that by replacing $\hat{\theta}(\xi)$ with $\hat{\theta}(\xi) = |\xi|^2 e^{-|\xi|^2}$ one obtains the usual characterization via the Gauss-Weierstrass kernel. We will use this fact further in the paper. Among various embeddings between these spaces and the Lebesgue and Sobolev ones, we recall that $L^p = \dot{F}_p^{0,2}$, for $1 < p < \infty$, and $\dot{B}_2^{s,2} = \dot{H}^s$.

Theorems and proofs

Let us start with the aforementioned result on the regularity of the solution.

THEOREM 1

Let $u(x,t)$ be a solution of (2) in $C([0,T], L^3)$, with initial data $u_0 \in L^3$ and denote by w the function $w = u - S(t)u_0$, then

$$(9) \qquad w \in C([0,T], \dot{F}_{3/2}^{1,2}).$$

In other words, the gradient of w is continuous in time with value in $L^{3/2}$.

We immediately remark that, via Sobolev embedding, $\dot{F}_{3/2}^{1,2} \hookrightarrow L^3$. Let's postpone the proof of the theorem for a moment, and comment on the meaning of this result. In [10], it was shown that for self-similar solutions (for which the initial data wasn't in L^3, but in some $\dot{B}_q^{-(1-\frac{3}{q}),\infty}$ with $q > 3$), the bilinear term was in $\dot{B}_{3/2}^{1,2}$, and it is a simple matter to obtain $\dot{F}_{3/2}^{1,2}$ instead. This is slightly better, as $\dot{F}_p^{s,q} \hookrightarrow \dot{B}_p^{s,q}$ for $p < q$. Now, in order to obtain this result, one makes use of the special structure of a self-similar solution. For such solutions, the time regularity is intimately related to the space regularity because of the scaling $u(x,t) = 1/\sqrt{t}\, U(x/\sqrt{t})$. On the other hand, using $L_t^p(L_x^q)$ estimates, it was proved in [10] that for a solution in $C_t(L^3)$ with initial data $u_0 \in L^3$ the function $w \in \dot{B}_3^{0,2}$. One remarks that $\dot{F}_{3/2}^{1,2} \hookrightarrow \dot{B}_3^{0,2}$. The proof of that result was a consequence of the following lemma applied with $p = 6$.

LEMMA 1

Let $3 \leq q \leq 6$. Then the bilinear operator $B(f,g)$ is bicontinuous from $L_t^{\frac{2}{1-3/q}}(L_x^q) \times L_t^{\frac{2}{1-3/q}}(L_x^q)$ into $L_t^\infty(\dot{B}_{q/2}^{\frac{6}{q}-1,\frac{q}{q-3}})$.

In particular, if $q = 3$, $B(f,g)$ is bicontinuous from $L_t^\infty(L_x^3) \times L_t^\infty(L_x^3)$ into $L_t^\infty(\dot{B}_{\frac{3}{2}}^{1,\infty})$. This last estimate for $q = 3$ was used in [4]. The proof we are giving here for $3 \leq q \leq 6$ is nothing but paraphrasing the case $q = 6$ dealt with in [10]. More precisely, we will prove the estimate by duality. To this end, let $0 < T < \infty$. By hypothesis,

$$(10) \qquad \int_0^T \|fg(x,t)\|_{q/2}^{\frac{1}{1-3/q}} dt < \infty,$$

where the integral in time is replaced by a $\sup_t = \sup_{t \in [0,T]}$ if $q = 3$. Then for an arbitrary test function $\varphi(x) \in C_0^\infty$, we consider the functional

(11)
$$I_t = \langle B(f, g), \varphi \rangle.$$

We find

(12)
$$
\begin{aligned}
I_t &= \int_0^t \left\langle \frac{1}{(t-s)^2} G\left(\frac{\cdot}{\sqrt{t-s}}\right) * (fg), \varphi \right\rangle ds \\
&= 2 \int_0^{\sqrt{t}} \left\langle fg(t - s^2), \frac{1}{s^3} \check{G}\left(\frac{\cdot}{s}\right) * \varphi \right\rangle ds
\end{aligned}
$$

where $\check{G}(x) = G(-x)$, and we made a change of variable. Applying Hölder inequality both in time and space variables, we get

(13)
$$
\begin{aligned}
|I_t| &\leq \left(\int_0^t \| fg(t-s) \|_{L^q}^{\frac{1}{1-3/q}} ds \right)^{1-3/q} \\
&\quad \times \left(\int_0^{\sqrt{t}} \left(\| s^{6/q-1} \check{G}_s(\cdot) * \varphi \|_{q/(q-2)}^{q/3} \right) \frac{ds}{s} \right)^{3/q}
\end{aligned}
$$

where $\check{G}_s = \frac{1}{s^3} \check{G}\left(\frac{\cdot}{s}\right)$. Using Definition 1, the second integral is found to be less than the norm of φ in $\dot{B}_{q/(q-2)}^{1-\frac{6}{q}, \frac{q}{3}}$, which is exactly the dual of $\dot{B}_{q/2}^{\frac{6}{q}-1, \frac{q}{q-3}}$ (The restriction $q \leq 6$ is mainly because we are interested in positive regularity indices). Let us stress again that the cancellation property of G plays a crucial role in this estimate (see [10]). We see that with this lemma we are far from the actual result of Theorem 1, because the third index $\frac{p}{p-3}$ is greater than 2. This is why we have to investigate further the behaviour of the bilinear operator $B(f, g)$. Let us start with the following lemma due to Furioli-Lemarié-Terraneo [4].

LEMMA 2 ([4])
The bilinear operator $B(f, g)$ is bicontinuous from $L_t^\infty(\dot{B}_2^{\frac{1}{2}, \infty}) \times L_t^\infty(L^3)$ into $L_t^\infty(\dot{B}_2^{\frac{1}{2}, \infty})$.

The lemma is actually true for more general Besov spaces, namely all the $\dot{B}_q^{\frac{3}{q}-1, \infty}$, provided that $3/2 < q < 3$. We give here a proof which is somewhat different from [4], for we hope it sheds some light on the behavior of the bilinear term and its particular structure, which makes it look like an approximate pseudo-identity applied to the product fg. This would be exactly the case, if not for the time dependency of f and g. We will use the duality technique again, so that keeping the same notation as in the proof of Lemma 1, we are faced with

(14)
$$|I_t| \leq C \int_0^{\sqrt{t}} \sup_t \| f \|_{\dot{B}_2^{\frac{1}{2}, \infty}} \| g(x, t - s^2) \check{G}_s * \varphi \|_{\dot{B}_2^{-\frac{1}{2}, 1}} ds.$$

Therefore, s being fixed, we have to estimate the quantity

$$\| g(x, t - s^2) \check{G}_s * \varphi \|_{\dot{B}_2^{-\frac{1}{2}, 1}}.$$

This is in turn equivalent to estimating

(15)
$$J = \int_0^\infty \| \theta_u * (g \check{G}_s * \varphi) \|_2 \frac{du}{\sqrt{u}}$$

where θ is a suitable Littlewood-Paley type function (and thus can be used to define the Besov spaces as in Definition 1). We assume that $\hat{\theta}$ is compactly supported, and that $0 \notin \text{supp}\,\hat{\theta}$. We consider first the case when $u < s$. Let

$$J_1 = \int_0^s \|\theta_u * (g\check{G}_s * \varphi)\|_2 \frac{du}{\sqrt{u}}$$

we have

$$\|\theta_u * (g\,\check{G}_s * \varphi)\|_2 \leq C\|g\check{G}_s * \varphi\|_2$$
$$\leq \|g\|_3 \|\check{G}_s * \varphi\|_6.$$

But

$$\|\check{G}_s * \varphi\|_6 \leq \frac{C}{s}\|\tilde{G}_s * \varphi\|_2$$

where $\hat{\tilde{G}} = |\xi|e^{-\frac{|\xi|^2}{2}}$ and we applied the Young inequality for the remaining part of $\check{G}$ (which is simply some sort of heat kernel). Then,

$$J_1 \leq C\frac{\|\tilde{G}_s * \varphi\|_2}{\sqrt{s}} \sup_t \|g\|_3.$$

Let us now consider the case $u > s$, more precisely the quantity

$$J_2 = \int_s^\infty \|\theta_u * (g\check{G}_s * \varphi)\|_2 \frac{du}{\sqrt{u}}.$$

Using Bernstein inequalities on θ,

$$\|\theta_u * (g\,\check{G}_s * \varphi)\|_2 \leq \frac{C}{u}\|g\,\check{G}_s * \varphi\|_{\frac{6}{5}}$$
$$\leq \frac{C}{u}\|\check{G}_s * \varphi\|_2 \sup_t \|g\|_3.$$

Which finally gives

$$J_2 \leq C\frac{\|\tilde{G}_s * \varphi\|_2}{\sqrt{s}} \sup_t \|g\|_3.$$

Therefore, we obtain an estimate on J. Furthermore

$$(16) \qquad \|g(x, t - s^2)\,\check{G}_s * \varphi\|_{\dot{B}_2^{-\frac{1}{2},1}} \leq C\frac{\|\tilde{G}_s * \varphi\|_2}{\sqrt{s}} \sup_t \|g\|_3$$

and we find
$$(17) \qquad |I_t| \leq C\sup_t(\|g\|_3 \|f\|_{\dot{B}_2^{\frac{1}{2},\infty}}) \int_0^\infty \sqrt{s}\|\tilde{G} * \varphi\|_2 \frac{ds}{s}.$$

The last integral is nothing but $\|\varphi\|_{\dot{B}_2^{-\frac{1}{2},1}}$, which, by invoking duality, achieves the proof.

Nevertheless, this is not sufficient to obtain the uniqueness of strong solutions in $C_t(L^3)$, except for small initial data. We actually need a third lemma (which is a modification of the previous one) in order to prove time-local uniqueness for arbitrary initial data.

LEMMA 3
Let $f(x, t) \in L_t^\infty(\dot{B}_2^{\frac{1}{2},\infty})$ and $g(x, t) = S(t)G$, where $G \in L^3$. Then the bilinear operator $B(f, g)$ belongs to $L_t^\infty(\dot{H}^{\frac{1}{2}})$.

As for uniqueness, we don't actually need $\dot{H}^{\frac{1}{2}}$ (which is included in $\dot{B}_2^{\frac{1}{2},\infty}$), but this result shows how the special structure of g can be usefully exploited here, and it is indeed such considerations which will lead to the main theorem of this paper. But let us deal with the lemma : as $G \in L^3$, g verifies all the usual estimates for a solution to the heat equation. For example, if we denote

$$\|g|_{4,T} = \sup_t t^{\frac{1}{8}}\|g(x,t)\|_4,$$

then

$$\sup_t t^{\frac{1}{8}}\|g(x,t)\|^4 \leq C\|G\|_3$$

and the left hand side goes to zero as T goes to zero (remember that $\sup_t = \sup_{t\in[0,T]}$ where T is to be chosen). We have therefore

$$\|g(x,t-s^2)\|_4 \leq \frac{\|g|_{4,T}}{(t-s^2)^{\frac{1}{8}}}.$$

Upon returning to the estimates of $\|g(x,t-s^2)\check{G}_s * \varphi\|_{\dot{B}_2^{-\frac{1}{2},1}}$ we obtain

$$
\begin{aligned}
J_1 &\leq \int_0^s \frac{\|g|_{4,T}}{(t-s^2)^{\frac{1}{8}}}\|\check{G}_s * \varphi\|_3 \frac{du}{\sqrt{u}} \\
&\leq \frac{C\|g|_{4,T}}{(t-s^2)^{\frac{1}{8}}s^{\frac{1}{4}}} \int_0^s \|\tilde{G}_s * \varphi\|_2 \frac{du}{\sqrt{u}} \\
&\leq C\frac{\|g|_{4,T}}{(t-s^2)^{\frac{1}{8}}} \frac{\|\tilde{G}_s * \varphi\|_2}{s^{\frac{1}{4}}}
\end{aligned}
$$

and,

$$
\begin{aligned}
J_2 &\leq \int_s^\infty \frac{C}{u^{\frac{3}{4}}}\|g\check{G}_s * \varphi\|_{\frac{4}{3}} \frac{du}{\sqrt{u}} \\
&\leq C\frac{\|g|_{4,T}}{(t-s^2)^{\frac{1}{8}}} \frac{\|\tilde{G}_s * \varphi\|_2}{s^{\frac{1}{4}}}.
\end{aligned}
$$

This yields

$$(18) \qquad \|g(x,t-s^2)\check{G}_s * \varphi\|_{\dot{B}_2^{-\frac{1}{2},1}} \leq C\frac{\|g|_{4,T}}{(t-s^2)^{\frac{1}{8}}} \frac{\|\tilde{G}_s * \varphi\|_2}{s^{\frac{1}{4}}},$$

so that

$$(19) \qquad |I_t| \leq C\sup_t(\|g|_{4,T}\|f\|_{\dot{B}_2^{\frac{1}{2},\infty}}) \int_0^{\sqrt{t}} \frac{\|\tilde{G} * \varphi\|_2}{(t-s^2)^{\frac{1}{8}}s^{\frac{1}{4}}}ds.$$

Applying Cauchy-Schwarz to this last integral, we bound it from above by

$$\left(\int_0^1 \frac{d\theta}{(1-\theta^2)^{1/4}\theta^{1/2}}\right)^{\frac{1}{2}} \left(\int_0^{\sqrt{t}} \|\tilde{G} * \varphi\|_2^2 ds\right)^{\frac{1}{2}} \leq C\|\varphi\|_{\dot{H}^{-\frac{1}{2}}}$$

which concludes the proof of Lemma 3.

From these three lemmata, we can deduce, as we announced, the uniqueness result of Furioli-Lemarié-Terraneo,

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THEOREM 2 ([4])

Let $u_0 \in L^3$. Then there exists a unique local strong solution of (2) in the class $C_t(L^3)$.

The existence of such a solution was proved in [12, 5, 6], where it is also shown that this solution is global if the initial data is small enough. As far as its uniqueness is concerned, consider two solutions $u(x,t)$ and $v(x,t)$ with the same initial data u_0 and for which u is actually the solution constructed via the fixed point method ([6]). We denote $w = u - S(t)u_0$ and $\tilde{w} = v - S(t)u_0$. Then, if we temporarily forget that the bilinear operator appearing in (1) is vectorial and non-commutative, we may abuse the notation and write (as in the scalar case)

$$w - \tilde{w} = 2B(S(t)u_0, w - \tilde{w}) + B(w + \tilde{w}, w - \tilde{w}).$$

We know from Lemma 1–via the embedding $\dot{B}_{3/2}^{1,\infty} \hookrightarrow \dot{B}_2^{1/2,\infty}$– that both w and $\tilde{w}$ belong to $\dot{B}_2^{\frac{1}{2},\infty}$.

By applying Lemma 3 to the first term, and Lemma 2 to the second, we obtain

$$\sup_t \|w - \tilde{w}\|_{\dot{B}_2^{\frac{1}{2},\infty}} \le C(\|S(t)u_0\|_{4,T} + \sup_t \|w + \tilde{w}\|_3) \sup_t \|w - \tilde{w}\|_{\dot{B}_2^{\frac{1}{2},\infty}}$$

so that $w = \tilde{w}$ at least on a small interval in time, as both quantities $\|S(t)u_0\|_{4,T}$ and $\sup_t \|w + \tilde{w}\|_3$ go to zero when T goes to zero (the first by Banach-Steinhaus and the second by the strong continuity in L^3 of the solutions). We conclude using the uniqueness part of the existence theorem, as the solution u obtained by fixed point is unique if it verifies $\lim_{t\to 0} \sqrt{t}\|u\|_\infty = 0$ ([1],[10]). Obviously v will verify the same estimate on the small interval on which we obtained $v = u$, and then both solutions are the same within this restricted class, on the interval they are defined. And this achieves the proof of Theorem 2.

We are finally in the position to prove Theorem 1. It is useful to rewrite B in a more suitable form, namely

$$(20) \qquad B(f,g) = 2 \int_0^{\sqrt{t}} G_s(x) * fg(x, t - s^2) ds$$

where, as usual, $G_s(x) = \frac{1}{s^3} G(\frac{x}{s})$. In addition, it is useful to work with the following operator, $A(f,g) = \Lambda B(f,g)$, where Λ is the Calderón operator, with symbol $|\xi|$. Then

$$(21) \qquad A(f,g) = 2 \int_0^{\sqrt{t}} \tilde{G}_s(x) * fg(x, t - s^2) \frac{ds}{s},$$

and $\mathcal{F}(\tilde{G})(\xi) = |\xi|^2 e^{-|\xi|^2}$. As we have already noted, if f and g were time independent, then A would reduced to $I - e^{t\Delta}$ applied to the product fg. Trying to do this when f and g are two continuous functions of time is bound to fail, and we saw that in fact it leads to Lemma 1. However, when those f and g are coordinates of u, the fixed point solution, the construction of u gives numerous estimates which are very useful in the same way as Lemma 3. Actually, we could directly apply a variant of Lemma 3 in order to obtain $B(f,g) \in \dot{H}^{\frac{1}{2}}$, but if we want the limiting exponents $s = 1$ and $p = \frac{3}{2}$, we have to be more cautious. We will proceed in two steps. First, we will deal with the bilinear operator applied to the linear part. Afterwards we will estimate the remaining part using a result similar to Lemma 2.

LEMMA 4

Let F and G be in L^3. If $f(x,t) = S(t)F(x)$ and $g(x,t) = S(t)G(x)$, then $A(f,g) \in C_t(L^{\frac{3}{2}})$.

In order to prove the lemma, we split A into two operators, $A_1 = \int_0^{\sqrt{\frac{t}{2}}}$ and $A_2 = \int_{\sqrt{\frac{t}{2}}}^{\sqrt{t}}$. This last part is easily dealt with, as

$$\|G_s * fg\|_{\frac{3}{2}} \leq \|fg\|_{\frac{3}{2}},$$

which gives

$$(22) \qquad \begin{aligned} \|A_2\|_{\frac{3}{2}} &\leq \int_{\sqrt{\frac{t}{2}}}^{\sqrt{t}} C \sup_t \|fg(t)\|_{\frac{3}{2}} \, ds \\ &\leq C \sup_t \|fg(t)\|_{\frac{3}{2}}. \end{aligned}$$

For the first part, we define A_1', as

$$(23) \qquad A_1' = \int_0^{\sqrt{\frac{t}{2}}} |G_s * D| \frac{ds}{s}$$

where $D = fg(x, t - s^2) - fg(x, t)$. Then,

$$(24) \qquad \|A_1'\|_{\frac{3}{2}} \leq \int_0^{\sqrt{\frac{t}{2}}} \|f(t - s^2) - f(t)\|_3 \frac{ds}{s} \sup_t \|g\|_3.$$

However, if $Op(\sigma(\xi))$ stands for the pseudo-differential operator of symbol σ,

$$(25) \qquad \begin{aligned} f(t - s^2) - f(t) &= Op(e^{-(t-s^2)|\xi|^2}(1 - e^{-s^2|\xi|^2}))F(x) \\ &= Op\left(s^2|\xi|^2 \int_0^1 e^{-s^2\lambda|\xi|^2} d\lambda \right) S(t - s^2)F(x), \end{aligned}$$

and therefore

$$(26) \qquad \begin{aligned} \|f(t - s^2) - f(t)\|_3 &\leq \int_0^1 \|Op(s^2|\xi|^2 e^{-(t-s^2)|\xi|^2} e^{-s^2\lambda|\xi|^2})F(x)\|_3 \, d\lambda \\ &\leq C\frac{s^2}{t - s^2} \|Op((t - s^2)|\xi|^2 e^{-(t-s^2)|\xi|^2})F(x)\|_3 \end{aligned}$$

which gives the desired estimate. The remaining term where $(fg)(x, t - s^2)$ is replaced by $(fg)(x, t)$ is easy to deal with, as we can integrate with respect to s, to obtain the operator $I - e^{t\Delta}$.

We will now prove another lemma, which will allow us to conclude.

LEMMA 5
Let $p > 3$. If $f(x, t) \in C_t(\dot{F}_{\frac{3}{2}}^{1,2})$ and $g(x, t)$ satisfies the estimate,

$$(27) \qquad |g\|_p = \sup_t(t^{\frac{1}{2} - \frac{3}{2p}} \|g(x, t)\|_p) < \infty$$

then

$$A(f, g) \in C_t(L^{\frac{3}{2}}).$$

In order to prove this, we write

$$A(f,g) = 2 \int_0^{\sqrt{t}} Op(s|\xi|e^{-s^2|\xi|^2})\Lambda(fg)(x, t-s^2)ds$$

As we control $\Lambda(fg)$ via $\Lambda(f)g + f\Lambda(g)$, we are left with the study of $B(\Lambda(f), g)$. We still have two different situations: either $f \in L_t^\infty(\dot{F}_{\frac{3}{2}}^{1,2})$ which gives $\Lambda(f) \in L^{\frac{3}{2}}$, or $f = S(t)u_0$ which leads to

$$t^{1-\frac{3}{2p}}\|\Lambda f(x,t)\|_p < \infty.$$

The first term is handled as follows:

$$(28) \qquad \|B(\Lambda(f), g)\| \le \int_0^t \frac{C}{(t-s)^{\frac{1}{2}+\frac{3}{2p}}s^{\frac{1}{2}-\frac{3}{2p}}}ds|g\|_p \sup_t \|\Lambda(f)\|_{\frac{3}{2}},$$

and we could just as well have chosen $p = 1/2$. The other term gives

$$(29) \qquad \|B(\Lambda(f), g)\| \le \int_0^t \frac{C}{(t-s)^{\frac{3}{2p}}s^{1-\frac{3}{2p}}}ds|\Lambda(g)\|_p \sup_t \|f\|_3.$$

We are then done with the lemma, and also with the theorem as the previous estimates allow us to preserve the property verified by $B(S(t)u_0, S(t)u_0)$ on all iterates of the fixed point scheme used to solve (2). Note that this fixed point scheme needs to be applied to the difference $w = u - S(t)u_0$, for the linear part doesn't verify the estimates.

References

[1] H. Brézis. Remarks on the preceding paper by M. Ben-Artzi. *Arch. Rat. Mech. Anal.*, 128:359–360, 1994.

[2] M. Cannone. *Ondelettes, Paraproduits et Navier-Stokes*. Diderot Editeurs, Paris, 1995.

[3] H. Fujita and T. Kato. On the Navier-Stokes initial value problem I. *Arch. Rat. Mech. Anal.*, 16:269–315, 1964.

[4] G. Furioli, P.G. Lemarié-Rieusset, and E. Terraneo. Sur l'unicité dans $L^3(\mathbb{R}^3)$ des solutions mild des équations de Navier-Stokes. preprint.

[5] Y. Giga and T. Miyakawa. Solutions in L^r of the Navier-Stokes initial value problem. *Arch. Rat. Mech. Anal.*, 89:267–281, 1985.

[6] T. Kato. Strong L^p solutions of the Navier-Stokes equations in $\mathbf{R}^m$ with applications to weak solutions. *Math. Zeit.*, 187:471–480, 1984.

[7] T. Kato and H. Fujita. On the non-stationary Navier-Stokes system. *Rend. Sem. Math. Univ. Padova*, 32:243–260, 1962.

[8] H. Kozono and M. Yamazaki. Semilinear heat equations and the Navier-Stokes equation with distributions as initial datas. *Comm. in Part. Diff. Eq.*, 19(5&6):959–1014, 1994.

[9] J. Peetre. *New thoughts on Besov Spaces*. Duke Univ. Math. Series. Duke University, Durham, 1976.

[10] F. Planchon. *Solutions globales et comportement asymptotique pour les équations de Navier-Stokes.* PhD thesis, Ecole Polytechnique, CMAT, Ecole Polytechnique, F-91128 Palaiseau Cedex, 1996.

[11] H. Triebel. *Theory of Function Spaces*, volume 78 of *Monographs in Mathematics.* Birkhauser, 1983.

[12] F. Weissler. The Navier-Stokes initial value problem in L^p. *Arch. Rat. Mech. Anal.*, 74:219–230, 1980.

Estimate on the Kac Transfer Operator with Applications*

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1. Introduction

The Kac transfer operator or Kac operator we refer to in this note is an operator of the kind $K(t) = e^{-tV/2}e^{t\Delta/2}e^{-tV/2}$ with integral kernel

$$K(x,y;t) = e^{-tV(x)/2}\frac{\exp\left(-\frac{(x-y)^2}{2t}\right)}{(\sqrt{2\pi t})^d}e^{-tV(y)/2}, \tag{1.1}$$

where Δ is the Laplacian and $V = V(x)$ a potential in d–dimensional space $\mathbb{R}^d$. The aim of this note is to survey some recent results obtained in [H3], [I-Tk1], [D-I-Tm], [I-Tk2] and [Tk] on a norm estimate for $K(t)$ compared with the Schrödinger semigroup $e^{-t(-\frac{1}{2}\Delta+V)}$ and to mention some applications as its consequences.

The operator $K(t)$ is regarded as a transfer matrix/operator for some lattice models in statistical mechanics which M. Kac [K] (cf. [K-Th1, 2]) studied to discuss a mathematical mechanism for a phase transition. To sketch how it comes out (cf. [H1, 2, 4]), consider a lattice model with exponential interaction, the simplest being a one-dimensional chain of N spins $\sigma_i = \pm 1$ with interaction energy

$$E = -\tfrac{t}{2}J \sum_{1\le i<j\le N} e^{-\frac{t}{2}|i-j|}\sigma_i\sigma_j, \tag{1.2}$$

where the interaction is assumed ferromagnetic, i.e. $J > 0$ and t^{-1} is interpreted as the effective number of spins interacting with a given spin. The partition function Q_N is given by

$$Q_N = e^{-\nu N t/4} \sum_{\{\sigma\}} \exp\left(\tfrac{\nu t}{4}\Sigma_{i=1}^N\Sigma_{j=1}^N e^{-\frac{t}{2}|i-j|}\sigma_i\sigma_j\right), \tag{1.3}$$

where $\nu = J/kT$ with k Boltzmann's constant and T the absolute temperature and the sum $\sum_{\{\sigma\}}$ is taken over all the spin configurations on the sites i ($i = 1, 2, \cdots, N$) of the lattice. The transfer operator $\tilde{K}(t)$ of this model turns out to be an integral operator with integral kernel

$$\tilde{K}(x,y;t) = e^{t/4}e^{-tV(x)/2}\frac{\exp\left(-\frac{(x-y)^2}{4\sinh(t/2)}\right)}{\sqrt{4\pi\sinh(t/2)}}e^{-tV(y)/2} \tag{1.4a}$$

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with

$$tV(x) = \tfrac{1}{2}(\tanh \tfrac{t}{4})x^2 - \log \cosh(\sqrt{\tfrac{\nu t}{2}}x).$$ (1.4b)

This operator $\tilde{K}(t)$ plays a crucial role, because the partition function Q_N is rewritten as

$$Q_N = e^{-\nu Nt/4} 2^N \iint \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tilde{K}^{(N-1)}(x,y;t) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dx dy,$$ (1.5)

where $\tilde{K}^{(N-1)}(x,y;t)$ is the integral kernel of the $(N-1)$ iterates $\tilde{K}(t)^{N-1}$ of $\tilde{K}(t)$. Moreover, for instance, the pair correlation $\rho(n)$ is given in terms of the eigenvalues μ_j $(\mu_1 > \mu_2 \geq \cdots)$ of $\tilde{K}(t)$ and the corresponding eigenfunctions $\varphi_j(x)$ by

$$\rho(n) = \lim_{N\to\infty} \langle \sigma_i \sigma_{i+n} \rangle_N$$

$$\equiv \lim_{N\to\infty} Q_N^{-1} \sum_{\{\sigma\}} \sigma_i \sigma_{i+n} \exp\left(\tfrac{\nu t}{4} \Sigma_{i=1}^N \Sigma_{j=1}^N e^{-\tfrac{t}{2}|i-j|} \sigma_i \sigma_j\right)$$ (1.6)

$$= \sum_{j=2}^\infty \left(\frac{\mu_j}{\mu_1}\right)^n \left(\int_{-\infty}^\infty \varphi_j(x)\varphi_1(x)\tanh(\sqrt{\tfrac{\nu t}{2}}x)dx\right)^2.$$

Therefore to know $\lim_{n\to\infty} \rho(n)$ is zero or positive, it is important to see whether the first eigenvalue $\mu_1(t)$ of $\tilde{K}(t)$ is asymptotically degenerate or to estimate the quotient $\mu_2(t)/\mu_1(t)$ of the first two eigenvalues of $\tilde{K}(t)$.

In this stage *notice* that our Kac operator $K(t)$ with $d=1$ is nearly equal to this $\tilde{K}(t)$ for sufficiently small $t > 0$, so that as $t \downarrow 0$, both $K(t)$ and $\tilde{K}(t)$ will have analogous asymptotic behavior. In fact, in order to estimate the quotient $\mu_2(t)/\mu_1(t)$ of the first two eigenvalues of $\tilde{K}(t)$, Kac himself took this approximation for granted and analyzed the eigenvalues λ of the Schrödinger equation $(-\tfrac{1}{2}\Delta + V(x))\varphi(x) = (\lambda + \tfrac{1}{2})\varphi(x)$ associated with our Kac operator $K(t)$.

In this note we consider both the nonrelativistic and relativistic Schrödinger operators

$$H = H_0 + V \equiv -\tfrac{1}{2}\Delta + V(x),$$ (1.7a)

$$H^r = H_0^r + V \equiv \sqrt{-\Delta + 1} - 1 + V(x),$$ (1.7b)

with mass 1, and the associated Kac operators

$$K(t) = e^{-tV/2} e^{-tH_0} e^{-tV/2},$$ (1.8a)

$$K^r(t) = e^{-tV/2} e^{-tH_0^r} e^{-tV/2},$$ (1.8b)

where $V(x)$ is a real-valued continuous function in $\mathbb{R}^d$ bounded below.

It is seen through the Feynman–Kac formula that both e^{-tH} and e^{-tH^r}, $t \geq 0$, define strongly continuous semigroups not only on $L^2(\mathbb{R}^d)$ but also on all $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, and on the Banach space $C_\infty(\mathbb{R}^d)$ of the continuous functions vanishing at infinity.

We estimate in L^p operator norm the difference between the Kac operator and the Schrödinger semigroup by a power $O(t^a)$ of small $t > 0$ with $a \geq 1$. This estimate, as an application, may be used to get an estimate of the eigenvalue of the Kac operator compared with that of the Schrödinger operator, such as done in [H3]. As a second application, it can give the Trotter product formula in L^p operator *norm*, when $a > 1$.

The method of proof is probabilistic based on the Feynman–Kac formula in [I-Tk1,2] and [Tk], while L^2 operator-theoretical in [D-I-Tm].

2. Results

In the following, $\| \cdot \|_p$ stands for the L^p operator norm on L^p for $1 \leq p < \infty$ and on C_∞ for $p = \infty$.

First we give from [I-Tk1, 2] the nonrelativistic result on $K(t)$ in (1.8a) (see [Tk] for more general potentials $V(x)$).

THEOREM 2.1. *(Nonrelativistic case) Let $0 < \delta \leq 1$ and m a nonnegative integer such that $m\delta \leq 1$. Suppose that $V(x)$ is a C^m-function in $\mathbb{R}^d$ bounded below by a constant b satisfying that*

$$|\partial^\alpha V(x)| \leq C(V(x) - b + 1)^{1 - |\alpha|\delta}, \quad 0 \leq |\alpha| \leq m, \tag{2.1a}$$

and further that $\partial^\alpha V(x), |\alpha| = m$, are Hölder-continuous:

$$|\partial^\alpha V(x) - \partial^\alpha V(y)| \leq C|x - y|^\kappa, \quad x, y \in \mathbb{R}^d, \tag{2.1b}$$

with constants $C > 0$ and $0 \leq \kappa \leq 1$ (By $\kappa = 0$ we understand $\partial^\alpha V(x), |\alpha| = m$, bounded). Then it holds that, as $t \downarrow 0$,

$$\|K(t) - e^{-tH}\|_p = \begin{cases} O(t^{1+\kappa/2}), & m = 0, \\ O(t^{1+2\delta\wedge\frac{1+\kappa}{2}}), & m = 1, \\ O(t^{1+2\delta}), & m \geq 2. \end{cases} \tag{2.2}$$

Here note that condition (2.1b) with $\kappa = 1$ is equivalent to that $\partial^\alpha V(x), |\alpha| = m+1$, are essentially bounded.

From Theorem 2.1 with telescoping follows immediately the Trotter product formula in L^p operator norm.

THEOREM 2.2. *(Nonrelativistic case) For the same function $V(x)$ as in Theorem 2.1, it holds that, as $n \to \infty$,*

$$\|(e^{-tV/2n}e^{-tH_0/n}e^{-tV/2n})^n - e^{-tH}\|_p, \quad \|(e^{-tV/n}e^{-tH_0/n})^n - e^{-tH}\|_p$$
$$= \begin{cases} n^{-\kappa/2}O(t^{1+\kappa/2}), & m = 0, 0 < \kappa \leq 1, \\ n^{-2\delta\wedge\frac{1+\kappa}{2}}O(t^{1+2\delta\wedge\frac{1+\kappa}{2}}), & m = 1, 0 \leq \kappa \leq 1, \\ n^{-2\delta}O(t^{1+2\delta}), & m \geq 2. \end{cases} \tag{2.3}$$

Examples. The following functions satisfy condition (2.1ab) for $V(x)$:

(i) $|x|^2$ (harmonic oscillator potential) with $(\delta, m, \kappa) = (\frac{1}{2}, 1, 1)$ or $(\frac{1}{2}, 2, 0)$,

(ii) $|x|^4 - |x|^2$ (double well potential) with $(\delta, m, \kappa) = (\frac{1}{4}, 3, 1)$ or $(\frac{1}{4}, 4, 0)$,

(iii) $|x|^\rho$ with $(\delta, m, \kappa) = (1, 0, \rho)$ for $0 < \rho \leq 1$ and $(\delta, m, \kappa) = (1/\rho, [\rho], \rho - [\rho])$ for $\rho > 1$, where $[\rho]$ is the maximal integer that is not greater than ρ.

But, for instance, $\exp(|x|^2 + 1)^a$, $a > 0$, and $\exp|x|^2$ do not satisfy the condition.

Remark 1. B. Helffer [H3] (also [H4], cf. [H2], [H1]) is the first who proved (2.2) in L^2 operator norm ($p = 2$), with $O(t^2)$ on the RHS of (2.2), by pseudo-differential operator calculus, when $V(x)$ is a C^∞-function bounded below by b and satisfying $|\partial^\alpha V(x)| \le C_\alpha(1 + x^2)^{(2-|\alpha|)+/2}$ for every multi-index α with constant C_α. In fact, as his condition implies that

$$|\partial^\alpha V(x)| \le C(V(x) - b + 1)^{(1-|\alpha|/2)+} \tag{2.4}$$

for the same α, so his result is included in the case $p = 2$ and $(\delta, m, \kappa) = (\frac{1}{2}, 1, 1)$ or $(\delta, m, \kappa) = (\frac{1}{2}, 2, 0)$ in Theorem 2.1.

Remark 2. Theorems 2.1 and 2.2 are valid with the operator H_0 replaced by the magnetic Schrödinger operator $H_0(A) = \frac{1}{2}(-i\partial - A(x))^2$ with vector potential $A(x)$ including the case of constant magnetic fields (see [I-Tk1], [D-I-Tm]).

Remark 3. As concerns the Trotter product formula in operator norm, Rogava [R] proved for nonnegative selfadjoint operators A and B in a Hilbert space that, if the domain $D[A]$ of A is a subset of the domain $D[B]$ of B and $C = A + B$ is selfadjoint on $D[C] = D[A] \cap D[B] = D[A]$, then, as $n \to \infty$,

$$\|(e^{-tA/2n}e^{-tB/n}e^{-tA/2n})^n - e^{-tC}\| = \|(e^{-tB/n}e^{-tA/n})^n - e^{-tC}\| = O(n^{-1/2}\ln n).$$

In this case, B is A-bounded. Notice that in our Theorems 2.1 and 2.2, neither V is H_0-bounded nor H_0 is V-bounded.

We refer to [I-Tm1] for some results complementary to Rogava's and to [I-Tm2] for the Trotter product formula in trace norm, both of which were proved by operator-theoretic methods.

Next we give from [I-Tk2] the relativistic result on $K^r(t)$ in (1.8b).

THEOREM 2.3. *(Relativistic case) Let $V(x)$ be the same function as in Theorem 2.1. Then it holds that, as $t \downarrow 0$,*

$$\|K^r(t) - e^{-tH^r}\|_p = \begin{cases} O(t^{1+\kappa}), & m = 0, 0 \le \kappa < 1, \\ O(t^2|\ln t|), & (m, \kappa) = (0, 1), \\ O(t(t^{2\delta} \vee t|\ln t|)), & (m, \kappa) = (1, 0), \\ O(t^{1+2\delta \wedge 1}), & m = 1, 0 < \kappa \le 1, \\ O(t^{1+2\delta}), & m \ge 2. \end{cases} \tag{2.5}$$

From Theorem 2.3 follows immediately the Trotter product formula in L^p operator norm.

THEOREM 2.4. *(Relativistic case) For the same function $V(x)$ as in Theorem 2.1, it holds that, as $n \to \infty$,*

$$\|(e^{-tV/2n}e^{-tH_0^r/n}e^{-tV/2n})^n - e^{-tH^r}\|_p, \quad \|(e^{-tV/n}e^{-tH_0^r/n})^n - e^{-tH^r}\|_p$$

$$= \begin{cases} n^{-\kappa}O(t^{1+\kappa}), & m = 0, 0 < \kappa < 1, \\ n^{-1}O(t^2|\ln(t/n)|), & (m, \kappa) = (0, 1), \\ O((n^{-2\delta}t^{1+2\delta}) \vee (n^{-1}t^2|\ln(t/n)|)), & (m, \kappa) = (1, 0), \\ n^{-2\delta \wedge 1}O(t^{1+2\delta \wedge 1}), & m = 1, 0 < \kappa \le 1, \\ n^{-2\delta}O(t^{1+2\delta}), & m \ge 2. \end{cases} \tag{2.6}$$

These results have been briefly announced in [I-Tk3].

REFERENCES

[D-I-Tm]. Atsushi Doumeki, Takashi Ichinose and Hideo Tamura, *Error bound on exponential product formulas for Schrödinger operators*, to appear in J. Math. Soc. Japan.

[H1]. Bernard Helffer, *Spectral properties of the Kac operator in large dimension*, Centre de Recherches Math., CRM Proceedings and Lecture Notes 8, 179–211 (1995).

[H2]. Bernard Helffer, *Correlation decay and gap of the transfer operator*, Algebra i Analiz (St. Petersburg Math. J.) 8, 192–210 (1996).

[H3]. Bernard Helffer, *Around the transfer operator and the Trotter–Kato formula*, Operator Theory: Advances and Appl. 78, 161–174 (1995).

[H4]. Bernard Helffer, *Recent results and open problems on Schrödinger operators, Laplace integrals, and transfer operators in large dimension*, "Schrödinger Operators, Markov Semigroups, Wavelet Analysis, Operator Algebras," Math. topics 11, pp. 11– 162, Akademie Verlag, Berlin, 1996.

[I-Tk1]. Takashi Ichinose and Satoshi Takanobu, *Estimate of the difference between the Kac operator and the Schrödinger semigroup*, Commun. Math. Phys. (in press).

[I-Tk2]. Takashi Ichinose and Satoshi Takanobu, *The norm estimate of the difference between the Kac operator and the Schrödinger semigroup: A unified approach to the nonrelativistic and relativistic cases*, to appear in Nagoya Math. J.

[I-Tk3]. Takashi Ichinose and Satoshi Takanobu, *On the norm estimate of the difference between the Kac operator and the Schrödinger semigroup*, "Differential Equations, Asymptotic Analysis and Mathematical Physics," Math. Research 100, pp. 164– 173, Akademie Verlag, Berlin, 1997.

[I-Tm1]. Takashi Ichinose and Hideo Tamura, *Error estimates in operator norm for Trotter–Kato product formula*, Integr. Equat. Oper. Theory 27, 195–207 (1997).

[I-Tm2]. Takashi Ichinose and Hideo Tamura, *Error bound in trace norm for Trotter–Kato product formula of Gibbs semigroups*, to appear in Asympt. Analysis.

[K]. Mark Kac, *Mathematical mechanism of phase transitions*, 1966 Brandeis Lecture, pp. 242–305, Gordon and Breach, New York, 1968.

[K-Th1]. Mark Kac and Colin J. Thompson, *On the mathematical mechanism of phase transition*, Proc. N. A. S. 55, 676–683 (1966); Erratum, *ibid.* 56, 1625 (1966).

[K-Th2]. Colin J. Thompson and Mark Kac, *Phase transition and eigenvalue degeneracy of a one dimensional anharmonic oscillator*, Studies in Appl. Math. 48, 257–264 (1969).

[R]. Dzh. L. Rogava, *Error bounds for Trotter–type formulas for self-adjoint operators*, Functional Analysis and Its Applications 27, 217–219 (1993).

[Tk]. Satoshi Takanobu, *On the error estimate of the integral kernel for the Trotter product formula for Schrödinger operators*, to appear in Ann. Probab.

A weighted equation approach to decay rate estimates for the Navier-Stokes equations

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Purposefulness   We consider the nonstationary incompressible Navier-Stokes equations in $\mathbf{R}^n$ ($n \geq 2$) with the zero initial data:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f, & \nabla \cdot u = 0, \quad \text{in } \mathbf{R}^n \times (0, \infty), \\ u(\cdot, 0) = 0, \quad |u(x,t)| \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{NS}$$

Here $u = (u_j(x,t))$ ($j = 1, \cdots, n$) and $p = p(x,t)$ denote unknown velocity and pressure, respectively, and $f = (f_j(x,t))$ denotes a given external force.

Knightly [1] constructed an *analytic, in space-time, solution* for the Navier-Stokes equations in $\mathbf{R}^n$ with zero external force and nonzero continuous initial data $g(x)$ with $|g(x)| \leq A\min(1, |x|^{-r})$ for some $r \geq 1$ and sufficiently small A. He assumed that for some constant vector u_∞ and some $T > 0$, u, u_t, ∇u, Δu, p and ∇p are all continuous on $\mathbf{R}^n \times (0, T)$ and $|u(x,t) - u_\infty| = o(1)$, $|\nabla u(x,t)| = o(|x|)$ and $|p(x,t)| = o(|x|)$ as $|x| \to \infty$, locally uniformly in t. He represented the solution by an integral equation with the fundamental solution of the Stokes operator and obtained that

$$\begin{aligned} |u(x,t)| &\leq C\min(1, |x|^{-r}, t^{-r/2}), & &\text{if } r \in [1, n), \\ |\nabla u(x,t)| &\leq Ct^{-1/2}\min(1, |x|^{-r}, t^{-r/2}), & &\text{if } r \in [1, n), \\ |p(x,t)| &\leq Ct^{-1/2}\min(1, |x|^{-(2r-1)}, t^{-(r-1/2)}), & &\text{if } r \in [1, n/2). \end{aligned} \tag{1}$$

Here $n \geq 3$ for $p(x,t)$ (See [2], [1] and references there in). In the first of (1), we see $|u(x,t)| \leq C|x|^{-r}$ in $\{(x,t) \in \mathbf{R}^n \times (0, \infty); |x| \geq \sqrt{t} \geq 1\}$ and $|u(x,t)| \leq Ct^{-r/2}$ in $\{(x,t) \in \mathbf{R}^n \times (0, \infty); \sqrt{t} \geq |x| \geq 1\}$. Thus (1.1) are not *uniform* decay estimates and actually yields $|u(x,t)| \leq C(1 + t + |x|^2)^{-r/2}$ (see proofs in [1] and Knightly [3]). The time decays of pressure in (1) are relatively small.

For the Oseen flows past bodies in $\mathbf{R}^3$, Mizumachi [4] showed decay rates in space-time for a classical solution, which indicates a *paraboloidal wake* behind the bodies. He imposed nonzero initial and (time-independent) boundary values and constant velocity u_∞ at the infinity. He assumed that (i) $u - u_\infty \in L^\infty(0, \infty; L^r(\Omega)^n)$ (Ω is the exterior domain of the bodies) for some $r \in [1, 3)$ and $\lim_{|x|, t \to \infty} |u(x, t) - u_\infty| = 0$, (ii) the existence of stationary solution $(\tilde{u}, \tilde{p})$ and (a) $\sup_x |u - \tilde{u}| + ||\nabla u - \nabla\tilde{u}||_{L^2(\Omega)} \leq M(1 + t)^{-\alpha}$, (b) $|| \, |p - \tilde{p}| + |\nabla u - \nabla\tilde{u}| \, ||_{L^1(\partial\Omega)} \leq M(1 + t)^{-\alpha}$, for some $\alpha > 0$. For initial data $a(x)$ with $a - \tilde{u} = O(|x|^{-(2+\beta)})$, he showed that

$$|u(x, t) - \tilde{u}(x)| \leq M|x|^{-1}(1 + s_x)^{-1}(1 + t)^{-\gamma}. \tag{2}$$

Here $s_x = |x| - (u_\infty/|u_\infty|) \cdot x$, $\gamma \leq 1/2$ for $\beta > 1$ and $\alpha > 3/2$ or $\gamma < \min(\beta/2, 1/2, \alpha/3)$ and constant M depends on u. His assumptions are followed by Heywood [5] and Masuda [6] as far as $\alpha \leq 1/4$. (Borchers and Miyakawa [7, Theorem 6.3 and 6.8] showed (ii)(a) with $\alpha < 5/4$ for $u_\infty = 0$ and suitable data. For suitable initial data and $u|_{\partial\Omega} = u_\infty = 0$, i.e., $\tilde{u} = 0$, we can derive (ii)(b) with $\alpha < 5/4$ by the trace theorem with [7], [8, Theorem 1.1 and Proposition 4.1] and [9, Corollary 2]).

As well-known, if a weak solution u of (NS) is in the class

$$L^s(0, \infty; L^q(\mathbf{R}^n)^n) \quad \text{with} \quad n/q + 2/s = 1 \tag{3}$$

(and if its norm there $||u||_{q,s}$ is sufficiently small when $(q, s) = (n, \infty)$), u is C^∞ regular in space after a finite time (See Serrin [10], Giga [11], Struwe [12] and Takahashi [13], [14]). Our goal is to show *almost optimal* uniform estimates, i.e., almost same decay rate estimates as those of the heat equations, for *weak solutions* of (NS) in the class (3) with $||u||_{q,s} \ll 1$, while the decay rates of external forces are prescribed. That is, for $f(x, t) = o(|x|^{-\mu} t^{-\lambda})$ as $|x|, t \to \infty$ for some $\mu, \lambda > 0$, we will show that

$$|u(x, t)| \leq C(1 + |x|)^{-\rho}(1 + t)^{-\sigma} \quad \text{with} \quad C = C(n, f, \rho, \sigma),$$

$$|\nabla u(x, t)| \leq C(1 + |x|)^{-\rho'}(1 + t)^{-\sigma'} \quad \text{with} \quad C = C(n, f, \rho', \sigma'),$$

$$|p(x, t)| \leq C|x|^{-\rho''}(1 + t)^{-\sigma''} \quad \text{with} \quad C = C(n, f, \rho'', \sigma'').$$

Here for any $c > 0$, (ρ, σ), (ρ', σ') and (ρ'', σ'') are given as follows.

(I)(a) For decays in space, $\{(\rho, \sigma), (\mu, \lambda)\} = \{(n - c, 0), (n, 1)\}$.

(b) For decays in time, $\{(\rho,\sigma),(\mu,\lambda)\} = \{(0,n/2-c),(n,n/2)\}$ for even n. For odd n, $\{(\rho,\sigma),(\mu,\lambda)\} = \{(0,n/2-c),(n,\frac{n}{2}+\frac{1}{n+1})\}$.

(c) For decays in space-time. For $n \geq 3$, $\{(\rho,\sigma),(\mu,\lambda)\} = \{(n-2k-c,k-c),(n,k)\}$ for $k \in \mathbf{N}$ with $k < n/2$. For $n = 2$, $\{(\rho,\sigma),(\mu,\lambda)\} = \{(2-2\delta,\delta),(2,1)\}$ for $\delta \in [1/2,3/4)$.

(II)(a) For decays in space, $\{(\rho',\sigma'),(\mu,\lambda)\} = \{(n+1-c,0),(n+1,1)\}$.

(b) For decays in time, $\{(\rho',\sigma'),(\mu,\lambda)\} = \{(0,\frac{n+1}{2}-c),(n,\frac{n+1}{2})\}$ for odd n and $\{(0,\frac{n+1}{2}-c),(n,\frac{n+1}{2}+\frac{2}{n+2})\}$ for even n.

(c) For decays in space-time, $\{(\rho',\sigma'),(\mu,\lambda)\} = \{(n+1-2k-c,k-c),(n,k)\}$ for $k \in \mathbf{N}$ with $k < \frac{n+1}{2}$.

(III) Let $\ell = 1,2,\cdots,n-1$. If $n-\ell$ is even, $\{(\rho'',\sigma''),(\mu,\lambda)\} = \{(n-\ell-c,\frac{n+\ell}{2}-c),(n,\frac{n+\ell}{4})\}$. If $n-\ell$ is odd, $\{(\rho'',\sigma''),(\mu,\lambda)\} = \{(n-\ell-c,\frac{n+\ell}{2}-c),(n,\frac{n+\ell+1}{4})\}$.

All estimates are almost optimal except (I)(b) for odd n, (II)(b) for even n and (III) for case $n-\ell$ is odd and $\lambda = (n+\ell+1)/4 > 1$, i.e., $(n,\ell) \neq (2,1)$.

The spatial decay in (II)(a) and the time decays in (III) are faster than those in (1.1). (I) and (II) yield the optimal decays even for the Stokes equations although the marginal cases $\rho + 2\sigma = n$ and $\rho' + 2\sigma' = n+1$ are not permitted. (III) also yields the optimal decays for (NS).

Since we consider the external forces in Lebesgue spaces and assume no regularities on them, the assumptions on properties at the infinity of the solutions such as Knightly's and Mizumachi's seem be too strong for such external forces. Functions in (3) may blowup at the infinity although they decay almost everywhere . Moreover external forces may not decay everywhere with prescribed decay rates to obtain (I)-(III). We note that if $u(x,t) = o(|x|^{-\rho}t^{-\sigma})$ with $\rho + 2\sigma = 1$, the solution can belong to (3), but decay rates in (I) can be much higher. There seem no literatures studying decay rates in the class (3) although the regularities there have been much studied in the whole space, in bounded domains, in the interior and up to the boundary of domains.

We first consider the linearized equations, so called the Stokes equations, which turns to the heat equations in $\mathbf{R}^n$ by applying the Helmholtz decomposition. We multiply a growing function $M(x)$ (resp. $H(t)$) to the equations and obtain the a priori estimates of the weighted solution Mu (resp. Hu). The embedding inequality in space-time

yields us the estimates in space $L^s(0, \infty; L^q(\mathbf{R}^n)^n)$. The Gagliado-Nirenberg inequality of parabolic type, which is proved by M.Yamazaki, gives the uniform bound of Mu (resp. Hu).

Since we can choose weighting functions $M(x)$ (resp. $H(t)$) as $|x|^\gamma$ (resp. t^δ) $(0 < \gamma, \delta < 1)$ far from the origin, we obtain the decay rates of the solution by applying this operation several times.

To estimate (NS) we treat the nonlinear terms $u \cdot \nabla u$ as external forces. Under the smallness condition $||u||_{q,s} \ll 1$ in (3), the nonlinear term is estimated by the parabolic derivatives, i.e., the time and spatial second derivatives. Hence we obtain the same estimates of Mu (resp. Hu) as those for the Stokes equations (This is the fundamental reason why weak solution u is regular if $||u||_{q,s} \ll 1$).

The basic idea of weighted equation approach   For instance, we first multiply a weighting function in space to the heat equation. Next we apply a priori estimates to the weighted solution and estimate the error terms, that is, the remainders of the weighted Laplacian, with the Hölder and the Sobolev inequalities.

Such an approach is applied in Takahashi [13], [14] for local regularity criteria for the Navier-Stokes system, in the interior and up to the boundary of domains, respectively. In local regularity problems, localizing functions, in other words, compactly supported (in space-time) weighting functions, $\varphi(x, t)$ are p-th integrable with all derivatives. To estimate the remainders of weighted Laplacian, Hölder's inequalities and Sobolev's inequalities make estimates worse and better, respectively, in bounded domains.

On the other hand, in decay property problems, weighting functions $M(x)$ must be growing functions with decaying derivatives. Thus the growing rates must be less than 1. Only the derivatives are p-th integrable. Hölder's and Sobolev's inequalities make estimates better and worse, respectively. Thus the situation is opposite to the interior regularity problems. But the weighted equation approach works on the decay properties in the same framework.

The above approach can be also available to the stationary problem. We can obtain the same estimates and decay rates. But one would expect faster decay *in space* on the instationary problem, if external forces decay *in time* fast. In fact, the embedding

inequalities of the form $||u||_{\alpha,\beta} + ||\nabla u||_{\alpha',\beta'} \leq C(||\partial_t u||_{q,s} + ||\nabla^2 u||_{q,s})$, instead of the Sobolev inequalities, enable us to obtain higher decay rates in the instationary case. By these inequalities, weighting in time is also applicable in parallel. Such an approach with the embedding inequality is applied in [14, Erratum] for the regularity up to the boundary.

To get spatial decay rates higher than $n - 2$ for the instationary Stokes system, we remove the pressure term from (NS) by the Helmholtz decomposition projector P. *The weighted equation approach is based on the commutation between the weights and the derivatives.* Now how does the weight $M(x)$ commute with P? We thus consider the commutator $(PM - MP)u \cdot \nabla u$. We will show the estimates of $(PM - MP)v$ in L^r for the spatial decays of u in (I). But moreover we need the estimates of $(PM - MP)\nabla v$ in L^r for the spatial decays of the gradient ∇u for $\rho' \geq n$ in (II).

Decays in space-time $|u(x,t)| \leq C(1 + |x|)^{-\rho}(1 + t)^{-\sigma}$ are directly given by spatial decays $|u(x,t)| \leq C(1+|x|)^{-a\rho}$ and time decays $|u(x,t)| \leq C(1+t)^{-b\sigma}$ with $1/a+1/b = 1$ by Young's inequality. But the optimality of estimates is lost in this way. Hence we adopt weighting functions in *space-time*. We cannot choose a growing functions of the form $\varphi(x,t)$ in general, since the partial derivatives are not integrable in space-time. We will estimate the weighted solution HMu for growing functions $M(x)$ and $H(t)$, although $\nabla(HM)$ and $\partial_t(HM)$ are not integrable.

Finally, our weighted equation approach does not lose the *probable* decay rates of solutions of the original equations. Namely, the weighted solutions are estimated by the weighted external forces and the estimates of weighted external forces are not worse than those of the original external force.

References

[1] Knightly G.H., A Cauchy problem for the Navier-Stokes equations in $\mathbf{R}^n$, *SIAM J. Anal.* **3**, 506-511 (1972).

[2] Knightly G.H., On a class of global solutions of the Navier-Stokes equations, *Arch. Rat. Mech. Anal.* **21**, 211-245 (1966).

[3] Knightly G.H., Some decay properties of solutions of the Navier-Stokes equations, *Lecture Notes in Mathematics No.771*, 287-298 (1980).

[4] Mizumachi R., On the asymptotic behavior of incompressible viscous fluid motions past bodies, *J. Math. Soc. Japan* **36**, 497-522 (1984).

[5] Heywood J.G., The Navier-Stokes equations: On the existence, regularity and decay of solutions, *Indiana Univ. Math. J.* **29**, 639-681 (1980).

[6] Masuda K., On the stability of incompressible viscous fluid motions past objects, *J. Math. Soc. Japan* **27**, 294-327 (1975).

[7] Borchers W. & Miyakawa T., On stability of exterior stationary Navier-Stokes flows, *Acta Math.* **174**, 311-382 (1995).

[8] Kozono H. & Ogawa T., Some L^p estimates for the exterior Stokes flow and an application to the non-stationary Navier-Stokes equations, *Indiana Univ. Math. J.* **41**, 789-808 (1992).

[9] Kozono H. & Ogawa T., Global strong solution and its decay properties for the Navier-Stokes equations in three dimensional domains with non-compact boundaries, *Math. Z.* **216**, 1-30 (1994).

[10] Serrin J., On the interior regularity of weak solutions of the Navier-Stokes equations, *Arch. Rational Mech. Anal.* **9**, 187-195 (1962).

[11] Giga Y., Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, *J. Differential Equations* **62**, 186-212 (1986).

[12] Struwe M., On partial regularity results for the Navier-Stokes equations, *Comm. Pure Appl. Math.* **41**, 437-458 (1988).

[13] Takahashi S., On interior regularity criteria for weak solutions of the Navier-Stokes equations, *Manuscripta Math.* **69**, 237-254 (1990).

[14] Takahashi S., On a regularity criterion up to the boundary for weak solutions, *Commun. Partial Differential Equations* **17**, 261-285 (1992) (Erratum, ibid. **19**, 1015-1017) (1994).

GLOBAL SOLUTIONS FOR WAVE EQUATIONS WITH NON-COERCIVE CRITICAL NONLINEARITY

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In this talk we consider unique global existence and asymptotic behaviour of solutions for the semilinear wave equation

(1)
$$\begin{cases} \Box u + f(u) = 0 \quad for \ (t,x) \in \mathbb{R}^{1+n}, n \geq 3, \\ u(0) = \varphi, \quad \dot{u}(0) = \psi, \end{cases}$$

for arbitrary initial data with (locally) finite energy. Assume that f satisfies

(2)
$$^\exists C_F > 0, \quad F(u) := 2 \int_0^u f(v)dv \geq -C_F |u|^2,$$

$$f(0) = 0, \quad |f(u) - f(v)| \leq A(|u| + |v|)|u - v|,$$

$$with \quad A(r) \leq C(1 + r^{2^*-2}) \quad for \ ^\exists C > 0,$$

where $2^* = 2n/(n-2)$.

As L. Kapitanski pointed out in [5, pp. 220, Remark], the known results about the unique global existence under the above assumption is limited to two separate classes of f's. The first class consists of the functions which are smaller than the critical power, namely those satisfying (2) with

(3)
$$\lim_{r \to \infty} \frac{A(r)}{r^{2^*-2}} = 0.$$

See, e.g., [3]. The second class consists of those which satisfy certain coerciveness assumptions, such as

(4)
$$\liminf_{|u| \to \infty} \frac{G(u)}{|u|^{2^*}} > 0,$$

where $G(u) := uf(u) - F(u)$. See [2, 8, 5]. Remark that the conditions (3) and (4) contradict each other. The situation about the asymptotic behaviour is quite similar. The existence of bijective wave operators is known for the

nonlinearities which are smaller than the critical power [4], and for the non-linearities which are coercive against the critical power [1].

We extend these known results for a natural class of nonlinearities which contains these two separate classes. Put $m = 2(n+1)/(n-2)$.

Theorem 1. *Assume (2), and that*

$$(5) \qquad \liminf_{|u| \to \infty} \frac{G(u)}{|u|^{2^*}} \geqq 0.$$

Then for any $(\varphi, \psi) \in H^1_{loc} \oplus L^2_{loc}$, (1) has a unique global solution u satisfying

$$(u, \dot{u}) \in C(\mathbb{R}; H^1_{loc} \oplus L^2_{loc}) \text{ and } u \in L^m_{loc}(\mathbb{R}^{1+n}).$$

Theorem 2. *Assume (2), (5) and $n \geq 4$. Let u be a solution of (1) satisfying*

$$(6) \qquad \begin{aligned} (u, \dot{u}) &\in C_w(\mathbb{R}; H^1 \oplus L^2), \\ u(t) &\text{ is spherically symmetric for any } t, \end{aligned}$$

where 'C_w' denotes the space of weak continuous functions. Then, this solution u also satisfies

$$(u, \dot{u}) \in C(\mathbb{R}; H^1 \oplus L^2) \text{ and } u \in L^m_{loc}(\mathbb{R}; L^m).$$

As a result, the uniqueness holds in the class (6).

Theorem 3. *Assume (2) with $A(r) \leq Cr^{2^*-2}$, and that $G \geq 0$. Then for any $(\varphi, \psi) \in \dot{H}^1 \oplus L^2$, there uniquely exists $(\varphi_+, \psi_+) \in \dot{H}^1 \oplus L^2$ such that*

$$(7) \qquad \lim_{t \to \infty} \left\{ \|\nabla(u(t) - u_+(t))\|_{L^2} + \|\dot{u}(t) - \dot{u}_+(t)\|_{L^2} \right\} = 0,$$

where u is the solution of (1) given by Theorem 1, and u_+ is the solution for the linear wave equation

$$(8) \qquad \begin{cases} \Box u_+ = 0, \\ u_+(0) = \varphi_+, \quad \dot{u}_+(0) = \psi_+. \end{cases}$$

Moreover, $u \in L^m(\mathbb{R}^{1+n})$. Here, $\dot{H}^1 = \{\varphi \in L^{2^*} \mid \nabla\varphi \in L^2\}$. In addition, assume that f satisfies

$$|f'(v) - f'(w)| \leq C|v - w|^{2^*-2}$$

if $n \geq 7$. Then, for $n \geq 3$, the above correspondence $(\varphi, \psi) \mapsto (\varphi_+, \psi_+)$ is bijective. In other words, we have the wave operators and the scattering operator, which are bijective from $\dot{H}^1 \oplus L^2$ onto itself.

Remark. Theorem 1 is a generalization of [5, Theorem 7] and [8]. The strict coerciveness (4) was assumed in [5]. The proof in [8] is valid under another (stronger) coerciveness assumption that

$$(9) \qquad \liminf_{|u| \to \infty} \frac{F(u)}{|u|^{2^*}} > 0 \quad and \quad \liminf_{|u| \to \infty} \frac{H(u)}{|u|^{2^*}} \geq 0,$$

where we put $H := (n-1)G/2 - F$. Theorem 2 is an improvement of [2, Proposition 4.3(1)], except the case $n = 3$. Their proof is valid under the assumption (4), though it is also valid for $n = 3$. Theorem 3 is a generalization of [4, Proposition 4.2] and [1, Corollary 2.5]. The proof in [1] is valid under the coerciveness assumption that for some $C > 0$, $F(z) \geq C|z|^{2^*}$ and $H(z) \geq 0$. The assumptions in [4] are as follows: $n \geq 4$, and there exist $C_1, C_2 > 0$, $p_1, p_2 \geq 2$ and $0 < p_3 < 2^* - 2 < p_4$ such that

$$G(u) \geq C_1 \min(|u|^{p_1}, |u|^{p_2}), \quad and \ (2) \ with \ A(r) \leq C_2 \min(r^{p_3}, r^{p_4}).$$

To explain the idea of the proofs of our theorems, we prepare some notations.

$$r = |x|, \quad \theta = \frac{x}{r}, \quad \lambda = \sqrt{t^2 + r^2}, \quad \omega = \frac{(-t, x)}{\lambda}, \quad \nabla_y = (\partial_t, \nabla),$$

$$u_r = \theta \cdot \nabla u, \quad u_\theta = \nabla u - \theta \, u_r, \quad u_\omega = \nabla_y u - \omega(\omega \cdot \nabla_y u).$$

In the critical case, it is the most important to estimate the quantity $|u|^{2^*}$, and in the previous papers [2, 8, 5, 1], to this end, certain Morawetz-type estimates were used. For example, multiplying the equation by $u_r + u\nabla \cdot \theta/2$ and integrating over $W = [S, T] \times \mathbb{R}^n$, one obtains the radial Morawetz estimate [9, (2.27)]:

$$(10) \qquad \int_W \frac{n\, G(u)}{2r} + R_1(u) \, dx \, dt \leq C < \infty,$$

where

$$R_1(u) = \frac{|u_\theta|^2}{r} + \frac{(n-1)(n-3)}{4} \frac{|u|^2}{r^3} \geq 0,$$

and C depends on only the energy and the time length. Similarly, multiplying the equation by $t\dot{u} + ru_r + (n-1)u/2$ and integrating over the light cone $K = \{\, y \mid |x| < |t| \,\}$, one obtains the (localized) dilation estimate [7, (1.5)], from which one can deduce that

$$\int_{\substack{t=T \\ |x|<T}} F(u) + R_2(u) \, dx + \int_{\substack{|x|<T \\ 0<t<T}} \frac{2H(u)}{T} \, dx \, dt \to 0 \qquad as \ T \searrow 0,$$

where

$$R_2(u) = |u_\theta|^2 + \frac{t-r}{t}|u_r|^2 + \frac{(n-1)^2}{4} \frac{|u|^2}{tr} \geq 0.$$

In the preceding works, such methods were used to estimate $G(u)$ and/or $F(u)$, and then the quantity $|u|^{2^*}$ were estimated by $G(u)$ and/or $F(u)$, so that the coerciveness assumptions (4) or (9) were indispensable. To overcome this difficulty, we notice in the Morawetz-type estimates those terms independent of f (in the above cases, $R_1(u)$ and $R_2(u)$), and by using certain Hardy-type inequalities, we prove that these terms can dominate the quantity $|u|^{2^*}$ with the aid of the energy; for example, we have

$$\int_{|x|<|t|} |u|^{2^*} dx \leq C\|\nabla u(t)\|_{L^2_x}^{2^*-2} \int_{|x|<|t|} R_2(u) \, dx.$$

However, the radial Morawetz estimate (10) is inadequate for such treatment in the case $n = 3$ because the coefficient of u^2/r^3 vanishes. To avoid this difficulty, we use a new multiplier: $\omega \cdot \nabla_y u + u \nabla_y \cdot \omega/2$. Multiplying the equation by this quantity and integrating over W, we obtain a new variant of the Morawetz estimate:

$$\int_W \frac{|u_\omega|^2 + 81G(u)}{\lambda} \, dx \, dt \leq C < \infty,$$

where C is dependent on only the energy and the time length. Using the following Sobolev-type inequality:

$$\int_W \frac{|u|^{2^*}}{\lambda} \, dx \, dt \leq C \|\nabla u\|_{L_t^\infty(L_x^2)}^{2^*-2} \int_W \frac{|u_\omega|^2}{\lambda} \, dx \, dt + C \int_{t=T} |u|^{2^*} dx,$$

we obtain under the assumption (5),

$$\int_W \frac{|u|^{2^*}}{\lambda} \, dx \, dt \leq C.$$

This estimate is a kind of decay estimate of the quantity $|u|^{2^*}$ at the tip of the cone K or at the space-time infinity, and this estimate is the starting point of the proofs of Theorems 1 and 3. Unfortunately, this new estimate seems insufficient to prove Theorem 2, which is the reason why we have to assume $n \geq 4$ in Theorem 2.

Detailed proofs of the above theorems are given in [6].

REFERENCES

1. H. Bahouri and P. Gérard, *High frequency approximation of solutions to critical non-linear wave equations*, Prépublications 97-34, Université de Paris-Sud , Mai 1997.
2. J. Ginibre, A. Soffer, and G. Velo, *The global Cauchy problem for the critical non-linear wave equation*, J. Funct. Anal. **110** (1992), 96–130.
3. J. Ginibre, and G. Velo, *The global Cauchy problem for the non-linear Klein-Gordon equation, II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **6** (1989), 15–35.
4. J. Ginibre, and G. Velo, *Scattering theory in the energy space for a class of non-linear wave equations*, Commun. Math. Phys. **123** (1989), 535–573.
5. L. Kapitanski, *Global and unique weak solutions of nonlinear wave equations*, Math. Res. Let. **1** (1994), 211–223.
6. K. Nakanishi, *Unique global existence and asymptotic behaviour of solutions for wave equations with non-coercive critical nonlinearity*, preprint.
7. J. Shatah and M. Struwe, *Regularity results for nonlinear wave equations*, Ann. of Math. **138** (1993), 503–518.
8. J. Shatah and M. Struwe, *Well-posedness in the energy space for semilinear wave equations with critical growth*, IMRN (1994), no. 7, 303–309.
9. W. Strauss, *Nonlinear wave equations*, CBMS Lecture Notes, vol. 73, Amer. Math. Soc. Providence, RI, 1989.

SELF-INTERSECTION OF CURVES DRIVEN BY SURFACE DIFFUSION LAW

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ABSTRACT: We give a rigorous proof for formation of self-intersection of evolving curves driven by surface diffusion.

1. INTRODUCTION

This is a joint work with Professor Yoshikazu Giga.

We study motion by surface diffusion which was first derived by Mullins [10]. Let $\Gamma_t \subset \mathbf{R}^2$ be a closed evolving curve depending on time t with initial data $\Gamma_t|_{t=0} = \Gamma_0$. The governing equation for evolving curves by surface diffusion is of the form

$$V = -\kappa_{ss}. \tag{1}$$

Here V denotes the outward normal velocity and κ denotes the outward curvature; s denotes the arclength parameter of Γ_t. There are several derivations of this equation other than Mullins [10]. See for example Cahn and Taylor [2] and Cahn, Elliott and Novick-Cohen [3]. In the latter paper, (1) is obtained as some formal limit of Cahn-Hilliard equations. A typical feature of Γ_t moved by (1) is that the area enclosed by Γ_t is preserved. Related equations to (1) are well explained in Elliott and Garcke [4] and Cahn and Taylor [2]. For physical background of these equations, see [2, 4] and references cited there.

In [4] local existence of solution for (1) was proved without uniqueness as well as for other equations. They proved that if initial data is close to a circle, then Γ_t exists globally in time and it converges to a circle with the same area enclosed by Γ_0 as t tends to infinity. They also conjectured that Γ_t moved by (1) may cease to be embedded for some embedded smooth initial data. After this work was completed, we were informed of a recent work of Escher, Mayer and Simonett [5] on unique local existence of solutions of (1). They proved the unique existence of local-in-time solutions even for higher dimensional version of (1) in small Hölder spaces by appealing abstract semigroup theory. They also studied the large time behavior of solutions of the higher dimensional version of (1) if initial data are close to a sphere. These results are regarded as a natural extension of the results of [4]

to higher dimensional setting. Moreover, they showed numerical evidence of existence of pinchings for various closed curves.

In this talk we give a rigorous proof to Elliott and Garcke's conjecture. Let us explain our idea. We consider a smooth closed embedded curve Γ_0 which is symmetric with respect to x-axis and y-axis. We assume that Γ_0 is of the form

$$\Gamma_0 = \{(x, y); y = \pm u_0(x)\},$$

where $u_0(x)$ is even and $u_0(x)$ takes the only local minimum at $x = 0$. If Γ_t is represented by $y = u(t, x)$, then (1) becomes a fourth order equation of $u(t, x)$. If we linearize (1) around $u = 0$, we obtain

$$u_t = -u_{xxxx}.$$

If we consider the Cauchy problem for this equation with $u_0(x) \geq 0$ and $u_0(x) = x^4 + \delta$ for small $\delta > 0$ near $x = 0$, then $u(t, 0)$ would be negative in a short time. In other words, the comparison principle does not hold. It is easy to guess this phenomenon since $u(t, x) = x^4 - 4!t + \delta$ solves $u_t = -u_{xxxx}$. In this talk we shall rigorously prove that for a good choice of $u_0(x)$, $u(t, 0)$ becomes negative in short time during the period that solution Γ_t: $y = u(t, x)$ of (1) exists as immersed curves. Since Γ_t is represented by $y = u(t, x)$, and symmetric with respect to $y = 0$, this means that Γ_t ceases to be embedded in short time even if Γ_0 is embedded. This is a rough idea of our proof. For this purpose we shall review unique local existence theorem for immersed curves with estimates of existence time interval as well as of solutions. Of course there are several versions of unique existence theorems which apply in this setting e.g. by Lunardi [9] but we presented a simple version with the aid of the classical Lax-Milgram type abstract existence theorem due to J. L. Lions [7] (see e.g. [8, 11]) which just needs Hilbert spaces; we do not use sophisticated abstract semigroup theory (e.g. [9]). Since we proved the uniqueness of solutions as well as higher derivative estimate near $t = 0$, our unique local existence theorem is not included in Elliott and Garcke [4] which is based on the method of X. Chen [1].

In the next section, we introduce a parametrized equation for (1). After showing a unique local existence theorem for this equation, we rigorously prove formation of self-intersection of evolving curves moved by surface diffusion.

2. SELF-INTERSECTION OF EVOLVING CLOSED CURVES

2.1. PARAMETRIZATION

We summarize here a parametrization of (1) by following Elliott and Garcke [4].

Let M^0 be a fixed reference C^∞ (or at least C^5) immersed closed curve with arclength $2L$. For $\mathbf{T} = \mathbf{R}/(2L\mathbf{Z})$, let

$$X^0: \quad \mathbf{T} \to M^0,$$
$$\eta \mapsto X^0(\eta)$$

be an arclength parametrization of M^0. Then, $\tau^0(\eta) = X^0_\eta(\eta)$ is the unit tangent vector of M^0 and the Frenet formula gives

$$\tau^0_\eta(\eta) = \kappa^0(\eta)n^0(\eta),$$
$$n^0_\eta(\eta) = -\kappa^0(\eta)\tau^0(\eta),$$

where $n^0(\eta)$ is the unit normal vector and $\kappa^0(\eta)$ is the curvature of M^0 with the sign convention that the curvature of a circle is negative.

Let $\Gamma_t \subset \mathbf{R}^2$ be a closed curve moved by surface diffusion law with respect to time $t \geq 0$ starting from initial closed curve Γ_0. For small $T > 0$ we expect that Γ_t is parametrized by

$$
\begin{aligned}
X : \quad [0,T) \times \mathbf{T} &\rightarrow \Gamma_t, \\
(t,\eta) &\mapsto X(t,\eta), \\
X(t,\eta) = X^0(\eta) &+ d(t,\eta)n^0(\eta)
\end{aligned}
$$

with some $d(t,\eta)$ defined on $[0,T) \times \mathbf{T}$. If Γ_0 is embedded and Γ_t is close to Γ_0, then $d(t,\eta)$ is the distance function from M^0. By this parametrization, (1) is equivalent to

$$
\frac{1 - d\kappa^0}{J} d_t = -\frac{1}{J}\partial_\eta\left(\frac{1}{J}\partial_\eta\kappa\right),
$$

where $J = |X_\eta| = \partial s/\partial\eta$ is the Jacobian and $\kappa(t,\eta)$ is the curvature of Γ_t in the direction of n^0. Their explicit forms are

$$
J = J(\eta,\alpha_0,\alpha_1)|_{(\alpha_0,\alpha_1)=(d,d_\eta)} = (d_\eta^2 + (1 - d\kappa^0)^2)^{1/2},
$$

$$
\kappa = \frac{1}{J^3}\{(1 - d\kappa^0)d_{\eta\eta} + 2\kappa^0 d_\eta^2 + \kappa_\eta^0 dd_\eta + \kappa^0(1 - d\kappa^0)^2\}.
$$

Thus, the equation (1) for $d(t,\eta)$ with initial data $\Gamma_t|_{t=0} = \Gamma_0$ is of the form:

$$
\begin{cases}
d_t + J^{-4}d_{\eta\eta\eta\eta} + Pd_{\eta\eta} + Q = 0, \ 0 < t < T, \ \eta \in \mathbf{T}, \\
d(0,\eta) = d_0(\eta), \ \eta \in \mathbf{T},
\end{cases}
\tag{2}
$$

where P and Q are polynomials with arguments $(1 - \kappa^0 d)^{-1}$, J^{-1}, κ^0, κ_η^0, $\kappa_{\eta\eta}^0$, $\kappa_{\eta\eta\eta}^0$, d, d_η and $d_{\eta\eta}$. We note that κ^0 toghther with its derivatives κ_η^0, $\kappa_{\eta\eta}^0$, $\kappa_{\eta\eta\eta}^0$ is continuous and bounded on $\mathbf{T}$ since M^0 is at least C^5.

2.2. LOCAL EXISTENCE

We state here a result of the unique local existence of smooth solutions of (2). To do this, we first treat a general framework.

We consider the equation:

$$
\begin{cases}
u_t + a(x,u,u_x)u_{xxxx} + b(x,u,u_x,u_{xx})u_{xxx} + c(x,u,u_x,u_{xx}) = 0, \\
u(0,x) = u_0(x),
\end{cases}
\tag{3}
$$

for $t > 0$ and $x \in \mathbf{T} = \mathbf{R}/(\omega\mathbf{Z})$ with $\omega > 0$. For (3), we assume:

(a) There are positive constants a_1 and Λ such that $a(x,\alpha_0,\alpha_1) \geq a_1$ for $|\alpha_0|$, $|\alpha_1| \leq \Lambda$.

(b) Let $M > 0$ be given. The functions $a(x,\alpha_0,\alpha_1)$, $b(x,\alpha_0,\alpha_1,\alpha_2)$ and $c(x,\alpha_0,\alpha_1,\alpha_2)$ are smooth in their all arguments but restricted for $|\alpha_0| \leq 2\mu M$ and ω-periodic in x, where $\mu = \mu(\mathbf{T}) > 0$ denotes the constant in the Sobolev inequality:

$$
\|f\|_{L^\infty(\mathbf{T})} \leq \mu\|f\|_{H^1(\mathbf{T})} \quad \text{for } f \in H^1(\mathbf{T}).
\tag{4}
$$

Theorem 1 *(Local existence for (3)).* *Let $M > 0$. Assume (a)-(b). Then, for any $u_0 \in H^4(\mathbf{T})$ with $\|u_0\|_{H^4(\mathbf{T})} \le M$, there is a $T_0(M) > 0$ such that there exists a unique solution $u(t,x)$ of (3) satisfying*

$$u \in L^2(0, T_0(M); H^6(\mathbf{T})), \quad u_t \in L^2(0, T_0(M); H^2(\mathbf{T})), \tag{5}$$

$$\|u\|_{H^4(\mathbf{T})}(t) \le 2M \quad \text{for } t \in [0, T_0(M)]. \tag{6}$$

Corollary 2 *Let $m \ge 4$ be integers and let $N \in (0, M]$. Then, for any $u_0 \in H^m(\mathbf{T})$ with $\|u_0\|_{H^m(\mathbf{T})} \le N$, there is a $T_1(N) > 0$ such that there exists a unique solution $u(t,x)$ of (3) satisfying*

$$u \in L^2(0, T_1(N); H^{m+2}(\mathbf{T})), \quad u_t \in L^2(0, T_1(N); H^{m-2}(\mathbf{T})), \tag{7}$$

$$\|u\|_{H^m(\mathbf{T})}(t) \le 2N \quad \text{for } t \in [0, T_1(N)]. \tag{8}$$

For the proofs of Theorem 1 and Corollary 2, see [6]. From Theorem 1 and Corollary 2, we obtain

Theorem 3 *(Local existence for (2)).* *Let ω and δ_0 be as*

$$\omega = 2L, \quad 4\delta_0 \|\kappa^0\|_{L^\infty(\mathbf{T})} \le 1. \tag{9}$$

(i) Let $M \in (0, \delta_0/\mu)$ where μ is in (4). Then, for any $d_0 \in H^4(\mathbf{T})$ with $\|d_0\|_{H^4(\mathbf{T})} \le M$, there is a $T_0(M) > 0$, which is nonincreasing in M, such that there exists a unique solution $d(t, \eta)$ of (2) satisfying

$$d \in L^2(0, T_0(M); H^6(\mathbf{T})), \quad d_t \in L^2(0, T_0(M); H^2(\mathbf{T})), \tag{10}$$

$$\|d\|_{H^4(\mathbf{T})}(t) \le 2M \quad \text{for } t \in [0, T_0(M)]. \tag{11}$$

(ii) Let $m \ge 4$ be integers and $N \in (0, \delta_0/\mu)$. Then, for any $d_0 \in H^m(\mathbf{T})$ with $\|d_0\|_{H^m(\mathbf{T})} \le N$, there is a $T_1(N) > 0$, which is nonincreasing in N, such that there exists a unique solution $d(t, \eta)$ of (2) satisfying

$$d \in L^2(0, T_1(N); H^{m+2}(\mathbf{T})), \quad d_t \in L^2(0, T_1(N); H^{m-2}(\mathbf{T})), \tag{12}$$

$$\|d\|_{H^m(\mathbf{T})}(t) \le 2N \quad \text{for } t \in [0, T_1(N)]. \tag{13}$$

Remark 1. Our local existence theorem in particular implies that for any immersed smooth curve Γ_0, there exists a unique local-in-time solution of (1) by taking Γ_0 as a reference curve M^0 with $d_0 = 0$.

Remark 2. It is easy to see from the above method of constructing of solutions that the curves Γ_t are uniquely determined by Γ_0 and are independent of parametrizations.

Remark 3. Since our main concern in this paper is pinching, we do not pursue higher regularity properties away from $t = 0$.

Proof of Theorem 3 admitting Theorem 1 and Corollary 2. It suffices to prove (b). For $|\alpha_0| \le 2\mu M$, $J(\eta, \alpha_0, \alpha_1)^{-4}$ has no singularities: in fact, it follows from (9) that

$$\alpha_1^2 + (1 - \alpha_0 \kappa^0(\eta))^2 \ge (1 - 2\mu M \|\kappa^0\|_{L^\infty(\mathbf{T})})^2$$

$$\ge (1 - 2\delta_0 \|\kappa^0\|_{L^\infty(\mathbf{T})})^2 \ge (\frac{1}{2})^2.$$

Similarly, we can show that P and Q have no singularities in the domain we are concerned. This gives (b). Applying Theorem 1 to (2), we obtain (i). The proof of (ii) is essentially with the same with M replaced by N, so we omit the proof of (ii). □

2.3. SELF-INTERSECTION OF EVOLVING CLOSED CURVES

We show that there is an evolving closed curve which ceases to be embedded in finite time, even if initial curve is embedded.

Let us explain our idea of the proof. Let $M^0 = \{X^0(\eta); \eta \in \mathbf{T} = \mathbf{R}/(2LZ)\}$ be a dumbbell like immeresed curve symmetric with respect to both x-axis and y-axis and its neck is so thin so that it is just a segment on the x-axis. It is normalized by setting $X^0(0) = X^0(L) =$ the origin $(0,0)$. Let $\Gamma_0 = \{X^0(\eta) + d_0(\eta)n^0(\eta); \eta \in \mathbf{T}\}$ with $d_0(\eta) > 0$ be symmetric with respect to both x-axis and y-axis and assume that $d_0(\eta)$ takes its global isolated minimun at $\eta = 0$ and L. Then, by symmetry of the equation (2), the solution $\Gamma_t = \{X^0(\eta) + d(t,\eta)n^0(\eta); \eta \in \mathbf{T}\}$ stays symmetric with respect to both x-axis and y-axis. In particular, $d_\eta(t,0) = 0$ and $d_{\eta\eta\eta}(t,0) = 0$. Thus if $d(t,\eta)$ solves (2), then

$$d_t(0,0) = -\partial_\eta^4 d(0,0) + 3\partial_\eta^2 d(0,0).$$

Thus, by the fundamental theorem of calculus,

$$
\begin{aligned}
d(t,0) &= d(0,0) + d_t(0,0)t + \int_0^t \int_0^\tau d_{ss}(s,0)dsd\tau \\
&\leq d(0,0) + (-\partial_\eta^4 d(0,0) + 3\partial_\eta^2 d(0,0))t + t^2 \cdot \sup_{t \in [0,\bar{t}], \eta \in \mathbf{T}} |d_{tt}(t,\eta)|,
\end{aligned}
\tag{14}
$$

where $\bar{t}$ is taken so that $d(t,\eta)$ exists for $[0,\bar{t}]$. Roughly speaking, if $d(0,0)$ is sufficiently small and $-\partial_\eta^4 d(0,0) + 3\partial_\eta^2 d(0,0) < 0$, then $d(t,0)$ may be nagative for t between two roots of the quadratic equation of t: the R.H.S. of (14)= 0, which will imply a pinching of Γ_t. We shall prove it more rigorously in the following.

To do this, we define a special (C^∞) reference curve M^0 (see Figure 1). This curve is
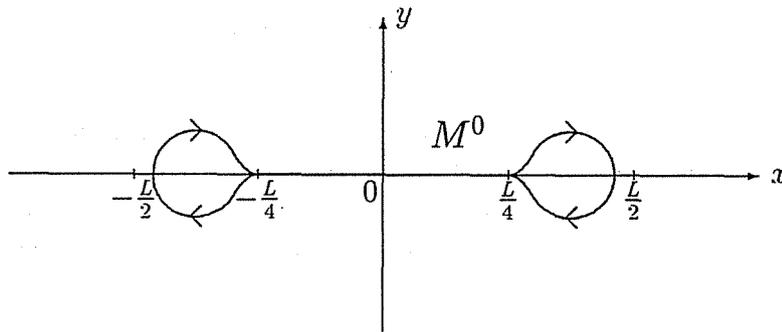

Figure 1: The reference curve M^0

immersed in $\mathbf{R}^2$. This is parametrized by

$$X^0(\eta) = (X_1^0(\eta), X_2^0(\eta)) \quad \text{for } \eta \in \mathbf{T} = \mathbf{R}/(2LZ)$$

satisfying

$$\begin{cases} X_1^0(\eta) = -X_1^0(-\eta), & 0 \le \eta \le L, \\ X_2^0(\eta) = X_2^0(-\eta), & 0 \le \eta \le L, \\ X^0(\eta) = (\eta, 0), & 0 \le \eta \le L/4, \\ (X_1^0)_\eta(\eta) > 0, & 0 \le \eta \le L/2, \\ X_1^0(L/2 + \eta) = X_1^0(L/2 - \eta), & 0 \le \eta \le L/2, \\ X_2^0(\eta) > 0, & L/4 < \eta < L/2, \\ X_2^0(L/2 + \eta) = -X_2^0(L/2 - \eta), & 0 \le \eta \le L/2, \end{cases}$$

where η is an arclength parameter. We define two sets of functions in $\mathbf{T}$ and in $[0, T] \times \mathbf{T}$ depending on positive parameters N, ε and T:

$$\begin{aligned} D_0(N, \varepsilon) &= \{d_0 \in H^9(\mathbf{T}); d_0(-\eta) = d_0(\eta) = d_0(L - \eta), \quad d_0(\eta) > 0 \quad (\forall \eta \in \mathbf{T}), \\ &\quad \|d_0\|_{H^9(\mathbf{T})} \le N, \quad d_0(0) \le \varepsilon, \quad -3d_0''(0)^3 + d_0^{(4)}(0) > 0, \\ &\quad d_0(\eta) \text{ attains its global minimum at } \eta = 0\}, \end{aligned}$$

$$\begin{aligned} D_T(N) &= \{d \in L^2(0, T; H^9(\mathbf{T})); d_t \in L^2(0, T; H^5(\mathbf{T})), \\ &\quad \|d\|_{H^9(\mathbf{T})}(t) \le 2N \quad (t \in [0, T])\}. \end{aligned}$$

Note that closed curves Γ_0 parametrized by $X(0, \eta) = X^0(\eta) + d_0(\eta) n^0(\eta)$ with $d_0 \in D_0(N, \varepsilon)$ are embedded in $\mathbf{R}^2$. Then, our main result is stated as follows.

Theorem 4 *(Self-intersection of evolving closed curves). For any $N \in (0, \delta_0/\mu)$, there is an $\varepsilon_0 > 0$; for any $\varepsilon \in (0, \varepsilon_0)$, there are $d_0 \in D_0(N, \varepsilon)$, $t_0 \in (0, T_1(N))$ (where $T_1(N)$ is in Theorem 3 (ii)) and $t_1(> t_0)$ such that for initial embedded closed curve Γ_0 with parametrization*

$$\Gamma_0 = \{X(0, \eta) = X^0(\eta) + d_0(\eta) n^0(\eta); \eta \in \mathbf{T}\},$$

the solution curve Γ_t with parametrization

$$\Gamma_t = \{X(t, \eta) = X^0(\eta) + d(t, \eta) n^0(\eta); \eta \in \mathbf{T}\}, \quad t \in [0, T_1(N)],$$

where $d \in D_{T_1(N)}(N)$ is the unique solution of (2) established in Theorem 3 (ii), ceases to be embedded for at least $t_0 < t < \min(t_1, T_1(N))$.

Proof. First step. We first show the procedure to choose an initial data $d_0(\eta)$ with which the solution Γ_t will be pinched in the existence time interval of solution Γ_t.

Take a $\tilde{d}_0 = \tilde{d}_0 \in D_0(N, \tilde{d}_0(0))$. For this $\tilde{d}_0$, Theorem 3 (ii) implies that there are $T_1(N) > 0$ and a unique solution $\tilde{d} \in D_{T_1(N)}(N)$ of (2) with initial data $d_0(\eta)$. Then, there is a $K(N) > 0$ such that

$$|-\partial_t \partial_\eta^4 \tilde{d}(t, 0) + 3\partial_t(\partial_\eta^2 \tilde{d}(t, 0))^3| \le K(N) \quad \text{for } t \in [0, T_1(N)].$$

Put

$$\sigma(\tilde{d}_0) = -3\tilde{d}_0''(0)^3 + \tilde{d}_0^{(4)}(0) > 0.$$

Take $\varepsilon_0 > 0$ as

$$\sigma(\tilde{d}_0)^2 - 4\varepsilon_0 K(N) > 0,$$

$$\frac{\sigma(\tilde{d}_0) - \sqrt{\sigma(\tilde{d}_0)^2 - 4\varepsilon_0 K(N)}}{2K(N)} < T_1(N).$$

Then, it holds

$$\sigma(\tilde{d}_0)^2 - 4\varepsilon K(N) > 0,$$

$$\frac{\sigma(\tilde{d}_0) - \sqrt{\sigma(\tilde{d}_0)^2 - 4\varepsilon K(N)}}{2K(N)} < T_1(N) \tag{15}$$

for any $\varepsilon \in (0, \varepsilon_0)$. Take θ as

$$\max\{0, \tilde{d}_0(0) - \frac{\sigma(\tilde{d}_0)^2}{4K(N)}, \tilde{d}_0(0) - \varepsilon\} < \theta < \tilde{d}_0(0).$$

Put $d_0(\eta) = \tilde{d}_0(\eta) - \theta$. Then, it is easy to check that $d_0 \in D_0(N, \varepsilon)$. Thus, it follows from Theorem 3 (ii) that there exists a unique solution $d \in D_{T_1(N)}(N)$ of (2) with initial data $d_0(\eta)$.

Second step. We shall show that for this $d_0(\eta)$, the solution Γ_t ceases to be embedded in finite time.

It follows from the uniqueness that

$$d_0(-\eta) = d_0(\eta) = d_0(L - \eta)$$

implies

$$d(t, -\eta) = d(t, \eta) = d(t, L - \eta). \tag{16}$$

Furthermore,

$$\sigma \equiv \sigma(d_0) = \sigma(\tilde{d}_0),$$

$$|-\partial_t \partial_\eta^4 d(t, 0) + 3\partial_t (\partial_\eta^2 d(t, 0))^3| \le K(N) \quad \text{for } t \in [0, T_1(N)].$$

Then, from (15),

$$t_0 \equiv \frac{\sigma - \sqrt{\sigma^2 - 4\varepsilon K(N)}}{2K(N)} < T_1(N)$$

for $0 < \varepsilon < \varepsilon_0$. Finally, from the first equality of (16), it follows that $d_\eta(t, 0) = d_{\eta\eta\eta}(t, 0) = 0$. Thus, using (2), we observe that

$$
\begin{aligned}
d(t, 0) &= d_0(0) + \int_0^t d_\tau(\tau, 0) d\tau \\
&= d_0(0) + \int_0^t (-\partial_\eta^4 d(\tau, 0) + 3(\partial_\eta^2 d(\tau, 0))^3) d\tau \\
&= d_0(0) - \sigma t + \int_0^t \int_0^\tau (-\partial_s \partial_\eta^4 d(s, 0) + 3\partial_s(\partial_\eta^2 d(s, 0))^3) ds d\tau \\
&\le \varepsilon - \sigma t + K(N) t^2,
\end{aligned}
$$

which and the second equality of (16) imply

$$d(t, 0) = d(t, L) < 0 \quad \text{for } t_0 < t < \min(t_1, T_1(N)),$$

where

$$t_1 \equiv \frac{\sigma + \sqrt{\sigma^2 - 4\varepsilon K(N)}}{2K(N)}.$$

This shows that Γ_t ceases to be embedded for $t_0 < t < \min(t_1, T_1(N))$. This completes the proof. $\qquad\square$

REFERENCES

[1] X. Chen, The Hele-Shaw problem and area-preserving curve-shortening motions, *Archive Rational Mech. Anal.*, **123** (1993), 117–151.

[2] J. W. Cahn and J. E. Taylor, Surface motion by surface diffusion, *Acta Metallurgica* **42** (1994), 1045–1063.

[3] J. W. Cahn, C. M. Elliott and A. Novick-Cohen, The Cahn-Hilliard equation: Surface motion by the Laplacian of the mean curvature, *Euro. J. Appl. Math.*, **7** (1996), 287–301.

[4] C. M. Elliott and H. Garcke, Existence results for diffusive surface motion laws, *Adv. Math. Sci. Appl.*, **7** (1997), 467–490.

[5] J. Escher, U. F. Mayer and G. Simonett, The surface diffusion flow for immersed hypersurfaces, preprint.

[6] Y. Giga and K. Ito, On pinching of curves moved by surface diffusion, preprint.

[7] J. L. Lions, *Équations différentielles opérationnelles et problèmes aux limites*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1961.

[8] J. L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Dunod, Gauthier-Villars, Paris, 1968.

[9] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser, 1995.

[10] W. W. Mullins, Theory of thermal grooving, *J. Appl. Phys.*, **28** (1957), 333–339.

[11] M. Renardy and R. C. Rogers, *An introduction to partial diiferential equations*, Springer-Verlag, 1993.