第５回 関数空間セミナー 報告集

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第５回　関数空間セミナー　報告集

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（会場：　東京理科大学 理学部)

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Topological property of an invariant set with respect to a family of functions II

Tomomi Hirasawa and Fukiko Takeo
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1 Introduction

For a family of contraction functions $\{f_1, f_2, \ldots, f_m\}$ $(m \geq 2)$ on $\mathbf{R}^{\mathbf{d}}$, there is an invariant set K satisfying the following

$$K = f_1(K) \cup \ldots \cup f_m(K).$$

When we consider the set $E^{(\omega)}$ of infinite sequences from $E = \{1, 2, \ldots, m\}$, there is a map ψ of $E^{(\omega)}$ onto K such that

$$\psi(x_1 x_2 \ldots) = \lim_{n \to \infty} f_{x_1} f_{x_2} \cdots f_{x_n}(K).$$

If ψ is not one to one, the equivalence relation $x \sim y$ is defined by $\psi(x) = \psi(y)$ and the quotient space $E^{(\omega)}/\sim$ is considered. We shall investigate the topological property of $E^{(\omega)}/\sim$ and give some results concerning the number of end points. In the last paper, we defined end points of type 1 by using a basis $\{\tilde{U}_n\}$. In this paper, we consider another basis in $E^{(\omega)}/\sim$ and define end points of type 2 by using it.

2 Preliminaries

- $E = \{1, 2, \ldots, m\}$ $(m \geq 2)$
- $E^{(\omega)}$: the set of infinite sequences from E
- $E^{(n)}$: the set of sequences from E of length n for $n \in \mathrm{N}$
- $E^{(0)}$: empty set
- $E^{(*)}$: the set of finite sequences from E, i.e. $E^{(*)} = \cup_{n=0}^{\infty} E^{(n)}$
- For $n \in \mathrm{N} \cup \{0\}$, let the map $P_n : E^{(\omega)} \to E^{(*)}$ be the projection such as

$$P_n x = x_1 x_2 \ldots x_n, \text{ where } x = x_1 x_2 \ldots \in E^{(\omega)}.$$

- For $s \in E$ and $x = x_1 x_2 \ldots,\ y = y_1 y_2 \ldots \in E^{(\omega)}$, let
  $sx = s x_1 x_2 \ldots$
  $(P_n x) y = x_1 x_2 \ldots x_n y_1 y_2 \ldots.$

- Let the map $\sigma : E^{(\omega)} \to E^{(\omega)}$ be a shift operator, i.e.

$$\sigma(x_1 x_2 \ldots) = x_2 x_3 \ldots.$$

- An equivalence relation $\sim$ on $E^{(\omega)}$ is called to be invariant if the following (1) and (2) are satisfied:
  (1) $x \sim y$ implies $sx \sim sy$ $(\forall s \in E)$
  (2) $sx \sim sy$ implies $x \sim y$ $(\forall s \in E)$.

- For $x \in E^{(\omega)}$, let Qx be the equivalence class of x,
  i.e. $Qx = \{y \in E^{(\omega)} | x \sim y\}$.

- $A := \{x \in E^{(\omega)} \,|\, \exists y \in Qx \text{ s.t. } P_1 x \neq P_1 y\}$

- $A_s := \{x \in A \,|\, P_1 x = s\}$

- $E_s := \{x \in E^{(\omega)} \,|\, P_1 x = s\}$

- Let the map $q : E^{(\omega)} \to E^{(\omega)}/\sim$ be the natural quotient map.

- $F_n := \{s \in E \,|\, \sharp(q(A_s)) = n\}$ where $\sharp(q(A_s))$ is the number of elements of $q(A_s)$.

Hereafter, we assume that the equivalence relation $\sim$ is invariant and $\sharp A < \infty$.

3 End points of type 1

Let $\tilde{U}_n(q(x))$ be the subset of the quotient space $E^{(\omega)}/\sim$ as follows:

$$\tilde{U}_n(q(x)) = \{q(y) \in E^{(\omega)}/\sim |P_n Qy \subset P_n Qx\}.$$

Proposition 1 *The family $\{\tilde{U}_n(q(x))|\ n \in \mathbf{N}, q(x) \in E^{(\omega)}/\sim\}$ is a fundamental neighborhood system for the quotient topology in $E^{(\omega)}/\sim$.*

Definition 1 *$q(x) \in E^{(\omega)}/\sim$ is called an end point of type 1 if there exists $N \in \mathbf{N}$ such that $\partial\tilde{U}_{n+1}(q(x))$ is a singleton for any $n \geq N$.*

Theorem 1 *1. The following (a) and (b) are equivalent.*

(a) $q(x) \in E^{(\omega)}/\sim$ is an end point of type 1.

(b) i. $Qx = \{x\}$ and

ii. There exists $N \in \mathbf{N}$ such that $n \geq N$ implies

$$x_n x_{n+1} \notin P_2 A,\ x_n \in F_1.$$

2. If $q(x)$ is an end point of type 1, then $q(\sigma x)$ is also an end point of type 1.

3. If $q(x)$ is an end point of type 1, then either $sx \in A$ or $q(sx)$ is an end point of type 1 for any $s \in E$.

Theorem 2 *The following are equivalent.*

1. *There exists an end point of type 1.*
2. $F_1 \neq \phi$ *and there exists* $\{s_1, s_2, \ldots, s_n\} \subset F_1$ $(n \geq 1)$ *such that*
$$s_j s_{j+1} \notin P_2 A \, (j = 1, 2, \ldots, n-1),\ s_n s_1 \notin P_2 A.$$

Theorem 3 *If* $\sharp A < \infty$, *then the number of end points of type 1 is 0, 1, 2 or infinity.*

4 End points of type 2

For a proper subset H of E, consider the following condition

$$q(\cup_{t\in H} E_t) \cap q(\cup_{t \in H^c} E_t) \quad \text{is a singleton.} \quad (*)$$

Let S_0 be the set consisting of $s \in E$ such that there exists the smallest set among those H which includes s and satisfies $(*)$.
For $s \in S_0$, let H_s be the smallest set and a_s be the element of $E^{(\omega)}$ satisfying

$$q(\cup_{t\in H_s} E_t) \cap q(\cup_{t \in H_s^c} E_t) = \{q(a_s)\}.$$

Let S_1 be the set $\{s \in S_0 \,|\, \text{the set } (P_1 Q a_s \cap H_s) \text{ is a singleton }\}$.
Let B_s and $W_n(q(x))$ be sets as follows:

$$B_s = \begin{cases} \{a \in A \,|\, P_1(Qa) \subset H_s\} & \text{if } s \in S_1 \\ \phi & \text{if } s \notin S_1 \end{cases}$$

$$W_n(q(x)) = \begin{cases} \cup\{\tilde{U}_n(q(wb)) \,|\, w = P_{n-1}x, b \in B_{x_n}\} & \text{if } Qx = \{x\} \text{ and } B_{x_n} \neq \phi \\ \tilde{U}_n(q(x)) & \text{otherwise.} \end{cases}$$

Proposition 2 *The family* $\{W_n(q(x)) | n \in \mathbf{N}, q(x) \in E^{(\omega)}/\sim\}$ *is a fundamental neighborhood system for the quotient topology in* $E^{(\omega)}/\sim$.

Definition 2 $q(x) \in E^{(\omega)}/\sim$ *is called an end point of type 2 if there exists* $N \in \mathbf{N}$ *such that* $\partial W_{n+1}(q(x))$ *is a singleton for any* $n \geq N$.

Theorem 4
1. *The following (a) and (b) are equivalent.*
   (a) $q(x) \in E^{(\omega)}/\sim$ *is an end point of type 2.*
   (b) i. $Qx = \{x\}$ *and*
   ii. *There exists* $N \in \mathbf{N}$ *such that* $n \geq N$ *implies*
   $$x_n H_{x_{n+1}} \cap P_2 A = \phi,\ x_n \in S_1.$$
2. *If* $q(x)$ *is an end point of type 2, then* $q(\sigma x)$ *is also an end point of type 2.*
3. *If* $q(x)$ *is an end point of type 2, then either* $sx \in A$ *or* $q(sx)$ *is an end point of type 2 for any* $s \in E$.

Theorem 5 *The following are equivalent.*

1. *There exists an end point of type 2.*

2. $S_1 \neq \phi$ *and there exists* $\{t^1, t^2, \ldots, t^n\} \subset S_1$ $(n \geq 1)$ *such that*

$$t^j H_{t^{j+1}} \cap P_2 A = \phi \, (j = 1, 2, \ldots, n-1), \; t^n H_{t^1} \cap P_2 A = \phi.$$

Definition 3 *Let* T *be the set*

$$\{(t^1, \ldots, t^k) \subset S_1 \,|\, k \geq 1, t^i \neq t^j (i \neq j), t^k H_{t^1} \cap P_2 A = \phi, t^j H_{t^{j+1}} \cap P_2 A = \phi \, (j = 1, 2, \ldots, k-1)\},$$

$EN(0) = \{\overline{t^1 \cdots t^k} \,|\, (t^1, \ldots, t^k) \in T\}$ *and*
$EN(1) = \{r\overline{t^1 \cdots t^k} \,|\, r \in E, (t^1, \ldots, t^k) \in T, r \neq t^k$ *and* $r\overline{t^1 \cdots t^k} \notin A\}$.
For $n \geq 2$, *let*
$EN(n) = \{re \,|\, r \in E, e \in EN(n-1),$ *and* $re \notin A\}$.

Proposition 3 1. *If* x *belongs to* $EN(j)$ $(j \geq 0)$, *then* $q(x)$ *is an end point of type 2.*

2. $EN(i) \cap EN(j) = \phi$ *folds for* $i \neq j$.

Lemma 1 *If* $q(x)$ *is an end point of type 2, then for any* $n \in \mathbf{N}$ *there exists* $y \in \cup_{j \geq 0} EN(j)$ *such that* $P_n x = P_n y$.

Theorem 6 *Let* EP *be the set of end points of type 2. Then the following holds:*

$$\cup_{j \geq 0} q(EN(j)) \subset EP \subset \overline{\cup_{j \geq 0} q(EN(j))}.$$

Proposition 4 *If the number of end points of type 2 is finite and* $q(x)$ *is an end point of type 2, then there exists* $n \in \mathbf{N} \cup \{0\}$ *such that* $\sigma^n x \in EN(0)$.

Theorem 7 *Suppose the number of end points of type 2 is finite. Then the set of end points of type 2 is exactly the set* $\cup_{j \geq 0} q(EN(j))$.

Theorem 8 *The following are equivalent.*

1. *The number of end points is finite.*

2. *There exists* $j \in \mathbf{N}$ *such that* $EN(j) = \phi$.

Theorem 9 *The number of end points of type 2 is the sum of the number of elements of* $EN(j)$, *i.e.*

$$\sharp(EP) = \sum_{j \geq 0} \sharp q(EN(j)).$$

Julia sets and laminations

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Abstract

We introduce an α - invariant equivalence relation on $\{0,1\}^{\infty}$ with $\alpha \in \{0,1\}^{\infty}$ and construct a lamination S_s^{α} using this relation ($s \in \{0,1\}^{\infty}$). We shall give the conditions for α and s that S_s^{α} corresponds to Julia sets.

1. Introduction

Julia sets play an important role on a complex dynamical system. Concerning these sets W.P.Thurston introduced "Invariant Lamination" on a circle [3]. A.Bandt and K.Keller showed the relationship between Thurston's invariant lamination and the symbolic dynamics represented by "itineraries" (infinite sequences of $\{0,1,*\}$), and they got an interesting result involving the correspondence between the dynamics of Julia sets and double-angle motion on a circle [1,2].

In this research we shall give a definition of another equivalence relation on $\{0,1\}^{\infty}$ not using itineraries. Next we construct a lamination with the equivalence relation which corresponds to a Julia set. According to this construction of the lamination, the calculations are much easier than that in the case of using itineraries, because we need only one chord and one boundary point. Besides, we can use this lamination not only for a non-periodic case but also for a periodic case. In general, the construction of a lamination for a periodic case is more complicated than that for a non-periodic case. So we mainly treat a periodic case and show the correspondence between Julia sets and laminations. We show the construction of a lamination only for a non-periodic case in this paper.

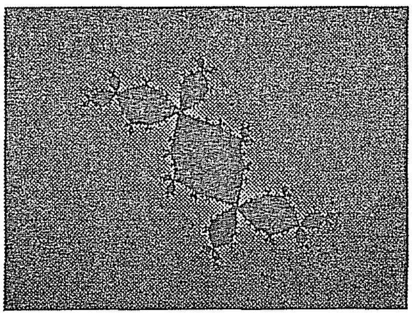


Fig1. **An example of Julia set**

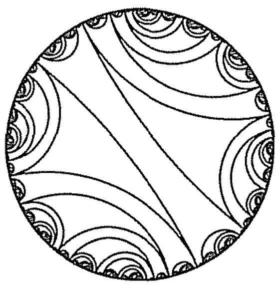


Fig2. **An example of Lamination**

2. locally connected Julia sets and binary sequences

We recall the definition of Julia sets of $g_c(z) = z^2 + c$. For $c, z \in C$, let $O_c(z) = \{z, g_c(z), g_c^2(z), \cdots\}$ denote the forward orbit of z, and call $K_c = \{z \in C \mid O_c(z)\text{is bounded}\}$ the filled-in Julia set. The boundery J_c of K_c is said to be the Julia set of g_c. $K_0 = D$ is the unit disk. Let I denote the closed set $[0, 1]$.

Definition 1.

- $$E_c \overset{\text{def}}{=} \left\{ h \in C_c^1[0,1] \;\middle|\; \begin{array}{l} h(0) = h(1) \\ h(I)\text{is a differentiable Jordan closed curve including } J_c \\ h(t) = -h(t+\frac{1}{2}) \quad (0 \le t \le \frac{1}{2}) \\ g_c(h(I)) \text{ includes } h(I) \end{array} \right\}.$$
- For $h \in E_c$, let $\qquad h(t) - c = r(t)e^{i\theta(t)} \qquad (-\pi \le \theta(0) < \pi,\ \theta(t) \in C[0,1])$.

  Define $f_0, f_1 \ : \ E_c \to C_c^1[0,1]$ by
  $$\begin{cases} f_0 \cdot h(t) \overset{\text{def}}{=} \ \ r(t)^{\frac{1}{2}} \cdot e^{i\frac{\theta(t)}{2}} \\ f_1 \cdot h(t) \overset{\text{def}}{=} -r(t)^{\frac{1}{2}} \cdot e^{i\frac{\theta(t)}{2}} \end{cases}.$$
- Define $S : E_c \to E_c$ by
  $$Sh(t) = \begin{cases} f_0 \cdot h(2t) & (0 \le t \le \frac{1}{2}) \\ f_1 \cdot h(2t-1) & (\frac{1}{2} \le t \le 1) \end{cases}.$$

Remark 1. "$h(I)$ includes J_c" means that if $h(I)$ is regarded as the boundary of a region S, then $J_c \subset S$.
2. We can show $r(0) = r(1)$ and $|\theta(0) - \theta(1)| = 2\pi$ because of $c \in K_c$. We can also show if $h \in E_c$ then $Sh \in E_c$.

We recall the next well-known fact. If J_c is connected, there is a unique conformal isomorphism $\Phi_c \ : \ C \setminus K_c \to C \setminus D$ with $\lim_{z\to\infty} \Phi_c(z)/z = 1$ satisfying $\Phi_c g_c \Phi_c^{-1} = g_0$. Let define field lines $\beta_c = \{z \in C \setminus K_c \mid \arg(\Phi_c(z)) = 2\pi\beta\}$. According to Caratheodory's theorem, each field line β_c has a continuous extension to a unique point z_β of J_c, and each point of J_c is obtained in this way, if and only if J_c is locally connected. We use the next lemma. The proof of it is shown in the reference [1].

Lemma 1. *J_c $(c \in C)$ is locally connected if and only if the functional equations* (1)
$$\varphi(2\beta) = \varphi(\beta)^2 + c \quad \textit{and} \quad -\varphi(\beta) = \varphi(\beta + \tfrac{1}{2}), \ \ \beta \in R \qquad (1)$$
have a continuous periodic solution. In this case, $J_c = \varphi(R)$. Moreover, every continuous solution of (1) *with minimal period 1 coincides with either φ_c^+ or φ_c^- where $\varphi_c^+(\beta) = z_{\beta mod1}$ and $\varphi_c^-(\beta) = \varphi_c^+(-\beta)$ for $\beta \in R$.*

Theorem 1. *If J_c is locally connected, there exists $\phi \in \bar{E}_c$ (the closure of E_c with sup norm) uniquely such that $S^n h(t) \to \phi(t)$ as $n \to \infty$ for $h \in E_c$ and for $t = \Sigma_{n=1}^{\infty} \frac{t_n}{2^n} \in I$ where $S^n h(t) = f_{t_1} \cdot f_{t_2} \cdots f_{t_n} \cdot h(\Sigma_{n=1}^{\infty} \frac{t_{n+1}}{2^n})$.*

Proof. By the property of the functions f_0, f_1, we can show the existence of the limit of $S^n h(t) = f_{t_1} \cdot f_{t_2} \cdots f_{t_n} \cdot h(\Sigma_{n=1}^{\infty} \frac{t_{n+1}}{2^n})$. So put $\phi(t) = \lim_{n\to\infty} S^n h(t)$. We can show

$$S^n h(t) = -S^n h(t+\tfrac{1}{2}) \quad \text{for all } n$$

by induction. We can also show two important equations

$$\phi(t) = -\phi(t+\tfrac{1}{2}) \qquad \text{(A)}$$

$$\begin{aligned}\phi(2t) &= \lim_{n\to\infty} S^n(gSh(t)) \\ &= g \lim_{n\to\infty} S^{n+1}h(t) \\ &= g\phi(t) \\ &= \phi(t)^2 + c \qquad \text{(B)}\end{aligned}$$

By Lemma 1, ϕ coincides with φ^+ or φ^-. By Caratheodory's theorem, $\phi(t)$ is in J_c and ϕ is uniquely determined. □

Let $f_{t_1\cdots t_n}$ denote $f_{t_1} \cdots f_{t_n}$. By the equation $\phi(t) = \lim_{n\to\infty} f_{t_1\cdots t_n} \cdot h(\Sigma_{n=1}^{\infty} \frac{t_{n+1}}{2^n})$, we can correspond points of J_c to binary sequences. But the correspondence is not one to one, so we introduce an equivalence relation such that the same points in the Julia set are in the same equivalent class.

Definition 2. For $\underline{x} = x_1x_2\cdots,\ \underline{y} = y_1y_2\cdots \in \Sigma = \{0,1\}^{\infty}$ (infinite sequences of 0 and 1), we define an equivalence relation $\approx$ as follows:

$$\underline{x} \approx \underline{y} \overset{\text{def}}{\Leftrightarrow} \psi(\underline{x}) = \psi(\underline{y}) \quad \text{where} \quad \psi(\underline{x}) = \phi(\Sigma_{n=1}^{\infty} \tfrac{x_n}{2^n}).$$

For $\alpha \in \{0,1\}^{\infty}$, we define the function τ_α as follows:

$\tau_\alpha : \{0,1\}^{\infty} \to \{0,1\}^{\infty}$

$$\tau_\alpha(s) \overset{\text{def}}{=} \begin{cases} 0s & k(s) \le k(\alpha) \\ 1s & k(s) > k(\alpha) \end{cases} \qquad \text{for} \quad s \in \{0,1\}^{\infty}$$

where $k(s) = \Sigma_{n=1}^{\infty} \frac{s_n}{2^n}$ with $s = s_1s_2\cdots$.

The next lemma for Theorem 2 is shown to study the properties of the equivalence relation.

Lemma 2. *If $\phi(t_1) = \phi(t_2) \neq \phi(t_3) = \phi(t_4)$ for $t_1 < t_2,\ t_3 < t_4 \in I$, then $t_1 < t_2 < t_3 < t_4$ or $t_1 < t_3 < t_4 < t_2$ or $t_3 < t_4 < t_1 < t_2$ or $t_3 < t_1 < t_2 < t_4$ holds.*

Theorem 2. *The equivalence relation $\approx$ satisfies the following.*

(1) $\underline{x} \approx \underline{y}$ *implies* $\sigma\underline{x} \approx \sigma\underline{y}$ $\ (\sigma(s_1x_2\cdots) = x_2x_3\cdots)$.

(2) $\underline{x} \approx \underline{y}$ *implies* $x_1'\sigma\underline{x} \approx y_1'\sigma\underline{y}$ $\ (x_1' = 1 - x_1, y_1' = 1 - y_1)$.

(3) $\exists\alpha \in \{0,1\}^{\infty}$ *s.t.* $\underline{x} \approx \underline{y}$ *implies* $\tau_\alpha(\underline{x}) \approx \tau_\alpha(\underline{y})$.

(4) $\underline{x} \approx \underline{u}, \underline{y} \approx \underline{v}, \underline{x} \not\approx \underline{y}$ *implies* $(k(\underline{x}), k(\underline{u})) \cap (k(\underline{y}), k(\underline{v})) = \phi$ *or* $(k(\underline{x}), k(\underline{u})) \supset (k(\underline{y}), k(\underline{v}))$ *or* $(k(\underline{x}), k(\underline{u})) \subset (k(\underline{y}), k(\underline{v}))$.

Proof. We can show (1),(2) by (A),(B) and (4) by Lemma 2. Put
$k(s_0) = \max\{k(s) \mid \exists k(t) > k(s);\ \theta(t) = \theta(s) + 2\pi\}$.
If $s \approx t$ $(k(s) \le k(t))$ then either (i) or (ii) holds.
(i) If $k(s) \le k(t) < k(s_0)$ then $0s \approx 0t$ and $1s \approx 1t$.
(ii) If $k(s) \le k(s_0) < k(t)$ then $0s \approx 1t$ and $1s \approx 0t$.
So put $\alpha = s_0$. Then $\underline{x} \approx \underline{y}$ implies $\tau_\alpha(\underline{x}) \approx \tau_\alpha(\underline{y})$. □

3. The constructure of α – invariant lamination

In this section we shall define an α - invariant equivalence relation satisfying (1) $\sim$ (3) in Theorem 2 and construct laminations by using this relation. We also give the conditions for α and s that S_s^α corresponds to Julia sets.

Let $\{0,1\}^\infty$ denote the set of one-sided sequences $s = s_1s_2s_3\cdots$. If $s = \overline{w}$ with $w \in \{0,1\}^n$, we call the sequence s to be n-periodic.

Definition 3. Let $\alpha = \alpha_1\alpha_2\cdots$ be an element of $\{0,1\}^\infty$.
An equivalence relation $\sim$ on $\{0,1\}^\infty$ is called to be *α-invariant* if it satisfies the following (1) and (2).

(1) For $s,\ t \in \{0,1\}^\infty$, $s \sim t$ implies $\sigma(s) \sim \sigma(t)$ where $\sigma(s_1s_2\cdots) = s_2s_3\cdots$.

(2) For $s,\ t \in \{0,1\}^\infty$, $s \sim t$ implies $\tau_\alpha(s) \sim \tau_\alpha(t)$ and $\tau_\alpha{}'(s) \sim \tau_\alpha{}'(t)$,

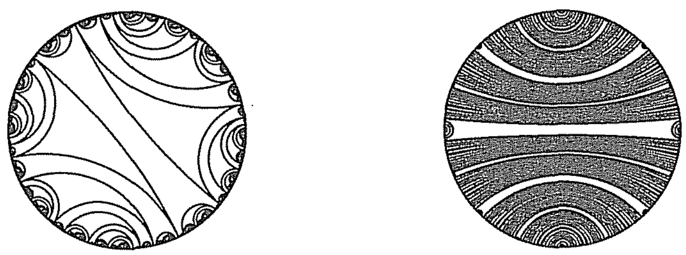
where $\tau_\alpha(s) \stackrel{\text{def}}{=} \begin{cases} 0s & \text{if } k(s) \le k(\alpha) \\ 1s & \text{if } k(s) > k(\alpha) \end{cases}$ and $\tau_\alpha{}'(s) \stackrel{\text{def}}{=} \begin{cases} 1s & \text{if } k(s) \le k(\alpha) \\ 0s & \text{if } k(s) > k(\alpha) \end{cases}$.

Let $\sim_{\bar{0}}$ be the smallest $\bar{0}$ - invariant equivalence relation satisfying $\bar{0} \sim \bar{1}$. Let $T = R/Z$. Then it is easy to show $\{0,1\}^\infty / \sim_{\bar{0}} \ \simeq\ T$. So let ϕ be the isomorphism from $\{0,1\}^\infty / \sim_{\bar{0}}$ onto T.

For $a, b \in \{0,1\}^\infty$, let $C_{a,b}$ be the chord connecting $\phi(a)$ and $\phi(b)$ on T, and let $C = \{C_{a,b} \mid a, b \in \{0,1\}^\infty\}$.

Definition 4. For $\alpha \in \{0,1\}^\infty$ and for chord $C_{a,b} \in C$, the equivalence relation $\sim_{a,b}^\alpha$ is defined as the smallest closed α-invariant equivalence relation on $\{0,1\}^\infty / \sim_{\bar{0}}$, satisfying $a \sim b$.
For $\alpha \in \{0,1\}^\infty$ and $C_{a,b} \in C$, let $S_{a,b}^\alpha$ be the collection of the chords $\left\{C_{\lambda_1\lambda_2\cdots\lambda_n(a),\lambda_1\lambda_2\cdots\lambda_n(b)} \mid \lambda_j \in \{\sigma, \tau_\alpha, \tau_\alpha{}'\}, n \in N \cup \{0\},\right\}$. We call the closure $\overline{S_{a,b}^\alpha}$ of $S_{a,b}^\alpha$ *α-invariant lamination.* For $s \in \{0,1\}^\infty$, we put $S_s^\alpha = S_{s,\sigma(s)}^\alpha$ if $s \sim \sigma(s)$.

Then the α - invariant lamination S_s^α (some examples are showm in Fig3.) is considered as the quotient space of T where points connected by a chord belong to the same class and we have the following.



$\alpha = s = \overline{001}$ $\qquad$ $\alpha = \overline{01111000},\ s = \overline{00001111}$

Fig3. **Some examples of S_s^α**

Thorem 3.
The quotient space $T/\sim^{\alpha}_{s,\sigma(s)}$ *is isomorphic to the* α *- invariant lamination* S^{α}_{s} *where points on* T *connected by a chord are considered as the same element.*

The α - invariant equivalence relation $\sim^{\alpha}_{s,\sigma(s)}$ satisfies (1) $\sim$ (3) in Theorem 2 but not necessarily (4). Since the equivalence relation $\approx$ induced from Julia sets satisfies (1) $\sim$ (4) in Theorem 2, the lamination S^{α}_{s} doesn't necessarily correspond to a Julia set. So we examine the condition for α and s that S^{α}_{s} corresponds to a Julia set and get the following theorem.

Thorem 4.
Let s *be an element of* $\{0,1\}^{\infty}$ *satisfying* $k(s) = \dfrac{1}{2^p-1}$ *with some* $p \geq 2$. *If the lamination* S^{α}_{s} *corresponds to a Julia set, then* α *satisfies the following*

$$\frac{1}{2^p-1} \leq k(\alpha) < \frac{2^{p-1}}{2^p-1}.$$

If s is p-periodic, $k(s) = \frac{q}{2^p-1}$ $(q \in N)$ holds. Theorem 4 is the case of $q = 1$. For an arbitrary q, there doesn't necessarily exist $\alpha \in \{0,1\}^{\infty}$ such that S^{α}_{s} corresponds to a Julia set. The next theorem shows another case of the existence of $\alpha \in \{0,1\}^{\infty}$ such that S^{α}_{s} corresponds to a Julia set.

Thorem 5.
(*i*) *Let* s *be an element of* $\{0,1\}^{\infty}$ *satisfying* $k(s) = \dfrac{\sum_{n=0}^{k_1} 2^{nj}}{2^p-1}$ *with* $p = jk_1 + j + 1$ *and* $k_1 \geq 0$. *If the lamination* S^{α}_{s} *corresponds to a Julia set, then* α *satisfies the following*

$$k(\sigma^{j+1}s) \leq k(\alpha) < k(\sigma s).$$

(*ii*) *Let* s *be an element of* $\{0,1\}^{\infty}$ *satisfying* $k(s) = 1 + 2^j + \dfrac{\sum_{n=1}^{k_2} 2^{(j+1)n+j}}{2^p-1}$ *with some* $p = (j+1)k_2 + j$ *and* $k_2 \geq 1$. *If the lamination* S^{α}_{s} *corresponds to a Julia set, then* α *satisfies the following*

$$k(\sigma^{(j+1)k_2}s) \leq k(\alpha) < k(\sigma s).$$

Reference

[1] Bandt,C. and Keller,K., *Symbolic dynamics for angle-doubling on the circle,I.* Ergodic Theory and Related Topics III, Springer-Verlag ,1990.

[2] Bandt,C. and Keller,K., *Symbolic dynamics for angle-doubling on the circle,II.* Nonlinearity 6 (1993) , 377-392

[3] Thurston,W.P., *On the combinatorics and dynamics of iterated rational maps.* Preprint , Princeton 1985

Toplogy of Moduli Spaces of Polynomial Maps

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Abstract

We investigate the geometric and topological aspects of the moduli space and singular locus of polynomial maps of degree n from a viewpoint of complex dynamical systems. Especially the cases of degree 3 and 4 are analyzed explicitly.

1 Introduction

In this paper, we study the geometry and topology of the space $\mathrm{Poly}_n(\mathbf{C})$ of polynomial maps of degree n from a viewpoint of complex dynamical systems, inspired by those of the quadratic rational maps due to J.Milnor ([6]). First, we investigate the moduli space $\mathrm{M}_n(\mathbf{C})$ consisting of all holomorphic (affine) conjugacy classes of polynomial maps of degree n. A polynomial map p from $\mathbf{C}$ to itself of degree n is monic and centered if it has the form $p(z) = z^n + c_{n-2}z^{n-2} + \cdots + c_1 z + c_0$. Every polynomial map from $\mathbf{C}$ to itself is conjugate under an affine change of variable to a monic centered one, and this is uniquely determined up to conjugacy under the action of the group $G(n-1)$ of $(n-1)$-st roots of unity. Hence the affine space $\mathcal{P}_1(n)$ of all monic centered polynomials of degree n with coordinate $(c_0, c_1, \cdots, c_{n-2})$ is regarded as an $(n-1)$-sheeted covering space of $\mathrm{M}_n(\mathbf{C})$. Thus we can use $\mathcal{P}_1(n)$ as the coordinate space for the moduli space $\mathrm{M}_n(\mathbf{C})$, though it remains the ambiguity up to the group $G(n-1)$. This coordinate space has the advantages of being easy to be treated and the singular locus of $\mathrm{M}_n(\mathbf{C})$ is described in a simplest manner (see Theorem 2). However, it would be also worthwhile to try to introduce another coordinate system having any merit different from $\mathcal{P}_1(n)$'s. In fact, J. Milnor successfully introduced coordinates in the moduli space $\mathcal{M}_2(\mathbf{C})$ in the case of the space $\mathrm{Rat}_2(\mathbf{C})$ of all quadratic rational maps using the elementary symmetric functions of the multipliers at the fixed points of a map ([6]). In the case of $\mathrm{Poly}_n(\mathbf{C})$, we try to explore an analogy to this. (The case of $\mathrm{Poly}_3(\mathbf{C})$ was also suggested in [6].)

Let $\sigma_{n,i}$ be the elementary symmetric functions of the multipliers μ_i at the fixed points of a map in $\mathrm{Poly}_n(\mathbf{C})$ $(i = 1, \cdots, n)$. Then $\sigma_{n,i}$'s are defined as functions on $\mathrm{M}_n(\mathbf{C})$ since μ_i's are invariant by affine conjugacy. From the Fatou index theorem([4]), we derive a linear relation among $\sigma_{n,i}$'s $(i = 1, 2, \cdots, n-1)$ (Theorem 1). (The case of $n = 3$ was mentioned in [6].) In view of this theorem, the affine space $\Sigma(n)$ with coordinate $(\sigma_{n,1}, \sigma_{n,2}, \cdots, \sigma_{n,n-2}, \sigma_{n,n})$ is expected to serve as a coordinate space (with singularity) for $M_n(C)$. For $n = 3$ and 4, we shall prove this is the case. In more detail, $M_3(C) \simeq C^2$ and $M_4(C)$ is a two-sheeted ramified covering space of C^3 (Propositions 1 and 4). We shall see that the affine structure imposed on $\Sigma(n)$ has certainly any preferred status different from $\mathcal{P}_1(n)$'s. For example, this goes well when we treat the locus $\mathrm{Per}_n(\mu)$ (see Proposition 2 and Corollary 1).

Next, we study the singular locus in the moduli space $M_n(C)$. By an automorphism of a polynomial map p we will mean an affine transformation g that commutes with p. The collection $\mathrm{Aut}(p)$ of all automorphisms of p forms a finite group. We obtain an characterization as Theorem 2. Let $\mathcal{S}_n$ ($\subset M_n$) be the set consisting of all conjugacy classes $\langle p \rangle$ of polynomial maps admitting non-trivial automorphisms. In the case of the quadratic rational maps, J.Milnor calls this singular locus $\mathcal{S}$ "symmetry locus". So, following him, we shall also call $\mathcal{S}_n$ symmetry locus. For the cases $n = 3$ and 4, we can give a defining equation of $\mathcal{S}_n$. Further, analogous to the case of quadratic rational maps, $\mathcal{S}_3$ coincides with the envelope of the family of straight lines $\mathrm{Per}_1(\mu)$. But for higher degree cases ($n \geq 4$), the symmetry locus $\mathcal{S}_n$ is properly contained in the envelope of the family $\mathrm{Per}_1(\mu)$.

Last, we express that for obtaining several defining equations of loci (affine algebraic curves), e.g., $Per_m(\mu)$, and the symmetry loci, we depend mainly on "Gröbner basis" of Risa/Asir, an experimental computer algebra system developed at FUJITSU LABORATORIES LIMITED.

2 Polynomials of degree n

2.1 Moduli space

Let $\mathrm{Poly}_n(C)$ be the space of all polynomial maps of degree n from C to itself. The group $A(C)$ of all affine transformations acts on $\mathrm{Poly}_n(C)$ by conjugation: $g \circ p \circ g^{-1} \in \mathrm{Poly}_n(C)$ for $g \in A(C)$, $p \in \mathrm{Poly}_n(C)$. Two maps $p_1, p_2 \in \mathrm{Poly}_n(C)$ are **holomorphically conjugate**, denoted by $p_1 \sim p_2$, if and only if there exists $g \in A(C)$ with $g \circ p_1 \circ g^{-1} = p_2$. The quotient space of $\mathrm{Poly}_n(C)$ under this action will be denoted by $M_n(C)$, and called the **moduli space** of holomorphic conjugacy classes $\langle p \rangle$ of polynomial maps p of degree n.

For each $\mu \in C$ let $\mathrm{Per}_n(\mu)$ be the set of all conjugacy classes $\langle p \rangle$ of maps p having a periodic point of period n and multiplier μ.

Under the conjugacy of the action of $A(C)$, it can be assumed that any map in $\mathrm{Poly}_n(C)$ is "monic" and "centered", i.e., $p(z) = z^n + c_{n-2}z^{n-2} + c_{n-3}z^{n-3} \cdots + c_0$. This p is determined up to the action of the group $G(n-1)$ of $(n-1)$-st roots of unity, where each $\eta \in G(n-1)$ acts on $p \in \mathrm{Poly}_n(C)$ by the transformation $p(z) \mapsto p(\eta z)/\eta$. For example, in the case of $n = 4$ the following three monic and centered polynomials belong to the same conjugacy class: $z^4 + az^2 + bz + c$, $z^4 + a\omega z^2 + bz + c\omega^2$ $z^4 + a\omega^2 z^2 + bz + c\omega$, where ω is a third root of unity.

Let $\mathcal{P}_1(n)$ be the affine space of all monic centered polynomials of degree n with coordinate $(c_0, c_1, \cdots, c_{n-2})$. Then we have an $(n-1)$ to one canonical projection $\Phi : \mathcal{P}_1(n) \to M_n(C)$ from $\mathcal{P}_1(n)$ onto $M_n(C)$. Thus we can use $\mathcal{P}_1(n)$ as coordinate space for $M_n(C)$ though there remains the ambiguity up to the group $G(n-1)$.

Now we intend to explore another coordinate space for $\mathrm{M}_n(\mathrm{C})$ which is "smaller" than $\mathrm{M}_n(\mathrm{C})$ in contrast with $\mathcal{P}_1(n)$: for each $p(z) \in \mathrm{Poly}_n(\mathrm{C})$, let $z_1, \cdots, z_n, z_{n+1}(=\infty)$ be the fixed points of p and μ_i the multipliers of z_i; $\mu_i = p'(z_i)$ $(1 \leq i \leq n)$, and $\mu_{n+1} = 0$. Consider the elementary symmetric functions of the n multipliers,

$$\begin{aligned}\sigma_{n,1} &= \mu_1 + \cdots + \mu_n,\\ \sigma_{n,2} &= \mu_1\mu_2 + \cdots + \mu_{n-1}\mu_n\\ &\cdots\\ \sigma_{n,n} &= \mu_1\mu_2\cdots\mu_n,\\ \sigma_{n,n+1} &= 0.\end{aligned}$$

Note that these are defined on the moduli space $\mathrm{M}_n(\mathrm{C})$, since μ_i's are invariant by affine conjugacy. The Fatou index theorem can be applied to these μ_i's;

$$\sum_{i=1}^{n} \frac{1}{1-\mu_i} + \frac{1}{1-0} = 1, \tag{1}$$

provided $\mu_i \neq 1 (1 < i < n)$. Arranging this equation for the form of elementary symmetric functions, we have $c_0 + c_1\sigma_{n,1} + c_2\sigma_{n,2} + \cdots + c_{n-1}\sigma_{n,n-1} = 0$, where

$$c_k = (-1)^k n \binom{n-1}{k} \Big/ \binom{n}{k} = (-1)^k (n-k).$$

Note that $\mu_i = 1$ $(1 \leq i \leq n)$ is allowable here. Then we have the following:

Theorem 1 *Among $\sigma_{n,i}$'s, there is a linear relation*

$$\sum_{k=0}^{n-1} (-1)^k (n-k) \sigma_{n,k} = 0, \tag{2}$$

where we put $\sigma_{n,0} = 1$.

Let $\Sigma(n)$ denote the affine space with coordinate $(\sigma_{n,1}, \sigma_{n,2}, \cdots, \sigma_{n,n-2}, \sigma_{n,n})$. In view of Theorem 1, we consider the map $\Psi : \mathrm{M}_n(\mathrm{C}) \to \Sigma(n)$ which is defined in an obvious manner. To investigate whether this map is surjective or not correspond to solve the inverse problem, that is, to determine a class of maps with the prescribed values of the multipliers satisfying the relation of the Fatou index theorem. As we see later, the case $n = 3$ is nicely solved: Ψ is surjective and $\mathrm{M}_n(\mathrm{C}) \simeq \Sigma(n) \simeq \mathrm{C}^2$. (This fact is mentioned in [6] without any details.) Using computer analysis, the case $n = 4$ is also solved: Ψ is surjective and $\mathrm{M}_4(\mathrm{C})$ is a two-sheeted ramified covering space of C^3. As for the cases of general n, we expect analogous results (see section 3).

2.2 Symmetry locus

By an automorphism of a polynomial p of degree n, we will mean $g \in \mathrm{A}(\mathrm{C})$ which commutes with p; $g \circ p \circ g^{-1} = p$. The collection $\mathrm{Aut}(p)$ of all automorphisms of p forms a finite group. It is clear that the following is well-defined:

Definition 1 (c.f. [6]) The set $\mathcal{S}_n = \{\langle f \rangle \in \mathrm{M}_n(\mathrm{C});\ \mathrm{Aut}(p) \text{ is non-trivial}\}$ is called the **symmetry locus**.

Theorem 2 *A polynomial of degree n has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form*

$$z^n + \sum_{1 \leq p \leq [n/k]} A_k(p) z^{kp+1} + Bz \tag{3}$$

where $k|(n-1), k \neq n-1$*,and* $A_k(p), B$ *are parameters in* C.

Proof.

Let $p(z) = a_n z^n + \cdots + a_0 \in \mathrm{Poly}_n(\mathrm{C})$ and $h(z) = \alpha z + \beta \in \mathrm{A}(\mathrm{C})$. Consider the identity $h \circ p \circ h^{-1} - p = 0$. The coefficient of the highest order term is: $a_n \alpha(\alpha^{n-1} - 1) = 0$. The coefficient of the second highest order term is: $n a_n \alpha^{n-1} \beta = 0$. Hence we have $\alpha^{n-1} = 1$ and $\beta = 0$. The rest is easy computations. ∎

3 Polynomial maps of lower degrees ($n = 3, 4$)

3.1 Moduli space $\mathrm{M}_3(\mathrm{C})$ and its symmetry locus $\mathcal{S}_3$

Here we abbreviate $\sigma_{3,i}$ as σ_i. Then these σ_i's are defined on $\mathrm{M}_3(\mathrm{C})$. Conversely $(\sigma_1, \sigma_2, \sigma_3)$ satisfying the relation given in Theorem 1, i.e.,$3 - 2\sigma_1 + \sigma_2 = 0$, uniquely determine $\langle p \rangle \in \mathrm{M}_3(\mathrm{C})$. A map in $\mathrm{Poly}_3(\mathrm{C})$ is conjugate to a map of the normal form $z^3 + az + b$, and its parameters (a, b^2) is used as a coordinate system of $\mathrm{M}_3(\mathrm{C})$ which is isomorphic to C^2 ([5]). These coordinates relates to (σ_1, σ_3) as follows:

$$\sigma_1 = -3a + 6, \quad \sigma_3 = 27b^2 + a(2a-3)^2, \tag{4}$$

or

$$a = (6 - \sigma_1)/3, \quad b^2 = (4\sigma_1^3 - 36\sigma_1^2 + 81\sigma_1 + 27\sigma_3 - 54)/729. \tag{5}$$

Hence the following is obtained:

Proposition 1 (σ_1, σ_3) *is a coordinate system of* $M_3(\mathrm{C})$.

Now the affine structure is imposed by the above coordinate system. With this structure thus imposed, the locus $\mathrm{Per}_1(\mu)$ is described finely:

Proposition 2 *The locus* $Per_1(\mu)$ *forms a straight line:*

$$Per_1(\mu) = \left\{(\sigma_1, \sigma_3); \sigma_3 = (-\mu^2 + 2\mu)\sigma_1 + \mu^3 - 3\mu\right\}.$$

From Theorem 2 a cubic polynomial map $z^3 + az + b$ has non-trivial automorphisms if and only if $b = 0$. From formulas (5), we obtain the following:

Proposition 3 *The symmetry locus* $\mathcal{S}_3$ *of cubic polynomials forms an irreducible algebraic curve:*

$$S_3(\sigma_1, \sigma_3) = 4\sigma_1^3 - 36\sigma_1^2 + 81\sigma_1 + 27\sigma_3 - 54 = 0. \tag{6}$$

Corollary 1 *The envelope of* $\{Per_1(\mu)\}_\mu$ *coincides with the symmetry locus* $\mathcal{S}_3$.

3.2 Moduli space $M_4(C)$ and its symmetry locus $\mathcal{S}_4$

In the case of $\mathrm{Poly}_4(C)$, we can go on further analysis by using a symbolic and algebraic computation systems. Here we write $\sigma_{4,i} = \sigma_i$ $(i = 1, \cdots, 4)$ for brevity. first, we investigate the inverse problem. This time we consider a polynomial $p(z) = a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$ that has at least two fixed points. After affine conjugation, we can assume they are 0 and 1. Then we will solve the following problem: *"Do the four multipliers $\mu_0 = p'(0)$, $\mu_1 = p'(1)$, $\mu_2 = p'(z_2)$, $\mu_3 = p'(z_3)$, where z_2, z_3 are also fixed points of $p(z)$, determine the five coefficients a_i $(0 \le i \le 4)$ of $p(z)$?"* In fact, the following relations hold;

$$\begin{array}{ll} a_0 = 0 & \text{because of } p(0) = 0, \\ a_1 = \mu_0 & \text{because of } p'(0) = \mu_0, \\ a_2 = a_4 + 3 - 2\mu_0 - \mu_1 & \text{because of } p'(1) = \mu_1, \\ a_3 = 1 - a_4 - a_2 - \mu_0 & \text{because of } p(1) = 1. \end{array}$$

Here, from the relations between coefficients and solutions, we obtain

$$z_2 + z_3 = \frac{\mu_0 + a_4 + 3 - 2\mu_0 - \mu_1 - 1}{a_4}, \quad z_2z_3 = \frac{1 - \mu_0}{a_4}. \tag{7}$$

To carry out the computation cleverly, we consider the following equalities:

$$\mu_2 + \mu_3 = p'(z_2) + p'(z_3), \quad \mu_2\mu_3 = p'(z_2)p'(z_3). \tag{8}$$

Using the relations (7), we can remove z_2 and z_3 from equations (8). Then a_4 is a common root of the derived equations as follows;

$$\begin{aligned} A_1 =&(\mu_2^2-2\mu_3\mu_2+\mu_3^2-\mu_0^2+2\mu_1\mu_0-\mu_1^2)a_4^4+(-4\mu_0^3+(4\mu_1+8)\mu_0^2+(-4\mu_1^2-8)\mu_0+4\mu_1^3- \\ &8\mu_1^2+8\mu_1)a_4^3+(-6\mu_0^4+(-4\mu_1+28)\mu_0^3+(4\mu_1^2+4\mu_1-44)\mu_0^2+(-4\mu_1^3+4\mu_1^2-8\mu_1+ \\ &32)\mu_0-6\mu_1^4+28\mu_1^3-44\mu_1^2+32\mu_1-16)a_4^2+(-4\mu_0^5+(-12\mu_1+32)\mu_0^4+(-8\mu_1^2+64\mu_1- \\ &96)\mu_0^3+(8\mu_1^3-96\mu_1+128)\mu_0^2+(12\mu_1^4-64\mu_1^3+96\mu_1^2-64)\mu_0+4\mu_1^5-32\mu_1^4+96\mu_1^3- \\ &128\mu_1^2+64\mu_1)a_4-\mu_0^6+(-6\mu_1+12)\mu_0^5+(-15\mu_1^2+60\mu_1-60)\mu_0^4+(-20\mu_1^3+120\mu_1^2- \\ &240\mu_1+160)\mu_0^3+(-15\mu_1^4+120\mu_1^3-360\mu_1^2+480\mu_1-240)\mu_0^2+(-6\mu_1^5+60\mu_1^4- \\ &240\mu_1^3+480\mu_1^2-480\mu_1+192)\mu_0-\mu_1^6+12\mu_1^5-60\mu_1^4+160\mu_1^3-240\mu_1^2+192\mu_1-64= \\ &0, \\ A_2 =&(\mu_2+\mu_3+\mu_0+\mu_1-4)a_4^2+(2\mu_0^2-4\mu_0-2\mu_1^2+4\mu_1)a_4+\mu_0^3+(3\mu_1-6)\mu_0^2+ \\ &(3\mu_1^2-12\mu_1+12)\mu_0+\mu_1^3-6\mu_1^2+12\mu_1-8=0. \end{aligned}$$

By computing the resultant, we see that the above two equations have common roots if and only if $\mu_0, \mu_1, \mu_2, \mu_3$ satisfy the equation (2) for $n = 4$. On the other hand, $\mu_0, \mu_1, \mu_2, \mu_3$ are the four multipliers of $p(z)$ and consequently they satisfy the equation (2). Hence the two equations always have common roots. Thus the five coefficients of $p(z)$ are calculated from its four multipliers, though a_4 is not decisive when the above two equations have distinct two common roots. Thus:

Proposition 4 *The moduli space $M_4(C)$ is a at most two-sheeted ramified covering of C^3 with coordinates $(\sigma_1, \sigma_2, \sigma_4)$.*

Next, we try to work out the explicit descliption of the symmetry locus $\mathcal{S}_4$.

Proposition 5 *The symmetry locus $\mathcal{S}_4$ in $M_4(C)$ forms the following algebraic curve:*

$$\begin{cases} \sigma_1 = s \\ \sigma_2 = 3(3s-4)(s+4)/32 \\ \sigma_4 = -(3s-4)^3(s-12)/4096. \end{cases}$$

Proof

Expressing the class $\langle z^4 + az \rangle$ by the coordinate system $(\sigma_1, \sigma_2, \sigma_4)$ on the moduli space, via Gröbner basis, we have the following:

$$\sigma_2^4 - 48\sigma_2^3 + 24\sigma_4\sigma_2^2 + 960\sigma_4\sigma_2 + 144\sigma_4^2 + 2304\sigma_4 = 0 \tag{9}$$

$$(72\sigma_4 - 648)\sigma_1 + \sigma_2^3 - 54\sigma_2^2 + (12\sigma_4 + 432)\sigma_2 + 504\sigma_4 + 864 = 0 \tag{10}$$

$$(6\sigma_2 + 36)\sigma_1 - \sigma_2^2 - 32\sigma_2 - 12\sigma_4 - 48 = 0. \tag{11}$$

Substituting (11) for (9) and (10), we obtain two cylindrical surfaces which turn out to have a common factor

$$32\sigma_2 - 9\sigma_1^2 - 24\sigma_1 + 48 = 0. \tag{12}$$

On the other hand, from the condition that the three multipliers are the same, we obtain the following cylindrical surface

$$\sigma_2^4 - 48\sigma_2^3 + 24\sigma_4\sigma_2^2 + 960\sigma_4\sigma_2 + 144\sigma_4^2 + 2304\sigma_4 = 0. \tag{13}$$

A defining equation of the symmetry locus $\mathcal{S}_4$ is the intersection of these two surfaces. Then, parameterizing (12) and (13) by σ_1, we obtain the desired result. ∎

References

[1] FUJIMURA, M., AND NISHIZAWA, K. Bifurcations and Algebraic Curves for Quadratic Rational Functions $\{mf(x)\}_m$. submitted.

[2] FUJIMURA, M., AND NISHIZAWA, K. Moduli Spaces and Symmetry Loci of Polynomial Maps. submitted.

[3] FUJIMURA, M., AND NISHIZAWA, K. Bifurcations of Nusse-Yorke's family in the quadratic rational functions. In *RIMS Kokyuroku 959: Complex Dynamics and Related Problems* (1996), T. Sugawa, Ed., Kyoto Univ., pp. 90–102.

[4] MILNOR, J. Dynamics in one complex variables: Introductory lectures. Preprint # 1990/5, SUNY Stony Brook,1990.

[5] MILNOR, J. Remarks on iterated cubic maps. *Experimental Mathematics 1* (1992), 5–24.

[6] MILNOR, J. Geometry and Dynamics of Quadratic Rational Maps. *Experimental Mathematics 2*, 1 (1993), 37–83.

[7] NISHIZAWA, K., AND NOJIRI, A. Center curves in the moduli space of the real cubic maps. *Proc. Japan Acad. Ser.A 69* (1993), 179–184.

もっと因数分解公式を

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Abstract.　多項式の因数分解についてお話しします．因数分解は中学校で教わります．式の展開の逆演算としてです．習った公式のいくつかを挙げてみれば，その一般化の存在が気に掛かります．本講演の目的は，数式処理を用いて因数分解公式の一つの一般化を追求した過程をお目に掛けることです．

1　動機

次の公式は何方も見覚えがおありでしょう：

$$a^2 \pm 2ab + b^2 = (a \pm b)^2$$
$$a^3 \pm 3a^2b + 3ab^2 \pm b^3 = (a + b)^3$$
$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

これらの公式を式の展開の逆演算として因数分解を教わったように思います．因数分解を学んで考えた事というのは，

- 他に因数分解公式はないのか
- 2 文字の公式に何故 2 があるか
- 3 文字の公式に何故 3 があるか
- 4 文字の公式を作れるか？ そこには 4 が現れるか？

ということでした．4 変数 4 次の公式はないのか，と言うのが当時の疑問です．父に尋ねたところ，対称式だからやってみれば，と言う答え．4 変数 4 次は面倒だな，と思ってそのままになりました．

月日が経ち，数式処理というこの疑問に格好の道具が出来ました．数式処理は Mathematica を使ってみました．目標は

- 4 変数 4 次の因数分解公式を発見する事
- n 変数 n 次の因数分解公式を発見する事
- それは，2 変数 2 次，3 変数 3 次の因数分解公式の自然な延長である事

です.

2 過程

対称式を扱うなら基本対称式が必要です．数式処理用に記号を導入します.

$$
\begin{aligned}
&E_{1,2}(a,b) = a+b,\\
&E_{2,2}(a,b) = ab,\\
&E_{1,3}(a,b,c) = a+b+c,\\
&E_{2,3}(a,b,c) = ab+bc+ca,\\
&E_{3,3}(a,b,c) = abc
\end{aligned}
$$

の様に，前の添字はオーダーを後の添字は変数の数を表します．この記号を用いると，2 次の因数分解公式は

$$
\begin{aligned}
a^2+b^2 &= (a+b)^2-2ab,\\
E_{1,2}(a^2,b^2) &= (E_{1,2}(a,b))^2-2E_{2,2}(a,b)
\end{aligned}
$$

と表すことが出来ます．3 次の因数分解公式も基本対称式で表してみます.

$$
\begin{aligned}
E_{1,3}(a^3,b^3,c^3) &= E_{1,3}(a,b,c)(E_{1,3}(a^2,b^2,c^2)-E_{2,3}(a,b,c))\\
&\quad +3E_{3,3}(a,b,c)
\end{aligned}
$$

これらから次のようなことが観察されます：

- 2 変数の 2 乗の 1 次基本対称式は 1 乗の 1 次及び 2 次の基本対称式で表現できる.
- 3 変数の 3 乗の 1 次基本対称式は 2 乗以下の 1 次，2 次，3 次基本対称式で表現できる.

更に，次の様な実験仮説をたてました.

- n 変数の n 乗の 1 次基本対称式は n-1 乗以下の 1 次，2 次，,...,n 次基本対称式で表現できる.
- この表現が因数分解公式を与える

ここで数式処理による実験を行いました．4 変数で考えます．変数の 4 乗のオーダー 4 の基本対称式をオーダーが 1 の基本対称式とオーダーが 3 の基本対称式との積で評価してみます.

```
E14[x1_,x2_,x3_,x4_]=x1+x2+x3+x4;
E24[x1_,x2_,x3_,x4_]=x1*x2+x1*x3+x1*x4+
                              x2*x3+x2*x4+
                                     x3*x4;
E34[x1_,x2_,x3_,x4_]=x1*x2*x3+
                     x1*x2*x4+
                     x1*x3*x4+
                     x2*x3*x4;
E44[x1_,x2_,x3_,x4_]=x1*x2*x3*x4;

ff=-Expand[E14[a^4,b^4,c^4,d^4]-
            E14[a,b,c,d]*E14[a^3,b^3,c^3,d^3]]
```

誤差は a^3b のような形をしています．次に誤差を評価します．誤差は

$$a^3b = a^3 \times b = a^2 \times ab = \cdots$$

と様々な因数に分解出来ます．それぞれの分解に応じて，基本対称式の積を対応させます．色々な可能性が考えられますが... 中でうまく行ったのが $a^3b = a^2 \times ab$ です．a^2 はオーダーが 1 の 2 乗の基本対称式に ab はオーダーが 2 の基本対称式に対応させます.

```
gg=-Expand[ff-
            E14[a^2,b^2,c^2,d^2]*E24[a,b,c,d]]
```

この過程ををもう一度実行すると次のものがうまく行きます：

```
Expand[gg-E14[a,b,c,d]*E34[a,b,c,d]]
```

これが示しているのは次の式です：

$$
\begin{aligned}
& a^4+b^4+c^4+d^4 \\
= \ & (a+b+c+d)(a^3+b^3+c^3+d^3+abc+abd+acd+bcd) \\
& -(a^2+b^2+c^2+d^2)(ab+ac+ad+bc+bd+cd)-4abcd.
\end{aligned}
$$

或いは因数分解らしく書くと，

$$
\begin{aligned}
& a^4+b^4+c^4+d^4 \\
& +(a^2+b^2+c^2+d^2)(ab+ac+ad+bc+bd+cd)+4abcd \\
= \ & (a+b+c+d)(a^3+b^3+c^3+d^3+abc+abd+acd+bcd).
\end{aligned}
$$

2 変数 3 次の因数分解公式

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

等が 3 変数 3 次の因数分解公式に於いて $c=0$ と置いて得られるのと同様に 4 変数 4 次の因数分解公式から次の様な因数分解公式もいろいろ得られます.

$$(a^4+b^4)+(a^2+b^2)ab=(a+b)(a^3+b^3).$$

3　結果

得られた式を 2 変数 2 次，3 変数 3 次のものと比較してみましょう.

$$
\begin{aligned}
a^2+b^2 &= (a+b)(a+b) \\
& \quad -2ab \\
a^3+b^3+c^3 &= (a^2+b^2+c^2)(a+b+c) \\
& \quad -(a+b+c)(ab+bc+ca) \\
& \quad +3abc \\
a^4+b^4+c^4+d^4 &= (a^3+b^3+c^3+d^3)(a+b+c+d) \\
& \quad -(a^2+b^2+c^2+d^2)(ab+ac+ad+bc+bd+cd) \\
& \quad +(a+b+c+d)(abc+abd+acd+bcd) \\
& \quad -4abcd.
\end{aligned}
$$

4 変数 4 次の因数分解される式はかなり長い式になりますが 3 次と比べてこれを因数分解公式に加えて悪いことはないでしょう.

これを一般化するには Hardy–Littlewood–Polya の不等式の本に則った記号を導入すると便利です.

$$\textstyle\sum_4 a = a+b+c+d$$
$$\textstyle\sum_4 ab = ab+ac+ad+bc+bd+cd$$
$$\textstyle\sum_4 a^2 = a^2+b^2+c^2+d^2$$

の様に総和記号は変数の数を添字に持ち，一般項として第 1 項を書くことにします．この記号で 2 次と 3 次の因数分解公式を書いてみましょう.

$$\begin{aligned}\textstyle\sum_2 a^2 &= \left(\textstyle\sum_2 a\right)^2 \\ &\quad -2ab \\ \textstyle\sum_3 a^3 &= \left(\textstyle\sum_3 a^2\right)\left(\textstyle\sum_3 a\right) \\ &\quad -\left(\textstyle\sum_3 a\right)\left(\textstyle\sum_3 ab\right) \\ &\quad +3abc\end{aligned}$$

4 次も同様に書きます.

$$\begin{aligned}\textstyle\sum_4 a^4 &= \left(\textstyle\sum_4 a^3\right)\left(\textstyle\sum_4 a\right) \\ &\quad -\left(\textstyle\sum_4 a^2\right)\left(\textstyle\sum_4 ab\right) \\ &\quad +\left(\textstyle\sum_4 a\right)\left(\textstyle\sum_4 abc\right) \\ &\quad -3abcd\end{aligned}$$

此処に規則性があることが分かります．それぞれの積では一方では一般項の次数が下がり，他方ではその反動で一般項のオーダーが上がっていきます.

従って，一般の等式は符号を交換しつつ次数の下がっていく総和とオーダーの上がっていく総和の積和になっています．$2ab, 3abc$ などの係数が $2, 3,$ と上がっていき，N変数の場合には Nであることは 1 の N個の和だからです.

$$\textstyle\sum_N a_1^N = \displaystyle\sum_{k=1}^{N}(-1)^{k-1}\left(\textstyle\sum_N a_1^{N-k}\right)\left(\textstyle\sum_N \displaystyle\prod_{j=1} ka_j\right).$$

この等式の証明は自明です．N=2,3 では中学校で習った公式です．また，最初の 2 項を除いてゼロだと思うと等比数列の和の公式になります.

4　プログラムと出力

```
E14[x1_,x2_,x3_,x4_]=x1+x2+x3+x4;
E24[x1_,x2_,x3_,x4_]=x1*x2+x1*x3+x1*x4+
                                x2*x3+x2*x4+
                                      x3*x4;
E34[x1_,x2_,x3_,x4_]=x1*x2*x3+
                     x1*x2*x4+
                     x1*x3*x4+
                     x2*x3*x4;
E44[x1_,x2_,x3_,x4_]=x1*x2*x3*x4;
E14[a,b,c,d]
E24[a,b,c,d]
E34[a,b,c,d]
E44[a,b,c,d]
E14[a^2,b^2,c^2,d^2]
E14[a^3,b^3,c^3,d^3]
E14[a^4,b^4,c^4,d^4]

ff=-Expand[E14[a^4,b^4,c^4,d^4]-
           E14[a,b,c,d]*E14[a^3,b^3,c^3,d^3]]
gg=-Expand[ff-
           E14[a^2,b^2,c^2,d^2]*E24[a,b,c,d]]
Expand[gg-E14[a,b,c,d]*E34[a,b,c,d]]
```

$$a + b + c + d$$

$$a b + a c + b c + a d + b d + c d$$

$$a b c + a b d + a c d + b c d$$

$$a b c d$$

$$a^2 + b^2 + c^2 + d^2$$

$$a^3 + b^3 + c^3 + d^3$$

$$a^4 + b^4 + c^4 + d^4$$

$$a^3 b + a b^3 + a^3 c + b^3 c + a c^3 + b c^3 + a^3 d + b^3 d + c^3 d + a d^3 + b d^3 + c d^3$$

$$a^2 b c + a b^2 c + a b c^2 + a^2 b d + a b^2 d + a^2 c d + b^2 c d + a c^2 d + b c^2 d + a b d^2 + a c d^2 + b c d^2$$

$$-4 a b c d$$

比較定理と進行波解の単調性
(COMPARISON THEOREM AND MONOTONICITY PROPERTY OF TRAVELLING WAVES)

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Abstract. Given an equation with certain symmetry it is important, from the point of view of applications, to study whether or not its solutions inherit the same type of symmetry. In the previous work with Prof. H. Matano (see [5]), we consider this problem in the class of equations in which the comparison principle holds. Such a class of equations form the so-called 'order-preserving dynamical systems'. We showed that, in an order-preserving dynamical system having a symmetry property corresponding to a connected group G, any stable equilibrium point is G-invariant. Furthermore, we applied our result to partial differential equations, and discussed the instability of stationary solutions for an evolution equation of surfaces and the monotonicity property of stable travelling waves for a competition system of nonlinear diffusion equations.

In this paper, we ristrict our attention to travelling waves and discuused the monotonicity property with respect to translation of stable travelling waves for a more general class of nonlinear diffusion equations —cooperation systems and degenerate diffusion equations.

1. はじめに

回転対称性や平行移動による不変性など, 何らかの意味で対称性をもつ方程式において, 解が方程式と同じ対称性をもつかどうかは, 生物学や物理学への応用上も興味深いことである.

第 3 回関数空間セミナーでは, ある種の比較定理の成り立つ系においては, 安定解への上で述べたような対称性の遺伝が常に起こることを報告した. 具体的には, 連結群の作用している順序保存力学系においては安定解は群の作用に関する「対称性」あるいは「ある種の単調性」をもつことを示した. また, その応用として, 曲面の発展方程式の定常解の不安定性 および 二種競争系の安定な進行波解の単調性を示した.

この結果を用いると, さらにさまざまな方程式系の解の性質を調べることができる. 今回は, 考察の対象を進行波解に限定し, 得られた結果を報告する. 詳しく述べると, n 種協

調系の安定な進行波解の平行移動に関する単調性を示す. その系として, 二種競争系の安定な進行波解の単調性が導かれる. また, 退化放物型方程式の進行波解に対する結果も併せて報告する.

2. 前回の結果

X は順序距離空間, すなわち, (閉) 半順序構造 ($\preceq$) をもつ完備距離空間とする. X の任意の元 u, v に対し, これらの最大下界 $u \wedge v$ が存在して, 写像 $(u, v) \mapsto u \wedge v : X \times X \to X$ は連続であると仮定する. X の距離を d で表し, $u \preceq v$ かつ $u \neq v$ を $u \prec v$ と表す.

$\{\Phi_t\}_{t \geq 0}$ は以下の仮定 (Φ1)-(Φ3) をみたす X から X への写像の半群とする.

(Φ1) 順序保存性 ($u \preceq v \Rightarrow \Phi_t u \preceq \Phi_t v$, $\forall t \geq 0$).

(Φ2) 上半連続性 (点列 $\{u_n\}_n$, $\{\Phi_t(u_n)\}_n$が収束するなら, $\lim_{n\to\infty} \Phi_t(u_n) \preceq \Phi_t(\lim_{n\to\infty} u_n)$, $\forall t \geq 0$).

(Φ3) 軌道 $\{\Phi_t u\}_{t \geq 0}$ が単調減少 ($t \leq t' \Rightarrow \Phi_t u \succeq \Phi_{t'} u$) でかつ有界なら, 相対コンパクト.

G は X の元に作用する連結で距離付け可能な位相群で次をみたすとする.

(G1) 順序保存性 ($u \preceq v \Rightarrow gu \preceq gv$, $\forall g \in G$).

(G2) 写像 Φ_t との可換性 ($g\Phi_t u = \Phi_t(gu), \forall g \in G$, $\forall t \geq 0$).

<u>定義.</u> $\{\Phi_t\}_{t \geq 0}$ の平衡点 $\overline{u} \in X$ が '下から安定' であるとは, 任意の $\epsilon > 0$ に対しある $\delta > 0$ が存在して, $d(v, \overline{u}) < \delta$ をみたす任意の $v \prec \overline{u}$ に対し, $d(\Phi_t v, \overline{u}) < \epsilon$, $\forall t \geq 0$ が成り立つことをいう.

以下, $B_\delta(e)$ は G の単位元 e の δ-近傍を表すものとする.

<u>定理 0.</u> $\{\Phi_t\}_{t \geq 0}$ の平衡点 $\overline{u}$ が, 以下をみたすとする. (1) $\overline{u}$ は下から安定. (2) $\{\Phi_t\}_{t \geq 0}$ の任意の平衡点 $u \prec \overline{u}$ に対し, ある $\delta > 0$ が存在して $gu \prec \overline{u}$, $\forall g \in B_\delta(e)$. このとき, 任意の $g \in G$ に対して $g\overline{u} \succeq \overline{u}$ または $g\overline{u} \preceq \overline{u}$ が成り立つ.

とくに, 群 G が平行移動群 $\mathbb{R}$ である場合には, $\mathbb{R}$ が全順序集合であることから次が導かれる.

<u>系.</u> $\overline{u}$ は上の定理の通りとし, さらに $G = \mathbb{R}$ とする. このとき, 次の (i)–(iii) のいずれかが成立.

(i) $\overline{u}$ は G-不変 ($g\overline{u} = \overline{u}$, $\forall g \in G$);

(ii) $g\overline{u}$ は $g \in \mathbb{R}$ の X 値関数として狭義単調増大 ($g < g'$ ならば $g\overline{u} \prec g'\overline{u}$);

(iii) $g\overline{u}$ は $g \in \mathbb{R}$ の X 値関数として狭義単調減少 ($g < g'$ ならば $g\overline{u} \succ g'\overline{u}$).

<u>注意.</u> 定理の結果は, 仮定 (a) を (a′) におきかえても正しい.

(a′) 任意の $\epsilon > 0$ に対しある $\delta > 0$ が存在して, $d(v, \overline{u}) < \delta$ ならばある $g \in G$ が存在し $d(\Phi_t v, g\overline{u}) < \epsilon$, $\forall t \geq 0$.

3. 安定な進行波解の単調性

3.1. まず, 簡単な例 (しかしながら重要例) として, 非線形拡散方程式

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \qquad x \in \mathbb{R},\ t > 0 \tag{1}$$

を考える. 方程式 (1) の解 u で

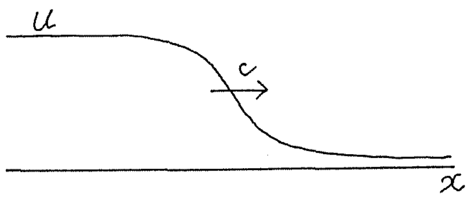
$$u(t, x) = \phi(x - ct) \quad (c \text{ は定数 })$$

の形に表せるものを進行波解という. ここでは以下の条件をみたすもののみを扱う.

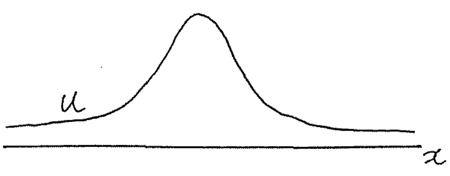
$$\lim_{x \to \pm\infty} \phi(x) = u^{\pm} \qquad \text{(複号同順)}$$

ただし, $u^{\pm}$ は常微分方程式の意味で安定な平衡解である (すなわち, $f(u^{\pm}) = 0$, $f'(u^{\pm}) < 0$) と仮定する. 関数 ϕ が単調増加 (または, 減少) 関数であるときに進行波解は '単調' であるという.

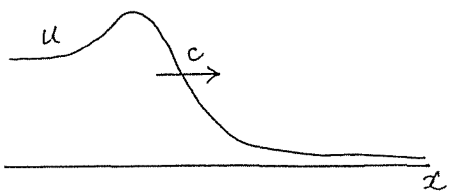
定理 1. 方程式 (1) の安定な進行波解は単調である.



単調な進行波解



孤立進行波



その他単調でない進行波解

これより, 孤立進行波 (ソリトン) やその他単調でない進行波解はすべて不安定であることがただちに導かれる. 定理 1 はよく知られた結果であるが, 2 節で述べた結果を適用しても簡単に示せる.

定理 1 の証明の方針.

速度 c の安定な進行波解 $\overline{u}$ に対し,

$$\text{順序距離空間 } X = C(\mathbb{R}) = \{\mathbb{R} \text{ 上有界で一様連続な関数全体の空間}\},$$

$$\text{写像 } \Phi_t u(x) = \Psi_t u(x+ct)$$

とおく. ただし, X は $u(x) \leq v(x)$, $\forall x \in \mathbb{R}$ のとき $u \preceq v$ とする順序構造をもち, Ψ_t は方程式 (1) の定める半流を表すものとする. 2 節の系を適用することにより, 関数 $\phi(x) = \overline{u}(x+ct)$ が単調減少関数または単調増加関数のいずれかであることがわかる.

3.2. 次に, n 種協調系

$$(2)\qquad \begin{cases} \frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} + f_1(u_1, \cdots, u_n), \qquad x \in \mathbb{R},\ t > 0, \\ &\vdots \\ \frac{\partial u_n}{\partial t} &= \frac{\partial^2 u_n}{\partial x^2} + f_n(u_1, \cdots, u_n), \qquad x \in \mathbb{R},\ t > 0 \end{cases}$$

を考える. 方程式系 (2) が協調系とは, $\partial f_i / \partial u_j \geq 0$ $(i \neq j)$ となることをいう. 方程式系 (2) の進行波解

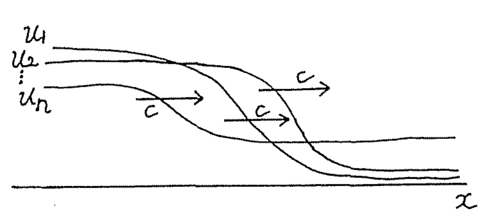
$$(u_1(t,x), \cdots, u_n(t,x)) = (\phi_1(x-ct), \cdots, \phi_n(x-ct)) \quad (c \text{ は定数 })$$

で, 以下をみたすものを考える.

$$\lim_{x \to \pm\infty} (\phi_1(x), \cdots, \phi_n(x)) = u^{\pm} = (u_1^{\pm}, \cdots, u_n^{\pm}) \qquad (\text{複号同順})$$

ただし, $u^{\pm}$ は常微分方程式の意味で安定な平衡解であるとする. 関数 ϕ_i がすべて単調増加 (または, すべて単調減少) 関数であるときに進行波解は '単調' であるという.

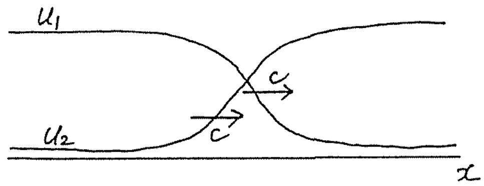
定理 2. n 種協調系の安定な進行波解は単調である.



n 種協調系の単調な進行波解

各 ϕ_i が高々有限個の臨界点をもつ場合には, 安定な進行波解の単調性が証明されている ([8]).

方程式系 (2) において $n = 2$, $\partial f_i/\partial u_j \le 0$ $(i \ne j)$ をみたすとき二種競争系という. このときには, 進行波解が単調とは $\phi_1, -\phi_2$ がともに単調増加 (または, ともに単調減少) と定義する. Lotka-Volterra 型二種競争系 $(f_i(u_1, u_2) = u_i(\alpha_i - \beta_i u_1 - \gamma_i u_2))$ に対しては, 孤立定常波の不安定性が知られていた ([3]) が, 第 3 回関数空間セミナー ([5]) では, 一般の二種競争系について安定な進行波解の単調性 (したがって, 単調でない進行波解はすべて不安定であること) を示した. (u_1, u_2) を $(u_1, -u_2)$ とおくと, 方程式系 (2) が協調系に書き直せることを用い, 定理 2 を適用しても結果は導ける.



二種協調系の単調な進行波解

<u>定理 2 の証明の方針.</u>

速度 c の進行波解 $\overline{u}$ に対し,

$$\text{順序距離空間 } X = \underbrace{C(\mathbb{R}) \times \cdots \times C(\mathbb{R})}_{n \text{ times}},$$

$$\text{写像 } \Phi_t u(x) = \Psi_t u(x + ct) \qquad (\Psi_t \text{ は方程式 (2) の定める半流})$$

とおく. ただし, X の順序構造は次により定める.

$$u = (u_1, \cdots, u_n) \preceq v = (v_1, \cdots, v_n) \Longleftrightarrow u_i \le v_i(x) \ \text{a.e.}\ x \in \mathbb{R}, \ \forall i.$$

2 節の系を適用することにより, 定理 2 を得る.

3.3. 最後に, 退化放物型方程式

$$\frac{\partial u}{\partial t} = \frac{\partial^2 (u^m)}{\partial x^2} + f(u), \qquad x \in \mathbb{R}, \ t > 0 \tag{3}$$

を考える. ここで $m > 1$ は定数で, 非負解のみを扱う. 方程式 (1) と同様, 以下を得る.

<u>**定理 3.**</u> 退化放物型方程式 (3) の安定な進行波解は単調である.

定理 3 の証明の方針.

速度 c の進行波解 $\overline{u}$ に対し, 順序距離空間 $X = \overline{\bigcup_{\alpha \in \mathbb{R}} X_\alpha}$

$$(\text{ここで}, X_\alpha = \{\phi + \xi \mid \xi \in L^1(\mathbb{R}) \text{ s.t. } 0 \leq \phi(x) + \xi(x) \leq \phi(x+\alpha) \text{ a.e. } x \in \mathbb{R}\})$$

とおき, 定理 1 と同様に証明する.

典型例 $f(u) = u(u-\alpha)(1-u)$ を含むあるクラスの非線形項 f に対して, 単調な進行波解は安定であることが証明されている ([2]) が, 定理 3 は逆の成立を主張する. また, 本講演の方法を用いると

$$\frac{\partial u}{\partial t} = \frac{\partial^2 a(u)}{\partial x^2} + f(u), \qquad x \in \mathbb{R},\ t > 0$$

(ただし, $a \in \mathcal{C}^{1+\beta}([0,\infty)) \cap \mathcal{C}^3((0,\infty))$ $(0 < \beta < 1)$, $a'(s) > 0, \forall s > 0$) という形の放物型方程式に結果を拡張することができる.

注意. 2 節で述べた注意を用いると, 定理 1–3 は, 安定な進行波解を '軌道安定' な進行波解としても成り立つことがわかる.

REFERENCES

1. Q. Fang and Y. Kan-on, Stability of monotone travelling waves for competition model with diffusion, in preparation.
2. Y. Hosono, Traveling wave solutions for some density dependent diffusion equations, Japan J. Appl. Math., **3** (1986), 163–196.
3. Y. Kan-on, Instability of stationary solution for Lotka-Volterra competition model with diffusion, in preparation.
4. H. Matano and T. Ogiwara, Stability analysis in order preserving systems in the presence of symmetry, in preparation.
5. T. Ogiwara, Stability analysis in order preserving systems in the presence of symmetry, Hokkaido Univ. Technical Report Series in Math., **140** (1995), 18–23.
6. T. Ogiwara, Monotonicity properties of stable travelling wave solusions for some degenerate diffusion equations, in preparation.
7. H. Protter and H. Weinberger, Maximum principles in differential equations, Prentice Hall, Englewood Cliffs, NJ, 1967.
8. A. I. Volpert, Vit. A. Volpert and Vl. A. Volpert, Traveling wave solutions of parabolic systems, Trans. Math. Monographs, **140**, Amer. Math. Soc., Providence, 1994.

LOCAL WELL-POSEDNESS FOR THE HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We establish the local well-posedness of the Cauchy problem for the higher-order nonlinear Schrödinger equation, employing a smoothing property of the KdV equation and a contraction argument, and local existence results of the periodic boundary value problem under the non resonance condition, applying a contraction argument for the specified space.

In this paper, we consider the following initial value problem for the higher-order nonlinear Schrödinger equation:

$$(1)\qquad \begin{cases} iu_t + i\alpha u_{xxx} + \beta u_{xx} + \gamma |u|^2 u + i\delta |u|^2 u_x + i\epsilon (|u|^2)_x u = 0, \\ (x,t) \in \Omega \times (-T, T), \\ u(x,0) = u_0(x), \end{cases}$$

where $\alpha, \beta, \gamma, \delta$ are real constants and ϵ is a complex constant, u is a complex valued function and T is a positive constant to be determined later. As for the domain Ω, we deal with two cases $\mathbb{R}$ and $\mathbb{T}$ where $\mathbb{T}$ is a one dimensional torus which implies the periodic boundary condition.

The equation (1) is modeled by A. Hasegawa and Y. Kodama [2,3] for a propagation of a signal in optical fibers in order to understand several phenomena which can not be explained by the following nonlinear Schrödinger equation:

$$iu_t + \frac{1}{2} u_{xx} + |u|^2 u = 0.$$

Our interest is to prove well-posedness results in a weak class for the initial value problem (1). The notion of the well-posedness here is that existence, uniqueness, persistence property (i.e., the solution describes a continuous curve in X whenever $u_0 \in X$) and continuous dependence of the solution upon the data. Particularly, the local well-posedness means that the fact mentioned above holds in the local time, on the other hand, the global well-posedness means the fact mentioned above holds in the global time.

The difficulty to deal with the equation (1) is that it has space derivatives in nonlinear terms. Then by rewriting the equation (1) as the integral equation and using a fix point argument to show the existence result, it occurs the so-called "loss of derivative". It is very difficult to regain derivatives by the dispersive of the Schrödinger, but due to the effect of the dispersive term of the KdV, we are able to overcome this difficulty. Hence, we assume $\alpha \neq 0$ for the equation (1) in this paper.

Recently, J. Bourgain [1] introduced a Fourier restriction norm and used a simple identity, then he regained space derivatives in nonlinear term of the KdV equation. Furthermore, C. E. Kenig, G. Ponce and L. Vega [4] used a similar norm and a smoothing effect of solution to the linear KdV equation, which allow us to regain derivatives up to one. The aim in this paper is to investigate the time local well-posedness in weak space for (1) by using the method due to Bourgain [1] and Kenig-Ponce-Vega [4,5].

For the case of $\Omega = \mathbb{R}$, the well-posedness results for the initial value problem (1) were studied by C. Laurey [6,7]. She showed that the initial value problem (1) is locally well-posed in $H^s(\mathbb{R})$ for $s > 3/4$. For the case of $\Omega = \mathbb{R}$, our result is an improvement of the result by Laurey.

We prepare the following notations.

Definition 1. We denote $\varphi_{\alpha,\beta}$ is $C^2(\mathbb{R})$ function satisfying

$$\varphi_{\alpha,\beta}(\xi) = \begin{cases} 1, & \text{if } |\xi| \geq 2|\beta|/|\alpha|, \\ 0, & \text{if } |\xi| \leq |\beta|/|\alpha|. \end{cases}$$

For $s, b, d_3 \in \mathbb{R}, s \geq 0$, define the spaces $X_{s,b}, \dot{X}_{s,b,d_3}, \dot{X}^{(1)}_{s,b}$ and $\dot{H}^s_{d_3}$ to be respectively the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R})$ with respect to the norms:

$$\|f\|_{X_{s,b}} = \left(\iint_{-\infty}^{\infty} (1 + |\tau - \alpha\xi^3 + \beta\xi^2|)^{2b} (1 + |\xi|)^{2s} |\widehat{f}(\xi,\tau)|^2 d\xi d\tau \right)^{1/2},$$

$$\|f\|_{\dot{X}_{s,b,d_3}} = \left(\iint_{-\infty}^{\infty} (1 + |\tau - \alpha\xi^3 + (3\alpha d_3 + (\beta - 3\alpha d_3)\varphi_{\alpha,\beta-3\alpha d_3}(\xi - d_3))\xi^2|)^{2b} \right.$$
$$\left. \times |\xi - d_3|^{2s} |\widehat{f}(\xi,\tau)|^2 d\xi d\tau \right)^{1/2},$$

$$\|f\|_{\dot{X}^{(1)}_{s,b}} = \left(\iint_{-\infty}^{\infty} (1 + |\tau - \alpha\xi^3 + \beta\xi^2|)^{2b} |\xi|^{2s} |\widehat{f}(\xi,\tau)|^2 d\xi d\tau \right)^{1/2},$$

$$\|f\|_{\dot{H}^s_{d_3}} = \left(\int_{-\infty}^{\infty} |\xi - d_3|^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2},$$

For $s, c_0 \in \mathbb{R}$, define the spaces Y_s and Y'_s for $f \in L^\infty_t(\mathbb{R})H^s_x(\mathbb{T})$ such that the following norms are finite:

$$\|f\|_{Y_s} = \left(\sum_n (1 + |n|)^{2s} \int_{-\infty}^{\infty} (1 + |\tau - \alpha n^3 + \beta n^2 + c_0 n|) |\widehat{f}(n,\tau)|^2 d\tau \right)^{1/2}$$
$$+ \left(\sum_n (1 + |n|)^{2s} \left(\int_{-\infty}^{\infty} |\widehat{f}(n,\tau)| d\tau \right)^2 \right)^{1/2},$$

$$\|f\|_{Y'_s} = \|f\|_{Y_s} + \left\| (1 + |n|)^{1/2} \left(\int_{-\infty}^{\infty} (1 + |\tau - \alpha n^3 + \beta n^2 + c_0 n|) |\widehat{f}(n,\tau)|^2 d\tau \right)^{1/2} \right\|_{l^\infty_n}$$
$$+ \left\| (1 + |n|)^{1/2} \left(\int_{-\infty}^{\infty} |\widehat{f}(n,\tau)|^2 d\tau \right)^{1/2} \right\|_{l^\infty_n}.$$

Our main results are the following two theorems.

Theorem 1. *Given $s \geq 1/4$. Then there exists $b > 1/2$ such that for any $u_0 \in H^s(\mathbb{R})$, there exists $T = T(\|u_0\|_{H^{1/4}}, \alpha, \beta, \gamma, \delta, \epsilon) > 0$ and a unique solution $u(t)$ of the initial value problem (1) for $\alpha \neq 0$ in the time interval $[-T, T]$ satisfying*

$$u \in C([-T,T] : H^s(\mathbb{R})), u \in X_{s,b}, \tag{2}$$

$$\gamma|u|^2u + i\delta|u|^2u_x + i\epsilon(|u|^2)_x u \in \dot{X}^{(1)}_{s,b-1}, u_t \in \dot{X}^{(1)}_{s-3,b-1}. \tag{3}$$

For any $T' \in (0,T)$, there exists a neighborhood $\mathcal{V}$ of $u_0 \in H^s(\mathbb{R})$ such that the map $\tilde{u}_0 \mapsto \tilde{u}(t)$ from $\mathcal{V}$ into the class defined by (2) with T' instead of T is Lipschitz.
In addition, when $s = 1/4$, the above result holds for $d_3 \in \mathbb{R}$ with $\dot{X}_{1/4,b,d_3}, \dot{X}_{1/4,b-1,d_3}, \dot{X}_{1/4-3,b-1,d_3}$ and $\dot{H}^{1/4}_{d_3}$ replaced by $X_{s,b}, \dot{X}^{(1)}_{s,b-1}, \dot{X}^{(1)}_{s-3,b-1}$ and H^s, respectively, with $T = T(\|u_0\|_{\dot{H}^{1/4}_{d_3}}, \alpha, \beta, \gamma, \delta, \epsilon, d_3)$ such that $T(d_3) \longrightarrow 0$ as $|d_3| \longrightarrow \infty$.

Theorem 2. *Given $s > 1/2$. For any $u_0 \in H^s(\mathbb{T})$, there exists $T = T(\|u_0\|_{H^s}, \alpha, \beta, \gamma, \delta, \epsilon) > 0$ and a unique solution $u(t)$ of the initial value problem (1) for $\alpha \neq 0$ and $2\beta/3\alpha \in \mathbb{R}\backslash\mathbb{Z}$ in the time interval $[-T, T]$ satisfying*

$$u \in C([-T,T] : H^s(\mathbb{T})),\ u \in Y_s,\ \int_{\mathbb{T}} |u(x,t)|^2 dx = \int_{\mathbb{T}} |u_0(x)|^2 dx, \tag{4}$$

for $c_0 = (\delta + \epsilon)\|u_0\|^2_{L^2_x}$.
For any $T' \in (0,T)$, there exists a neighborhood $\mathfrak{V}$ of $u_0 \in H^s(\mathbb{T})$such that the map $\tilde{u}_0 \mapsto \tilde{u}(t)$ from $\mathfrak{V}$ into the class defined by (4) with T' instead of T is Lipschitz.
Moreover, if $u_0 \in H^{s'}(\mathbb{T})$ with $s' > s$, then the above result holds with s' instead of s in the same time interval $[-T, T]$.
Furthermore, if $\epsilon = 0$, then the above result holds for any $\alpha \neq 0$ and β.
In addition, if $\beta/\alpha \in \mathbb{Q}$, then the above result holds in $H^{1/2}(\mathbb{T})$.
Moreover, if $\beta/\alpha \in \mathbb{Q}$, $\delta = \epsilon$ and $2\beta/3\alpha \in \mathbb{R}\backslash\mathbb{Z}$, then the above result holds with Y'_s for $s > 1/4$ instead of Y_s, when the initial value satisfies $\|u_0\|_{H^s} + \|(1+|n|)^{1/2}\widehat{u_0}(n)\|_{l^\infty_n} < \infty$.

The main method to prove Theorem 1 and Theorem 2 is based on the argument of the mKdV and the KdV equation similar to Bourgain [1] and Kenig-Ponce-Vega [4,5], respectively. First, we define $u(x,t) = v(x - d_1 t, d_2 t)e^{i(d_3 x + d_4 t)}$.

To prove the local well-posedness in $H^s(\mathbb{R})$ for $s \geq 1/4$, we put $d_1 = \beta^2/3\alpha, d_2 = 1, d_3 = \beta/3\alpha, d_4 = -2\beta^3/27\alpha^2$, then the equation (1) becomes the complex valued modified KdV type equation. Hence we can assume that $\beta = 0$ in (1). By using a smoothing effect of solution to the linear KdV equation and the contraction argument in $X_{s,b}$, we prove the local well-posedness. On the other hand, when we work in $\dot{H}^{1/4}_{d_3}$ where $d_3 \in \mathbb{R}$ is a parameter which indicates the position of the singularity point of $\widehat{u_0}(\xi)$, there is some choices of d_1, d_2 and d_4. In order to simplify the Fourier restriction norm of the solution to the equation (1), we put $d_1 = 3\alpha d_3^2, d_2 = 1, d_4 = -2\alpha d_3^3$. But different from the modified KdV equation, the dispersive of the linear part of the equation is not similar to the KdV. In order to make use of the dispersion of the linear part similar to the KdV, we rewrite the equation as follows:

$$\begin{aligned} &iv_t + i\alpha v_{xxx} + (\beta - 3\alpha d_3)((Pv)_{xx} + 2id_3(Pv)_x - d_3^2 Pv) \\ &\quad + (\beta - 3\alpha d_3)(((1-P)v)_{xx} + 2id_3((1-P)v)_x - d_3^2(1-P)v) \\ &\quad + (\gamma - \delta d_3)|v|^2 v + i\delta|v|^2 v_x + i\epsilon(|v|^2)_x v = 0. \end{aligned}$$

where $\mathfrak{F}_x(Pf)(\xi) = \widehat{f}(\xi)\varphi_{\alpha,\beta-3\alpha d_3}(\xi)$. Then same as in $H^s(\mathbb{R})$ for $s \geq 1/4$, we prove the local well-posedness. This proof is completely different from the previous proof in [6,7].

On the other hand, for the periodic boundary condition case, we also use a Fourier restriction norm and the identity as follows:

$$
\begin{aligned}
(5) \qquad & (\tau - \alpha n^3 + \beta n^2) - (\tau - \tau_1 - \tau_2 - \alpha(n - n_1 - n_2)^3 + \beta(n - n_1 - n_2)^2) \\
& - (\tau_1 - \alpha n_1^3 + \beta n_1^2) - (\tau_2 - \alpha n_2^3 - \beta n_2^2) \\
= & - 3\alpha(n_1 + n_2)(n - n_1)(n - n_2 - \frac{2\beta}{3\alpha}).
\end{aligned}
$$

which is used in [1] for $\alpha = 1$ and $\beta = 0$. But there is something different from the usual modified KdV equation. By using the L^2 conservation law, we rewrite the linear part of the equation (1) and dividing the nonlinear part that satisfies where the right-hand side of (5) is zero or not zero. When $\epsilon \neq 0$, the two cases of $n_1 + n_2 = 0$ and $n - n_1 = 0$ can be avoided, but the case of $n - n_2 - 2\beta/3\alpha = 0$ can not be avoided. So the assumption that $2\beta/3\alpha$ is not integer will be necessary in our proof. However by also dividing the nonlinear terms, even if $2\beta/3\alpha$ is integer, we can avoid the case of $n - n_2 - 2\beta/3\alpha = 0$ for $\epsilon = 0$. This is a different point from the real valued modified KdV case. For the proof in $H^s(\mathbb{T})$ for $s > 1/2$, we use the method Kenig-Ponce-Vega [5]. This method is valid for general α and β. On the other hand, for the proof in $H^{1/2}(\mathbb{T})$, we use the Strichartz type estimate. To prove it, we need to assume $\beta/\alpha \in \mathbb{Q}$. But the author does not know whether that the above Strichartz type estimate holds for general α and β or not. Accordingly, we need to assume $\beta/\alpha \in \mathbb{Q}$. For the proof in $H^s(\mathbb{T})$ for $1/4 < s < 1/2$, by the result in [5], we have to do more strong restriction to the class of the solution. In our proof, the assumption that $\|(1 + |n|)^{1/2}\widehat{u_0}(n)\|_{l_n^\infty} < \infty$ will be necessary for the data.

References

[1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, Part I: Schorödinger Equation, Part II: The KDV-Equation*, Geometric and Functional Anal. **3** (1993), 107–156, 209–262.

[2] A. Hasegawa and Y. Kodama, *Nonlinear pulse propagation in a monomode dielectric guide*, I. E. E. E. J. Quant. Elect. **23** (1987), 510–524.

[3] Y. Kodama, *Optical solitons in a monomode fiber*, J. Phys. Stat. **39** (1985), 597–614.

[4] C. E. Kenig, G. Ponce and L. Vega, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke. Math. J. **71** (1993), 1–21.

[5] ———, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), 573–603.

[6] C. Laurey, *Le problème de Cauchy pour une équation de Schrödinger non linéaire d'order 3*, C. R. Acad. Sc. Paris. t. 315 Séri **1** (1992), 165–168.

[7] ———, *The Cauchy problem for a third order nonlinear Schrödinger equations*, Nonlinearity (to appear).

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Super-reflexive Banach spaces and norms of the admissible matrices

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Definitions and notations

Let us recall the classical definition of the Banach-Mazur distance. For two isomorphic Banach spaces E and F, put

$$d(E,F) = \inf\{\|T\| \cdot \|T^{-1}\| \ ; \ T \text{ is an isomorphism from E onto F}\}.$$

For Banach spaces X and Y, Y is said to be finitely representable (f.r.) in X if for each (some) $\lambda > 1$ and for each finite-dimensional subspace F of Y, there is a finite-dimensional subspace E of X with dim E = dim F such hat $d(E,F) < \lambda$.

Super-reflexivity : X is said to be super-reflexive if any Banach space f.r. in X is reflexive.

Remark. Consider (P), a property of Banach spaces. We say that (P) is a super-property provided that if X has (P) and Y is f.r. in X, then Y has (P). It is easy to see that uniform convexity, uniform non-squareness, B-convexity, type p and cotype q are super-properties. On the oter hand, reflexivity and Radon-Nikodym property (RNP) are not super-properties.

A finite sequence of signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is said to be admissible if all + signs are before all - signs (so there are n different sequences of admissible signs of length n, counting the sequence +,+......,+).

J_n-convexity and J-convexity : X is said to be J_n-convex ($n \geqq 2$) if there is a $\delta > 0$ such that for any $x_1, x_2, \ldots, x_n$ in the unit ball B_X,

$$\min\{\|\Sigma \varepsilon_j x_j\| \ ; \ (\varepsilon_j) \text{ is admissible}\} \leqq (1 - \delta)n.$$

X is said to be J-convex if it is J_n-convex for some $n \geqq 2$.

Admissible matrices : An $n \times n$ matrix $A_n = (a_{ij})$ is said to be admissible if $a_{ij} = 1$ for $j \leqq n - i + 1$ and $a_{ij} = -1$ for $j > n - j + 1$.

Main results

We first recall the following facts (cf. [1], [2], [5], [6], [7]):

Hilbert space (von Neumann-Jordan equality) ⇔ (2,2')-Clarkson inequality ⇒ (p,p')-Clarkson inequality ($1<p\leqq 2$, $1/p+1/p'=1$) ⇒ p-uniformly smooth (p'-uniformly convex) ⇒ uniformly smooth (uniformly convex) ⇒ uniformly non-square ⇔ J_2-convex ⇒ J_n-convex for all $n\geqq 2$ ⇒ J_n-convex for some $n\geqq 2$ ⇔ J-convex ⇔ super-reflexive ⇔ super-RNP ⇒ B-convex (type p for some p>1) and reflexive ⇒ cotype q for some $q<\infty$.

Remarks. (1) Any Hilbert space has all super-properties (cf. [3]).
(2) The property of finite cotype (cotype q for some $q<\infty$) is the weakest super-property.
(3) The (p,p')-Clarkson inequality holds in X iff X is of type p (resp. cotype p') and its type (resp. cotype) constant is one (cf. [8]).

Theorem 1. (1) If X is J_n-convex, then it is J_{n+1}-convex.
(2) If X is J_n-convex and Y is a subspace of X, then Y and X/Y is J_n-convex.

Theorem 2. For a Banach space X, the following assertions are equivalent.
(1) X is J_n-convex.
(2) $\| A_n : \ell_p^n(X) \to \ell_p^n(x) \| < n$ for any (some) $p > 1$.
(3) $\| A_n : \ell_p^n(X) \to \ell_{p'}^n(X) \| < n^{2/p'}$ for any (some) $p > 1$.
(4) $\| A_n : \ell_r^n(X) \to \ell_s^n(X) \| < n^{1/r'+1/s}$ for any (some) $r > 1$ and $s > 1$.

Remark. The above theorem is false for p = 1 or r = 1. More precisely, it is shown that for any $n \geqq 2$ and for any Banach space X with dim $X \geqq 1$, the equality

$$\| A_n : \ell_1^n(X) \to \ell_s^n(X) \| = n^{1/s}$$

holds for all $1 \leqq s \leqq \infty$.

Corollary 3. (1) X is J_n-convex if and only if X' is.
(2) X is uniformly non-square if and only if X' is.
(3) X is super-reflexive if and only if X' is.

Corollary 4. Let $1 < p < \infty$.
(1) The Lebesgue-Bochner space $L_p(X)$ is J_n-convex if and only if X is.
(2) $L_p(X)$ is super-reflexive if and only if X is.

References

[1] B. Beauzamy, Introduction to Banach spaces and their geometry, 2nd Ed., North Holland, Amsterdam-New York-Oxford, 1985.

[2] J. A. Clarkson, Uniformly convex spaces, Trans. Amer. Math. Soc. 40 (1936), 396-414.

[3] A. Dvoretzky, Some results on convex bodies and Banach spaces, Proc. Symp. on Linear Spaces, Jerusalem 1961, 123-160.

[4] P. Enflo, Banach spaces which can be given as equivalent uniformly convex norm, Israel J. Math. 13 (1972).

[5] R. C. James, Uniformly non-square Banach spaces, Ann. of Math. 80 (1964), 542-550.

[6] R. C. James, Super-reflexive Banach spaces, Canad. J. Math. 24 (1972), 896-904.

[7] P. Jordan and J. von Neumann, On inner products in linear metric spaces, Ann. of Math. 36 (1935), 719-723.

[8] M. Kato and Y. Takahashi, Type, cotype constants and Clarkson's inequalities for Banach space, to appear in Math. Nachr.

[9] M. Kato and Y. Takahashi, On the von Neumann-Jordan constant for Banach spaces, to appear in Proc. Amer. Math. Soc.

THE SAZONOV TOPOLOGY AND MEASURABLE NORMS

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Abstract

In this short note, we define some kinds of topology determined by μ-measurable seminorms. If cylindrical measure μ is of cotype 1, then the aforesaid topology becomes a quasi-sufficient Sazonov topology of real separable reflexive Banach spaces.

1　序

この小論では可測ノルムを用いて Sazonov 位相を定義することを目指す。初めにここで用いる主な記号を説明する。 Eは実可分 Banach 空間、E'は Eの位相的双対とし $C(E)$ で E上のシリンダー測度の全体、$P(E)$ で E 上の Radon 確率測度の全体を表わす。$P(E)\subset C(E)$ なることは明らかである。また$\Phi(E')$ で E' 上で定義された複素数値関数で次の条件を満たすようなϕ の全体を表わす。

(1)$\phi(0)=1$ 、(2) ϕ は正型、(3) E' の有限次元部分空間上へのϕ の制限は連続。

ここで$\mu\in C(E)$ のとき μの特性関数を$\hat{\mu}$ で表わすと

$$\Phi(E')=\{\hat{\mu};\mu\in C(E)\}$$

となる。

次に Sazonov 位相に関する定義を述べる ([6,12])。

定義　E' 上のベクトル位相を τ とする。

(1)「$\hat{\mu}\in\Phi(E')$ が τー連続 $\Rightarrow\mu\in P(E)$」が成り立つときにτ は十分 Sazonov 位相 (SS 位相) と言う。

(2)「$\mu\in P(E)\Rightarrow\hat{\mu}$ はτー連続」が成り立つときにτ は必要 Sazonov 位相 (NS 位相) と言う。

(3) τ が SS 位相 で NS 位相でもあるときに Sazonov 位相 (S 位相) といい、E' 上に S位相が存在するときに Eは S 空間であるという。

S 空間の例を挙げてみよう。

(1) $\mathbf{R}^n$ (Bochner の定理)

(2) Hilbert 空間 (Sazonov の 定理 [9])

S 位相 としては τ_{HS} とτ_m ([2,3,4]) がある。 τ_{HS} は Hilbert-Schmidt 作用素をすべて連続にするような最弱位相、τ_m はγー可測なセミノルムをすべて連続にするような最弱位相のことである。 ここでγ は標準 Gauss シリンダー測度である。

(3) Banach 空間はすべてが S 空間というわけには行かない。Mouchtari の結果 ([7]) で S 空間ならばL^0に埋め込み可能で cotype2 になるということが知られている。また逆にL^0に埋め込み可能で metric approximation property (m.a.p.) を持てば S 空間になることも知られている。 この場合 S 位相はτ_0である。これについては更に具体的に $L^p(1 \le p \le 2)$ 空間 は S 空間で、S 位相は $\tau_q(0 \le q < p)$ ということが知られている ([5,8])。

ここでの $\tau_q(0 \le q < 2)$ については次の節で詳しく述べる。

この小論では例の 2 で得られた結果を $L^p(1 < p \le 2)$ 空間上に拡張して考えることを目的としている。

2 準備と記法

初めに Radon 確率測度の order、シリンダー測度の type, cotype について述べる ([10,11])。ここで $-\infty < p \le +\infty$ とする。

$\mu \in P(E)$ のとき

$\|\mu\|_p = \{\int_E \|x\|^p d\mu(x)\}^{1/p}$ $(p \ne +\infty, p \ne 0)$

$\|\mu\|_\infty = ess.sup\|x\|$ (μ に関して本質的上限)

$\|\mu\|_0 = \exp \int_E \log \|x\| d\mu(x)$

$\|\mu\|_p < +\infty$ ならば μは order p であるという。

$\mu \in C(E), \xi \in E'$ のとき $\mu_\xi = \xi(\mu)$ とする。

$$\|\mu\|_p^\star = \sup_{\|\xi\| \le 1} \|\mu_\xi\|_p, \ \|\mu\|_p^\circ = [\inf_{\|\xi\| \ge 1} \|\mu_\xi\|_p]^{-1}$$

$\|\mu\|_p^\star < +\infty$ のときμ は type p であると言い、$\|\mu\|_p^\circ < +\infty$ のときμ は cotype p であると言う。

定義 $1 \le p \le 2$ のとき p' を conjugate index (i.e. $1/p + 1/p' = 1$) とする。ただし $p = 1$ のときは $p' = +\infty$ と考える。このとき $L^{p'}$ ($p = 1$ のときは $\sigma(L^\infty, L^1)$) 上の標準 p-stable symmetric シリンダー測度 γ_pとは、これの特性関数が$\hat{\gamma}_p(\xi) = \exp(-\|\xi\|_p^p)$ となるもののことである。 γ_2 は L^2 上での前述のγ と等しい。

命題 γ_p は type q であり且つ cotype q でもある。ここで q は $p \ne 2$ のときは $\forall q < p$ で $p = 2$ のときは $\forall q < +\infty$ である。

次に一般の Banach 空間上の p-stable symmetric シリンダー測度とΛ_pー作用素 ([5]) の定義を述べる。

定義 $\mu \in C(E)$ で $\hat{\mu}(\xi) = \exp(-\|T\xi\|_p^p)$ となるような E' から L^p への作用素 T が存在するときにμ を p-stable symmetric シリンダー測度という。 更にこのようなμ が Radon 測度であるときに 、この作用素 T を Λ_pー作用素 という。

$\mu \in C(E)$ のときに $E' \to L^0(\Omega, P)$ への次のような線形写像 T が対応する。

$$\mu_{\xi_1,\xi_2,\cdots,\xi_n}(Z) = P\{\omega \in \Omega; (T(\xi_1)(\omega), T(\xi_2)(\omega), \cdots, T(\xi_n)(\omega)) \in Z\}$$

特にμ が type p ($p \geq 0$) のときは Tは $E' \to L^p(\Omega, P)$ への連続な写像として定まる。この T をμに対する linear random function と言う。$\mu \in P(E)$ のとき対応する T を decomposed 作用素 と言う。

定義 $0 \leq p \leq 2$ のとき 位相τ_pは次のように定められる。
τ_0 : decomposed 作用素をすべて連続にするような最弱位相。
$\tau_p(0 < p \leq 2)$: Λ_pー 作用素をすべて連続にする最弱位相。

命題 位相 の 強弱を 大小関係 $<$ で表わすと $0 \leq q < p \leq 2$ ならば $\tau_2 \leq \tau_p \leq \tau_q \leq \tau_0$ となる。

この節の最後に 可測ノルムについて述べよう。$\mu \in C(E)$ とし、$\|\cdot\|$ を E上で定義されたセミノルムとする。

定義 次の条件が成り立つときに $\|\cdot\|$ は μー可測であるという。
「$\forall\epsilon > 0$ に対して $\exists G \in FD(E)$ ($FD(E)$ は Eの有限次元部分空間の全体) で、 $F \in FD(E')$,$F \perp G$ なる Fについては

$$\mu(\{x; \|x - F^{\perp}\| < \epsilon\}) \geq 1 - \epsilon$$

が成り立つ。」

上の定義で $F^{\perp} = \{y \in E; <x, y>= 0 \ for \ \forall x \in F\}$ とする。
E のセミノルム $\|\cdot\|$ に関する associated Banach 空間 を $E_{\|\cdot\|}$で表し $E \to E_{\|\cdot\|}$ への canonical map を i とする。 このとき $i(\mu)$ が $E_{\|\cdot\|}$上で Radon 拡大可能になるための必要十分条件が、$\|\cdot\|$ がμ-可測であるということである。
この定義は Dudley-Feldman-Le Cam ([1]) によるものであるがもう 1 つ可測ノルムというと Gross の定義がある。わかりやすく比較するために E が Hilbert 空間の場合にふたつの定義を書いてみよう。
$N_\epsilon = \{x \in E; \|x\| \leq \epsilon\}$ とおくと今のμー可測という定義は次のように書ける。

$$\forall\epsilon > 0 \ \exists G \in FD(E); \ \mu(P_F(N_\epsilon) + F^{\perp}) \geq 1 - \epsilon \ for \ \forall F \perp G, F \in FD(E)$$

(P_F は E から F への 直交射影)
これに対して Gross の定義は

$$\forall\epsilon > 0 \ \exists G \in FD(E); \ \mu(N_\epsilon \cap F + F^{\perp}) \geq 1 - \epsilon \ for \ \forall F \perp G, F \in FD(E)$$

である。

命題 (1) Hilbert 空間上で Gross の意味で γー可測なセミノルム $\|\cdot\|$ に対しては $\|\cdot\|$ より強くて m.a.p. を持つようなγー可測なセミノルムが存在する ([4])。
(2)Hilbert 空間上では γー可測という概念は Gross の意味でも Dudley-Feldman -Le Cam の意味でも同じである。

3 Quasi-Sazonov 位相 と 位相 $\mathbf{M}_\mu$

$\mu \in C(E)$ であるときに $\hat{\mu}$ と、μ に対応する linear random function T との E' 上での連続性は同等なので 、SS 位相というのは T が $(E',\tau) \to L^0$ への連続性 から $\mu \in P(E)$ をだすときの位相τ という形で言い換えられる。ここではこれを少し変えた次のような定義を与える。

定義 $\mu \in C(E)$ で type 1 であるとし、μ に対応する linear random function T が (E',τ) から L^1 への連続写像 であるときに $\mu \in P(E)$ となるならば、このベクトル位相 τ を quasi-SS (QSS) 位相と言う。又 逆に $\mu \in P(E)$ で order 1 であるときに μ に対する T が $(E',\tau) \to L^1$ へ連続であるときに τ を quasi-NS (QNS) 位相であると言う。

τ が QSS で 且つ QNS であるときに quasi-S 位相 (QS 位相) であると言う。
ここで 新しい位相 $\mathbf{M}_\mu$ を導入する。

定義 $\mu \in C(E')$ のとき

$$\mathbf{S} = \{\|\cdot\|; E'\text{上で定義された連続で}\mu\text{ー可測なセミノルム}\}$$

とする。このとき $\mathbf{S}$ に属するすべて の セミノルム を 連続にするような最弱位相 を $\mathbf{M}_\mu$ とする。

このとき 次のような 結果を得る。

定理 1 $1 < p \leq 2$、p' は p の conjugate index とする。μ が $L^{p'}$上のシリンダー測度で cotype q $(0 < q < p)$ であれば位相 $\mathbf{M}_\mu$は QSS ー位相になる。

(注) $p = 2$ の場合は SS 位相になる。

定理 2 $\mathbf{M}_{\gamma_p}$ は L^p の QS 位相である。

(注) $p = 2$ の場合は S 位相である。

更に次の結果が得られる。

定理 3 Eが 回帰的な実可分 Banach 空間であるならば cotype 1 の シリンダー測度 μ に対して $\mathbf{M}_\mu$は QSS 位相になる。

これらの証明に言及する前に、その考え方のもとになっている Schwartz の定理と p-summing 作用素、 p-Radonifying 作用素の関係について述べる。

Schwartz の定理 F, G は Banach 空間、u は F から G への連続線形写像とし、$p > -1$ とする。ρ が $\sigma(G', G)$ 上の cotype p のシリンダー測度で $u'(\rho)$ (u' は u の dual operator) は$\sigma(E', E)$ 上の order p の Radon 測度であるとする。このとき u は p-summing 作用素である。

命題 次の条件のいずれかを満たす時、$F \to G$への p-summing 作用素は p-Radonifying 作用素になる。

(1) $1 < p < +\infty$

(2) G が回帰的 で F' が m.a.p. を持つ

(3) $p = 1$ で G が回帰的

[定理 1 の 証明の概略] λを L^p上の type 1 のシリンダー測度とする。対応する linear random function を Tとすると Tは $L^{p'} \to L^1(\Omega, P)$ への線形写像で位相 $\mathbf{M}_\mu$ に関して連続になる。$\mathbf{M}_\mu$ の定義から Tは次のように分解される。

$$T = \phi \circ i;\ i : L^{p'} \to L^{p'}_{\|\cdot\|};\ \phi : L^{p'}_{\|\cdot\|} \to L^1$$

ここで $\|\cdot\| \in \mathbf{S}$ である。

Schwartz の定理から i の双対作用素 i' は 1-summing 作用素 となる。L^pは回帰的であるから i' は 1-Radonifying 作用素 になる。従って $L^1 \to L^1$ への恒等写像に対応するシリンダー測度をν とすると$\phi'(\nu)$ はϕ に対応するシリンダー測度になってこれは type 1 になる。更に $i'(\phi'(\nu)) = \lambda$ となって Radon 測度になることが導かれる。以上が証明の概略である。

定理 2 については L^pが stable type q $(0 < \forall q < p)$ の Banach 空間であることから $\tau_0 = \tau_q$で、これが NS 位相のみならず QNS 位相にもなっていることが解る。また $\mathbf{M}_{\gamma_p}$ はτ_qより強い (このことについては後述する) ので $\mathbf{M}_{\gamma_p}$が QNS-位相になることから得られる。

定理 3 は、 定理 1 の証明が L^p が回帰的ということにのみかかるので成立がいえる。

$\tau_0 = \tau_q \leq \mathbf{M}_{\gamma_p}$ について述べる。$\Lambda_p(E', L^p)$ でΛ_pー作用素の全体、$\Pi_p(F, G)$ で Fから G への p-summing 作用素の全体を表わす。$\Lambda_p^{dual}(L^{p'}, E) = \{S; \exists T \in \Lambda_p(E', L^p)$ で $S = T'\}$ とすると $S(\gamma_p)$ は E で Radon 拡大可能となる。τ_q $(0 \leq q < p)$ は L^pの S 位相で $L^{p'}$ 上で$\tau_0 = \tau_q$となる。τ_qは$\Lambda_q(L^{p'}, L^q)$ に属するすべての作用素を連続にする最弱位相であるが $1 < q < p$ のとき$\Lambda_q(L^{p'}, L^q) = \Pi_q(L^{p'}, L^q)$ である。また $1 < q < p$ ならば q-summing は q-Radonifying になりγ_p は type q なのでこの作用素は$\Lambda_p^{dual}(L^{p'}, L^q)$ に属する。これより $\Pi_q(L^{p'}, L^q) \subset \Lambda_p^{dual}(L^{p'}, L^q)$ となり$\tau_0 = \tau_q \leq \mathbf{M}_{\gamma_p}$ がいえる。

L^2上では$\tau_{HS} < \tau_m$なる S 位相が存在する。ここで $\tau_{HS} = \tau_2$ であり $\tau_m = \mathbf{M}_{\gamma_2}$ である。$L^p(1 < p < 2)$ では $\tau_0 = \tau_q(0 \leq q < p)$ と、これより強い $\mathbf{M}_{\gamma_p}$ が QS 位相であることが解っている。L^p は stable type 1 なので type 1 のシリンダー測度について Radon 拡大可能な条件を示す QS 位相が実質上、連続なすべてのシリンダー測度の Radon 拡大可能になる条件を示していることになる。そこで残っている問題は S 位相と QS 位相の関係と、 $\tau_0 \neq \mathbf{M}_{\gamma_p}$ であることを具体的に示すことである。この結果は Linde の予想からも推測されるがまだはっきりした例をもって示されてはいない。

参考文献

[1] R.M.Dudley, J.Feldman, and L.Le Cam, On seminorms and probabilities, and ab-

stract Wiener spaces, Ann. Math. 93 (1971), 390-408.

[2] L.Gross, Measurable functions on Hilbert space, Trans. Amer. Math. Soc. 105 (1962), 372-390.

[3] L. Gross, Harmonic analysis on Hilbert space, Mem. Amer. Math. Soc. 46 (1963).

[4] H. H. Kuo, Gaussian measures in Banach spaces, Lecture Notes in Math. 463, Springer-Verlag, New York, 1975.

[5] W. Linde, Probability in Banach spaces - Stable and Infinitely Divisible Distributions, Wiley, New York, 1983.

[6] M. Maeda, Some remarks on the Sazonov topology, Proceedings of the Fourteenth Symposium on Applied Functional Analysis (1995), 82-87.

[7] D. H. Mouchtari, Certain general questions of the theory of probability measures in linear spaces, Theor. Probab. Appl. 18 (1973), 64-75.

[8] Y. Okazaki, Harmonic analysis in a Banach space, Mem. Fac. Sci. Kyushu Univ. 34 (1980), 27-69.

[9] V. V. Sazonov, On characteristic functionals, Theor. Verojatnost. i Primenen 3 (1958), 201-205.

[10] L. Schwartz, Cylindrical probabilities and p-summing and p-Radonifying maps, Seminar Schwartz, Notes on Pure Mathematics 7 (1973), 1-64.

[11] L. Schwartz, Geometry and Probability in Banach Spaces, Lecture Notes in Math. 852,Springer-Verlag, New York, 1981.

[12] Y. Takahashi, Some results on Bochner-type theorems, 数理解析研究所講究録 887「無限次元空間上の測度論、無限次元群の表現および関連した話題」(1994), 121-140.

Some inequalities associated with operator radii and Schur product

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Abstract For $\rho > 0$, let $w_\rho(\cdot)$ be ρ-operator radius on $\mathbb{M}_n$. For $A, B \in \mathbb{M}_n$, we denote the Schur product of A and B by $A \circ B$. In this paper we deal with the inequalities associated with $w_\rho(\cdot)$ and the Schur product of matrices. We show that for $\rho, \sigma > 0$, $0 < \alpha < 1$ and $A, B \in \mathbb{M}_n$ with non-negative entries,

$$w_{\alpha\rho+(1-\alpha)\sigma}(A^{(\alpha)} \circ B^{(1-\alpha)}) \le w_\rho(A)^\alpha w_\sigma(B)^{1-\alpha}.$$

Here $A^{(s)}$ means the s^{th} Schur power of A, that is, $A^{(s)} = (a_{jk}^s)$ for $A = (a_{jk})$ and positive number s.

1. はじめに

M_n 上には次のノルムが考えられる。即ち、Schatten の *p-norm* $\|\cdot\|_p$ $(1 \le p \le \infty)$

$$\|A\|_p \stackrel{def}{=} \{\sum_{k=1}^n s_k(A)^p\}^{1/p} \quad (1 \le p < \infty) \tag{1}$$

$$\|A\|_\infty \stackrel{def}{=} s_1(A) = \max_{k=1,\ldots,n} s_k(A), \tag{2}$$

ここで、$s_1(A) \ge \cdots \ge s_n(A)$ は A の特異値 (A^*A の固有値の平方根) を非増加順序に並べたものとする。$\|\cdot\|_p$ は次の意味で (*strong*) *unitary invariance* である。

$$\|UAV\|_p = \|A\|_p \quad (\text{unitary } U, V). \tag{3}$$

一方で、余りなじみのないノルムの族として ρ-*radii* $w_\rho(\cdot)$ $(0 < \rho \le 2)$ がある。$X \in M_n$ が ρ-*contraction* とは、$\mathcal{K} \supset \mathbb{C}^n$ なるヒルベルト空間 $\mathcal{K}$ と $\mathcal{K}$ 上のユニタリ作用素 U があって、

$$A^k = \rho P U^k|_{\mathbb{C}^n} \quad (k = 1, 2, \ldots) \tag{4}$$

を満たすこととする。ただし、P は $\mathcal{K}$ から $\mathbb{C}^n$ への直交射影である。このとき、$A \in M_n$ に対して、$w_\rho(A)$ は

$$w_\rho(A) \stackrel{def}{=} \inf\{\lambda > 0\ ;\ \frac{1}{\lambda}A \text{ is a } \rho\text{-contraction}\} \tag{5}$$

で定義される。定義 (4),(5) から、$w_\rho(A)$ は $2 < \rho < \infty$ に広げられる。しかし、$w_\rho(\cdot)$ $(2 < \rho < \infty)$ は 以下のように quasi-norm (凸性がない) にはなるがノルムではない。

$$w_\rho(A+B) \le \frac{\rho}{2}\{w_\rho(A) + w_\rho(B)\} \tag{6}$$

また、ρ-radius は次の性質をもつ。([1],[6] 参照)

$\cdot\ w_1(A) = \|A\|_\infty, \quad w_2(A) = w(A):$ the numerical radius
$\cdot\ \lim_{\rho \longrightarrow \infty} w_\rho(A) = r(A):$ the spectral radius
$\cdot\ \rho \longrightarrow w_\rho(\cdot)$ は 凸である。即ち、

$$w_{\lambda\rho+(1-\lambda)\sigma}(A) \le \lambda w_\rho(A) + (1-\lambda) w_\sigma(A)$$

$\cdot\ 1 \le \sigma \le \rho$ ならば $w_\rho(A) \le w_\sigma(A)$
$\cdot\ 1 \le \sigma \le \rho$ ならば $\sigma w_\sigma(A) \le \rho w_\rho(A)$
$\cdot\ w_\rho(\cdot)$ は *(weak) unitary invariance* を満たす

$$w_\rho(UAU^*) = w_\rho(A) \quad (\text{unitary } U)$$

$\|\cdot\|_p$ と $w_\rho(\cdot)$ とでは

(7) $$\|A\|_\infty = w_1(A) = \text{spectral norm } \|A\|$$

が成り立つ。
さらに、$w_\rho(A) \le 1$ の特徴づけとして次のことが知られている。

Theorem A. *(B.Sz.-Nagy and C. Foiaş, J.A.R.Holbrook)* $A \in M_n$, $\rho > 0$ とする。このとき、次の条件は同値である。
(i) $w_\rho(A) \le 1$
(ii) $r(A) \le \frac{\rho}{|\rho-1|}$, $\|zA\{\rho - z(\rho-1)A\}^{-1}\| \le 1 \ (|z| < 1)$
(iii) $-2Re[zA(I - zA)^{-1}] \le \rho I \ (|z| < 1)$

$|A| = (A^*A)^{1/2}$ とすると (3) から、

(8) $$\|A\|_p = \||A|\|_p$$

(9) $$0 \le B \le A \implies \|B\|_p \le \|A\|_p$$

がでる。ρ-radius は性質 (14) をもたない。

2. Hölder 型不等式
$A = (a_{ij})$ に対して、$[\![A]\!] = (|a_{ij}|)$ とする。また、実数を成分とする $A = (a_{ij}), B = (b_{ij}) \in M_n$ に対して $A \preceq B$ を $a_{ij} \le b_{ij}$ for all $1 \le i, j \le n$ で定義するとき、Theorem A を用いて次のことがいえる。

Theorem 1. $A \in \mathbb{M}_n$ と $0 < \rho < \infty$ に対して

(10) $$w_\rho(A) \le w_\rho([\![A]\!]),$$

(11) $$0 \preceq B \preceq A \implies w_\rho(B) \le w_\rho(A)$$

がいえる。

証明の概略. 定理は次のことがいえるとよい。

$$w_\rho([\![A]\!]) = 1 \Longrightarrow w_\rho(A \circ D) \le 1$$

for every matrix $D = (d_{jk})$ with $|d_{jk}| \leq 1$ $(j, k = 1, 2, \ldots, n)$。また、$0 < \rho < 1$ に対して $w_\rho(X) = \frac{2-\rho}{\rho} w_{2-\rho}(X) \quad (0 < \rho < 1)$ が成り立つ ([1] 参照) から、$\rho > 1$ としてよい。Theorem A を用いると、

$$\| \sum_{k=1}^{\infty} \frac{(\rho-1)^k |z|^{k-1}}{\rho^k} \| A \|^k \| \leq \rho - 1 \qquad (|z| < 1)$$

がいえ、D の各成分の絶対値が 1 以下であることから、

$$\left\| \frac{(\rho-1)^k z^{k-1}}{\rho^k} (A \circ D)^k \right\| \preceq \frac{(\rho-1)^k |z|^{k-1}}{\rho^k} \| A \|^k \quad k = 1, 2, \ldots$$

がいえる。したがって、

$$\begin{aligned} \|(A \circ D)\{\rho I - z(\rho-1)(A \circ D)\}^{-1}\| &= \| \sum_{k=1}^{\infty} \frac{(\rho-1)^k z^{k-1}}{\rho^k} (A \circ D)^k \| \\ &\leq \| \sum_{k=1}^{\infty} \frac{(\rho-1)^k |z|^{k-1}}{\rho^k} \| A \|^k \| \\ &\leq \rho - 1 \qquad (|z| < 1) \end{aligned}$$

が示される。また、$r(A \circ D) \leq r(\| A \|) \leq \frac{\rho}{\rho-1}$ だから、再び Theorem A より、$w_\rho(A \circ D) \leq 1$ となる。 ■

$\| \cdot \|_p$ $(0 < p < \infty)$ は次の意味で *normal interpolation scale* であることが知られている。即ち、$0 < p_0 < p_1 < \infty$, $z \longmapsto C(z) \in \mathbb{M}_n$ が $\{z \ ; \ 0 \leq \mathrm{Re}(z) \leq 1\}$ 上の bounded analytic matrix-valued fucniton で

$$\sup_{t \in \mathbb{R}} \|C(it)\|_{p_0} \leq 1, \quad \sup_{t \in \mathbb{R}} \|C(1+it)\|_{p_1} \leq 1$$

を満たすとき、

$$\|C(\alpha)\|_p \leq 1 \quad (0 < \alpha < 1; \frac{1}{p} = \frac{\alpha}{p_0} + \frac{1-\alpha}{p_1})$$

である ([5] 参照)。このことから 2 つの非負定値行列 $A, B \geq 0$ に対して Hölder 型不等式

(12) $$\|A^\alpha B^{1-\alpha}\|_p \leq \|A\|_{p_0}^\alpha \|B\|_{p_1}^{1-\alpha} \quad (0 < \alpha < 1, \frac{1}{p} = \frac{\alpha}{p_0} + \frac{1-\alpha}{p_1})$$

が成り立つことがわかる。これと (8) から、

(13) $$\|AB\|_r \leq \||A|^p\|_r^{1/p} \cdot \||B|^q\|_r^{1/q} \quad (r > 0; 1/p + 1/q = 1)$$

が成り立つ。

M_n 上にはこの通常の積に対して、Schur 積 (または Hadamard 積) と呼ばれる次の積が考えられる。$A = (a_{ij})$, $B = (b_{ij})$ に対して

$$A \circ B = (a_{ij} \cdot b_{ij})$$

このとき、(13) の類似な不等式として次が知られている。([7] 参照)

$$\|A \circ B\|_r \leq \||A|^p\|_r^{1/p} \cdot \||B|^q\|_r^{1/q} \quad (r > 0; 1/p + 1/q = 1) \tag{14}$$

このことを $w_\rho(\cdot)$ に置き換えることはすぐにできる。

$w_\rho(\cdot)$ $(0 < \rho < \infty)$ は次の意味で *normal interpolation scale* である。即ち、$0 < \rho_0 \leq \rho_1 < \infty$ として、$z \longmapsto C(z) \in \mathbb{M}_n$ が $\{z\ ;\ 0 \leq \mathrm{Re}(z) \leq 1\}$ 上の有界な行列を値にとる解析関数として、

$$\sup_{t\in\mathbb{R}} w_{\rho_0}(C(it)) \leq 1, \quad \sup_{t\in\mathbb{R}} w_{\rho_1}(C(1+it)) \leq 1$$

を満たすとき

$$w_\rho(C(\alpha)) \leq 1 \quad (0 < \alpha < 1; \rho = \alpha\rho_0 + (1-\alpha)\rho_1)$$

である ([1] 参照) 。
$A = (a_{jk}) \succeq 0$ と任意の複素数 $z \in \mathbb{C}$ に対して *z-power* $A^{(z)} \stackrel{def}{=} (a_{jk}^z)$ とすると (12) と類似な次の性質が上の性質を使うことにより導かれる。

Theorem 2. $0 < \rho_0 \leq \rho_1 < \infty$ とする。このとき、$A, B \succeq 0$ に対して

$$w_\rho(A^{(\alpha)} \circ B^{(1-\alpha)}) \leq w_{\rho_0}(A)^\alpha w_{\rho_1}(B)^{1-\alpha} \qquad (0 < \alpha < 1;\ \rho = \alpha\rho_0 + (1-\alpha)\rho_1) \tag{15}$$

がいえる。

証明の概略. $A, B \succeq 0, 0 < \alpha < 1,\ 0 < \rho_0 < \rho_1 < \infty$ and $w_{\rho_0}(A) = w_{\rho_1}(B) = 1$ とする。$\{z\ ;\ 0 \leq \mathrm{Re}(z) \leq 1\}$ から $\mathbb{M}_n$ 上への bounded analytic matrix-valued function を

$$C(z) \stackrel{def}{=} A^{(z)} \circ B^{(1-z)}$$

で定義する。このとき、

$$|C(1+it)| = A, \quad |C(it)| = B \quad (t \in \mathbb{R}),$$

となり、Theorem 1 から、

$$\sup_{t\in\mathbb{R}} w_{\rho_0}(C(1+it)) \leq w_{\rho_0}(A) = 1 \quad \sup_{t\in\mathbb{R}} w_{\rho_1}(C(it)) \leq w_{\rho_0}(B) = 1$$

となる。$\{w_\rho(\cdot)\ ;\ 0 < \rho < \infty\}$ が normal interpolation scale だから、

$$w_\rho(A^{(\alpha)} \circ B^{(1-\alpha)}) \leq 1$$

がいえ、定理が示される。 ■
このことから、Schur 積に関する次の Hölder 型の不等式がいえる。

Theorem 3. $A, B \in \mathbb{M}_n$, $0 < \rho < \infty$ に対して

(16) $$w_\rho(A \circ B) \leq w_\rho(\|A\|^{(p)})^{1/p} \cdot w_\rho(\|B\|^{(q)})^{1/q} \quad (p, q > 0;\ 1/p + 1/q = 1)$$

が成り立つ。

3. ある最良定数

κ_ρ を

$$\kappa_\rho = \min\{\kappa \mid w_\rho(\|A\|) \leq \kappa w_\rho(A) \text{ for all } A \in M_n\}$$

で定義する。κ_ρ を見つけることは興味深い。$A \in M_n$ に対し、S_A(Schur multiplier operator) on M_n を

$$S_A(X) = A \circ X \ (X \in M_n)$$

とする。また、

$$\|S_A\|_{w_\rho} = \sup\{w_\rho(A \circ X) \mid w_\rho(X) \leq 1\},$$

$$\mathbb{M} = \{M = (m_{ij}) \in M_n \mid |m_{ij}| = 1 \text{ for every } i, j\}$$

とする。さらに、

$$\lambda_\rho = \max\{\|S_A\|_{w_\rho} \mid A \in \mathbb{M}\},$$

and

$$\mu_\rho = \min\{\mu \mid w_\rho(\|A\|) \leq \mu w_\rho(A) \text{ for all } A \in \mathbb{M}\}$$

とすると、次のことがいえる。

Theorem 4. $\rho \geq 1$ に対して

$$\kappa_\rho = \lambda_\rho = \mu_\rho$$

また、$0 < \rho < 1$ に対して$\kappa_\rho = \kappa_{2-\rho}$ である。

このことから、

Corollary 5. $\rho \geq 1$ に対して$\rho \longrightarrow \kappa_\rho$ は *nondecreasing* である。

がわかる。

References

1. T. Ando and K. Nishio, *Convexity properties of operator radii associated with unitary ρ-dilations*, Michigan Math. J. **20** (1973), 303–307.
2. C. Davis, *The shell of a Hilbert-space operator*, Acta Sci. Math.(Szeged) **29** (1968), 69–86.
3. L. Elsner, C. R. Johnson and J. A. D. Silva, *The Perron root of a weighted germetric mean of non-negative matrices*, Linear and Multilinear Alg. **24** (1988), 1–13.
4. C. K. Fong and J.A.R.Holbrook, *Unitarily invariant operator norms*, Can. J. Math. **35** (1983), 274–299.
5. I. C. Gohberg and M.G.Kreĭn, *Introduction to the theory of linear nonselfadjoint operators, Translations of Mathematical monographs, vol 18*, American Mathematical Society, 1969.
6. J. A. R. Holbrook, *On the power-bounded operators of Sz.-Nagy and Foiaş*, Acta Sci. Math. **29** (1968), 299–310.
7. K. Okubo, *Hölder-type norm inequalities for Schur products of matrices*, Linear Alg. and Appli. **91** (1987), 13–28.
8. B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North–Holland Publishing Company, 1970.

Duality Relation between the q-Numerical Ranges and the Higher Dimensional Numerical Range of a Matrix

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Abstract

In this talk I give the following theorem: Suppose that A is a $n \times n$ matrix, B is a $m \times m$ matrix. Then the folloewing two conditions are mutually equivalent (i) $W(A) \subseteq W(B)$ and $\max\{||A\xi|| : \xi \in \mathbf{C}^n, \quad ||\xi|| = 1, \xi^* A\xi = z\} \leq \max\{||B\eta|| : \eta \in \mathbf{C}^m, ||\eta|| = 1, \eta^* B\eta = z\}$ for every $z \in W(A)$ (ii) $W(q : A) \subseteq W(q : B)$ for every $q \in [0, 1]$. In the above $W(A)$ denotes the classical numerical range of A and $W(q : A)$ denotes the $q-$ numerical range of A.

1　q-数域　および　Davis-Wielandt shell　の定義

$n \times n$ 複素行列 A　および　複素数　$q, |q| \leq 1$　に対して　A の q-数域 $W(q : A) \subset \mathbf{C}$ を次のように定める：

$$W(q : A) = \{\eta^* A\xi : \xi, \eta \in \mathbf{C}^n, \xi^*\xi = \eta^*\eta = 1, \eta^*\xi = q\}.$$

また、A の　Davis-Wielandt shell $W(A_h, A_{sh}, A^*A) \subset \mathbf{R}^3$ を次のように定める：

$$W(A_h, A_{sh}, A^*A) = \{(\xi^* A_h\xi, \xi^* A_{sh}\xi, \xi^* A^* A\xi) \in \mathbf{R}^3 : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}.$$

但し、$A_h = (1/2)(A + A^*), A_{sh} = (-i/2)(A - A^*)$.

さて、(古典的) 数域　$W(A) = W(1 : A)$ の 研究においては、Toeplitz=Hausdorff の定理：$W(A) = \{\xi^* A\xi : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}$ は凸集合である、が基本となる。q-数域に対しては、1984 年に丁 南僑 (Tsing, Nam-Kiu) が、次の公式 およびそこで登場する関数　ϕ_A が数域　$W(A)$ 上で凹であることを証明し、それを用いて　$W(q : A)$　が一般に凸となることを示した ([5])：

$$W(q : A) = \{\quad q\,z + \zeta\sqrt{1 - |q|^2}\phi_A(z) : z \in W(A), \zeta \in \mathbf{C}, |\zeta| \leq 1\},$$

$$\phi_A(z) = \max\{\sqrt{||A\xi||^2 - |(A\xi, \xi)|^2} : \xi \in \mathbf{C}^n, \xi^*\xi = 1, (A\xi, \xi) = z\}\ (z \in W(A)).$$

また、A が、正規行列のときは、$W(q:A)$ の境界は、円弧と離心率 $|q|$ の楕円弧からなることも、わかっている（中里 [3]）。$q-$数域 $W(q:A)$ の持つ基本性質としては、(i)$n\times n$ ユニタリー行列 U に対して $W(q:A)=W(q:U^*AU)$ となること、(ii) A の転置行列 A^t に対して $W(q:A^t)=W(q:A)$ となることなどが、挙げられる。上記のような弱ユニタリー不変量として、$q-$ 数域を古典的数域を一般化するものとして導入する意義は、次のような、例から解る。$0\leq\theta\leq\pi/2$ に対して、$N(\theta)$ を、次のような 3×3 巾零行列とする：

$$N(\theta)=\begin{pmatrix}0 & \cos\theta & 0\\ 0 & 0 & \sin\theta\\ 0 & 0 & 0\end{pmatrix}.$$

このとき、$W(N(\theta))=\{z\in\mathbf{C}:|z|\leq 1/2\}$ が、任意の $0\leq\theta\leq\pi/2$ に対してなりたつが、$W(0:N(\theta))=\{z\in\mathbf{C}:|z|\leq\max\{\cos\theta,\sin\theta\}\}$ となり、$0\leq\theta_1<\theta_2\leq\pi/4$ に対して、$W(0:\theta_1)\neq W(0:\theta_2)$ となる。ここで、$0<\theta\leq\pi/4$ に対して、$N(\pi/4+\theta)$ は、$N(\pi/4-\theta)$ の転置行列とユニタリー同値である。

最近になって、上記の Tsing の結果のより簡潔な証明が、李　志光 (Li, Chi-Kwong) によって発見され ([2])、そのなかで q-数域と、Davis-Wielandt shell　が、密接に関係していることが、わかってきた。

また、Davis-Wielandt shell については、1983年の Y.H.Au-Yeung と　N.K.Tsing の　論文 [1] により　$n\geq 3$ のとき　$W(A_h,A_{sH},A^*A)$ が凸となること、$n=2$　のとき、そのアフィン包が、(実) 2次元ならば凸、3次元ならば、$W(A_h,A_{sh},A^*A)$ の境界が、凸曲面になることが、証明されている。

2　双対定理およびその他の話題

Davis-Wielandt shell $W(A_h,A_{sh},A^*A)$ に対してその　" 上蓋"（うわぶた）を次のような $W(A)$ 上の関数　ψ_A　で表わす：

$$\psi_A(z)=\max\{\xi^*A^*A\xi:\xi\in\mathbf{C}^n,\xi^*\xi=1,\xi^*A\xi=z\}\ (z\in W(A)).$$

このとき、関数 ψ_A もまた、$W(A)$　上で凹となり、次の関係　$\phi_A(z)=\sqrt{\psi_A(z)-|z|^2}$ が、成り立つ。このような、関係を通じて、q-数域と Davis-Wielandt shell　（の上蓋）の関連が想像されるが、実際次の定理が成り立つ。(特に、$(II)\to(I)$　が、より問題となる)

定理 2.1.　A,B をそれぞれ、$n\times n, m\times m$　の行列とするとき、　次の2条件は、互いに同値である：　(I) $W(A)\subseteq W(B)$ かつ　$\psi_A(z)\leq\psi_B(z)$ が、　任意の　$z\in W(A)$ に対して成立する；　(II) $W(q:A)\subseteq W(q:B)$ が、任意の $q\in\mathbf{C},|q|\leq 1$ に対して成立する.

この定理の証明の基礎となるのは、双対凸関数に関する次のようなよく知られた結果である：有限閉区間　$I=[\alpha,\beta]$ 上で定義された、連続凸関数 f に対して、実数直線上の連続関数　f^* を、

$$f^*(x)=\max\{x\,t-f(t):t\in[\alpha,\beta]\}$$

で、定める。このとき、f^* もまた、凸関数となる。さらに、f,g が、（必ずしも同一でない）有限閉区間　I,J 上で定義された、連続凸関数であって、　$f^*(x)\leq g^*(x)$

が任意の　$x \in \mathbf{R}$ に対して成り立つならば、$I \subseteq J$ であって、$f(t) \geq g(t)$ が、任意の　$t \in I$ に対して成立する ([4], 34 頁参照)。

最近の話題. 正方行列 A の q-数域　$W(q:A)$ の直径を　$d(q:A)$ と表わすことにする：$d(q:A) = \max\{|z_1 - z_2| : z_1, z_2 \in W(q:A)\}$.

M. Aleksiejczyk は次の問題 を提起している：A を　$n \times n$ 行列とするとき、関数 $q \mapsto d(q:A)$　は、区間 $[0,1]$ で、単調減少かつ凹か？

[部分解答] A の数域　$W(A)$ が、0 を中心とする閉円板であって、　関数　ψ_A が、半径　$|z|$ だけに依存する関数のとき、　(例えば、この条件は、A が、weighted unilateral shift のとき成立)　$q \mapsto d(q:A)$　は、区間 $[0,1]$ で、単調減少かつ凹である。この特殊な場合には、問題の解決は、概略次のようになされる。A の数域半径を $w(A)$ とする。このとき、$\phi(x) = \phi_A(x)\ (-w(A) \leq x \leq w(A))$ に対して、$x \mapsto \phi(x)^2 + x^2$ も凹関数であり、このことより、$\phi''(x)$ が、存在し、連続であるような　x に対しては、

$$\psi''(x) = 2\,(\phi''(x)\phi(x) + \phi'(x) + 1) \leq 0$$

となり、またこのことより　$\phi''(x) < 0$ となる。故に、

$$\frac{\phi''(x)\phi(x) + \phi'(x) + 1}{\phi''(x)} \geq 0$$

が成り立ち、関数　ϕ のグラフの縮閉線 (evolute),　即ち接触円の中心の軌跡は、常に領域　$\{(x,y) \in \mathbf{R}^2 : y \geq 0\}$ にある。一般に、C^2- 関数 $t \mapsto f(t)$ に対して、そのグラフの縮閉線の parametrization $\{(x(t), y(t)) : \ldots\}$ が、　$y(t) = f(t) + \frac{f'(t)^2+1}{f''(t)} = \frac{f''(t)f(t)+f'(t)^2+1}{f''(t)}$, $x(t) = t - f'(t)\left(\frac{f'(t)^2+1}{f''(t)}\right)$ で与えられる。

上記のような 縮閉線の位置に関する結果より、$d(q:A)$ が、q に関して凹であることが言え、さらにこのことより、$0 \leq q \leq 1$ において、$d(q:A)$ が単調減少することが言える。

付記： 定理 2.1. については、筆者は'96 年 8 月の札幌の第 3 回 WONRA で、講演している。この定理については、最近　双対凸関数をつかわず、凸体に対する分離定理に基づいより平易で簡潔な証明が、C.K. Li 氏によりなされている。

References

[1] Y.H. Au-Yeung, N.K. Tsing : An extension of the Hausdorff -Toeplitz theorm on the numerical range, Proc. AMS **89** (1983), pp, 215-218.

[2]C.K. Li : Some convexity theorems for the generalized numerical ranges, Linear and Mltilinear Algebra, **40** (1996), pp. 235-240.

[3]H. Nakazato:The boundary of the range of a constrained sesquilinear form, Linear and Mltilinear Algebra,**40**(1995), pp.37-43.

[4] A.W. Roberts, D.E. Varberg :"Convex Functions", Academic Press, 1973, New York.

[5]N.K. Tsing : The constrained bilinear form and the C-numerical range, Linear Algebra and its Applications, **56**(1984), pp.195-206.

Monotone properties of operator functions associated with the Furuta inequality

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For positive operators on a Hilbert space A and B, the Furuta inequality is given as follows:

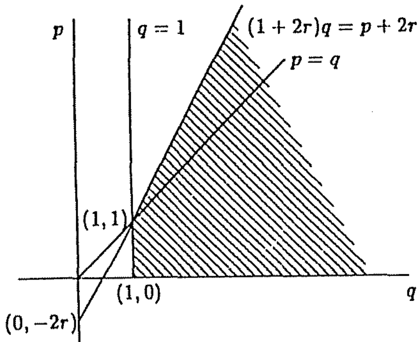
Furuta inequality:([11],cf.[12]) If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(B^r A^p B^r)^{\frac{1}{q}} \geq (B^r B^p B^r)^{\frac{1}{q}}$$

and

$$(A^r A^p A^r)^{\frac{1}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}}$$

hold for p and q such that $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.
The best possibility of this condition is assured by Tanahashi[18].



Figure

In this inequality, if we take $r = 0$, then the following Löwner-Heinz inequality is obtained.

Löwner-Heinz inequality: If $A \geq B \geq 0$, then

$$A^{\alpha} \geq B^{\alpha} \text{ for } \alpha \in [0,1].$$

We can regard the Furuta inequality as an operator function[13], in this case its expression is the following:

For $A \geq B \geq 0$ and $r \geq 0$, the operator functions

$$F(p) = (B^r A^p B^r)^{\frac{1+2r}{p+2r}} \text{ ; monotone increasing function for } p \geq 1 \text{ ,}$$

and

$$G(p) = (A^r B^p A^r)^{\frac{1+2r}{p+2r}} \text{ ; monotone decreasing function for } p \geq 1 \text{ .}$$

In [14], Furuta evolved his inequality more genaral form by which some results on *log majorization* due to Ando-Hiai[2] were extended. We call it the *grand Furuta inequality*.

Grand Furuta inequality : If $A \geq B \geq 0$ and A is invertible, then for each $t \in [0,1]$,

$$A^{1-t} \geq A^{-\frac{r}{2}}\{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{-\frac{r}{2}}$$

for $r \geq t, p \geq 1$ and $s \geq 1$.

When $t = 1, s = r$, this inequality shows the following main inequality of [2];

$$A^r \geq \{A^{\frac{r}{2}}(A^{-\frac{1}{2}}B^p A^{-\frac{1}{2}})^r A^{\frac{r}{2}}\}^{\frac{1}{p}},$$

and when $t = 0$, it is the Furuta inequalty.

The grand Furuta inequality is also considered as an operator function [14] by putting

$$F_{p,t}(A,B,r,s)=A^{-\frac{r}{2}}\{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{-\frac{r}{2}}$$

which is monotone decreasing for $r(\geq t)$ and $s(\geq 1)$.

Now, the purpose of this note is to review the above results from the view points of operator mean, which is established by Kubo-Ando[17]. Especially we use the *α-power mean* which is given as

$$A\ \sharp_\alpha\ B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}},\ for\ \alpha\in[0,1]$$

By using this operator mean, we can rewrite the Furuta inequality as flows;

$$A\ \geq A^t\ \sharp_{\frac{1-t}{p-t}}\ B^p,\ for\ p\geq 1\ and\ t\leq 0.$$

The arguments of the Furuta inequality by using the *α-power mean* presented us the following[16](cf.[3],[15]):

Satellite theorem of the Furuta inequality. If $A\geq B\geq 0$,then for $p\geq 1$ and $0\leq 1$,

$$A^t\ \sharp_{\frac{1-t}{p-t}}\ B^p\ \leq B\leq A\leq B^t\ \sharp_{\frac{1-t}{p-t}}\ A^p.$$

Moreover, the Grand Furuta inequality is also rewritten by using the *α-power mean* as follows[8]:

Grand Furuta inequality by operator mean. If $A\geq B\geq 0$ and A is invertible, then for $p\geq 1, s\geq 1$ and $1\geq t\geq 0$,

$$A^{-r+t}\ \sharp_{\frac{1-t+r}{(p-t)s+r}}(A^t\ \natural_s\ B^p)\leq B\leq A.$$

Here the notation $\natural$ is given as an extension of the *α-power mean*, which is defined by

$$A\ \natural_s\ B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^sA^{\frac{1}{2}}, for\ s\in R.$$

In the case of $s\in[0,1]$, this coincides with the *α-power mean* .

To prove this theorem from our view point, we have to know the behavior of $A^t\ \natural_s\ B^p$. The following is our fundamental theorem.

Theorem 1. If $A\geq B\geq 0$ and A is invertible, then for $p\geq 1, s\geq 1$ and $1\geq t\geq 0$

$$(A^t\ \natural_s\ B^p)^{\frac{1}{(p-t)s+t}}\ \leq B.$$

Using this result and the Furuta inequality, we can show the grand Furuta inequality as follows; if $(A^t\ \natural_s\ B^p)^{\frac{1}{(p-t)s+t}}\ \leq B$, by putting $B_1=(A^t\ \natural_s\ B^p)^{\frac{1}{(p-t)s+t}}$, $p_1=(p-t)s+t$ and $-r+t=t_1$, we have

$$\begin{aligned}A^{-r+t}\ \sharp_{\frac{1-t+r}{(p-t)s+r}}(A^t\ \natural_s\ B^p) &= A^{t_1}\ \natural_{\frac{1-t_1}{p_1-t_1}}\ B_1^{p_1}\\ &\leq B_1\leq B\leq A.\end{aligned}$$

The next result[10] gives us a fine view in our following arguments, although it is an easy application of the Furuta inequality.

Theorem 2. Let $A \geq B \geq 0$ and A, B be positive invertible. If $\delta \in [0,1]$, then for $p \geq \delta$ and $t \leq 0$,

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \leq B^\delta \leq A^\delta$$

In a similar form to the satellite theorem, Theorm 2 is described as follows;

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \leq B^\delta \leq A^\delta \leq B^t \natural_{\frac{\delta-t}{p-t}} A^p.$$

As $\delta \longrightarrow +0$, we have the following;

$$A^t \natural_{\frac{-t}{p-t}} B^p \leq 1 \leq B^t \natural_{\frac{-t}{p-t}} A^p$$

This is known as a characterization for $\log A \geq \log B$ by Ando[1]. So we call this *chaotic order* and use the following notation:

$$A \gg B \stackrel{\text{def}}{\Longleftrightarrow} \log A \geq \log B$$

Theorem (Ando)([1]). Let A, B be positive invertible.Then the following are equivalent.

(1) $A \gg B$,
(2) $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$, *for* $p \geq 0$,
(3) $A^{-p} \natural_{\frac{1}{2}} B^p$; *decreasing for* $p \geq 1$.

As an extension of Ando's theorem, we obtained the following:

Theorem (FFK)([5].cf.[6]). If A, B are positive and invertible, then $A \gg B$ if and only if for $p \geq 0, t \leq 0$, $A^t \natural_{\frac{-t}{p-t}} B^p \leq I$ holds.

Recently we have found a characterization for *strictly chaotic order* (i.e. $\log A > \log B$) .

Theorem (FJK)([9]). If A and B are positive invertible, then $\log A > \log B$ if and only if there exists an $\alpha \in (0, 1]$ such that $A^\alpha > B^\alpha$.

Corollary (FJK)([9]). For positive invertible operators A and B, $A \gg B$ if and only if for any $\delta \in (0, 1]$ there exists an $\alpha = \alpha_\delta$ such that $(e^\delta A)^\alpha > B^\alpha$.

Let's return to the monotone properties of the Furuta inequality. These properties are also held under the chaotic order.

Theorem (FFK)([4]). Let A and B be positive invertible operators. If $A \gg B$, then $A^t \sharp_{\frac{\delta-t}{p-t}} B^p$ is decreasing for $p \geq \delta \geq 0$ and increasing for $t \leq 0$.

Here we review Theorem 1 by the form $A^t \natural_{\frac{\delta-t}{p-t}} B^p$. Then it is reformed as follows:

Theorem 1'. If $A \geq B \geq 0$ and A is invertible, then for $\delta \geq p \geq 1$ and $0 \leq t$,

$$(A^t \natural_{\frac{\delta-t}{p-t}} B^p)^{\frac{1}{\delta}} \leq B \leq A \textit{ and decreasing on } \delta.$$

Now if $p \geq 1, p \geq \delta \geq t$ for $1 \geq t \geq 0$ and $A \geq B \geq 0$, then the next is easily seen by the use of the Löwner-Heinz inequality and the property of operator mean:

$$A^t \sharp_{\frac{\delta-t}{p-t}} B^p \geq B^t \sharp_{\frac{\delta-t}{p-t}} B^p = B^\delta.$$

On the other hand, we have seen in Theorem 2 that for $p \geq \delta, 1 \geq \delta \geq 0, t \leq 0$,

$$A^p \sharp_{\frac{\delta-t}{p-t}} B^p \leq B^\delta \leq A^\delta.$$

The next relation is a use of the Löwner-Heinz inequality to the Furuta inequality:
For $p \geq 1, t \leq 0,\ 0 \leq \delta \leq 1$,

$$(A^t \sharp_{\frac{1-t}{p-t}} B^p)^\delta \leq B^\delta \leq A^\delta.$$

So we see here the order between $A^t \sharp_{\frac{\delta-t}{p-t}} B^p$ and $(A^t \sharp_{\frac{1-t}{p-t}} B^p)^\delta$.

Theorem 3. If $A \geq B \geq 0$ and A is invertible,then for $p \geq 1, t \leq 0$ and $1 \geq \delta \geq 0$,

$$A^t \sharp_{\frac{\delta-t}{p-t}} B^p \leq (A^t \sharp_{\frac{1-t}{p-t}} B^p)^\delta \leq B^\delta \leq A^\delta.$$

In [7], we have shown a ginkgo leaf structure in the Furuta inequality. The following is a result relating to this.

Theorem (FK)([7]). If $A \geq B \geq 0$ and A is invertible,then for each $\alpha \in [0,1]$,

$$A^t \natural_{\frac{(p-t-n)\alpha+n}{p-t}} B^p \leq B^{(p-t-n)\alpha+t+n}$$

holds for $p \geq 1$ and $n+1 \geq -t \geq n$ for some $n \geq 0$, integer.

When $p > 1$, there is a gap between $\frac{1-t}{p-t}$ and 1. This theorm buries the gap and the case of $\alpha = \frac{1-t-n}{p-t-n}$ is just the Furuta inequality. We can describe also this theorem by using $A^t \sharp_{\frac{\delta-t}{p-t}} B^p$ as follows:

Theorem 4. If $A \geq B \geq 0$ and A is invertible,then for $1 \leq \delta \leq p$ and $t \leq 0$,

$$A^t \natural_{\frac{\delta-t}{p-t}} B^p \leq B^\delta.$$

Combining Theorem 2 to Theorem 4, we have $A^t \natural_{\frac{\delta-t}{p-t}} B^p \leq B^\delta$ for $p \geq 1,\ t \leq 0$ and $0 \leq \delta \leq p$.

Finally we give the case of $B^\delta \leq A^t \natural_{\frac{\delta-t}{p-t}} B^p$.

Theorem 5. If $A \geq B \geq 0$ and A, B are invertible, then $B^\delta \leq A^t \natural_{\frac{\delta-t}{p-t}} B^p$ holds for

(1) $-1 \leq t \leq 0,$ *and* $p \leq \delta \leq 2p - t,$

or

(2) $t \leq -1,$ *and* $p \leq \delta \leq 2p + 1.$

References

[1] T.Ando, On some operator inequalities,Math.Ann.,279(1987),157-159.

[2] T.Ando and F.Hiai, Log majorization and complementary Golden-Thompson type inequalities,Linear Algebra Appl.,197,198(1994),113-131.

[3] M.Fujii, Furuta's inequality and its mean theoretic approach,J.Operator Theory,23(1990),67-72.

[4] M.Fujii,T.Furuta and E.Kamei, Operator functions associated with Furuta's inequality,Linear Alg. and Its Apll.,149(1991),91-96.

[5] M.Fujii,T.Furuta and E.Kamei, Furuta's inequality and its application to Ando's theorem,Linear Alg. and Its Apll.,179(1993),161-169.

[6] M.Fujii and E.Kamei, Furuta's inequality and a generalization of Ando's theorem,Proc.Amer.Math. Soc.,115(1992),409-413.

[7] M.Fujii and E.Kamei, A geometrical structure in the Furuta inequality,Math.Japon.,43(1996),83-90.

[8] M.Fujii and E.Kamei, Mean theoretic approach to the grand Furuta inequality,Proc.Amer.Math.Soc., 124(1996),2751-2756.

[9] M.Fujii,J.F.Jiang and E.Kamei, Characterization of chaotic order to Furuta inequality,Proc.Amer. Math.Soc.,to appear.

[10] M.Fujii,J.F.Jiang and E.Kamei, A characterization of orders defined by $A^\delta \geq B^\delta$ via Furuta inequality,Math.Japon.,to appear.

[11] T.Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{(p+2r)}{q}}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$,Proc.Amer.Math.Soc.,101(1987),85-88.

[12] T.Furuta, Elementary proof of an order preserving inequality,Proc.Japan Acad.,65(1989),126.

[13] T.Furuta, Two operator functions with monotone property,Proc.Amer.Math.Soc.,111(1991),511-516.

[14] T.Furuta, Extension of the Furuta inequality and Ando-Hiai log-majorization,Linear Alg. and Its Appl.,219(1995),139-155.

[15] E.Kamei, Furuta's inequality via operator means,Math.Japon.,33(1988),737-739.

[16] E.Kamei, A satellite to Furuta's inequality,Math.Japon.,33(1988),883-886.

[17] F.Kubo and T.Ando, Means of positive linear operators,Math.Ann.,246(1980),205-224.

[18] K.Tanahashi, Best possibility of the Furuta inequality,Proc.Amer.Math.Soc.,124(1996),141-146.

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COVARIANCE IN NONCOMMUTATIVE PROBABILITY

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1. Introduction. H.Umegaki [9] founded, about forty years ago, the noncommutative probability theory as an application of the theory of von Neumann algebras. A (bounded linear) operator T on a Hilbert space H plays the role of a random variable and (Tx, x) does the *mean* of T *at a state* x (with $\|x\| = 1$).

In 1994, J.I.Fujii introduced the *covariance* of (not necessarily commutative) operators S and T at a state x in his seminor talk by

$$\text{Cov}(T, S) = (S^*Tx, x) - (S^*x, x)(Tx, x), \tag{1}$$

and the *variance* of T at a state x by

$$\text{Var}(T) = \|Tx\|^2 - |(Tx, x)|^2. \tag{2}$$

The following inequality is fundamental in this note, which is shown by the Schwarz inequality because the covariance is semi-inner product:

The covariance-variance inequality. *The square of the absolute of the covariance of two operators S and T is not greater than the product of the variances of S and T:*

$$|\text{Cov}(S, T)|^2 \leq \text{Var}(S) \cdot \text{Var}(T). \tag{3}$$

In this note, we point out that several known operator inequalities are unified by the covariance-variance inequality, e.g. the Kantorovich inequality and the Heinz-Kato-Furuta inequality. These are based on our joint papers [2] and [3].

2. Estimations. The following is a known fact; we cite a simple proof.

Lemma 1. *If a selfadjoint operator A on H satisfying $m \leq A \leq M$ for some scalars m and M, then*

$$\text{Var}(A) \leq \frac{1}{4}(M - m)^2 \quad \textit{for any state} \quad x \in H. \tag{4}$$

Proof. We first note that

$$(M - \alpha)(\alpha - m) \leq (\frac{M - m}{2})^2$$

for all real numbers α. Hence it follows that for each unit vector x

$$\begin{aligned}\text{Var}(A) &= (A^2x, x) - (Ax, x)^2 \\ &= (M - (Ax, x))((Ax, x) - m) - ((M - A)(A - m)x, x) \\ &\leq (M - (Ax, x))((Ax, x) - m) \\ &\leq \frac{1}{4}(M - m)^2.\end{aligned}$$

The following estimation is obtained by the covariance-variance inequality and the above lemma.

Theorem 2. *If A and B are selfadjoint operators on H such that $m_1 \le A \le M_1$ and $m_2 \le B \le M_2$ for some m_i and M_i, then*

$$|\mathrm{Cov}(A,B)| \le \frac{1}{4}(M_1 - m_1)(M_2 - m_2) \quad \text{for any state} \quad x \in H. \tag{5}$$

Remark. Theorem 2 is a noncommutative extension of the following inequality due to Grüss [7]: If $f_i (i = 1, 2)$ be continuous (or Riemann integrable) functions on the interval $[a, b]$ such that $0 < m_i \le f_i \le M_i$ for some m_i and M_i, then

$$\left|\frac{1}{b-a}\int_a^b f_1(x)f_2(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f_1(x)\,dx \cdot \int_a^b f_2(x)\,dx\right| \le \frac{1}{4}(M_1 - m_1)(M_2 - m_2).$$

3. Applications. In this section, we give some applications of the above inequalities. First of all, we begin with the following celebrated inequality due to Kantorovich :

The Kantorovich inequality. *If a positive operator A on H satisfies $0 < m \le A \le M$ for some $m < M$, then for each unit vector $x \in H$*

$$(Ax, x)(A^{-1}x, x) \le \frac{(M+m)^2}{4Mm}. \tag{6}$$

It should be recognized as an estimation of the covariance of a selfadjoint operator and its inverse; namely we take $B = A^{-1}$ in Theorem 2. Then we have

$$|1 - (Ax, x)(A^{-1}x, x)| \le \frac{1}{4}(M - m)(m^{-1} - M^{-1}) = \frac{(M-m)^2}{4Mm}.$$

Hence it follows that

$$(Ax, x)(A^{-1}x, x) - 1 \le \frac{(M-m)^2}{4Mm},$$

which is nothing but the Kantorovich inequality (6).

The Hölder-McCarthy inequality [8] says that if A is a positive operator on H, then for each unit vector $x \in H$

$$(A^r x, x) \ge (Ax, x)^r \quad \text{for } r \ge 1.$$

We estimate, as an application of Theorem 3, the difference in the Hölder-McCarthy inequality:

Theorem 3. *If a positive operator A on H satisfies $0 < m \le A \le M$ for some $m < M$, then for each unit vector $x \in H$*

$$0 \le (A^{k+1}x, x) - (Ax, x)^{k+1} \le \frac{1}{4}(M - m)^2 \cdot \sum_{p=1}^{k}(k - p + 1)m^{p-1}M^{k-p} \tag{7}$$

for all natural numbers k.

Proof. We show it by induction; since the case $k = 1$ is true by Lemma 1, we assume that (7) holds for k. Putting $B = A^{k+1}$ in (5), we have

$$0 \le |(A^{k+2}x, x) - (Ax, x)(A^{k+1}x, x)| \le \frac{1}{4}(M - m)(M^{k+1} - m^{k+1}). \tag{7'}$$

Hence it implies that

$$
\begin{aligned}
0 &\le (A^{k+2}x,x) - (Ax,x)^{k+2} \\
&\le (A^{k+2}x,x) - (Ax,x)(A^{k+1}x,x)| + |(Ax,x)||(A^{k+1}x,x) - (Ax,x)^{k+1}| \\
&\le \frac{1}{4}(M-m)(M^{k+1}-m^{k+1}) + M\frac{1}{4}(M-m)^2 \cdot \sum_{p=1}^{k}(k-p+1)m^{p-1}M^{k-p} \\
&\qquad \text{by (7') and the assumption of induction} \\
&= \frac{1}{4}(M-m)\{(k+1)M^k + kmM^{k-1} + \cdots + 2m^{k-1}M + m^k\},
\end{aligned}
$$

which completes the proof.

Somehow the same constant $(M_1-m_1)(M_2-m_2)/4$ is appeared in Strang's theorem on an estimation of the imaginary part of the product of two selfadjoint operators, cf. [1; Cor. 4.3]; we point out that Theorem 2 implies it:

Theorem 4. (Strang) *If A and B are selfadjoint operators such that $m_1 \le A \le M_1$ and $m_2 \le B \le M_2$ for some m_i and M_i, then*

$$
|\mathrm{Im}\ AB| \le \frac{1}{4}(M_1-m_1)(M_2-m_2). \tag{8}
$$

Proof. As a matter of fact, we have for each unit vector x

$$
\begin{aligned}
|\mathrm{Im}(ABx,x)| &= \frac{1}{2}|(ABx,x) - (BAx,x)| \\
&\le \frac{1}{2}\{|(ABx,x) - (Ax,x)(Bx,x)| + |(Ax,x)(Bx,x) - (BAx,x)|\} \\
&= \frac{1}{2}\{|\ \mathrm{Cov}(A,B)| + |\ \mathrm{Cov}(B,A)|\} \\
&\le\ \mathrm{Var}(A) \cdot\ \mathrm{Var}(B) \quad \text{by (3)}.
\end{aligned}
$$

Incidentally, we note that Theorem 4 has an alternative simple proof as follows: For simplicity, we prove that if A and B are positive contractions, then $|\mathrm{Im}\ AB| \le \frac{1}{4}$. We can easily checked the following equations;

$$
i(BA-AB) + \frac{1}{2} = \{(A-\frac{1}{2}) + i(B-\frac{1}{2})\}\{(A-\frac{1}{2}) - i(B-\frac{1}{2})\} + A - A^2 + B - B^2 \ge 0
$$

and

$$
i(AB-BA) + \frac{1}{2} = \{(A-\frac{1}{2}) - i(B-\frac{1}{2})\}\{(A-\frac{1}{2}) + i(B-\frac{1}{2})\} + A - A^2 + B - B^2 \ge 0.
$$

The former implies

$$
\mathrm{Im}\ AB = \frac{AB-BA}{2i} = \frac{i(BA-AB)}{2} \ge -\frac{1}{4},
$$

and the latter does

$$
\mathrm{Im}\ AB = \frac{AB-BA}{2i} \le \frac{1}{4},
$$

as required.

Next we look at the Heinz-Kato-Furuta inequality [5], from the covariance-variance inequality.

The Heinz-Kato-Furuta inequality. *Let R be an operator on H. If A and B are positive operators on H such that $R^*R \le A^2$ and $RR^* \le B^2$, then for each $x, y \in H$*

$$|(R|R|^{\alpha+\beta-1}x, y)| \le \|A^\alpha x\| \|B^\beta y\| \tag{9}$$

holds for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \ge 1$.

Now let $R = U|R|$ be the polar decomposition of R. We choose a unit vector u such that $(|R|^\alpha x, u) = 0 = (|R|^\beta U^* y, u)$ and define operators S and T by

$$S = |R|^\alpha x \otimes u \quad \text{and} \quad T = |R|^\beta U^* y \otimes u,$$

where $(x \otimes y)z = (z, y)x$ for $x, y, z \in H$. Then the covariance and variances at the state u are actually determined by

$$\begin{aligned} |\mathrm{Cov}(S,T)| &= |(R|R|^{\alpha+\beta-1}x, y)|, \\ \mathrm{Var}(S) &= \||R|^\alpha x\|^2 \le \|A^\alpha x\|^2, \\ \mathrm{Var}(T) &= \||R^*|^\beta y\|^2 \le \|B^\beta y\|^2. \end{aligned}$$

Here the final inequalities are ensured by the Löwner-Heinz inequality.

Anyway the covariance-variance inequality implies the desired inequality (9).

In [6], Furuta showed the following theorem which is an improvement of Bernstein's one.

Theorem A. *If e is a unit eigenvector corresponding to an eigenvalue λ in a dominant operator A on a Hilbert space H , then*

$$|(g,e)|^2 \le \frac{\|g\|^2\|Ag\|^2 - |(g,Ag)|^2}{\|(A-\lambda)g\|^2} \tag{10}$$

for all g in H for which $Ag \ne \lambda g$.

Here an operator A is called dominant if for each λ there is a real number $M_\lambda \ge 1$ such that $\|(A-\lambda)^*x\| \le M_\lambda \|(A-\lambda)x\|$ for all x in H . We have to remark that $(A-\lambda)^*e = o$ under the dominance of A, that is, λ is a normal eigenvalue of A, i.e., there is a nonzero vector x in H such that $(A-\lambda)x = 0$ and $(A-\lambda)^*x = 0$. Under this consideration, we weakened the assumption of Theorem A to normality of the eigenvalue in [3]. More precisely,

Theorem B. *If e is a unit eigenvector corresponding to a normal eigenvalue λ of A on a Hilbert space H, then (10) holds for all g in H for which $Ag \ne \lambda g$.*

We mention the following improvement of Theorem B by the covariance variance inequality, [3].

Theorem 5. *If e is a unit eigenvector corresponding to an eigenvalue $\overline{\lambda}$ of A^* on a Hilbert space H, then (10) holds for all g in H for which $Ag \ne \lambda g$.*

Proof. First of all, we note that the covariance is translation-invariant, i.e.,

$$\mathrm{Cov}(A-a, B-b) = \mathrm{Cov}(A, B)$$

for $a, b \in \mathbb{C}$, and so is the variance. We put $B = A - \lambda$ and may assume that $\|g\| = 1$. Now (10) can be rephrased as

$$|(g,e)|^2 \|Bg\|^2 \le \mathrm{Var}_g(B). \tag{11}$$

To prove (11), it suffices to take the projection E corresponding to the eigenvector e, i.e., $Ex = (x, e)e$ for $x \in H$. That is, we apply the covariance-variance inequality to E and B. Then we have

$$|\mathrm{Cov}_g(E, B)|^2 \leq \mathrm{Var}_g(E)\mathrm{Var}_g(B). \tag{12}$$

Noting that $B^*e = 0$ by the assumption on λ, (12) is rewritten by

$$|(g, e)|^2|(Bg, g)|^2 \leq \mathrm{Var}_g(B)(1 - |(g, e)|^2),$$

so that

$$|(g, e)|^2\|Bg\|^2 = |(g, e)|^2(|(Bg, g)|^2 + \mathrm{Var}_g(B)) \leq \mathrm{Var}_g(B),$$

as desired.

In addition, Theorem 5 is generalized as follows:

Theorem 6. *If $\{e_n\}$ is a sequence of unit vectors corresponding to an approximate eigenvalue $\overline{\lambda}$ of A^*, then*

$$\overline{\lim}\, |(g, e_n)|^2 \leq \frac{\|g\|^2\|Ag\|^2 - |(g, Ag)|^2}{\|(A - \lambda)g\|^2} \tag{13}$$

for all g in H for which $Ag \neq \lambda g$.

References

1. J.I.Fujii, M.Fujii, S.Izumino, F.Kubo and R.Nakamoto, *Strang's inequality*, Math. Japon., **37** (1992), 479-486.
2. M.Fujii, T.Furuta, R.Nakamoto and S.-E.Takahasi, *Operator inequalities and covariance in noncommutative probability,*, Math. Japon., to appear.
3. M.Fujii, T.Furuta and Y.Seo, *Covariance in Bernstein's inequality for operators*, Nihonkai Math. J., to appear.
4. M.Fujii, R.Nakamoto and Y.Seo, *An inequality for some nonnormal operators -Extension to normal approximate eigenvalues*, Proc.Amer.Math.Soc. **118** (1993), 899-902..
5. T.Furuta, *An extension of the Heinz-Kato theorem*, Proc. Amer. Math. Soc., **120** (1994), 785-787.
6. T.Furuta, *An inequality for some nonnormal operators*, Proc.Amer.Math.Soc. **104** (1988), 1216-1217.
7. G.Grüss, *Über das maximum des absoluten betrages von* $\frac{1}{b-a}\int_a^b f_1(x)f_2(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f_1(x)\,dx \cdot \int_a^b f_2(x)\,dx$., Math. Z., **39** (1935), 215-226.
8. C.A.McCarthy, c_p, Israel J. Math., **5** (1967), 249-271.
9. H.Umegaki, *Conditional expectation in an operator algebra*, Tohoku Math. J., **6** (1954), 177-181.

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Small Bound Isomorphisms of the Domain of a Closed $*$-Derivation

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§1．序

Banach-Stone の定理は、次のように述べられる。“ X, Y を コンパクト・ハウスドルフ空間、T を $C(X)$ から $C(Y)$ の上への線形等距離作用素とする。このとき、Y から X への同相写像 τ と、$w \in C(Y)$: $|w| \equiv 1$ が存在して、

$$Tf(y) = w(y)f(\tau(y)) \qquad (\forall f \in C(X),\ \forall y \in Y)$$

となる。” この定理は、作用素論的に見ると、T が荷重合成作用素であることを示している。あるいは、T の極分解定理であると考えることもできる。一方、このような T の存在が、X と Y の間の同相写像 τ を誘導するともいえる。D. Amir と M. Cambern は、この定理の後者の側面に注目して、次の定理を示した。

<u>**定理**</u>（D. Amir, M. Cambern） X と Y を コンパクト・ハウスドルフ空間、$T : C(X) \to C(Y)$ を 線形同型作用素で、$\|T\|\|T^{-1}\| < 2$ とする。このとき、X と Y は位相同型である。

この結果は、その後 種々の方向へ拡張、一般化されている。

X を $\mathbb{R}$ のコンパクト部分集合、$C^1(X)$ を X 上の連続微分可能関数全体、$C^1(X) \ni f$ のノルムを $\|f\| = \|f\|_\infty + \|\delta(f)\|_\infty$ で与える。このとき、K. Jarosz は [4] で次の問題を提出した。

問題 “X と Y を $\mathbf{R}$ のコンパクト部分集合、$T : C^1(X) \to C^1(Y)$ が線形同型作用素で $\|T\|\|T^{-1}\| < \epsilon$ ならば、X と Y は位相同型である。” が、$\mathbf{R}$ の任意のコンパクト部分集合 X, Y に対して成り立つような (X と Y に無関係な) ϵ が存在するか。

これに対して、K-W. Jun と Y-H. Lee は、次の結果を示した。

定理 (K-W. Jun and Y-H. Lee) X, Y を $\mathbf{R}$ のコンパクト部分集合で $X \subset [a, b], Y \subset [c, d]$ とする。$T : C^1(X) \to C^1(Y)$ を線形同型作用素で、次の (1) ~ (4) を満たすとする。

(1) $f'(t) \equiv 0$ ならば $(Tf)'(t) \equiv 0.$

(2) $\|fg\| \leq \|TfTg\| \leq (1+\epsilon)^2\|fg\|$

(3) $\|f\| \leq \|Tf\| \leq (1+\epsilon)\|f\|$

(4) $\epsilon < \min\{\frac{1}{49}, \frac{1}{2(b-a)+1}, \frac{1}{2(d-c)+1}\}$

このとき、X と Y は位相同型である。

定理 (K-W. Jun and Y-H. Lee) X, Y を $\mathbf{R}$ の部分集合で $X \subset \bigcup_{i=1}^{n} [a_i, b_i]$ $(a_i < b_i < a_{i+1})$, $Y \subset \bigcup_{j=1}^{m} [c_j, d_j]$ $(c_j < d_j < c_{j+1})$, $\max_i\{|b_i - a_i|\} < k$, $\max_j\{|d_j - c_j|\} < k$ とする。$T : C^1(X) \to C^1(Y)$ を線形同型作用素で、次の (1) ~ (3) を満たすとする。

(1) $(Tf)'(t) \equiv 0 \Longleftrightarrow f'(t) \equiv 0,$

(2) $\|f\| \leq \|Tf\| \leq (1+\epsilon)\|f\|,$

(3) $k < \frac{4-\sqrt{10}}{6},\ \ \epsilon < 6k^2 - 8k + 1$

このとき、X と Y は位相同型である。

$\tau(K_2(\delta_2)) = K_1(\delta_1)$,

$$(Tf)(y) = w_1(y)f(\tau(y)) \qquad (\forall f \in \mathfrak{D}(\delta_1), \forall y \in K_2)$$

$$\delta_2(Tf)(y) = w_2(y)\delta_1(f)(\tau(y)) \qquad (\forall f \in \mathfrak{D}(\delta_1), \forall y \in K_2)$$

となる。

したがって、$K_1(\delta_1)$ と $K_2(\delta_2)$, K_1 と K_2 は位相同型である。この定理の観点から K. Jarosz の問題を考える。

定理 K_i $(i = 1, 2)$ は第一可算公理をみたす コンパクト・ハウスドルフ空間, δ_i を $C(K_i)$ における 閉 $*$-微分とする。$T : \mathfrak{D}(\delta_1) \to \mathfrak{D}(\delta_2)$: $\|T\|\|T^{-1}\| < 2$ であるような線形同型作用素が存在し、さらに、T と T^{-1} は supnorm で有界とする。このとき、$K_1(\delta_1)$ と $K_2(\delta_2)$ は位相同型である。

例　$f \in C[a, b]$ に対して $f\dfrac{d}{dt}$ は閉化可能であるので、その閉包を δ_f とおく。さらに、$K_f = \{x\,;\, f(x) \neq 0\}$ とおく。

$f, g \in C[a, b]$ に対して、$T : \mathfrak{D}(\delta_f) \to \mathfrak{D}(\delta_g)$ を線形同型作用素で $\|T\|\|T^{-1}\| < 2$ であり、さらに、T と T^{-1} は supnorm で有界とする。このとき、K_f と K_g は位相同型である。

(証明)

$\|T^{-1}\| \leq 1$, $\|T\| < 2$ としてよい。$\mathfrak{D}(\delta_1) \ni f$ に対して、

$$\tilde{f}(x, x', z) = zf(x) + \delta_1(f)(x'), \qquad (x, x', z) \in K_1 \times K_1 \times \mathbf{T} \equiv \mathbf{W_1}$$

$\mathfrak{D}(\delta_2) \ni g$ に対して、

$$\tilde{g}(y, y', z) = zg(y) + \delta_2(g)(y'), \qquad (y, y', z) \in K_2 \times K_2 \times \mathbf{T} \equiv \mathbf{W_2}$$

この 2 つの定理の証明は、いずれも初等的証明であるが、区間の巾が大きい場合は適用できない等、もっと改良する余地があるように思われる。つぎの第 2 節で、我々は別の観点から この問題を考える。

§2. 閉 $*$-微分の定義域

K を コンパクト・ハウスドルフ空間、$C(K) \supset \mathfrak{D}(\delta)$ を $C(K)$ のノルム稠密 $*$-部分代数とし、線形作用素 $\delta : \mathfrak{D}(\delta) \to C(K)$ は

$$\delta(fg) = f\delta(g) + \delta(f)g \qquad (f, g \in \mathfrak{D}(\delta)),$$

$$\delta(\bar{f}) = \overline{\delta(f)} \qquad (f \in \mathfrak{D}(\delta))$$

をみたし、かつ閉作用素であるとする。このとき、δ を $C(K)$ における閉 $*$-微分という。$\mathfrak{D}(\delta) \ni f$ に対して、f のノルムを

$$\|f\| = \|f\|_\infty + \|\delta(f)\|_\infty$$

により与える。$K \ni x$ に対して、$\mathfrak{D}(\delta)$ 上の有界線形作用素 $\eta_x \circ \delta$ を

$$\eta_x \circ \delta(f) = \delta(f)(x) \qquad (\forall f \in \mathfrak{D}(\delta))$$

により定義する。$K(\delta) = \{x \in K : \eta_x \circ \delta \neq 0\}$ とおく。$K(\delta)$ は K の開集合である。このとき、T. Matsumoto と S. Watanabe は 次の結果を示した。

定理 (T. Matsumoto and S. Watanabe) K_i $(i = 1, 2)$ をコンパクト・ハウスドルフ空間、δ_i を $C(K_i)$ における閉 $*$-微分とする。$T : \mathfrak{D}(\delta_1) \to \mathfrak{D}(\delta_2)$ を上への等距離線形作用素とする。このとき、K_2 から K_1 上への同相写像 τ と $w_1 \in Ker(\delta_1)$: $|w_1| \equiv 1$, 及び、$w_2 \in C(K_2(\delta_2))$: $|w_2| \equiv 1$ が存在して、

とおく。このとき、

$$\mathfrak{D}(\delta_1) \ni f \longrightarrow \tilde{f} \in C(W_1), \qquad \mathfrak{D}(\delta_2) \ni g \longrightarrow \tilde{g} \in C(W_2)$$

は線形等距離作用素である。$S_1 = \{\tilde{f}\,;\, f \in \mathfrak{D}(\delta_1)\}$, $S_2 = \{\tilde{g}\,;\, g \in \mathfrak{D}(\delta_2)\}$ とおく。$\tilde{T} : S_1 \ni \tilde{f} \longrightarrow \tilde{T}\tilde{f} \in S_2$ を $\tilde{T}\tilde{f} = \widetilde{Tf}$ により定義すると、$\|\tilde{T}^{-1}\| \leq 1$, $\|\tilde{T}\| < 2$ である。$W_2 \ni (y, y', z)$ に対して、

$$L_{(y,y',z)}(\tilde{g}) = \tilde{g}(y, y', z) \qquad (\forall \tilde{g} \in S_2)$$

とおく。$\tilde{T}^* L_{(y,y',z)}$ を $C(W_1)$ へノルムを変えないで拡張したものに対して、正則測度 $\mu^{y,y',z}$ が存在して、

$$\tilde{T}^* L_{(y,y',z)}(f) = \int_{W_1} f \, d\mu^{y,y',z} \qquad (\forall f \in C(W_1))$$

となる。次に、$\|T\| < 2M < 2$ である M をとる。$\forall z \in \mathbf{T}$ と、$\tilde{T}^* L_{(y,y,z)}$ のすべてのノルムを変えない拡張に対して、

$$|\mu^{y,y,z}(K_1 \times \{x\} \times \mathbf{T})| > M$$

となる x が存在する y の全体を $\tilde{K_2}$ とする。このとき、$\tilde{K_2} \ni y$ に対して、x は存在すれば一意に決まるので、$\rho(y) = x$ とする。このとき、$\tilde{K_2} \supset K_2(\delta_2)$ かつ、$\rho : K_2(\delta_2) \to K_1(\delta_1)$ は同相写像であることがわかる。

[注意]

(1) $\mathbf{D}$ をカントール集合とすると、$C(\mathbf{D})$ におけるノンゼロ閉微分は存在しない。一方、K (コンパクト・ハウスドルフ空間) の閉部分集合 E と、$C(K)$ における閉微分 δ に対して,

$$\mathfrak{D}(\delta_E) = \{f|_E \,:\, f \in \mathfrak{D}(\delta)\}$$

とおく。もし、

$$f|_E = 0, \qquad f \in \mathfrak{D}(\delta) \Rightarrow \delta(f)|_E = 0$$

ならば、

$$\delta_E(f|_E) = \delta(f)|_E \qquad (f|_E \in \mathfrak{D}(\delta_E))$$

により、$C(E)$ における (閉とは限らない) 微分 δ_E が定義できる。この枠組み $(E, C(E), \mathfrak{D}(\delta_E), \delta_E)$ に対して、上の定理と同様の議論を行うことができる。
(2) $C^1(X)$ (X : $\mathbf{R}$ のコンパクト部分集合) に対しても、定理の証明と同様の方法で類似の結果が得られる。

参考文献

[1] D. Amir, *On isomorphisms of function spaces*, Israel J. Math., **3**(1965), pp.205–210.

[2] M. Cambern, *On isomorphisms with small bound*, Proc. Amer. Math. Soc., **18**(1967), pp.1062–1066.

[3] M. Cambern and V. D. Pathak, *Isomorphisms of spaces of differentiable functions*, Rev. Roum. Math. Pures et Appl., **27**(1982), pp.737–743.

[4] K. Jarosz, *Perturbations of Banach algebras*, Lecture Notes in Math., **1120**, Springer-Verlag, 1985.

[5] K-W. Jun and Y-H. Lee, *Isometries with small bound on $C^1(X)$ spaces*, Bull. Korean Math. Soc., **92**(1995), pp.85–91.

[6] T. Matsumoto and S. Watanabe, *Extreme points and linear isometries of the domain of a closed $*$-derivation in $C(K)$*, J. Math. Soc. Japan, **48**(1996), pp.229–254.

Jensen の逆不等式とその応用

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よく知られた Jensen の不等式は多方面に渡って重要であるが、その逆不等式を考察することもまた然りであろう(cf. [1, p. 29]). 我々は Jensen の逆不等式と考えられる（特に $m=1,\ \beta=0$ の場合は）次の様な不等式を示す。

Theorem 1. Let $\varphi, \varphi_1, \dots, \varphi_m$ be a strictly convex and strictly positive C^1–functions on a convex domain D in R^k and π a hyperplane in R^{k+1} defined by $y = a_1x_1 + \dots + a_kx_k + b$. Let $f_1, \dots, f_k$ be measurable functions on a probability space (Ω, μ) such that $(f_1(\omega), \dots, f_k(\omega)) \in D_\pi$ for all $\omega \in \Omega$, where $D_\pi = \{x \in D : h(x_1, \dots, x_k) \le a_1x_1 + \dots + a_kx_k + b \ (\forall h = \varphi, \varphi_1, \dots, \varphi_m)\}$. Then we have

$$(1)\quad \int_\Omega \varphi(f_1(\omega), \dots, f_k(\omega))d\mu(\omega) \le \sum_{j=1}^m \alpha_j\varphi_j\Big(\int_\Omega f_1(\omega)d\mu(\omega), \dots, \int_\Omega f_k(\omega)d\mu(\omega)\Big) + \beta$$

for all real numbers $\alpha_1, \dots, \alpha_m$ and β satisfying $\alpha_1, \dots, \alpha_m > 0$ and

$$(2)\quad \sum_{j=1}^m \alpha_j\varphi_j(x_1, \dots, x_k) + \beta = a_1x_1 + \dots + a_kx_k + b,$$

$$\begin{bmatrix} \partial\varphi_1/\partial x_1 & \dots & \partial\varphi_m/\partial x_1 \\ \vdots & \vdots & \vdots \\ \partial\varphi_1/\partial x_k & \dots & \partial\varphi_m/\partial x_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

for some $x = (x_1, \dots, x_k) \in D_\pi$.

略証。我々は [1, Theorem 1, p. 27] の証明を参考にする。凸曲面：$y = \sum_{j=1}^m \alpha_j\varphi_j(x_1, \dots, x_k) + \beta\ (\alpha_j > 0 ; j = 1, \dots, m)$ が超平面：$y = a_1x_1 + \dots + a_kx_k + b$ に接する条件が、ある $x = (x_1, \dots, x_k) \in D_{\pi,\varphi}$ に対して、定理の条件 (2) を満たすことに注意すれば，グラフの位置関係を考えることにより，欲しい不等式 (1) が得られる。

証明終

同様の考察をすれば、凹関数に関する次の様な不等式が得られる。但しこの場合は、$m=1$ で、評価する関数も元のものと同じものとしないと、それ以外はなかなか面白みが見えない。

Theorem 2. Let ψ be a strictly concave and strictly positive C^1 –function on a convex domain D in R^k and π a hyperplane in R^{k+1} defined by $y = a_1x_1 + \ldots + a_kx_k + b$. Let $f_1, \ldots, f_k$ be measurable functions on a probability space (Ω, μ) such that $(f_1(\omega), \ldots, f_k(\omega)) \in D_\pi$ for all $\omega \in \Omega$, where $D_\pi = \{x \in D : \psi(x_1, \ldots, x_k) \geq a_1x_1 + \ldots + a_kx_k + b\}$. Then we have

$$\psi(\int_\Omega f_1(\omega)d\mu(\omega), \ldots, \int_\Omega f_k(\omega)d\mu(\omega)) \leq \alpha\int_\Omega \psi(f_1(\omega), \ldots, f_k(\omega))d\mu(\omega) + \beta$$

for all real numbers α and β satisfying $\alpha > 0$ and

$$\beta = -\alpha(a_1x_1 + \ldots + a_kx_k + b) + \psi(x_1, \ldots, x_k)\ , \quad \frac{\partial\psi(x)}{\partial x_i} = \alpha a_i \ \ (i = 1, \ldots, k)$$

for some $x = (x_1, \ldots, x_k) \in D_\pi$.

Theorem 1 において、$k = m = 1$ の場合を考えると次の系を得る。

Corollary 3. Let $\varphi, \psi : [m, M] \to R, \varphi(t), \psi(t) > 0, \varphi''(t), \psi''(t) > 0$ for $t \in [m, M]$ and f a measurable function on a probability space (X, μ) with $f(X) \subseteq [m, M]$. Then

$$\int_X \varphi \circ f \, d\mu \leq \alpha\psi(\int_X f \, d\mu) + \beta$$

for all real numbers α and β such that $\alpha > 0$ and $\beta = -\alpha\psi(\psi'^{-1}(\frac{a}{\alpha})) + a\psi'^{-1}(\frac{a}{\alpha}) + b$,
where $a = \dfrac{\varphi(M) - \varphi(m)}{M - m}$, $b = \dfrac{M\varphi(m) - m\varphi(M)}{M - m}$.

注意：このとき、$\dfrac{d\beta}{d\alpha} = \psi(\psi'^{-1}(\frac{a}{\alpha})) < 0$, $\dfrac{d^2\beta}{d\alpha^2} = \dfrac{a^2}{\alpha^3\psi''(\psi'^{-1}(\frac{a}{\alpha}))} > 0$ が得られるので、β は α の単調減少する凸関数となっている。我々は β の零点の値の評価に興味があり、今後の課題である。

また Theorem 2 において、$k = 1$ の場合を考えると次の系を得る。これは [8] のなかで既に得られたものでもある。

Corollary 4. Let $\psi : [m, M] \to R, \psi(t) > 0, \psi''(t) < 0$ for $t \in [m, M]$ and f a measurable function on a probability space (X, μ) with $f(X) \subseteq [m, M]$. Then

$$\psi(\int_X f \, d\mu) \leq \alpha\int_X \psi \circ f \, d\mu + \beta$$

for all real numbers α and β such that $\alpha > 0$ and $\beta = -\alpha(a\psi'^{-1}(a\alpha) + b) + \psi(\psi'^{-1}(a\alpha)))$,
where $a = \dfrac{\psi(M) - \psi(m)}{M - m}$, $b = \dfrac{M\psi(m) - m\psi(M)}{M - m}$.

注意：このとき、$\frac{d\beta}{d\alpha}=-(a\psi'^{-1}(\alpha a)+b)<0$, $\frac{d^2\beta}{d\alpha^2}=\frac{-a^2}{\psi''(\psi'^{-1}(\alpha a))}>0$ が得られるので、やはり β は α の単調減少する凸関数となっている。我々はこの場合も β の零点の値の評価に興味があり、今後の課題である。

重要：Corollary 4 と可換 Banach 環の Gelfand 理論を組み合わせると、Kantorovich [6] の不等式の一般化である Ky Fan [2] 及び Mond-Pečarić [7] の行列不等式を導くことができるのであるが (cf. [8])、実は、 Ky Fan 及び Mond-Pecaric の行列不等式の一般化は、『Furuta inequality』の創始者で有名な古田孝之先生によって、この小論を本質的に含む深い議論がすでになされているのである。それ故、読者に文献：

T. Furuta [3, 4]

を読まれることを薦めたい。

実は、筆者は古田先生からすでに上の文献を頂いていることに気付き、まさに汗顔のいたりであったことを最後に述べたい。

References

1. P. S. Bullen, D. S. Mitrinovic and P. M. Vasic, "Means and Their Inequalities, " Reidel, Dordrecht / Boston / Lancaster / Tokyo, 1988.
2. K. Fan, Some matrix inequalities, Math. Sem. Univ. Hamburg, 29 (1966), 185-196.
3. T. Furuta, Extensions of Mond-Pecaric generalization of Kantorovich inequality, preprint.
4. T. Furuta, Two extensions of Ky Fan generalization and Mond-Pecaric matrix version generalization of Kantorovich inequality, preprint.
5. W. Greub and W. Rheinbold, On a generalization of an inequality of L. V. Kantrovich, Proc. Amer. Math. Soc. 10(1954), 407-415.
6. L. V. Kantorovich, Functional analysis and applied mathematics (in Russian), Uspechi Mat. Nauk, 3(1948), 89-185.
7. B. Mond and J. E. Pečarić, Generalization of a matrix inequality of Ky Fan, J. Math. Anal. Appl. 190 (1995), 244-247.
8. 高橋眞映、Ky Fan の行列不等式に関する一考察、実解析学シンポジューム大分、1996, 近刊。

$C(X)$ の恒等作用素近似と Scheffold の条件

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Approximation of the identity operator on $C(X)$ and Scheffold's conditions

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次が有名な Korovkin の定理である [4]。

定理. $\{T_n\}_n$ が $C([0,1])$ 上の有界線形作用素列で $\|T_n\| \leq 1$ かつ $\|T_x^j - x^j\|_\infty \to 0$, $j = 0, 1, 2,$ を満たすとき、$\|T_n f - f\|_\infty \to 0\ \forall f \in C([0,1])$.

この定理は、$\{T_n\}_n$ が I に強収束するかどうかは、$\{1, x, x^2\}$ に対してテストすれば良いというものである。一般のコンパクト空間 X 上の連続関数空間 $C(X)$ で考えたとき、どうのような部分集合、部分空間の S に対してテストすればに強収束が示せるかという問題が生ずる。

$S \subset C(X)$ とする。

定義。S に対して (sequence type) Korovkin の定理成立
$\Longleftrightarrow$ $T_n : C(X) \to C(X), \|T_n\| \leq 1, \|T_n f - f\|_\infty \to 0\ \forall f \in S$ ならば
$\|T_n g - g\|_\infty \to 0\ \forall g \in C(X)$.

列を net に代えても同様に Korovkin の定理を考えることができる。これに関しては次の特徴付けが知られている [6, 7]。

定理B. S に対して net type Korovkin の定理成立
$\Longleftrightarrow$ $\forall x \in X, x \in \forall U$ open $\exists f \in S$ s.t. $\|f\|_\infty = 1, f(x) = 1, |f| < 1$ on U^c.

- net type で成立 $\Rightarrow$ sequence type で成立。
- S を C^*-subalgebra とする。$S \neq C(X)$ $\Longleftrightarrow$ S に対して net type が成立せず。
- S が separable ならば、net type で成立 $\Longleftrightarrow$ sequence type で成立 [2]。

Scheffold [5] は次の S に対して Korovkin の定理が成立することを示した。

例 1. $L^\infty([0,1]) \cong C(X), x_0 \in X, S = \{f \in C(X); f(x_0) = 0\}$.

例 2. $l^\infty \cong C(\beta N), x_0 \in \beta N \setminus N, S = \{f \in C(\beta N); f(x_0) = 0\}$.

上の例は共に、S は closed ideal である。これに関連して Scheffold は 2 つの条件を考えた。

$E \subset X$ closed が条件 (α) を満たすとは

(α) $\qquad \|f_n\|_{U^c} \to 0,\ E \subset \forall U \subset X$ open, ならば $\|f_n\|_{E^c} \to 0.$

$Z = \beta(X \times N), Y = Z \setminus (X \times N)$ とする。$f \in C(X)$ に対して、$\bar{f}(x,n) = f(x)\ \forall x \in X,\ \forall n \in N$ と定義する。

$$\phi : Y \to X,\ f(\phi(y)) = \bar{f}(y) \quad \text{continuous}$$

が定義できる。

$A \subset X$ に対して（closed とは限らない）、

$$\hat{A} = \phi^{-1}(A) \subset Y,\ \tilde{A} = Y \cap \overline{(\bar{A}^X \times N)}^Z$$

とする。次が条件 (β) である。

(β) $\qquad \tilde{A} \subset \overline{\hat{A}}^Y.$

次の定義は [3] で導入された概念である。

定義。$\emptyset \neq E \subset X$ closed が quasi G_δ-集合とは

$$\exists U_n \text{ open } s.t.\ U_n \downarrow,\ E = \bigcap_{n=1}^{\infty} \overline{U_n}.$$

$\Gamma \subset X$ を closed とし、

$$S = \{f \in C(X); f = 0 \text{ on } \Gamma\}$$

とする。次の 2 つの結果は Scheffold [5] による。

命題 1. Γ^c は (β) を満たし、Γ^c dense $\Rightarrow$ S に対して Korovkin 成立。

命題 2. Γ は (α) を満たす $\Rightarrow$ Γ^c は (β) を満たす。

次が S が closed ideal のときの特徴付けである [3]。

定理. 次は同値。

1) S に対して Korovkin 成立。

2) Γ は quasi G_δ-集合を含まない。

3) Γ は (α) を満たし、Γ^c dense.

4) $\hat{\Gamma}$ は Y で内点を持たない。

注意。それぞれ、2) は位相空間的性質、3) は関数列を使って、4) は Stone-Čech compactification の中での特徴付けとなっている。

REFERENCES

[1] F. Altomare and M. Campiti, Korovkin-type approximation theory and its applications, W. de Gruyter, 1994.

[2] K. Izuchi, H. Takagi and S. Watanabe, Sequential BKW-operators and function algebras, J. Approx. theory 85(1996), 185-200.

[3] K. Izuchi, H. Takagi and S. Watanabe, Sequential Korovkin type theorems and weighted composition operators, Acta Sci. Math., 62(1996), 161-174.

[4] P. P. Korovkin, Linear operators and approximation theory, Hindustan Publishing, 1960.

[5] E. Scheffold, Über die punktweise Konvergenz von Operatoren in $C(X)$, Rev. Acad. Ci. Zaragoza, 28(1973), 5-12.

[6] S. -E. Takahasi, (T, E)-Korovkin closures in normed spaces and BKW-operators, J. Approx. Theory 82(1995), 340-351.

[7] D. E. Wulbert, Convergence of operators and Korovkin's theorem, J. Approx. Theory 1(1968), 381-390.

L^p-Lipschitz Classes and Weighted Bergman Spaces

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ABSTRACT. We introduce and study a class of functions on the unit disk D which satisfy

$$\rho_{p,\alpha,r}(f) = \left(\int_D \left\{ \sup_{w \in D_r(z)} \frac{|f(z) - f(w)|}{|z - w|^\alpha} \right\}^p d\mu(z) \right)^{1/p} < \infty,$$

where $D_r(z)$ is a Bergman disk and μ is a σ-finite positive Borel measure. We obtain equivalent quantities of $\rho_{p,\alpha,r}(f)$, and investigate linear functionals on this class. Particularly, we estimate Taylor coefficients of these functions for some measures.

§1. Introduction

Let D be the open unit disk in the complex plane $\mathbb{C}$. H denotes the set of all harmonic functions and A denotes the set of all analytic functions on D. For each $a \in D$, let ϕ_a be the Möbius function on D, that is,

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z} \qquad (z \in D)$$

and put

$$\beta(a, z) = \frac{1}{2} \log \frac{1 + |\phi_a(z)|}{1 - |\phi_a(z)|} \quad (a, z \in D).$$

For $0 < r < \infty$ and $a \in D$,

$$D_r(a) = \{z \in D \ ; \ \beta(a, z) < r\}$$

is the Bergman disk with "center" a and "radius" r. Particularly, the Euclidean center and radius of $D_r(a)$ are

$$C_{a,r} = \frac{1 - s^2}{1 - s^2|a|^2} a \qquad R_{a,r} = \frac{1 - |a|^2}{1 - s^2|a|^2} s$$

respectively, where $s = \tanh r$. For $0 \leq \alpha \leq 1$ and a continuous function f on D, we define a function $O_r^\alpha(f, z)$ by

$$O_r^\alpha(f, z) = \sup_{w \in D_r(z)} \frac{|f(z) - f(w)|}{|z - w|^\alpha}.$$

We say that $f \in A$ satisfies a Lipschitz condition of order α if $O_r^\alpha(f, z)$ is bounded on D. It is well-known fact that $f \in A$ satisfies this condition if and only if $(1 - |z|^2)^{1-\alpha}|f'(z)|$ is bounded on D. These classes are very familiar to classical analysis. Especially when $\alpha \neq 0$ it is also well-known fact that above these conditions are equivalent that $\dfrac{|f(z)-f(w)|}{|z-w|^{\alpha}}$ is bounded on $D \times D$.

We will introduce new classes of analytic or harmonic functions on the unit disk D, which are generalizations of a Lipschitz class. Let μ be a σ-finite positive Borel measure on D and $0 < r < \infty$ be fixed. For $0 \leq \alpha \leq 1$ and $1 \leq p < \infty$, $B_h^p(\mu,\alpha) = B_h^p(\mu,\alpha,r)$ consists of functions $f \in H$ such that $f(0) = 0$ and

$$\rho_{p,\alpha}(f) = \rho_{p,\alpha,r}(f) = \left(\int_D \{O_r^{\alpha}(f,z)\}^p \, d\mu(z)\right)^{1/p} < \infty,$$

and let $B_a^p(\mu,\alpha) = B_a^p(\mu,\alpha,r) = B_h^p(\mu,\alpha,r) \cap A$. We are interested in several properties of these spaces. Let $\alpha = p = 1$. For each $f \in A$, clearly we have that $|f'(z)| \leq O_r^1(f,z)$ for all $z \in D$. Therefore an inequality $\int |f'(z)| d\mu \leq \int O_r^1(f,z) d\mu$ is valid for all $f \in A$. However, the left hand side of the inequality is not always comparable to $\int O_r^1(f,z) d\mu$ except for special measures, that is, which is too small. Moreover, it is clear that $\rho_{p,\alpha}(\cdot)$ is a norm on $B_a^p(\mu,\alpha)$ and $B_h^p(\mu,\alpha)$. Therefore, we are also interested in the properties of them as normed linear spaces.

§2. Estimations of L^p-Lipschitz norms

Let m be the normalized Lebesgue area measure, that is, $dm = dxdy/\pi$. For $1 \leq p < \infty$ a Bergman space L_a^p is defined by $L_a^p = L^p(D,m) \cap A$ and the $L^p(m)$-norm is denoted by $\|\cdot\|_p$. For fixed $0 < r < \infty$, we define the arithmetic mean on the Bergman disk $D_r(a)$ of a σ-finite positive Borel measure μ by

$$\hat{\mu}_r(a) = \frac{1}{m(D_r(a))} \int_{D_r(a)} d\mu \qquad (a \in D).$$

If there exists a non-negative function u such that $d\mu = udm$, then we may write it $\hat{u}_r$ instead of $\hat{\mu}_r$. For a differentiable complex function f on D, we define a differential operator $\mathcal{D}$ by $\mathcal{D}f = \partial f/\partial x = (\partial/\partial z + \partial/\partial \bar{z}) f$. If $f \in H$ then there are functions $g, h \in A$ such that $f = g + \bar{h}$, and thus $\mathcal{D}f = g' + \overline{h'}$. In this section, we give lower and upper estimations of the norm $\rho_{p,\alpha}(f)$ in $B_h^p(\mu,\alpha)$ or $B_a^p(\mu,\alpha)$.

Theorem 1. *Let $0 < r < \infty$ be fixed, and suppose μ is a σ-finite positive Borel measure on D. Then the following are valid.*

(1) *For $0 \leq \alpha \leq 1$ and $1 \leq p < \infty$ there exists a constant $C = C_{r,\alpha,p} > 0$ (independent of f) such that*

$$\begin{aligned} & C^{-1} \int_D \left\{(1-|z|^2)^{1-\alpha} |\mathcal{D}f(z)|\right\}^p \hat{\mu}_{r/2}(z) dm \\ \leq \; & \int_D \{O_r^{\alpha}(f,z)\}^p \, d\mu \\ \leq \; & C \int_D \left\{(1-|z|^2)^{1-\alpha} |\mathcal{D}f(z)|\right\}^p \hat{\mu}_{2r}(z) dm \end{aligned}$$

for all $f \in H$.

(2) *For* $0 \leq \alpha \leq 1$ *and* $1 \leq p < \infty$ *there exists a constant* $C = C_{r,\alpha,p} > 0$ *such that*

$$\int_D \left\{(1-|z|^2)^{1-\alpha}|\mathcal{D}f(z)|\right\}^p d\mu \leq C \int_D \{O_r^\alpha(f,z)\}^p d\mu$$

for all $f \in H$.

We note that if we replace $f \in H$ by $f \in A$ in the inequalities then Theorem 1 is valid for $0 < p < 1$. Moreover, if we replace $\mathcal{D} = \partial/\partial y$ or $\mathcal{D} = \nabla$ instead of $\partial/\partial x$, then Theorem 1 is also valid.

Corollary 1. *Let* $0 < r < \infty$, $0 \leq \alpha \leq 1$, *and* $1 \leq p < \infty$ *be fixed, then the following are valid.*

(1) *Suppose there exists* $\gamma > 0$ *such that* $t^\gamma \sigma(t)$ *is a non-decreasing function on* $[1, \infty)$ *and* v *is a positive superharmonic function on* D, *then there are constants* $C = C_{r,\alpha,p} > 0$ *and* $\eta = \eta_r > 1$ *such that*

$$\begin{aligned} & C^{-1} \int_D \left\{(1-|z|^2)^{1-\alpha}|\mathcal{D}f(z)|\right\}^p v(z)\sigma\left(\frac{1}{1-|z|^2}\right) dm \\ \leq\ & \int_D \{O_r^\alpha(f,z)\}^p v(z)\sigma\left(\frac{1}{1-|z|^2}\right) dm \\ \leq\ & C \int_D \left\{(1-|z|^2)^{1-\alpha}|\mathcal{D}f(z)|\right\}^p v(z)\sigma\left(\frac{\eta}{1-|z|^2}\right) dm \end{aligned}$$

for all $f \in H$.

(2) *Let* $\{b_j\} \subset \bar{D}$ *be a finite sequence of complex numbers with* $b_i \neq b_j$ $(i \neq j)$, *and let* $\{l_j\}$ *be a finite sequence of arbitrary real numbers with* $0 \leq l_j$, *then there is a constant* $C = C_{r,\alpha,p} > 0$ *such that*

$$\begin{aligned} & C^{-1} \int_D \left\{(1-|z|^2)^{1-\alpha}|\mathcal{D}f(z)|\right\}^p \log \Pi_j \frac{2^{l_j}}{|z-b_j|^{l_j}} dm \\ \leq\ & \int_D \{O_r^\alpha(f,z)\}^p \log \Pi_j \frac{2^{l_j}}{|z-b_j|^{l_j}} dm \\ \leq\ & C \int_D \left\{(1-|z|^2)^{1-\alpha}|\mathcal{D}f(z)|\right\}^p \log \Pi_j \frac{2^{l_j}}{|z-b_j|^{l_j}} dm \end{aligned}$$

for all $f \in H$.

(3) *Let* $\{b_j\} \subset \bar{D}$ *be a finite sequence of complex numbers with* $b_i \neq b_j$ $(i \neq j)$, *and let* $\{l_j\}$ *be a finite sequence of arbitrary real numbers with* $-2 < l_j$ *for* $j \in \Lambda^c$, *where* $\Lambda = \{j; b_j \in \partial D\}$. *Then there is a constant* $C = C_{r,\alpha,p} > 0$ *such that*

$$\begin{aligned} & C^{-1} \int_D \left\{(1-|z|^2)^{1-\alpha}|\mathcal{D}f(z)|\right\}^p \Pi_{j\in\Lambda}|z-b_j|^{l_j} dm \\ \leq\ & \int_D \{O_r^\alpha(f,z)\}^p \Pi_j|z-b_j|^{l_j} dm \\ \leq\ & C \int_D \left\{(1-|z|^2)^{1-\alpha}|\mathcal{D}f(z)|\right\}^p \Pi_{j\in\Lambda}|z-b_j|^{l_j} dm \end{aligned}$$

for all $f \in H$.

§3. Linear functionals on weighted Bergman spaces

Let ν be a finite positive Borel measure on D and suppose $0 < p < \infty$. A weighted Bergman space $L^p_a(\nu)$ is defined by $L^p_a(\nu) = L^p(D,\nu) \cap A$. Let $\Psi_\nu = \Psi^p_\nu$ be the set of all functions $\psi \in A$ such that

$$r(\psi) = r(\nu,p,\psi) = \sup\left\{ \left| \lim_{t\to 1-} \int_D f(tz)\bar{\psi}(z)dm \right|^p ; f \in A, \int_D |f|^p d\nu \le 1 \right\}$$

is finite. Moreover, put

$$s(\psi) = s(\nu,p,\psi) = \inf\left\{ \int_D |f|^p d\nu ; f \in A, \lim_{t\to 1-} \int_D f(tz)\bar{\psi}(z)dm = 1 \right\},$$

which is the reciprocal of $r(\psi)$. In [11], a Riesz's function $r(a) = \sup\left\{ |f(a)|^p ; \int_D |f|^p d\nu \le 1 \right\}$ was introduced and studied its behavior near the boundary of D. By the definition of $r(\psi)$, if we put $\psi_a = (1-\bar{a}z)^{-2}$ then $\int_D f\bar{\psi}_a dm = f(a)$, and thus $r(\psi_a) = r(a)$. Moreover, if $\psi_n = (n+1)z^n$ then $\int_D f\bar{\psi}_n dm = \hat{f}(n)$, where the numbers $\hat{f}(n)$ are the Taylor coefficients of f. The functions $r(\psi)$ and $s(\psi)$ are also called Riesz's functions.

In section 2, (1) of Theorem 1 shows that the norm $\rho_{p,\alpha}(f)$ on $B^p_a(\mu,\alpha)$ is estimated by the $L^p(\nu)$-norm of f', where $d\nu = (1-|z|^2)^{(1-\alpha)p}\hat{\mu}_r(z)dm$. Therefore, in order to study the space $B^p_a(\mu,\alpha)$ we will investigate the properties of the weighted Bergman space. Particularly, we will estimate $r(\psi_n)$, which is the norm of a linear functional $f \mapsto \hat{f}(n)$.

Theorem 2. *Let $0 < p < \infty$ be fixed, and suppose ν is a finite positive Borel measure on D. Then the following are valid.*

(1) *If there is a constant $C > 0$ such that $\int |f|^p d\tau \le C \int |f|^p d\nu$ for all $f \in A$ then $\Psi_\tau \subset \Psi_\nu$, where τ is a finite positive Borel measure on D.*

(2) *If $\{|\psi_\lambda|^p / \| \psi_\lambda \|_2^{2p} \, d\nu\}$ is uniformly absolutely continuous with respect to ν and there exists $\gamma > -1$ such that $\int |\psi_\lambda|^p / \| \psi_\lambda \|_2^{2p} \, (1-|z|^2)^\gamma dm \to 0$ then $s(\psi_\lambda) \to 0$.*

(3) $r(\psi_a) \to \infty$ $(|a| \to 1)$ *and* $r(\psi_n) \to \infty$ $(n \to \infty)$.

(4) *If $1 < p < \infty$ and there exists a compact set $K \subset D$ such that* $\mathrm{supp}\nu \cap K$ *is not a finite set, then $\Psi_\nu \cap L^q_a$ is dense in L^q_a, where $1/p + 1/q = 1$.*

In [11], using $r(a) = r(\psi_a)$, a completeness of $L^p_a(\nu)$ was characterized. We will give another characterization of it. A subset E of D is a uniqueness set for $L^p_a(\nu)$ if E satisfies the following : If $f \in L^p_a(\nu)$ is zero on E, then $f \equiv 0$ on D.

Proposition 3. *Let $1 \le p < \infty$ be fixed, and suppose ν is a finite positive Borel measure on D such that* $\mathrm{supp}\nu \cap D$ *is a uniqueness set for $L^p_a(\nu)$. Then the following are mutually equivalent.*

(1) $L^p_a(\nu)$ *is closed in* $L^p(D,\nu)$.

(2) *For each compact set* K *in* D, $\int_K \log r(\nu,p,\psi_a)dm(a) < \infty$.

(3) *For each* $0 < c < \infty$ *there is a constant* $\gamma_c > 0$ *such that* $r(\nu,p,\psi_n) \le \gamma_c \exp(cn)$ *for all* n.

We say that a positive function v on D has the generalized subharmonic property if for each $0 < r < \infty$ there is a constant $C = C_r > 0$ such that $v(z) \le \frac{C}{m(D_r(z))}\int_{D_r(z)} vdm$ for all $z \in D$. Moreover, we define the geometric mean on the circle of radius ρ of v by

$$V(\rho) = \exp\left[\int_0^{2\pi} \log v(\rho e^{i\theta})d\theta/2\pi\right] \qquad (0 < \rho < 1).$$

We note that if v is subharmonic then it has the generalized subharmonic property (see the proof of Proposition 4.3.8 in [15, p.62]). The following Theorem 4 gives more precise estimations for some special measures. We note that Theorem 4 is closely related with a estimation of the Taylor coefficients of the functions in $B^p_a(\mu,\alpha)$, which is defined in section 2. The relation will be described in section 4.

Theorem 4. *Let* $0 < p < \infty$ *be fixed. Suppose there exists* $\gamma > 0$ *such that* $t^\gamma\sigma(t) \downarrow 0$ $(t \to 0)$ *and* $d\nu = v(\rho e^{i\theta})\sigma(1-\rho)d\theta/2\pi d\rho$, *then the following are valid.*

(1) *If* $v(\rho e^{i\theta})\sigma(1-\rho)$ *is bounded on* D *and* $V(\rho)$ *is non-increasing on* $[0,1)$, *then there exist* $n_0 \in \mathbb{N}$ *and constants* $0 < C_1, C_2 < \infty$ *such that*

$$r(\nu,p,\psi_n) \le C_1 \exp\left[\sqrt{n}\log\frac{C_2 n^\gamma}{V(1-1/\sqrt{n})\sigma(1/\sqrt{n})^2}\right]$$

for all $n \ge n_0$.

(2) *If* v *has the generalized subharmonic property, then there exist* $n_0 \in \mathbb{N}$ *and constants* $0 < C_1, C_2 < \infty$ *such that*

$$r(\nu,p,\psi_n) \le C_1 \exp\left[\sqrt{n}\log\frac{C_2 n^{\gamma+2}}{\inf\limits_{n\le t\le 2n} V(1-1/\sqrt{t})\sigma(1/8\sqrt{n})^2}\right]$$

for all $n \ge n_0$.

§4. Equivalent norms and Taylor coefficients

In section 2, (1) of Theorem 1 shows that the norm $\rho_{p,\alpha}(f)$ on $B^p_a(\mu,\alpha)$ is estimated by the $L^p(\nu)$-norm of f', where $d\nu = (1-|z|^2)^{(1-\alpha)p}\hat{\mu}_r(z)dm$. In this section, using the estimations and Theorem 4, we will study the growth of the Taylor coefficients of functions in $B^p_a(\mu,\alpha)$.

Let I_n be the interval in $[0,1)$ such that $I_n = [1-1/\sqrt{n}, 1-1/2\sqrt{n}]$. We will give estimations of the Taylor coefficients of L^p-Lipschitz functions.

Theorem 5. *Let* $1 \le p < \infty$, $0 \le \alpha \le 1$ *be fixed, and suppose that* $(1-|z|^2)^{(1-\alpha)p}d\mu$ *is a finite positive Borel measure on* D. *If there exist a compact set* $K \subset D$ *and* $0 < r < \infty$ *such that* $\operatorname{supp}\mu \cap D_r(a) \ne \phi$ *for all* $a \in K^c$, *then the following are valid.*

(1) *For each* $0 < c < \infty$, *there exists a constant* $0 < \gamma_c < \infty$ *such that*

$$\sup\{|\hat{f}(n)|^p \ ; \ \rho_{p,\alpha,r}(f) \leq 1\} \leq \gamma_c \exp(cn)$$

for all $n \geq 1$.

(2) *Let* $\delta > 0$ *be given, then for each* $0 < c < \infty$, *there exists a constant* $0 < \gamma_c < \infty$ *such that*

$$\sup\{|\hat{f}(n)|^p \ ; \ \rho_{p,\alpha,r}(f) \leq 1\} \leq \gamma_c \exp(cn^{1/2+\delta}) \sup_{\rho \in I_n} \exp\left[-\sqrt{n}\int_0^{2\pi} \log \hat{\mu}_r(\rho e^{i\theta}) d\theta/2\pi\right]$$

for all $n \geq 1$.

Corollary 2. *Let* $1 \leq p < \infty$, $0 \leq \alpha \leq 1$ *be fixed. Suppose that* $(1-|z|^2)^{(1-\alpha)p} d\mu$ *is a finite positive Borel measure on* D *and* $d\mu = udm$, *then the following are valid.*

(1) *Suppose that* u *is a bounded superharmonic function on* D *and let* $\delta > 0$ *be given, then for each* $0 < c < \infty$, *there exists a constant* $0 < \gamma_c < \infty$ *such that*

$$\sup\{|\hat{f}(n)|^p \ ; \ \rho_{p,\alpha,r}(f) \leq 1\} \leq \gamma_c \exp(cn^{1/2+\delta}) \exp\left[-\sqrt{n}\int_0^{2\pi} \log u(\rho e^{i\theta}) d\theta/2\pi\right]$$

for all $n \geq 1$, *where* $\rho = 1 - 1/\sqrt{n}$.

(2) *Suppose that* u *has the generalized subharmonic property. If there exist a compact set* $K \subset D$ *and* $0 < r < \infty$ *such that* $\mathrm{supp}\mu \cap D_r(a) \neq \phi$ *for all* $a \in K^c$, *and let* $\delta > 0$ *be given, then for each* $0 < c < \infty$, *there exists a constant* $0 < \gamma_c < \infty$ *such that*

$$\sup\{|\hat{f}(n)|^p \ ; \ \rho_{p,\alpha,r}(f) \leq 1\} \leq \gamma_c \exp(cn^{1/2+\delta}) \sup_{\rho \in I_n} \exp\left[-\sqrt{n}\int_0^{2\pi} \log u(\rho e^{i\theta}) d\theta/2\pi\right]$$

for all $n \geq 1$.

(3) *Suppose that* $h \in A$ *and* p *is an analytic polynomial. If* $u = |h + \bar{p}|$, *and let* $\delta > 0$ *be given, then for each* $0 < c < \infty$, *there exists a constant* $0 < \gamma_c < \infty$ *such that*

$$\sup\{|\hat{f}(n)|^p \ ; \ \rho_{p,\alpha,r}(f) \leq 1\} \leq \gamma_c \exp(cn^{1/2+\delta})$$

for all $n \geq 1$.

References

[1] J.B.Conway, Subnormal Operators, Research Notes in Mathematics, Pitman Advanced Publishing Program, 1981.

[2] P.L.Duren, Theory of H^p Spaces, Academic Press, New York, 1970.

[3] T.W.Gamelin, Uniform Algebras, Prentice-Hall, Englewood Cliffs, New Jersey, 1969.

[4] J.Garnett, Bounded Analytic Functions, Academic Press, New York, 1981.

[5] T.Kriete and B.MacCluer, Composition operators and weighted polynomial approximation, Proc. of Symposia in Pure Math. vol.51(1990), 175-182.

[6] T.Kriete and T.Trent, Growth near the boundary in $H^2(\mu)$ spaces, Proc. Amer. Math. Soc. 62(1977), 83-88.

[7] J.McCarthy, Weighted Bergman spaces, Proceedings of Conference on Function Spaces(Edwardsville, 1990).

[8] J.McCarthy, Topologies on the Smirnov class, J. of Funct. Anal. 104(1992), 229-241.

[9] J.McCarthy, Coefficient estimates on weighted Bergman spaces, Duke Math. J. (1994), 751-760.

[10] T.Nakazi and M.Yamada, (A_2)-conditions and Carleson inequalities in Bergman spaces, Pacific J. Math. 173(1996), 151-171.

[11] T.Nakazi and M.Yamada, Riesz's functions in weighted Hardy and Bergman spaces, to appear in Canad. J. Math.

[12] N.Yanagihara, Mean growth and Taylor coefficients of some classes of functions, Ann. Polon. Math. 30(1974), 37-48.

[13] K.Zhu, Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains, J. Operator Theory 20(1988), 329-357.

[14] K.Zhu, Analytic Besov spaces, J. Math. Anal. Appl. 157(1991), 318-336.

[15] K.Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York, 1990.

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Lattice structure on partially ordered linear space

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Abstract

In this short note, we shall explain the notion of lattice order structure on partially ordered linear space E . In general, partially ordered linear space E is not lattice ordered *i.e.* it is not necessary to have least upper bound for arbitrary two elements $x, y \in E$. But in some partially ordered linear space , we can define least upper bounds denoted by $x \vee y$ which is a subset and no more single element in general. The subset $x \vee y$ is defined as totality of minimal elements z for a set of upper bounds of x and y in the sense that

(1) $z \geq x$ and $z \geq y$, (2) if $z \geq w$ and $w \geq x, w \geq y$, then $w = z$.

We shall clearify properties of these subsets $x \vee y$. Also, we shall discuss greatest lower bounds denoted by $x \wedge y$ for $x, y \in E$.

Sometimes we shall consider normed partially ordered linear space, since we can expect nice properties in this space. The content of this note is almost same as a note presented in Real Analysis Seminar held on November at Oita University. More precise and extended results with proofs will be found in another paper in near future.

1 Partially ordered linear space and supremum of subsets

Let E be a linear space of real coefficients. We shall conider a convex cone with (1) $P \cap -P = \{0\}$ and (2) $P - P = E$.

We shall define $x \leq y$ if $x - y \in P$ ($y \geq x$ in the same meaning) , then we have (a) $x \leq y$ and $y \leq x$ imply $x = y$; (b) $x \leq y$ and $y \leq z$ imply $x \leq z$; (c) $x \leq y$ implies $x + z \leq y + z$ for all $z \in E$; (d) for $x \leq y$ and a positive number α we have $\alpha x \leq \alpha y$; (e) for every element $x \in E$ we find $x_1, x_2 \geq 0$ with $x = x_1 - x_2$.

Conversely if there is a relation $\leq$ satisfying (a), (b), (c), (d), (e), then the set $P = \{x; 0 \leq x\}$ is a convex cone satisfying above condition (1) and (2). A convex cone P with

(1) and (2) is called an *order* and E is called a *partially ordered linear space.*
We shall define subset $U\{x, y\} = \{z; z \leq x \text{ and } z \leq y\}$, then $x \vee y$ is a subset of all minimal elements of $U\{x, y\}$. We have $U\{x, y\} \neq \emptyset$, but it may happen that $x \vee y = \emptyset$. We shall show such example.

Example Let E be 2-dimensional Euclidean space $\mathbf{R^2}$, and let $P = \{(x, y); x > 0, y > 0\} \cup (0, 0)$ be an order in E . Then $x \vee 0 = \emptyset$ for $x = (-1, 1)$. More generally, if $x \notin P \cup \{-P\}$, then $x \vee 0 = \emptyset$

For a subset $A = \{a_{\lambda \in \Lambda}\}$ of E , we can define $\vee A = \vee_{\lambda \in \Lambda} a_\lambda$ and $\vee A$ is a subset of all minimal elements of $U(A)$, where $U(A) = \{z; z \geq a, \text{ for all } a \in A\}$. Sometimes , we use notation $supA$ instead of $\vee A$. We can define also $x \wedge y$ and $L\{x, y\} = \{z; z \leq a \text{ for all } a \in A\}$, $x \wedge y$ is a set of all maximal elements of $L\{x, y\}$. We have $L\{x, y\} \neq \emptyset$, it may happen that $x \wedge y = \emptyset$. We can define also $\wedge A = \wedge_{\lambda \in \Lambda}$ for $A = \{a_\lambda; \lambda \in \Lambda\}$. Sometimes we use notation $infA$ instead of $\wedge A$.

We have the following propositions :

Proposition 1

(1) $-supA = inf(-A)$

(2) $\alpha \; supA = sup \; \alpha A$ for $\alpha > 0$

$\alpha \; infA = inf \alpha A$ for $\alpha > 0$

Proposition 2 For every $b \in E$ and a suset A of E. we have

(1) $supA + b = sup\{A + b\}$

(2) $infA + b = inf\{A + b\}$

Proposition 3 If $a \in supA$ and $b \in supB$, then $a + b \in sup\{A + B\}$.

Proposition 4 $a \vee b \neq \emptyset$ for every $a, b \in E$ if and only if $a \wedge b \neq \emptyset$ for every $a, b \in E$.

Theorem 1. If $a \vee b \neq \emptyset$ for some $a, b \in E$, then we have
$a \vee b = a + b - (a \wedge b)$.

Proof. Since $a \wedge b = -\{(-a) \wedge (-b)\}$, we have

$\{(-a) \vee (-b)\} + a + b = a \vee b = b \vee a$.

Hence, we have $-(a \wedge b) + a + b = a \wedge b$.

Let E be a partially ordered linear space with an order P. If $a \vee b$ is always a single element i.e. supremum of two elements a and b of E exists always, then E is called a *Riesz space* or *vecor lattice.* Usual function spaces are considered as a Riesz space with usual function order.

2 Monotone order complete partially ordered linear space

Let E be a partially ordered linear space with an order P. We shall consider suitable condition that $a \vee b$ is always not empty for all pair of elements a and $b \in E$.

For this purpose, we prepare some definitions.
A subset A of E is called *linear ordered* (or A is called a linear subset) if every elements of A is comparable with respect to the order P.

A subset A of E is called *upper bounded (lower bounded)* if there exists $a \in E$ with $a \geq x$ for all $x \in E$ ($a \in E$ with $x \leq a$ for all $x \in A$). a is called an *upper bound* of A (a *lower bound* of A).

We say that a partially ordered linear space E with an order P is *monotone order complete* if every upper bounded linear ordered subset A has a least upper bound i.e. *sup* A is a single element. It is easy to see that if E is a monotone order complete partially complete ordered linear space, then every lower bounded linear ordered subset A has greatest lower bounded i.e. *inf* A is a single element.

If every countable linear ordered and upper bounded subset A of E has least upper bound, then E is called *sequentially monotone order complete.* Every monotone order complete partially ordered linear space is naturally sequentially monotone complete.

A normed partially ordered linear space E is called a partially ordered Banach space if the norm on E is complete i.e. every Cauchy sequence is convergent by this norm.

Let E be a partially ordered Banach space with a closed order P. We shall consider the dual space E^* of E. We shall define an order P^* as follows : Let x^* and $y^* \in E^*$, and $x^* \geq y^*$ if and only if $x^*(x) \geq y^*(x)$ for all $x \in P$.

By T. Ando [2], we have the following fact under the condition that norm is ordered norm :

Theorem 2 P^* generates E^* i.e. P^* is an order in E^* if and only if every order interval $[a, b] = \{x \quad ; a \geq x \geq b\}$ for every pair $a, b \in E$ with $a \geq b$ is norm bounded.

It is known : let E be a partially ordered Banach space with a closed order P , if E is monotone order complete by the order, then every order interval of E is norm bounded.

Related topics are discussed in [3].

Theorem 3 Let E be a partially ordered Banach space with a closed order , and let every order intervals are norm bounded. Then, every upper bounded linear subset of E^* is a partially ordered Banach space with an order P^*, and every upper bounded linear subset of E^* has least upper bound in E^* i.e. E^* is monotone order complete.

Proof. Let A^* be an upper bounded linear subset of E^* . Let us define a linear functional x^* as follows : $x^*(x) = sup\{y^*(x); y^* \in A^*\}$ for $x \in P$. For $x = x_1 - x_2$ with arbitrary $x \in E$ and $x_1, x_2 \in P$, we can define $x^*(x) = x^*(x_1) - x^*(x_2)$ without any contradiction. Then, by the theorem of Banach-Steinhaus x^* is in E^* and it is easy to see that $supA^* = x^*$.

Theorem 4 Let E be a partially ordered Banach space in which every order interval is norm bounded. Then , for every non-empty subset A^* of dual space E^* which is upper bounded by the order P^*, we find that $sup\ A^* \neq \emptyset$

Let P be an order in a partially ordered reflexive Banach space E. If P is closed, then $P = P^{**}$ by the bipolar thorem on convex sets.

Hence we have the following theorem as follows:

Theorem 6 Let E be a finite-dimensional partially ordered linear space with closed order. Then, for every non-empty upper bounded linear set A of E we have $sup\ A \neq \emptyset$. Moreover, E is monotone order complete and every upper bounded linear set A has least upper bound $sup\ A$ which consists of a single element.

If the order P in E is not closed, above theorem is not true in general. We see for example lexicographic order P is not closed for more than 2-dimensional Euclidean space and by this order Theorem 6 is not true.

More interesting topics on partially ordered linear spaces will be found in another paper.

References

[1] I.Amemiya, A generalization of Riesz-Fisher's theorem. Jour. Math. Soc. Japan 5(1953) 353-354.

[2] T.Ando, On fundamental properties of a Banach space with cone. Pacific J. Math. 12(1962) 1163-1169.

[3] S.Koshi,S.Dimiev, R.Lazov, Partially ordered normed linear spaces with weak Fatou property. to appear

[4] W.A.J.Luxemburg, A.C.Zaanen, Riesz spaces I, North Holland Math. Libr. 1(1971).

[5] H.Nakano, Linear topologies on semi-ordered linear spaces. Jour. Faculty of Sciences, Hokkaido University,12(1953)87-104.

[6] I.Namioka, Partially ordered linear topological spaces. Memoirs Amer. Math. Soc.,24(1957).

[7] H.H. Schaefer, Banach lattice and positive operators. Grundlehre der Math. Wiss. Springer Verlag. Berlin-Heidelberg-New York,215(1974).

[8] A.C.Zaanen, Riesz spaces II, North Holland Math. Libr. 30(1983).

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On the zeros of functions in the solution sets of an extremal problem in H^1

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This note is an anouncement of the resusts obtained in the resent work ; J. Inoue and T. Nakazi [1].

For a non-zero function f in H^1, the classical Hardy space on the unit circle, we put

$$\mathcal{S}^f = \{g \in H^1 : \arg f(e^{i\theta}) = \arg g(e^{i\theta}) \ \ a.e.\ \theta\}$$

The intersection of $\mathcal{S}^f$ and the unit sphere in H^1 is just a set of solutions of some extremal problem in H^1. It is known that $\mathcal{S}^f = \mathcal{S}^{\mathcal{B}} \times g_0$, where $\mathcal{B}$ is a Blaschke product and g_0 is a function in H^1 with $\mathcal{S}^{g_0} = \{\lambda \cdot g_0 : \lambda > 0\}$ (E. Hayashi [1]). Also it is known that the linear span of $\mathcal{S}^f$ is of finite dimensional if and only if $\mathcal{B}$ is a finite Blaschke product, and when $\mathcal{B}$ is a finite Blaschke product, we can describe completely the set $\mathcal{S}^{\mathcal{B}}$ and the zeros of an arbitrary nember in $\mathcal{S}^{\mathcal{B}}$ (T. Nakazi [3]).

For each f in H^1, sing(g) denotes the set of points of ∂D on which g cannot be analytically extended.

In this note, we assume that the Blaschke product $\mathcal{B}$ under the consideration has the property that sing$(\mathcal{B})$ has only a finite number of accumulation points, say $\{e^{i\theta_j} : j = 1, ..., N\}$. Under this condition, we define and investigate the local convergence order of $Z(f : \overline{D})$ at $e^{i\theta_j}$ from the left(resp. right) for $f \in \mathcal{S}^{\mathcal{B}}$ and $j \in \{1, ..., N\}$, and get Theorem 1 which is the main theorem of [2].

Definition 1 Let f be an element in $H^1(D)$ such that

$$\text{sing}(f) = \{e^{\theta_1}, ..., e^{\theta_N}\} : \ 0 \le \theta_1 < ... < \theta_N < 2\pi.$$

Let us denote $Z(f; \overline{D}) = \{\alpha_k\}$, and we define the local convergence order of $Z(f; \overline{D})$ at $e^{i\theta_j}$ $(1 \le j \le N)$ from the left by (1) and from the right by (2) below.

$$\mathbf{Ord}^{(l)}[e^{i\theta_j}; \{\alpha_j\}] = \inf\{\sigma > 0 : \sum_{n \in I_j^{(l)}} | e^{i\theta_j} - \alpha_n |^\sigma < \infty\} \quad (1)$$

$$\mathbf{Ord}^{(r)}[e^{i\theta_j}; \{\alpha_j\}] = \inf\{\sigma > 0 : \sum_{n \in I_j^{(r)}} | e^{i\theta_j} - \alpha_n |^\sigma < \infty\} \quad (2)$$

where $I_j^{(l)} = \{n : \alpha_n \in \{re^{i\theta} \in \overline{D} : (\theta_{j-1} + \theta_j)/2 < \theta < \theta_j\}\}$, $I_j^{(r)} = \{n : \alpha_n \in \{re^{i\theta} \in \overline{D} : (\theta_j < \theta < (\theta_j + \theta_{j+1})/2\}\}$, with the convension that $\theta_{n+1} = \theta_1 + 2\pi$, and $\theta_0 = \theta_n - 2\pi$.

Lemma 1. Let $\mathcal{B}$ be an infinite Blaschke product such that

$$\text{sing}(\mathcal{B}) = \{e^{i\theta_1}, ..., e^{i\theta_N}\}; \ 0 \le \theta_1 < ... < \theta_N < 2\pi.$$

For each $j \in \{1, ..., N\}$ and $f \in \mathcal{S}^{\mathcal{B}}$, we put

$$\{\alpha_n\} = \{\alpha_n^{(f;j)}\} = Z(f; \overline{D}) \cap \{re^{i\theta} \in \overline{D} : \theta_j < \theta < (\theta_j + \theta_{j+1})/2\}, \quad (3)$$

$$\{\beta_n\} = \{\beta_n^{(f;j)}\} = Z(f; \overline{D}) \cap \{re^{i\theta} \in \overline{D} : (\theta_{j-1} + \theta_j)/2 < \theta < \theta_j\}, \quad (4)$$

where we mean $\theta_{N+1} = \theta_1 + 2\pi$, $\theta_0 = \theta_n - 2\pi$.

(i) If $\sum_n | \alpha_n^{(f;j)} - e^{i\theta_j} |^\sigma < \infty$ for some $f \in \mathcal{S}^{\mathcal{B}}$ and $\sigma > 1$, then we have $\sum_n | \alpha_n^{(g;j)} - e^{i\theta_j} |^\rho < \infty$ for every $g \in \mathcal{S}^{\mathcal{B}}$ and $\rho > \sigma$.

(ii) If $\sum_n | \beta_n^{(f;j)} - e^{i\theta_j} |^\sigma < \infty$ for some $f \in \mathcal{S}^{\mathcal{B}}$ and $\sigma > 1$, then we have $\sum_n | \beta_n^{(g;j)} - e^{i\theta_j} |^\rho < \infty$ for every $g \in \mathcal{S}^{\mathcal{B}}$ and $\rho > \sigma$.

By Lemma 2, we have at once the following theorem.

Theorem 1. Let $\mathcal{B}$ be an infinite Blaschke product such that

$$\mathrm{sing}(\mathcal{B}) = \{e^{i\theta_1}, ..., e^{i\theta_N}\}; 0 \leq \theta_1 < ... < \theta_N < 2\pi,$$

and suppose $\mathbf{Ord}^{(\varepsilon)}[e^{i\theta_j}; Z(f; \overline{D})] = \sigma < \infty$ for some $f \in \mathcal{S}^{\mathcal{B}}$, $j \in \{1, ..., N\}$ and $\varepsilon \in \{r, l\}$.

(i) If $\sigma > 1$, we have $\mathbf{Ord}^{(\varepsilon)}[e^{i\theta_j}; Z(g; \overline{D})] = \sigma$ for every $g \in \mathcal{S}^{\mathcal{B}}$.

(ii) If $\sigma \leq 1$, we have $\mathbf{Ord}^{(\varepsilon)}[e^{i\theta_j}; Z(g; \overline{D})] \leq 1$ for every $g \in \mathcal{S}^{\mathcal{B}}$.

Remark 1. In Theorem 1 (i), we can not replace $\sigma > 1$ by $\sigma \geq 1$ as the following example shows.

Example 1. Let $\mathcal{B}$ be an infinite Blaschke product with the zeros

$$\{\alpha_n\}_{n=1}^{\infty}; \alpha_n = \Big(1 - \frac{1}{n(\log(n+1))^2}\Big) e^{i/n^2}, \quad n = 1, 2, ...$$

Then , $\mathbf{Ord}^{(l)}[1; Z(\mathcal{B}; \overline{D})] = 1$. On the other hand we have

$$g_{\mathcal{B}}(z) := \prod_{n=1}^{\infty} \frac{(1 - e^{i/n^2})^2 z}{(1 - \overline{\alpha_n} z)^2} \in \mathcal{S}^{\mathcal{B}} \cap H^{\infty},$$

and $\mathbf{Ord}^{(l)}[1; Z(g_{\mathcal{B}}; \overline{D})] = 1/2$.

[1] E. Hayashi, The solution sets of extremal problems in H^1 , Proc. Amer. Math. Soc., 93(1985), 690-696

[2] J. Inoue and T. Nakazi, On the zeros of solutions of an extremal problem in H^1.

[3] T. Nakazi, Exposed points and extremal problems in H^1, J. Funct. Analysis, 53(1983), 224-230

Multipliers And Common Zero Sets Of Invariant Subspaces In The Polydisk

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Abstract. For any nonzero invariant subspace M in $H^2(D^n)$, put $\mathcal{M}(M) = \{\phi \in L^\infty(T^n) \ ; \ \phi M \subseteq H^2(D^n)\}$. If $n = 1$, by Beurling's theorem $M = qH^2(D^n)$ for some inner function q and so $\mathcal{M}(M) = \bar{q}H^\infty(D^n)$. Hence the mapping from M to $\mathcal{M}(M)$ is one-to-one. However for $n \neq 1$, it is not one-to-one. It is believed by many people that the description of arbitrary invariant subspace M is impossible. In this article, we describe $\mathcal{M}(M)$ for arbitrary M. If $M = H^2(D^n)$ then $\mathcal{M}(M) = H^\infty(D^n)$. We give a necessary and sufficient condition for $\mathcal{M}(M) = H^\infty(D^n)$. If M is of finite codimension in $H^2(D^n)$ then $\mathcal{M}(M) = H^\infty(D^n)$ and the common zero set $Z(M)$ of M is of finite points. We show $\mathcal{M}(M) = H^\infty(D^n)$ when M satisfies some condition on $Z(M)$. When h is a nonzero function in $H^2(D^n), M_h$ denotes an invariant subspace generated by h. If h is an outer function, then $\mathcal{M}(M_h) = H^\infty(D^n)$. We give two necessary and sufficient conditions for $\mathcal{M}(M_h) = H^\infty(D^n)$. Moreover a good sufficient condition is given.

This is a survey article on the author's three papers [4],[5] and [6] and contains some new result.

§0. Introduction

D^n は $\mathbf{C}^n$ の open unit polydisk, T^n はその distingwshed boundary of D^n かつ m は T^n 上の normalized Lebegue measure を示す。$1 \leq p \leq \infty$ に対して、$H^p(D^n) = H^p(T^n)$ はそれぞれ D^n と T^n 上の Hardy space を示すが、同一視されている。$L^p(T^n)$ は T^n 上の Lebesgue space を示すが、このとき $H^p(T^n)$ は $L^p(T^n)$ の closed subspace となる。

M が $H^2(D^n)$ の invariant subspace とは、M が $H^2(D^n)$ の closed subspace かつ $z_jM \subseteq M$ $(1 \leq j \leq n)$ なるものである。ここで z_j は座標関数である。零でない $h \in H^2(D^n)$ に対して、M_h は h で生成される invariant subspace を示す。$q \in H^p(D^n)$ が inner function とは T^n 上で $m - a.e.$ 絶対値 1 となるものであり、$h \in H^p(D^n)$ が outer function とは

$$\int_{T^n} \log|h|dm = \log|\int_{T^n} hdm| > -\infty$$

となるものである。

定理 (Beurling, 1949) $n=1$ とする。M が $H^2(D)$ の零でない invariant subspace である必要十分条件は $M=qH^2(D)$ となる inner function q が存在することである。また $h\in H^2(D)$ が outer function である必要十分条件は $M_h=H^2(D)$ となることである。

$n\neq 1$ のとき、上の定理は成立しなく、また invariant subspace を一般に描くことは不可能であると多くの人に信じられている。それで、より描くのにやさしく思われるその multiplier の集合を研究する。

$H^2(D)$ の零でない invariant subspace M に対して、

$$\mathcal{M}(M)=\{\phi\in L^\infty(T^n)\ ;\ \phi M\subseteq H^2(D^n)\}$$

とおき、$\mathcal{M}(M)$ を M の multiplier の集合と呼ぶ。Beurling の定理により、$n=1$ のとき M と $\mathcal{M}(M)$ は 一対一に対応する。$n\neq 1$ のとき一対一とならないことを見るのはやさしい。

問題 $\mathcal{M}(M)=H^\infty(D^n)$ となる M を決定せよ。$\mathcal{M}(M_h)=H^\infty(D^n)$ となる h を決定せよ。

上の問題は $n=1$ のときには、Beurling の定理により、$M=H^2(D)$ と h は outer function として解決されている。この小論では $N=2$ のときにのみ考える。得られる多くの結果は $n>2$ に対しても正しい。

§1. $\mathcal{M}(M)=M^\times$

$H^p=\{f\in L^p(T^2)\ ;\ \hat{f}(\ell,n)=0\ \text{if}\ \ell<0\ \text{or}\ n<0\}$ となる。$H^p_z=\{f\in L^p(T^2)\ ;\ \hat{f}(\ell,n)=0\ \text{if}\ \ell<0\}$ かつ $H^p_w=\{f\in L^p(T^2)\ ;\ \hat{f}(\ell,n)=0\ \text{if}\ n<0\}$ とする。$K^p_0=\{f\in L^p(T^2)\ ;\ \hat{f}(\ell,n)=0\ \text{if}\ \ell\le 0\ \text{and}\ n\le 0\}$ とする。このとき $H^p=H^p_z\cap H^p_w$ かつ $[zH^p_z+wH^p_w]_p=K^p_0$ である。$[\,\cdot\,]_p$ は $L^p(T^2)$ での closure を示す。H^2 の零でない invariant subspace に対して

$$M^\times=\left[\bigcup_{n=0}^{\infty}\bar{w}^nM\right]_2\bigcap\left[\bigcup_{n=0}^{\infty}\bar{z}^nM\right]_2$$

とする。このとき $M^\times$ は M と H^2 の間の invariant subspace となる。inner function q に対して $M=qH^2$ ならば $M^\times=M$ となる。$M=\{f\in H^2\ ;\ \hat{f}(0,0)=0\}$ ならば $M^\times=H^2$ となる。また $\mathcal{M}(M)=\mathcal{M}(M^\times)$。もっと一般に、inner function $\{q_j\}_{j=1}^n$ に対して、$M=\cap_{j=1}^n q_jH^2$ ならば $M^\times=M$ となることを示すことができる。

定理 1 M を 零でない H^2 の invariant subspace とする。

(1) $M^\times=Q_zH^2_z\cap Q_wH^2_w$。ここで $Q_z\in H^2_z$ かつ $Q_w\in H^2_w$ は unimodular function である。$M^\times=H^2\iff Q_z=Q_w=1$ *a.e.*。

(2) $\mathcal{M}(M) = \bar{Q}_z H_z^\infty \cap \bar{Q}_w H_w^\infty$ ここで Q_z と Q_w は (1) の unimodular function である。$\mathcal{M}(M) = H^\infty \iff [Q_z(zH_z^1) + Q_w(wH_w^1)]_1 = K_0^1$。

$Q_z(zH_z^1) \subseteq zH_z^1$ かつ $Q_w(wH_w^1) \subseteq wH_w^1$ であり、$[zH_z^1 + wH_w^1]_1 = K_0^1$ となることに注意せよ。定理 1 より $M^\times$ と $\mathcal{M}(M)$ は 1 対 1 であろうか？ 残念ながら、そうはならないことが §3 で示される。

§2. Slice map

$(\alpha, \beta) \in \overline{D^2}$ のとき、$f \in H^2(D^2)$ に対して

$$(\Phi_{\alpha\beta} f)(\lambda) = f(\alpha\lambda, \beta\lambda) \quad (\lambda \in D)$$

と定義する。$\Phi_{\alpha\beta}$ は slice map と呼ばれ、polydisk 上の関数論の研究において重要で、$H^2(D^2)$ を Bergman 空間 $L_a^2(D)$ へ移すことが知られている。このとき $\ker \Phi_{\alpha\beta}$ は $H^2(D^2)$ の invariant subspace となる。

$f \in H^p(D^2)$ に対して、$\log|f|$ の least n-harmonic majorant を $u(\log|f|)$ で示すとき、T^2 上の非負 singular measure が存在して $u(\log|f|)(\zeta) = P_\zeta(\log|f^*| + d\sigma_f)$ と書ける。ここで f^* は f の radial limit を示し、P_ζ は Poisson 積分である。$H^2(D^2)$ の invariant subspace M に対して、

$$Z(M) = \{\zeta \in D^2 \ ; \ f(\zeta) = 0 \ \ \text{for} f \in M\},$$
$$Z_\partial(M) = \inf\{-d\sigma_f \ ; \ f \in M, f \neq 0\}$$

とする。h_2 を real 2-dimensional Hausdorff measure とする。Douglas-Yan [2] は、もし $h_2(Z(M)) = 0$ かつ $Z_\partial(M) = 0$ ならば $\mathcal{M}(M) = H^\infty(D^2)$ を示した。しかし 、もし $(\alpha, \beta) \neq (0, 0)$ ならば $Z_\partial(\ker \Phi_{\alpha\beta}) = 0$ であるが $h_2(Z(\ker \Phi_{\alpha\beta})) > 0$ である。

定理 2　$(\alpha, \beta) \in \overline{D^2}$ とする。

(1) $\mathcal{M}(\ker \Phi_{\alpha\beta}) = H^\infty(D^2)$ となる必要十分条件は $0 \leq {}^\exists r \leq 1$ 、$(\alpha, \beta) \in rT^2$ となることである。

(2) M が $\ker \Phi_\alpha \subsetneq M \subseteq H^2(D^2)$ となる invariant subspace ならば $\mathcal{M}(M) = H^\infty(D^2)$ 。

(3) M を $Z(M) = Z(\ker \Phi_{\alpha\beta})$ かつ $Z_\partial(M) = 0$ となる invariant subspace とするとき、$\mathcal{M}(M) = H^\infty(D^2)$ となる必要十分条件は $0 \leq {}^\exists r \leq 1$ 、$(\alpha, \beta) \in rT^2$ となることである。

§3. Outer function

m_z と m_w をそれぞれ $T = T_z$ と $T = T_w$ 上の normalized Lebesgue measure とすると、$m = m_z \times m_w$ かつ $T^2 = T_z \times T_w$ 。h が z-outer for $E \subset T_w$ とは $\displaystyle\int_{T \times E} \log|h| dm =$

$\int_E (\log|\int_T h dm_z|) dm_w$ を、h が w-outer for $E \subset T_z$ とは $\int_{E\times T} \log|h| dm =$ $\int_E (\log|\int_T h dm_w|) dm_z$ を満足するときをいう。$h \in H^p(T^2)$ について不等式はつねに成立する。h が z-outer for $E = T_w$ のとき z-outer, w-outer for $E = T_z$ のとき w-outer と呼ぶ。

$h(z, w) = w - g(z)$, $g \in H^\infty(T_z)$, $\|g\|_\infty \le 1$ かつ $E = \{\zeta \in T_z\ ;\ |g(\zeta)| = 1\}$ のとき、h は z-outer かつ w-outer for $E \subset T_z$ 。故 高橋氏は、この場合に $\mathcal{M}(M_h) = H^\infty(T^2)$ となる必要十分条件は $\int_T \log(1 - |g|) dm_z = -\infty$ であることを私に指摘しました。泉池氏 [3] は h が outer ならば $\mathcal{M}(M_h) = H^\infty(T^2)$ を示した。同様にして、もっと一般に h が z-outer かつ w-outer ならば $\mathcal{M}(M_h) = H^\infty(T^2)$ となる。次の定理の (3) は z-outer ならば必ずしも w-outer でなくても $\mathcal{M}(M_h) = H^\infty(T^2)$ となることを示している。またこのとき $(M_h)^\times \ne H^2(T^2)$ なので $M^\times$ と $\mathcal{M}(M^\times)$ は1対1でない。定理の (2) と関係して、$[\mathrm{hK}_0^\infty]_1 = K_0^1$ ならば $[hK_0^\infty]_2 = K_0^2$ となることを知ることは興味ある。

定理 3 $h \in H^\infty(T^2)$ かつ $h \not\equiv 0$ とする。

(1) $\mathcal{M}(M_h) = H^\infty(T^2)$ である必要十分条件は「$|h| \ge |g|$ *a.e.* on T^2 かつ $g \in H^2(T^2)$ ならば $|h| \ge |g|$ on D^2」となること。

(2) $\mathcal{M}(M_h) = H^\infty(T^2)$ となる必要十分条件は $[hK_0^\infty]_1 = K_0^1$。

(3) h が z-outer かつ w-outer for E with $m_z(E) > 0$ ならば $\mathcal{M}(M_h) = H^\infty(T^2)$ 。

参考文献

1. A.Beurling ; On two problems concerning linear transformations in Hilbert space, Acta Math. 81(1949), 239-255

2. R.G.Douglas and K.Yan ; On the rigidity of Hardy submodules, Integ. Eq. Op. Th. 13(1990), 350-363

3. K.Izuchi ; Unitary equivalence of invariant subspaces in the polydisk, Pacific J. Math. 130(1987), 351-358

4. T.Nakazi ; Multipliers of invariant subspaces in the bidisc, Proc. Edinburgh Math. Soc. 37(1994), 193-199

5. T.Nakazi ; Slice maps and multipliers of invariant subspaces, Canad. Math. Bull. 39(1996), 219-226

6. T.Nakazi ; An outer function and several important functions in two variable, Arch. Math. 66(1996), 490-498