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SAPPORO GUEST HOUSE SYMPOSIUM  
ON MATHEMATICS 20

# Nonlinear Wave Equations

Edited by

H. Kubo, T. Ozawa, K. Yamauchi

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SAPPORO GUEST HOUSE SYMPOSIUM  
ON MATHEMATICS 20

# Nonlinear Wave Equations

Edited by  
H. Kubo, T. Ozawa, K. Yamauchi

Sapporo, 2005

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## PREFACE

This volume is intended as the proceedings of Sapporo Guest House Symposium on Mathematics 20, Nonlinear Wave Equations, held on November 22 and 23 in 2005 at Sapporo Guest House.

The first Sapporo Guest House Symposium was held in 1999 by Y. Giga. We keep the size of each meeting relatively small but international. The complete list of symposia at the Sapporo Guest House is in our website:

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Hideo Kubo  
Tohru Ozawa  
Kazuyuki Yamauchi

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## Nonlinear Wave Equations

日時：2005年11月22日(火曜日)13:00～11月23日(水曜日)17:00

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<http://www.plaza-sapporo.or.jp/sgh/>

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# The KP I equation : new results and open problems

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The Kadomtsev-Petviashvili (KP) equations [4]

$$(u_t + uu_x + u_{xxx})_x \pm uu_{yy} = 0, \quad u = u(x, t) \quad (1)$$

are “universal” models for the propagation of long, weakly nonlinear dispersive waves with weak transverse effect. The “+” sign in (1) corresponds to the KP II equation, the “-” sign, to the KP I equation. For instance, in the context of surface water waves, KP II rules the purely gravity waves, and KP I incorporates strong surface tension effects.

The KP equations were in fact introduced to analyze the transverse instability of the KdV solitary wave (the “line soliton”) with profile

$$\psi_c(x) = 3c \cosh^{-2}\left(\frac{\sqrt{2}x}{2}\right)$$

with respect to perturbations depending on  $y$ .

Note that (1) can be formally written in the “integrated form”

$$u_t + uu_x + u_{xxx} \pm \partial_x^{-1} u_{yy} = 0. \quad (2)$$

This form is meaningful provided a certain constraint is satisfied on  $u$ . It was recently observed [8] that this zero mass (in  $x$ ) constraint is satisfied at any non zero time even if it is not satisfied at initial time zero.

On the other hand constraint has to be imposed when using the Hamiltonian form of the equation, namely

$$u_t + \partial_x H(u) = 0 \quad (3)$$

where

$$H(u) = \int \left[ \frac{1}{2} u_x^2 \mp \frac{1}{2} (\partial_x^{-1} u_y)^2 - \frac{1}{6} u^3 \right]. \quad (4)$$

In (4) the “+” sign corresponds to KP I and “-” to KP II. Together with the (formal) conservation of the  $L^2$  norm, (4) suggest a natural energy space for the KP equations:

$$Y = \{v \in L^2, u_x, \partial_x^{-1} u_y \in L^2\}.$$

Note that  $Y \subset L^6(\mathbb{R}^2)$ , and (4) makes sense for  $u \in Y$ .

A first step before looking for qualitative properties is the study of the well-posedness of the Cauchy problem.

Considering the “bad” sign in the Hamiltonian, the only way to obtain global well-posedness for KP II is to prove local well-posedness in  $L^2$ . This was achieved by Bourgain who proved :

**Theorem 1** [2] *The Cauchy problem for KP II is locally (thus globally) well-posed for data in  $L^2(\mathbb{T}^2)$  or  $L^2(\mathbb{R}^2)$ .*

The strategy of the proof is to use an iterative method on the Duhamel (integral) formulation. As usual this implies that the flow map is smooth.

The next result proves that such a strategy is impossible for the KP I equation.

**Theorem 2** [6] *Let  $S(t) = e^{t(\partial_x^3 - \partial_x^{-1} \partial_y^2)}$  the KP I group. Let  $Z$  be any of the spaces  $H^s(\mathbb{R}^2)$ ,  $\forall s \in \mathbb{R}$ ;  $H^{s_1, s_2}(\mathbb{R}^2)$ ,  $\forall s_1, s_2 \in \mathbb{R}^2$ ; or  $Y$ .*

*Let  $T > 0$ . There exists no space  $X_T \hookrightarrow C([-T, T]; Z)$  such that*

$$(i) \quad \|S(t)\varphi\|_{X_T} \lesssim \|\varphi\|_Z$$

$$(ii) \quad \left\| \int_0^t S(t-t')[u(t')u_x(t')] dt' \right\|_{X_T} \lesssim \|u\|_{X_T}^2 .$$

**Remarks :**

1. (ii) is the estimate needed to perform the second iteration in  $X_T$  in the Picard iteration on the Duhamel formulation.
2. Theorem 2 is essentially equivalent to the fact that the flow map cannot be  $C^2$  in  $Z$ . It has been recently observed by Koch and Tzvetkov that the flow map cannot be even uniformly continuous on bounded sets of  $Z$ .
3. This displays the “quasilinear” character of the KP I equation (while the KP II equation is “semilinear”).

Thus, in order to get the global well-posedness for KP I we need to use a compactness method. The choice of the space  $X$  of initial data is dictated by the conservation laws. Choosing  $X = L^2$  seems out of reach for the moment. The natural choice  $X = Y$  (the energy space) is open so far (global existence of weak solutions in  $Y$  is easy however, see [10]).

Since KP I is (formally) integrable, a natural idea is to use the expected next invariants to obtain suitable *a priori* bounds. As noticed in [7] there is a serious analytic obstruction to give sense of the conservation law governing  $\|u_{xxx}(\cdot, t)\|_{L^2}$ . This difficulty can be overcome by using instead a “truncated” conservation law (which has to be justified rigorously) and then by estimating the remainders *via* suitable dispersive estimates. This leads to the following theorem which states the first global well-posedness result for KP I.

**Theorem 3** [7] *Let  $Z = \{\varphi \in L^2(\mathbb{R}^2), \varphi_y, \varphi_{xxx}, \varphi_{xy}, \partial_x^{-2} \varphi_{yy} \in L^2(\mathbb{R}^2)\}$ . Then, for any  $\varphi \in Z$ , there exists a unique global solution  $u$  of KP I with initial data  $\varphi$  which satisfies  $u \in L_{\text{loc}}^\infty(\mathbb{R}; Z)$ . Moreover, the  $L^2$ -norm and the Hamiltonian are conserved and  $u, u_y, u_{xx} \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^2))$ . In particular,  $u \in L^\infty(\mathbb{R} \times \mathbb{R}^2)$ .*

**Remarks 1.** The corresponding global bounds are not known to be true for smooth solutions of the KP II equation.

2. It can be shown that the  $Y$ -solitary wave solutions of KP I (see [1]) belongs to  $Z$ .

This result has been improved by C. Kenig [5] who proved

**Theorem 4** [5] *Let  $\varphi \in Z_0 = \{\varphi \in L^2(\mathbb{R}^2), \varphi_{xx}, \partial_x^{-2} \varphi_{yy} \in L^2(\mathbb{R}^2)\}$ . Then there exists a unique solution  $u$  of KP I with initial data  $\varphi$  such  $u \in L_{\text{loc}}^\infty(\mathbb{R}; Z_0)$ .*

Kenig's improvement is based on new dispersive estimates on the linearized KP I equation which have the interesting by-product :

**Theorem 5** [5] *For  $s \in \mathbb{R}_+$ , let*

$$Y_s = \{\varphi \in L^2(\mathbb{R}^2); |D_x|^s \varphi, \partial_x^{-1} \partial_y \varphi \in L^2(\mathbb{R}^2)\}.$$

*Then the Cauchy problem for KP I is locally well-posed for data in  $Y_s$ ,  $s > \frac{3}{2}$ . More precisely, for any  $\varphi \in Y_s$ ,  $s > \frac{3}{2}$ , there exists  $T = T(\|\varphi\|_{Y_s}) > 0$  and a unique solution*

$$u \in C([0, T]; Y_s) \quad u, \partial_x u \in L_T^1(L_{xy}^\infty)$$

*of the Cauchy problem for KP I. Moreover, the map  $\varphi \mapsto u$  is continuous from  $Y_s$  to  $C([0, T]; Y_s)$ .*

We turn now to the transverse stability/instability properties of the KdV line soliton, which was at the origin of the KP equation. To justify rigorously any stability theory one needs first to solve the Cauchy problem in an adapted framework (the line soliton has infinite energy). A natural setting is to consider initial data of the type (where  $\varphi$  is "localized")

$$u(0, x, y) = \varphi(x, y) + \psi_c(x, y) \tag{5}$$

where  $\psi_c$  is the profile of a non-localized (*i.e.* not decaying in all spatial directions) travelling wave of the KP I equation moving with velocity  $c$ . This  $\psi_c$  could be the line soliton of the KdV equation or more complicated objects (for instance the Zaitsev solitons which are decaying in  $x$  and periodic in  $y$  or conversely, see [9]).

The Cauchy problem is in fact globally well-posed for a large class of  $\psi_c$  as shows the

**Theorem 6** [7] *Let  $\psi_c(x - ct, y)$  be a solution of the KP I equation such that*

$$\psi_c : \mathbb{R}^2 \rightarrow \mathbb{R}$$

*is smooth and bounded with all its derivatives. Then, for any*

$$\varphi \in Z_0 = \{u \in L^2(\mathbb{R}^2), \partial_x^{-2} u_{yy}, u_{xx} \in L^2(\mathbb{R}^2)\},$$

*there exists a unique solution  $u$  of KP I with initial data (5) satisfying for all  $T > 0$*

$$[u(t, x, y) - \psi_c(x - ct, y)] \in C([0, T]; Z_0),$$

$$\partial_x [u(t, x, y) - \psi_c(x - ct, y)] \in L_T^1 L_{xy}^\infty.$$

*Furthermore, for all  $T > 0$ ,  $\varphi \mapsto u$  is continuous from  $Z$  into  $C([0, T]; Z)$ .*

The strategy is to follow the proof of Theorem 3 and 4. New terms appear with respect to the proof of Theorem 3 but they can be controlled since  $\psi_c$  and all its derivatives are bounded. Among others, we use a new dispersive estimate for  $|D_x|^\varepsilon S(t)$ ,  $0 \leq \varepsilon \leq \frac{1}{2}$  which has the independent interest to lead to the (modest) extension of Theorem 5 :

**Theorem 7** [7] *The Cauchy problem for KP I is locally well-posed for data in  $\{u \in L^2(\mathbb{R}^2); |D_x^s u \in L^2(\mathbb{R}^2), s > \frac{3}{2}\}$ .*

Note that  $y$  derivative is needed. Another possible framework to study the stability of non localized objects would be to work on  $\mathbb{R} \times \mathbf{T}$ . This poses serious difficulties since the crucial Strichartz estimates for the free KP I group on  $L^2(\mathbb{R}^2)$  fails to be true in the periodic setting. Nevertheless, replacing these Strichartz estimates with certain time-frequency localized estimates, Ionescu and Kenig [3] were able to prove the global well-posedness of KP I on  $\mathbb{R} \times \mathbf{T}$  or  $\mathbf{T} \times \mathbf{T}$ .

To conclude we review briefly a list of open problems.

- Global well-posedness of the Cauchy problem for KP I in the natural energy space  $Y$ .
- Rigorous theory for the (conjectured) instability of the KdV line soliton in the context of KP I, in particular description of the mechanism of instability. While there exist various (formal or rigorous) results on the linear theory (see references in [9]) nothing seems to be known on the nonlinear theory.
- This question is marginally related to the Cauchy problem but has its own interest : is the explicit lump solitary wave of KP I with profile

$$\psi_c(x, y) = \frac{8c(1 - \frac{c}{3}x^2 + \frac{c^2}{3}y^2)}{(1 + \frac{c}{3}x^2 + \frac{c^2}{3}y^2)^2}$$

a ground state solitary wave (see [1]).

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# LONG TIME BEHAVIOR OF SMALL AMPLITUDE SOLUTIONS OF GENERALIZED BOUSSINESQ AND MODIFIED IMPROVED BOUSSINESQ EQUATIONS

YONGGEUN CHO AND TOHRU OZAWA

## 1. INTRODUCTION

We consider one dimensional Cauchy problems for Boussinesq (Bq) and modified improved Boussinesq (imBq) equations defined by the following equations:

$$(1.1) \quad \begin{aligned} \partial_t^2 u_1 - \partial_x^2 u_1 + \partial_x^4 u_1 &= \partial_x^2 f(u_1), & (x, t) \in \mathbb{R}^{1+1}, \\ u_1(x, 0) = \varphi_1(x), \quad \partial_t u_1(x, 0) &= \psi_1(x), & x \in \mathbb{R} \end{aligned}$$

and

$$(1.2) \quad \begin{aligned} \partial_t^2 u_2 - \partial_t^2 \partial_x^2 u_2 - \partial_x^2 u_2 &= \partial_x^2 f(u_2), & (x, t) \in \mathbb{R}^{1+1}, \\ u_2(x, 0) = \varphi_1(x), \quad \partial_t u_2(x, 0) &= \psi_2(x), & x \in \mathbb{R}, \end{aligned}$$

where  $f \in C^k(\mathbb{R})$  in the real sense and  $|f^{(l)}(v)| \lesssim |v|^{p-l}$  for  $0 \leq l \leq k \leq p$  and  $p > 1$ .

By the Duhamel's principle both equations can be rewritten as for  $i = 1, 2$

$$(1.3) \quad u_i(x, t) = (\partial_t S_i(t))\varphi_i(x) + (S_i(t)\psi_i)(x) - \int_0^t T_i(t-t')f(u_i)(t') dt'.$$

Here

$$\begin{aligned} \partial_t S_i(t)\varphi_i &= \frac{1}{2\pi} \int e^{ix\xi} \cos(t\omega_i(\xi)) \widehat{\varphi}_i(\xi) d\xi \\ S_i(t)\psi_i &= \frac{1}{2\pi} \int e^{ix\xi} \sin(t\omega_i(\xi)) \omega_i^{-1}(\xi) \widehat{\psi}_i(\xi) d\xi, \\ T_1(t) &= S_1(t)(-\partial_x^2), \quad T_2(t) = S_2(t)(1 - \partial_x^2)^{-1}(-\partial_x^2) \\ \omega_1(\xi) &= \xi\sqrt{1 + \xi^2}, \quad \omega_2(\xi) = \frac{\xi}{\sqrt{1 + \xi^2}}, \end{aligned}$$

where  $\widehat{\varphi}(\xi) = \mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}} e^{-ix \cdot \xi} \varphi(x) dx$  is the Fourier transform of  $\varphi$ .

The equation (1.1) describes the shallow-water waves as KdV equation and also other physical phenomena like ion-sound waves in a plasma [7]. The equation (1.2) is a modification of Boussinesq equation analogous to the MKdV equations.

Our aim is to find minimal power  $p$  which makes the global existence and scattering of small amplitude solution to the Cauchy problems (1.1) and (1.2) possible. In general, the value of  $p$  is conjectured to be greater than 3 for the scattering like Schrödinger equation and KdV equations [1, 6, 8]. Up to now, there has been few

results for these problems. The best known results were  $p > 2 + \sqrt{7} \sim 4.646$  and  $p > 8$  for the case of Bq [6] and imBq [5] equations respectively. We will see slight improvements ( $p > \frac{3+\sqrt{17}}{2} \sim 3.562$  for Bq and  $p > 4$  for imBq) on the previous results under the vanishing condition near zero frequency of initial data (see Theorems 3.1, 3.2 and 3.3). For these purposes, we will investigate dispersive properties of linear evolution groups (see Lemmas 2.1, 2.3 and Remark 2.2).

## 2. PRELIMINARIES

Let us first introduce linear dispersive estimates on  $\partial_t S_i(t)$  and  $S_i(t)$ .

**Lemma 2.1.** (1) *If  $\varphi$  and  $\psi$  are Schwartz functions, then we have for any  $s_1 > \frac{5}{6}$  and  $s_2 > \frac{3}{2}$*

$$\begin{aligned} \|\partial_t S_i(t)\varphi\|_{L^\infty} &\lesssim (1+|t|)^{-\frac{1}{3}} \left( \|\varphi\|_{L^1} + \|\varphi\|_{W^{s_i, \frac{6}{5}}} \right), \\ \|S_i(t)\psi\|_{L^\infty} &\lesssim (1+|t|)^{-\frac{1}{3}} \left( \|\omega_i(D)^{-1}\psi\|_{L^1} + \|\omega_i(D)^{-1}\psi\|_{W^{s_i, \frac{6}{5}}} \right), \end{aligned}$$

where  $\omega_i(D) = \mathcal{F}^{-1}\omega_i(|\xi|)\mathcal{F}$ .

(2) *If  $\varphi$  and  $\psi$  are Schwartz functions with their Fourier supports not containing zero, then for any  $s_1 > \frac{1}{r'}$  and  $s_2 > 2 - \frac{3}{r}$  with  $2 < r < \infty$*

$$\begin{aligned} \|\partial_t S_i(t)\varphi\|_{L^\infty} &\lesssim (1+|t|)^{-\left(\frac{1}{2}-\frac{1}{r}\right)} \|\varphi\|_{W^{s_i, r'}}, \\ \|S_i(t)\psi\|_{L^\infty} &\lesssim (1+|t|)^{-\left(\frac{1}{2}-\frac{1}{r}\right)} \|\omega_i(D)^{-1}\psi\|_{W^{s_i, r'}}. \end{aligned}$$

**Remark 2.2.** *Applying the Sobolev embedding  $W_{r', 1}^1 \hookrightarrow L^{r'}$  to the second part of Lemma 2.1, we can obtain for any  $s_1 > 1$  and  $s_2 > 2$*

$$\begin{aligned} \|\partial_t S_i(t)\varphi\|_{L^\infty} &\lesssim (1+|t|)^{-\frac{1}{2}} \|\varphi\|_{W^{s_i, 1}}, \\ \|S_i(t)\psi\|_{L^\infty} &\lesssim (1+|t|)^{-\frac{1}{2}} \|\omega_i(D)^{-1}\psi\|_{W^{s_i, 1}}. \end{aligned}$$

Thus by complex interpolation, for any  $2 \leq q \leq \infty$

$$(2.1) \quad \begin{aligned} \|\partial_t S_i(t)\varphi\|_{L^q} &\lesssim (1+|t|)^{-\left(\frac{1}{2}-\frac{1}{q}\right)} \|\varphi\|_{W^{s_i(1-\frac{2}{q}), q'}}, \\ \|S_i(t)\psi\|_{L^q} &\lesssim (1+|t|)^{-\left(\frac{1}{2}-\frac{1}{q}\right)} \|\omega_i(D)^{-1}\psi\|_{W^{s_i(1-\frac{2}{q}), q'}}. \end{aligned}$$

The next is on the retarded estimate.

**Lemma 2.3.** (1) *Let  $2 < r < \infty$ ,  $s_1 > \frac{1}{r'}$  and  $s_2 > 2 - \frac{3}{r}$ . Then we have for  $i = 1, 2$*

$$(2.2) \quad \left\| \int_0^t T_i(t-t')g(t') dt' \right\|_{L^\infty} \lesssim \int_0^t (1+|t-t'|)^{-\left(\frac{1}{2}-\frac{1}{r}\right)} \|g(t')\|_{W^{s_i, r'}} dt'.$$

(2) *Let  $2 \leq q \leq \infty$  and  $\gamma \in \mathbb{R}$ . Then we have*

$$(2.3) \quad \left\| \int_0^t T_1(t-t')g(t') dt' \right\|_{W^{\gamma, q}} \lesssim \int_0^t |t-t'|^{-\left(\frac{1}{2}-\frac{1}{q}\right)} \|g(t')\|_{W^{\gamma, q'}} dt'.$$

One can show the above lemmas by a dyadic decomposition and the stationary and non-stationary phase estimates (see [2, 3]). For the time decay rate like  $(1 + |t|)^{-\theta}$ ,  $W^{s,r'}$  regularity is necessary. Since the second derivative of  $\omega_2$  goes to zero as  $\xi \rightarrow \infty$ , for the application of stationary phase estimate, higher regularity is needed than  $\omega_1$ . By the same reason, the operator  $T_2$  cannot have the estimate like the part (2) of Lemma 2.3. For the Strichartz estimate concerned with  $\partial_t S_1$ ,  $S_1$  and  $T_1$ , see [4].

### 3. MAIN RESULTS

The first result is the global existence of small amplitude solution with initial data without vanishing condition near the zero frequency.

**Theorem 3.1.** *Let  $\varphi_i$  and  $\psi_i$  be Schwartz functions with*

$$\|\varphi_i\|_{L^1} + \|\varphi_i\|_{W^{s_i, \frac{6}{5}} \cap H^{s_i}} + \|\omega_i(D)^{-1}\psi_i\|_{L^1} + \|\omega_i(D)^{-1}\psi_i\|_{W^{s_i, \frac{6}{5}} \cap H^{s_i}} \leq \delta.$$

*If  $p > 5$  and  $\delta$  is sufficiently small, then there exist unique solutions  $u_1, u_2 \in C(\mathbb{R}; H^{s_i})$  to the Cauchy problems (1.1) and (1.2) respectively such that*

$$\sup_{t \in \mathbb{R}} (1 + |t|)^{\frac{1}{3}} \|u_i(t)\|_{L^\infty} + \|u_i(t)\|_{H^{s_i}} \lesssim \delta.$$

*Moreover, there exist four pairs of functions  $(\varphi_i^\pm, \psi_i^\pm) \in H^{s_i} \times \omega_i(D)H^{s_i}$  such that*

$$(3.1) \quad \|u_i(t) - u_i^\pm(t)\|_{H^{s_i}} \rightarrow 0 \text{ as } t \rightarrow \pm\infty,$$

*where  $u_i^\pm$  is the solution to the linear problem (1.1) and (1.2) with  $f = 0$  and initial data  $(\varphi_i^\pm, \psi_i^\pm)$ .*

If we assume the vanishing condition near the zero frequency on the initial data, then we have the following result.

**Theorem 3.2.** *Let  $\varphi_i$  and  $\psi_i$  be Schwartz functions with their Fourier supports not containing zero frequency and*

$$\|\varphi_i\|_{W^{s_i, r'} \cap H^{s_i}} + \|\omega_i(D)^{-1}\psi_i\|_{W^{s_i, r'} \cap H^{s_i}} \leq \delta$$

*for some  $2 < r < \infty$  and  $s_1 > \frac{1}{r}, s_2 > 2 - \frac{3}{r}$ . If  $p > 4$  and  $\delta$  is sufficiently small, then there exist unique solutions  $u_1, u_2 \in C(\mathbb{R}; H^{s_i})$  to the Cauchy problems (1.1) and (1.2) respectively such that*

$$\sup_{t \in \mathbb{R}} (1 + |t|)^{\frac{1}{2} - \frac{1}{r}} \|u_i(t)\|_{L^\infty} + \|u_i(t)\|_{H^{s_i}} \lesssim \delta.$$

*Moreover, there exist four pairs of functions  $(\varphi_i^\pm, \psi_i^\pm) \in H^{s_i} \times \omega_i(D)H^{s_i}$  satisfying (3.1).*

Now if we use the dispersive property (2.1) and (2.3) of the Boussinesq equation, then we can make the power  $p$  smaller than 4.

**Theorem 3.3.** *Let  $\varphi_1, \psi_1$  be Schwartz functions with their Fourier supports not containing zero frequency and*

$$\|(\varphi_1, \omega_1(D)^{-1}\psi_1)\|_{W^{s_1, p+1} \cap H^\gamma} \leq \delta$$

for some  $s_1 > 1 - \frac{2}{p+1} + \gamma$  and  $\gamma > \frac{1}{p+1}$ . If  $p > \frac{3+\sqrt{17}}{2}$  and  $\delta$  is sufficiently small, then there exist a unique solution  $u_1 \in C(\mathbb{R}; W^{\gamma, p+1} \cap H^\gamma)$  to the Cauchy problems (1.1) such that

$$\sup_{t \in \mathbb{R}} (1 + |t|)^{\frac{1}{2} - \frac{1}{p+1}} \|u_1(t)\|_{W^{\gamma, p+1}} + \|u_1(t)\|_{H^\gamma} \lesssim \delta.$$

Moreover, there exist two pairs of functions  $(\varphi_1^\pm, \psi_1^\pm) \in H^\gamma \times \omega_i(D)H^\gamma$  satisfying (3.1).

*Outline of proof.* Since the proof of Theorems 3.2 and 3.3 is similar to that of Theorem 3.1, we introduce the outline of proof only for Theorem 3.1. The strategy is to use the contraction mapping theorem. We define a nonlinear functional  $N$  by

$$N(u_i) = \partial_t S_i(t)\varphi_i + S_i(t)\psi_i - \int_0^t T_i(t-t')f(u_i)(t') dt'.$$

Then from Lemma 2.1 in the previous section, we can easily show that  $N$  is a contraction map from  $X_\rho$  to itself, where

$$X_\rho = \{v : \sup_{t \in \mathbb{R}} (1 + |t|)^{\frac{1}{3}} \|v(t)\|_{L^\infty} + \|v(t)\|_{H^{s_i}} \leq \rho\},$$

provided  $\delta + \rho^p \ll \rho$  and  $\delta$  is sufficiently small.

In fact, using generalized chain and Leibniz rule:

**Lemma 3.4** (Lemma 3.1 in [2]). *For any  $s \geq 0$ , we have*

$$\begin{aligned} \|D^s f(u)\|_{L^r} &\lesssim \|u\|_{L^{(p-1)r_1}}^{p-1} \|D^s\|_{L^{r_2}}, \\ &\left( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad r_1 \in (1, \infty], r_2 \in (1, \infty) \right) \\ \|D^s(uv)\|_{L^r} &\lesssim \|D^s u\|_{L^{r_1}} \|v\|_{L^{q_2}} + \|u\|_{L^{q_1}} \|D^s v\|_{L^{r_2}}. \\ &\left( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{r_2}, \quad r_i \in (1, \infty), q_i \in (1, \infty] \quad (i = 1, 2) \right) \end{aligned}$$

one can show that for any  $u, v \in X_\rho$

$$\begin{aligned} \|N(u)\|_X &\lesssim \delta + \rho^p, \\ \|N(u) - N(v)\|_X &\lesssim \rho^{p-1} \|u - v\|_X, \end{aligned}$$

if  $\frac{1}{3}(p-2) > 1$ . The uniqueness and time continuity follow immediately from the contraction argument and integral equation (1.3).

Now for the proof of the existence of asymptotically free state, we define functions  $\varphi_i^\pm, \psi_i^\pm$  by

$$\begin{aligned}\widehat{\varphi}_1^\pm(\xi) &= \widehat{\varphi}_1(\xi) + \int_0^{\pm\infty} \frac{\xi}{\sqrt{1+\xi^2}} \sin \frac{t'\xi}{\sqrt{1+\xi^2}} \widehat{f(u_1)}(\xi, t') dt', \\ \widehat{\psi}_1^\pm(\xi) &= \widehat{\psi}_1(\xi) - \int_0^{\pm\infty} \xi^2 \cos \frac{t\xi}{\sqrt{1+\xi^2}} \widehat{f(u_1)}(\xi, t) dt\end{aligned}$$

and

$$\begin{aligned}\widehat{\varphi}_2^\pm(\xi) &= \widehat{\varphi}_2(\xi) + \int_0^{\pm\infty} \frac{\xi}{\sqrt{1+\xi^2}} \sin \frac{t\xi}{\sqrt{1+\xi^2}} \widehat{f(u_2)}(\xi, t) dt, \\ \widehat{\psi}_2^\pm(\xi) &= \widehat{\psi}_2(\xi) - \int_0^{\pm\infty} \frac{\xi^2}{1+\xi^2} \cos \frac{t\xi}{\sqrt{1+\xi^2}} \widehat{f(u_2)}(\xi, t) dt.\end{aligned}$$

Then the solution  $u_i^\pm$  to the linear problem (1.1) and (1.2) with  $f = 0$  and initial data  $(\varphi_i^\pm, \psi_i^\pm)$  is represented by

$$u_i^\pm(x, t) = (\partial_t S_i(t)\varphi_i)(x) + (S_i(t)\psi_i)(x) + \int_0^{\pm\infty} T_i(t-t')f(u_i(t')) dt'.$$

Now we have from Lemma 3.4

$$\begin{aligned}\|u_i(\cdot, t) - u_i^\pm(\cdot, t)\|_{H^{s_i}} &\lesssim \int_t^{\pm\infty} \|f(u_i(\cdot, t'))\|_{H^{s_i}} dt' \\ &\lesssim \rho^p \int_t^{\pm\infty} (1+|t'|)^{-\frac{1}{3}(p-1)} dt' \\ &= O(|t|^{-\frac{1}{3}(p-1)+1})\end{aligned}$$

as  $t \rightarrow \pm\infty$ . This proves the theorem. □

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# ASYMPTOTICS OF SOLUTIONS TO DISPERSIVE EQUATIONS AND NONLINEAR PROBLEMS

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## 1. Abstract

We study the asymptotic behavior in time of solutions of the Cauchy problem for the following dispersive equation of the Schrödinger type

$$(1) \quad \begin{aligned} u_t - \frac{i}{\rho} (-\partial_x^2)^{\frac{\rho}{2}} u &= 0, \quad (t, x) \in \mathbf{R} \times \mathbf{R}, \\ u(t_0, x) &= u_0(x), \quad x \in \mathbf{R}, \end{aligned}$$

where  $\rho > 3$ . Then we apply estimates of solutions to linear problem to the non-linear problems:

$$(2) \quad \begin{aligned} u_t - \frac{i}{\rho} (-\partial_x^2)^{\frac{\rho}{2}} u &= f(u, \bar{u}), \quad x \in \mathbf{R}, \\ \lim_{t \rightarrow \infty} U(-t)u(t) &= u_+, \quad x \in \mathbf{R}. \end{aligned}$$

In what follows we let

$$\max \left\{ 0, \frac{\rho}{2} - 2 \right\} \leq \beta < \rho - 2, \max \left\{ 0, \frac{\rho}{2} - 1 \right\} \leq \delta < \rho - 1$$

and

$$\begin{aligned} 0 &\leq \tilde{\beta} = \beta - \frac{\rho - 1}{2} < \frac{\rho}{2} - \frac{3}{2}, \\ 0 &\leq \tilde{\delta} = \delta - \frac{\rho - 1}{2} < \frac{\rho}{2} - \frac{1}{2} \end{aligned}$$

**Theorem 1.1.** *We assume that*

$$\|v\|_{\infty, \rho-2, \rho-1} = \|\hat{v}\|_{\infty} + \|\partial_{\xi} \hat{v}\|_{\infty} + \left\| |\xi|^{-(\rho-2)} \partial_{\xi} \hat{v} \right\|_{\infty} + \left\| |\xi|^{-(\rho-1)} \hat{v} \right\|_{\infty} < \infty.$$

Then

$$\|U(t)v\|_{\infty} \leq C \|v\|_{\infty, \rho-2, \rho-1} t^{-\frac{1}{2}}$$

for all  $t > 0$ , where  $U(t) = \mathcal{F}^{-1} \exp \left( \frac{i}{\rho} t |\xi|^{\rho} \right) \mathcal{F}$ . Furthermore there exist a constant  $C_1$  such that the asymptotic formula for large time  $t$  hold

$$u(t, x) = u_v(t, q) + R_1(t, x),$$

where

$$u_v(t, q) = C_1 t^{-\frac{1}{2}} |q|^{\frac{2-\rho}{2}} e^{-i(1-\frac{1}{\rho})t|q|^{\rho}} \hat{v}(q), q = - \left( \frac{|x|}{t} \right)^{\frac{1}{\rho-1}} \frac{x}{|x|}$$

and the reminder

$$\begin{aligned}\|R_1(t)\|_\infty &\leq C \max\left(t^{-\frac{1}{\rho}(\beta+2)}, t^{-\frac{1}{\rho}(\delta+1)}\right) \|v\|_{\infty, \rho-2, \rho-1}, \\ \|R_1(t)\| &\leq C \max\left(t^{-\frac{1}{\rho}(\tilde{\beta}+\frac{3}{2})}, t^{-\frac{1}{\rho}(\tilde{\delta}+\frac{1}{2})}\right) \|v\|_{\infty, \rho-2, \rho-1}.\end{aligned}$$

We now state our results of nonlinear problems in this paper.

**Theorem 1.2.** *Let  $u_+ \in \mathbf{H}^{1,0}$  satisfy  $\|u_+\|_{\infty, \rho-2, \rho-1} + \|u_+\|_{1,0} < \infty$ , and  $f(u, \bar{u})$  satisfy the growth conditions*

$$|D^j f(u, \bar{u})| \leq C |u|^{p-j}, \quad j = 0, 1, 2, 3$$

where  $D = (\partial_u, \partial_{\bar{u}})$  and  $1 + 2\rho > p > p(\rho)$ ,  $p(\rho)$  is a positive root of

$$p^2 + 2\left(\frac{1}{\rho} - 2\right)p - \left(\frac{2}{\rho} + 1\right) = 0.$$

Then there exists a positive time  $T > 1$  and a unique global solution  $u$  of (2) such that

$$\|u(t) - U(t)u_+\| + \left(\int_t^\infty \|u(t) - U(t)u_+\|_\infty^{2\rho} dt\right)^{\frac{1}{2\rho}} \leq Ct^{-b},$$

for all  $t > 0$ , where

$$\frac{2\rho + 1 - p}{2(p-1)\rho} < b < \min\left\{\frac{1}{2}, \frac{1}{4}(p-3)\right\}.$$

In the case of  $p \geq 1 + 2\rho$ ,  $L^\infty - L^1$  estimates of linear problem obtained in [1] yield the time global existence of solutions of (2) in the usual order Sobolev norms. Hence we concentrate our attention to the case  $1 + 2\rho > p$ .

Next result is concerns with lower order nonlinearity. We assume that

$$\frac{\rho-1}{2\rho} < \frac{1}{\rho}\left(\tilde{\beta} + \frac{3}{2}\right), \quad \frac{\rho-1}{2\rho} < \frac{1}{\rho}\left(\tilde{\delta} + \frac{1}{2}\right)$$

which imply that

$$\rho - \frac{5}{2} < \beta < \rho - 2, \quad \rho - \frac{3}{2} < \delta < \rho - 1.$$

**Theorem 1.3.** *Let  $u_+ \in \mathbf{H}^{1,0}$  satisfy  $\|u_+\|_{\infty, \rho-2, \rho-1} + \|u_+\|_{1,0} < \infty$ , and let  $f(u, \bar{u}) = i\lambda |u|^{p-1}u$ ,  $3 < p < 5$ ,  $\lambda \in \mathbf{R}$ , then there exists a positive time  $T > 1$  and a unique global solution  $u$  of (2) such that*

$$\|u(t) - U(t)u_+\| + \left(\int_t^\infty \|u(t) - U(t)u_+\|_\infty^{2\rho} dt\right)^{\frac{1}{2\rho}} \leq Ct^{-b},$$

for all  $t > T$ , where

$$\frac{\rho-1}{2\rho} < b < \frac{1}{2}.$$

Furthermore there exist positive constants  $c_1, c_2$

$$c_2 t^{-\frac{1}{2}(p-3)} \leq \|u(t) - U(t)u_+\| \leq c_1 t^{-\frac{1}{2}(p-3)}$$

for large  $t$ . Let  $p = 3$ , and  $\|u_+\|_\infty + \|\partial_x u_+\|_\infty + \|u_+\|_{1,0}$  is sufficiently small, then there exists a unique global solution  $u$  of (2) such that

$$\|u(t) - u_w(t)\| + \left( \int_t^\infty \|u(\tau) - u_w(\tau)\|_\infty^{2\rho} d\tau \right)^{\frac{1}{2\rho}} \leq Ct^{-b}, \frac{\rho-1}{2\rho} < b < \frac{1}{2},$$

where

$$u_w(t, q) = C_1 t^{-\frac{1}{2}} |q|^{\frac{2-\rho}{2}} e^{-i(1-\frac{1}{\rho})t|q|^\rho} \widehat{u}_+(q) e^{-i\lambda|C_1|^2|q|^{2-\rho}|\widehat{u}_+(q)|^2 \log t}, q = - \left( \frac{|x|}{t} \right)^{\frac{1}{\rho-1}} \frac{x}{|x|},$$

$$w(t, \xi) = \mathcal{F}^{-1} \widehat{u}_+(q) e^{-i\lambda|C_1|^2|q|^{2-\rho}|\widehat{u}_+(q)|^2 \log t}.$$

**Theorem 1.4.** Let  $u_+ \in \mathbf{H}^{1,0}$  satisfy  $\|u_+\|_{\infty, \rho-2, \rho-1} + \|u_+\|_{1,0} < \infty$ , and let  $f(u, \bar{u}) = i\lambda|u|^2 u, \text{Im}\lambda < 0$ , then there exists a positive time  $T > 1$  and a unique global solution  $u$  of (2) such that

$$\|u(t) - u_W(t)\| + \left( \int_t^\infty \|u(\tau) - u_W(\tau)\|_\infty^{2\rho} d\tau \right)^{\frac{1}{2\rho}} \leq Ct^{-b}, \frac{\rho-1}{2\rho} < b < \frac{1}{2},$$

where

$$u_W(t, q) = C_1 t^{-\frac{1}{2}} |q|^{\frac{2-\rho}{2}} e^{-i(1-\frac{1}{\rho})t|q|^\rho} \widehat{u}_+(q) e^{-i\frac{(\text{Re}\lambda)}{(\text{Im}\lambda)}\varphi(t,q)+\varphi(t,q)}, q = - \left( \frac{|x|}{t} \right)^{\frac{1}{\rho-1}} \frac{x}{|x|},$$

$$W(t, \xi) = \mathcal{F}^{-1} e^{-i\frac{(\text{Re}\lambda)}{(\text{Im}\lambda)}\varphi(t,q)+\varphi(t,q)} \widehat{u}_+(q)$$

$$e^{2\varphi(t,q)} = \frac{1}{1 - 2(\text{Im}\lambda)|C_1|^2|q|^{(2-\rho)}|\widehat{u}_+(q)|^2 \log t}.$$

Next result implies the nonexistence of usual wave operator.

**Theorem 1.5.** Let  $u_+ \in \mathbf{L}^2$  satisfy  $\|u_+\| + \|u_+\|_{\infty, \rho-2, \rho-1} < \infty$ , and  $f(u, \bar{u}) = \lambda|u|^2 u$ . (1) We assume that  $\text{Im}\lambda = 0$  and there exists a solution of  $u$  of (2) such that

$$\lim_{t \rightarrow \infty} \|u(t) - U(t)u_+\| = 0,$$

then  $u = 0$ . (2) We assume that  $\text{Im}\lambda = 0$  and there exists a solution of  $u$  of (2) such that

$$\lim_{t \rightarrow \infty} \|u(t) - U(t)u_+\| = 0,$$

then  $u = 0$ .

In [7], the modified wave operator of (2) is constructed when  $f(u, \bar{u}) = i\lambda|u|^2 u, \lambda \in \mathbf{R}, \rho = 4$  under the condition that

$$\|\widehat{u}_+\|_{4,0} + \sum_{k=0}^4 \left\| |\xi|^{-12+k} \partial_\xi^k \widehat{u}_+ \right\|$$

is sufficiently small. His method is based on the method by T.Ozawa [6]. More precisely, it was checked

$$R = L_4 u_w - i\lambda|u_w|^2 u_w$$

is remainder term, where  $L_\rho = \partial_t - \frac{i}{4}(-\partial_x^2)^{\frac{\rho}{2}}$ . Computation of  $L_4 u_w$  means that the condition  $\|\widehat{u}_+\|_{4,0} < \infty$  is required at least. His method can be applied to higher order  $\rho$  if we assume  $\|\widehat{u}_+\|_{\rho,0} < \infty$ , however a long computation is needed to check  $R$  is remainder since we have to do  $\rho$  times differentiation on  $u_w$ . In [8], it was checked

$$R = L_4 u_w - i\lambda |u_w|^2 u_w$$

is remainder term by combining the method of [6] and [9] under the condition that

$$\|\widehat{u}_+\|_{4,0} + \sum_{k=0}^4 \left\| |\xi|^{-12+k} \partial_\xi^k \widehat{u}_+ \right\|$$

is finite. Therefore our results are improvements of the previous works and our method can be applied to non integer order  $\rho$ . Similar result of Theorem 1.1 was shown in [4] in the case of Benjamin-Ono type equation  $u_t - \frac{i}{2}\partial_x(-\partial_x^2)^{\frac{1}{2}}u = 0$  and in [3], [5] in the case of Kortweg-de Vries type equation  $u_t - \frac{i}{3}\partial_x(-\partial_x^2)u = 0$ .

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# Global existence for damped nonlinear Schrödinger equations

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This note is based on a joint work with Grozdna Todorova (University of Tennessee). We consider the Cauchy problem for the damped nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u + |u|^{p-1}u + i\delta u = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n \quad (1)$$

with initial data  $u(0) = u_0 \in H^1(\mathbb{R}^n)$ , where  $\delta \geq 0$ ,  $n \in \mathbb{N}$ ,  $p > 1$  and  $p < 1 + 4/(n-2)$  if  $n \geq 3$ . It is known that the Cauchy problem for (1) is locally well-posed in  $H^1(\mathbb{R}^n)$ . So, the problem addressed in this note is the global existence of solutions for (1). We denote the maximal existence time of local solution of (1) with initial data  $u_0$  by  $T_\delta^*(u_0)$ . So,  $T_\delta^*(u_0) = \infty$  means that the solution  $u(t)$  of (1) with  $u(0) = u_0$  exists globally.

First, let us recall some known results for the case  $\delta = 0$  (see textbooks [1, 4] for more details). When  $p < 1 + 4/n$ , we have  $T_0^*(u_0) = \infty$  for any  $u_0 \in H^1(\mathbb{R}^n)$ . When  $p \geq 1 + 4/n$ , we have  $T_0^*(u_0) = \infty$  if the initial data  $u_0$  is sufficiently small in  $H^1(\mathbb{R}^n)$ , and  $T_0^*(u_0) < \infty$  if  $u_0 \in H^1(\mathbb{R}^n)$  satisfies  $xu_0 \in L^2(\mathbb{R}^n)$  and  $E(u_0) < 0$ . Here, the energy  $E$  is defined by

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}}^{p+1}.$$

The global existence result follows from the local well-posedness in  $H^1(\mathbb{R}^n)$ , the conservation laws of energy  $E$  and charge  $\|u\|_{L^2}^2$ , and the Gagliardo-Nirenberg inequality:

$$\|v\|_{L^q} \leq C \|v\|_{L^2}^{1-\alpha} \|\nabla v\|_{L^2}^\alpha, \quad v \in H^1(\mathbb{R}^n),$$

where  $2 \leq q \leq \infty$ ,  $0 \leq \alpha \leq 1$ ,  $\alpha < 1$  if  $n = 2$ , and

$$\frac{1}{q} = \frac{1}{2}(1 - \alpha) + \frac{n - 2}{2n}\alpha.$$

The blowup result follows from the virial identity:

$$\begin{aligned} \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 &= 16P(u(t)), \\ P(u) &= \frac{1}{2} \|\nabla u\|_2^2 - \frac{N(p-1)}{4(p+1)} \|u\|_{p+1}^{p+1}. \end{aligned}$$

Next, we consider the case  $\delta > 0$  in (1). In this case, we have

$$\|u(t)\|_{L^2} = e^{-\delta t} \|u_0\|_{L^2}, \quad t \in [0, T_\delta^*(u_0)), \quad (2)$$

but the energy  $E$  is no longer conserved nor decreasing. In fact, we have

$$\begin{aligned} \frac{d}{dt} E(u(t)) &= -\delta K(u(t)), \\ K(u) &= \|\nabla u\|_{L^2}^2 - \|u\|_{L^{p+1}}^{p+1}. \end{aligned} \quad (3)$$

M. Tsutsumi [5] proved that when  $p > 1 + 4/n$ , we have  $T_\delta^*(u_0) < \infty$  if  $u_0 \in H^1(\mathbb{R}^n)$  satisfies  $xu_0 \in L^2(\mathbb{R}^n)$ ,  $E(u_0) \leq 0$  and

$$\frac{2(p-1)\delta}{(p-1)n-4} \|xu_0\|_{L^2}^2 + V(u_0) < 0,$$

where

$$V(u) = -4 \operatorname{Im} \int_{\mathbb{R}^n} x \cdot \nabla u(x) \overline{u(x)} dx.$$

The proof in [5] is based on the identities:

$$\frac{d}{dt} \|xu(t)\|_{L^2}^2 + 2\delta \|xu(t)\|_{L^2}^2 = V(u(t)), \quad (4)$$

$$\frac{d}{dt} V(u(t)) + 2\delta V(u(t)) = 16P(u(t)). \quad (5)$$

For the case  $p = 1 + 4/n$  and  $\delta > 0$ , numerical simulations suggest the existence of finite time blowup solutions of (1) (see [3]), but it is an open problem whether there exist finite time blowup solutions for this case.

We state our main results in this note.

**Theorem 1** *Assume that  $1 + 4/n \leq p < 1 + 4/(n-2)$ . For any  $u_0 \in H^1(\mathbb{R}^n)$  there exists  $\delta^*(u_0) \in [0, \infty)$  such that  $T_\delta^*(u_0) = \infty$  for all  $\delta \geq \delta^*(u_0)$ .*

**Remark 1** A similar result to Theorem 1 is mentioned in [4, p.98] without proof.

**Remark 2** By reading the proof in [5] carefully, we have the following: When  $p > 1 + 4/n$ , for any  $u_0 \in H^1(\mathbb{R}^n)$  satisfying  $xu_0 \in L^2(\mathbb{R}^n)$  and  $E(u_0) < 0$ , there exists  $\delta_1(u_0) > 0$  such that  $T_\delta^*(u_0) < \infty$  for all  $\delta \in [0, \delta_1(u_0)]$ .

**Remark 3** It is proved by Cazenave and Weissler (see [2, Corollary 2.5]) that when  $p > p_0(n)$ , for any  $u_0 \in H^1(\mathbb{R}^n)$  satisfying  $xu_0 \in L^2(\mathbb{R}^n)$  there exists  $b^*(u_0) \in [0, \infty)$  such that  $T_0^*(e^{ib|x|^2}u_0) = \infty$  for all  $b \geq b^*(u_0)$ , where

$$p_0(n) = \frac{n + 2 + \sqrt{n^2 + 12n + 4}}{2n}.$$

Note that  $1 < p_0(n) < 1 + 4/n$ . The proof of Theorem 1 is based on the argument in the proofs of Proposition 2.4 and Corollary 2.5 of [2].

**Proof of Theorem 1 (Outline)** Let  $U_\delta(t)$  be the propagator for the linear equation:

$$i\partial_t u + \Delta u + i\delta u = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n. \quad (6)$$

The Cauchy problem for (1) with  $u(0) = u_0 \in H^1(\mathbb{R}^n)$  is equivalent to the integral equation:

$$u(t) = U_\delta(t)u_0 + i \int_0^t U_\delta(t-s)|u(s)|^{p-1}u(s) ds. \quad (7)$$

Since  $U_\delta(t) = e^{-\delta t}U_0(t)$ , as in Proposition 2.4 of [2], we have  $T_\delta^*(u_0) = \infty$  if  $\|U_\delta(\cdot)u_0\|_{L^\alpha(0, \infty; L^{p+1})}$  is sufficiently small, where

$$a = \frac{2(p-1)(p+1)}{4 - (n-2)(p-2)}.$$

Moreover, by the dominated convergence theorem, for any  $u_0 \in H^1(\mathbb{R}^n)$  we have

$$\lim_{\delta \rightarrow \infty} \|U_\delta(\cdot)u_0\|_{L^\alpha(0, \infty; L^{p+1})} = 0.$$

This completes the proof. □

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# Global Existence and Uniqueness of Solutions to the Maxwell-Schrödinger Equations

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## §1. Introduction

The Maxwell-Schrödinger system (MS) in space dimension 3 describes the time evolution of a charged nonrelativistic quantum mechanical particle interacting with the (classical) electro-magnetic field it generates. We can write down this system in usual vector notation as follows:

$$i\partial_t u = (-\Delta_A + \phi)u, \quad (1.1)$$

$$-\Delta\phi - \partial_t \operatorname{div} A = \rho, \quad (1.2)$$

$$\square A + \nabla(\partial_t \phi + \operatorname{div} A) = J, \quad (1.3)$$

where  $(u, \phi, A) : \mathbf{R}^{1+3} \rightarrow \mathbf{C} \times \mathbf{R} \times \mathbf{R}^3$ ,  $\nabla_A = \nabla - iA$ ,  $\Delta_A = \nabla_A^2$ ,  $\rho = |u|^2$ ,  $J = 2 \operatorname{Im} \bar{u} \nabla_A u$ . Physically,  $u$  is the wave function of the particle,  $(\phi, A)$  is the electro-magnetic potential,  $\rho$  is the charge density, and  $J$  is the current density. The system (MS) formally conserves at least two quantities, namely the total charge  $\mathcal{Q} \equiv \|u\|_2^2$  and the total energy

$$\mathcal{E} \equiv \|\nabla_A u\|_2^2 + \frac{1}{2} \|\nabla\phi + \partial_t A\|_2^2 + \frac{1}{2} \|\operatorname{rot} A\|_2^2.$$

The system (MS) is gauge invariant and we study it in the Coulomb gauge  $\operatorname{div} A = 0$ , in which we can treat the system most easily. In this gauge, (1.2) and (1.3) become

$$-\Delta\phi = \rho, \quad \square A + \nabla\partial_t \phi = J. \quad (1.4)$$

The first equation of (1.4) is solved as

$$\phi = \phi(u) = (-\Delta)^{-1} \rho = (4\pi|x|)^{-1} * |u|^2$$

and the term  $\nabla\partial_t\phi$  in the second equation is dropped by operating the Helmholtz projection  $P = 1 - \nabla\operatorname{div}\Delta^{-1}$  to the both sides of the equation. Therefore in the Coulomb gauge the system (MS) is rewritten as

$$i\partial_t u = (-\Delta_A + \phi(u))u, \quad (1.5)$$

$$\square A = PJ, \quad (1.6)$$

which is referred to as (MS-C). To solve (MS-C) we should give the initial condition

$$(u(0), A(0), \partial_t A(0)) = (u_0, A_0, A_1) \quad (1.7)$$

in a direct sum of Sobolev spaces

$$X^{s,\sigma} = \{(u_0, A_0, A_1) \in H^s \oplus H^\sigma \oplus H^{\sigma-1}; \operatorname{div} A_0 = \operatorname{div} A_1 = 0\}.$$

Several authors study the Cauchy problem and the scattering theory for (MS-C). Nakamitsu-M. Tsutsumi [10] showed the time local well-posedness for (MS-C) in  $X^{s,\sigma}$  with  $s = \sigma = 3, 4, 5, \dots$ . In fact, they treated the case of Lorentz gauge, but the Coulomb gauge case can be treated analogously. We can easily refine their condition as  $s = \sigma > 5/2$  by the use of fractional order Sobolev spaces and the commutator estimate by Kato-Ponce [8]. Recently Nakamura-Wada [11] showed the time local well-posedness for wider class of  $(s, \sigma)$  including the case  $s = \sigma \geq 5/3$  (precisely see the remark for Theorem 1). On the other hand, Guo-Nakamitsu-Strauss [6] constructed a time global (weak) solution in  $X^{1,1}$  although they did not show the uniqueness. Indeed, in the Coulomb gauge the energy takes the form

$$\mathcal{E} = \|\nabla_A u\|_2^2 + \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{1}{2}\|\partial_t A\|_2^2 + \frac{1}{2}\|\nabla A\|_2^2,$$

and hence  $\|(u, A, \partial_t A); X^{1,1}\|$  does not blow up. Therefore the global existence is proved by parabolic regularization and compactness method. For the scattering theory, the existence of modified wave operators was proved by Y. Tsutsumi [14], Shimomura [12], and Ginibre-Velo [4, 5]. However these results dose not mean the existence of global strong solution since their solution to (MS-C) exist only for  $t \geq 0$  [12, 14] or for  $t \geq T$  [4, 5], where  $T$  is a sufficiently large positive number.

As we summarize above, there are many results for the Cauchy problem both at  $t = 0$  or  $t = \infty$ . However there are no results concerning the global existence or blow up of strong solutions even for small data. The aim of this talk is to answer this problem. Shortly, we prove the global existence of unique strong solutions. To do this, we use a priori estimates derived from the conservation laws of charge and energy, and hence it is desirable to show the local well-posedness in lower regularity. The following theorem does not cover the result for the energy class  $H^1$ , but it is sufficient for our aim.

**Theorem 1.** Let  $s \geq 11/8$ ,  $\sigma > 1$  and

$$\max\{s - 2; (2s - 1)/4\} \leq \sigma \leq \min\{s + 1; 3s/2; 7(2s - 1)/6\}$$

with  $(s, \sigma) \neq (2, 3), (7/2, 3/2)$ . Then for any  $(u_0, A_0, A_1) \in X^{s, \sigma}$ , there exists  $T > 0$  such that (MS-C) with initial condition (1.7) has a unique solution  $(u, A)$  satisfying  $(u, A, \partial_t A) \in C([0, T]; X^{s, \sigma})$ . Moreover if  $\sigma \geq \max\{(s - 1), (2s + 1)/4\}$  with  $(s, \sigma) \neq (5/2, 3/2)$ , then the map  $(u_0, A_0, A_1) \mapsto (u, A, \partial_t A)$  is continuous as a map from  $X^{s, \sigma}$  to  $C([0, T]; X^{s, \sigma})$ .

*Remark.* (1)  $T$  depends only on  $s, \sigma$  and  $\|(u_0, A_0, A_1); X^{s, \sigma}\|$ .

(2) For any  $s$  and  $\sigma$  satisfying the assumption above for the unique existence of the solution, the map  $(u_0, A_0, A_1) \mapsto (u, A, \partial_t A)$  is continuous in  $w^*$ -sense. Namely if a sequence of initial data strongly converges in  $X^{s, \sigma}$ , then corresponding sequence of solutions also converges star-weakly in  $L^\infty(0, T; X^{s, \sigma})$ .

(3) In [11], we also assume  $s \geq 5/3$  and  $4/3 \leq \sigma \leq (5s - 2)/3$  with  $(s, \sigma) \neq (5/2, 7/2)$ .

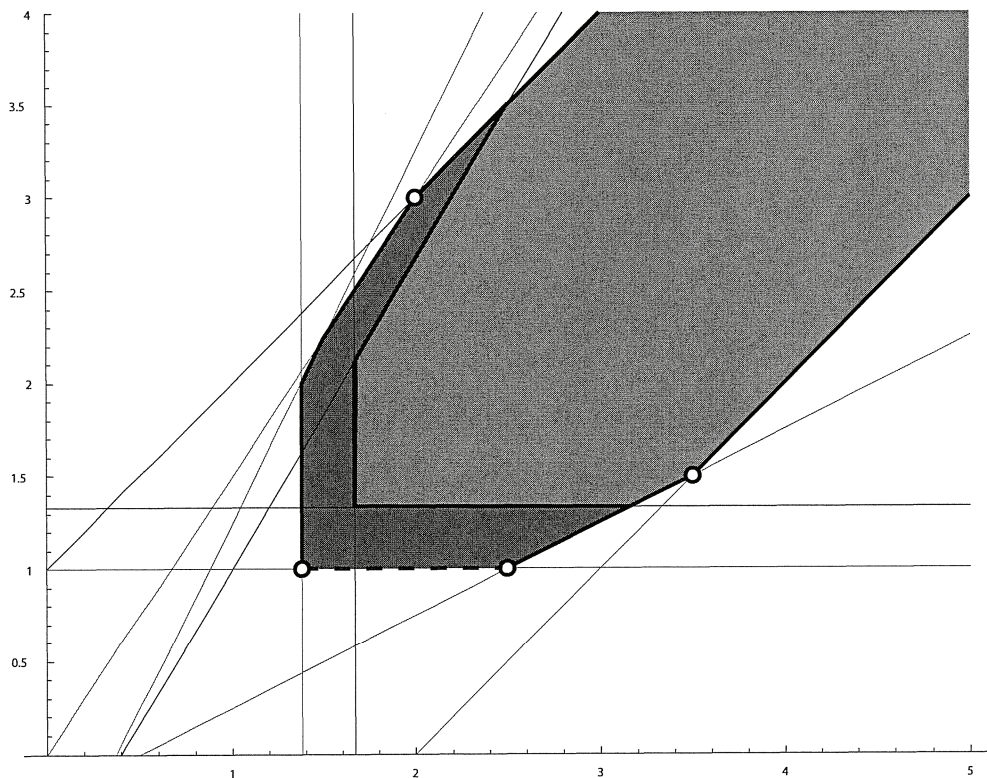


FIGURE 1. For  $(s, \sigma)$  in the blue region the unique existence of solutions has already proved in [11] and in the present work the result is extended in the red region.

By relaxing the assumption for the local theory, we can show the global existence.

**Theorem 2.** *The solution obtained in Theorem 1 exists time globally.*

In this abstract we use the following notation: For a Banach space  $X$  we put  $L_T^q X = L^q(0, T; X)$ . Similarly we use the abbreviation  $C_T^m X = C^m([0, T]; X)$ .  $M_T^{m, \sigma} = \bigcap_{j=0}^m C_T^j H^{\sigma-j}$  and its norm is defined as  $\|A; M_T^{m, \sigma}\| = \max_{0 \leq j \leq m} \|\partial_t^j A; L_T^\infty H^{\sigma-j}\|$ .

## §2. Preliminaries

In this section we summarize lemmas used in the proof of Theorems 1 and 2. The first one is a covariant derivative estimate, whose proof is done by the use of the Gagliardo-Nirenberg inequality.

**Lemma 1.** *Let  $A \in \dot{H}^1 \cap L^6$  satisfy  $\operatorname{div} A = 0$ . Then the following estimates hold for any  $v \in H^2$ :*

$$\|\nabla_A v; H^1\| \lesssim \langle \|A; \dot{H}^1\| \rangle \|v; H^2\|, \quad (2.1)$$

$$\|v; H^2\| + \langle \|A; \dot{H}^1\| \rangle^4 \|v\|_2 \simeq \|\Delta_A v\|_2 + \langle \|A; \dot{H}^1\| \rangle^4 \|v\|_2. \quad (2.2)$$

Next we introduce Strichartz type estimates for Klein-Gordon equations (see for example [1, 2, 3, 13]).

**Lemma 2.** *Let  $T > 0$ ,  $\sigma \in \mathbf{R}$  and let  $(q_j, r_j)$ ,  $j = 0, 1$ , satisfy  $0 \leq 2/q_j - 1 - 2/r_j < 1$ . Then a solution  $A$  to the equation  $(\square + 1)A = F$  satisfies the estimate*

$$\max_{k=0,1} \|\partial_t^k A; L_T^{q_0} H_{r_0}^{\sigma-k-2/q_0}\| \lesssim \|(A(0), \partial_t A(0)); H^\sigma \oplus H^{\sigma-1}\| + \|F; L_T^{q_1'} H_{r_1'}^{\sigma-1+2/q_1}\|. \quad (2.3)$$

Usual Strichartz estimates for Schrödinger equations does not match the equation (1.5) since we cannot avoid the loss of derivative coming from  $2iA \cdot \nabla u$ . In the present work we use a variation of Strichartz estimates introduced by Kenig-Koenig [9] for Benjamin-Ono type equations and by J. Kato [7] for Schrödinger equations, with a slight improvement (They need  $u \in L_T^\infty H^{s+\epsilon}$  and the present authors removed  $\epsilon$ ).

**Lemma 3.** *Let  $T > 0$ ,  $\alpha > 0$  and  $s \in \mathbf{R}$ . Then a solution  $u$  to the equation*

$$i\partial_t u = -\Delta u + f, \quad 0 < t < T,$$

*satisfies the estimate*

$$\|u; L_T^2 H_6^{s-\alpha}\| \lesssim \|u; L_T^\infty H^s\| + T^{1/2} \|f; L_T^2 H^{s-2\alpha}\|. \quad (2.4)$$

## §3. Sketch of proof

In this section we shall sketch the proof of Theorem 2, from which we can also understand the essence of the proof of Theorem 1. For simplicity, we restrict our attention to the case  $s = 2$ ,  $0 < \sigma - 1 \leq 1/9$ . In this section we fix a positive number  $\delta$  so that  $0 < \delta \leq (\sigma - 1)/2$  and put  $1/q = 1/2 - 2\delta/3$ ,  $1/r = 2\delta/3$ . We begin with the following a priori estimates.

**Lemma 4.** *Let  $(u, A, \partial_t A) \in C_T X^{s, \sigma}$  be a solution to (MS-C) obtained in Theorem 1. Then the following estimates hold.*

$$\|(u, A, \partial_t A); L_T^\infty(H^1 \oplus \dot{H}^1 \oplus L^2)\| \leq C, \quad (3.1)$$

$$\|A; L_T^\infty L^2\| \leq C \langle T \rangle, \quad (3.2)$$

$$\|A; L_T^q L^r\| \leq C \langle T \rangle^2, \quad (3.3)$$

$$\|u; L_T^2 H_6^{1/2-\delta}\| \leq C \langle T \rangle^3, \quad (3.4)$$

$$\|A; M_T^{1, \sigma}\| \leq C \langle T \rangle^5. \quad (3.5)$$

The constants  $C$  depend only on  $\|(u_0, A_0, A_1); H^1 \oplus H^1 \oplus L^2\|$ .

*Proof.* We easily obtain (3.1) by the conservation laws of charge and energy. We obtain (3.2) by the energy inequality for the wave equation and (3.1). We obtain (3.3) by Lemma 2 together with (3.1)-(3.2). We obtain (3.4) by the use of Lemma 3, for

$$\begin{aligned} \|u; L_T^2 H_6^{1/2-\delta}\| &\lesssim \|u; L_T^\infty H^1\| + T^{1/2} \|2iA \cdot \nabla u + |A|^2 u + \phi u; L_T^2 H^{-2\delta}\| \\ &\lesssim \langle T \rangle \|u; L_T^\infty H^1\| \langle \|A; L_T^q L^r\| + \|A; L_T^\infty \dot{H}^1\|^2 + \|u; L_T^\infty H^1\|^2 \rangle. \end{aligned}$$

Finally we obtain (3.5) by Lemma 2:

$$\|A; M_T^{1, \sigma}\| \lesssim \|(A_0, A_1); H^\sigma \oplus H^{\sigma-1}\| + \|A; L_T^1 H^{\sigma-1}\| + \|PJ; L_T^{6/5} H_{3/2}^{\sigma-2/3}\|$$

and the last term in the right-hand side is estimated by

$$\langle T \rangle \|u; L_T^\infty H^1\| \langle \|A; L_T^\infty H^1\| \rangle \|u; L_T^2 H_6^{1/2-\delta}\|.$$

Thus we have obtained the lemma.  $\square$

We proceed to the estimate of solutions to the following linear Schrödinger equation, namely in the following lemmas we regard  $A$  and  $u$  as known functions defined on  $0 \leq t \leq T$ :

$$i\partial_t v = (-\Delta_A + \phi(u))v, \quad 0 < t < T, \quad (3.6)$$

$$v(0) = v_0. \quad (3.7)$$

**Lemma 5.** *Let  $A \in M_T^{1, \sigma}$  with  $\operatorname{div} A = 0$  and let  $u \in C_T^\infty H^1$ . Let  $v \in C_T H^2$  be a solution to (3.6). Then  $v \in L_T^2 H_6^{3/2-\delta}$  and satisfies the estimate*

$$\|v; L_T^2 H_6^{3/2-\delta}\| \lesssim \langle T \rangle^m \langle \|A; L_T^\infty \dot{H}^1\| \rangle^m \langle \|A; L_T^q L^r\| \vee \|v; L_T^\infty H^1\|^2 \rangle \|v; L_T^\infty H^2\|.$$

*Proof.* Applying Lemma 3 to (3.6), we obtain

$$\|v; L_T^2 H_6^{3/2-\delta}\| \lesssim \|v; L_T^\infty H^2\| + T^{1/2} \|2iA \cdot \nabla v + |A|^2 v + \phi v; L_T^2 H^{1-2\delta}\|. \quad (3.8)$$

By the Leibniz rule we have  $\|A \cdot \nabla v; H^{1-2\delta}\| \lesssim \|A; H_q^{1-2\delta}\| \|\nabla v\|_r + \|A\|_r \|\nabla v; H_q^{1-2\delta}\|$ . Applying the estimate  $\|\nabla v\|_r \lesssim \|v; H^2\|^\alpha \|v; H_6^{3/2-\delta}\|^{1-\alpha}$ ,  $\alpha = 2\delta/(1-2\delta)$ , derived from

the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned} T^{1/2} \|A \cdot \nabla v; L_T^2 H^{1-2\delta}\| &\lesssim \epsilon T^{1/2} \|v; L_T^2 H_6^{3/2-\delta}\| + \epsilon^{(\alpha-1)/\alpha} T \|A; L_T^\infty \dot{H}^1\|^{1/\alpha} \|v; L_T^\infty H^2\| \\ &\quad + T^{1-1/q} \|A; L_T^q L^r\| \|v; L_T^\infty H^2\|. \end{aligned}$$

We choose  $\epsilon > 0$  so small that the first term in the right-hand side is absorbed in the left-hand side of (3.8). Another terms can be treated more easily. Thus we obtain the lemma.  $\square$

**Lemma 6.** *Let  $A \in M_T^{1,\sigma}$  with  $\operatorname{div} A = 0$  and let  $u \in C_T^\infty H^1$ . Then there exists a unique solution to (3.6)-(3.7) belonging to  $C_T H^2 \cap C_T^1 L^2$ . Moreover the solution  $v$  to (3.6)-(3.7) satisfies the following estimates:*

$$\begin{aligned} \|v; L_T^\infty H^2\| &\leq C \|v_0; H^2\| \langle \|A; L_T^\infty \dot{H}^1\| \rangle^4 \\ &\quad \times \exp\{C \langle T \rangle^l \langle \|A; M_T^{1,\sigma}\| \vee \|A; L_T^q L^r\| \vee \|u; L_T^\infty H^1\| \rangle^m\}. \end{aligned} \quad (3.9)$$

Here  $l$  and  $m$  are positive numbers.

*Proof.* For simplicity we only prove the estimate (3.9). The conservation law  $\|v(t)\|_2 = \|v_0\|_2$  immediately follows from the equation (3.6). Taking Lemma 1 into account, we estimate

$$\|v; H_A^2\| \equiv \|\Delta_A v\|_2 + \langle R \rangle^4 \|v\|_2$$

instead of  $\|v; H^2\|$ , where  $R \equiv \|A; L_T^\infty \dot{H}^1\|$ . Taking the time derivative of  $\Delta_A v$  and using the equation (3.6), we find the equation for  $\Delta_A v$ :

$$i \partial_t \Delta_A v = (-\Delta_A + \phi) \Delta_A v + 2 \partial_t A \cdot \nabla_A v + [\Delta_A, \phi] v. \quad (3.10)$$

Therefore standard energy method shows that

$$\|v; L_T H_A^2\| \leq \|v_0; H_{A_0}^2\| + \|2 \partial_t A \cdot \nabla_A v + [\Delta_A, \phi] v; L_T^1 L^2\|.$$

Similarly as in the proof of Lemma 5, we have

$$\begin{aligned} \|\partial_t A \cdot \nabla_A v\|_2 &\lesssim \epsilon \|v; H_6^{3/2-\delta}\| \\ &\quad + \{\epsilon^{(\alpha-1)/\alpha} \|\partial_t A; H^{\sigma-1}\|^{1/\alpha} + \|\partial_t A; H^{\sigma-1}\| \| \|A\|_r \} \|v; H_A^2\|. \end{aligned} \quad (3.11)$$

We can easily handle the term  $[\Delta_A, \phi] v$  by the Hardy-Littlewood-Sobolev inequality. Therefore

$$\begin{aligned} \|v; L_T^\infty H_A^2\| &\lesssim \|v_0; H_{A_0}^2\| \\ &\quad + \int_0^T \{\epsilon^{(\alpha-1)/\alpha} \|\partial_t A; H^{\sigma-1}\|^{1/\alpha} + \|\partial_t A; H^{\sigma-1}\| \| \|A\|_r + \|u; H^1\|^2\} \|v; H_A^2\| \\ &\quad + T^{1/2} \|v; L_T^2 H_6^{3/2-\delta}\| dt. \end{aligned}$$

Taking Lemma 5 into account, we choose the positive number  $\epsilon$  so small that the last term in the right-hand side is absorbed in the left-hand side. Then we obtain an integral inequality for  $\|v; H_A^2\|$ . Applying the Gronwall inequality we obtain (3.9).  $\square$

*Proof of Theorem 2.* The solution  $(u, A)$  to (MS-C) clearly satisfies the estimate (3.9) with  $v = u, v_0 = u_0$ . Therefore the global existence follows from Lemma 4.  $\square$

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# On the existence of standing waves for the Maxwell-Schrödinger equations

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## 1 Introduction and main results

In this talk we consider the existence and the multiplicity of solutions to the following elliptic system:

$$-\frac{1}{2}\Delta u + \omega u + e\Phi u - |u|^{p-1}u = 0, \quad (1)$$

$$-\Delta\Phi = 4\pi e u^2 \quad \text{in } \mathbb{R}^3, \quad (2)$$

where  $u, \Phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $\omega, e > 0$ , and  $1 < p < 5$ .

The motivation to study the system (1)–(2) stems from the following Maxwell-Schrödinger equations:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\left(\nabla - ie\mathbf{A}\right)^2\psi + e\phi\psi - |\psi|^{p-1}\psi, \quad (3)$$

$$-\operatorname{div}\left(\nabla\phi + \frac{\partial\mathbf{A}}{\partial t}\right) = 4\pi e|\psi|^2, \quad (4)$$

$$\frac{e}{2}\left(\langle i\nabla\psi, \psi \rangle + e\mathbf{A}|\psi|^2\right) + \frac{1}{8\pi}\left(\frac{\partial}{\partial t}\left(\nabla\phi + \frac{\partial\mathbf{A}}{\partial t}\right) + \nabla \times (\nabla \times \mathbf{A})\right) = 0 \quad (5)$$

in  $\mathbb{R} \times \mathbb{R}^3$ ,

where  $\psi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$ ,  $\phi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ , and  $\mathbf{A} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ .

We are interested in the standing wave solutions in the electrostatic case, that is,

$$\psi(t, x) = e^{i\omega t}u(x), \quad \phi(t, x) = \Phi(x), \quad \mathbf{A}(t, x) = 0.$$

Then the equation (5) is automatically satisfied and the equations (3) and (4) become the system (1)–(2).

We give several notation. We define the function space  $H^1$  by the usual Sobolev space equipped with the norm

$$\|u\|_{H^1} = \left( \int |\nabla u|^2 + |u|^2 dx \right)^{1/2}.$$

We define the function space  $D^{1,2}$  by the completion of  $C_0^\infty(\mathbb{R}^3)$  with respect to the norm

$$\|u\|_{D^{1,2}} = \left( \int |\nabla u|^2 dx \right)^{1/2}.$$

Many authors have studied the existence of the standing wave solutions for the Klein-Gordon, Dirac or Schrödinger equation coupled with the Maxwell equations in the electrostatic case (see e.g. [3], [4], [5], [7], [8], [9], [10], [11], [12], [15], and [17]). In the case where the vector potential  $A$  is not identically zero, Georgiev and Visciglia [15] prove a non-existence result for the Klein-Gordon-Maxwell equations. We recall several known results about the system (1)–(2). Coclite [8] proves that the system (1)–(2) has infinitely many radially symmetric solutions in  $H^1 \times D^{1,2}$  when  $3 \leq p < 5$ . D’Aprile and Mugunai [10] show that the system (1)–(2) has no nontrivial solution in  $H^1 \times D^{1,2}$  when  $0 \leq p < 1$  or  $5 \leq p$ . On the other hand, D’Avenia [12] shows the existence of non-radially symmetric solution in  $H^1 \times D^{1,2}$  when  $3 \leq p < 5$ . Recently, Ruiz [17] shows that there exists a constant  $\bar{e} > 0$  sufficiently small such that the system (1)–(2) has a family of solutions  $(u(e), \Phi(e), e)$  bifurcating from  $(u_0, \Phi_0, 0)$  if  $e < \bar{e}$ . Here, the function  $u_0$  is the unique positive radially symmetric solution to the following scalar field equation:

$$-\frac{1}{2}\Delta u + u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N,$$

and the function  $\Phi_0$  is the unique positive radially symmetric solution to the equation (2) with  $u = u_0$ . In this talk we consider the existence and the multiplicity of solutions to the system (1)–(2) when  $1 < p < 3$ . Our main results are the following:

**Theorem 1.1.** *Assume that  $\omega > 0$  and  $1 < p \leq 3$ . For each  $m \in \mathbb{N}$ , there exists a positive number  $e_m > 0$  such that if  $0 < e < e_m$ , the system (1)–(2) has at least  $m$  radially symmetric solutions in  $H^1 \times D^{1,2}$ . In particular, we can take  $e_1 = +\infty$  when  $2 < p < 3$ .*

**Theorem 1.2.** *Assume that  $\omega > 0$ . For each  $p$  with  $1 < p \leq 2$ , there exists a positive number  $e_0 > 0$  such that if  $e > e_0$ , the system (1)–(2) has no nontrivial solution in  $H^1 \times D^{1,2}$ .*

We shall prove our theorems by the variational method. We define a functional  $J_e \in C^1(H^1, \mathbb{R})$  by

$$J_e(u) = \frac{1}{4} \int |\nabla u|^2 dx + \frac{\omega}{2} \int |u|^2 dx + \frac{1}{4} \int (|x|^{-1} * u^2) |u|^2 dx - \frac{1}{p+1} \int |u|^{p+1} dx.$$

Then we deduce that a pair of functions  $(u, \Phi)$  is a solution to the system (1)–(2) if and only if the function  $u$  is a critical point of the functional  $J_e$  and the function  $\Phi$  satisfies  $\phi = e|x|^{-1} * u^2$  (see Section 2). Using the mountain pass theorem, Coclite [8] shows that there exist infinitely many critical points of the functional  $J_e$ . We need the following Palais-Smale condition to use the mountain pass theorem.

**Definition.** Let  $E$  be a Banach space and assume that  $I \in C^1(E, \mathbb{R})$ .

- (i) We say that a sequence  $\{u_n\}$  is a *Palais-Smale sequence* (PS sequence, for short) associated with the functional  $I$  if and only if there exists a constant  $M > 0$  such that  $|I(u_n)| \leq M$  and  $I'(u_n) \rightarrow 0$  in  $E^*$  as  $n \rightarrow \infty$ . Here,  $I'(\cdot)$  is the Fréchet derivative of  $I(\cdot)$  and  $E^*$  is the dual of  $E$ .
- (ii) We say that *the functional  $I$  satisfies the Palais-Smale condition* (PS condition, for short) if any PS sequence has a convergent subsequence.

To show that the functional  $J_e$  satisfies the PS condition, we need to obtain the boundedness of a PS sequence. In fact, if  $\{u_n\}$  is a bounded PS sequence for the functional, we deduce that there exists a convergent subsequence (see Lemma 3.3). We find that every PS sequence is bounded in  $H^1$  when  $3 \leq p < 5$ . Indeed, if  $\{u_n\}$  is a PS sequence for the functional  $J_e$ , then we have

$$\begin{aligned} (p+1)M + \|u_n\|_{H^1} &\geq (p+1)J_e(u_n) - \langle J'_e(u_n), u_n \rangle \\ &\geq \frac{p-1}{4} \int |\nabla u_n|^2 dx + \frac{p-1}{2} \int |u_n|^2 dx \\ &\quad + \frac{p-3}{4} \int (|x|^{-1} * u^2) |u|^2 dx \\ &\geq \frac{p-1}{4} \int |\nabla u_n|^2 dx + \frac{p-1}{2} \int |u_n|^2 dx \\ &\geq c \|u_n\|_{H^1}^2, \end{aligned}$$

since  $3 \leq p < 5$ . It follows that  $\{u_n\}$  is bounded in  $H^1$ .

However, it seems difficult to obtain the boundedness of a PS sequence when  $1 < p < 3$ . To overcome this difficulty, we invoke Struwe's argument

(see e.g. [19], [20], and [21]). For  $\lambda \in [1/2, 1]$ , we consider a family of functionals  $J_{e,\lambda}$  defined by

$$J_{e,\lambda}(u) = \frac{1}{4} \int |\nabla u|^2 dx + \frac{\omega}{2} \int |u|^2 dx + \frac{1}{4} \int (|x|^{-1} * u^2) |u|^2 dx - \frac{\lambda}{p+1} \int |u|^{p+1} dx.$$

Using Jeanjean's result [16], which is based on Struwe's argument, there exists a sequence  $\{(u_j, \lambda_j)\} \subset H^1 \times [1/2, 1]$  such that

- (i)  $\lambda_j \rightarrow 1$  as  $j \rightarrow \infty$ ,
- (ii)  $u_j$  is a critical point of the functional  $J_{e,\lambda_j}$ .

In other words, a pair of functions  $(u_j, e|x|^{-1} * u_j^2)$  is a solution to the following parameterized system:

$$-\frac{1}{2} \Delta u + \omega u + e\Phi u - \lambda_j |u|^{p-1} u = 0, \quad (6)$$

$$-\Delta \Phi = 4\pi e u^2 \quad \text{in } \mathbb{R}^3. \quad (7)$$

From the fact that  $(u_j, e|x|^{-1} * u_j^2)$  is a solution to the system (6)–(7), we know that  $u_j$  satisfies the Pohozaev identity (see Lemma 3.5). Using the Pohozaev identity, we show that the sequence  $\{u_j\}$  is bounded in  $H^1$ . Then there exists a subsequence  $\{u_j\}$  (we still denote by  $\{u_j\}$ ) and a function  $u$  such that  $u_j \rightarrow u$  in  $H^1$  as  $j \rightarrow \infty$ . We deduce that the function  $u$  is a critical point of the functional  $J_{e,1} = J_e$ . In a similar way, we find that for any  $m \in \mathbb{N}$ , if the constant  $e > 0$  is sufficiently small, then there exist solutions  $\{u^k\}_{k=1}^m$  satisfying  $J_e(u^1) < J_e(u^2) < \dots < J_e(u^k)$ . Then we can prove the multiplicity of solutions to the system (1)–(2).

To prove Theorem 1.2, it is enough to show that there exists a constant  $\bar{e}$  such that if  $e > \bar{e}$  then  $\langle J'_e(u), u \rangle > 0$  for all  $u \in H^1$ . To show this, we use the fact that the function  $|x|^{-1} * u^2$ , which is the unique positive solution to the equation (2) with  $e = 1$ , is a minimizer of

$$\inf_{\phi \in D^{1,2}} \left\{ \frac{1}{2} \int |\nabla \phi|^2 dx - 4\pi \int u^2 \phi dx \right\}.$$

## 2 Variational setting

In this section we prove some preliminary results concerning the variational structure for the system (1)–(2). We may assume that  $\omega = 1$  without loss

of generality. The system (1)–(2) is the Euler-Lagrange equations of the functional  $F_e : H^1 \times \mathcal{D}^{1,2} \rightarrow \mathbb{R}$  defined by

$$F_e(u, \phi) = \frac{1}{4} \int |\nabla u|^2 dx - \frac{1}{16\pi} \int |\nabla \phi|^2 dx + \frac{e}{2} \int \phi |u|^2 dx + \frac{1}{2} \int |u|^2 dx - \frac{1}{p+1} \int |u|^{p+1} dx.$$

In other words, a pair of the function  $(u, \phi)$  is a critical point of the functional  $F_e$  if and only if  $(u, \phi)$  is a solution to the system (1)–(2). For each  $u \in H^1$ , the function  $e|x|^{-1} * u^2 \in H^1$  is the unique solution to the equation (2). We define a functional  $J_e : H^1 \rightarrow \mathbb{R}$  by

$$\begin{aligned} J_e(u) &= F_e(u, e|x|^{-1} * u^2) \\ &= \frac{1}{4} \int |\nabla u|^2 dx + \frac{1}{2} \int |u|^2 dx + \frac{e^2}{4} \int (|x|^{-1} * u^2) |u|^2 dx \\ &\quad - \frac{1}{p+1} \int |u|^{p+1} dx, \end{aligned}$$

Then we find a relationship between a critical point of the functional  $F_e$  and that of the functional  $J_e$ :

**Proposition 2.1.** *The following statements are equivalent.*

- (i) *A pair of functions  $(u, \phi) \in H^1 \times \mathcal{D}^{1,2}$  is a critical point of the functional  $F_e$ .*
- (ii) *A function  $u \in H^1$  is a critical point of the functional  $J_e$  and a function  $\phi$  satisfies  $\phi = e|x|^{-1} * u^2$*

For the proof of Proposition 2.1, see Proposition 3.5 in [4]. From Proposition 2.1, it is enough to find a critical point of the functional  $J_e$  to prove the existence of a solution to the system (1)–(2). Since the functional  $J_e$  is invariant under the transformation  $u(x) \rightarrow u(x+a)$  for any  $a \in \mathbb{R}^3$ , there is a lack of compactness. To overcome this difficulty, we restrict ourselves to the space radial functions  $u(x) = u(r)$ ,  $r = |x|$ . More precisely, we consider the functional  $J_e$  on the subspace  $H_r^1 = \{u \in H^1 \mid u(x) = u(|x|)\}$ . We recall the subspace  $H_r^1$  is compactly embedded in  $L^q$  when  $2 < q < 6$  (see e.g. [6], [18]). Then, we can show the following lemma.

**Lemma 2.2.** *Any critical point  $u \in H_r^1$  of the functional  $J_e|_{H_r^1}$  is that of the functional  $J_e$ .*

For the proof of Lemma 2.2, see Lemma 4.2 in [4].

### 3 Proof of Theorem 1.1

We first show the existence of solution. To show the existence, we make an approximate solution for the system (1)–(2). We consider a family of functionals  $J_{e,\lambda} \in C^1(H^1, \mathbb{R})$  defined by

$$J_{e,\lambda}(u) = \frac{1}{4} \int |\nabla u|^2 dx + \frac{\omega}{2} \int |u|^2 dx + \frac{e^2}{4} \int (|x|^{-1} * u^2) |u|^2 dx - \frac{\lambda}{p+1} \int |u|^{p+1} dx,$$

for  $\lambda \in [1/2, 1]$ . We prove that there exists a critical point of the functional  $J_{e,\lambda}$  for almost all  $\lambda \in [1/2, 1]$ . To prove this, we use the following theorem which is due to Jeanjean [16].

**Theorem 3.1.** *Let  $X$  be a Banach space equipped with a norm  $\|\cdot\|_X$  and let  $L \subset \mathbb{R}^+$  be an interval. We consider a family  $\{I_\lambda\}_{\lambda \in L}$  of  $C^1$  functionals on  $X$  of the form*

$$I_\lambda(u) = A(u) - \lambda B(u) \quad \text{for all } \lambda \in L,$$

where  $B(u) \geq 0$  for all  $u \in X$  and either  $A(u) \rightarrow +\infty$  or  $B(u) \rightarrow +\infty$  as  $\|u\| \rightarrow +\infty$ . We assume that there exist two functions  $v_1, v_2 \in X$  such that

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\} \quad \text{for all } \lambda \in L,$$

where  $\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = v_1, \gamma(1) = v_2\}$ . Then, for almost all  $\lambda \in L$ , there exists a subsequence  $\{v_n^\lambda\} \subset X$  such that

- (i)  $\{v_n^\lambda\}$  is bounded in  $X$ ,
- (ii)  $I_\lambda(v_n^\lambda) \rightarrow c_\lambda$ ,
- (iii)  $I'_\lambda(v_n^\lambda) \rightarrow 0$  in  $X^*$ , where  $X^*$  is the dual of  $X$ .

From Ekeland's principle [13], we know that if the functional  $I \in C^1(X, \mathbb{R})$  has the mountain pass geometry, there exists a PS sequence for the functional  $I$ . On the other hand, Theorem 3.1 implies that if the functionals  $I_\lambda$  satisfies the assumption of the theorem, there exists a bounded PS sequence for the  $\{I_\lambda\}$  for almost all  $\lambda \in L$ .

We explain the reason why there exists a bounded PS sequence for almost all  $\lambda$ . Since the map  $\lambda \mapsto c_\lambda$  is non-increasing, the derivative  $c'_\lambda = \partial c_\lambda / \partial \lambda$  exists for almost all  $\lambda \in L$ . Roughly speaking, there exists a constant  $M(c_\lambda, c'_\lambda)$  such that  $\|v_n^\lambda\| \leq M(c_\lambda, c'_\lambda)$ .

We use Theorem 3.1 with  $X = H_r^1$ ,  $L = [1/2, 1]$ , and  $I_\lambda = J_\lambda$ . We can find that  $B(u) = (p+1)^{-1} \int |u|^{p+1} dx \geq 0$  for all  $u \in H_r^1$  and that  $A(u) = 4^{-1} \int |\nabla u|^2 + 2^{-1} \int |u|^2 dx + e^2 4^{-1} \int (|x|^{-1} * u^2) |u|^2 dx \rightarrow +\infty$  as  $\|u\|_{H^1} \rightarrow \infty$ . From the next lemma we know that the functional  $J_e$  satisfies the assumption of Theorem 3.1.

**Lemma 3.2.** (i) Let  $2 < p < 3$ . For any  $e > 0$ , there exist functions  $v_1, v_2 \in H_r^1$  satisfying  $c_\lambda > \max\{J_{e,\lambda}(v_1), J_{e,\lambda}(v_2)\}$  for all  $\lambda \in [1/2, 1]$ .

(ii) Let  $1 < p \leq 2$ . There exists  $\bar{e} > 0$  such that if  $e < \bar{e}$  then there exist two functions  $v_1, v_2 \in H_r^1$  satisfying  $c_\lambda > \max\{J_{e,\lambda}(v_1), J_{e,\lambda}(v_2)\}$  for all  $\lambda \in [1/2, 1]$ .

From Theorem 3.1 and Lemma 3.2, we deduce that there exists a bounded PS sequence  $\{u_n^\lambda\}$  for the functional  $J_{e,\lambda}$  for almost all  $\lambda \in [1/2, 1]$ . The next lemma follows from the fact that  $H_r^1$  is compactly embedded in  $L^q$  when  $2 < q < 6$ .

**Lemma 3.3.** Let  $\lambda \in [1/2, 1]$ . Assume that  $\{u_n\}$  is a bounded PS sequence for the functional  $J_{e,\lambda}$ . Then there exists a convergent subsequence of  $\{u_n\}$ .

For the proof of Lemma 3.3, see Theorem 3.3 in [10].

From Lemma 3.3, there exists a subsequence of  $\{u_n^\lambda\}$  (still denoted by  $\{u_n^\lambda\}$ ) and a function  $u^\lambda$  such that  $u_n^\lambda \rightarrow u^\lambda$  strongly in  $H_r^1$  as  $n \rightarrow \infty$ . Since the functional  $J_{e,\lambda} \in C^1(H_r^1, \mathbb{R})$ , we have  $J_{e,\lambda}(u^\lambda) = c_\lambda$  and  $J'_{e,\lambda}(u^\lambda) = 0$  in  $H_r^{-1}$ . That is,  $u^\lambda$  is a critical point of the functional  $J_{e,\lambda}$  and its critical value is  $c_\lambda$ . Therefore, we can choose  $\{u_j, \lambda_j\} \subset H_r^1 \times [1/2, 1]$  such that  $u_j$  is a critical point of the functional  $J_{e,\lambda_j}$  and  $\lambda_j \rightarrow 1$ . Using the fact that  $u_j$  is a critical point of the functional  $J_{e,\lambda_j}$ , we can show the following lemma.

**Lemma 3.4.** The sequence  $\{u_j\}$  above is bounded in  $H_r^1$ .

To prove Lemma 3.4, we use the following the Pohozaev identity which is due to D'Aprile and Mugnai [10].

**Lemma 3.5.** Assume that  $u$  is a critical point of the functional  $J_{e,\lambda}$  with  $\lambda \in [1/2, 1]$ . Then  $u$  satisfies

$$\frac{1}{4} \int |\nabla u|^2 dx + \frac{3}{2} \int |u|^2 dx + \frac{5e^2}{4} \int (|x|^{-1} * u^2) |u|^2 dx - \frac{3\lambda}{p+1} \int |u|^{p+1} dx = 0.$$

*Proof of Theorem 1.1.* We prove only the existence of a solution. We claim that the sequence  $\{u_j\}$  is bounded PS sequence for the functional  $J_{e,1} = J_e$ . Indeed, we have

$$\begin{aligned} |J_e(u_j)| &\leq |J_{e,\lambda}(u_j)| + (1 - \lambda_j) \int |u_j|^{p+1} dx \\ &\leq c_{\lambda_j} + c(1 - \lambda_j) \|u_j\|_{H^1}. \end{aligned}$$

Since the sequence  $\{u_j\}$  is bounded in  $H^1$  and  $\lambda_j \rightarrow 1$  as  $j \rightarrow \infty$ , we have  $|J_e(u_j)| \leq c_{\frac{1}{2}} + 1$ . Similarly, we find that  $J'_e(u_j) \rightarrow 0$  in  $H^{-1}$ . Therefore,  $\{u_j\}$  is a bounded PS sequence. From Lemma 3.3, there exists a function  $u_0 \in H_r^1$  such that  $u_j \rightarrow u_0$  strongly in  $H_r^1$ . Therefore  $u_0$  is a critical point of the functional  $J_e$ .  $\square$

We show the multiplicity of solutions to the system (1)–(2) following Zou [23]. Let  $\{e_j\}$  be an orthonormal basis of  $H_r^1$ . We set  $Y_k = \bigoplus_{j=1}^k \text{span}\{e_j\}$  and set  $B_k = \{u \in Y_k \mid \|u\|_{H^1} \leq \rho_k\}$  for  $\rho_k > 0$ . By a argument similar to that used in the proof of the existence, we deduce that for each  $m \in \mathbb{N}$ , there exists a constant  $e_m$  such that if  $e < e_m$  then  $J_e$  has a critical points  $\{u^k\}_{k=1}^m$ . Furthermore we find that for each  $k = 1, 2, \dots, m$ ,

$$J_e(u^k) = \inf_{\gamma \in \Gamma} \max_{u \in B_k} J_e(\gamma(u)),$$

where  $\Gamma = \{\gamma \in C(B_k, H_r^1) \mid \gamma \text{ is odd and } \gamma|_{\partial B_k} = id\}$ . We define a functional  $\tilde{J}$  by

$$\tilde{J}(u) = \frac{1}{4} \int |\nabla u|^2 dx + \frac{1}{2} \int |u|^2 dx - \frac{1}{p+1} \int |u|^{p+1} dx.$$

From the definition of the functional  $\tilde{J}$ , we find that  $J_e(u) \geq \tilde{J}(u)$  for all  $u \in H^1$ . The positive number  $\inf_{\gamma \in \Gamma} \max_{u \in B_k} \tilde{J}(\gamma(u))$  is independent of the constant  $e (< e_m)$  and we know that  $\inf_{\gamma \in \Gamma} \max_{u \in B_k} \tilde{J}(\gamma(u)) \rightarrow +\infty$  as  $k \rightarrow +\infty$  (see e.g. [23], [22]). Therefore we deduce that for any  $m \in \mathbb{N}$  there exists a  $m$  radially symmetric solutions  $\{u^k\}_{k=1}^m$  satisfying  $J_e(u^1) < J_e(u^2) < \dots, < J_e(u^m)$ .

## 4 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. To prove Theorem 1.2, we need the following well-known lemma.

**Lemma 4.1.** *For every  $u \in H^1$ , the function  $|x|^{-1} * u^2 (\in D^{1,2})$  achieves the minimum:*

$$\inf_{\phi \in D^{1,2}} \left( \frac{1}{2} \int |\nabla \phi|^2 dx - 4\pi \int u^2 \phi dx \right).$$

The proof can be found in [14]. This lemma follows from the fact that the function  $|x|^{-1} * u^2$  is a solution to the equation (2) with  $e = 1$  for each  $u \in H^1$ . Now we are in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* It is enough to show that there exists a positive number  $e_0$  for which

$$\langle J'_e(u), u \rangle > 0 \quad \text{holds for all } u \in H^1 \setminus \{0\} \quad (8)$$

if  $e > e_0$ . We shall show (8). For each  $u \in H^1 \setminus \{0\}$ , we deduce that

$$\begin{aligned} \langle J'_e(u), u \rangle &= \frac{1}{2} \int |\nabla u|^2 dx + \int |u|^2 dx + e^2 \int (|x|^{-1}) |u|^2 dx - \int |u|^{p+1} dx \\ &\geq \frac{1}{2} \int |\nabla u|^2 dx + \int |u|^2 dx \\ &\quad + e^2 \left( -\frac{1}{4\pi} \inf_{\phi \in D^{1,2}} \left\{ \frac{1}{2} \int |\nabla \phi|^2 - 4\pi \int u^2 \phi dx \right\} \right) - \int |u|^{p+1} dx, \end{aligned}$$

where we used Lemma 4.1. Setting  $\phi = e^{-1}u \in H^1$ , we have

$$\begin{aligned} \langle J'_e(u), u \rangle &= \frac{1}{2} \int |\nabla u|^2 dx + \int |u|^2 dx - \frac{1}{8\pi} \int |\nabla u|^2 dx + e \int |u|^3 dx - \int |u|^{p+1} dx \\ &= \left( \frac{1}{4} - \frac{1}{8\pi} \right) \int |\nabla u|^2 dx + \int |u|^2 dx + e \int |u|^3 dx - \int |u|^{p+1} dx. \end{aligned}$$

Since  $1 < p \leq 2$ , there exists a positive number  $e_0$  such that  $s^2 + es^3 - s^{p+1} > 0$  for all  $s > 0$  and all  $e > e_0$ . Therefore we have proved (8). This completes the proof.  $\square$

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# Scattering for the Gross-Pitaevskii equation<sup>1</sup>

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## ABSTRACT

The Gross-Pitaevskii (or Ginzburg-Landau-Schrödinger) equation models the Bose-Einstein condensation. It is formally equivalent to the usual nonlinear Schrödinger equation with the defocusing cubic nonlinearity, but physically natural solutions are perturbation from equilibria that do not decay at the spatial infinity. We investigate behavior of such solutions for large time from the view point of the scattering theory.

## 1. INTRODUCTION

We consider time-global behavior of a class of solutions to the defocusing cubic nonlinear Schrödinger equation:

$$i\dot{\varphi} + \Delta\varphi = |\varphi|^2\varphi, \quad (1)$$

where  $\varphi(t, x) : \mathbb{R}^{1+d} \rightarrow \mathbb{C}$  and  $d \in \mathbb{N}$ . There are a lot of works on the time asymptotic behavior for this equation, where the solutions are usually supposed to decay at the spatial infinity, such as  $u(t) \in H^s(\mathbb{R}^d)$  for some  $s \in \mathbb{R}$ . In other words, the solutions are close to the trivial solution 0. However this assumption is not always relevant to the physical phenomena which the solutions are expected to describe.

A typical example is the Bose-Einstein condensation, where the other trivial solution  $e^{-it}$  corresponds to the equilibrium of perfect condensation. Hence the natural class of solutions are small perturbation of the equilibrium given by

$$\varphi = e^{-it}\psi, \quad \psi \rightarrow 1 \quad (|x| \rightarrow \infty) \quad (2)$$

The equation for  $\psi$

$$i\dot{\psi} + \Delta\psi = (|\psi|^2 - 1)\psi \quad (3)$$

is called the Gross-Pitaevskii equation in such context as the superfluid and the superconductors. Further substituting  $\psi = 1 + u$ , we obtain the equation for  $u$ :

$$iu + \Delta u = 2 \operatorname{Re} u + (u + 2\bar{u} + |u|^2)u. \quad (4)$$

The global existence of solution  $u$  in  $H^1$  has been proved in [3] for  $d \leq 3$ . It is also known that there exist traveling vortex rings of the form  $v(t, x) = v(x - ct)$  for  $d \geq 2$  [3, 2, 4, 11]. Thus the global behavior of general solution could be quite complicated, but it can be expected that small solutions  $u$  would behave like linearized solutions.

We would like to study the asymptotic behavior of small perturbation  $u$  by the scattering theory. For that purpose, we further transform the equation by

$$u = Vv := \sqrt{\frac{-\Delta}{2 - \Delta}} \operatorname{Re} u + i \operatorname{Im} u, \quad (5)$$

such that the linearized flow preserves the  $L^2$  norm. The equation for  $v$  is given by

$$i\dot{v} - Hv = -iV^{-1}i(u + 2\bar{u} + |u|^2)u, \quad (6)$$

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<sup>1</sup>This talk is based on the joint work with Stephen Gustafson and Tai-Peng Tsai [23].

where  $H = \sqrt{-\Delta(2 - \Delta)}$ . The main difficulties compared with the usual Schrödinger equation are

1. invariances destroyed (gauge, dilation and Galilean invariances are all fixed by the boundary condition)
2. singularity at the Fourier origin due to  $V^{-1}$

We can resolve the second difficulty by the Strichartz estimate with additional gain at the Fourier origin, and a sort of normal form in the low frequency.

## 2. MAIN THEOREMS

For  $d \geq 4$ , we obtain small data scattering in some Sobolev spaces.

**Theorem 1.** *Let  $d \geq 4$ ,  $s = d/2 - 1$  and  $|\sigma| \leq (d - 3)/2 - 1/d$ . There exists  $\delta > 0$  such that if  $\|U^\sigma V^{-1}u(0)\|_{H^s} \leq \delta$  then the solution  $u$  of (4) exists globally in time and satisfies*

$$\|U^\sigma V^{-1}u(t)\|_{H^s} \lesssim \|U^\sigma V^{-1}u(0)\|_{H^s} \quad (7)$$

Moreover, there exist  $v_\pm \in U^{-\sigma}H^s$  such that

$$\|U^\sigma(V^{-1}u(t) - e^{-iHt}v_\pm)\|_{H^s} \rightarrow 0, \quad (t \rightarrow \pm\infty) \quad (8)$$

For  $d = 3$ , we can construct the wave operators.

**Theorem 2.** *Let  $d = 3$ . There exists  $\delta > 0$  such that if  $v_+ \in H^1$  satisfies*

$$\sup_{t \geq 1} \|e^{-iHt}v_+\|_{L^2 \cap t^{-1/2}L^3} \leq \delta \quad (9)$$

then there exists a global solution  $u$  of (4) satisfying

$$\|V^{-1}u(t) - e^{-iHt}v_+\|_{L^2} \rightarrow 0 \quad (t \rightarrow \infty) \quad \sup_{t \geq 1} t^{1/2} \|V^{-1}u(t)\|_{L^3} \lesssim \delta \quad (10)$$

(9) is satisfied for example if  $v_+$  is sufficiently small in  $L^{3/2}$ . The key ingredients of our proofs are the  $L^p$  decay estimate

$$\|e^{-iHt}\varphi\|_{B_{p,2}^0} \lesssim t^{-d\alpha} \|U^{(d-2)\alpha}\varphi\|_{B_{q',2}^0}, \quad (11)$$

where  $2 \leq p \leq \infty$ ,  $\alpha = 1/2 - 1/p$  and  $U = \sqrt{-\Delta/(2 - \Delta)}$ , and the normal form transform  $u \mapsto w = u + P|u|^2/2$ , where  $P$  denotes a high frequency cut-off.

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# ALMOST NONEXISTENCE OF ASYMPTOTICALLY FREE SOLUTIONS FOR SOME QUADRATIC NONLINEAR SCHRÖDINGER EQUATION IN TWO DIMENSIONS

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## 1. INTRODUCTION AND THE MAIN RESULT

This talk is based on [9]. We study the asymptotic behavior on time of solutions for the nonlinear Schrödinger equation with the quadratic nonlinearity in two space dimensions:

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^2, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad (1.1)$$

where  $u$  is a complex valued unknown function of  $(t, x)$ , and  $\lambda \in \mathbb{C} \setminus \{0\}$ . In this talk, we show the almost nonexistence of asymptotically free solutions for the equation (1.1). More precisely, we show that if there exist a  $T \geq 0$ , a final state  $u_+ \in H^{0,2}$  and a solution  $u \in C([T, \infty); L^2)$  for this equation such that  $u$  approaches the free solution  $U(t)u_+$  in  $L^2$  with a suitable convergence rate, where  $U(t) = e^{it\Delta/2}$ , as  $t \rightarrow \infty$ , then  $u_+$  is identically zero. This implies the almost nonexistence of wave operators for this equation on  $L^2$ .

There are several results on the large-time behavior of solutions to the nonlinear Schrödinger equation with non-gauge-invariant quadratic nonlinearities

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda_1 u^2 + \lambda_2 \bar{u}^2 + \lambda_3 |u|^2, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad (1.2)$$

where  $\lambda_1, \lambda_2$  and  $\lambda_3$  are complex constants. In four space dimensions, the scattering operator to the equation (1.2) exists for small data in the energy class (see, e.g., Kato [6]). In three space dimensions, by introducing similar function spaces as in Ozawa [8] and Ginibre-Ozawa [1], we can show the existence of a unique asymptotically free solution to this equation. Furthermore, recently, Hayashi and Naumkin [3] obtain a sharp asymptotic behavior of solution to the final value problem of this equation and time decay rate  $t^{-3/2}$  of this solution in  $L_x^\infty$ . In three-dimensional case, Hayashi, Mizumachi and Naumkin [2] showed the existence of unique small global solution to the initial value problem to (1.2) with  $\lambda_3 = 0$  which decays like  $t^{-3/2}$  in  $L_x^\infty$ , and proved that this solution has a free profile. In two space dimensions, when  $\lambda_3 = 0$ , the existence of wave operators to the equation (1.2) was shown in

[5, 7, 10] for small final states  $u_+$  satisfying  $u_+ \in H^{0,2} \cap \dot{H}^{-\delta}$  with suitable  $\delta > 0$ . Recently, in two-dimensional case, Hayashi and Naumkin [4] showed that when

$$\lambda_1 = \frac{\lambda_3}{2}, \quad \lambda_2 = \frac{\lambda_3^2}{2\lambda_3}, \quad (1.3)$$

for a given small final state  $u_+ \in H^{0,2} \cap H_1^2$  with  $\hat{u}_+(0) = 0$ , there exists a unique solution  $u$  to the equation (1.2) satisfying

$$\left\| u(t) - U(t)u_+ - i\lambda_3 \int_t^\infty |U(s)u_+|^2 ds \right\|_{H^2} = O(t^{-1}),$$

as  $t \rightarrow \infty$ . Because their method depends on special properties of their nonlinearity, that is, the condition (1.3), it seems that we can not apply their method to the general case directly.

According to the author's knowledge, there is no result on the large time behavior of solutions for the equation (1.2) with  $\lambda_1 = \lambda_2 = 0$  and  $\lambda_3 \in \mathbb{C} \setminus \{0\}$ , that is, the equation (1.1), in two space dimensions. It seems to be difficult to investigate it for the equation (1.1) in two space dimensions, because the nonlinearity  $\lambda|u|^2$  does not have an oscillation as the nonlinear terms  $u^2$  and  $\bar{u}^2$ . In this talk, we show the almost nonexistence of asymptotically free solutions for the equation (1.1), that is, the almost nonexistence of wave operators to this equation. More precisely, we show that if there exist a  $T \geq 0$ , a final state  $u_+ \in H^{0,2}$  and a solution  $u \in C([T, \infty); L^2)$  such that  $u$  approaches the free solution  $U(t)u_+$  in  $L^2$  as  $t \rightarrow \infty$ , then  $u_+$  is identically zero.

Before stating our main result, we introduce several notation.

**Notation.** For  $\psi \in \mathcal{S}'$ , we denote the Fourier transform of  $w$  by  $\hat{\psi}$ . For  $\psi \in L^1(\mathbb{R}^n)$ ,  $\hat{\psi}$  is represented as

$$\hat{\psi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \psi(x) e^{-ix \cdot \xi} dx.$$

For  $m, s \in \mathbb{R}$ , we introduce the weighted Sobolev spaces:

$$H^{m,s} = \{\psi \in \mathcal{S}' : \|\psi\|_{H^{m,s}} \equiv \|(1 + |x|^2)^{s/2} (1 - \Delta)^{m/2} \psi\|_{L^2} < \infty\}.$$

We denote  $H^{m,0}$  by  $H^m$ . We introduce the following propagator:

$$U(t) = e^{it\Delta/2}.$$

$B(a, R)$  denotes the closed ball in  $\mathbb{R}^2$  centered at  $a$  with radius  $R$ .

The main result is the following:

**Theorem 1.1** ([9]). *Assume that there exist a constant  $T \geq 0$ , a final state  $u_+ \in H^{0,2}$  and a solution  $u \in C([T, \infty); L_x^2)$  to the equation (1.1)*

satisfying

$$\|u(t) - U(t)u_+\|_{L_x^2} = O(t^{-\varepsilon}), \quad (1.4)$$

$$\left( \int_t^\infty \|u(s) - U(s)u_+\|_{L_x^4}^4 ds \right)^{1/4} = O(t^{-\varepsilon-1/4}), \quad (1.5)$$

as  $t \rightarrow \infty$ , for some  $\varepsilon > 0$ . Then  $u_+$  is identically zero.

**Remark 1.1.** We recall that the free solution  $U(t)\phi$  conserves the  $L^2$ -norm and decays as  $t^{-1/2}$  in  $L^4$  in two space dimensions. In particular, we see  $\|U(\cdot)\phi\|_{L^4((t,\infty);L_x^4)} = O(t^{-1/4})$ . The assumptions (1.4) and (1.5) in Theorem 1.1 mean that the solution  $u$  for the equation (1.1) approaches the free solution  $U(t)u_+$  slightly faster than the time decay rates of the free solution in  $L_x^2$  and  $L^4((t,\infty);L_x^4)$ , respectively.

**Remark 1.2.** The same result holds under the assumption

$$\|u(t)\|_{L_x^\infty} = O(t^{-1}), \quad t \rightarrow \infty \quad (1.6)$$

on the time decay rate of the solution  $u$  for the equation (1.1), instead of the assumption (1.5) on a convergence rate in the space-time with the admissible exponent of the Strichartz estimate for the two dimensional Schrödinger equation. The condition (1.6) means that the solution  $u$  for the equation (1.1) decays as fast as the free solution.

## 2. THE STRATEGY OF THE PROOF

We briefly explain the idea of the proof of Theorem 1.1. For the detailed proof, see [9]. We prove Theorem 1.1 by the contradiction argument. Let the assumptions in Theorem 1.1 be satisfied. We assume that  $\hat{u}_+$  is not identically zero, and we derive a contradiction. We set  $f(u) = \lambda|u|^2$  with  $\lambda \in \mathbb{C} \setminus \{0\}$ . Let  $t \geq 1$ . The final state  $u_+$  and the solution  $u$  in Theorem 1.1 satisfies the following integral equation:

$$u(t) - U(t)u_+ = i \int_t^\infty U(t-s)f(u(s)) ds.$$

We can rewrite integral equation as

$$u(t) - U(t)u_+ = I_{1,1}(t) + I_{1,2}(t) + I_2(t) + I_3(t),$$

where

$$\begin{aligned} I_{1,1}(t) &= i \int_t^\infty f(u_a(\tau)) d\tau, \\ I_{1,2}(t) &= \frac{1}{2} \int_t^\infty U(t-s) \left( \int_s^\infty \Delta f(u_a(\tau)) d\tau \right) ds, \\ I_2(t) &= i \int_t^\infty U(t-s)(f(U(s)u_+) - f(u_a(s))) ds, \\ I_3(t) &= i \int_t^\infty U(t-s)(f(u(s)) - f(U(s)u_+)) ds, \end{aligned}$$

$$u_a(t, x) = (U(t)e^{-i|\cdot|^2/2t}u_+)(x) = \frac{1}{it}\hat{u}_+\left(\frac{x}{t}\right)e^{i|x|^2/2t}.$$

(It is well-known that  $u_a$  is an asymptotics of the free solution  $U(t)u_+$ .)  
Therefore we have

$$\begin{aligned} & \|u(t) - U(t)u_+\|_{L^2} \\ & \geq \|I_{1,1}(t)\|_{L^2} - (\|I_{1,2}(t)\|_{L^2} + \|I_2(t)\|_{L^2} + \|I_3(t)\|_{L^2}). \end{aligned} \quad (2.1)$$

$I_{1,1}$  is a principal term in the right hand side in (2.1). We will estimate  $I_{1,1}$  by a strictly positive constant from below, and we will evaluate  $I_{1,2}$ ,  $I_2$  and  $I_3$  to show that they converge to zero in  $L^2$ . We write the principal part  $I_{1,1}$  as an integral which does not include the propagator  $U(t-s)$  in order to be able us to estimate the  $L^2$ -norm of it from below.

Since  $u_a$  is an asymptotics of the free solution  $U(t)u_+$  and  $u$  converges to  $U(t)u_+$  by the assumption, it is easy to see that  $\|I_2(t)\|_{L^2}$  and  $\|I_3(t)\|_{L^2}$  converge zero as  $t \rightarrow \infty$ . Because  $f(u_a(t)) = \lambda t^{-2}|\hat{u}_+(x/t)|^2$  does not oscillate,  $\Delta f(u_a(t))$  decays sufficiently, which implies  $\|I_{1,2}(t)\|_{L^2} \rightarrow 0$  as  $t \rightarrow \infty$ . We estimate  $I_{1,1}$  from below in the following way. Since we assumed that  $\hat{u}_+$  is not identically zero, there exists  $x_0 \in \mathbb{R}^2$  such that  $\hat{u}_+(x_0) \neq 0$ . Since  $\hat{u}_+ \in H^2$  and hence  $\hat{u}_+$  is continuous, there exist  $M > 0$  and  $r > 0$  such that  $|\hat{u}_+(x)| \geq M$  if  $x \in B(x_0, 2r)$ , where  $B(x_0, 2r)$  is a closed ball in  $\mathbb{R}^2$  centered at  $x_0$  with radius  $2r$ . We restrict the integral region over  $B(tx_0, tr)$  with respect to the space variable  $x$  and over  $(t, (1+r/|x_0|)t)$  with respect to the time variable  $\tau$  so that  $|\hat{u}_+(x/\tau)| \geq M$  in the integrand in  $\|I_{1,1}\|_{L^2}$ . Using this estimate, we can show that  $\|I_{1,1}(t)\|_{L^2}$  is bounded by a strictly positive constant  $K$  from below. As mentioned above, we can estimate  $\|I_{1,1}(t)\|_{L^2}$  from below because  $I_{1,1}$  is represented as an integral which does not include the propagator  $U(t-s)$ . Therefore we see that when  $t \rightarrow \infty$ , then the right hand side of (2.1) converges to the strictly positive constant  $K$ . On the other hand, by the assumption (1.4), the left hand side of (2.1) converges to zero as  $t \rightarrow \infty$ . This is a contradiction. Therefore  $u_+$  is identically zero.

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# Nonlinear Schrödinger equation with triple $\delta$ -functions as initial data

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## Abstract

We consider the Cauchy problem of the  $n$ -dimensional nonlinear Schrödinger equation with superposed  $\delta$ -functions as initial data. We treat this problem case by case, i.e., the cases in which the initial data consists of single, double and triple  $\delta$ -functions, respectively. In particular, when the initial data consists of double or triple  $\delta$ -functions, we observe that the generation of new modes appears in the expression of the solution, which is visible only in the nonlinear problem. As for the global existence in triple  $\delta$  case, the location of  $\delta$  functions causes some difficulty in the a priori estimate.

## 1 Introduction and Main Results

We consider the initial value problem of the nonlinear Schrödinger equation like

$$(1.1) \quad \begin{cases} i\partial_t u = -\Delta u + \lambda \mathcal{N}(u) \\ u(0, x) = (\text{the superposition of } \delta \text{ functions}) \end{cases}$$

where  $(t, x) \in \mathbf{R} \times \mathbf{R}^n$  ( $n \geq 1$ ),  $\partial_t = \partial/\partial t$  and  $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_n^2$ . The unknown variable  $u = u(t, x)$  takes a complex number. The nonlinearity  $\mathcal{N}(u)$  is of the gauge invariant power type given by

$$\mathcal{N}(u) = |u|^{p-1}u \quad \text{with } 1 < p < 1 + 2/n.$$

The nonlinear coefficient  $\lambda$  belongs to  $\mathbf{C}$  (the set of complex numbers). In particular, if  $\text{Im}\lambda < 0$ , the nonlinear term causes dissipative effect. We mainly treat this initial value problem by assuming that  $u(0, x) = \mu\delta_0$ ,  $u(0, x) = \mu_0\delta_0 + \mu_1\delta_a$  or  $u(0, x) = \mu_{00}\delta_0 + \mu_{10}\delta_a + \mu_{01}\delta_b$ , where  $\delta_a$  denotes the well-known point mass measure supported at  $x = a \in \mathbf{R}^n$  and  $\mu, \mu_k, \mu_{jk}$  ( $j, k = 0, 1$ ) are complex numbers.

From the physical point of view, the cubic nonlinearity (i.e.,  $p = 3$  which is excluded in our assumption for mathematical reason explained later) frequently appears. For example, (1.1) with  $\lambda \in \mathbf{R}$ ,  $n = 1$  and  $p = 3$  is said to govern the motion of vortex filament in the ideal fluid [10]. In fact, letting  $\kappa(t, x)$  be the curvature of the filament and  $\tau(t, x)$  the torsion, we observe that  $u(t, x) = \kappa(t, x) \exp(i \int_0^x \tau(t, y) dy)$  (which is called "Hasegawa transform" [10]) satisfies (1.1), where  $x$  stands for the position parameter along

the filament. Our assumption is slightly away from the cubic nonlinear case because of the mathematical reason given below. However, if one allows us to treat (1.1) as a fine approximation of the physically important case, one may imagine the time evolution of vortex filament with a locally bended singular initial state, e.g.,  $\kappa(0, x) = \delta_a$ .

The Cauchy problems with measures as initial data are extensively studied for various kinds of nonlinear evolution equations. As for the nonlinear parabolic equation, i.e.,  $\partial_t u - \Delta u + |u|^{p-1}u = 0$  with  $u(0, x) = \delta_0$ , Brezis-Friedman [2] specify the critical nonlinear power determining the solvability of the equation. They prove that, if  $1 + 2/n \leq p$ , there exists no solution continuously connected with the  $\delta$ -function at  $t = 0$  in the distribution sense and that, if  $1 < p < 1 + 2/n$ , it is possible to construct a solution with a general measure as initial data. Their argument relies on the comparison principle and smoothing property of the linear diffusion. For the KdV equation, Tsutsumi [23] constructs a solution by imposing measure on initial data. In his work, he makes use of Miura transformation [17] which reduces the original KdV equation into the modified one with cubic nonlinearity. Recently, Abe-Okazawa [1] have studied this kind of problem for the complex Ginzburg-Landau equation. The ideas to construct solutions in these known results are based on the strong smoothing effect of linear semi-group or the nonlinear transformation of unknown functions into the suitably handled equation. In the present case, however, the nonlinear Schrödinger equation have neither the useful smoothing properties like the heat equation nor the crucial transformation like Miura type, which makes it so difficult to study (1.1) with general measure as initial data. It is still open whether (NLS) is solvable when the initial data is arbitrary measure except for  $\delta$ -functions.

We here remark Kenig-Ponce-Vega's work [15]. They proved the ill-posedness of the nonlinear Schrödinger equation when  $u(0, x) = \delta_0$  and  $1 + 2/n \leq p$  — This is the main reason the physically important case is excluded. The situation is very similar to the nonlinear parabolic case introduced above. They proved that (1.1) possesses either no solution or more than one in  $C([0, T]; \mathcal{S}'(\mathbf{R}^n))$ , where  $\mathcal{S}'(\mathbf{R}^n)$  denotes the class of tempered distributions. In their work, the Galilean invariance plays an important role, where the Galilean invariance means the property that, if  $u(t, x)$  is a solution to (1.1), then  $u_N(t, x) = \exp(-itN^2) \exp(iNx \cdot e)u(t, x - 2tNe)$  also satisfies (1.1) where  $e$  denotes a unit vector in  $\mathbf{R}^n$ . Then, the obvious identity  $\delta_0 = e^{iNx \cdot e} \delta_0$  determines the concrete representation of  $u$  via  $u(t, x) = u_N(t, x)$  and the super critical power yields the divergence of the phase at  $t = 0$ . This rough sketch of their argument lets us expect that, for the subcritical case, it is possible to construct a solution continuous at  $t = 0$ .

There are large amount of articles concerning the local or global well-posedness for the nonlinear Schrödinger equations in the  $L^2(\mathbf{R}^n)$  or  $H^s(\mathbf{R}^n)$  ( $s > 0$ ) frame work (see [5, 6, 8, 11, 12, 13, 18, 19, 21, 22] and references therein). Roughly speaking, this is because these function spaces works so well via the conservation laws, energy estimates and Strichartz' estimates [20, 24]. On the other hand, since our present setting is away from these frame works, we employ another method to construct a solution. Our idea is based on the reduction of (1.1) into the ordinary differential equation (ODE) system by making use of the special structure of linear solution with  $\delta$ -function at  $t = 0$ . To see the detail, we state our main results case by case (The main concern in this talk is the triple  $\delta$  case, but, for reader's convenience, the single and double  $\delta$  cases are also given in this abstract).

**The case  $u(0, x) = \mu\delta_0$ .** Let us first consider very simple case in which the initial data consists of a single  $\delta$ -function. This case gives an explicit solution. Actually, the solution to (1.1) is given by

$$(1.2) \quad u(t, x) = A(t)U(t)\delta_0,$$

where  $U(t)\delta_0 = \exp(it\Delta)\delta_0 = (4\pi it)^{-n/2} \exp(i|x|^2/4t)$  and the modified amplitude  $A(t)$  is evaluated as

$$(1.3) \quad A(t) = \begin{cases} \mu \exp\left(\frac{2\lambda|\mu|^{p-1}}{i(n+2-np)}|4\pi t|^{-n(p-1)/2}t\right) & \text{if } \text{Im}\lambda = 0, \\ \mu \left(1 - \frac{2(p-1)\text{Im}\lambda|\mu|^{p-1}}{n+2-np}|4\pi t|^{-n(p-1)/2}t\right)^{\frac{i\lambda}{(p-1)\text{Im}\lambda}} & \text{if } \text{Im}\lambda \neq 0. \end{cases}$$

In fact, by substituting (1.2) into (1.1), we have the ODE of  $A(t)$  :

$$(1.4) \quad \begin{cases} i\frac{dA}{dt} = \lambda|4\pi t|^{-(p-1)/2}\mathcal{N}(A), \\ A(0) = \mu_0. \end{cases}$$

To solve (1.4), we first multiply  $\overline{A(t)}$  on both hand sides. Then, it is easy to see that  $\frac{d}{dt}|A|^2 = 2|4\pi t|^{-n(p-1)/2}\text{Im}\lambda|A|^{p+1}$  and so

$$(1.5) \quad |A(t)| = \left(|\mu|^{-(p-1)} - (p-1)\text{Im}\lambda \int_0^t |4\pi\tau|^{-n(p-1)/2} d\tau\right)^{-1/(p-1)}.$$

The integral in the parenthesis of (1.5) makes a sense since  $p < 1 + 2/n$ . Substituting (1.5) into (1.4) and solving the simple ODE, we obtain the explicit formula (1.3). Note that  $\text{Im}\lambda > 0$  implies blowing-up of  $A(t)$  in positive finite time.

**The case  $u(0, x) = \mu_0\delta_0 + \mu_1\delta_a$ .** Our next concern is to consider the slightly complicated case in which the initial data consists of double  $\delta$ -functions. In this case, we observe that the superposition of  $\delta$ -functions causes "the mode generation" at  $t \neq 0$ . Before stating our main results, let us present several notations. Let  $\ell_\alpha^2$  be the weighted sequence space defined by

$$\ell_\alpha^2 = \left\{ \{A_k\}_{k \in \mathbf{Z}}; \|\{A_k\}_{k \in \mathbf{Z}}\|_{\ell_\alpha^2}^2 = \sum_{k \in \mathbf{Z}} (1 + |k|)^{2\alpha} |A_k|^2 < \infty \right\}.$$

For the simplicity of description, we often use  $\{A_k\}$  in place of  $\{A_k\}_{k \in \mathbf{Z}}$ . Then our first result is

**Theorem 1.1 (local result)** *Let  $\lambda \in \mathbf{C}$ . Then, for some  $T > 0$ , there exists a unique solution to (1.1) described as*

$$(1.6) \quad u(t, x) = \sum_{j \in \mathbf{Z}} A_j(t)U(t)\delta_{ja},$$

where  $\{A_j(t)\} \in C([0, T]; \ell_1^2) \cap C^1((0, T]; \ell_1^2)$  with  $A_j(0) = \mu_j$  if  $j = 0, 1$  and  $A_j(0) = 0$  otherwise.

**Remark 1.1.** Let us call  $A_k(t)U(t)\delta_{ka}$  the  $k$ -th mode at  $t$ . Then, (1.6) suggests that new modes away from 0-th and first ones appear in the solution though the initial data contains only the two modes. This special property is visible only in the nonlinear problem.

**Remark 1.2.** Reading the proof of Theorem 1.1, we see that it is possible to generalize the initial data. Namely, (1.1) is solvable even when point masses are distributed on a line at equal intervals, i.e.,  $u(0, x) = \sum_{j \in \mathbf{Z}} \mu_j \delta_{ja}$ , where  $\{\mu_k\} \in \ell_1^2$ . In this generalized case, the solution is described similarly to (1.6) but  $A_j(0) = \mu_j$  for  $j \in \mathbf{Z}$ . The decay condition on the coefficients  $\{\mu_j\}$  is required to estimate the nonlinearity.

**Remark 1.3.** The infinite summation of (1.6) converges in  $L_{loc}^\infty((0, T]; L^\infty(\mathbf{R}^n))$ , since, for any  $\tau \in (0, T)$ ,

$$\begin{aligned} \sup_{\tau \leq t \leq T} \|u(t, \cdot)\|_{L^\infty(\mathbf{R}^n)} &\leq (4\pi\tau)^{-n/2} \sup_{\tau \leq t \leq T} \sum_j |A_j(t)| \\ &\leq C(4\pi\tau)^{-1/2} \|\{A_j(t)\}\|_{L^\infty([0, T]; \ell_1^2)} \\ &< \infty. \end{aligned}$$

This implies that the nonlinearity  $\mathcal{N}(u(t, x))$  makes a sense as a function for almost every  $t \in (0, T]$ . We also note that  $u(t, x) \in C([0, T]; \mathcal{S}'(\mathbf{R}^n))$ .

**Remark 1.4.** The representation (1.6) is heuristically derived by the following rough consideration. Since the nonlinear solution is firstly well-approximated by the linear solution  $u_1(t, x) = U(t)(\mu_0\delta_0 + \mu_1\delta_a)$  around  $t = 0$ , the second approximation  $u_2(t, x)$  is given by solving

$$\begin{aligned} (1.7) \quad (i\partial_t + \Delta)u_2 &= \mathcal{N}(u_1) \\ &= \mathcal{N}((2\pi)^{-n/2} e^{i|x|^2/4t} D(\mu_0 + \mu_1 e^{-ia \cdot x} e^{i|a|^2/4t})) \\ &= |4\pi t|^{-n(p-1)/2} (2\pi)^{-n/2} e^{i|x|^2/4t} D\mathcal{N}(\mu_0 + \mu_1 e^{-ia \cdot x} e^{i|a|^2/4t}), \end{aligned}$$

where we have used  $u_1 = e^{i|x|^2/4t} D\mathcal{F}e^{i|x|^2/4t}u(0, x)$ ,  $Df(t, x) = (2it)^{-n/2}f(t, x/2t)$  and  $\mathcal{F}$  denotes the Fourier transform. Let us replace  $a \cdot x$  by  $\theta$ . Then,  $\mathcal{N}(\mu_0 + \mu_1 e^{-i\theta} e^{i|a|^2/4t})$  in (1.7) is regarded as a  $2\pi$ -periodic function of  $\theta$ , and hence the Fourier series expansion yields

$$\begin{aligned} &(\text{the right hand side of (1.7)}) \\ &= |4\pi t|^{-n(p-1)/2} (2\pi)^{-n/2} e^{i|x|^2/4t} D \sum_{j \in \mathbf{Z}} B_j(t) e^{i|ja|^2/4t} e^{-ij\theta} \\ &= |4\pi t|^{-n(p-1)/2} \sum_{j \in \mathbf{Z}} B_j(t) U(t) \delta_{ja}, \end{aligned}$$

where  $B_j(t)e^{i|ja|^2/4t}$  plays a roll of the Fourier coefficient. Accordingly, the Duhamel principle leads us to the description (1.6).

Our next interest is to see the global solvability of (1.1). The sign of  $\text{Im}\lambda$  determines the blowing-up or global existence.

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**Theorem 1.2 (blowing-up result)** *Let  $\text{Im}\lambda > 0$ . Then, the solution in Theorem 1.1 blows up in positive finite time. Precisely speaking,  $\lim_{t \uparrow T^*} \|\{A_j(t)\}\|_{\ell_0^2} = \infty$  for some  $T^* > 0$ .*

**Theorem 1.3 (global result)** *Let  $\text{Im}\lambda \leq 0$ . Then, there exists a unique global solution to (1.1) described as in Theorem 1.1, where  $\{A_j(t)\} \in C([0, \infty); \ell_1^2) \cap C^1((0, \infty); \ell_1^2)$ .*

**The case  $u(0, x) = \mu_{00}\delta_0 + \mu_{10}\delta_a + \mu_{01}\delta_b$  ( $a \neq qb$  for any  $q \in \mathbf{Q}$ ).** We next consider the case in which the initial data consists of triple  $\delta$ -functions supported at  $x = 0, a$  and  $b$ . If  $a = qb$  for some  $q \in \mathbf{Q}$  ( $\mathbf{Q}$  denotes the quotient number field), the location of  $\delta$ -functions is the special one mentioned in Remark 1.2 and thus (1.1) is solved as in Theorem 1.1–1.3. Therefore, our concern is to observe the case  $a \neq qb$  for any  $q \in \mathbf{Q}$ . Before stating our next main results, we introduce several new notations. The weighted sequence space  $\ell_\alpha^2(\mathbf{Z}^2)$  is defined by

$$\ell_\alpha^2(\mathbf{Z}^2) = \{\{A_{jk}\}_{j,k \in \mathbf{Z}}; \|\{A_{jk}\}_{j,k \in \mathbf{Z}}\|_{\ell_\alpha^2(\mathbf{Z}^2)} < \infty\},$$

where  $\|\{A_{jk}\}_{j,k \in \mathbf{Z}}\|_{\ell_\alpha^2(\mathbf{Z}^2)}^2 = \sum_{j,k \in \mathbf{Z}} (1 + |j| + |k|)^{2\alpha} |A_{jk}|^2$ . For the simplicity of description, we often use  $\{A_{jk}\}$  in place of  $\{A_{jk}\}_{j,k \in \mathbf{Z}}$ . Then our next result is

**Theorem 1.4 (local result)** *Let  $\lambda \in \mathbf{C}$  and  $1 < \alpha < p$ . Then, for some  $T > 0$ , there exists a unique solution to (1.1) described as*

$$(1.8) \quad u(t, x) = \sum_{j,k \in \mathbf{Z}} A_{jk}(t) U(t) \delta_{ja+kb},$$

where  $\{A_{jk}(t)\} \in C([0, T]; \ell_\alpha^2(\mathbf{Z}^2)) \cap C^1((0, T]; \ell_\alpha^2(\mathbf{Z}^2))$  with  $A_{jk}(0) = \mu_{jk}$  if  $(j, k) = (0, 0), (1, 0), (0, 1)$  and  $A_{jk}(0) = 0$  otherwise.

**Remark 1.5.** As mentioned in Remark 1.1, the solution in Theorem 1.4 also causes the generation of new modes. The point remarkably different from Theorem 1.1 is that, for  $t \neq 0$ ,  $U(-t)u$  looks like the point mass measures supported at the lattice points if  $a \not\parallel b$ , and densely distributed on the line along vector  $a$  if  $a \parallel b$  and  $a \neq qb$  for any  $q \in \mathbf{Q}$ . Reading the proof of Theorem 1.4, we see that it is possible to construct a solution even when the initial data consists of infinitely many  $\delta$ -functions such as  $u(0, x) = \sum_{j,k \in \mathbf{Z}} \mu_{jk} \delta_{ja+kb}$  with  $\{\mu_{jk}\} \in \ell_\alpha^2(\mathbf{Z}^2)$  and  $\alpha > 1$ .

Similarly to Theorem 1.2 and 1.3, the sign of  $\text{Im}\lambda$  determines the blowing-up or global existence of the solution.

**Theorem 1.5 (blowing-up result)** *Let  $\text{Im}\lambda > 0$ . Then, the solution in Theorem 1.4 blows up in positive finite time. Precisely speaking,  $\lim_{t \uparrow T^*} \|\{A_{jk}(t)\}\|_{\ell_0^2(\mathbf{Z}^2)} = \infty$  for some  $T^* > 0$ .*

Note that the difficulty concerning the global existence largely depends on whether  $a$  and  $b$  are parallel or not.

**Theorem 1.6 (global result)** (1) *Let  $a \not\parallel b$ . Then, if  $\text{Im}\lambda \leq 0$ , there exists a unique global solution to (1.1) described as in Theorem 1.4, where  $\{A_{jk}(t)\} \in C([0, \infty); \ell_\alpha^2(\mathbf{Z}^2)) \cap C^1((0, \infty); \ell_\alpha^2(\mathbf{Z}^2))$ .*

(2) *Let  $a \parallel b$  and  $a \neq qb$  for any  $q \in \mathbf{Q}$ . Then, if  $\text{Im}\lambda \leq 0$  and additionally  $|\text{Re}\lambda| \leq \frac{2\sqrt{p}}{p-1}|\text{Im}\lambda|$ , there exists a unique global solution to (1.1) described as in Theorem 1.4, where  $\{A_{jk}(t)\} \in C([0, \infty); \ell_\alpha^2(\mathbf{Z}^2)) \cap C^1((0, \infty); \ell_\alpha^2(\mathbf{Z}^2))$ .*

**Remark 1.6.** When  $a \not\parallel b$ , the important matter is the equivalence of  $\|\{(ja+kb)A_{jk}\}\|_{\ell_0^2(\mathbf{Z}^2)}$  and  $\|\{jA_{jk}\}\|_{\ell_0^2(\mathbf{Z}^2)} + \|\{kA_{jk}\}\|_{\ell_0^2(\mathbf{Z}^2)}$ , which makes the proof of Theorem 1.6 (1) quite similar to that of Theorem 1.3. However, this is not the case if  $a \parallel b$ . As for the global result in Theorem 1.6 (2), it is still open whether the additional condition  $|\text{Re}\lambda| \leq \frac{2\sqrt{p}}{p-1}|\text{Im}\lambda|$  is removed or not. In our proof, this condition will be applied to obtain the time global estimate of  $\|\{A_{jk}(t)\}\|_{\ell_1^2(\mathbf{Z}^2)}$ . The key to derive this estimate is Liskevich-Perelmuter's inequality [16], i.e., if  $\text{Im}\lambda \leq 0$  and  $|\text{Re}\lambda| \leq \frac{2\sqrt{p}}{p-1}|\text{Im}\lambda|$ , then it follows that  $\text{Im}(\lambda(\mathcal{N}(v_1) - \mathcal{N}(v_2))\overline{(v_1 - v_2)}) \leq 0$ .

We close this abstract by giving some more notations used in this talk. Let  $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$  where  $\mathbf{Z}$  stands for the integer set. The Lebesgue space  $L^q(\mathbf{T})$  denotes the class of measurable functions on  $\mathbf{T}$  with  $\|f\|_{L^q(\mathbf{T})}^q \equiv \int_0^{2\pi} |f(\theta)|^q d\theta < \infty$  if  $1 < q < \infty$  and  $\|f\|_{L^\infty(\mathbf{T})} \equiv \text{ess. sup}_{\theta \in \mathbf{T}} |f(\theta)| < \infty$  if  $q = \infty$ . The Sobolev space  $H^s(\mathbf{T})$  is defined by  $H^s(\mathbf{T}) = \{f(\theta) \in L^2(\mathbf{T}); \|f\|_{H^s(\mathbf{T})} < \infty\}$ , where  $\|f\|_{H^s(\mathbf{T})}^2 = \sum_{j \in \mathbf{Z}} (1 + |j|)^{2s} |C_j|^2$  with  $C_j = (2\pi)^{-1} \int_0^{2\pi} f(\theta) e^{-ij\theta} d\theta$ . We often use  $L^q$  (resp.  $H^s$ ) in place of  $L^q(\mathbf{T})$  (resp.  $H^s(\mathbf{T})$ ) if there is no risk of misleading. Also, the inner product of  $f(\theta)$  and  $g(\theta) \in L^2$  is defined by  $\langle f, g \rangle_\theta = \int_0^{2\pi} f(\theta) \overline{g(\theta)} d\theta$ .

The quantity  $\|f\|_{L^q(\mathbf{T}^2)}$  denotes  $\left(\int_{\mathbf{T}^2} |f(\theta_1, \theta_2)|^q d\theta_1 d\theta_2\right)^{1/q}$ . We next define the Besov space for periodic functions. Let  $[s]$  be the greatest integer not exceeding  $s$ . Then, if  $s$  is not integer and  $1 < q, r < \infty$ , the Besov space  $B_{q,r}^s(\mathbf{T}^2)$  is defined by

$$B_{q,r}^s(\mathbf{T}^2) = \{f \in L^q(\mathbf{T}^2); \|f\|_{B_{q,r}^s(\mathbf{T}^2)} < \infty\},$$

where

$$\|f\|_{B_{q,r}^s(\mathbf{T}^2)} \equiv \|f\|_{L^q(\mathbf{T}^2)} + \|f\|_{\dot{B}_{q,r}^s}$$

$$\equiv \|f\|_{L^q(\mathbf{T}^2)} + \left( \int_0^\infty \tau^{-rs-1} \sup_{|h|<\tau} \|d_h^{[s]+1} f\|_{L^q(\mathbf{T}^2)}^r d\tau \right)^{1/r}$$

with  $h = (h_1, h_2)$  and  $d_h^N f(\theta_1, \theta_2) = \sum_{j=0}^N \binom{N}{j} (-1)^k f(\theta_1 + jh_1, \theta_2 + jh_2)$ . We remark that, if  $0 \leq \sigma \leq 1$  and  $1/q = \sigma/q_1 + (1 - \sigma)/q_0$  with  $1 \leq q_1, q_0 \leq \infty$ , then the Gagliardo-Nirenberg type inequality  $\|f\|_{\dot{B}_{q,r/\sigma}^{\sigma s}(\mathbf{T}^2)} \leq C \|f\|_{\dot{B}_{q_1,r}^{\sigma}(\mathbf{T}^2)}^\sigma \|f\|_{L^{q_0}(\mathbf{T}^2)}^{1-\sigma}$  follows from the above definition. We also note that  $\|f\|_{B_{2,2}^s(\mathbf{T}^2)}$  is equivalent to

$$\|f\|_{H^s(\mathbf{T}^2)} \equiv \left( \sum_{j,k \in \mathbf{Z}} (1 + |j| + |k|)^{2s} |C_{jk}|^2 \right)^{1/2},$$

where  $C_{jk}$  is the Fourier coefficient of  $f$  given by  $(2\pi)^{-2} \int_{\mathbf{T}^2} f(\theta_1, \theta_2) e^{-i(j\theta_1 + k\theta_2)} d\theta_1 d\theta_2$ . For more detail about Besov space, see [4]. Also, the inner product of  $f(\theta_1, \theta_2)$  and  $g(\theta_1, \theta_2) \in L^2(\mathbf{T}^2)$  is defined by  $\langle f, g \rangle_{\theta_1, \theta_2} = \int_{\mathbf{T}^2} f(\theta_1, \theta_2) \overline{g(\theta_1, \theta_2)} d\theta_1 d\theta_2$ .

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# GLOBAL EXISTENCE OF SCHRÖDINGER MAPS WITH SMALL DATA

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ABSTRACT. The global existence of Schrödinger maps with small initial data is proved in dimension  $n \geq 3$ . The uniqueness and behaviour of the solution as  $t \rightarrow \infty$  is obtained.

## 1. Introduction

Consider the Schrödinger map

$$U(t, x) = (U_1(t, x), U_2(t, x), U_3(t, x)) : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow S^2$$

$$(1.1) \quad \partial_t U = U \times \Delta U, \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^n.$$

By the transformation

$$(1.2) \quad \begin{aligned} u(t, x) &:= \mathbb{P}U := \frac{U_1(t, x) + iU_2(t, x)}{1 + U_3(t, x)}, \\ U_1(t, x) &= \frac{2\operatorname{Re}u(t, x)}{1 + |u(t, x)|^2} \\ U_2(t, x) &= \frac{2\operatorname{Im}u(t, x)}{1 + |u(t, x)|^2} \\ U_3(t, x) &= \frac{1 - |u(t, x)|^2}{1 + |u(t, x)|^2}. \end{aligned}$$

(1.1) is equivalent to

$$(1.3) \quad \frac{i}{2}\partial_t u + \frac{1}{2}\Delta u = \frac{\bar{u} \sum_{j=1}^n (\partial_{x_j} u)^2}{1 + |u|^2}, \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^n.$$

Changing  $t$  to  $2t$ , instead of (1.3) we consider

$$(1.4) \quad i\partial_t u + \frac{1}{2}\Delta u = \frac{\bar{u} \sum_{j=1}^n (\partial_{x_j} u)^2}{1 + |u|^2}, \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^n.$$

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This is a joint work with Prof. Tohru Ozawa.

**Definition 1.1.** A function  $f$  is called  $\Phi$ -type, if for some  $\alpha_j \in \mathbb{N}$  ( $j = 1, 2$ ) and  $\beta_j \in \mathbb{N}$  ( $j = 1, 2, 3$ ),

$$\begin{aligned} f(u) &= u^{\alpha_1} \bar{u}^{\alpha_2} \left( \sum_{j=1}^n |\partial_{x_j} u|^2 \right)^{\beta_1} \left( \sum_{j=1}^n (\partial_{x_j} u)^2 \right)^{\beta_2} \left( \sum_{j=1}^n (\partial_{x_j} \bar{u})^2 \right)^{\beta_3} \\ &= \Phi(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3)(u). \end{aligned}$$

Let

$$(1.5) \quad \begin{aligned} F(u) &= \frac{1}{1 + |u|^2} F_1(u), \\ F_1(u) &= \Phi(0, 1, 0, 1, 0)(u) = \bar{u}(\partial_r u)^2. \end{aligned}$$

Then  $F_1$  is a  $\Phi$ -type function, and (1.4) equals to

$$(1.6) \quad i\partial_t u + \frac{1}{2}\Delta u = F(u), \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^n.$$

The Schrödinger map (1.1) is considered as the continuous limit of the classical model for an isotropic ferromagnet provided by a collection of 3-dimensional spin vectors with unit length and arbitrary directions. The local existence of regular solutions and global existence of weak solutions of (1.1) were proved in [SSB]. For the Schrödinger maps from a Riemannian manifold to a complete Kähler manifold, the local existence was obtained in [DW]. On the other hand, the local existence and (or) global existence for small data to the Schrödinger equations with the generalized derivative nonlinear terms were considered in [C], [H], [HO], [HMN] and [K], where the smoothing operator method were developed and applied to their proofs. In this paper, we shall consider the possible improvement for the special nonlinear term of the equation (1.4) and apply to the Schrödinger map (1.1).

## 2. Main theorem

**Theorem 2.1.** Let  $n \geq 3$  and  $g \in B^{2m}$  with  $2m \geq n + 3$  and  $g$  be radially symmetric. Then there exists a unique global solution

$$(2.1) \quad u \in C(\mathbb{R}; H^{2m-1}) \cap L^\infty(\mathbb{R}; H^{2m}), \quad \sup_{t \in \mathbb{R}} \|u(t)\|_{X_t^{2m}} \leq \infty$$

to (1.4) with

$$(2.2) \quad u(0, x) = \epsilon g(|x|), \quad \forall x \in \mathbb{R}^n,$$

provided that  $\epsilon$  is small enough.

**Definition 2.2.**

$$\begin{aligned}
& \|\phi(t)\|_{X_t^{2m}} \\
& := \|\phi(t)\|_{Y_t^{2m}} + \|Q\phi(t)\|_{Y_t^{2(m-1)}} + (1+|t|)^{-\left(\frac{4-n}{2}\right)_+} \|Q^2\phi(t)\|_{Y_t^{2(m-2)_+}}; \\
& \|\phi(t)\|_{Y_t^{2m}} = \sum_{0 \leq l \leq m} \|P^l \phi(t)\|_{H^{2(m-l)}}; \\
& B^{2m} = \{g \in L^2(\mathbb{R}^n) : \|g\|_{B^{2m}} = \sum_{0 \leq l \leq m} \|(r\partial_r)^l g\|_{H^{2(m-l)}} < \infty\}. \\
& Z^m = \{\phi \in C(\mathbb{R}, L^2(\mathbb{R}^n)) : \|\phi\|_{Z^m} \leq \infty\}, \\
& \|\phi\|_{Z^m}^2 \\
& := \sup_t (\|\Gamma^2 \phi\|_{H^{m-4}}^2 + \|\Gamma\Theta\phi\|_{H^{m-4}}^2 + \|\Theta\phi\|_{H^{m-2}}^2 + \langle t \rangle^{-1} \|Q^2\phi\|_{H^{m-4}}^2 \\
& + \sum_{1 \leq j, k \leq n} \int_0^t \|\omega_k S_k |\partial_k|^{1/2} \Gamma^2 \partial_j^{m-4} \phi(\tau)\|^2 \frac{d\tau}{\langle \tau \rangle^{\frac{1}{2} + 2\sigma}}); \\
& O^m = \{\phi : \|\phi\|_{O^m} = \sum_{|\alpha|+|\beta| \leq 2} \|\Omega^\alpha (x \cdot \nabla)^\beta \phi\|_{H^{m-2(|\alpha|+|\beta|)}} < \infty\}.
\end{aligned}$$

Here  $\langle t \rangle := (1+t^2)^{1/2}$  and

$$\begin{aligned}
P &= x \cdot \nabla + 2t\partial_t, \quad Q = x \cdot \nabla + it\Delta, \\
J &= x + it\nabla, \quad \Omega = \{\Omega_{j,k}\}_{1 \leq j, k \leq n}, \quad \Omega_{j,k} = x_j \partial_k - \partial_k(x_j \cdot), \\
\Gamma &= (P, \Omega, \Delta, 1) \\
\Theta &= (Q, \Omega, \Delta, 1).
\end{aligned}$$

Notice that

$$\begin{aligned}
[Q, \nabla] &= [P, \nabla] = -\nabla, \quad [Q, J] = [P, J] = J, \\
[P, Q] &= [\Omega, P] = [\Omega, Q] = 0, \\
[\partial_j, J_k] &= \delta_{jk}, \quad [\Omega_{jk}, \partial_l] = \delta_{kl} \partial_j - \delta_{jl} \partial_k.
\end{aligned}$$

In general case, we can prove the following result.

**Theorem 2.3.** *Suppose  $n \geq 3$  and  $g \in O^m$  with  $m = \lfloor \frac{n}{2} \rfloor + 3$ . If  $\epsilon$  is small, then there is unique global solution  $u \in Z^m$  to (1.4) with the Cauchy data  $u(0, x) = \epsilon g(x)$ . Moreover, there exists  $u_+ \in H^m$  such that*

$$\|u(t) - (2\pi it)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{i\frac{|x-y|^2}{2t}} u_+(y) dy\|_{L^2(\mathbb{R}^n)} \leq C \langle t \rangle^{-\frac{1}{4}}.$$

The direct corollary of theorem 2.3 is

**Theorem 2.4.** *Suppose  $n \geq 3$  and  $\mathbb{P}U_0 \in O^m$  with  $m = [\frac{n}{2}] + 3$ . If  $\|\mathbb{P}U_0\|_{O^m}$  is small, then there is a unique global Schrödinger map  $U$  ( $\mathbb{P}U \in Z^m$ ) to (1.1) with the Cauchy data*

$$U(0, x) = U_0(x), \quad \text{on } \mathbb{R}^n.$$

Moreover,

$$\lim_{t \rightarrow \infty} \|\mathbb{P}U(t) - (2\pi it)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{i\frac{|x-y|^2}{2t}} u_+(y) dy\|_{L^2(\mathbb{R}^n)} = 0.$$

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