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Large Time Behavior of Solutions to the Viscous Conservation Law with Dispersion

(分散効果を伴う粘性保存則方程式の解の長時間挙動)

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Contents

Chapter 1

Introduction

1.1 Viscous Conservation Law with Dispersion

The initial value problem for the viscous conservation law is one of the nonlinear model for complicated systems governing viscous compressible flow in fluid and gas dynamics (cf. [2, 38, 39, 45, 61]):

$$\begin{aligned} u_t + \sum_{i=1}^{n} (f_i(u))_{x_i} &= \sum_{i,j=1}^{n} \mu_{ij} u_{x_i x_j}, \quad x \in \mathbb{R}^n, \ t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.1.1}$$

where $u(x,t)$ denotes scalar unknown function for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$, and $f(u) = (f_1(u), \cdots, f_n(u))$ is given smooth flux vector function. The coefficients μ_{ij} are positive constants for all integers $1 \leq i, j \leq n$. The subscripts x_i and t denote the partial derivatives with respect to x_i and t, respectively. The terms $\mu_{ij} u_{x_i x_j}$ are called the viscosity terms which describe the dissipation effect and lead to the decay of solutions. To understand the fluid and gas dynamics theoretically, it is very important to construct the mathematical theory for (1.1.1). Especially, in this study, we are interested in the large time behavior of the solutions to (1.1.1). Concerning the large time behavior in several space dimensional case, we refer to the papers [12, 13, 24, 36, 56, 57, 65]. However, it is very difficult to understand the detailed asymptotic profiles of the solutions to (1.1.1) in high space dimensions due to the nonlinear convection terms $(f_i(u))_{x_i}$. To discuss the detailed large time behavior, let us consider (1.1.1) in the one-dimensional case:

$$\begin{aligned} u_t + (f(u))_x &= \mu u_{xx}, \quad x \in \mathbb{R}, \ t > 0, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.1.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given smooth flux function, and $\mu > 0$. Concerning this problem, the solutions to (1.1.2) take several types of asymptotic profiles, depending on the far field states of the initial data. Indeed, if we assume

$$\lim_{x \to \pm\infty} u_0(x) = u_\pm,$$

then the different asymptotic profiles can be obtained in the following three cases:

$$\text{(i) } u_+ = u_-, \quad \text{(ii) } u_+ < u_-, \quad \text{(iii) } u_+ > u_-.$$

Roughly speaking, in the case of (i), it is known that the solution goes to a self-similar solution of the Burgers equation called a nonlinear diffusion wave. This case will be explained in details, later on. In case (ii), the solution converges to the traveling wave solution called viscous shock wave. There are a lot of works related to the traveling wave solution (cf. [6, 11, 19, 20, 23, 26, 34, 35, 41, 42, 48, 53, 55, 56, 62]). On the other hand, in the case of (iii), the asymptotic stability of the rarefaction wave which is a weak solution of the inviscid Burgers equation have been studied (cf. [15, 23, 24, 43, 44, 46, 49, 57]).

In this thesis, we study the large time behavior of solutions toward a nonlinear diffusion wave, i.e., we focus on the case of (i). Especially, we consider the viscous conservation law (1.1.2) with dispersion term of the following form:

$$\begin{aligned} u_t + (f(u))_x + k u_{xxx} = \mu u_{xx}, \quad x \in \mathbb{R}, \ t > 0, \\ u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.1.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given smooth flux function, $k \in \mathbb{R}$ and $\mu > 0$. The equation (1.1.3) is called the generalized Korteweg-de Vries-Burgers equation (we call it generalized KdV-Burgers equation for short). This is a generalized KdV equation with the viscosity term, as well as a viscous conservation law with the term $k u_{xxx}$ describing a dispersive perturbation. Therefore, it can be regarded not only as the equation of viscous compressible flow but also as a general nonlinear model for taking into account the dispersive and dissipative processes as well as the convection effect. Hence, it is applicable in various fields of physics and engineering, see e.g. [29, 30, 51] and also references therein. For this equation, there are several results corresponding to the large time behavior mentioned in the above paragraph (see e.g. [1, 5, 7, 8, 17, 27, 29, 30, 54, 59, 60, 66]). By analyzing the large time behavior of the solutions to this equation, we would like to understand the effects of the nonlinear term and the dispersion term on the solutions.

The first topic of this thesis is due to Fukuda [7]. We shall introduce the result briefly. Firstly, we consider (1.1.3) with the the small initial data decaying at spatial infinity (i.e., $u_+ = u_- = 0$ in (i)) and satisfying $u_0 \in L^1(\mathbb{R})$. In the following, we set the flux function $f(u) = (b/2)u^2 + (c/3)u^3$ with $b \neq 0$ and $c \in \mathbb{R}$. For simplicity, we set $\mu = 1$ in (1.1.3). When $k = 0$, (1.1.3) becomes (1.1.2). In this case, it is known that the solution of this problem converges to a nonlinear diffusion wave which is essentially the self-similar solution of the Burgers equation

$$\chi_t + \left(\frac{b}{2}\chi^2\right)_x = \chi_{xx}, \quad \int_{\mathbb{R}} \chi(x, 0) dx = \delta$$

and is defined by

$$\chi(x, t) \equiv \frac{1}{\sqrt{1+t}} \chi_* \left(\frac{x}{\sqrt{1+t}}\right), \quad x \in \mathbb{R}, \ t > 0,$$

where

$$\chi_*(x) \equiv \frac{1}{b} \frac{(e^{b\delta/2} - 1)e^{-x^2/4}}{\sqrt{\pi} + (e^{b\delta/2} - 1)\int_{x/2}^{\infty} e^{-y^2} dy}, \quad \delta \equiv \int_{\mathbb{R}} u_0(x) dx$$

(cf. [33, 39, 45, 52]). Especially, Matsumura and Nishihara [45] showed that the original solution tends to $\chi(x,t)$ at the rate of $t^{-1}\log t$ in the L^∞-sense, under suitable assumptions. Moreover, Kato [31] proved that the above asymptotic rate is optimal, by constructing the second asymptotic profile under the assumption $u_0 \in L^1_1(\mathbb{R}) \cap H^1(\mathbb{R})$. Here, we define $L^1_1(\mathbb{R})$ is a subset of $L^1(\mathbb{R})$ whose elements satisfy $\int_{\mathbb{R}}(1+|x|)|u_0(x)|dx < \infty$. On the other hand, in the case of $b = k = 1$ and $c = 0$ in (1.1.3) (KdV-Burgers equation), the similar result to Kato [31] was also obtained by Kaikina and Ruiz-Paredes in [27]. For the general case, in [7] the author considered (1.1.3) for all $b, c, k \in \mathbb{R}$ with $b \neq 0$ and obtained a result about the large time behavior of the solution of (1.1.3) under the assumption $u_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$, which unifies the results due to Kato [31] and Kaikina and Ruiz-Paredes [27]. Indeed, if $\delta \neq 0$ and $(b^2k)/8 + c/3 \neq 0$, then the following estimate holds:

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} = (\tilde{C} + o(1))(1+t)^{-1}\log(1+t) \quad as \quad t \to \infty,$$

where $\tilde{C}$ is a certain positive constant. In Chapter 2, we also construct the second asymptotic profile of the solution. For details, see Theorem 2.2.1 and Corollary 2.2.2 below.

The second topic of this thesis is due to Fukuda [8]. We consider (1.1.3) with the same flux function and the following condition on the initial data:

$$\exists \alpha > 1, \quad \exists C > 0 \quad s.t. \quad |u_0(x)| \le C(1+|x|)^{-\alpha}, \quad x \in \mathbb{R}. \tag{1.1.4}$$

In the above paragraph, we considered (1.1.3) under the assumption $u_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$. The above result tells us that the optimal asymptotic rate to the nonlinear diffusion wave is $t^{-1}\log t$ under the above assumptions. Here, we note that the condition $u_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$ is realized when $u_0 \in H^3(\mathbb{R})$ and $\alpha > 2$ in (1.1.4). Therefore, we can regard that the above previous result is corresponding to the case where the decay rate of the initial data u_0 is rapid. In Chapter 3, we are interested in the large time behavior of the solution to (1.1.3) in the case that the initial data decays more slowly. To the best of our knowledge, there were no results about the large time behavior for (1.1.3) with slowly decaying data. Because of this, we would like to study the asymptotic profile for the solution to (1.1.3) when $1 < \alpha \le 2$ in (1.1.4). Indeed, it is natural to assume that $u_0 \in L^1(\mathbb{R})$ which is indicated by (1.1.4), because the equation (1.1.3) leads to the conservation law: $\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx$. For this reason, we restrict our attention to the case of $1 < \alpha \le 2$. Actually, if we assume $u_0 \in H^3(\mathbb{R})$, applying the method used in [4, 18, 37, 45], we obtain the following estimate:

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \le C \begin{cases} (1+t)^{-\alpha/2}, & t \ge 0, \ 1 < \alpha < 2, \\ (1+t)^{-1}\log(2+t), & t \ge 0, \ \alpha = 2. \end{cases}$$

Furthermore, we prove that the asymptotic rate given in the above estimate is optimal under the additional assumptions. By applying the technique used in [50], we construct the second asymptotic profile corresponding to the decay rate of the initial data. Indeed, under the suitable assumptions on the initial data, the following lower bound estimate

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \ge \begin{cases} c_0(1+t)^{-\alpha/2}, & 1 < \alpha < 2, \\ c_1(1+t)^{-1}\log(1+t), & \alpha = 2 \end{cases}$$

holds for sufficiently large t, where c_0 and c_1 are certain positive constants. For details, see Theorem 3.2.1, Theorem 3.2.2 and Corollary 3.2.3 in Chapter 3.

In Appendix, we explain the derivation of the asymptotic profiles. First, we derive the nonlinear diffusion wave $\chi(x,t)$ as a self-similar solution of the Burgers equation. For this derivation, we refer to [4, 18, 45]. Next, we introduce the idea of the derivation of the second asymptotic profiles of the solutions to (1.1.3) defined in (2.2.3) and (3.2.5) below, by heuristic discussion.

1.2 Notations

In this thesis, for $1 \le p \le \infty$, $L^p(\mathbb{R})$ denotes the usual Lebesgue spaces. Moreover, for $\gamma \ge 0$, we define the following weighted Lebesgue spaces:

$$L^1_\gamma(\mathbb{R}) \equiv \left\{ f \in L^1(\mathbb{R});\ \|f\|_{L^1_\gamma} \equiv \int_{\mathbb{R}} |f(x)|(1+|x|)^\gamma dx < \infty \right\}.$$

In the following, for $f, g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, we denote the Fourier transform of f and the inverse Fourier transform of g as follows:

$$\hat{f}(\xi) \equiv \mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)dx,$$
$$\check{g}(x) \equiv \mathcal{F}^{-1}[g](x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} g(\xi)d\xi.$$

Then, for $s \ge 0$, we define the Sobolev spaces by

$$H^s(\mathbb{R}) \equiv \left\{ f \in L^2(\mathbb{R});\ \|f\|_{H^s} \equiv \left(\int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty \right\}.$$

To express Sobolev spaces, for $1 \le p \le \infty$, we also set

$$W^{m,p}(\mathbb{R}) \equiv \left\{ f \in L^p(\mathbb{R});\ \|f\|_{W^{m,p}} \equiv \left(\sum_{n=0}^{m} \|\partial_x^n f\|_{L^p}^p \right)^{1/p} < \infty \right\}.$$

Let T be a positive constant and B a Banach space. Here, for any natural number $m \ge 0$, $C^m([0,T];B)$ denotes the space of B-valued m-times continuously differentiable functions on $[0,T]$. The corresponding space of B-valued functions on $[0,\infty)$ is denoted by $C^m([0,\infty);B)$

Throughout this thesis, C denotes various positive constants, which may vary from line to line during computations. Also, it may depends on norms of the initial data or other parameters. However, we note that it does not depend on the space and time variables x and t.

Chapter 2

Construction of the Asymptotic Profiles for the Solutions

In this chapter, we introduce the result given by the author in [7].

2.1 Introduction

In this chapter, we consider the following initial value problem for the generalized KdV-Burgers equation:

$$\begin{gathered} u_t + (f(u))_x + k u_{xxx} = u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{gathered} \tag{2.1.1}$$

where $u_0 \in L^1(\mathbb{R})$, $f(u) = (b/2)u^2 + (c/3)u^3$ and $b, c, k \in \mathbb{R}$. The main purpose of this chapter is to obtain an asymptotic profile of the solution $u(x,t)$ and to examine the optimality of its asymptotic rate.

First of all, we recall known results concerning this problem. When $k = 0$, (2.1.1) becomes the generalized Burgers equation:

$$\begin{gathered} u_t + (f(u))_x = u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{2.1.2}$$

It is known that the solution of (2.1.2) converges to a nonlinear diffusion wave which is essentially the self-similar solution of the Burgers equation defined by

$$\chi(x,t) \equiv \frac{1}{\sqrt{1+t}} \chi_* \left(\frac{x}{\sqrt{1+t}} \right), \quad x \in \mathbb{R},\ t > 0, \tag{2.1.3}$$

where

$$\chi_*(x) \equiv \frac{1}{b} \frac{(e^{b\delta/2} - 1) e^{-x^2/4}}{\sqrt{\pi} + (e^{b\delta/2} - 1) \int_{x/2}^{\infty} e^{-y^2} dy}, \quad \delta \equiv \int_{\mathbb{R}} u_0(x) dx, \quad b \neq 0 \tag{2.1.4}$$

(cf. [33, 37, 39, 45, 52]). By the Hopf-Cole transformation (cf. [4, 18]), we can see that $\chi(x,t)$ satisfies the following Burgers equation

$$\chi_t + \left(\frac{b}{2} \chi^2 \right)_x = \chi_{xx}, \quad \int_{\mathbb{R}} \chi(x,0) dx = \delta. \tag{2.1.5}$$

Especially, concerning the convergence rate to the nonlinear diffusion wave $\chi(x,t)$, it was shown in Matsumura and Nishihara [45] that if we set

$$w_0(x) \equiv \exp\Big(-\frac{b}{2}\int_{-\infty}^{x} u_0(y)dy\Big) - \exp\Big(-\frac{b}{2}\int_{-\infty}^{x} \chi_*(y)dy\Big) \tag{2.1.6}$$

and assume that $w_0 \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\|w_0\|_{H^2} + \|w_0\|_{L^1} + \|u_0\|_{L^1}$ is sufficiently small, then the following estimate holds:

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1}\log(2+t), \quad t \ge 0. \tag{2.1.7}$$

Moreover, if $u_0 \in L^1_1(\mathbb{R}) \cap H^1(\mathbb{R})$ and $\|u_0\|_{L^1_1} + \|u_0\|_{H^1}$ is sufficiently small, then the optimality of the asymptotic rate to the nonlinear diffusion wave obtained in (2.1.7) was examined by Kato [31] by constructing the second asymptotic profile $V_1(x,t)$ which is the leading term of $u - \chi$. Here, we recall that $L^1_\gamma(\mathbb{R})$ is a subset of $L^1(\mathbb{R})$ whose elements satisfy $\|u_0\|_{L^1_\gamma} \equiv \int_{\mathbb{R}}(1+|x|)^\gamma|u_0(x)|dx < \infty$. Under these assumptions, Kato [31] showed that if $\delta \ne 0$ and $c \ne 0$, then the asymptotic rate to the nonlinear diffusion wave $\chi(x,t)$ given by (2.1.7) is optimal with respect to the time decaying order. More precisely, the following decay estimate was established:

$$\|u(\cdot,t) - \chi(\cdot,t) - V_1(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1}, \quad t \ge 1, \tag{2.1.8}$$

where

$$V_1(x,t) \equiv -\frac{cd}{12\sqrt{\pi}}V_*\Big(\frac{x}{\sqrt{1+t}}\Big)(1+t)^{-1}\log(1+t), \quad x \in \mathbb{R}, \ t \ge 0, \tag{2.1.9}$$

$$V_*(x) \equiv (b\chi_*(x) - x)e^{-x^2/4}\eta_*(x) = 2\frac{d}{dx}(\eta_*(x)e^{-x^2/4}), \tag{2.1.10}$$

$$\eta_*(x) \equiv \exp\Big(\frac{b}{2}\int_{-\infty}^{x}\chi_*(y)dy\Big), \quad d \equiv \int_{\mathbb{R}}(\eta_*(y))^{-1}(\chi_*(y))^3dy. \tag{2.1.11}$$

From (2.1.8), the triangle inequality and (2.1.9), one can obtain the following optimal decay estimate:

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} = (\tilde{C} + o(1))(1+t)^{-1}\log(1+t) \ \ as \ \ t \to \infty, \tag{2.1.12}$$

where $\tilde{C} = |cd|\|V_*\|_{L^\infty}/(12\sqrt{\pi})$. Therefore, we see that the original solution $u(x,t)$ tends to the nonlinear diffusion wave $\chi(x,t)$ at the rate of $t^{-1}\log t$, and in addition, if $\delta \ne 0$ and $c \ne 0$, then this asymptotic rate is optimal with respect to the time decaying order. Also, we see that $u - \chi$ tends to the second asymptotic profile $V_1(x,t)$ at the rate of t^{-1}.

Next, we consider the case where $b = k = 1$ and $c = 0$ in (2.1.1):

$$\begin{gathered} u_t + uu_x + u_{xxx} = u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{2.1.13}$$

This equation is called the KdV-Burgers equation. The large time behavior of the solutions to (2.1.13) is also studied by many researchers (cf. [1, 17, 27, 28, 30]): In particular, it was shown in Kaikina and Ruiz-Paredes [27] that if $u_0 \in L^1_1(\mathbb{R}) \cap H^s(\mathbb{R})$ with $s > -1/2$, then the following estimate

$$\|u(\cdot,t) - \chi(\cdot,t-1) - V_2(\cdot,t)\|_{L^\infty} \le Ct^{-1}\sqrt{\log t} \tag{2.1.14}$$

holds for sufficiently large t, where

$$V_2(x,t) \equiv -\frac{d}{32\sqrt{\pi}} V_*\left(\frac{x}{\sqrt{t}}\right) t^{-1}\log t$$

with $V_*(x)$ being defined by (2.1.10). We see from this result that the solution of (2.1.13) also tends to the nonlinear diffusion wave $\chi(x,t)$ at the rate of $t^{-1}\log t$ and this rate is optimal. On the other hand, the asymptotic rate given by (2.1.14) is rougher than (2.1.8) by $\sqrt{\log t}$, although they mentioned in [27] that the term $\sqrt{\log t}$ in the estimate (2.1.14) could be removed by more delicate consideration, without any proof.

For the general case, in [7] the author considered (2.1.1) for all $b, c, k \in \mathbb{R}$ with $b \neq 0$ and obtained a result about the large time behavior of the solution of (2.1.1), which unifies the results due to Kato [31] and Kaikina and Ruiz-Paredes [27]. We state the result in the next section.

2.2 Main Result

In this section, we introduce the main result of [7]. We consider (2.1.1) for all $b, c, k \in \mathbb{R}$ with $b \neq 0$. For $s \ge 1$ and $\gamma \ge 0$, we set

$$E_{s,\gamma} \equiv \|u_0\|_{H^s} + \|u_0\|_{L^1_\gamma}. \tag{2.2.1}$$

Then, we obtain the following result:

Theorem 2.2.1. *Assume that $u_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $E_{3,0}$ is sufficiently small. Then (2.1.1) has a unique global solution $u(x,t) \in C^0([0,\infty); H^3)$. Moreover if $u_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $E_{3,1}$ is sufficiently small, then we have*

$$\|u(\cdot,t) - \chi(\cdot,t) - V(\cdot,t)\|_{L^\infty} \le CE_{3,1}(1+t)^{-1}, \quad t \ge 1, \tag{2.2.2}$$

where $\chi(x,t)$ is defined by (2.1.3), while $V(x,t)$ is defined by

$$V(x,t) \equiv -\frac{d}{4\sqrt{\pi}}\left(\frac{b^2k}{8} + \frac{c}{3}\right) V_*\left(\frac{x}{\sqrt{1+t}}\right)(1+t)^{-1}\log(1+t), \quad x \in \mathbb{R}, \ t \ge 0, \tag{2.2.3}$$

with $V_*(x)$ being defined by (2.1.10).

In view of the second asymptotic profile, we are able to obtain the optimal asymptotic rate to the nonlinear diffusion wave $\chi(x,t)$ as follows:

Corollary 2.2.2. *Under the same assumptions in Theorem 2.2.1, if $\delta \neq 0$ and $(b^2k)/8 + c/3 \neq 0$, the following estimate holds:*

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} = (\tilde{C} + o(1))(1+t)^{-1}\log(1+t) \quad \text{as} \quad t \to \infty, \tag{2.2.4}$$

where $\tilde{C} \equiv |d((b^2k)/8 + c/3)| \|V_*\|_\infty / (4\sqrt{\pi})$ is a positive constant.

Remark 2.2.3. From (2.2.4), if $\delta \neq 0$ and $(b^2k)/8+c/3 \neq 0$, we see that the original solution $u(x,t)$ tends to the nonlinear diffusion wave $\chi(x,t)$ at the rate of $t^{-1}\log t$ and this asymptotic rate is optimal with respect to the time decaying order. On the other hand, from (2.2.2), if $(b^2k)/8 + c/3 = 0$, then we find the asymptotic rate to the nonlinear diffusion wave is t^{-1}, because $V(x,t)$ vanishes identically.

Remark 2.2.4. It seems that our regularity assumption on the initial data is stronger than previous works [27] and [31]. However, even if we assume the same regularity as those works, since the solution of (2.1.1) becomes smooth $u(x,t) \in C^\infty((0,\infty); H^\infty)$ by virtue of the smoothing effect for the parabolic type equations, we may assume that the initial data $u_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R})$ by changing the initial time. Taking this fact into account, we can say that (2.1.14) is improved, and that the results due to Kato [31] and Kaikina and Ruiz-Paredes [27] are unified.

The rest of this chapter is organized as follows. In Section 2.3, we prove the global existence and L^p-decay estimates of the solutions to (2.1.1). In Section 2.4 and Section 2.5, we prepare a couple of lemmas and introduce an auxiliary problem. Finally, we give the proof of our main result Theorem 2.2.1 in Section 2.6.

2.3 Global Existence and Decay Estimates

In this section we shall prove the global existence and decay estimates for the solutions to (2.1.1). First, we introduce the Green function associated with the linear part of the equation in (2.1.1):

$$S(x,t) \equiv \mathcal{F}^{-1}[e^{-t|\xi|^2+itk\xi^3}](x).$$

By a direct calculation, we can show the estimates of $S(x,t)$ and its derivatives. The following estimates were given by Karch in [30] as Lemma A.1 and Lemma A.2.

Lemma 2.3.1. *Let l be a non-negative integer. Then, for $p \in [2,\infty]$, we have*

$$\|\partial_x^l S(\cdot,t)\|_{L^1} \le Ct^{-l/2}(1+t^{-1/4}), \quad t>0, \tag{2.3.1}$$

$$\|\partial_x^l S(\cdot,t)\|_{L^p} \le Ct^{-(1/2)(1-1/p)-l/2}, \quad t>0. \tag{2.3.2}$$

Proof. First, we consider the case of $p=2$ in (2.3.2). By using Plancherel's theorem and the change of variables, we have

$$\begin{aligned}\|\partial_x^l S(\cdot,t)\|_{L^2}^2 &= \|(i\xi)^l e^{-t|\xi|^2+itk\xi^3}\|_{L^2}^2 \\ &= \int_{\mathbb{R}} \xi^{2l} e^{-2t|\xi|^2} d\xi = t^{-1/2-l}\int_{\mathbb{R}} z^{2l}e^{-2z^2}dz \le Ct^{-1/2-l}.\end{aligned}$$

Therefore, we have the L^2-decay estimate

$$\|\partial_x^l S(\cdot,t)\|_{L^2} \le Ct^{-1/4-l/2}, \quad t>0. \tag{2.3.3}$$

For the L^∞-estimate, from the Sobolev inequality, we see that

$$\|\partial_x^l S(\cdot,t)\|_{L^\infty} \le \|\partial_x^l S(\cdot,t)\|_{L^2}^{1/2}\|\partial_x^{l+1} S(\cdot,t)\|_{L^2}^{1/2} \le Ct^{-1/2-l/2}, \quad t>0. \tag{2.3.4}$$

Then, for $p \in [2, \infty]$, by the interpolation inequality, combining (2.3.3) and (2.3.4), we arrive at (2.3.2):

$$\|\partial_x^l S(\cdot,t)\|_{L^p} \le \|\partial_x^l S(\cdot,t)\|_{L^\infty}^{1-2/p} \|\partial_x^l S(\cdot,t)\|_{L^2}^{2/p} \le Ct^{-1/2+1/2p-l/2}, \quad t > 0.$$

To handle the L^1-norm, we repeat an argument from [28] involving the inequality

$$\|\hat{F}\|_{L^1} \le C\|F\|_{L^2}^{1/2} \|\partial_x F\|_{L^2}^{1/2}, \quad F \in H^1(\mathbb{R}). \tag{2.3.5}$$

Putting $F(\xi, t) = (i\xi)^l e^{-t|\xi|^2 + itk\xi^3}$ in (2.3.5) and using (2.3.3), we obtain (2.3.1) as follows

$$\begin{aligned}
\|\partial_x^l S(\cdot,t)\|_{L^1} &\le C \left(\int_{\mathbb{R}} \xi^{2l} e^{-2t\xi^2} d\xi \right)^{1/4} \left(\int_{\mathbb{R}} |\partial_\xi ((i\xi)^l e^{-t|\xi|^2 + itk\xi^3})|^2 d\xi \right)^{1/4} \\
&\le C(t^{-1/2-l})^{1/4} \left(\int_{\mathbb{R}} \{ l\xi^{l-1} e^{-t\xi^2} + \xi^l | -2t\xi + 3itk\xi^2 | e^{-t\xi^2} \}^2 d\xi \right)^{1/4} \\
&\le C(t^{-1/2-l})^{1/4} \left(\int_{\mathbb{R}} \{ \xi^{2(l-1)} + \xi^{2l}(t\xi + t\xi^2)^2 \} e^{-2t\xi^2} d\xi \right)^{1/4} \\
&\le C(t^{-1/2-l})^{1/4} \left(\int_{\mathbb{R}} \{ \xi^{2(l-1)} + t^2(\xi^{2(l+1)} + \xi^{2(l+2)}) \} e^{-2t\xi^2} d\xi \right)^{1/4} \\
&\le C(t^{-1/2-l})^{1/4} \left(t^{-(1/2)-l+1} + t^2 \left(t^{-(1/2)-l-1} + t^{-(1/2)-l-2} \right) \right)^{1/4} \\
&\le Ct^{-l/2}(1 + t^{-1/4}).
\end{aligned}$$

This completes the proof. $\square$

Moreover, for the convolution $S(t) * g$, we obtain the following estimate:

Lemma 2.3.2. *Let s be a positive integer. Suppose $g \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$. Then the estimate*

$$\|\partial_x^l (S(t) * g)\|_{L^2} \le C(1+t)^{-1/4-l/2} \|g\|_{L^1} + Ce^{-t} \|\partial_x^l g\|_{L^2}, \quad t \ge 0 \tag{2.3.6}$$

holds for any integer $0 \le l \le s$.

Proof. By using Plancherel's theorem, we have

$$\begin{aligned}
\|\partial_x^l (S(t) * g)\|_{L^2}^2 &= 2\pi \|e^{-t|\xi|^2 + itk\xi^3} (i\xi)^l \hat{g}(\xi)\|_{L^2}^2 = 2\pi \int_{\mathbb{R}} e^{-2t|\xi|^2} |(i\xi)^l \hat{g}(\xi)|^2 d\xi \\
&= 2\pi \left(\int_{|\xi| \ge 1} + \int_{|\xi| \le 1} \right) e^{-2t|\xi|^2} |(i\xi)^l \hat{g}(\xi)|^2 d\xi \\
&\equiv I_1 + I_2.
\end{aligned}$$

First, we evaluate I_1. By Plancherel's theorem, we have

$$I_1 \le 2\pi e^{-2t} \int_{|\xi| \ge 1} |(i\xi)^l \hat{g}(\xi)|^2 d\xi \le 2\pi e^{-2t} \int_{\mathbb{R}} |\widehat{\partial_x^l g}(\xi)|^2 d\xi = 2\pi e^{-2t} \|\partial_x^l g\|_{L^2}^2. \tag{2.3.7}$$

Next, we evaluate I_2. Since $|\hat{g}(\xi)| \leq C\|g\|_{L^1}$ for all $\xi \in \mathbb{R}$, we have

$$\begin{aligned}I_2 &= 2\pi \int_{|\xi|\leq 1} e^{-2t|\xi|^2} |(i\xi)^l|^2 |\hat{g}(\xi)|^2 d\xi \leq 2\pi \Big(\sup_{|\xi|\leq 1} |\hat{g}(\xi)|\Big)^2 \int_{|\xi|\leq 1} e^{-2t|\xi|^2} |\xi|^{2l} d\xi \\ &\leq C\|g\|_{L^1}^2 \int_0^1 e^{-2t\xi^2} \xi^{2l} d\xi \leq C(1+t)^{-1/2-l} \|g\|_{L^1}^2.\end{aligned} \tag{2.3.8}$$

From (2.3.7) and (2.3.8) we obtain (2.3.6). □

Now we turn back to (2.1.1). By using Lemma 2.3.1 and Lemma 2.3.2, we are able to obtain the global existence and the decay estimates for the solutions to (2.1.1) as follows:

Proposition 2.3.3. *Let s be a positive integer. Assume that $u_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ and $E_{s,0}$ is sufficiently small. Then (2.1.1) has a unique global solution $u(x,t) \in C^0([0,\infty); H^s)$. Moreover the solution satisfies the following estimates:*

$$\|\partial_x^l u(\cdot,t)\|_{L^2} \leq CE_{s,0}(1+t)^{-1/4-l/2}, \quad t \geq 0, \tag{2.3.9}$$
$$\|\partial_x^l u(\cdot,t)\|_{L^1} \leq CE_{s,0} t^{-l/2}(1+t^{-1/4}), \quad t > 0 \tag{2.3.10}$$

hold for any integer $0 \leq l \leq s$. In particular, we get

$$\|\partial_x^l u(\cdot,t)\|_{L^\infty} \leq CE_{s,0}(1+t)^{-1/2-l/2}, \quad t \geq 0 \tag{2.3.11}$$

for any integer $0 \leq l \leq s-1$.

Proof. We consider the following integral equation associated with the initial value problem (2.1.1):

$$\begin{aligned}u(t) &= S(t) * u_0 - \int_0^t S(t-\tau) * (f(u)_x)(\tau) d\tau \\ &= S(t) * u_0 - \int_0^t (\partial_x S(t-\tau)) * (f(u))(\tau) d\tau.\end{aligned} \tag{2.3.12}$$

We solve this integral equation by using the contraction mapping principle for the mapping

$$N[u] \equiv S(t) * u_0 - \int_0^t (\partial_x S(t-\tau)) * (f(u))(\tau) d\tau. \tag{2.3.13}$$

We set $N_0 \equiv S(t) * u_0$. Let us introduce the Banach space X as follows:

$$X \equiv \left\{ u \in C^0([0,\infty); H^s);\ \|u\|_X \equiv \sum_{l=0}^{s} \sup_{t\geq 0} (1+t)^{1/4+l/2} \|\partial_x^l u(\cdot,t)\|_{L^2} < \infty \right\}. \tag{2.3.14}$$

From Lemma 2.3.2, we have

$$\exists C_0 > 0 \ \ s.t. \ \ \|N_0\|_X \leq C_0 E_{s,0}. \tag{2.3.15}$$

We apply the contraction mapping principle to (2.3.13) on a closed subset Y of X below:

$$Y \equiv \{u \in X;\ \|u\|_X \leq 2C_0 E_{s,0}\}.$$

Then it is sufficient to show the following estimates:

$$\|N[u]\|_X \le 2C_0 E_{s,0}, \tag{2.3.16}$$

$$\|N[u]-N[v]\|_X \le \frac{1}{2}\|u-v\|_X \tag{2.3.17}$$

for $u, v \in Y$. If we have shown (2.3.16) and (2.3.17), by using Banach fixed-point theorem, we see that (2.3.12) has a unique global solution in Y.

Here and later $E_{s,0}$ is assumed to be small. First, from the Sobolev inequality

$$\|F\|_{L^\infty} \le \sqrt{2}\|F\|_{L^2}^{1/2}\|F'\|_{L^2}^{1/2}, \quad F \in H^1(\mathbb{R}),$$

for $0 \le l \le s-1$, we have

$$\|\partial_x^l u(\cdot,t)\|_{L^\infty} \le \|u\|_X(1+t)^{-1/2-l/2}. \tag{2.3.18}$$

Before proving (2.3.16) and (2.3.17), we prepare the following estimates for $0 \le l \le s$ and $u, v \in Y$:

$$\|\partial_x^l(f(u)-f(v))(t)\|_{L^1} \le C(\|u\|_X+\|v\|_X)\|u-v\|_X(1+t)^{-1/2-l/2}, \tag{2.3.19}$$
$$\|\partial_x^l(f(u)-f(v))(t)\|_{L^2} \le C(\|u\|_X+\|v\|_X)\|u-v\|_X(1+t)^{-3/4-l/2}. \tag{2.3.20}$$

We shall prove only (2.3.19), since we can prove (2.3.20) in the same way. We have from (2.3.14) and (2.3.18)

$$\begin{aligned}
&\|\partial_x^l(f(u)-f(v))(t)\|_{L^1}\\
&\le C\|\partial_x^l(u^2-v^2)(\cdot,t)\|_{L^1} + C\|\partial_x^l(u^3-v^3)(\cdot,t)\|_{L^1}\\
&\le C\|\partial_x^l((u+v)(u-v))(\cdot,t)\|_{L^1} + C\|\partial_x^l((u-v)(u^2+uv+v^2))(\cdot,t)\|_{L^1}\\
&\le C\sum_{m=0}^{l}\left(\|\partial_x^{l-m}u(\cdot,t)\|_{L^2}+\|\partial_x^{l-m}v(\cdot,t)\|_{L^2}\right)\|\partial_x^m(u-v)(\cdot,t)\|_{L^2}\\
&\quad + C\|\partial_x^l(u-v)(\cdot,t)\|_{L^2}\big(\|u(\cdot,t)\|_{L^2}\|u(\cdot,t)\|_{L^\infty}\\
&\quad + \|u(\cdot,t)\|_{L^2}\|v(\cdot,t)\|_{L^\infty}+\|v(\cdot,t)\|_{L^2}\|v(\cdot,t)\|_{L^\infty}\big)\\
&\quad + C\sum_{m=0}^{l-1}\sum_{n=0}^{l-m}\|\partial_x^m(u-v)(\cdot,t)\|_{L^\infty}\big(\|\partial_x^n u(\cdot,t)\|_{L^2}\|\partial_x^{l-m-n}u(\cdot,t)\|_{L^2}\\
&\quad + \|\partial_x^n u(\cdot,t)\|_{L^2}\|\partial_x^{l-m-n}v(\cdot,t)\|_{L^2}+\|\partial_x^n v(\cdot,t)\|_{L^2}\|\partial_x^{l-m-n}v(\cdot,t)\|_{L^2}\big)\\
&\le C\sum_{m=0}^{l}(\|u\|_X+\|v\|_X)(1+t)^{-1/4-(l-m)/2}\|u-v\|_X(1+t)^{-1/4-m/2}\\
&\quad + C\|u-v\|_X(1+t)^{-1/4-l/2}(\|u\|_X^2+\|u\|_X\|v\|_X+\|v\|_X^2)(1+t)^{-1/4}(1+t)^{-1/2}\\
&\quad + C\sum_{m=0}^{l-1}\sum_{n=0}^{l-m}\|u-v\|_X(1+t)^{-1/2-m/2}\\
&\quad \times(\|u\|_X^2+\|u\|_X\|v\|_X+\|v\|_X^2)(1+t)^{-1/4-n/2}(1+t)^{-1/4-(l-m-n)/2}\\
&\le C(\|u\|_X+\|v\|_X)\|u-v\|_X(1+t)^{-1/2-l/2}.
\end{aligned}$$

Now we prove (2.3.16) and (2.3.17). Using (2.3.13), we obtain

$$(N[u]-N[v])(t) = -\int_0^t (\partial_x S(t-\tau)) * (f(u)-f(v))(\tau)d\tau \equiv I(x,t). \tag{2.3.21}$$

By Plancherel's Theorem, we have

$$\|\partial_x^l I(\cdot,t)\|_{L^2} \le \|(i\xi)^l \hat{I}(\xi,t)\|_{L^2(|\xi|\le 1)} + \|(i\xi)^l \hat{I}(\xi,t)\|_{L^2(|\xi|\ge 1)} \equiv I_1 + I_2. \tag{2.3.22}$$

Since

$$\int_{|\xi|\le 1} |\xi|^j e^{-2(t-\tau)|\xi|^2} d\xi \le C(1+t-\tau)^{-j/2-1/2}, \quad j \ge 0,$$

and (2.3.19), we have

$$\begin{aligned}
I_1 &\le C\int_0^t \|(i\xi)^{l+1} e^{-(t-\tau)|\xi|^2+ik(t-\tau)\xi^3}\mathcal{F}[f(u)-f(v)](\xi,\tau)\|_{L^2(|\xi|\le 1)}d\tau \\
&\le C\int_0^{t/2} \sup_{|\xi|\le 1}|\mathcal{F}[f(u)-f(v)](\xi,\tau)|\left(\int_{|\xi|\le 1}|\xi|^{2(l+1)}e^{-2(t-\tau)|\xi|^2}d\xi\right)^{1/2} d\tau \\
&\quad + C\int_{t/2}^t \sup_{|\xi|\le 1}|(i\xi)^l\mathcal{F}[f(u)-f(v)](\xi,\tau)|\left(\int_{|\xi|\le 1}|\xi|^2 e^{-2(t-\tau)|\xi|^2}d\xi\right)^{1/2} d\tau \\
&\le C\int_0^{t/2}(1+t-\tau)^{-3/4-l/2}\|(f(u)-f(v))(\tau)\|_{L^1}d\tau \\
&\quad + C\int_{t/2}^t (1+t-\tau)^{-3/4}\|\partial_x^l(f(u)-f(v))(\tau)\|_{L^1}d\tau \\
&\le C(\|u\|_X+\|v\|_X)\|u-v\|_X\left(\int_0^{t/2}(1+t-\tau)^{-3/4-l/2}(1+\tau)^{-1/2}d\tau\right. \\
&\quad \left. + \int_{t/2}^t (1+t-\tau)^{-3/4}(1+\tau)^{-1/2-l/2}d\tau\right) \\
&\le C(\|u\|_X+\|v\|_X)\|u-v\|_X(1+t)^{-1/4-l/2}.
\end{aligned} \tag{2.3.23}$$

For $|\xi|\ge 1$, by using the Schwarz inequality, we have

$$\begin{aligned}
&|(i\xi)^l\hat{I}(\xi,t)| \\
&= \left|(i\xi)^{l+1}\int_0^t e^{-(t-\tau)|\xi|^2+ik(t-\tau)\xi^3}\mathcal{F}[f(u)-f(v)](\xi,\tau)d\tau\right| \\
&\le C\int_0^t |\xi|e^{-(t-\tau)|\xi|^2}|(i\xi)^l\mathcal{F}[f(u)-f(v)](\xi,\tau)|d\tau \\
&\le C\left(\int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2}d\tau\right)^{1/2}\left(\int_0^t e^{-(t-\tau)|\xi|^2}|(i\xi)^l\mathcal{F}[f(u)-f(v)](\xi,\tau)|^2 d\tau\right)^{1/2} \\
&\le C\left(\int_0^t e^{-(t-\tau)|\xi|^2}|(i\xi)^l\mathcal{F}[f(u)-f(v)](\xi,\tau)|^2 d\tau\right)^{1/2}.
\end{aligned}$$

Therefore we have from (2.3.20)

$$
\begin{aligned}
I_2 &\le C\left(\int_{|\xi|\ge 1}\int_0^t e^{-(t-\tau)|\xi|^2}|(i\xi)^l\mathcal{F}[f(u)-f(v)](\xi,\tau)|^2 d\tau d\xi\right)^{1/2}\\
&\le C\left(\int_0^t e^{-(t-\tau)}\int_{|\xi|\ge 1}|(i\xi)^l\mathcal{F}[f(u)-f(v)](\xi,\tau)|^2 d\xi d\tau\right)^{1/2}\\
&\le C\left(\int_0^t e^{-(t-\tau)}\|\partial_x^l(f(u)-f(v))(\tau)\|_{L^2}^2 d\tau\right)^{1/2} \qquad (2.3.24)\\
&\le C(\|u\|_X+\|v\|_X)\|u-v\|_X\left(\int_0^t e^{-(t-\tau)}(1+\tau)^{-3/2-l}d\tau\right)^{1/2}\\
&\le C(\|u\|_X+\|v\|_X)\|u-v\|_X(1+t)^{-3/4-l/2}.
\end{aligned}
$$

Combining (2.3.21) through (2.3.24), we obtain

$$
\|\partial_x^l(N[u]-N[v])(t)\|_{L^2}\le C(\|u\|_X+\|v\|_X)\|u-v\|_X(1+t)^{-1/4-l/2},\quad t\ge 0
$$

for $0\le l\le s$. Thus, there exists a positive constant $C_1>0$ such that

$$
\|N[u]-N[v]\|_X\le C_1(\|u\|_X+\|v\|_X)\|u-v\|_X\le 4C_0C_1E_{s,0}\|u-v\|_X,\quad u,v\in Y.
$$

Choosing $E_{s,0}$, which satisfies $4C_0C_1E_{s,0}\le 1/2$, then we have (2.3.17). Moreover, taking $v=0$ in (2.3.17), it follows that

$$
\|N[u]-N[0]\|_X\le C_0E_{s,0}.
$$

Since $N[0]=N_0$, we obtain from (2.3.15) that

$$
\|N[u]\|_X\le\|N_0\|_X+\|N[u]-N[0]\|_X\le 2C_0E_{s,0}.
$$

Therefore, we get (2.3.16). This completes the proof of the global existence and L^2-dacay estimate (2.3.9).

Finally, from Young's inequality and Lemma 2.3.1, (2.3.9), (2.3.14) and (2.3.19), we have

$$
\begin{aligned}
\|\partial_x^l u(\cdot,t)\|_{L^1} &\le \|\partial_x^l (S(t) * u_0)\|_{L^1} + \int_0^{t/2} \|(\partial_x^{l+1} S(t-\tau)) * (f(u))(\tau)\|_{L^1} d\tau \\
&\quad + \int_{t/2}^t \|(\partial_x S(t-\tau)) * (\partial_x^l f(u))(\tau)\|_{L^1} d\tau \\
&\le \|\partial_x^l S(\cdot,t)\|_{L^1} \|u_0\|_{L^1} + \int_0^{t/2} \|\partial_x^{l+1} S(\cdot,t-\tau)\|_{L^1} \|(f(u))(\tau)\|_{L^1} d\tau \\
&\quad + \int_{t/2}^t \|\partial_x S(\cdot,t-\tau)\|_{L^1} \|(\partial_x^l f(u))(\tau)\|_{L^1} d\tau \\
&\le C E_{s,0} t^{-l/2}(1+t^{-1/4}) \\
&\quad + C E_{s,0} \int_0^{t/2} (t-\tau)^{-1/2-l/2}(1+(t-\tau)^{-1/4})(1+\tau)^{-1/2} d\tau \\
&\quad + C E_{s,0} \int_{t/2}^t (t-\tau)^{-1/2}(1+(t-\tau)^{-1/4})(1+\tau)^{-1/2-l/2} d\tau \\
&\le C E_{s,0} \Big(t^{-l/2}(1+t^{-1/4}) + t^{-1/2-l/2}(1+t^{-1/4}) t^{1/2} \\
&\quad + (1+t)^{-1/2-l/2}(t^{1/2}+t^{1/4}) \Big) \\
&\le C E_{s,0} t^{-l/2}(1+t^{-1/4}), \quad t>0.
\end{aligned}
$$

Thus we get (2.3.10). This completes the proof. □

2.4 Basic Lemmas

In order to prove the main theorem, we prepare a couple of lemmas. First, we treat the nonlinear diffusion wave $\chi(x,t)$ defined by (2.1.3), and the heat kernel

$$
G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}. \tag{2.4.1}
$$

A direct calculation yields

$$
|\chi(x,t)| \le C|\delta|(1+t)^{-1/2} e^{-x^2/4(1+t)}, \quad x \in \mathbb{R},\ t \ge 0. \tag{2.4.2}
$$

Moreover, we can estimate derivatives of $\chi(x,t)$ and $G(x,t)$. For the following estimates, we refer to the paper [32].

Lemma 2.4.1. *Let l and m be non-negative integers. Then, for $p \in [1,\infty]$, we have*

$$
\|\partial_x^l \partial_t^m \chi(\cdot,t)\|_{L^p} \le C|\delta|(1+t)^{-(1/2)(1-1/p)-l/2-m}, \quad t \ge 0, \tag{2.4.3}
$$

$$
\|\partial_x^l \partial_t^m G(\cdot,t)\|_{L^p} \le C t^{-(1/2)(1-1/p)-l/2-m}, \quad t > 0. \tag{2.4.4}
$$

Proof. Since (2.4.4) is a well known estimate, we shall prove only (2.4.3). Because $|\chi_*(x)| \le C|\delta| e^{-x^2/4}$, which comes from (2.1.4), we first obtain $\|\chi_*\|_{L^p} \le C|\delta|$. It is easy to see from (2.1.5) and (2.1.4) that χ_* satisfies

$$
\chi_*' + \frac{x}{2}\chi_* = \frac{b}{2}\chi_*^2.
$$

Thus, we can estimate

$$|\chi_*'| \le C(|x| + |\chi_*|)|\chi_*| \le C|\delta| e^{-x^2/8},$$

and hence $\|\partial_x \chi_*\|_{L^p} \le C|\delta|$. Therefore, applying the induction argument with respect to l, we obtain $\|\partial_x^l \chi_*\|_{L^p} \le C|\delta|$ for $l \ge 0$. From this fact and (2.1.3), we can get

$$\|\partial_x^l \chi(\cdot,t)\|_{L^p} \le C|\delta|(1+t)^{-(1/2)(1-1/p)-l/2}. \tag{2.4.5}$$

Furthermore, using

$$|\partial_t \partial_x^l \chi| \le C\left(|\partial_x^{l+2}\chi| + \sum_{j=0}^{l} |\partial_x^j \chi||\partial_x^{l-j+1}\chi|\right),$$

which comes from (2.1.5), we can get from (2.4.5)

$$\|\partial_t \partial_x^l \chi(\cdot,t)\|_{L^p} \le C|\delta|(1+t)^{-(1/2)(1-1/p)-l/2-1}.$$

Therefore, similarly as before, we use the induction argument with respect to m and derive the desired estimate (2.4.3). □

Next, for the latter sake, we define

$$\eta(x,t) \equiv \eta_*\left(\frac{x}{\sqrt{1+t}}\right) = \exp\left(\frac{b}{2}\int_{-\infty}^{x} \chi(y,t)dy\right). \tag{2.4.6}$$

For this function, we can easily show

$$\min\{1, e^{b\delta/2}\} \le \eta(x,t) \le \max\{1, e^{b\delta/2}\}, \tag{2.4.7}$$

$$\min\{1, e^{-b\delta/2}\} \le \eta(x,t)^{-1} \le \max\{1, e^{-b\delta/2}\}. \tag{2.4.8}$$

Moreover, we have the following estimates by using Lemma 2.4.1 (the proof is given by Kato as Corollary 2.3 in [31]).

Lemma 2.4.2. *Let l be a positive integer and $p \in [1,\infty]$. If $|\delta| \le 1$, then we have*

$$\|\partial_x^l \eta(\cdot,t)\|_{L^p} + \|\partial_x^l(\eta(\cdot,t)^{-1})\|_{L^p} \le C|\delta|(1+t)^{-(1/2)(1-1/p)-l/2+1/2}, \quad t \ge 0. \tag{2.4.9}$$

Proof. We shall prove only the estimate for η, since we can prove the estimate for η^{-1} in the same way. We put $q_1(x) = e^{(b/2)x}$, $q_2(x,t) = \int_{-\infty}^{x} \chi(y,t)dy$. Then, for $l \ge 1$, we have

$$\begin{aligned}
\partial_x^l \eta(x,t) &= \partial_x^l \exp\left(\frac{b}{2}\int_{-\infty}^{x} \chi(y,t)dy\right) \\
&= \partial_x^l((q_1 \circ q_2)(x,t)) \\
&= \sum C_{l,m_1,\cdots,m_l} \frac{d^p q_1}{dx^p}(q_2(x,t))(\partial_x q_2(x,t))^{m_1} \cdots (\partial_x^l q_2(x,t))^{m_l} \\
&= \sum C_{l,m_1,\cdots,m_l} \left(\frac{b}{2}\right)^p \eta(x,t)(\chi(x,t))^{m_1} \cdots (\partial_x^{l-1}\chi(x,t))^{m_l},
\end{aligned} \tag{2.4.10}$$

where the sum is taken for all $(m_1, \cdots, m_l) \in \mathbb{N}^l$ such that $m_1+2m_2+\cdots+lm_l = l$, and $p = m_1 + \cdots + m_l$, $1 \le p \le l$.

First, we shall prove the L^1-estimate for η. Since $m_1 + 2m_2 + \cdots + lm_l = l$, there exists $j \in \{1, 2, \cdots, l\}$ such that $m_j \ge 1$. Therefore, from Lemma 2.4.1 and (2.4.10), we have

$$\begin{aligned}
\|\partial_x^l \eta(\cdot,t)\|_{L^1} &\le C \sum \|\chi(\cdot,t)\|_{L^\infty}^{m_1} \|\partial_x \chi(\cdot,t)\|_{L^\infty}^{m_2} \cdots \|\partial_x^{j-1}\chi(\cdot,t)\|_{L^{m_j}}^{m_j} \cdots \|\partial_x^{l-1}\chi(\cdot,t)\|_{L^\infty}^{m_l} \\
&\le C|\delta| \sum (1+t)^{-m_1/2}(1+t)^{-2m_2/2} \cdots (1+t)^{-(jm_j-1)/2} \cdots (1+t)^{-lm_l/2} \\
&\le C|\delta|(1+t)^{-(l-1)/2}.
\end{aligned} \tag{2.4.11}$$

Next, we shall prove L^∞-estimate. From Lemma 2.4.1 and (2.4.10), we have

$$\begin{aligned}
\|\partial_x^l \eta(\cdot,t)\|_{L^\infty} &\le C \sum \|\chi(\cdot,t)\|_{L^\infty}^{m_1} \|\partial_x \chi(\cdot,t)\|_{L^\infty}^{m_2} \cdots \|\partial_x^{l-1}\chi(\cdot,t)\|_{L^\infty}^{m_l} \\
&\le C|\delta| \sum (1+t)^{-m_1/2}(1+t)^{-2m_2/2} \cdots (1+t)^{-lm_l/2} \\
&\le C|\delta|(1+t)^{-l/2}.
\end{aligned} \tag{2.4.12}$$

Finally, by using the interpolation inequality, we have,

$$\begin{aligned}
\|\partial_x^l \eta(\cdot,t)\|_{L^p} \le& \|\partial_x^l \eta(\cdot,t)\|_{L^\infty}^{1-1/p} \|\partial_x^l \eta(\cdot,t)\|_{L^1}^{1/p} \\
\le& C|\delta|^2 (1+t)^{-l/2+l/(2p)}(1+t)^{-(l-1)/(2p)} \\
\le& C|\delta|(1+t)^{-(1/2)(1-1/p)-l/2+1/2}, \quad 1 \le p \le \infty.
\end{aligned}$$

This completes the proof. □

2.5 Auxiliary Problem

In order to prove the main theorem, we introduce an auxiliary problem. We set

$$\psi(x,t) \equiv u(x,t) - \chi(x,t),$$

where $u(x,t)$ is the original solution to (2.1.1) and $\chi(x,t)$ is the nonlinear diffusion wave defined by (2.1.3). Then, $\psi(x,t)$ satisfies the following equation:

$$\psi_t + (b\chi\psi)_x + \left(\frac{b}{2}\psi^2\right)_x + \left(\frac{c}{3}(\psi+\chi)^3\right)_x + k\psi_{xxx} + k\chi_{xxx} - \psi_{xx} = 0.$$

Based on the observation given in the paper [27] and [31], one may guess that the leading term of asymptotic expansion of $\psi(x,t)$ at $t \to \infty$ is governed by the solution to the following equation:

$$v_t + (b\chi v)_x + \left(\frac{c}{3}\chi^3\right)_x + k\chi_{xxx} - v_{xx} = 0.$$

This observation leads to the following auxiliary problem:

$$\begin{gathered}
z_t + (b\chi z)_x - z_{xx} = \partial_x \lambda(x,t), \quad x \in \mathbb{R}, \ t > 0, \\
z(x,0) = z_0(x), \quad x \in \mathbb{R},
\end{gathered} \tag{2.5.1}$$

where $\lambda(x,t)$ is a given regular function decaying fast enough at spatial infinity. The explicit representation formula (2.5.3) below plays an important role in the proof of the main theorem, especially in the proofs of Proposition 2.6.2 and Proposition 2.6.3 below. If we set

$$U[h](x,t,\tau) \equiv \int_{\mathbb{R}} \partial_x \left(G(x-y,t-\tau)\eta(x,t)\right)(\eta(y,\tau))^{-1}\left(\int_{-\infty}^{y} h(\xi)d\xi\right)dy, \quad 0 \le \tau < t, \quad x \in \mathbb{R}, \tag{2.5.2}$$

then we have:

Lemma 2.5.1. *Let $z_0(x)$ be a sufficiently regular function decaying at spatial infinity. Then we can get the smooth solution of* (2.5.1) *which satisfies the following formula:*

$$z(x,t) = U[z_0](x,t,0) + \int_0^t U[\partial_x\lambda(\tau)](x,t,\tau)d\tau, \quad x \in \mathbb{R},\ t > 0. \tag{2.5.3}$$

Proof. We set

$$r(x,t) = \int_{-\infty}^{x} z(y,t)dy,$$

and integrate both sides of the equation (2.5.1). Then, we get

$$\begin{aligned} r_t + b\chi r_x - r_{xx} &= \lambda, \quad x \in \mathbb{R},\ t > 0, \\ r(x,0) &= \int_{-\infty}^{x} z_0(y)dy. \end{aligned} \tag{2.5.4}$$

Multiplying $\eta(x,t)^{-1}$ both sides of (2.5.4), we have

$$\eta^{-1}r_t - 2(\partial_x(\eta^{-1}))r_x - \eta^{-1}r_{xx} = \eta^{-1}\lambda, \tag{2.5.5}$$

since $\partial_x(\eta(x,t)^{-1}) = -(b/2)\chi(x,t)\eta(x,t)^{-1}$. Now, we put $E(x,t) = \eta(x,t)^{-1}r(x,t)$. Since $\partial_t(\eta^{-1}) - \partial_x^2(\eta^{-1}) = 0$, (2.5.5) leads to

$$E_t - E_{xx} = \eta^{-1}\lambda.$$

Therefore, we obtain

$$E(x,t) = \int_{\mathbb{R}} G(x-y,t)E(y,0)dy + \int_0^t \int_{\mathbb{R}} G(x-y,t-\tau)(\eta(y,\tau))^{-1}\lambda(y,\tau)dyd\tau,$$

or

$$\begin{aligned} \eta(x,t)^{-1}r(x,t) = &\int_{\mathbb{R}} G(x-y,t)(\eta(y,0))^{-1}r(y,0)dy \\ &+ \int_0^t \int_{\mathbb{R}} G(x-y,t-\tau)(\eta(y,\tau))^{-1}\lambda(y,\tau)dyd\tau. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \int_{-\infty}^{x} z(y,t)dy = &\ \eta(x,t)\int_{\mathbb{R}} G(x-y,t)(\eta(y,0))^{-1}\left(\int_{-\infty}^{y} z_0(\xi)d\xi\right)dy \\ &+ \eta(x,t)\int_0^t \int_{\mathbb{R}} G(x-y,t-\tau)(\eta(y,\tau))^{-1}\lambda(y,\tau)dyd\tau. \end{aligned}$$

Thus we get (2.5.3). □

To prove the main theorem, we shall prepare some decay estimates for the solution $z(x,t)$ to the auxiliary problem. First, for the first term of (2.5.3), the following two estimates were given by Kato [31]:

Lemma 2.5.2. *Let $\beta \in [0,1]$, m be a positive integer and $p \in [1,\infty]$. Assume that $|\delta| \le 1$, $z_0 \in L^1_\beta(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then, the estimate*

$$\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^p} \le Ct^{-(1-1/p+\beta+l)/2}\|z_0\|_{L^1_\beta}, \quad t>0 \tag{2.5.6}$$

holds for any integer $0 \le l \le m$.

Proof. We shall prove the following estimates:

$$\begin{aligned}\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^\infty} \le& Ct^{-(1+\beta+l)/2}\|z_0\|_{L^1_\beta}, \quad t>0, &(2.5.7)\\ \|\partial_x^l U[z_0](\cdot,t,0)\|_{L^1} \le& Ct^{-(\beta+l)/2}\|z_0\|_{L^1_\beta}, \quad t>0. &(2.5.8)\end{aligned}$$

If we have shown these estimates, from the interpolation inequality, we have

$$\begin{aligned}\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^p} \le& C\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^\infty}^{1-1/p}\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^1}^{1/p}\\ \le& Ct^{-(1-1/p)(1+\beta+l)/2}t^{-(1/p)(\beta+l)/2}\|z_0\|_{L^1_\beta}\\ \le& Ct^{-(1-1/p+\beta+l)/2}\|z_0\|_{L^1_\beta}.\end{aligned}$$

First, we shall prove (2.5.7). Since $\int_{\mathbb{R}} z_0(x)dx = 0$, by the integration by parts with respect to y, we have from (2.5.2)

$$U[z_0](x,t,0) = -\int_{\mathbb{R}} z_0(y)\left(\int_{-\infty}^{y} \partial_x(G(x-\xi,t)\eta(x,t))\eta(\xi,0)^{-1}d\xi\right)dy.$$

Therefore, we get

$$\begin{aligned}\partial_x^l U[z_0](x,t,0) =& -\int_{\mathbb{R}} z_0(y)\int_{-\infty}^{y} \partial_x^{l+1}(G(x-\xi,t)\eta(x,t))\eta(\xi,0)^{-1}d\xi dy\\ =& -\int_{\mathbb{R}} z_0(y)dy\left(\int_{-\infty}^{0} \partial_x^{l+1}(G(x-\xi,t)\eta(x,t))\eta(\xi,0)^{-1}d\xi\right)\\ &-\int_{\mathbb{R}} z_0(y)\left(\int_{0}^{y} \partial_x^{l+1}(G(x-\xi,t)\eta(x,t))\eta(\xi,0)^{-1}d\xi\right)dy\\ =& -\int_{\mathbb{R}} z_0(y)\left(\int_{0}^{y} \partial_x^{l+1}(G(x-\xi,t)\eta(x,t))\eta(\xi,0)^{-1}d\xi\right)dy.\end{aligned} \tag{2.5.9}$$

From (2.5.9), we have

$$\begin{aligned}|\partial_x^l U[z_0](x,t,0)| \le& \int_{\mathbb{R}} |z_0(y)|\left|\int_{0}^{y} \partial_x^{l+1}(G(x-\xi,t)\eta(x,t))\eta(\xi,0)^{-1}d\xi\right|dy\\ \equiv& \int_{\mathbb{R}} |z_0(y)|I_1(x,y,t)dy.\end{aligned} \tag{2.5.10}$$

From Lemma 2.4.2, we have

$$
\begin{aligned}
I_1 &\le C\left|\int_0^y \sum_{m=0}^{l+1} \partial_x^{l+1-m}\eta(x,t)\partial_x^m G(x-\xi,t)\eta(\xi,0)^{-1}d\xi\right| \\
&\le C\sum_{m=0}^{l+1} \|\partial_x^{l+1-m}\eta(\cdot,t)\|_{L^\infty}\left|\int_0^y \partial_x^m G(x-\xi,t)\eta(\xi,0)^{-1}d\xi\right| \\
&\le C\sum_{m=0}^{l+1}(1+t)^{-(l+1-m)/2}\left|\int_0^y \partial_x^m G(x-\xi,t)\eta(\xi,0)^{-1}d\xi\right|.
\end{aligned}
\tag{2.5.11}
$$

If $y>0$, from (2.4.8) and the Hölder inequality, we have

$$
\begin{aligned}
\left|\int_0^y \partial_x^m G(x-\xi,t)\eta(\xi,0)^{-1}d\xi\right| &\le \|\partial_x^m G(x-\cdot,t)\|_{L^{1/(1-\beta)}}\left(\int_0^y \eta(\xi,0)^{-1/\beta}d\xi\right)^\beta \\
&\le C\|\partial_x^m G(\cdot,t)\|_{L^{1/(1-\beta)}}y^\beta.
\end{aligned}
$$

Similarly, if $y<0$, we have

$$
\left|\int_0^y \partial_x^m G(x-\xi,t)\eta(\xi,0)^{-1}d\xi\right|\le C\|\partial_x^m G(\cdot,t)\|_{L^{1/(1-\beta)}}(-y)^\beta.
$$

Therefore, from Lemma 2.4.1, we get

$$
\begin{aligned}
I_1 &\le C\sum_{m=0}^{l+1}(1+t)^{-(l+1-m)/2}\|\partial_x^m G(\cdot,t)\|_{L^{1/(1-\beta)}}|y|^\beta \\
&\le C\sum_{m=0}^{l+1}(1+t)^{-(l+1-m)/2}t^{-\beta/2-m/2}|y|^\beta \\
&\le Ct^{-(1+\beta+l)/2}|y|^\beta.
\end{aligned}
\tag{2.5.12}
$$

From (2.5.10) and (2.5.12), we have

$$
\begin{aligned}
|\partial_x^l U[z_0](x,t,0)| &\le C\int_{\mathbb{R}}|z_0(y)|t^{-(1+\beta+l)/2}|y|^\beta dy \\
&\le Ct^{-(1+\beta+l)/2}\int_{\mathbb{R}}|z_0(y)|(1+|y|^\beta)dy.
\end{aligned}
\tag{2.5.13}
$$

Hence we have (2.5.7).

Next, we shall prove (2.5.8). From (2.5.9), we have

$$
\begin{aligned}
\|\partial_x^l U[z_0](x,t,0)\|_{L^1} &\le \int_{\mathbb{R}}\int_{\mathbb{R}}|z_0(y)|\left|\int_0^y \partial_x^{l+1}(G(x-\xi,t)\eta(x,t))\eta(\xi,0)^{-1}d\xi\right|dydx \\
&= \int_{\mathbb{R}}|z_0(y)|\int_{\mathbb{R}}\left|\int_0^y \partial_x^{l+1}(G(x-\xi,t)\eta(x,t))\eta(\xi,0)^{-1}d\xi\right|dxdy \\
&\equiv \int_{\mathbb{R}}|z_0(y)|I_2(y,t)dy.
\end{aligned}
$$

In order to show (2.5.8), it suffices to get the following estimates.

$$I_2 \leq Ct^{-l/2}, \tag{2.5.14}$$

$$I_2 \leq C|y|t^{-(1+l)/2}. \tag{2.5.15}$$

Indeed, if we have shown these estimates, then we get

$$\begin{aligned}
\|\partial_x^l U[z_0](x,t,0)\|_{L^1} &\leq C\int_{\mathbb{R}} |z_0(y)| I_2^{1-\beta} I_2^{\beta} dy \\
&\leq C\int_{\mathbb{R}} |z_0(y)| t^{-l(1-\beta)/2} t^{-(1+l)\beta/2} |y|^{\beta} dy \\
&\leq Ct^{-(\beta+l)/2} \|z_0\|_{L^1_\beta}.
\end{aligned}$$

First, we shall prove (2.5.14). It follows that

$$\begin{aligned}
I_2 &\leq C\int_{\mathbb{R}} \left| \eta(x,t) \int_0^y \partial_x^{l+1} G(x-\xi,t)\eta(\xi,0)^{-1} d\xi \right| dx \\
&\quad + C\sum_{m=0}^{l} \int_{\mathbb{R}} \left| \partial_x^{l+1-m}\eta(x,t) \int_0^y \partial_x^m G(x-\xi,t)\eta(\xi,0)^{-1} d\xi \right| dx \\
&\equiv I_{2.1} + I_{2.2}.
\end{aligned} \tag{2.5.16}$$

By integretion by parts with respect to ξ, Lemma 2.4.1 and (2.4.8), we have

$$\begin{aligned}
I_{2.1} &\leq C\int_{\mathbb{R}} |\partial_x^l G(x-y,t)\eta(y,0)^{-1} - \partial_x^l G(x,t)\eta(0,0)^{-1}| dx \\
&\quad + C\int_{\mathbb{R}} \left| \int_0^y \partial_x^l G(x-\xi,t)\partial_\xi(\eta(\xi,0)^{-1}) d\xi \right| dx \\
&\leq C\int_{\mathbb{R}} |\partial_x^l G(x,t)| dx + C\int_{\mathbb{R}} |\partial_\xi(\eta(\xi,0)^{-1})| \int_{\mathbb{R}} |\partial_x^l G(x-\xi,t)| dx d\xi \\
&\leq Ct^{-l/2}.
\end{aligned} \tag{2.5.17}$$

On the other hand, from Lemma 2.4.1, Lemma 2.4.2 and (2.4.8), we have

$$\begin{aligned}
I_{2.2} &\leq C\sum_{m=0}^{l} \int_{\mathbb{R}} |\partial_x^{l+1-m}\eta(x,t)| dx \int_{\mathbb{R}} |\partial_\xi^m G(\xi,t)| d\xi \\
&\leq C\sum_{m=0}^{l} \|\partial_x^{l+1-m}\eta(\cdot,t)\|_{L^1} \|\partial_x^m G(\cdot,t)\|_{L^1} \\
&\leq C\sum_{m=0}^{l} (1+t)^{-(l-m)/2} t^{-m/2} \\
&\leq Ct^{-l/2}.
\end{aligned} \tag{2.5.18}$$

Therefore, we have (2.5.14).

Next we prove (2.5.15). From (2.4.8), we get

$$
\begin{aligned}
I_2 &\leq C \sum_{m=0}^{l+1} \int_{\mathbb{R}} \left| \partial_x^{l+1-m} \eta(x,t) \int_0^y \partial_x^m G(x-\xi,t) \eta(\xi,0)^{-1} d\xi \right| dx \\
&\leq C \sum_{m=0}^{l+1} \|\partial_x^{l+1-m} \eta(\cdot,t)\|_{L^\infty} \int_{\mathbb{R}} \left| \int_0^y \partial_x^m G(x-\xi,t) d\xi \right| dx.
\end{aligned} \tag{2.5.19}
$$

If $y > 0$, we have

$$
\begin{aligned}
\int_{\mathbb{R}} \left| \int_0^y \partial_x^m G(x-\xi,t) d\xi \right| dx &\leq \int_{\mathbb{R}} \int_0^y |\partial_x^m G(x-\xi,t)| d\xi dx \\
&= \int_0^y \int_{\mathbb{R}} |\partial_x^m G(x-\xi,t)| dx d\xi \\
&= \|\partial_x^m G(\cdot,t)\|_{L^1} y.
\end{aligned}
$$

Similarly, if $y < 0$, we have

$$
\int_{\mathbb{R}} \left| \int_0^y \partial_x^m G(x-\xi,t) d\xi \right| dx \leq \|\partial_x^m G(\cdot,t)\|_{L^1} (-y).
$$

Therefore, from Lemma 2.4.1 and Lemma 2.4.2, we have

$$
\begin{aligned}
I_2 \leq& C \sum_{m=0}^{l+1} \|\partial_x^{l+1-m} \eta(\cdot,t)\|_{L^\infty} \|\partial_x^m G(\cdot,t)\|_{L^1} |y| \\
\leq& C \sum_{m=0}^{l+1} (1+t)^{-(l+1-m)/2} t^{-m/2} |y| \\
\leq& C t^{-(1+l)/2} |y|.
\end{aligned} \tag{2.5.20}
$$

Hence we get (2.5.15). This completes the proof. □

Lemma 2.5.3. *Let m be a positive integer. Assume that $|\delta| \leq 1$, $z_0 \in H^m(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x) dx = 0$. Then, the estimate*

$$
\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^2} \leq C(1+t)^{-(1/4+l/2)} \|z_0\|_{L^1} + Ce^{-t} \|z_0\|_{H^l}, \quad t > 0 \tag{2.5.21}
$$

holds for any integer $0 \leq l \leq m$.

Proof. From (2.5.2), we have

$$\begin{aligned}
&\partial_x^l U[z_0](x,t,0) \\
&= \int_{\mathbb{R}} \partial_x^{l+1}(G(x-y,t)\eta(x,t))\eta(y,0)^{-1}\int_{-\infty}^y z_0(\xi)d\xi dy \\
&= \sum_{m=0}^{l+1}\binom{l+1}{m}\partial_x^{l+1-m}\eta(x,t)\int_{\mathbb{R}}\partial_x^m G(x-y,t)\eta(y,0)^{-1}\int_{-\infty}^y z_0(\xi)d\xi dy \\
&= \partial_x^{l+1}\eta(x,t)\int_{\mathbb{R}} G(x-y,t)\eta(y,0)^{-1}\int_{-\infty}^y z_0(\xi)d\xi dy \\
&\quad + \sum_{m=1}^{l+1}\binom{l+1}{m}\partial_x^{l+1-m}\eta(x,t)\int_{\mathbb{R}}\partial_x^m G(x-y,t)\eta(y,0)^{-1}\int_{-\infty}^y z_0(\xi)d\xi dy \\
&= \partial_x^{l+1}\eta(x,t)\int_{\mathbb{R}} G(x-y,t)\eta(y,0)^{-1}\int_{-\infty}^y z_0(\xi)d\xi dy \\
&\quad + \sum_{m=1}^{l+1}\binom{l+1}{m}\partial_x^{l+1-m}\eta(x,t)\int_{\mathbb{R}}\partial_x^{m-1} G(x-y,t)\partial_y J(y)dy,
\end{aligned} \tag{2.5.22}$$

where we set

$$J(y) = \eta(y,0)^{-1}\int_{-\infty}^y z_0(\xi)d\xi. \tag{2.5.23}$$

Therefore, we have from Lemma 2.4.1, Lemma 2.4.2 and (2.4.8)

$$\begin{aligned}
&\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^2} \\
&\le C\|\partial_x^{l+1}\eta(\cdot,t)\|_{L^2}\|G(\cdot,t)\|_{L^1}\|z_0\|_{L^1} \\
&\quad + C\sum_{m=1}^{l+1}\|\partial_x^{l+1-m}\eta(\cdot,t)\|_{L^\infty}\|\partial_x^{m-1}e^{t\Delta}[\partial_x J]\|_{L^2} \\
&\le C(1+t)^{-l/2-1/4}\|z_0\|_{L^1} + C\sum_{m=1}^{l+1}(1+t)^{-(l+1-m)/2}\|\partial_x^{m-1}e^{t\Delta}[\partial_x J]\|_{L^2},
\end{aligned} \tag{2.5.24}$$

where we have used the notation $e^{t\Delta}g = \mathcal{F}^{-1}[e^{-t|\xi|^2}\hat{g}(\xi)]$. From (2.5.23), Lemma 2.4.1 and Lemma 2.4.2, we have

$$\begin{aligned}
\|\partial_x J\|_{L^1} &= \left\|\partial_x(\eta(x,0)^{-1})\int_{-\infty}^x z_0(\xi)d\xi + \eta(x,0)^{-1}z_0(x)\right\|_{L^1(\mathbb{R}_x)} \\
&\le C\left\|\chi(x,0)\int_{-\infty}^x z_0(\xi)d\xi\right\|_{L^1(\mathbb{R}_x)} + C\|z_0\|_{L^1} \\
&\le C\|z_0\|_{L^1},
\end{aligned} \tag{2.5.25}$$

and for $m \geq 1$

$$\begin{aligned}\|\partial_x^m J\|_{L^2} \leq & C\sum_{n=0}^{m}\left\|\partial_x^{m-n}(\eta(x,0)^{-1})\partial_x^n\int_{-\infty}^{x} z_0(\xi)d\xi\right\|_{L^2(\mathbb{R}_x)} \\ \leq & C\|\partial_x^m(\eta(\cdot,0)^{-1})\|_{L^2}\|z_0\|_{L^1} + C\sum_{n=1}^{m}\|\partial_x^{m-n}(\eta(\cdot,0)^{-1})\|_{L^2}\|\partial_x^{n-1}z_0\|_{L^2} \\ \leq & C(\|z_0\|_{L^1} + \|z_0\|_{H^{m-1}}).\end{aligned} \tag{2.5.26}$$

From Lemma 2.3.2 with $k = 0$ and (2.5.25) for $1 \leq m \leq l+1$, we have

$$\begin{aligned}\|\partial_x^{m-1}e^{t\Delta}[\partial_x J]\|_{L^2} \leq & C(1+t)^{-1/4-(m-1)/2}\|\partial_x J\|_{L^1} + Ce^{-t}\|\partial_x^m J\|_{L^2} \\ \leq & C(1+t)^{-1/4-(m-1)/2}\|z_0\|_{L^1} + Ce^{-t}(\|z_0\|_{L^1} + \|z_0\|_{H^{m-1}}) \\ \leq & C(1+t)^{-1/4-(m-1)/2}\|z_0\|_{L^1} + Ce^{-t}\|z_0\|_{H^{m-1}}.\end{aligned}$$

Hence we get (2.5.21). □

Lemma 2.5.2 and Lemma 2.5.3 lead to the following uniform estimate:

Corollary 2.5.4. *Let m be a positive integer. Assume that $|\delta| \leq 1$, $z_0 \in H^m(\mathbb{R}) \cap L^1_1(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then the estimate*

$$\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^2} \leq C(\|z_0\|_{H^l} + \|z_0\|_{L^1_1})(1+t)^{-3/4-l/2}, \quad t > 0 \tag{2.5.27}$$

holds for any integer $0 \leq l \leq m$.

For the second term of (2.5.3), the following estimate was also given by Kato [31]:

Lemma 2.5.5. *Let m be a positive integer. Assume that $|\delta| \leq 1$ and $\lambda \in C^0(0,\infty;H^m) \cap C^0(0,\infty;W^{m,1})$. Then the estimate*

$$\begin{aligned}&\left\|\partial_x^l\int_0^t U[\partial_x\lambda(\tau)](\cdot,t,\tau)d\tau\right\|_{L^2} \\ &\leq C\int_0^{t/2}(1+t-\tau)^{-3/4-l/2}\|\lambda(\cdot,\tau)\|_{L^1}d\tau \\ &\quad + C\sum_{n=0}^{l}\int_{t/2}^{t}(1+t-\tau)^{-3/4}(1+\tau)^{-(l-n)/2}\|\partial_x^n\lambda(\cdot,\tau)\|_{L^1}d\tau \\ &\quad + C\sum_{n=0}^{l}\left(\int_0^t e^{-(t-\tau)}(1+\tau)^{-(l-n)}\|\partial_x^n\lambda(\cdot,\tau)\|_{L^2}^2 d\tau\right)^{1/2}\end{aligned} \tag{2.5.28}$$

holds for any integer $0 \leq l \leq m$.

Proof. Since

$$\begin{aligned}
&\partial_x^l \int_0^t U[\partial_x \lambda(\tau)](x,t,\tau)d\tau \\
&= \int_0^t \sum_{n=0}^{l+1} \binom{l+1}{n} \partial_x^{l+1-n}\eta(x,t) \int_{\mathbb{R}} \partial_x^n G(x-y,t-\tau)(\eta(y,\tau))^{-1}\lambda(y,\tau)dyd\tau,
\end{aligned}$$

from Lemma 2.4.1, we have

$$\begin{aligned}
\left\| \partial_x^l \int_0^t U[\partial_x \lambda(\tau)](\cdot,t,\tau)d\tau \right\|_{L^2} &\le C \sum_{n=0}^{l+1} \|\partial_x^{l+1-n}\eta(\cdot,t)\|_{L^\infty} \|\partial_x^n I(\cdot,t)\|_{L^2} \\
&\le C \sum_{n=0}^{l+1} (1+t)^{-(l+1-n)/2} \|\partial_x^n I(\cdot,t)\|_{L^2},
\end{aligned} \tag{2.5.29}$$

where we put

$$\begin{aligned}
I(x,t) &\equiv \int_0^t \int_{\mathbb{R}} G(x-y,t-\tau)(\eta(y,\tau))^{-1}\lambda(y,\tau)dyd\tau \\
&= \int_0^t e^{(t-\tau)\Delta}[(\eta^{-1}\lambda)(\tau)](x)d\tau.
\end{aligned} \tag{2.5.30}$$

By Lemma 2.3.2 and the Schwarz inequality, we have

$$\begin{aligned}
\|I(\cdot,t)\|_{L^2} &\le C \int_0^t \|e^{(t-\tau)\Delta}[(\eta^{-1}\lambda)(\tau)](\cdot)\|_{L^2} d\tau \\
&\le C \int_0^t \left((1+t-\tau)^{-1/4} \|\lambda(\cdot,\tau)\|_{L^1} + e^{-(t-\tau)} \|\lambda(\cdot,\tau)\|_{L^2} \right) d\tau \\
&\le C \int_0^t (1+t-\tau)^{-1/4} \|\lambda(\cdot,\tau)\|_{L^1} d\tau \\
&\quad + C \left(\int_0^t e^{-(t-\tau)} d\tau \right)^{1/2} \left(\int_0^t e^{-(t-\tau)} \|\lambda(\cdot,\tau)\|_{L^2}^2 d\tau \right)^{1/2} \\
&\le C \int_0^t (1+t-\tau)^{-1/4} \|\lambda(\cdot,\tau)\|_{L^1} d\tau \\
&\quad + C \left(\int_0^t e^{-(t-\tau)} \|\lambda(\cdot,\tau)\|_{L^2}^2 d\tau \right)^{1/2}.
\end{aligned} \tag{2.5.31}$$

In the following, let $1 \le n \le l+1$. By Plancherel's theorem, we have

$$\begin{aligned}
\|\partial_x^n I(\cdot,t)\|_{L^2} &= \|(i\xi)^n \hat{I}(\xi,t)\|_{L^2} \\
&\le \|(i\xi)^n \hat{I}(\xi,t)\|_{L^2(|\xi|\le 1)} + \|(i\xi)^n \hat{I}(\xi,t)\|_{L^2(|\xi|\ge 1)} \\
&\equiv I_1 + I_2.
\end{aligned}$$

First, we evaluate I_1. From (2.5.30), we have

$$\begin{aligned}
I_1 &\le \int_0^t \|(i\xi)^n e^{-(t-\tau)|\xi|^2} \mathcal{F}[\eta^{-1}\lambda](\xi,\tau)\|_{L^2(|\xi|\le 1)} d\tau \\
&= \left(\int_0^{t/2} + \int_{t/2}^t \right) \|(i\xi)^n e^{-(t-\tau)|\xi|^2} \mathcal{F}[\eta^{-1}\lambda](\xi,\tau)\|_{L^2(|\xi|\le 1)} d\tau \\
&\equiv I_{1.1} + I_{1.2}.
\end{aligned} \tag{2.5.32}$$

Since

$$\int_{|\xi|\le 1} |\xi|^j e^{-2(t-\tau)|\xi|^2} d\xi \le C(1+t-\tau)^{-(j+1)/2}, \quad j \ge 0, \tag{2.5.33}$$

from (2.4.8), we have

$$\begin{aligned} I_{1.1} &\le C\int_0^{t/2} \sup_{|\xi|\le 1} |\mathcal{F}[\eta^{-1}\lambda](\xi,\tau)| \Big(\int_{|\xi|\le 1} |\xi|^{2n} e^{-2(t-\tau)|\xi|^2} d\xi\Big)^{1/2} d\tau \\ &\le C\int_0^{t/2} (1+t-\tau)^{-n/2-1/4} \|\lambda(\cdot,\tau)\|_{L^1} d\tau. \end{aligned} \tag{2.5.34}$$

It follows from Lemma 2.4.2 and (2.5.33) that

$$\begin{aligned} I_{1.2} &\le C\int_{t/2}^{t} \sup_{|\xi|\le 1} |(i\xi)^{n-1}\mathcal{F}[\eta^{-1}\lambda](\xi,\tau)| \Big(\int_{|\xi|\le 1} |\xi|^{2} e^{-2(t-\tau)|\xi|^2} d\xi\Big)^{1/2} d\tau \\ &\le C\int_{t/2}^{t} (1+t-\tau)^{-3/4} \|\partial_x^{n-1}(\eta^{-1}\lambda)(\cdot,\tau)\|_{L^1} d\tau \\ &\le C\int_{t/2}^{t} (1+t-\tau)^{-3/4} \Big\|\sum_{m=0}^{n-1} \binom{n-1}{m} \partial_x^{n-1-m}(\eta(\cdot,\tau)^{-1})\partial_x^m\lambda(\cdot,\tau)\Big\|_{L^1} d\tau \\ &\le C\int_{t/2}^{t} (1+t-\tau)^{-3/4} \sum_{m=0}^{n-1} (1+\tau)^{-(n-1-m)/2} \|\partial_x^m\lambda(\cdot,\tau)\|_{L^1} d\tau. \end{aligned} \tag{2.5.35}$$

Therefore, we have from (2.5.34) and (2.5.35)

$$\begin{aligned} I_1 &\le C\int_0^{t/2} (1+t-\tau)^{-n/2-1/4} \|\lambda(\cdot,\tau)\|_{L^1} d\tau \\ &\quad + C\int_{t/2}^{t} (1+t-\tau)^{-3/4} \sum_{m=0}^{n-1} (1+\tau)^{-(n-1-m)/2} \|\partial_x^m\lambda(\cdot,\tau)\|_{L^1} d\tau. \end{aligned} \tag{2.5.36}$$

Next, we evaluate I_2. For $|\xi| \ge 1$, by the Schwarz inequality, we have

$$\begin{aligned} |(i\xi)^n \hat{I}(\xi,t)| &= \Big|(i\xi)^n \int_0^t e^{-(t-\tau)|\xi|^2} \mathcal{F}[\eta^{-1}\lambda](\xi,\tau) d\tau\Big| \\ &\le C\int_0^t |\xi| e^{-(t-\tau)|\xi|^2} |(i\xi)^{n-1}\mathcal{F}[\eta^{-1}\lambda](\xi,\tau)| d\tau \\ &\le C\Big(\int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} d\tau\Big)^{1/2} \Big(\int_0^t e^{-(t-\tau)|\xi|^2} |(i\xi)^{n-1}\mathcal{F}[\eta^{-1}\lambda](\xi,\tau)|^2 d\tau\Big)^{1/2} \\ &\le C\Big(\int_0^t e^{-(t-\tau)|\xi|^2} |(i\xi)^{n-1}\mathcal{F}[\eta^{-1}\lambda](\xi,\tau)|^2 d\tau\Big)^{1/2}. \end{aligned}$$

Therefore, from Lemma 2.4.2, we have

$$
\begin{aligned}
I_2 &\le C\biggl(\int_{|\xi|\ge 1}\int_0^t e^{-(t-\tau)|\xi|^2}|(i\xi)^{n-1}\mathcal{F}[\eta^{-1}\lambda](\xi,\tau)|^2 d\tau d\xi\biggr)^{1/2} \\
&\le C\biggl(\int_0^t e^{-(t-\tau)}\int_{|\xi|\ge 1}|(i\xi)^{n-1}\mathcal{F}[\eta^{-1}\lambda](\xi,\tau)|^2 d\xi d\tau\biggr)^{1/2} \\
&\le C\biggl(\int_0^t e^{-(t-\tau)}\|\partial_x^{n-1}(\eta^{-1}\lambda)(\cdot,\tau)\|_{L^2}^2 d\tau\biggr)^{1/2} \\
&= C\biggl(\int_0^t e^{-(t-\tau)}\biggl\|\sum_{m=0}^{n-1}\binom{n-1}{m}\partial_x^{n-1-m}(\eta(\cdot,\tau)^{-1})\partial_x^m\lambda(\cdot,\tau)\biggr\|_{L^2}^2 d\xi d\tau\biggr)^{1/2} \\
&\le C\biggl(\int_0^t e^{-(t-\tau)}\sum_{m=0}^{n-1}(1+\tau)^{-(n-1-m)}\|\partial_x^m\lambda(\cdot,\tau)\|_{L^2}^2 d\tau\biggr)^{1/2}.
\end{aligned}
\tag{2.5.37}
$$

Summing up (2.5.29), (2.5.31), (2.5.36) and (2.5.37), we obtain (2.5.28). This completes the proof. □

Moreover, for the latter sake, we prepare another estimate for the second term of (2.5.3). We will use the following estimate in the next chapter to prove Theorem 3.2.2 below.

Lemma 2.5.6. *Assume that $|\delta| \le 1$. Then the following estimate*

$$
\begin{aligned}
&\left\|\int_0^t U[\partial_x\lambda(\tau)](\cdot,t,\tau)d\tau\right\|_{L^\infty} \\
&\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2} \\
&\quad\times\biggl(\int_0^{t/2}(t-\tau)^{-1/2-n/2}\|\lambda(\cdot,\tau)\|_{L^1}d\tau + \int_{t/2}^t (t-\tau)^{-n/2}\|\lambda(\cdot,\tau)\|_{L^\infty}d\tau\biggr)
\end{aligned}
\tag{2.5.38}
$$

holds for $t > 0$.

Proof. From (2.5.2), we obtain

$$
\begin{aligned}
&\int_0^t U[\partial_x\lambda(\tau)](x,t,\tau)d\tau \\
&= \int_0^t\int_{\mathbb{R}}\partial_x(G(x-y,t-\tau)\eta(x,t))(\eta(y,\tau))^{-1}\lambda(y,\tau)dyd\tau \\
&= \sum_{n=0}^{1}\partial_x^{1-n}\eta(x,t)\biggl(\int_0^{t/2}+\int_{t/2}^t\biggr)\int_{\mathbb{R}}\partial_x^n G(x-y,t-\tau)(\eta(y,\tau))^{-1}\lambda(y,\tau)dyd\tau.
\end{aligned}
$$

By using Young's inequality, Lemma 2.4.1, Lemma 2.4.2 and (2.4.8), we obtain

$$
\begin{aligned}
&\left\| \int_0^t U[\partial_x \lambda(\tau)](\cdot,t,\tau)d\tau \right\|_{L^\infty} \\
&\le C \sum_{n=0}^{1} \|\partial_x^{1-n}\eta(\cdot,t)\|_{L^\infty} \\
&\quad \times \left(\int_0^{t/2} \|\partial_x^n G(t-\tau) * (\eta^{-1}\lambda)(\tau)\|_{L^\infty} d\tau + \int_{t/2}^t \|\partial_x^n G(t-\tau) * (\eta^{-1}\lambda)(\tau)\|_{L^\infty} d\tau \right) \\
&\le C \sum_{n=0}^{1} (1+t)^{-1/2+n/2} \\
&\quad \times \left(\int_0^{t/2} (t-\tau)^{-1/2-n/2} \|\lambda(\cdot,\tau)\|_{L^1} d\tau + \int_{t/2}^t (t-\tau)^{-n/2} \|\lambda(\cdot,\tau)\|_{L^\infty} d\tau \right).
\end{aligned}
$$

□

2.6 Proof of the Main Theorem

Finally in this section, we shall prove Theorem 2.2.1. First, we consider

$$
\begin{aligned}
v_t + (b\chi v)_x + \left(\frac{c}{3}\chi^3\right)_x + k\chi_{xxx} - v_{xx} = 0, \quad x \in \mathbb{R},\ t > 0, \\
v(x,0) = 0, \quad x \in \mathbb{R}.
\end{aligned}
\tag{2.6.1}
$$

The solution of this problem satisfies the following estimates.

Lemma 2.6.1. *Let l be a non-negative integer. Assume that $|\delta| \le 1$. Then we have*

$$
\|\partial_x^l v(\cdot,t)\|_{L^2} \le C|\delta|(1+t)^{-3/4-l/2}\log(2+t), \quad t \ge 0, \tag{2.6.2}
$$

where $v(x,t)$ is the solution to (2.6.1). *In particular, we get*

$$
\|\partial_x^l v(\cdot,t)\|_{L^\infty} \le C|\delta|(1+t)^{-1-l/2}\log(2+t), \quad t \ge 0. \tag{2.6.3}
$$

Proof. Applying Lemma 2.5.1, the solution $v(x,t)$ to (2.6.1) is given by

$$
v(x,t) = \int_0^t U\left[\left(\partial_x\left(-\frac{c}{3}\chi^3\right) - k\chi_{xxx}\right)(\tau)\right](x,t,\tau)d\tau. \tag{2.6.4}
$$

By Lemma 2.5.5, we have

$$
\begin{aligned}
&\|\partial_x^l v(\cdot,t)\|_{L^2} \\
&\le C\int_0^{t/2} (1+t-\tau)^{-3/4-l/2}\left(\|\chi^3(\cdot,\tau)\|_{L^1} + \|\chi_{xx}(\cdot,\tau)\|_{L^1}\right)d\tau \\
&\quad + C\sum_{m=0}^{l}\int_{t/2}^t (1+t-\tau)^{-3/4}(1+\tau)^{-(l-m)/2}\left(\|\partial_x^m(\chi^3(\cdot,\tau))\|_{L^1} + \|\partial_x^{m+2}\chi(\cdot,\tau)\|_{L^1}\right)d\tau \\
&\quad + C\sum_{m=0}^{l}\left(\int_0^t e^{-(t-\tau)}(1+\tau)^{-(l-m)}\left(\|\partial_x^m(\chi^3(\cdot,\tau))\|_{L^2}^2 + \|\partial_x^{m+2}\chi(\cdot,\tau)\|_{L^2}^2\right)d\tau\right)^{1/2} \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}
\tag{2.6.5}
$$

For $p \in [1, \infty]$, we obtain

$$
\begin{aligned}
&\|\partial_x^m(\chi^3(\cdot,\tau))\|_{L^p} \\
&\le C \sum_{n=0}^{m} \sum_{i=0}^{m-n} \|\partial_x^n \chi(\cdot,\tau)\|_{L^\infty} \|\partial_x^i \chi(\cdot,\tau)\|_{L^\infty} \|\partial_x^{m-n-i} \chi(\cdot,\tau)\|_{L^p} \\
&\le C|\delta|^3 \sum_{n=0}^{m} \sum_{i=0}^{m-n} (1+\tau)^{-1/2-n/2}(1+\tau)^{-1/2-i/2}(1+\tau)^{-1/2+1/2p-(m-n-i)/2} \\
&\le C|\delta|^3 (1+\tau)^{-(3-1/p+m)/2},
\end{aligned}
\tag{2.6.6}
$$

where we used Lemma 2.4.1. Thus we get

$$
\begin{aligned}
I_1 &\le C \int_0^{t/2} (1+t-\tau)^{-3/4-l/2} \big(|\delta|^3 (1+\tau)^{-1} + |\delta|(1+\tau)^{-1}\big) d\tau \\
&\le C|\delta|(1+t)^{-3/4-l/2} \log(2+t).
\end{aligned}
\tag{2.6.7}
$$

Moreover, we have from (2.6.6) and Lemma 2.4.1

$$
\begin{aligned}
I_2 &\le C \sum_{m=0}^{l} \int_{t/2}^{t} (1+t-\tau)^{-3/4}(1+\tau)^{-(l-m)/2} \\
&\qquad \times \big(|\delta|^3(1+\tau)^{-1-m/2} + |\delta|(1+\tau)^{-1-m/2}\big) d\tau \\
&\le C|\delta| \int_{t/2}^{t} (1+t-\tau)^{-3/4}(1+\tau)^{-1-l/2} d\tau \\
&\le C|\delta|(1+t)^{-3/4-l/2}
\end{aligned}
\tag{2.6.8}
$$

and

$$
\begin{aligned}
I_3 &\le C \sum_{m=0}^{l} \left(\int_0^t e^{-(t-\tau)} (1+\tau)^{-(l-m)} \big(|\delta|^6 (1+\tau)^{-5/2-m} + |\delta|^2 (1+\tau)^{-5/2-m}\big) d\tau \right)^{1/2} \\
&\le C|\delta| \left(\int_0^t e^{-(t-\tau)} (1+\tau)^{-5/2-l} d\tau \right)^{1/2} \\
&\le C|\delta|(1+t)^{-5/4-l/2}.
\end{aligned}
\tag{2.6.9}
$$

Summing up (2.6.5), (2.6.7), (2.6.8) and (2.6.9), we obtain (2.6.2). $\square$

Our first step to prove Theorem 2.2.1 is to show the following proposition:

Proposition 2.6.2. *If $u_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $E_{3,1}$ is sufficiently small, then the estimate*

$$
\|\partial_x^l (u(\cdot,t) - \chi(\cdot,t) - v(\cdot,t))\|_{L^2} \le C E_{3,1} (1+t)^{-3/4-l/2}, \quad t \ge 0 \tag{2.6.10}
$$

holds for $l = 0, 1$, where $\chi(x,t)$ is defined by (2.1.3) and $v(x,t)$ is the solution to (2.6.1), while $E_{3,1}$ is defined by (2.2.1). In particular, we get

$$
\|u(\cdot,t) - \chi(\cdot,t) - v(\cdot,t)\|_{L^\infty} \le C E_{3,1} (1+t)^{-1}, \quad t \ge 0. \tag{2.6.11}
$$

Proof. We set
$$w(x,t) \equiv u(x,t) - \chi(x,t) - v(x,t).$$
Then $w(x,t)$ satisfies the following equation:
$$\begin{aligned} w_t + (b\chi w)_x - w_{xx} &= g(w,\chi,v)_x - kw_{xxx} - kv_{xxx}, \quad x \in \mathbb{R},\ t > 0, \\ w(x,0) &= w_0(x), \quad x \in \mathbb{R}, \end{aligned}$$
where we have set
$$g(w,\chi,v) \equiv -\frac{b}{2}(w+v)^2 - \frac{c}{3}\Big(w^3 + v^3 + 3(w+v)(w+\chi)(\chi+v)\Big),$$
$$w_0(x) \equiv u_0(x) - \chi(x,0).$$
By the assumption on the initial data, (2.1.4) and (2.4.3), we get $w_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$ and $\int_{\mathbb{R}} w_0(x)dx = 0$. From Lemma 2.5.1, we obtain
$$\begin{aligned} w(x,t) &= U[w_0](x,t,0) + \int_0^t U[\partial_x g(w,\chi,v)(\tau)](x,t,\tau)d\tau \\ &\quad - k\int_0^t U[(w_{xxx} + v_{xxx})(\tau)](x,t,\tau)d\tau \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{2.6.12}$$
Now, we define $N(T)$ by
$$N(T) \equiv \sup_{0\le t\le T} \sum_{n=0}^{1} (1+t)^{3/4+n/2} \|\partial_x^n w(\cdot,t)\|_{L^2}. \tag{2.6.13}$$
Then, from the Sobolev inequality, we have
$$\|w(\cdot,t)\|_{L^\infty} \le N(T)(1+t)^{-1}. \tag{2.6.14}$$
Before evaluating I_1, I_2 and I_3, we prepare the following estimates for $l = 0, 1$:
$$\|\partial_x^l g(\cdot,t)\|_{L^1} \le C(1+t)^{-3/2-l/2}((|\delta|\log(2+t))^2 + N(T)^2), \tag{2.6.15}$$
$$\|\partial_x^l g(\cdot,t)\|_{L^2} \le C(1+t)^{-7/4-l/2}((|\delta|\log(2+t))^2 + N(T)^2). \tag{2.6.16}$$
We shall prove only (2.6.15), since we can prove (2.6.16) in the same way. Here and later, $|\delta|$ and $N(T)$ are assumed to be small. We put $h_1(x,t) = w(x,t) + v(x,t)$, $h_2(x,t) = w(x,t) + \chi(x,t)$ and $h_3(x,t) = \chi(x,t) + v(x,t)$. Also, let $m = 0, 1$ and $0 \le t \le T$. Then we have from Lemma 2.6.1 and (2.6.13)
$$\|\partial_x^m h_1(\cdot,t)\|_{L^2} \le C(1+t)^{-3/4-m/2}(|\delta|\log(2+t) + N(T)). \tag{2.6.17}$$
In particular, from the Sobolev inequality, we get
$$\|h_1(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1}(|\delta|\log(2+t) + N(T)). \tag{2.6.18}$$
Moreover, we get from Lemma 2.4.1 and (2.6.13)
$$\|\partial_x^m h_2(\cdot,t)\|_{L^2} \le C(1+t)^{-1/4-m/2}(|\delta| + N(T)). \tag{2.6.19}$$

In particular, we get

$$\|h_2(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1/2}(|\delta| + N(T)). \tag{2.6.20}$$

Moreover, from Lemma 2.4.1 and (2.6.3), we have

$$\|\partial_x^m h_3(\cdot,t)\|_{L^\infty} \le C|\delta|(1+t)^{-1/2-m/2}. \tag{2.6.21}$$

Hence, for $l = 0, 1$, we have from Lemma 2.6.1 (2.6.13), (2.6.14), (2.6.17), (2.6.19) and (2.6.21) that

$$\begin{aligned}
\|\partial_x^l((w+v)^2(\cdot,t))\|_{L^1} &\le C\sum_{m=0}^{l} \|\partial_x^m h_1(\cdot,t)\|_{L^2}\|\partial_x^{l-m} h_1(\cdot,t)\|_{L^2} \\
&\le C(1+t)^{-3/2-l/2}((|\delta|\log(2+t))^2 + N(T)^2),
\end{aligned} \tag{2.6.22}$$

$$\begin{aligned}
&\|\partial_x^l(w^3(\cdot,t))\|_{L^1} \\
&\le C\|\partial_x^l w(\cdot,t)\|_{L^2}\|w(\cdot,t)\|_{L^2}\|w(\cdot,t)\|_{L^\infty} \\
&\quad + C\sum_{m=0}^{l-1}\sum_{n=0}^{l-m} \|\partial_x^m w(\cdot,t)\|_{L^\infty}\|\partial_x^n w(\cdot,t)\|_{L^2}\|\partial_x^{l-m-n} w(\cdot,t)\|_{L^2} \\
&\le C(1+t)^{-3/4-l/2}(1+t)^{-3/4}(1+t)^{-1}N(T)^3 \\
&\quad + C\sum_{m=0}^{l-1}\sum_{n=0}^{l-m}(1+t)^{-1-m/2}(1+t)^{-3/4-n/2}(1+t)^{-3/4-l/2+m/2+n/2}N(T)^3 \\
&\le C(1+t)^{-5/2-l/2}N(T)^3,
\end{aligned} \tag{2.6.23}$$

$$\begin{aligned}
&\|\partial_x^l(v^3(\cdot,t))\|_{L^1} \\
&\le C\sum_{m=0}^{l}\sum_{n=0}^{l-m} \|\partial_x^m v(\cdot,t)\|_{L^\infty}\|\partial_x^n v(\cdot,t)\|_{L^2}\|\partial_x^{l-m-n} v(\cdot,t)\|_{L^2} \\
&\le C\sum_{m=0}^{l}\sum_{n=0}^{l-m}(1+t)^{-1-m/2}(1+t)^{-3/4-n/2}(1+t)^{-3/4-l/2+m/2+n/2}(|\delta|\log(2+t))^3 \\
&\le C(1+t)^{-5/2-l/2}(|\delta|\log(2+t))^3 \\
&\le C(1+t)^{-3/2-l/2}(|\delta|\log(2+t))^2
\end{aligned} \tag{2.6.24}$$

and

$$\begin{aligned}
&\|\partial_x^l(h_1h_2h_3)(\cdot,t)\|_{L^1} \\
&\le C\sum_{m=0}^{l}\sum_{n=0}^{l-m} \|\partial_x^m h_1(\cdot,t)\|_{L^2}\|\partial_x^n h_2(\cdot,t)\|_{L^2}\|\partial_x^{l-m-n} h_3(\cdot,t)\|_{L^\infty} \\
&\le C\sum_{m=0}^{l}\sum_{n=0}^{l-m}(1+t)^{-3/4-m/2}(|\delta|\log(2+t) + N(T)) \\
&\quad \times (1+t)^{-1/4-n/2}(|\delta| + N(T))(1+t)^{-1/2-l/2+m/2+n/2}|\delta| \\
&\le C(1+t)^{-3/2-l/2}(|\delta|\log(2+t) + N(T))(|\delta| + N(T)) \\
&\le C(1+t)^{-3/2-l/2}((|\delta|\log(2+t))^2 + N(T)^2).
\end{aligned} \tag{2.6.25}$$

We note that the second term in (2.6.23) does not appear for $l = 0$. Summing up (2.6.22) through (2.6.25), we obtain (2.6.15).

Now, we start with evaluation of I_1, I_2 and I_3. By using Lemma 2.5.4 and $|\delta| \le \|u_0\|_{L^1_1}$, we get

$$\|\partial_x^l I_1(\cdot,t)\|_{L^2} \le C(\|u_0\|_{H^l} + \|u_0\|_{L^1_1})(1+t)^{-3/4-l/2}, \quad l = 0,1,2,3. \tag{2.6.26}$$

From Lemma 2.5.5, for $l = 0, 1$, we have

$$\begin{aligned}\|\partial_x^l I_2(\cdot,t)\|_{L^2} \le{}& C\int_0^{t/2}(1+t-\tau)^{-3/4-l/2}\|g(\cdot,\tau)\|_{L^1}d\tau\\&+C\sum_{m=0}^{l}\int_{t/2}^{t}(1+t-\tau)^{-3/4}(1+\tau)^{-(l-m)/2}\|\partial_x^m g(\cdot,\tau)\|_{L^1}d\tau\\&+C\sum_{m=0}^{l}\Big(\int_0^t e^{-(t-\tau)}(1+\tau)^{-(l-m)}\|\partial_x^m g(\cdot,\tau)\|^2_{L^2}d\tau\Big)^{1/2}\\ \equiv{}& I_{2.1}+I_{2.2}+I_{2.3}.\end{aligned} \tag{2.6.27}$$

We have from (2.6.15) and (2.6.16) that

$$\begin{aligned}I_{2.1} &\le C\int_0^{t/2}(1+t-\tau)^{-3/4-l/2}(1+\tau)^{-3/2}\big((|\delta|\log(2+\tau))^2+N(T)^2\big)d\tau\\&\le C(1+t)^{-3/4-l/2}(|\delta|^2+N(T)^2),\end{aligned} \tag{2.6.28}$$

$$\begin{aligned}I_{2.2} &\le C\sum_{m=0}^{l}\int_{t/2}^{t}(1+t-\tau)^{-3/4}(1+\tau)^{-3/2-l/2}\big((|\delta|\log(2+\tau))^2+N(T)^2\big)d\tau\\&\le C(1+t)^{-3/2-l/2}((|\delta|\log(2+t))^2+N(T)^2)\int_{t/2}^{t}(1+t-\tau)^{-3/4}d\tau\\&\le C(1+t)^{-5/4-l/2}((|\delta|\log(2+t))^2+N(T)^2)\end{aligned} \tag{2.6.29}$$

and

$$\begin{aligned}I_{2.3} \le{}& C\sum_{m=0}^{l}\Big(\int_0^t e^{-(t-\tau)}(1+\tau)^{-7/2-l}\big((|\delta|\log(2+\tau))^4+N(T)^4\big)d\tau\Big)^{1/2}\\ \le{}& C((|\delta|\log(2+t))^2+N(T)^2)\Big(\int_0^t e^{-(t-\tau)}(1+\tau)^{-7/2-l}d\tau\Big)^{1/2}\\ \le{}& C(1+t)^{-7/4-l/2}((|\delta|\log(2+t))^2+N(T)^2).\end{aligned} \tag{2.6.30}$$

Summarizing (2.6.27) through (2.6.30), we obtain

$$\|\partial_x^l I_2(\cdot,t)\|_{L^2} \le C(|\delta|^2+N(T)^2)(1+t)^{-(3/4+l/2)}. \tag{2.6.31}$$

Finally, we evaluate I_3. First, since

$$\psi(x,t) = u(x,t) - \chi(x,t) = w(x,t) + v(x,t),$$

from Proposition 2.3.3, Lemma 2.4.1 and Lemma 2.6.1, we have

$$\|\partial_x^l \psi(\cdot,t)\|_{L^1} \le CE_{3,0} t^{-l/2}(1+t^{-1/4}), \quad t>0, \quad l=0,1,2,3, \tag{2.6.32}$$

$$\|\partial_x^l \psi(\cdot,t)\|_{L^2} \le CE_{3,0}(1+t)^{-1/4-l/2}, \quad t\ge 0, \quad l=0,1,2,3, \tag{2.6.33}$$

$$\|\partial_x^l \psi(\cdot,t)\|_{L^2} \le C(|\delta|\log(2+t)+N(T))(1+t)^{-3/4-l/2}, \quad l=0,1. \tag{2.6.34}$$

From the definition of I_3, it follows that

$$\begin{aligned}
\partial_x^l I_3(x,t) =& -k\partial_x^l \int_0^t U[\psi_{xxx}(\tau)](x,t,\tau)d\tau \\
=& -k\int_0^t\int_{\mathbb{R}} \partial_x^{l+1}(G(x-y,t-\tau)\eta(x,t))(\eta(y,\tau))^{-1}\psi_{yy}(y,\tau)dyd\tau \\
=& -k\sum_{n=0}^{l+1}\binom{l+1}{n}\partial_x^{l+1-n}\eta(x,t)\partial_x^n J(x,t),
\end{aligned}$$

where we put

$$J(x,t) = \int_0^t\int_{\mathbb{R}} G(x-y,t-\tau)(\eta(y,\tau))^{-1}\psi_{yy}(y,\tau)dyd\tau.$$

Therefore, from Lemma 2.4.2, we have

$$\|\partial_x^l I_3(\cdot,t)\|_{L^2} \le C\sum_{n=0}^{l+1}(1+t)^{-(l+1-n)/2}\|\partial_x^n J(\cdot,t)\|_{L^2}. \tag{2.6.35}$$

By making the integration by parts, we have

$$\begin{aligned}
J(x,t) =& \int_0^t\int_{\mathbb{R}} \partial_x^2 G(x-y,t-\tau)(\eta(y,\tau))^{-1}\psi(y,\tau)dyd\tau \\
&-2\int_0^t\int_{\mathbb{R}} \partial_x G(x-y,t-\tau)\partial_y(\eta(y,\tau)^{-1})\psi(y,\tau)dyd\tau \\
&+\int_0^t\int_{\mathbb{R}} G(x-y,t-\tau)\partial_y^2(\eta(y,\tau)^{-1})\psi(y,\tau)dyd\tau.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
\|\partial_x^n J(\cdot,t)\|_{L^2} \le& C\sum_{r=0}^{2}\left\|\partial_x^n\int_0^t\int_{\mathbb{R}}\partial_x^{2-r}G(x-y,t-\tau)\partial_y^r(\eta(y,\tau)^{-1})\psi(y,\tau)dyd\tau\right\|_{L^2} \\
\equiv& C\sum_{r=0}^{2} J_r.
\end{aligned} \tag{2.6.36}$$

First, we shall evaluate J_r for $r=0,1$. By Plancherel's Theorem, we have

$$\begin{aligned}
J_r \le& \left\|(i\xi)^{n+2-r}\int_0^t e^{-(t-\tau)|\xi|^2}\mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau)d\tau\right\|_{L^2(|\xi|\le 1)} \\
&+\left\|(i\xi)^{n+2-r}\int_0^t e^{-(t-\tau)|\xi|^2}\mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau)d\tau\right\|_{L^2(|\xi|\ge 1)} \\
\equiv& J_{r.1}+J_{r.2}
\end{aligned} \tag{2.6.37}$$

and

$$J_{r.1} \le \left(\int_0^{t/2} + \int_{t/2}^t\right) \|(i\xi)^{n+2-r} e^{-(t-\tau)|\xi|^2} \mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau)\|_{L^2(|\xi|\le 1)} d\tau \\ \equiv J_{r.1.1} + J_{r.1.2}. \tag{2.6.38}$$

Since

$$\int_{|\xi|\le 1} |\xi|^j e^{-2(t-\tau)|\xi|^2} d\xi \le C(1+t-\tau)^{-j/2-1/2}, \quad j \ge 0, \tag{2.6.39}$$

and (2.6.32), (2.4.8) and Lemma 2.4.2, we have

$$\begin{aligned} J_{r.1.1} &\le C \int_0^{t/2} \sup_{|\xi|\le 1} |\mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau)| \left(\int_{|\xi|\le 1} |\xi|^{2n+4-2r} e^{-2(t-\tau)|\xi|^2} d\xi\right)^{1/2} d\tau \\ &\le C \int_0^{t/2} (1+t-\tau)^{-n/2-5/4+r/2} \|((\partial_x^r(\eta^{-1}))\psi)(\cdot,\tau)\|_{L^1} d\tau \\ &\le C E_{3,0} (1+t)^{-n/2-5/4+r/2} \int_0^{t/2} (1+\tau)^{-r/2}(1+\tau^{-1/4}) d\tau \\ &\le C E_{3,0} (1+t)^{-n/2-1/4}. \end{aligned} \tag{2.6.40}$$

While, we have from Lemma 2.4.2, (2.4.8), (2.6.32), (2.6.33) and (2.6.39)

$$\begin{aligned} J_{r.1.2} &\le C \int_{t/2}^t \sup_{|\xi|\le 1} |(i\xi)^{n+1-r} \mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau)| \left(\int_{|\xi|\le 1} |\xi|^2 e^{-2(t-\tau)|\xi|^2} d\xi\right)^{1/2} d\tau \\ &\le C \int_{t/2}^t (1+t-\tau)^{-3/4} \|\partial_x^{n+1-r}((\partial_x^r(\eta^{-1}))\psi)(\cdot,\tau)\|_{L^1} d\tau \\ &\le C \sum_{m=0}^{n+1-r} \int_{t/2}^t (1+t-\tau)^{-3/4} \|\partial_x^{n+1-m}(\eta^{-1})(\cdot,\tau)\partial_x^m \psi(\cdot,\tau)\|_{L^1} d\tau \\ &\le C E_{3,0} \sum_{m=0}^{n+1-r} \int_{t/2}^t (1+t-\tau)^{-3/4}(1+\tau)^{-n/2-1/2+m/2}\tau^{-m/2}(1+\tau^{-1/4}) d\tau \\ &\le C E_{3,0}(1+t)^{-n/2-1/4}, \quad t \ge 1. \end{aligned} \tag{2.6.41}$$

For $|\xi| \ge 1$, by using the Schwarz inequality, we have

$$\begin{aligned} &\left|(i\xi)^{n+2-r} \int_0^t e^{-(t-\tau)|\xi|^2} \mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau) d\tau\right| \\ &\le C \int_0^t |\xi| e^{-(t-\tau)|\xi|^2} |(i\xi)^{n+1-r} \mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau)| d\tau \\ &\le C \left(\int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} d\tau\right)^{1/2} \left(\int_0^t e^{-(t-\tau)|\xi|^2} |(i\xi)^{n+1-r} \mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau)|^2 d\tau\right)^{1/2} \\ &\le C \left(\int_0^t e^{-(t-\tau)|\xi|^2} |(i\xi)^{n+1-r} \mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau)|^2 d\tau\right)^{1/2}. \end{aligned}$$

Therefore we have from Lemma 2.4.2 and (2.6.33)

$$
\begin{aligned}
J_{r.2} &\le C\Big(\int_{|\xi|\ge 1}\int_0^t e^{-(t-\tau)|\xi|^2}|(i\xi)^{n+1-r}\mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau)|^2 d\tau d\xi\Big)^{1/2}\\
&\le C\Big(\int_0^t e^{-(t-\tau)}\int_{|\xi|\ge 1}|(i\xi)^{n+1-r}\mathcal{F}[(\partial_x^r(\eta^{-1}))\psi](\xi,\tau)|^2 d\xi d\tau\Big)^{1/2}\\
&\le C\sum_{m=0}^{n+1-r}\Big(\int_0^t e^{-(t-\tau)}\|\partial_x^{n+1-m}(\eta^{-1})(\cdot,\tau)\partial_x^m\psi(\cdot,\tau)\|_{L^2}^2 d\tau\Big)^{1/2}\\
&\le C\sum_{m=0}^{n+1-r}\Big(\int_0^t e^{-(t-\tau)}|\delta|^2(1+\tau)^{-n-1+m}E_{3,0}^2(1+\tau)^{-1/2-m}d\tau\Big)^{1/2}\\
&\le CE_{3,0}\Big(\int_0^t e^{-(t-\tau)}(1+\tau)^{-3/2-n}d\tau\Big)^{1/2}\\
&\le CE_{3,0}(1+t)^{-n/2-3/4}.
\end{aligned}
\tag{2.6.42}
$$

From (2.6.37), (2.6.38), (2.6.40), (2.6.41) and (2.6.42), we obtain

$$
J_r \le CE_{3,0}(1+t)^{-n/2-1/4}, \quad t\ge 1, \quad r=0,1. \tag{2.6.43}
$$

Next, we evaluate J_2. For $n=0$, we have from Lemma 2.3.2 with $k=0$

$$
\begin{aligned}
J_2 &\le C\int_0^t \|e^{(t-\tau)\Delta}[((\partial_x^2(\eta^{-1}))\psi)(\tau)](\cdot)\|_{L^2}d\tau\\
&\le C\int_0^t \big((1+t-\tau)^{-1/4}\|((\partial_x^2(\eta^{-1}))\psi)(\cdot,\tau)\|_{L^1}+e^{-(t-\tau)}\|((\partial_x^2(\eta^{-1}))\psi)(\cdot,\tau)\|_{L^2}\big)d\tau\\
&\equiv J_{2.0.1}+J_{2.0.2}.
\end{aligned}
\tag{2.6.44}
$$

From Lemma 2.4.2 and (2.6.34), we have

$$
\begin{aligned}
J_{2.0.1} &= C\Big(\int_0^{t/2}+\int_{t/2}^t\Big)(1+t-\tau)^{-1/4}\|((\partial_x^2(\eta^{-1}))\psi)(\cdot,\tau)\|_{L^1}d\tau\\
&\le C(1+t)^{-1/4}\int_0^{t/2}|\delta|(1+\tau)^{-3/4}(|\delta|\log(2+\tau)+N(T))(1+\tau)^{-3/4}d\tau\\
&\quad + C\int_{t/2}^t(1+t-\tau)^{-1/4}|\delta|(1+\tau)^{-3/4}(|\delta|\log(2+\tau)+N(T))(1+\tau)^{-3/4}d\tau\\
&\le C(1+t)^{-1/4}(|\delta|^2+N(T)^2)+C(1+t)^{-3/4}(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2)\\
&\le C(1+t)^{-1/4}(|\delta|^2+N(T)^2)
\end{aligned}
\tag{2.6.45}
$$

and

$$
\begin{aligned}
J_{2.0.2} &\le C\int_0^t e^{-(t-\tau)}|\delta|(1+\tau)^{-1}(|\delta|\log(2+\tau)+N(T))(1+\tau)^{-3/4}d\tau\\
&\le C(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2)\int_0^t e^{-(t-\tau)}(1+\tau)^{-7/4}d\tau\\
&\le C(1+\tau)^{-7/4}(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2).
\end{aligned}
\tag{2.6.46}
$$

Therefore, we obtain from (2.6.44) through (2.6.46)

$$J_2 \le C(1+t)^{-1/4}(|\delta|^2 + N(T)^2), \quad n=0. \tag{2.6.47}$$

In the following, let $n \ge 1$. By using Plancherel's theorem, we have

$$\begin{aligned} J_2 \le & \left\| (i\xi)^n \int_0^t e^{-(t-\tau)|\xi|^2} \mathcal{F}[(\partial_x^2(\eta^{-1}))\psi](\xi,\tau)d\tau \right\|_{L^2(|\xi|\le 1)} \\ & + \left\| (i\xi)^n \int_0^t e^{-(t-\tau)|\xi|^2} \mathcal{F}[(\partial_x^2(\eta^{-1}))\psi](\xi,\tau)d\tau \right\|_{L^2(|\xi|\ge 1)} \\ \equiv & J_{2.1} + J_{2.2}. \end{aligned} \tag{2.6.48}$$

First, we estimate $J_{2.1}$. It follows that

$$\begin{aligned} J_{2.1} \le & \left(\int_0^{t/2} + \int_{t/2}^t \right) \|(i\xi)^n e^{-(t-\tau)|\xi|^2} \mathcal{F}[(\partial_x^2(\eta^{-1}))\psi](\xi,\tau)\|_{L^2(|\xi|\le 1)} d\tau \\ \equiv & J_{2.1.1} + J_{2.1.2}. \end{aligned} \tag{2.6.49}$$

From Lemma 2.4.2, (2.6.34) and (2.6.39), we have

$$\begin{aligned} J_{2.1.1} \le & C \int_0^{t/2} \sup_{|\xi|\le 1} |\mathcal{F}[(\partial_x^2(\eta^{-1}))\psi](\xi,\tau)| \left(\int_{|\xi|\le 1} |\xi|^{2n} e^{-2(t-\tau)|\xi|^2} d\xi \right)^{1/2} d\tau \\ \le & C \int_0^{t/2} (1+t-\tau)^{-n/2-1/4} \|((\partial_x^2(\eta^{-1}))\psi)(\cdot,\tau)\|_{L^1} d\tau \\ \le & C(1+t)^{-n/2-1/4} \int_0^{t/2} |\delta|(1+\tau)^{-3/4}(|\delta|\log(2+\tau) + N(T))(1+\tau)^{-3/4} d\tau \\ \le & C(1+t)^{-n/2-1/4}(|\delta|^2 + N(T)^2) \end{aligned} \tag{2.6.50}$$

and

$$\begin{aligned} J_{2.1.2} \le & C \int_{t/2}^t \sup_{|\xi|\le 1} |(i\xi)^{n-1} \mathcal{F}[(\partial_x^2(\eta^{-1}))\psi](\xi,\tau)| \left(\int_{|\xi|\le 1} |\xi|^2 e^{-2(t-\tau)|\xi|^2} d\xi \right)^{1/2} d\tau \\ \le & C \int_{t/2}^t (1+t-\tau)^{-3/4} \|\partial_x^{n-1}((\partial_x^2(\eta^{-1}))\psi)(\cdot,\tau)\|_{L^1} d\tau \\ \le & C \sum_{m=0}^{n-1} \int_{t/2}^t (1+t-\tau)^{-3/4} \|\partial_x^{n+1-m}(\eta^{-1})(\cdot,\tau)\partial_x^m \psi(\cdot,\tau)\|_{L^1} d\tau \\ \le & C \sum_{m=0}^{n-1} \int_{t/2}^t (1+t-\tau)^{-3/4} |\delta|(1+\tau)^{-n/2-1/2+m/2+1/4} \\ & \times (|\delta|\log(2+\tau) + N(T))(1+\tau)^{-m/2-3/4} d\tau \\ \le & C(1+t)^{-n/2-1}(|\delta|^2 \log(2+t) + |\delta|^2 + N(T)^2) \int_{t/2}^t (1+t-\tau)^{-3/4} d\tau \\ \le & C(1+t)^{-n/2-3/4}(|\delta|^2 \log(2+t) + |\delta|^2 + N(T)^2). \end{aligned} \tag{2.6.51}$$

In the same way as (2.6.42), we have from Lemma 2.4.2 and (2.6.34)

$$
\begin{aligned}
J_{2.2} &\leq C\biggl(\int_{|\xi|\geq 1}\int_0^t e^{-(t-\tau)|\xi|^2}|(i\xi)^{n-1}\mathcal{F}[(\partial_x^2(\eta^{-1}))\psi](\xi,\tau)|^2 d\tau d\xi\biggr)^{1/2} \\
&\leq C\biggl(\int_0^t e^{-(t-\tau)}\int_{|\xi|\geq 1}|(i\xi)^{n-1}\mathcal{F}[(\partial_x^2(\eta^{-1}))\psi](\xi,\tau)|^2 d\xi d\tau\biggr)^{1/2} \\
&\leq C\sum_{m=0}^{n-1}\biggl(\int_0^t e^{-(t-\tau)}\|\partial_x^{n+1-m}(\eta^{-1})(\cdot,\tau)\partial_x^m\psi(\cdot,\tau)\|_{L^2}^2 d\tau\biggr)^{1/2} \\
&\leq C\sum_{m=0}^{n-1}\biggl(\int_0^t e^{-(t-\tau)}|\delta|^2(1+\tau)^{-n-1+m} \\
&\qquad \times (|\delta|\log(2+\tau)+N(T))^2(1+\tau)^{-3/2-m}d\tau\biggr)^{1/2} \\
&\leq C(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2)\biggl(\int_0^t e^{-(t-\tau)}(1+\tau)^{-5/2-n}d\tau\biggr)^{1/2} \\
&\leq C(1+t)^{-n/2-5/4}(|\delta|^2\log(2+t)+|\delta|^2+N(T)^2).
\end{aligned} \tag{2.6.52}
$$

Therefore, summing up (2.6.47) through (2.6.52), for $n \geq 0$, we obtain

$$
J_2 \leq C(1+t)^{-n/2-1/4}(|\delta|^2+N(T)^2). \tag{2.6.53}
$$

Therefore, from (2.6.35), (2.6.36), (2.6.43) and (2.6.53), we have

$$
\|\partial_x^l I_3(\cdot,t)\|_{L^2} \leq C(E_{3,0}+|\delta|^2+N(T)^2)(1+t)^{-(3/4+l/2)}, \quad t\geq 1, \quad l=0,1. \tag{2.6.54}
$$

Since $|\delta| \leq E_{3,1}$, summarizing (2.6.12), (2.6.26), (2.6.31) and (2.6.54), we obtain

$$
\|\partial_x^l w(\cdot,t)\|_{L^2} \leq C(1+t)^{-3/4-l/2}(E_{3,1}+N(T)^2), \quad 1\leq t\leq T, \quad l=0,1. \tag{2.6.55}
$$

For $0 \leq t \leq 1$, from (2.3.9), (2.4.3) and (2.6.2), we obtain

$$
\begin{aligned}
\|\partial_x^l w(\cdot,t)\|_{L^2} &\leq C\|\partial_x^l u(\cdot,t)\|_{L^2}+C\|\partial_x^l \chi(\cdot,t)\|_{L^2}+C\|\partial_x^l v(\cdot,t)\|_{L^2} \\
&\leq CE_{3,0}+C|\delta|, \quad 0\leq t\leq 1, \quad l=0,1.
\end{aligned} \tag{2.6.56}
$$

Finally, combining (2.6.55) and (2.6.56), we get

$$
(1+t)^{3/4+l/2}\|\partial_x^l w(\cdot,t)\|_{L^2} \leq C(E_{3,1}+N(T)^2), \quad 0\leq t\leq T, \quad l=0,1.
$$

Since $E_{3,1}$ is small, we obtain the desired estimate

$$
N(T) \leq CE_{3,1}.
$$

This completes the proof. □

In order to complete the proof of Theorem 2.2.1, it is sufficient to show Proposition 2.6.3 below. Here we need to improve the proof of Lemma 3 in [27] to avoid the factor $\sqrt{\log t}$.

Proposition 2.6.3. *Assume that $|\delta| \le 1$. Then the estimate*

$$\|v(\cdot,t) - V(\cdot,t)\|_{L^\infty} \le C|\delta|(1+t)^{-1}, \ t \ge 1 \tag{2.6.57}$$

holds. Here $v(x,t)$ is the solution to (2.6.1) *and $V(x,t)$ is defined by* (2.2.3).

Proof. We set

$$\begin{aligned}\Lambda(x,y,t,\tau) &\equiv \partial_x(G(x-y,t-\tau)\eta(x,t))\\ &= \partial_x G(x-y,t-\tau)\eta(x,t) + \frac{b}{2}\chi(x,t)G(x-y,t-\tau)\eta(x,t),\\ F(y,\tau) &\equiv \eta(y,\tau)^{-1}\left(-\frac{c}{3}\chi(y,\tau)^3 - k\chi_{yy}(y,\tau)\right).\end{aligned} \tag{2.6.58}$$

By Lemma 2.5.1 and (2.5.2), we have

$$\begin{aligned}v(x,t) &= \int_0^t \int_{\mathbb{R}} \Lambda(x,y,t,\tau)F(y,\tau)dyd\tau\\ &= \int_{t/2}^t \int_{\mathbb{R}} \Lambda(x,y,t,\tau)F(y,\tau)dyd\tau + \int_0^{t/2} \int_{\mathbb{R}} \Lambda(x,y,t,\tau)F(y,\tau)dyd\tau\\ &\equiv J_1 + J_2.\end{aligned} \tag{2.6.59}$$

First, we evaluate J_1. We shall show

$$\|J_1(\cdot,t)\|_{L^\infty} \le C|\delta|(1+t)^{-1}. \tag{2.6.60}$$

It follows that from (2.6.58)

$$\begin{aligned}J_1 = &-\frac{c}{3}\eta(x,t)\int_{t/2}^t \int_{\mathbb{R}} \left(\partial_x G(x-y,t-\tau) + \frac{b}{2}\chi(x,t)G(x-y,t-\tau)\right)\\ &\times (\eta(y,\tau))^{-1}\chi(y,\tau)^3 dyd\tau\\ &- k\eta(x,t)\int_{t/2}^t \int_{\mathbb{R}} \left(\partial_x G(x-y,t-\tau) + \frac{b}{2}\chi(x,t)G(x-y,t-\tau)\right)\\ &\times (\eta(y,\tau))^{-1}\chi_{yy}(y,\tau)dyd\tau\\ \equiv &\ J_{1.1} + J_{1.2}.\end{aligned}$$

For $J_{1.1}$, by making the integration by parts, we have

$$\begin{aligned}J_{1.1}(x,t) = &-\frac{c}{3}\eta(x,t)\int_{t/2}^t \int_{\mathbb{R}} G(x-y,t-\tau)\Big(\partial_y\left((\eta(y,\tau))^{-1}\chi(y,\tau)^3\right)\\ &+ \frac{b}{2}\chi(x,t)(\eta(y,\tau))^{-1}\chi(y,\tau)^3\Big)dyd\tau\\ = &-\frac{c}{3}\eta(x,t)\int_{t/2}^t \int_{\mathbb{R}} G(x-y,t-\tau)\Big(-\frac{b}{2}\chi(y,\tau)^4(\eta(y,\tau))^{-1}\\ &+ 3(\eta(y,\tau))^{-1}\chi(y,\tau)^2\chi_y(y,\tau) + \frac{b}{2}\chi(x,t)(\eta(y,\tau))^{-1}\chi(y,\tau)^3\Big)dyd\tau.\end{aligned}$$

Therefore, from Lemma 2.4.1, we obtain

$$\begin{aligned}\|J_{1.1}(\cdot,t)\|_{L^\infty} &\le C\int_{t/2}^{t}\|G(\cdot,t-\tau)\|_{L^1}\big(\|\chi(\cdot,\tau)\|_{L^\infty}^4+\|\chi(\cdot,\tau)\|_{L^\infty}^2\|\chi_x(\cdot,\tau)\|_{L^\infty}\\&\quad+\|\chi(\cdot,t)\|_{L^\infty}\|\chi(\cdot,\tau)\|_{L^\infty}^3\big)d\tau\\&\le C|\delta|^3\int_{t/2}^{t}((1+\tau)^{-2}+(1+t)^{-1/2}(1+\tau)^{-3/2})d\tau\\&\le C|\delta|^3(1+t)^{-1}.\end{aligned}\tag{2.6.61}$$

For $J_{1.2}$, by making the integration by parts, we have

$$\begin{aligned}J_{1.2}(x,t)&=-k\eta(x,t)\int_{t/2}^{t}\int_{\mathbb{R}}G(x-y,t-\tau)\bigg(\partial_y\left((\eta(y,\tau))^{-1}\chi_{yy}(y,\tau)\right)\\&\quad+\frac{b}{2}\chi(x,t)(\eta(y,\tau))^{-1}\chi_{yy}(y,\tau)\bigg)dyd\tau.\\&=-k\eta(x,t)\int_{t/2}^{t}\int_{\mathbb{R}}G(x-y,t-\tau)\bigg(-\frac{b}{2}\chi(y,\tau)\chi_{yy}(y,\tau)(\eta(y,\tau))^{-1}\\&\quad+(\eta(y,\tau))^{-1}\chi_{yyy}(y,\tau)+\frac{b}{2}\chi(x,t)(\eta(y,\tau))^{-1}\chi_{yy}(y,\tau)\bigg)dyd\tau.\end{aligned}$$

Therefore, from Lemma 2.4.1, we have

$$\begin{aligned}\|J_{1.2}(\cdot,t)\|_{L^\infty} &\le C\int_{t/2}^{t}\|G(\cdot,t-\tau)\|_{L^1}\big(\|\chi(\cdot,\tau)\|_{L^\infty}\|\chi_{xx}(\cdot,\tau)\|_{L^\infty}+\|\chi_{xxx}(\cdot,\tau)\|_{L^\infty}\\&\quad+\|\chi(\cdot,t)\|_{L^\infty}\|\chi_{xx}(\cdot,\tau)\|_{L^\infty}\big)d\tau\\&\le C|\delta|\int_{t/2}^{t}((1+\tau)^{-2}+(1+t)^{-1/2}(1+\tau)^{-3/2})d\tau\\&\le C|\delta|(1+t)^{-1}.\end{aligned}\tag{2.6.62}$$

Hence, from (2.6.61) and (2.6.62), we have (2.6.60).

Next, we evaluate J_2. Spliting the y-integral at $y=0$ and making the integration by parts, we obtain

$$\begin{aligned}J_2&=\int_0^{t/2}\int_{\mathbb{R}}\Lambda(x,y,t,\tau)F(y,\tau)dyd\tau\\&=\int_0^{t/2}\int_0^{\infty}\Lambda_y(x,y,t,\tau)\int_y^{\infty}F(q,\tau)dqdyd\tau\\&\quad-\int_0^{t/2}\int_{-\infty}^{0}\Lambda_y(x,y,t,\tau)\int_{-\infty}^{y}F(q,\tau)dqdyd\tau\\&\quad+\int_0^{t/2}\Lambda(x,0,t,\tau)\int_{\mathbb{R}}F(q,\tau)dqd\tau\\&\equiv J_3+J_4+J_5.\end{aligned}\tag{2.6.63}$$

First, we note that Lemma 2.4.1 yields

$$\begin{aligned}
&\sup_{0\le\tau\le t/2}\sup_{x\in\mathbb{R}}\sup_{y\in\mathbb{R}}|\Lambda_y(x,y,t,\tau)| \\
&\le C\sup_{0\le\tau\le t/2}(\|\partial_x^2 G(\cdot,t-\tau)\|_{L^\infty}+\|\chi(\cdot,t)\|_{L^\infty}\|\partial_x G(\cdot,t-\tau)\|_{L^\infty}) \\
&\le C\sup_{0\le\tau\le t/2}((t-\tau)^{-3/2}+(1+t)^{-1/2}(t-\tau)^{-1})\le Ct^{-3/2},
\end{aligned} \tag{2.6.64}$$

since

$$\Lambda_y(x,y,t,\tau)=-\eta(x,t)\biggl(\partial_x^2 G(x-y,t-\tau)+\frac{b}{2}\chi(x,t)\partial_x G(x-y,t-\tau)\biggr).$$

By making the integration by parts, we have

$$\begin{aligned}
&\int_y^\infty(\eta(q,\tau))^{-1}\chi_{qq}(q,\tau)dq \\
&=-(\eta(y,\tau))^{-1}\chi_y(y,\tau)+\frac{b}{4}\int_y^\infty(\eta(q,\tau))^{-1}(\chi(q,\tau)^2)_q dq \\
&=-(\eta(y,\tau))^{-1}\chi_y(y,\tau)-\frac{b}{4}(\eta(y,\tau))^{-1}\chi(y,\tau)^2+\frac{b^2}{8}\int_y^\infty(\eta(q,\tau))^{-1}\chi(q,\tau)^3 dq.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_y^\infty F(q,\tau)dq&=\int_y^\infty(\eta(q,\tau))^{-1}\biggl(-\frac{c}{3}\chi(q,\tau)^3-k\chi_{qq}(q,\tau)\biggr)dq \\
&=k(\eta(y,\tau))^{-1}\biggl(\chi_y(y,\tau)+\frac{b}{4}\chi(y,\tau)^2\biggr) \\
&\quad-\biggl(\frac{b^2k}{8}+\frac{c}{3}\biggr)\int_y^\infty(\eta(q,\tau))^{-1}\chi(q,\tau)^3 dq.
\end{aligned} \tag{2.6.65}$$

Similarly, we obtain

$$\begin{aligned}
\int_{-\infty}^y F(q,\tau)dq&=-k(\eta(y,\tau))^{-1}\biggl(\chi_y(y,\tau)+\frac{b}{4}\chi(y,\tau)^2\biggr) \\
&\quad-\biggl(\frac{b^2k}{8}+\frac{c}{3}\biggr)\int_{-\infty}^y(\eta(q,\tau))^{-1}\chi(q,\tau)^3 dq,
\end{aligned} \tag{2.6.66}$$

because

$$\begin{aligned}
&\int_{-\infty}^y(\eta(q,\tau))^{-1}\chi_{qq}(q,\tau)dq \\
&=(\eta(y,\tau))^{-1}\chi_y(y,\tau)+\frac{b}{4}\int_{-\infty}^y(\eta(q,\tau))^{-1}(\chi(q,\tau)^2)_q dq \\
&=(\eta(y,\tau))^{-1}\chi_y(y,\tau)+\frac{b}{4}(\eta(y,\tau))^{-1}\chi(y,\tau)^2+\frac{b^2}{8}\int_{-\infty}^y(\eta(q,\tau))^{-1}\chi(q,\tau)^3 dq.
\end{aligned}$$

From Lemma 2.4.1, (2.6.64) and (2.6.65), we have

$$
\begin{aligned}
&|J_3(x,t)| \\
&\le Ct^{-3/2}\int_0^{t/2}\int_0^{\infty}\left(|\chi_y(y,\tau)|+|\chi(y,\tau)|^2+\int_y^{\infty}|\chi(q,\tau)|^3dq\right)dyd\tau \\
&\le Ct^{-3/2}\left(\int_0^{t/2}\int_{\mathbb{R}}(|\chi_y(y,\tau)|+|\chi(y,\tau)|^2)dyd\tau+\int_0^{t/2}\int_0^{\infty}\int_0^{q}|\chi(q,\tau)|^3dydqd\tau\right) \\
&\le Ct^{-3/2}\left(\int_0^{t/2}(\|\chi_x(\cdot,\tau)\|_{L^1}+\|\chi(\cdot,\tau)\|_{L^2}^2)d\tau+\int_0^{t/2}\int_0^{\infty}q|\chi(q,\tau)|^3dqd\tau\right) \\
&\le Ct^{-3/2}\left(\int_0^{t/2}(|\delta|(1+\tau)^{-1/2}+|\delta|^2(1+\tau)^{-1/2})d\tau+\int_0^{t/2}\int_{\mathbb{R}}|y||\chi(y,\tau)|^3dyd\tau\right) \\
&\le C|\delta|(1+t)^{-1}+Ct^{-3/2}\int_0^{t/2}\int_{\mathbb{R}}|y||\chi(y,\tau)|^3dyd\tau.
\end{aligned}
$$

By using (2.4.2), we have

$$
\begin{aligned}
\int_{\mathbb{R}}|y||\chi(y,\tau)|^3dy &\le C|\delta|^3\int_{\mathbb{R}}(1+\tau)^{-3/2}e^{-3y^2/4(1+\tau)}|y|dy \\
&\le C|\delta|^3\int_{\mathbb{R}}(1+\tau)^{-1/2}|y|(1+\tau)^{-1}e^{-y^2/4(1+\tau)}dy \\
&\le C|\delta|^3\|\partial_x G(\cdot,1+\tau)\|_{L^1} \\
&\le C|\delta|^3(1+\tau)^{-1/2}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\|J_3(\cdot,t)\|_{L^\infty} &\le C|\delta|(1+t)^{-1}+Ct^{-3/2}\int_0^{t/2}|\delta|^3(1+\tau)^{-1/2}d\tau \\
&\le C|\delta|(1+t)^{-1},\quad t\ge 1.
\end{aligned}
\tag{2.6.67}
$$

Similarly, we have from (2.6.64) and (2.6.66)

$$
\begin{aligned}
&|J_4(x,t)| \\
&\le Ct^{-3/2}\int_0^{t/2}\int_{-\infty}^{0}\left(|\chi_y(y,\tau)|+|\chi(y,\tau)|^2+\int_{-\infty}^{y}|\chi(q,\tau)|^3dq\right)dyd\tau \\
&\le Ct^{-3/2}\left(\int_0^{t/2}\int_{\mathbb{R}}(|\chi_y(y,\tau)|+|\chi(y,\tau)|^2)dyd\tau+\int_0^{t/2}\int_{-\infty}^{0}\int_q^{0}|\chi(q,\tau)|^3dydqd\tau\right) \\
&\le Ct^{-3/2}\left(\int_0^{t/2}(\|\chi_x(\cdot,\tau)\|_{L^1}+\|\chi(\cdot,\tau)\|_{L^2}^2)d\tau+\int_0^{t/2}\int_{-\infty}^{0}(-q)|\chi(q,\tau)|^3dqd\tau\right) \\
&\le C|\delta|(1+t)^{-1}+Ct^{-3/2}\int_0^{t/2}\int_{\mathbb{R}}|y||\chi(y,\tau)|^3dyd\tau.
\end{aligned}
$$

Therefore, we have

$$
\|J_4(\cdot,t)\|_{L^\infty}\le C|\delta|(1+t)^{-1},\quad t\ge 1. \tag{2.6.68}
$$

Finally, we evaluate J_5. By the integration by parts, we get

$$\begin{aligned}\int_{\mathbb{R}} F(q,\tau)dq &= -\frac{c}{3}\int_{\mathbb{R}}(\eta(q,\tau))^{-1}\chi(q,\tau)^3dq - k\int_{\mathbb{R}}(\eta(q,\tau))^{-1}\chi_{qq}(q,\tau)dq \\ &= -\frac{c}{3}\int_{\mathbb{R}}(\eta(q,\tau))^{-1}\chi(q,\tau)^3dq - \frac{bk}{4}\int_{\mathbb{R}}(\eta(q,\tau))^{-1}(\chi(q,\tau)^2)_q dq \\ &= -\left(\frac{c}{3}+\frac{b^2k}{8}\right)\int_{\mathbb{R}}(\eta(q,\tau))^{-1}\chi(q,\tau)^3dq.\end{aligned}$$

From the definition of $\eta(x,t)$ and $\chi(x,t)$, and (2.1.11), we have

$$\begin{aligned}\int_{\mathbb{R}}(\eta(q,\tau))^{-1}\chi(q,\tau)^3dq &= \int_{\mathbb{R}}\eta_*\left(\frac{q}{\sqrt{1+\tau}}\right)^{-1}(1+\tau)^{-3/2}\chi_*\left(\frac{q}{\sqrt{1+\tau}}\right)^3 dq \\ &= (1+\tau)^{-1}\int_{\mathbb{R}}\eta_*(z)^{-1}\chi_*(z)^3dz \\ &= d(1+\tau)^{-1}.\end{aligned}$$

Thus, we obtain

$$\int_{\mathbb{R}} F(q,\tau)dq = -d\left(\frac{b^2k}{8}+\frac{c}{3}\right)(1+\tau)^{-1}.$$

Therefore, we have

$$\begin{aligned}J_5 &= -d\left(\frac{b^2k}{8}+\frac{c}{3}\right)\eta(x,t)\int_0^{t/2}(1+\tau)^{-1}\left(\partial_x G(x,t-\tau)+\frac{b}{2}\chi(x,t)G(x,t-\tau)\right)d\tau \\ &= -d\left(\frac{b^2k}{8}+\frac{c}{3}\right)\eta(x,t)\int_0^{t/2}(1+\tau)^{-1}\bigg(\partial_x G(x,t-\tau)-\partial_x G(x,t+1) \\ &\quad + \frac{b}{2}\chi(x,t)(G(x,t-\tau)-G(x,t+1))\bigg)d\tau \\ &\quad - d\left(\frac{b^2k}{8}+\frac{c}{3}\right)\eta(x,t)\left(\partial_x G(x,t+1)+\frac{b}{2}\chi(x,t)G(x,t+1)\right)\log\left(1+\frac{t}{2}\right) \\ &\equiv J_{5.1}+J_{5.2}.\end{aligned} \tag{2.6.69}$$

Since

$$\partial_x^l G(x,t-\tau)-\partial_x^l G(x,t+1) = -(1+\tau)\int_0^1(\partial_t\partial_x^l G)(x,1+t-\theta(1+\tau))d\theta$$

for $l=0,1$, we have

$$|\partial_x^l G(x,t-\tau)-\partial_x^l G(x,t+1)| \le C(1+\tau)(t-\tau)^{-3/2-l/2}.$$

From (2.1.11), (2.4.8) and Lemma 2.4.1, we obtain

$$|d| \le \int_{\mathbb{R}}|\eta_*(y)|^{-1}|\chi_*(y)|^3dy \le C\|\chi(\cdot,0)\|_{L^3}^3 \le C|\delta|^3.$$

Therefore, we obtain

$$
\begin{aligned}
&\|J_{5.1}(\cdot,t)\|_{L^\infty} \\
&\le C|\delta|^3 \int_0^{t/2} (1+\tau)^{-1}\Big((1+\tau)(t-\tau)^{-2} + (1+t)^{-1/2}(1+\tau)(t-\tau)^{-3/2}\Big)d\tau \\
&\le C|\delta|^3(1+t)^{-1}, \quad t \ge 1.
\end{aligned}
\tag{2.6.70}
$$

Finally, by the definition of $V(x,t)$, $J_{5.2}$ can be written as follows:

$$
\begin{aligned}
J_{5.2}(x,t) &= -d\left(\frac{b^2k}{8}+\frac{c}{3}\right)\eta(x,t)\left(\partial_x G(x,t+1) + \frac{b}{2}\chi(x,t)G(x,t+1)\right)\log\left(1+\frac{t}{2}\right) \\
&= V(x,t) + \frac{d}{4\sqrt{\pi}}\left(\frac{b^2k}{8}+\frac{c}{3}\right)V_*\left(\frac{x}{\sqrt{1+t}}\right)(1+t)^{-1}\left(\log(1+t) - \log\left(1+\frac{t}{2}\right)\right).
\end{aligned}
$$

Since $V_*(x)$ is bounded, we obtain

$$
\begin{aligned}
\|J_{5.2}(\cdot,t) - V(\cdot,t)\|_{L^\infty} &\le C|\delta|^3 \left\| V_*\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^\infty} \log\left(1+\frac{t/2}{1+t/2}\right)(1+t)^{-1} \\
&\le C|\delta|^3(1+t)^{-1}, \quad t \ge 1.
\end{aligned}
\tag{2.6.71}
$$

Therefore, summarizing (2.6.59), (2.6.60), (2.6.63) and (2.6.67) through (2.6.71), we obtain (2.6.57). □

Finally, we can complete the proof of the Main Theorem.

End of the proof of Theorem 2.2.1. Combining Proposition 2.6.2 and Proposition 2.6.3, we immediately have Theorem 2.2.1. □

Chapter 3

Consideration of the Effect of the Decay Rate of the Initial Data

In this chapter, we introduce the results given by the author in [8].

3.1 Introduction

In this chapter, we study the large time behavior of the global solutions to the generalized KdV-Burgers equation under a different assumption on the initial data from that of the previous chapter:

$$\begin{aligned} u_t + (f(u))_x + k u_{xxx} = u_{xx}, \quad x \in \mathbb{R}, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{2.1.1}$$

where $f(u) = (b/2)u^2 + (c/3)u^3$ and $b \neq 0$, $c, k \in \mathbb{R}$. Here and in the following, the mathematical formula that appeared so far is numbered with the same reference number. Concerning the initial data, we assume the following condition:

$$\exists \alpha > 1, \quad \exists C > 0 \ \ s.t. \ \ |u_0(x)| \le C(1+|x|)^{-\alpha}, \quad x \in \mathbb{R}. \tag{3.1.1}$$

The purpose of this study is to obtain an asymptotic profile of the solution $u(x,t)$ and to examine the optimality of its asymptotic rate. Especially, we investigate how the change of the decay rate of the initial value affects its asymptotic rate.

In the previous chapter, we studied the large time behavior of the solution to (2.1.1) under the assumption $u_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$. Indeed, we showed that the solution to (2.1.1) converges to the nonlinear diffusion wave defined by

$$\chi(x,t) \equiv \frac{1}{\sqrt{1+t}} \chi_* \left(\frac{x}{\sqrt{1+t}} \right), \quad x \in \mathbb{R}, \ t > 0, \tag{2.1.3}$$

where

$$\chi_*(x) \equiv \frac{1}{b} \frac{(e^{b\delta/2}-1)e^{-x^2/4}}{\sqrt{\pi} + (e^{b\delta/2}-1)\int_{x/2}^{\infty} e^{-y^2} dy}, \quad \delta \equiv \int_{\mathbb{R}} u_0(x) dx, \quad b \neq 0. \tag{2.1.4}$$

More precisely, if $\delta \neq 0$ and $(b^2k)/8 + c/3 \neq 0$, the following estimate is established:

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} = (\tilde{C} + o(1))(1+t)^{-1} \log(1+t) \quad as \quad t \to \infty,$$

where $\tilde{C}$ is a positive constant. This result tells us that the optimal asymptotic rate to the nonlinear diffusion wave is $t^{-1}\log t$ under the above assumptions. Here, we note that the condition $u_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$ is realized when $u_0 \in H^3(\mathbb{R})$ and $\alpha > 2$ in (3.1.1). Therefore, we can regard that the above previous result is corresponding to the case where the decay rate of the initial data u_0 is rapid. In the present chapter, we are interested in the large time behavior of the solution to (2.1.1) in the case that the initial data decays more slowly. However for (2.1.1) in the case of $1 < \alpha \leq 2$ in (3.1.1), it was not known that the optimal asymptotic rate to the nonlinear diffusion wave, up to the author's knowledge. On the other hand, it was studied that the asymptotic profile for the solution to the semilinear heat equation with slowly decaying data. Actually, Narazaki and Nishihara [50] studied the following initial value problem:

$$\begin{aligned} &u_t - u_{xx} = |u|^{p-1}u, \quad x \in \mathbb{R},\ t > 0, \\ &u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3.1.2}$$

where the data $u_0 \in C^0(\mathbb{R})$ satisfies $|u_0(x)| \leq C(1+|x|)^{-\gamma}$ for $0 < \gamma \leq 1$. In this case, if $p > 1 + 2/\gamma$ (supercritical case) and the data is sufficiently small, then the asymptotic profile is given by

$$\Psi(x,t) = c_\gamma \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}(1+|y|)^{-\gamma}dy,$$

provided that the data satisfies $\lim_{|x|\to\infty}(1+|x|)^\gamma u_0(x) = c_\gamma$. More precisely, they proved

$$\lim_{t\to\infty} a_\gamma(t)\|u(\cdot,t) - \Psi(\cdot,t)\|_{L^\infty} = 0,\ a_\gamma(t) = \begin{cases} (1+t)^{\gamma/2}, & 0 < \gamma < 1, \\ \frac{(1+t)^{1/2}}{\log(1+t)}, & \gamma = 1. \end{cases}$$

Moreover, in [50], the semilinear damped wave equation

$$\begin{aligned} &u_{tt} - u_{xx} + u_t = |u|^{p-1}, \quad x \in \mathbb{R},\ t > 0, \\ &u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \mathbb{R} \end{aligned} \tag{3.1.3}$$

with slowly decaying data is also treated. Furthermore, the damped wave equation (3.1.3) in two and three space dimensional cases were also studied. For the related results concerning (3.1.3), we also refer to [21, 22]. However, as we mentioned in the above, there are no results about the large time behavior for the generalized KdV-Burgers equation with slowly decaying data. Because of this, we would like to study the asymptotic profile for the solution to (2.1.1) when $1 < \alpha \leq 2$ in (3.1.1), which corresponds to the case of $0 < \gamma \leq 1$. Indeed, it is natural to assume that $u_0 \in L^1(\mathbb{R})$ which is indicated by (3.1.1), because the equation (2.1.1) leads to the conservation law: $\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx$. For this reason, we restrict our attention to the case of $1 < \alpha \leq 2$.

3.2 Main Results

In this section, we introduce the main results of [8]. In the following, we consider the initial value problem (2.1.1) with the condition (3.1.1) when $1 < \alpha \leq 2$. Our main results are as follows:

Theorem 3.2.1. *Assume the condition* (3.1.1) *with* $1 < \alpha \le 2$. *Let* $u_0 \in H^3(\mathbb{R})$ *and* $\|u_0\|_{L^1} + \|u_0\|_{H^3}$ *is sufficiently small. Then* (2.1.1) *has a unique global solution* $u(x,t) \in C^0([0,\infty); H^3)$. *Moreover, the solution satisfies the following estimate:*

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} \le C \begin{cases} (1+t)^{-\alpha/2}, & t \ge 0,\ 1 < \alpha < 2, \\ (1+t)^{-1}\log(2+t), & t \ge 0,\ \alpha = 2, \end{cases} \tag{3.2.1}$$

where $\chi(x,t)$ *is defined by* (2.1.3).

Furthermore, we examine the optimality of the asymptotic rate given in (3.2.1). Actually, we have constructed the second asymptotic profile corresponding to the decay rate of the initial data:

Theorem 3.2.2. *Assume the same conditions on* u_0 *in Theorem 3.2.1 are valid. We set*

$$z_0(x) \equiv \eta_*(x)^{-1} \int_{-\infty}^{x} (u_0(y) - \chi_*(y))dy. \tag{3.2.2}$$

If there exists $\lim_{x\to\pm\infty}(1+|x|)^{\alpha-1} z_0(x) \equiv c_\alpha^\pm$, *then the solution to* (2.1.1) *satisfies*

$$\lim_{t\to\infty}(1+t)^{\alpha/2}\|u(\cdot,t) - \chi(\cdot,t) - Z(\cdot,t)\|_{L^\infty} = 0,\ 1 < \alpha < 2, \tag{3.2.3}$$

$$\lim_{t\to\infty}\frac{(1+t)}{\log(1+t)}\|u(\cdot,t) - \chi(\cdot,t) - Z(\cdot,t) - V(\cdot,t)\|_{L^\infty} = 0,\ \alpha = 2, \tag{3.2.4}$$

where $\chi(x,t)$ *and* $V(x,t)$ *are defined by* (2.1.3) *and* (2.2.3), *respectively, while* $Z(x,t)$ *is defined by*

$$Z(x,t) \equiv \int_{\mathbb{R}} \frac{c_\alpha(y)\partial_x(G(x-y,t)\eta(x,t))}{(1+|y|)^{\alpha-1}}dy, \quad c_\alpha(y) \equiv \begin{cases} c_\alpha^+, & y \ge 0, \\ c_\alpha^-, & y < 0, \end{cases} \tag{3.2.5}$$

with $G(x,t)$ *and* $\eta(x,t)$ *being defined by* (2.4.1) *and* (2.4.6), *respectively. Moreover, if* $\delta \ne 0$, *there are positive constants* ν_0 *and* ν_1 *such that*

$$\|Z(\cdot,t)\|_{L^\infty} \ge \nu_0|\beta_0|(1+t)^{-\alpha/2}, \quad 1 < \alpha < 2, \tag{3.2.6}$$

$$\|Z(\cdot,t) + V(\cdot,t)\|_{L^\infty} \ge \nu_1|\beta_1|(1+t)^{-1}\log(1+t), \quad \alpha = 2 \tag{3.2.7}$$

holds for sufficiently large t, *where*

$$\begin{aligned} \beta_0 &\equiv (c_\alpha^+ - c_\alpha^-)\Gamma\left(\frac{3-\alpha}{2}\right) + \frac{(c_\alpha^+ + c_\alpha^-)b\chi_*(0)}{2-\alpha}\Gamma\left(2 - \frac{\alpha}{2}\right), \\ \beta_1 &\equiv \frac{c_\alpha^+ + c_\alpha^-}{2} - d\left(\frac{b^2k}{8} + \frac{c}{3}\right), \quad \Gamma(s) \equiv \int_0^\infty e^{-x}x^{s-1}dx, \quad s > 0, \end{aligned} \tag{3.2.8}$$

while δ *and* d *are defined by* (2.1.4) *and* (2.1.11), *respectively.*

From (3.2.3), (3.2.4), (3.2.6) and (3.2.7), we get the following lower bound estimate of $u - \chi$:

Corollary 3.2.3. *Under the same assumptions in Theorem 3.2.2, there are positive constants μ_0 and μ_1 such that the following estimate*

$$\|u(\cdot,t)-\chi(\cdot,t)\|_{L^\infty} \geq \begin{cases} \mu_0|\beta_0|(1+t)^{-\alpha/2}, & 1<\alpha<2, \\ \mu_1|\beta_1|(1+t)^{-1}\log(1+t), & \alpha=2 \end{cases} \tag{3.2.9}$$

holds for sufficiently large t.

Remark 3.2.4. From (3.2.1) and (3.2.9), if $\delta \neq 0$ and $\beta_0 \neq 0$, we see that the solution $u(x,t)$ converges to the nonlinear diffusion wave $\chi(x,t)$ at the optimal rate of $t^{-\alpha/2}$ when $1<\alpha<2$. In the case of $\alpha=2$, the optimal asymptotic rate to $\chi(x,t)$ is $t^{-1}\log t$ under the assumptions $\delta \neq 0$ and $\beta_1 \neq 0$.

Remark 3.2.5. In the case of $k=0$, the similar estimate to (3.2.1) and (3.2.9) were obtained by evaluating $u-\chi$ directly (cf. Kitagawa [37]). However, the proof of the lower bound estimate of $u-\chi$ in [37] is only valid for the concrete perturbations such as $w_0(x)=(1+x^2)^{-(\alpha-1)/2}$ $(1<\alpha\leq 2)$ or $w_0(x)=0$ $(\alpha=2)$, where $w_0(x)$ is defined by (2.1.6). For this reason, we can say that our results are generalization of [37].

Remark 3.2.6. In Narazaki and Nishihara [50], they assumed that there exists $\lim_{|x|\to\infty}(1+|x|)^\gamma u_0(x)=c_\gamma$. However, it seems a bit restrictive to assume the limit of $(1+|x|)^\gamma u_0(x)$ coinside each other. In fact, by introducing the function $c_\alpha(y)$ the definition of asymptotic profile $Z(x,t)$ (3.2.5), we are able to handle the case where the limits are different. Besides, such a modification can be applied in the case of the heat equation and the damped wave equation treated in [50].

The rest of this chapter is organized as follows. First, in Section 3.3, we give the proof of Theorem 3.2.1. Next, we prove Theorem 3.2.2 in last two sections (Section 3.4 is for $1<\alpha<2$, Section 3.5 is for $\alpha=2$).

3.3 Proof of Theorem 3.2.1

In this section, we shall prove Theorem 3.2.1. First, in order to obtain the upper bound estimates of $u-\chi$, we transform the equation (2.1.1) and (2.1.5) by using the Hopf-Cole transformation used in [4, 18, 37, 45]. We set

$$\rho(x,t) \equiv \exp\biggl(-\frac{b}{2}\int_{-\infty}^{x} u(y,t)dy\biggr), \tag{3.3.1}$$

then we have the following equation:

$$\rho_t-\rho_{xx}=\frac{b}{2}\rho\biggl(\frac{c}{3}u^3+ku_{xx}\biggr). \tag{3.3.2}$$

Also, for $\eta(x,t)$ defined by (2.4.6), we have

$$(\eta^{-1})_t-(\eta^{-1})_{xx}=0. \tag{3.3.3}$$

Therefore, if we set

$$w(x,t) = \rho(x,t) - \eta(x,t)^{-1}, \tag{3.3.4}$$

then we have the following initial value problem for $w(x,t)$ from (3.3.2) and (3.3.3):

$$\begin{aligned} w_t - w_{xx} &= \frac{b}{2}\rho\left(\frac{c}{3}u^3 + ku_{xx}\right), \quad x \in \mathbb{R}, \ t > 0, \\ w(x,0) &= w_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3.3.5}$$

where $w_0(x)$ is defined by (2.1.6). Our first step to prove Theorem 3.2.1 is to show the following proposition:

Proposition 3.3.1. *Assume the same conditions on u_0 in Theorem 3.2.1 are valid. Then we have*

$$\|\partial_x^l w(\cdot,t)\|_{L^\infty} \le C \begin{cases} (1+t)^{-(\alpha-1)/2-l/2}, & t \ge 1, \ 1 < \alpha < 2, \\ (1+t)^{-1/2-l/2}\log(2+t), & t \ge 1, \ \alpha = 2 \end{cases} \tag{3.3.6}$$

for $l = 0, 1$, where $w(x,t)$ is the solution to (3.3.5).

Proof. Applying the Duhamel principle to (3.3.5), we have the following integral equation:

$$\begin{aligned} w(t) &= G(t) * w_0 + \frac{bc}{6}\int_0^t G(t-\tau) * (\rho u^3)(\tau) d\tau \\ &\quad + \frac{bk}{2}\int_0^t G(t-\tau) * (\rho u_{xx})(\tau) d\tau \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{3.3.7}$$

Before evaluating I_1, I_2 and I_3, we prove the following estimate for the initial data $w_0(x)$:

$$|w_0(x)| \le C(1+|x|)^{-(\alpha-1)}, \quad x \in \mathbb{R}. \tag{3.3.8}$$

By the mean value theorem, we obtain

$$|w_0(x)| \le C\left|\int_{-\infty}^x (u_0(y) - \chi_*(y)) dy\right|.$$

If $x < 0$, from (3.1.1) and (2.1.4), we have

$$\begin{aligned} |w_0(x)| &\le C\int_{-\infty}^x (|u_0(y)| + |\chi_*(y)|) dy \\ &\le C\int_{-\infty}^x (1+|y|)^{-\alpha} dy + C\int_{-\infty}^x (1+|y|)^{-N} dy \quad (\forall N \ge 0) \\ &\le C\int_{-\infty}^x (1-y)^{-\alpha} dy \le C(1-x)^{-(\alpha-1)} = C(1+|x|)^{-(\alpha-1)}. \end{aligned}$$

On the other hand, since $\int_{\mathbb{R}} u_0(x)dx = \int_{\mathbb{R}} \chi_*(x)dx = \delta$, if $x > 0$, similarly we have

$$
\begin{aligned}
|w_0(x)| &\le C\left|\int_{-\infty}^{x}(u_0(y)-\chi_*(y))dy\right| \\
&= C\left|\int_{-\infty}^{x}u_0(y)dy-\int_{\mathbb{R}}u_0(y)dy-\int_{-\infty}^{x}\chi_*(y)dy+\int_{\mathbb{R}}\chi_*(y)dy\right| \\
&\le C\left|\int_{x}^{\infty}u_0(y)dy\right|+\left|\int_{x}^{\infty}\chi_*(y)dy\right| \\
&\le C\int_{x}^{\infty}(1+|y|)^{-\alpha}dy+C\int_{x}^{\infty}(1+|y|)^{-N}dy \quad (\forall N\ge 0) \\
&\le C\int_{x}^{\infty}(1+y)^{-\alpha}dy\le C(1+x)^{-(\alpha-1)}=C(1+|x|)^{-(\alpha-1)}.
\end{aligned}
$$

Thus we get (3.3.8).

Next, we prove the uniform boundedness of $\rho(x,t)$:

$$
\|\rho(\cdot,t)\|_{L^\infty}\le C,\quad t\ge 0. \tag{3.3.9}
$$

To prove (3.3.9), it suffices to show

$$
\|w(\cdot,t)\|_{L^\infty}\le C,\quad t\ge 0, \tag{3.3.10}
$$

since $\rho=w+\eta^{-1}$ and (2.4.8). By using (3.3.7), (3.3.4), Young's inequality, Lemma 2.4.1, (3.3.8), (2.4.8) and Proposition 2.3.3, we have

$$
\begin{aligned}
&\|w(\cdot,t)\|_{L^\infty} \\
&\le \|G(\cdot,t)\|_{L^1}\|w_0\|_{L^\infty} \\
&\quad + C\int_0^t\|G(\cdot,t-\tau)\|_{L^1}(\|w(\cdot,\tau)\|_{L^\infty}+\|\eta(\cdot,\tau)^{-1}\|_{L^\infty})\|u^3(\cdot,\tau)\|_{L^\infty}d\tau \\
&\quad + C\int_0^t\|G(\cdot,t-\tau)\|_{L^1}(\|w(\cdot,\tau)\|_{L^\infty}+\|\eta(\cdot,\tau)^{-1}\|_{L^\infty})\|u_{xx}(\cdot,\tau)\|_{L^\infty}d\tau \\
&\le C+C'\int_0^t\|w(\cdot,\tau)\|_{L^\infty}(1+\tau)^{-3/2}d\tau,\quad t\ge 0.
\end{aligned}
$$

Therefore, by using Gronwall's Lemma, we obtain

$$
\|w(\cdot,t)\|_{L^\infty}\le C\exp\left(C'\int_0^t(1+\tau)^{-3/2}d\tau\right)\le C,\quad t\ge 0.
$$

Now, we start with evaluation of I_1, I_2 and I_3. From Lemma 2.4.1, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} |\partial_x^l G(x-y,t)|(1+|y|)^{-(\alpha-1)}dy \\
&= C\Big(\int_{|y|\geq\sqrt{1+t}-1} + \int_{|y|\leq\sqrt{1+t}-1}\Big)|\partial_x^l G(x-y,t)|(1+|y|)^{-(\alpha-1)}dy \\
&\leq C\Big(\sup_{|y|\geq\sqrt{1+t}-1}(1+|y|)^{-(\alpha-1)}\Big)\int_{|y|\geq\sqrt{1+t}-1}|\partial_x^l G(x-y,t)|dy \\
&\quad + C\Big(\sup_{|y|\leq\sqrt{1+t}-1}|\partial_x^l G(x-y,t)|\Big)\int_{|y|\leq\sqrt{1+t}-1}(1+|y|)^{-(\alpha-1)}dy \\
&\leq C(1+t)^{-(\alpha-1)/2}\|\partial_x^l G(\cdot,t)\|_{L^1} + C\|\partial_x^l G(\cdot,t)\|_{L^\infty}\int_{|y|\leq\sqrt{1+t}-1}(1+|y|)^{-(\alpha-1)}dy \\
&\leq C(1+t)^{-(\alpha-1)/2-l/2} + Ct^{-1/2-l/2}\int_0^{\sqrt{1+t}-1}(1+y)^{-(\alpha-1)}dy \\
&\leq C(1+t)^{-(\alpha-1)/2-l/2} + Ct^{-1/2-l/2}\begin{cases}(1+t)^{-(\alpha-1)/2+1/2}, & 1<\alpha<2, \\ \log(2+t), & \alpha=2\end{cases} \\
&\leq C\begin{cases}(1+t)^{-(\alpha-1)/2-l/2}, & t\geq 1,\ 1<\alpha<2, \\ (1+t)^{-1/2-l/2}\log(2+t), & t\geq 1,\ \alpha=2.\end{cases}
\end{aligned}
\tag{3.3.11}
$$

Therefore, we obtain from (3.3.8) and (3.3.11)

$$
\|\partial_x^l I_1(\cdot,t)\|_{L^\infty} \leq C\begin{cases}(1+t)^{-(\alpha-1)/2-l/2}, & t\geq 1,\ 1<\alpha<2, \\ (1+t)^{-1/2-l/2}\log(2+t), & t\geq 1,\ \alpha=2\end{cases}
\tag{3.3.12}
$$

for $l=0,1$. By Young's inequality, (3.3.9), Lemma 2.4.1 and Proposition 2.3.3, we have

$$
\begin{aligned}
\|\partial_x^l I_2(\cdot,t)\|_{L^\infty} &\leq C\Big(\int_0^{t/2} + \int_{t/2}^t\Big)\|\partial_x^l G(t-\tau) * (\rho u^3)(\tau)\|_{L^\infty}d\tau \\
&\leq C\int_0^{t/2}\|\partial_x^l G(\cdot,t-\tau)\|_{L^\infty}\|u^3(\cdot,\tau)\|_{L^1}d\tau \\
&\quad + C\int_{t/2}^t\|\partial_x^l G(\cdot,t-\tau)\|_{L^1}\|u^3(\cdot,\tau)\|_{L^\infty}d\tau \\
&\leq C\int_0^{t/2}\|\partial_x^l G(\cdot,t-\tau)\|_{L^\infty}\|u(\cdot,\tau)\|_{L^\infty}\|u(\cdot,t)\|_{L^2}^2d\tau \\
&\quad + C\int_{t/2}^t\|\partial_x^l G(\cdot,t-\tau)\|_{L^1}\|u^3(\cdot,\tau)\|_{L^\infty}d\tau \\
&\leq C\int_0^{t/2}(t-\tau)^{-1/2-l/2}(1+\tau)^{-1}d\tau + C\int_{t/2}^t(t-\tau)^{-l/2}(1+\tau)^{-3/2}d\tau \\
&\leq Ct^{-1/2-l/2}\log(2+t) + C(1+t)^{-1/2-l/2} \\
&\leq C(1+t)^{-1/2-l/2}\log(2+t), \quad t\geq 1
\end{aligned}
\tag{3.3.13}
$$

for $l = 0, 1$. Finally, we evaluate I_3. Since $\rho_x = -(b/2)\rho u$ from (3.3.1), making the integration by parts for the first part of the integral, it follows that

$$
\begin{aligned}
I_3(x,t) &= \frac{bk}{2}\left(\int_0^{t/2} + \int_{t/2}^t\right)\int_{\mathbb{R}} G(x-y,t-\tau)\rho(y,\tau)u_{yy}(y,\tau)dyd\tau \\
&= \frac{bk}{2}\int_0^{t/2}\int_{\mathbb{R}} \partial_x G(x-y,t-\tau)\rho(y,\tau)u_y(y,\tau)dyd\tau \\
&\quad + \frac{b^2k}{4}\int_0^{t/2}\int_{\mathbb{R}} G(x-y,t-\tau)\rho(y,\tau)u(y,\tau)u_y(y,\tau)dyd\tau \\
&\quad + \frac{bk}{2}\int_{t/2}^t\int_{\mathbb{R}} G(x-y,t-\tau)\rho(y,\tau)u_{yy}(y,\tau)dyd\tau.
\end{aligned}
\tag{3.3.14}
$$

Therefore, in the same way to get (3.3.13), we have from (3.3.14)

$$
\begin{aligned}
\|\partial_x^l I_3(\cdot,t)\|_{L^\infty} &\le C\int_0^{t/2} \|\partial_x^{l+1}G(t-\tau)*(\rho u_x)(\tau)\|_{L^\infty}d\tau \\
&\quad + C\int_0^{t/2} \|\partial_x^{l}G(t-\tau)*(\rho u u_x)(\tau)\|_{L^\infty}d\tau \\
&\quad + C\int_{t/2}^t \|\partial_x^{l}G(t-\tau)*(\rho u_{xx})(\tau)\|_{L^\infty} \\
&\le C\int_0^{t/2} \|\partial_x^{l+1}G(\cdot,t-\tau)\|_{L^\infty}\|u_x(\cdot,\tau)\|_{L^1}d\tau \\
&\quad + C\int_0^{t/2} \|\partial_x^{l}G(\cdot,t-\tau)\|_{L^\infty}\|u(\cdot,\tau)\|_{L^2}\|u_x(\cdot,\tau)\|_{L^2}d\tau \\
&\quad + C\int_{t/2}^t \|\partial_x^{l}G(\cdot,t-\tau)\|_{L^1}\|u_{xx}(\cdot,t)\|_{L^\infty}d\tau \\
&\le C\int_0^{t/2}(t-\tau)^{-1-l/2}\tau^{-1/2}(1+\tau^{-1/4})d\tau \\
&\quad + C\int_0^{t/2}(t-\tau)^{-1/2-l/2}(1+\tau)^{-1}d\tau \\
&\quad + C\int_{t/2}^t(t-\tau)^{-l/2}(1+\tau)^{-3/2}d\tau \\
&\le C(1+t)^{-1/2-l/2} + C(1+t)^{-1/2-l/2}\log(2+t) + C(1+t)^{-1/2-l/2} \\
&\le C(1+t)^{-1/2-l/2}\log(2+t), \quad t \ge 1
\end{aligned}
\tag{3.3.15}
$$

for $l = 0, 1$. Summing up (3.3.7), (3.3.12), (3.3.13) and (3.3.15), we obtain (3.3.6). $\square$

End of the proof of Theorem 3.2.1. From (3.3.1) and (2.4.6), it follows that

$$u = -\frac{2}{b}(\log \rho)_x, \quad \chi = \frac{2}{b}(\log \eta)_x.$$

Therefore, we have

$$\begin{aligned} u - \chi &= -\frac{2}{b}(\log \rho + \log \eta)_x = -\frac{2}{b}(\log(w + \eta^{-1}) + \log \eta) \\ &= -\frac{2}{b}(\log(\eta w + 1))_x = -\frac{2}{b}\frac{\eta_x w + \eta w_x}{\eta w + 1} \\ &= -\frac{2}{b}\frac{w}{w + \eta^{-1}}\frac{\eta_x}{\eta} - \frac{2}{b}\frac{w_x}{w + \eta^{-1}} = -\frac{w\chi}{\rho} - \frac{2}{b}\frac{w_x}{\rho}, \end{aligned} \tag{3.3.16}$$

since $\eta_x = (b/2)\eta\chi$ and $\rho = w + \eta^{-1}$. By using (2.3.10), we have

$$\left| \int_{-\infty}^{x} u(y,t)dy \right| \le \|u(\cdot,t)\|_{L^1} \le C(1 + t^{-1/4}) \le C =: C_0, \quad t \ge 1.$$

Thus, from (3.3.1), we get the lower bound estimate of $\rho(x,t)$:

$$\inf_{t\ge1, x\in\mathbb{R}} \rho(x,t) \ge e^{-|b|C_0/2}. \tag{3.3.17}$$

Therefore, using (3.3.16), Lemma 2.4.1, Proposition 3.3.1 and (3.3.17), if $1 < \alpha < 2$, we have

$$\begin{aligned} \|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} &\le C\|w(\cdot,t)\|_{L^\infty}\|\chi(\cdot,t)\|_{L^\infty} + C\|\partial_x w(\cdot,t)\|_{L^\infty} \\ &\le C(1+t)^{-(\alpha-1)/2}(1+t)^{-1/2} + C(1+t)^{-\alpha/2} \\ &\le C(1+t)^{-\alpha/2}, \ t \ge 1. \end{aligned} \tag{3.3.18}$$

On the other hand, if $\alpha = 2$, it follows that

$$\begin{aligned} \|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} &\le C\|w(\cdot,t)\|_{L^\infty}\|\chi(\cdot,t)\|_{L^\infty} + C\|\partial_x w(\cdot,t)\|_{L^\infty} \\ &\le C(1+t)^{-1/2}\log(2+t)(1+t)^{-1/2} \\ &\quad + C(1+t)^{-1}\log(2+t) \\ &\le C(1+t)^{-1}\log(2+t), \ t \ge 1. \end{aligned} \tag{3.3.19}$$

For $0 \le t \le 1$, from Proposition 2.3.3 and Lemma 2.4.1, we obtain

$$\begin{aligned} \|u(\cdot,t) - \chi(\cdot,t)\|_{L^\infty} &\le C\|u(\cdot,t)\|_{L^\infty} + C\|\chi(\cdot,t)\|_{L^\infty} \\ &\le CE_s(1+t)^{-1/2} + C|\delta|(1+t)^{-1/2} \\ &\le C, \quad 0 \le t \le 1. \end{aligned} \tag{3.3.20}$$

Combining (3.3.18) through (3.3.20), we get (3.2.1) for all $t \ge 0$. This completes the proof. □

In the rest of this section, we will show the L^p-decay estimate of $u - \chi$ for $p \in [2, \infty]$. Especially, the following L^2-decay estimate will be used in the proof of Theorem 3.2.2.

Proposition 3.3.2. *Assume the same conditions on u_0 in Theorem 3.2.1 are valid. Then, the following estimate holds:*

$$\|u(\cdot,t)-\chi(\cdot,t)\|_{L^2} \le C\begin{cases}(1+t)^{-\alpha/2+1/4}, & t\ge 0,\ 1<\alpha<2,\\ (1+t)^{-3/4}\log(2+t), & t\ge 0,\ \alpha=2.\end{cases} \tag{3.3.21}$$

Proof. It suffices to derive the following L^2-decay estimate of $\partial_x w(x,t)$:

$$\|\partial_x w(\cdot,t)\|_{L^2} \le C\begin{cases}(1+t)^{-\alpha/2+1/4}, & t\ge 1,\ 1<\alpha<2,\\ (1+t)^{-3/4}\log(2+t), & t\ge 1,\ \alpha=2.\end{cases} \tag{3.3.22}$$

Indeed, if we have shown (3.3.22), then from (3.3.16), Proposition 3.3.1, Lemma 2.4.1 and (3.3.17), it follows that

$$\begin{aligned}\|u(\cdot,t)-\chi(\cdot,t)\|_{L^2} &\le C\|w(\cdot,t)\|_{L^\infty}\|\chi(\cdot,t)\|_{L^2}+C\|\partial_x w(\cdot,t)\|_{L^2}\\ &\le C(1+t)^{-(\alpha-1)/2}(1+t)^{-1/4}+C(1+t)^{-\alpha/2+1/4}\\ &\le C(1+t)^{-\alpha/2+1/4},\ t\ge 1\end{aligned} \tag{3.3.23}$$

for $1<\alpha<2$. While, the following estimate holds in the case of $\alpha=2$:

$$\begin{aligned}\|u(\cdot,t)-\chi(\cdot,t)\|_{L^2} &\le C\|w(\cdot,t)\|_{L^\infty}\|\chi(\cdot,t)\|_{L^2}+C\|\partial_x w(\cdot,t)\|_{L^2}\\ &\le C(1+t)^{-1/2}\log(2+t)(1+t)^{-1/4}\\ &\quad +C(1+t)^{-3/4}\log(2+t)\\ &\le C(1+t)^{-3/4}\log(2+t),\ t\ge 1.\end{aligned} \tag{3.3.24}$$

For $0\le t\le 1$, from Proposition 2.3.3 and Lemma 2.4.1, we obtain

$$\begin{aligned}\|u(\cdot,t)-\chi(\cdot,t)\|_{L^2} &\le C\|u(\cdot,t)\|_{L^2}+C\|\chi(\cdot,t)\|_{L^2}\\ &\le CE_s(1+t)^{-1/4}+C|\delta|(1+t)^{-1/4}\\ &\le C,\quad 0\le t\le 1.\end{aligned} \tag{3.3.25}$$

Combining (3.3.23) through (3.3.25), we get (3.3.21) for all $t\ge 0$.

To prove (3.3.22), we consider the following integral equation again:

$$\begin{aligned}w(t) &= G(t)*w_0+\frac{bc}{6}\int_0^t G(t-\tau)*(\rho u^3)(\tau)d\tau\\ &\quad +\frac{bk}{2}\int_0^t G(t-\tau)*(\rho u_{xx})(\tau)d\tau\\ &\equiv I_1+I_2+I_3.\end{aligned} \tag{3.3.7}$$

For the initial data, using (2.1.6), (3.1.1) and (2.1.4), we have

$$\begin{aligned}&|w_0'(x)|\\ &\le C\left|\exp\left(-\frac{b}{2}\int_{-\infty}^x u_0(y)dy\right)\right||u_0(x)|+C\left|\exp\left(-\frac{b}{2}\int_{-\infty}^x \chi_*(y)dy\right)\right||\chi_*(x)|\\ &\le C\exp\left(\frac{|b|\|u_0\|_{L^1}}{2}\right)(1+|x|)^{-\alpha}+C\exp\left(\frac{|b|\|\chi_*\|_{L^1}}{2}\right)(1+|x|)^{-N}\quad(\forall N\ge 0)\\ &\le C(1+|x|)^{-\alpha},\quad x\in\mathbb{R}.\end{aligned} \tag{3.3.26}$$

Now we shall evaluate I_1, I_2 and I_3. Splitting the y-integral and making the integration by parts, we have

$$
\begin{aligned}
\partial_x I_1(x,t) &= \int_{\mathbb{R}} \partial_x G(x-y,t) w_0(y)dy \\
&= \left(\int_{-\infty}^{1-\sqrt{1+t}} + \int_{\sqrt{1+t}-1}^{\infty} \right) \partial_x G(x-y,t) w_0(y)dy \\
&\quad + \int_{|y| \le \sqrt{1+t}-1} \partial_x G(x-y,t) w_0(y)dy \\
&= -G(x-1+\sqrt{1+t},t) w_0(1-\sqrt{1+t}) \\
&\quad + G(x-\sqrt{1+t}-1,t) w_0(\sqrt{1+t}-1) \\
&\quad + \left(\int_{-\infty}^{1-\sqrt{1+t}} + \int_{\sqrt{1+t}-1}^{\infty} \right) G(x-y,t) w_0'(y)dy \\
&\quad + \int_{|y| \le \sqrt{1+t}-1} \partial_x G(x-y,t) w_0(y)dy \\
&\equiv I_{1.1} + I_{1.2} + I_{1.3} + I_{1.4}.
\end{aligned}
\tag{3.3.27}
$$

Using Lemma 2.4.1 and (3.3.8), we have

$$
\begin{aligned}
\|I_{1.1}(\cdot,t)\|_{L^2} &\le C\|G(\cdot,t)\|_{L^2} |w_0(1-\sqrt{1+t})| \\
&\le Ct^{-1/4}(1+t)^{-(\alpha-1)/2} \\
&\le C(1+t)^{-\alpha/2+1/4}, \ \ t \ge 1.
\end{aligned}
\tag{3.3.28}
$$

Similarly, we obtain

$$
\|I_{1.2}(\cdot,t)\|_{L^2} \le C(1+t)^{-\alpha/2+1/4}, \ \ t \ge 1. \tag{3.3.29}
$$

From Lemma 2.4.1 and (3.3.26), we obtain

$$
\begin{aligned}
\|I_{1.3}(\cdot,t)\|_{L^2} &\le C\left(\int_{-\infty}^{1-\sqrt{1+t}} + \int_{\sqrt{1+t}-1}^{\infty} \right) \|G(\cdot - y,t)\|_{L^2} |w_0'(y)|dy \\
&\le Ct^{-1/4} \int_{-\infty}^{1-\sqrt{1+t}} (1-y)^{-\alpha}dy + Ct^{-1/4} \int_{\sqrt{1+t}-1}^{\infty} (1+y)^{-\alpha}dy \\
&= Ct^{-1/4} \left[\frac{-1}{1-\alpha}(1-y)^{1-\alpha} \right]_{-\infty}^{1-\sqrt{1+t}} + Ct^{-1/4} \left[\frac{1}{1-\alpha}(1+y)^{1-\alpha} \right]_{\sqrt{1+t}-1}^{\infty} \\
&\le Ct^{-1/4}(1+t)^{1/2-\alpha/2} \\
&\le C(1+t)^{-\alpha/2+1/4}, \ \ t \ge 1.
\end{aligned}
\tag{3.3.30}
$$

On the other hand, we have from Lemma 2.4.1 and (3.3.8)

$$\begin{aligned}
\|I_{1.4}(\cdot,t)\|_{L^2} &\le \int_{|y|\le\sqrt{1+t}-1} \|\partial_x G(\cdot - y,t)\|_{L^2}|w_0(y)|dy \\
&\le Ct^{-3/4}\int_{|y|\le\sqrt{1+t}-1}(1+|y|)^{-(\alpha-1)}dy \\
&\le Ct^{-3/4}\int_0^{\sqrt{1+t}-1}(1+y)^{-(\alpha-1)}dy \\
&\le C\begin{cases}(1+t)^{-\alpha/2+1/4}, & t\ge 1,\ 1<\alpha<2,\\ (1+t)^{-3/4}\log(2+t), & t\ge 1,\ \alpha=2.\end{cases}
\end{aligned} \tag{3.3.31}$$

Therefore, combining (3.3.27) through (3.3.31), we obtain

$$\|\partial_x I_1(\cdot,t)\|_{L^2} \le C\begin{cases}(1+t)^{-\alpha/2+1/4}, & t\ge 1,\ 1<\alpha<2,\\ (1+t)^{-3/4}\log(2+t), & t\ge 1,\ \alpha=2.\end{cases} \tag{3.3.32}$$

By Young's inequality, (3.3.9), Lemma 2.4.1 and Proposition 2.3.3, we have

$$\begin{aligned}
\|\partial_x I_2(\cdot,t)\|_{L^2} &\le C\left(\int_0^{t/2}+\int_{t/2}^t\right)\|\partial_x G(t-\tau)*(\rho u^3)(\tau)\|_{L^2}d\tau \\
&\le C\int_0^{t/2}\|\partial_x G(\cdot,t-\tau)\|_{L^2}\|u(\cdot,\tau)\|_{L^\infty}\|u(\cdot,t)\|_{L^2}^2 d\tau \\
&\quad + C\int_{t/2}^t \|\partial_x G(\cdot,t-\tau)\|_{L^1}\|u(\cdot,t)\|_{L^\infty}^2\|u(\cdot,\tau)\|_{L^2}d\tau \\
&\le C\int_0^{t/2}(t-\tau)^{-3/4}(1+\tau)^{-1}d\tau + C\int_{t/2}^t (t-\tau)^{-1/2}(1+\tau)^{-5/4}d\tau \\
&\le Ct^{-3/4}\log(2+t) + C(1+t)^{-3/4} \\
&\le C(1+t)^{-3/4}\log(2+t), \quad t\ge 1.
\end{aligned} \tag{3.3.33}$$

Finally, we evaluate I_3. Using (3.3.14), in the same way to get (3.3.15), we have

$$\begin{aligned}
&\|\partial_x I_3(\cdot,t)\|_{L^2} \\
&\le C\int_0^{t/2}\|\partial_x^2 G(\cdot,t-\tau)\|_{L^2}\|u_x(\cdot,t)\|_{L^1}d\tau \\
&\quad + C\int_0^{t/2}\|\partial_x G(\cdot,t-\tau)\|_{L^2}\|u(\cdot,\tau)\|_{L^2}\|u_x(\cdot,\tau)\|_{L^2}d\tau \\
&\quad + C\int_{t/2}^{t}\|\partial_x G(\cdot,t-\tau)\|_{L^1}\|u_{xx}(\cdot,t)\|_{L^2}d\tau \\
&\le C\int_0^{t/2}(t-\tau)^{-5/4}\tau^{-1/2}(1+\tau^{-1/4})d\tau + C\int_0^{t/2}(t-\tau)^{-3/4}(1+\tau)^{-1}d\tau \\
&\quad + C\int_{t/2}^{t}(t-\tau)^{-1/2}(1+\tau)^{-5/4}d\tau \\
&\le C(1+t)^{-3/4} + C(1+t)^{-3/4}\log(2+t) + C(1+t)^{-3/4} \\
&\le C(1+t)^{-3/4}\log(2+t), \quad t\ge 1.
\end{aligned} \tag{3.3.34}$$

Summing up (3.3.7) and (3.3.32) through (3.3.34), we obtain (3.3.22). This completes the proof. □

Finally, by the interpolation inequality, it follows that

$$\|u(\cdot,t)-\chi(\cdot,t)\|_{L^p} \le \|u(\cdot,t)-\chi(\cdot,t)\|_{L^\infty}^{1-2/p}\|u(\cdot,t)-\chi(\cdot,t)\|_{L^2}^{2/p}$$

for $p\in[2,\infty]$. Therefore, (3.2.1) and (3.3.21) lead to the following estimate:

Corollary 3.3.3. *Assume the same conditions on u_0 in Theorem 3.2.1 are valid. Then, for $p\in[2,\infty]$, we have*

$$\|u(\cdot,t)-\chi(\cdot,t)\|_{L^p} \le \begin{cases}(1+t)^{-\alpha/2+1/2p}, & t\ge 0,\ 1<\alpha<2,\\ (1+t)^{-1+1/2p}\log(2+t), & t\ge 0,\ \alpha=2.\end{cases}$$

3.4 Proof of Theorem 3.2.2 for $1<\alpha<2$

In this section, we shall prove Theorem 3.2.2 in the case of $1<\alpha<2$. Namely, we prove (3.2.3) and (3.2.6). First, we set

$$\psi(x,t)\equiv u(x,t)-\chi(x,t), \quad \psi_0(x)\equiv u_0(x)-\chi_*(x). \tag{3.4.1}$$

Then we have the following equation:

$$\begin{aligned}
&\psi_t + (b\chi\psi)_x - \psi_{xx} = -\left(\frac{b}{2}\psi^2\right)_x - \left(\frac{c}{3}u^3\right)_x - ku_{xxx}, \quad x\in\mathbb{R},\ t>0, \\
&\qquad\qquad \psi(x,0)=\psi_0(x), \quad x\in\mathbb{R}.
\end{aligned} \tag{3.4.2}$$

From Lemma 2.5.1, we obtain

$$
\begin{aligned}
\psi(x,t) = U[\psi_0](x,t,0) - \frac{b}{2}\int_0^t U[\partial_x \psi^2(\tau)](x,t,\tau)d\tau \\
- \frac{c}{3}\int_0^t U[\partial_x u^3(\tau)](x,t,\tau)d\tau - k\int_0^t U[\partial_x^3 u(\tau)](x,t,\tau)d\tau.
\end{aligned}
\tag{3.4.3}
$$

For the first term of the above equation (3.4.3), we have the following asymptotic formula. This formula plays an essential role in the proof of (3.2.3) and (3.2.4). The idea of the proof is based on the method used in Narazaki and Nishihara [50].

Proposition 3.4.1. *Assume the same conditions on u_0 in Theorem 3.2.2 are valid. Then we have*

$$
\lim_{t\to\infty}(1+t)^{\alpha/2}\|U[\psi_0](\cdot,t,0) - Z(\cdot,t)\|_{L^\infty} = 0,\ 1<\alpha<2, \tag{3.4.4}
$$

$$
\lim_{t\to\infty}\frac{(1+t)}{\log(1+t)}\|U[\psi_0](\cdot,t,0) - Z(\cdot,t)\|_{L^\infty} = 0,\ \alpha=2, \tag{3.4.5}
$$

where $Z(x,t)$ is defined by (3.2.5).

Proof. From the definition of U given by (2.5.2) we have

$$
\begin{aligned}
U[\psi_0](x,t,0) &= \int_{\mathbb{R}} \partial_x(G(x-y,t)\eta(x,t))\eta_*(y)^{-1}\left(\int_{-\infty}^y (u_0(\xi)-\chi_*(\xi))d\xi\right)dy \\
&= \int_{\mathbb{R}} \partial_x(G(x-y,t)\eta(x,t))z_0(y)dy,
\end{aligned}
\tag{3.4.6}
$$

where $z_0(y)$ is defined by (3.2.2). Now, from (2.4.8), (3.1.1) and (2.1.4), we can check the following estimate in the same way to get (3.3.8):

$$
|z_0(y)| \le C(1+|y|)^{-(\alpha-1)},\ y\in\mathbb{R}.
$$

Thus, we obtain the boundedness of $(1+|y|)^{\alpha-1}z_0(y)$. Moreover, from the assumption on $z_0(y)$, for any $\varepsilon>0$ there is a constant $R=R(\varepsilon)>0$ such that

$$
\begin{aligned}
|z_0(y) - c_\alpha^+(1+|y|)^{-(\alpha-1)}| \le \varepsilon(1+|y|)^{-(\alpha-1)},\ y\ge R, \\
|z_0(y) - c_\alpha^-(1+|y|)^{-(\alpha-1)}| \le \varepsilon(1+|y|)^{-(\alpha-1)},\ y\le -R.
\end{aligned}
$$

From (3.2.5) and (3.4.6), we have the following estimate

$$
\begin{aligned}
&|U[\psi_0](x,t,0) - Z(x,t)| \\
&\le \int_{\mathbb{R}} |\partial_x(G(x-y,t)\eta(x,t))||z_0(y) - c_\alpha(y)(1+|y|)^{-(\alpha-1)}|dy \\
&\le \int_{|y|\le R} |\partial_x(G(x-y,t)\eta(x,t))||z_0(y) - c_\alpha(y)(1+|y|)^{-(\alpha-1)}|dy \\
&\quad + \varepsilon \int_{|y|\ge R} |\partial_x(G(x-y,t)\eta(x,t))|(1+|y|)^{-(\alpha-1)}dy \\
&\le C\sum_{n=0}^{1} \|\partial_x^{1-n}\eta(\cdot,t)\|_{L^\infty}\|\partial_x^n G(\cdot,t)\|_{L^\infty} \int_{|y|\le R} |z_0(y) - c_\alpha(y)(1+|y|)^{-(\alpha-1)}|dy \\
&\quad + \varepsilon C\sum_{n=0}^{1} \|\partial_x^{1-n}\eta(\cdot,t)\|_{L^\infty} \int_{\mathbb{R}} |\partial_x^n G(x-y,t)|(1+|y|)^{-(\alpha-1)}dy.
\end{aligned}
\tag{3.4.7}
$$

Therefore, by using (3.4.7), Lemma 2.4.2, Lemma 2.4.1 and (3.3.11), we get

$$
\begin{aligned}
\|U[\psi_0](\cdot,t,0) - Z(\cdot,t)\|_{L^\infty} &\le C(1+t)^{-1} \\
&\quad + \varepsilon C \begin{cases} (1+t)^{-\alpha/2}, & t\ge 1,\ 1<\alpha<2, \\ (1+t)^{-1}\log(2+t), & t\ge 1,\ \alpha = 2. \end{cases}
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty}(1+t)^{\alpha/2}\|U[\psi_0](\cdot,t,0) - Z(\cdot,t)\|_{L^\infty} \le \varepsilon C,\ 1<\alpha<2, \\
&\limsup_{t\to\infty}\frac{(1+t)}{\log(1+t)}\|U[\psi_0](\cdot,t,0) - Z(\cdot,t)\|_{L^\infty} \le \varepsilon C,\ \alpha = 2.
\end{aligned}
$$

Therefore, we get (3.4.4) and (3.4.5), because $\varepsilon > 0$ can be chosen arbitrarily small. □

End of the proof of Theorem 3.2.2 for $1 < \alpha < 2$. First, we shall prove (3.2.3). By using (3.2.5) and (3.4.3), we have

$$
\begin{aligned}
&u(x,t) - \chi(x,t) - Z(x,t) \\
&= U[\psi_0](x,t,0) - Z(x,t) - \frac{b}{2}\int_0^t U[\partial_x\psi^2(\tau)](x,t,\tau)d\tau \\
&\quad - \frac{c}{3}\int_0^t U[\partial_x u^3(\tau)](x,t,\tau)d\tau - k\int_0^t U[\partial_x^3 u(\tau)](x,t,\tau)d\tau \\
&\equiv U[\psi_0](x,t,0) - Z(x,t) + J_1 + J_2 + J_3.
\end{aligned}
\tag{3.4.8}
$$

We shall evaluate J_1, J_2 and J_3. Using Lemma 2.5.6, Proposition 3.3.2 and

Theorem 3.2.1, we obtain

$$
\begin{aligned}
&\|J_1(\cdot,t)\|_{L^\infty}\\
&\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2}\\
&\quad\times\left(\int_0^{t/2}(t-\tau)^{-1/2-n/2}\|\psi^2(\cdot,\tau)\|_{L^1}d\tau+\int_{t/2}^{t}(t-\tau)^{-n/2}\|\psi^2(\cdot,\tau)\|_{L^\infty}d\tau\right)\\
&\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2}\\
&\quad\times\left(\int_0^{t/2}(t-\tau)^{-1/2-n/2}(1+\tau)^{-\alpha+1/2}d\tau+\int_{t/2}^{t}(t-\tau)^{-n/2}(1+\tau)^{-\alpha}d\tau\right)\\
&\le C\begin{cases}(1+t)^{-1}, & t\ge 1,\ 3/2<\alpha<2,\\ (1+t)^{-1}\log(1+t), & t\ge 1,\ \alpha=3/2,\\ (1+t)^{-\alpha+1/2}, & t\ge 1,\ 1<\alpha<3/2.\end{cases}
\end{aligned}
\tag{3.4.9}
$$

Next we evaluate J_2. In the same way to get (3.4.9), from Lemma 2.5.6 and Proposition 2.3.3, we get

$$
\begin{aligned}
&\|J_2(\cdot,t)\|_{L^\infty}\\
&\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2}\\
&\quad\times\left(\int_0^{t/2}(t-\tau)^{-1/2-n/2}\|u^3(\cdot,\tau)\|_{L^1}d\tau+\int_{t/2}^{t}(t-\tau)^{-n/2}\|u^3(\cdot,\tau)\|_{L^\infty}d\tau\right)\\
&\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2}\\
&\quad\times\left(\int_0^{t/2}(t-\tau)^{-1/2-n/2}(1+\tau)^{-1}d\tau+\int_{t/2}^{t}(t-\tau)^{-n/2}(1+\tau)^{-3/2}d\tau\right)\\
&\le C(1+t)^{-1}\log(1+t),\ t\ge 1.
\end{aligned}
\tag{3.4.10}
$$

Finally, we evaluate I_3. We define $J_{3.1}$ and $J_{3.2}$ are as follows.

$$
\begin{aligned}
J_3=&-k\int_0^t\int_{\mathbb{R}}\partial_x(G(x-y,t-\tau)\eta(x,t))(\eta(y,\tau))^{-1}\partial_y^2u(y,\tau)dyd\tau\\
=&-k\sum_{n=0}^{1}\partial_x^{1-n}\eta(x,t)\\
&\times\left(\int_0^{t/2}+\int_{t/2}^{t}\right)\int_{\mathbb{R}}\partial_x^nG(x-y,t-\tau)(\eta(y,\tau))^{-1}\partial_y^2u(y,\tau)dyd\tau\\
\equiv&\ J_{3.1}+J_{3.2}.
\end{aligned}
\tag{3.4.11}
$$

Then, by making the integration by parts for $J_{3.1}$, we obtain

$$
\begin{aligned}
J_{3.1} =& - k \sum_{n=0}^{1} \partial_x^{1-n} \eta(x,t) \int_0^{t/2} \int_{\mathbb{R}} \partial_x^{n+1} G(x-y,t-\tau)(\eta(y,\tau))^{-1} \partial_y u(y,\tau) \\
&+ \partial_x^n G(x-y,t-\tau) \partial_y (\eta(y,\tau)^{-1}) \partial_y u(y,\tau) dy d\tau.
\end{aligned}
$$

Using Young's inequality, Lemma 2.4.2, Lemma 2.4.1, Proposition 2.3.3 and (2.4.8), we have

$$
\begin{aligned}
&\|J_{3.1}(\cdot,t)\|_{L^\infty} \\
&\le C \sum_{n=0}^{1} \|\partial_x^{1-n}\eta(\cdot,t)\|_{L^\infty} \int_0^{t/2} \big(\|\partial_x^{n+1} G(t-\tau) * (\eta^{-1} u_x)(\tau)\|_{L^\infty} \\
&\quad + \|\partial_x^n G(t-\tau) * ((\eta^{-1})_x u_x)(\tau)\|_{L^\infty} \big) d\tau \\
&\le C \sum_{n=0}^{1} (1+t)^{-1/2+n/2} \int_0^{t/2} \big(\|\partial_x^{n+1} G(\cdot,t-\tau)\|_{L^\infty} \|u_x(\cdot,\tau)\|_{L^1} \\
&\quad + \|\partial_x^n G(\cdot,t-\tau)\|_{L^\infty} \|u_x(\cdot,\tau)\|_{L^2} \|\partial_x(\eta(\cdot,\tau)^{-1})\|_{L^2} \big) d\tau \qquad (3.4.12) \\
&\le C \sum_{n=0}^{1} (1+t)^{-1/2+n/2} \\
&\quad \times \int_0^{t/2} \left((t-\tau)^{-1-n/2} \tau^{-1/2} (1+\tau^{-1/4}) + (t-\tau)^{-1/2-n/2} (1+\tau)^{-1} \right) d\tau \\
&\le C(1+t)^{-1} + C(1+t)^{-1} \log(1+t) \\
&\le C(1+t)^{-1} \log(1+t), \ \ t \ge 1.
\end{aligned}
$$

On the other hand, for $J_{3.2}$, the following estimate is obtained:

$$
\begin{aligned}
\|J_{3.2}(\cdot,t)\|_{L^\infty} &\le C \sum_{n=0}^{1} \|\partial_x^{1-n}\eta(\cdot,t)\|_{L^\infty} \int_{t/2}^{t} \|\partial_x^n G(\cdot,t-\tau)\|_{L^1} \|u_{xx}(\cdot,\tau)\|_{L^\infty} d\tau \\
&\le C \sum_{n=0}^{1} (1+t)^{-1/2+n/2} \int_{t/2}^{t} (t-\tau)^{-n/2} (1+\tau)^{-3/2} d\tau \\
&\le C(1+t)^{-1}, \ \ t \ge 1.
\end{aligned} \tag{3.4.13}
$$

Therefore, from (3.4.11) through (3.4.13), we get

$$
\|J_3(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1} \log(1+t), \ \ t \ge 1. \tag{3.4.14}
$$

By using (3.4.8), (3.4.9), (3.4.10) and (3.4.14), we have

$$
\begin{aligned}
&\|u(\cdot,t) - \chi(\cdot,t) - Z(\cdot,t)\|_{L^\infty} \\
&\le \|U[\psi_0](\cdot,t,0) - Z(\cdot,t)\|_{L^\infty} + C(1+t)^{-1} \log(1+t) \\
&\quad + C \begin{cases} (1+t)^{-1}, & t \ge 1, \ \ 3/2 < \alpha < 2, \\ (1+t)^{-1} \log(1+t), & t \ge 1, \ \ \alpha = 3/2, \\ (1+t)^{-\alpha+1/2}, & t \ge 1, \ \ 1 < \alpha < 3/2. \end{cases}
\end{aligned}
$$

Therefore, from (3.4.4), we obtain

$$\limsup_{t\to\infty}(1+t)^{\alpha/2}\|u(\cdot,t)-\chi(\cdot,t)-Z(\cdot,t)\|_{L^\infty}=0.$$

This completes the proof of (3.2.3).

Next, we shall prove (3.2.6). First, we take $x=0$ in (3.2.5), then we have from (2.1.3) and (2.4.1)

$$\begin{aligned}
&Z(0,t)\\
&=\int_{\mathbb{R}} c_\alpha(y)\left((\partial_y G)(-y,t)\eta(0,t)+G(-y,t)\frac{b}{2}\chi(0,t)\eta(0,t)\right)(1+|y|)^{-(\alpha-1)}dy\\
&=c_\alpha^+\eta_*(0)\int_0^\infty(\partial_y G)(-y,t)(1+y)^{-(\alpha-1)}dy\\
&\quad+c_\alpha^-\eta_*(0)\int_{-\infty}^0(\partial_y G)(-y,t)(1-y)^{-(\alpha-1)}dy\\
&\quad+\frac{bc_\alpha^+}{2}\eta_*(0)\chi_*(0)(1+t)^{-1/2}\int_0^\infty G(y,t)(1+y)^{-(\alpha-1)}dy\\
&\quad+\frac{bc_\alpha^-}{2}\eta_*(0)\chi_*(0)(1+t)^{-1/2}\int_{-\infty}^0 G(y,t)(1-y)^{-(\alpha-1)}dy\\
&=(c_\alpha^+-c_\alpha^-)\eta_*(0)\int_0^\infty(\partial_y G)(-y,t)(1+y)^{-(\alpha-1)}dy\\
&\quad+\frac{b(c_\alpha^++c_\alpha^-)}{2}\eta_*(0)\chi_*(0)(1+t)^{-1/2}\int_0^\infty G(y,t)(1+y)^{-(\alpha-1)}dy\\
&\equiv L_1(t)+L_2(t),
\end{aligned}\tag{3.4.15}$$

since $\partial_y G(y,t)$ and $G(y,t)$ are the odd and even functions for y-variable, respectively.

From the mean value theorem, there exists $\theta_j\in(0,1)$ such that

$$(1+y)^{-(\alpha-j)}-y^{-(\alpha-j)}=-(\alpha-j)(y+\theta_j)^{-(\alpha-j+1)}.\tag{3.4.16}$$

Therefore, we obtain from (2.4.1)

$$\begin{aligned}
L_1(t)&=\frac{(c_\alpha^+-c_\alpha^-)\eta_*(0)}{4\sqrt{\pi}}t^{-3/2}\int_0^\infty e^{-y^2/4t}y(1+y)^{-(\alpha-1)}dy\\
&=\frac{(c_\alpha^+-c_\alpha^-)\eta_*(0)}{4\sqrt{\pi}}t^{-3/2}\\
&\quad\times\left(\int_0^\infty e^{-y^2/4t}y^{2-\alpha}dy+\int_0^\infty e^{-y^2/4t}y((1+y)^{-(\alpha-1)}-y^{-(\alpha-1)})dy\right)\\
&=\frac{(c_\alpha^+-c_\alpha^-)\eta_*(0)}{4\sqrt{\pi}}t^{-3/2}\\
&\quad\times\left(2^{2-\alpha}t^{(3-\alpha)/2}\Gamma\left(\frac{3-\alpha}{2}\right)-(\alpha-1)\int_0^\infty e^{-y^2/4t}y(y+\theta_1)^{-\alpha}dy\right),
\end{aligned}\tag{3.4.17}$$

since

$$\int_0^\infty e^{-y^2/4t} y^{j-\alpha} dy = 2^{j-\alpha} t^{(j+1-\alpha)/2} \Gamma\left(\frac{j+1-\alpha}{2}\right), \quad j \geq 1, \tag{3.4.18}$$

where $\Gamma(s)$ is the Gamma function for $s > 0$. Similarly, by making the integration by parts, we have from (3.4.16) and (3.4.18)

$$\begin{aligned}
L_2(t) &= \frac{b(c_\alpha^+ + c_\alpha^-)}{4\sqrt{\pi}} \eta_*(0)\chi_*(0)(1+t)^{-1/2} t^{-1/2} \int_0^\infty e^{-y^2/4t}(1+y)^{-(\alpha-1)} dy \\
&= \frac{b(c_\alpha^+ + c_\alpha^-)}{4\sqrt{\pi}} \eta_*(0)\chi_*(0)(1+t)^{-1/2} t^{-1/2} \\
&\quad \times \left(\frac{1}{\alpha-2} + \frac{1}{2(2-\alpha)} t^{-1} \int_0^\infty e^{-y^2/4t} y(1+y)^{-(\alpha-2)} dy\right) \\
&= \frac{b(c_\alpha^+ + c_\alpha^-)}{4\sqrt{\pi}} \eta_*(0)\chi_*(0)(1+t)^{-1/2} t^{-1/2} \Bigg(\frac{1}{\alpha-2} + \frac{1}{2(2-\alpha)} t^{-1} \\
&\quad \times \left(2^{3-\alpha} t^{2-(\alpha/2)} \Gamma\left(2-\frac{\alpha}{2}\right) + (2-\alpha)\int_0^\infty e^{-y^2/4t} y(y+\theta_2)^{-(\alpha-1)} dy\right)\Bigg).
\end{aligned} \tag{3.4.19}$$

Therefore, from (3.4.15), (3.4.17) and (3.4.19), we obtain

$$\begin{aligned}
&\|Z(\cdot,t)\|_{L^\infty} \geq |Z(0,t)| = |L_1(t) + L_2(t)| \\
&= \frac{|\eta_*(0)|}{4\sqrt{\pi}} \Bigg| 2^{2-\alpha}(c_\alpha^+ - c_\alpha^-)\Gamma\left(\frac{3-\alpha}{2}\right) t^{-\alpha/2} \\
&\quad - (c_\alpha^+ - c_\alpha^-)(\alpha-1) t^{-3/2} \int_0^\infty e^{-y^2/4t} y(y+\theta_1)^{-\alpha} dy \\
&\quad - \frac{(c_\alpha^+ + c_\alpha^-) b\chi_*(0)}{2-\alpha}(1+t)^{-1/2} t^{-1/2} \\
&\quad + \frac{2^{2-\alpha}(c_\alpha^+ + c_\alpha^-) b\chi_*(0)}{2-\alpha} \Gamma\left(2-\frac{\alpha}{2}\right)(1+t)^{-1/2} t^{1/2-\alpha/2} \\
&\quad + \frac{(c_\alpha^+ + c_\alpha^-) b\chi_*(0)}{2}(1+t)^{-1/2} t^{-3/2} \int_0^\infty e^{-y^2/4t} y(y+\theta_2)^{-(\alpha-1)} dy \Bigg| \\
&= \frac{|\eta_*(0)|}{4\sqrt{\pi}} \Bigg| 2^{2-\alpha}\left((c_\alpha^+ - c_\alpha^-)\Gamma\left(\frac{3-\alpha}{2}\right) + \frac{(c_\alpha^+ + c_\alpha^-) b\chi_*(0)}{2-\alpha} \Gamma\left(2-\frac{\alpha}{2}\right)\right) t^{-\alpha/2} \\
&\quad - \frac{(c_\alpha^+ + c_\alpha^-) b\chi_*(0)}{2-\alpha}(1+t)^{-1/2} t^{-1/2} \\
&\quad - (c_\alpha^+ - c_\alpha^-)(\alpha-1) t^{-3/2} \int_0^\infty e^{-y^2/4t} y(y+\theta_1)^{-\alpha} dy \\
&\quad + \frac{2^{2-\alpha}(c_\alpha^+ + c_\alpha^-) b\chi_*(0)}{2-\alpha} \Gamma\left(2-\frac{\alpha}{2}\right)((1+t)^{-1/2} - t^{-1/2}) t^{1/2-\alpha/2} \\
&\quad + \frac{(c_\alpha^+ + c_\alpha^-) b\chi_*(0)}{2}(1+t)^{-1/2} t^{-3/2} \int_0^\infty e^{-y^2/4t} y(y+\theta_2)^{-(\alpha-1)} dy \Bigg| \\
&\equiv \frac{|\eta_*(0)|}{4\sqrt{\pi}} |M_1(t) + M_2(t) + M_3(t) + M_4(t) + M_5(t)|.
\end{aligned} \tag{3.4.20}$$

For $M_1(t)$, we get

$$|M_1(t)| \geq 2^{2-\alpha}|\beta_0|(1+t)^{-\alpha/2}, \tag{3.4.21}$$

where β_0 is defined by (3.2.8). Obviously, we have

$$|M_2(t)| \leq \frac{|(c_\alpha^+ + c_\alpha^-)b\chi_*(0)|}{2-\alpha}t^{-1}. \tag{3.4.22}$$

On the other hand, for $M_3(t)$, we have from (3.4.18)

$$\begin{aligned} |M_3(t)| &\leq (\alpha-1)|c_\alpha^+ - c_\alpha^-|t^{-3/2}\int_0^\infty e^{-y^2/4t}y^{1-\alpha}dy \\ &= \frac{(\alpha-1)|c_\alpha^+ - c_\alpha^-|}{2^{\alpha-1}}\Gamma\left(1-\frac{\alpha}{2}\right)t^{-1/2-\alpha/2}. \end{aligned} \tag{3.4.23}$$

It is easy to see that

$$\begin{aligned} |M_4(t)| &\leq \frac{2^{2-\alpha}|(c_\alpha^+ + c_\alpha^-)b\chi_*(0)|}{2-\alpha}\Gamma\left(2-\frac{\alpha}{2}\right)(t^{-1/2}-(1+t)^{-1/2})t^{1/2-\alpha/2} \\ &\leq \frac{2^{2-\alpha}|(c_\alpha^+ + c_\alpha^-)b\chi_*(0)|}{2-\alpha}\Gamma\left(2-\frac{\alpha}{2}\right)t^{-1-\alpha/2}. \end{aligned} \tag{3.4.24}$$

Finally, for $M_5(t)$, we obtain from (3.4.18)

$$\begin{aligned} |M_5(t)| &\leq \frac{|(c_\alpha^+ + c_\alpha^-)b\chi_*(0)|}{2}(1+t)^{-1/2}t^{-3/2}\int_0^\infty e^{-y^2/4t}y^{-(\alpha-2)}dy \\ &\leq \frac{|(c_\alpha^+ + c_\alpha^-)b\chi_*(0)|}{2^{\alpha-1}}\Gamma\left(\frac{3-\alpha}{2}\right)t^{-1/2-\alpha/2}. \end{aligned} \tag{3.4.25}$$

Therefore, combining (3.4.20) through (3.4.25), we have

$$\begin{aligned} \|Z(\cdot,t)\|_{L^\infty} &\geq \frac{|\eta_*(0)|}{4\sqrt{\pi}}(|M_1(t)|-|M_2(t)|-|M_3(t)|-|M_4(t)|-|M_5(t)|) \\ &\geq \frac{|\eta_*(0)|}{4\sqrt{\pi}}\Bigg(2^{2-\alpha}|\beta_0|(1+t)^{-\alpha/2} - \frac{|(c_\alpha^+ + c_\alpha^-)b\chi_*(0)|}{2-\alpha}t^{-1} \\ &\quad - \frac{(\alpha-1)|c_\alpha^+ - c_\alpha^-|}{2^{\alpha-1}}\Gamma\left(1-\frac{\alpha}{2}\right)t^{-1/2-\alpha/2} \\ &\quad - \frac{2^{2-\alpha}|(c_\alpha^+ + c_\alpha^-)b\chi_*(0)|}{2-\alpha}\Gamma\left(2-\frac{\alpha}{2}\right)t^{-1-\alpha/2} \\ &\quad - \frac{|(c_\alpha^+ + c_\alpha^-)b\chi_*(0)|}{2^{\alpha-1}}\Gamma\left(\frac{3-\alpha}{2}\right)t^{-1/2-\alpha/2}\Bigg). \end{aligned}$$

Hence, there is a positive constant ν_0 such that (3.2.6) holds. This completes the proof of Theorem 3.2.2 for $1<\alpha<2$. □

3.5 Proof of Theorem 3.2.2 for $\alpha = 2$

In this last section, we shall completes the proof of Theorem 3.2.2.

End of the proof of Theorem 3.2.2 for $\boldsymbol{\alpha = 2}$. First, we shall prove (3.2.4). We set

$$\begin{aligned}\phi(x,t) &\equiv u(x,t) - \chi(x,t) - v(x,t)\\ &= \psi(x,t) - v(x,t),\end{aligned}$$

where $v(x,t)$ is the solution to

$$\begin{aligned} v_t + (b\chi v)_x - v_{xx} = -\left(\frac{c}{3}\chi^3\right)_x &-k\chi_{xxx}, \quad x\in\mathbb{R},\ t>0,\\ v(x,0) = 0, \quad x\in\mathbb{R}.& \end{aligned} \tag{2.6.1}$$

Then, from (2.1.1), (2.1.5), (3.4.1) and (2.6.1), $\phi(x,t)$ satisfies the following equation:

$$\begin{aligned} \phi_t + (b\chi\phi)_x - \phi_{xx} = -\left(\frac{b}{2}\psi^2\right)_x &-\left(\frac{c}{3}\psi^3\right)_x -c(u\chi\psi)_x - k\psi_{xxx}, \quad x\in\mathbb{R},\ t>0,\\ \phi(x,0) = \psi_0(x) = u_0(x) - \chi_*(x),&\ x\in\mathbb{R}. \end{aligned} \tag{3.5.1}$$

From Lemma 2.5.1, we obtain

$$\begin{aligned} \phi(x,t) = U[\psi_0](x,t,0) &- \frac{b}{2}\int_0^t U[\partial_x\psi^2(\tau)](x,t,\tau)d\tau\\ &-\frac{c}{3}\int_0^t U[\partial_x\psi^3(\tau)](x,t,\tau)d\tau - c\int_0^t U[\partial_x(u\chi\psi)(\tau)](x,t,\tau)d\tau\\ &-k\int_0^t U[\partial_x^3\psi(\tau)](x,t,\tau)d\tau\\ \equiv U[\psi_0]&(x,t,0) + K_1 + K_2 + K_3 + K_4. \end{aligned} \tag{3.5.2}$$

Thus, we have

$$\begin{aligned} &u(x,t) - \chi(x,t) - Z(x,t) - V(x,t)\\ &= U[\psi_0](x,t,0) - Z(x,t) + v(x,t) - V(x,t) + K_1 + K_2 + K_3 + K_4, \end{aligned} \tag{3.5.3}$$

where $Z(x,t)$ and $V(x,t)$ are defined by (3.2.5) and (2.2.3), respectively.

We evaluate K_1, K_2, K_3 and K_4. In the same way to get (3.4.9), from Lemma 2.5.6, Theorem 3.2.1 and Proposition 3.3.2 for $\alpha = 2$, we obtain

$$
\begin{aligned}
&\|K_1(\cdot,t)\|_{L^\infty} \\
&\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2} \\
&\quad \times \left(\int_0^{t/2}(t-\tau)^{-1/2-n/2}\|\psi^2(\cdot,\tau)\|_{L^1}d\tau + \int_{t/2}^{t}(t-\tau)^{-n/2}\|\psi^2(\cdot,\tau)\|_{L^\infty}d\tau\right) \\
&\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2}\left(\int_0^{t/2}(t-\tau)^{-1/2-n/2}(1+\tau)^{-3/2}\log(2+\tau)^2 d\tau\right. \\
&\quad \left. + \int_{t/2}^{t}(t-\tau)^{-n/2}(1+\tau)^{-2}\log(2+\tau)^2 d\tau\right) \\
&\le C(1+t)^{-1}, \ t \ge 1.
\end{aligned}
\tag{3.5.4}
$$

We omit the evaluation of K_2, since we can easily evaluate it in the same way as the above calculation. Actually, it follows that

$$
\|K_2(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1}, \ t \ge 1. \tag{3.5.5}
$$

Next, we evaluate K_3. Using Lemma 2.5.6, the Schwarz inequality, Proposition 2.3.3, Lemma 2.4.1 and Theorem 3.2.1 for $\alpha = 2$, we obtain

$$
\begin{aligned}
&\|K_3(\cdot,t)\|_{L^\infty} \\
&\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2}\left(\int_0^{t/2}(t-\tau)^{-1/2-n/2}\|(u\chi\psi)(\cdot,\tau)\|_{L^1}d\tau\right. \\
&\quad \left. + \int_{t/2}^{t}(t-\tau)^{-n/2}\|(u\chi\psi)(\cdot,\tau)\|_{L^\infty}d\tau\right) \\
&\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2}\left(\int_0^{t/2}(t-\tau)^{-1/2-n/2}\|u(\cdot,t)\|_{L^2}\|\chi(\cdot,t)\|_{L^2}\|\psi(\cdot,\tau)\|_{L^\infty}d\tau\right. \\
&\quad \left. + \int_{t/2}^{t}(t-\tau)^{-n/2}\|u(\cdot,t)\|_{L^\infty}\|\chi(\cdot,t)\|_{L^\infty}\|\psi(\cdot,\tau)\|_{L^\infty}d\tau\right) \\
&\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2}\left(\int_0^{t/2}(t-\tau)^{-1/2-n/2}(1+\tau)^{-3/2}\log(2+\tau)d\tau\right. \\
&\quad \left. + \int_{t/2}^{t}(t-\tau)^{-n/2}(1+\tau)^{-2}\log(2+\tau)d\tau\right) \\
&\le C(1+t)^{-1}, \ t \ge 1.
\end{aligned}
\tag{3.5.6}
$$

Finally, we evaluate K_4. From the definition of K_4, it follows that

$$
\begin{aligned}
K_4 &= -k\int_0^t\int_{\mathbb{R}}\partial_x(G(x-y,t-\tau)\eta(x,t))(\eta(y,\tau))^{-1}\partial_y^2\psi(y,\tau)dyd\tau \\
&= -k\sum_{n=0}^{1}\partial_x^{1-n}\eta(x,t)g_n(x,t),
\end{aligned}
$$

where we put

$$g_n(x,t) \equiv \int_0^t \int_{\mathbb{R}} \partial_x^n G(x-y,t-\tau)(\eta(y,\tau))^{-1}\partial_y^2\psi(y,\tau)dyd\tau.$$

Then, we have from Lemma 2.4.2

$$\begin{aligned} \|K_4(\cdot,t)\|_{L^\infty} &\le C\sum_{n=0}^{1} \|\partial_x^{1-n}\eta(\cdot,t)\|_{L^\infty}\|g_n(\cdot,t)\|_{L^\infty} \\ &\le C\sum_{n=0}^{1}(1+t)^{-1/2+n/2}\|g_n(\cdot,t)\|_{L^\infty}. \end{aligned} \tag{3.5.7}$$

By making the integration by parts for the first part of the τ-integral of $g_n(x,t)$, we have

$$\begin{aligned} g_n(x,t) &= \left(\int_0^{t/2} + \int_{t/2}^t\right)\int_{\mathbb{R}} \partial_x^n G(x-y,t-\tau)(\eta(y,\tau))^{-1}\partial_y^2\psi(y,\tau)dyd\tau \\ &= \int_0^{t/2}\int_{\mathbb{R}} \partial_x^{n+2} G(x-y,t-\tau)(\eta(y,\tau))^{-1}\psi(y,\tau)dyd\tau \\ &\quad - 2\int_0^{t/2}\int_{\mathbb{R}} \partial_x^{n+1} G(x-y,t-\tau)\partial_y(\eta(y,\tau)^{-1})\psi(y,\tau)dyd\tau \\ &\quad + \int_0^{t}\int_{\mathbb{R}} \partial_x^{n} G(x-y,t-\tau)\partial_y^2(\eta(y,\tau)^{-1})\psi(y,\tau)dyd\tau \\ &\quad + \int_{t/2}^{t}\int_{\mathbb{R}} \partial_x^{n} G(x-y,t-\tau)(\eta(y,\tau))^{-1}\partial_y^2\psi(y,\tau)dyd\tau. \end{aligned}$$

Therefore, from Young's inequality, the Schwarz inequality, Proposition 2.3.3, Lemma 2.4.1, Lemma 2.4.2 and Proposition 3.3.1, we have

$$\begin{aligned} \|g_n(\cdot,t)\|_{L^\infty} &\le C\int_0^{t/2} \|\partial_x^{n+2}G(\cdot,t-\tau)\|_{L^\infty}(\|u(\cdot,\tau)\|_{L^1} + \|\chi(\cdot,t)\|_{L^1})d\tau \\ &\quad + C\int_0^{t/2} \|\partial_x^{n+1}G(\cdot,t-\tau)\|_{L^\infty}\|\partial_x(\eta(\cdot,\tau)^{-1})\|_{L^2}\|\psi(\cdot,\tau)\|_{L^2}d\tau \\ &\quad + C\int_0^{t/2} \|\partial_x^{n}G(\cdot,t-\tau)\|_{L^\infty}\|\partial_x^2(\eta(\cdot,\tau)^{-1})\|_{L^2}\|\psi(\cdot,\tau)\|_{L^2}d\tau \\ &\quad + C\int_{t/2}^{t} \|\partial_x^{n}G(\cdot,t-\tau)\|_{L^1}(\|\partial_x^2 u(\cdot,\tau)\|_{L^\infty} + \|\partial_x^2\chi(\cdot,\tau)\|_{L^\infty})d\tau \\ &\le C\int_0^{t/2}(t-\tau)^{-3/2-n/2}(1+\tau^{-1/4})d\tau \\ &\quad + C\int_0^{t/2}(t-\tau)^{-1-n/2}(1+\tau)^{-1}\log(2+\tau)d\tau \\ &\quad + C\int_0^{t/2}(t-\tau)^{-1/2-n/2}(1+\tau)^{-3/2}\log(2+\tau)d\tau \\ &\quad + C\int_{t/2}^{t}(t-\tau)^{-n/2}(1+\tau)^{-3/2}d\tau \\ &\le C(1+t)^{-1/2-n/2},\ t \ge 1. \end{aligned} \tag{3.5.8}$$

Combining (3.5.7) and (3.5.8), we get

$$\|K_4(\cdot,t)\|_{L^\infty} \le C(1+t)^{-1}, \ \ge 1. \tag{3.5.9}$$

Summing up (2.6.57), (3.5.4), (3.5.5), (3.5.6) and (3.5.9), we have

$$\begin{aligned}&\|u(\cdot,t)-\chi(\cdot,t)-Z(\cdot,t)-V(\cdot,t)\|_{L^\infty}\\ &\le \|U[\psi_0](\cdot,t,0)-Z(\cdot,t)\|_{L^\infty}+C(1+t)^{-1}, \quad t\ge 1.\end{aligned}$$

Therefore, from (3.4.5), we obtain

$$\limsup_{t\to\infty}\frac{(1+t)}{\log(1+t)}\|u(\cdot,t)-\chi(\cdot,t)-Z(\cdot,t)-V(\cdot,t)\|_{L^\infty}=0.$$

Thus, we completes the proof of (3.2.4).

Finally, we shall derive the lower bound estimate of $Z(x,t)+V(x,t)$, that is (3.2.7). First, we take $x=0$ in (2.2.3). Then, from (2.1.10), it follows that

$$V(0,t) = -\frac{bd}{4\sqrt{\pi}}\left(\frac{b^2k}{8}+\frac{c}{3}\right)\chi_*(0)\eta_*(0)(1+t)^{-1}\log(1+t). \tag{3.5.10}$$

Combining (3.4.15) and (3.5.10), we have from the triangle inequality and (2.4.1)

$$\begin{aligned}&\|Z(\cdot,t)+V(\cdot,t)\|_{L^\infty} \ge |Z(0,t)+V(0,t)|\\ &=\Bigg|\frac{(c_2^+-c_2^-)\eta_*(0)}{4\sqrt{\pi}}t^{-3/2}\int_0^\infty e^{-y^2/4t}y(1+y)^{-1}dy\\ &\quad+\frac{(c_2^++c_2^-)b\chi_*(0)\eta_*(0)}{4\sqrt{\pi}}(1+t)^{-1/2}t^{-1/2}\int_0^\infty e^{-y^2/4t}(1+y)^{-1}dy\\ &\quad-\frac{bd}{4\sqrt{\pi}}\left(\frac{b^2k}{8}+\frac{c}{3}\right)\chi_*(0)\eta_*(0)(1+t)^{-1}\log(1+t)\Bigg|\\ &\ge\frac{|b|}{4\sqrt{\pi}}|\chi_*(0)||\eta_*(0)|(1+t)^{-1/2}\Bigg|(c_2^++c_2^-)t^{-1/2}\int_0^\infty e^{-y^2/4t}(1+y)^{-1}dy\\ &\quad-d\left(\frac{b^2k}{8}+\frac{c}{3}\right)(1+t)^{-1/2}\log(1+t)\Bigg|-\frac{|(c_2^+-c_2^-)\eta_*(0)|}{4\sqrt{\pi}}t^{-3/2}\int_0^\infty e^{-y^2/4t}dy\\ &\equiv\frac{|b|}{4\sqrt{\pi}}|\chi_*(0)||\eta_*(0)|(1+t)^{-1/2}W(t)-\frac{|(c_2^+-c_2^-)\eta_*(0)|}{4}t^{-1}.\end{aligned} \tag{3.5.11}$$

Now, we evaluate $W(t)$ from below. Splitting the y-integral and using the triangle inequality, we obtain

$$
\begin{aligned}
W(t) &= \Bigg|(c_2^+ + c_2^-)t^{-1/2}\bigg(\int_0^{\sqrt{1+t}-1} + \int_{\sqrt{1+t}-1}^{\infty}\bigg)e^{-y^2/4t}(1+y)^{-1}dy \\
&\qquad - d\bigg(\frac{b^2k}{8} + \frac{c}{3}\bigg)(1+t)^{-1/2}\log(1+t)\Bigg| \\
&\geq \Bigg|(c_2^+ + c_2^-)t^{-1/2}\int_0^{\sqrt{1+t}-1} e^{-y^2/4t}(1+y)^{-1}dy \\
&\qquad - d\bigg(\frac{b^2k}{8} + \frac{c}{3}\bigg)(1+t)^{-1/2}\log(1+t)\Bigg| \\
&\qquad - \Bigg|(c_2^+ + c_2^-)t^{-1/2}\int_{\sqrt{1+t}-1}^{\infty} e^{-y^2/4t}(1+y)^{-1}dy\Bigg| \\
&\equiv W_1(t) - W_2(t).
\end{aligned} \tag{3.5.12}
$$

From the mean value theorem, there exists $\theta \in (0,1)$ such that

$$
e^{-y^2/4t} = 1 - \frac{y^2}{4t}e^{-\theta y^2/4t}.
$$

Therefore, we obtain

$$
\begin{aligned}
W_1(t) &= \Bigg|(c_2^+ + c_2^-)t^{-1/2}\int_0^{\sqrt{1+t}-1}\bigg(1 - \frac{y^2}{4t}e^{-\theta y^2/4t}\bigg)(1+y)^{-1}dy \\
&\qquad - d\bigg(\frac{b^2k}{8} + \frac{c}{3}\bigg)(1+t)^{-1/2}\log(1+t)\Bigg| \\
&\geq \Bigg|(c_2^+ + c_2^-)t^{-1/2}\int_0^{\sqrt{1+t}-1}(1+y)^{-1}dy \\
&\qquad - d\bigg(\frac{b^2k}{8} + \frac{c}{3}\bigg)(1+t)^{-1/2}\log(1+t)\Bigg| \\
&\qquad - \Bigg|\frac{c_2^+ + c_2^-}{4}t^{-3/2}\int_0^{\sqrt{1+t}-1} e^{-\theta y^2/4t}y^2(1+y)^{-1}dy\Bigg| \\
&\equiv W_{1.1}(t) - W_{1.2}(t).
\end{aligned} \tag{3.5.13}
$$

For $W_{1.1}(t)$, we have

$$
\begin{aligned}
W_{1.1}(t) &= \Bigg|\frac{c_2^+ + c_2^-}{2}t^{-1/2}\log(1+t) - d\bigg(\frac{b^2k}{8} + \frac{c}{3}\bigg)(1+t)^{-1/2}\log(1+t)\Bigg| \\
&= \Bigg|\frac{c_2^+ + c_2^-}{2}t^{-1/2}\log(1+t) - \frac{c_2^+ + c_2^-}{2}(1+t)^{-1/2}\log(1+t) \\
&\qquad + \bigg(\frac{c_2^+ + c_2^-}{2} - d\bigg(\frac{b^2k}{8} + \frac{c}{3}\bigg)\bigg)(1+t)^{-1/2}\log(1+t)\Bigg| \\
&\geq |\beta_1|(1+t)^{-1/2}\log(1+t) - \frac{|c_2^+ + c_2^-|}{2}\log(1+t)(t^{-1/2} - (1+t)^{-1/2}) \\
&\geq |\beta_1|(1+t)^{-1/2}\log(1+t) - \frac{|c_2^+ + c_2^-|}{2}t^{-3/2}\log(1+t),
\end{aligned} \tag{3.5.14}
$$

where β_1 is defined by (3.2.8). On the other hand, for $W_{1.2}(t)$, we obtain

$$\begin{aligned} W_{1.2}(t) &\leq \frac{|c_2^+ + c_2^-|}{4} t^{-3/2} \int_0^{\sqrt{1+t}-1} y \, dy \\ &= \frac{|c_2^+ + c_2^-|}{8} t^{-3/2} (\sqrt{1+t}-1)^2 \leq \frac{|c_2^+ + c_2^-|}{8} t^{-1/2}. \end{aligned} \tag{3.5.15}$$

Analogously, for $W_2(t)$, it follows that

$$\begin{aligned} W_2(t) &\leq |c_2^+ + c_2^-| t^{-1/2} \left(\sup_{y \geq \sqrt{1+t}-1} (1+|y|)^{-1} \right) \int_{y \geq \sqrt{1+t}-1} e^{-y^2/4t} dy \\ &\leq \sqrt{\pi} |c_2^+ + c_2^-| (1+t)^{-1/2}. \end{aligned} \tag{3.5.16}$$

Therefore, summing up (3.5.11) through (3.5.16), we get

$$\begin{aligned} &\|Z(\cdot,t) + V(\cdot,t)\|_{L^\infty} \\ &\geq \frac{|b|}{4\sqrt{\pi}} |\chi_*(0)| |\eta_*(0)| \left(|\beta_1| (1+t)^{-1} \log(1+t) - \frac{|c_2^+ + c_2^-|}{2} t^{-2} \log(1+t) \right. \\ &\quad \left. - |c_2^+ + c_2^-| \left(\sqrt{\pi} + \frac{1}{8} \right) t^{-1} \right) - \frac{|(c_2^+ - c_2^-)\eta_*(0)|}{4} t^{-1}. \end{aligned}$$

Hence, there is a positive constant ν_1 such that (3.2.7) holds. This completes the proof of Theorem 3.2.2 for $\alpha = 2$. □

Remark 3.5.1. Our method can be applied not only to (2.1.1) but also to other equations. For example, we can treat the following damped wave equation with a convection term in the same way:

$$u_{tt} - u_{xx} + u_t + (f(u))_x = 0,$$

where $f(u) = au + \frac{b}{2}u^2 + \frac{c}{3!}u^3$ with $|a| < 1$, $b \neq 0$ and $c \in \mathbb{R}$. Actually, for the above equation, we are able to obtain the the similar results for Theorem 3.2.1 and Theorem 3.2.2 (for details, see [9]).

Concluding Remark. We studied the large time behavior of the solutions for the initial value problem for the generalized KdV-Burgers equation:

$$\begin{aligned} u_t + (f(u))_x + k u_{xxx} = u_{xx}, \quad x \in \mathbb{R}, \ t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{2.1.1}$$

where $f(u) = (b/2)u^2 + (c/3)u^3$ and $b \neq 0$, $c, k \in \mathbb{R}$. In Chapter 2, we studied (2.1.1) with the data $u_0 \in L^1_1(\mathbb{R}) \cap H^3(\mathbb{R})$ and constructed the asymptotic profiles. Then, we have seen that the effects of the nonlinear term and the dispersion term on the asymptotic profiles of the solutions. In Chapter 3, we studied (2.1.1) with slowly decaying data satisfying (3.1.1) with $1 < \alpha \leq 2$, and investigated the effect of the shape of the initial value on the asymptotic profiles of the solutions. However, as we mentioned in Chapter 1, if the far field states of the data ($\lim_{x \to \pm\infty} u_0(x) = u_\pm$) are different, we can get the different asymptotic profiles. When $u_+ < u_-$, under the suitable assumptions on the initial perturbation, the solution to (2.1.1) converges to the traveling wave solution (cf. [54, 59, 60, 66]). On the other hand, when $u_+ > u_-$, the asymptotic profile is given by the rarefaction wave (see e.g. [5]). However, the optimal asymptotic rates to these functions and second asymptotic profiles of the solutions have not yet been obtained, and we leave these as future works.

Chapter 4

Appendix

The purpose of this appendix is to explain the idea of the derivation of the asymptotic profiles.

4.1 Self-Similar Solution of the Burgers Equation

We have already shown that the asymptotic profile of the solution to (2.1.1) is given by the nonlinear diffusion wave $\chi(x,t)$ which is essentially the self-similar solution of the Burgers equation. In this section, we shall derive the exact self-similar solution.

We consider the following Burgers equation:

$$\chi_t + \left(\frac{b}{2}\chi^2\right)_x = \mu\chi_{xx}, \quad \int_{\mathbb{R}} \chi(x,t)dx = \delta \neq 0, \tag{4.1.1}$$

where $b \neq 0$ and $\mu > 0$. If $\chi(x,t)$ is a solution of (4.1.1),

$$\chi^\lambda(x,t) \equiv \lambda\chi(\lambda x, \lambda^2 t), \quad \lambda > 0 \tag{4.1.2}$$

is also a solution of (4.1.1). We suppose $\chi = \chi^\lambda$. Then we have $\chi = \dfrac{1}{\sqrt{t}}\chi\left(\dfrac{x}{\sqrt{t}}, 1\right)$. Therefore, our task is reduced to find the solution of (4.1.1) of the form:

$$\chi(x,t) = \frac{1}{\sqrt{t}}\chi_*\left(\frac{x}{\sqrt{t}}\right), \quad \int_{\mathbb{R}} \chi_*(\xi)d\xi = \delta. \tag{4.1.3}$$

Here and later, we assume that $\chi_*(\xi)$ is smooth and decaying fast enough at spatial infinity. Substituting (4.1.3) to (4.1.1), we have

$$t^{-3/2}\left(-\frac{1}{2}\chi_* - \frac{1}{2}\frac{x}{\sqrt{t}}(\chi_*)' + \frac{b}{2}((\chi_*)^2)' - \mu(\chi_*)''\right) = 0.$$

Therefore, it follows that

$$-\frac{1}{2}(\xi\chi_*)' + \frac{b}{2}((\chi_*)^2)' = \mu(\chi_*)'', \quad \xi = \frac{x}{\sqrt{t}}, \quad {}' = \frac{d}{d\xi}.$$

Integrate both sides, we obtain Bernoulli's differential equation

$$(\chi_*)' + \frac{\xi}{2\mu}\chi_* = \frac{b}{2\mu}(\chi_*)^2.$$

We can rewrite it to the following equation:

$$\left(\frac{1}{\chi_*}\right)' - \frac{\xi}{2\mu}\frac{1}{\chi_*} = -\frac{b}{2\mu}.$$

If we set $\dfrac{1}{\chi_*} = \psi$, it follows that

$$\psi' - \frac{\xi}{2\mu}\psi = -\frac{b}{2\mu}.$$

Therefore, we can solve this equation and get

$$\psi(\xi) = e^{\xi^2/4\mu}\left(\psi(0) - \frac{b}{2\mu}\int_0^\xi e^{-\eta^2/4\mu}d\eta\right).$$

Thus we obtain

$$\chi_*(\xi) = \frac{e^{-\xi^2/4\mu}}{\frac{1}{\chi_*(0)} - \frac{b}{2\mu}\int_0^\xi e^{-\eta^2/4\mu}d\eta} = \frac{e^{-\xi^2/4\mu}}{\kappa + \frac{b}{\sqrt{\mu}}\int_{\xi/\sqrt{4\mu}}^\infty e^{-y^2}dy}.$$

where $\kappa = \dfrac{1}{\chi_*(0)} - \dfrac{b\sqrt{\pi}}{2\sqrt{\mu}}$. Since $\int_{\mathbb{R}} \chi_*(\xi)d\xi = \delta$, we have

$$\delta = \left[-\frac{2\mu}{b}\log\left|\kappa + \frac{b}{\sqrt{\mu}}\int_{\xi/\sqrt{4\mu}}^\infty e^{-y^2}dy\right|\right]_{-\infty}^{\infty} = -\frac{2\mu}{b}\log\left|\frac{\kappa}{\kappa + b\sqrt{\pi/\mu}}\right|,$$

which yields

$$\kappa = b\sqrt{\frac{\pi}{\mu}}(e^{b\delta/2\mu} - 1)^{-1}.$$

Hence, the self-similar solution of (4.1.1) is given by

$$\chi(x,t) = \frac{1}{b}\sqrt{\frac{\mu}{t}}\frac{(e^{b\delta/2\mu} - 1)e^{-x^2/4\mu t}}{\sqrt{\pi} + (e^{b\delta/2\mu} - 1)\int_{x/\sqrt{4\mu t}}^\infty e^{-y^2}dy}. \tag{4.1.4}$$

The nonlinear diffusion wave defined by (2.1.3) is obtained from (4.1.4) by a small shift in time, so that we can avoid the singularity at $t = 0$.

4.2 Idea of the Derivation of the Second Asymptotic Profiles

We have seen that the second asymptotic profile of the solution to (2.1.1) is given by $V(x,t)$ (defined by (2.2.3)) and $Z(x,t)$ (defined by (3.2.5)) in Theorem 2.2.1 and

Theorem 3.2.2, respectively. But it might be useful if we introduce the idea of the derivation of the second asymptotic profile by heuristic discussion.

First we consider the linear heat equation:

$$\begin{aligned} u_t - u_{xx} = 0, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{4.2.1}$$

where $u_0 \in C^0(\mathbb{R})$ satisfying the following condition:

$$\exists \gamma > 0, \quad \exists C > 0 \ \ s.t.\ \ |u_0(x)| \le C(1+|x|)^{-\gamma}, \quad x \in \mathbb{R}. \tag{4.2.2}$$

If $\gamma > 1$ in (4.2.2), then $u_0 \in L^1(\mathbb{R})$, so that we can pose the condition $\int_{\mathbb{R}} u_0(x)dx \equiv \delta$. In this case, it is known that the solution to (4.2.1) tends to $\delta G(x,t)$ (for details, see e.g. pp.10-11 of [10]). Formally, we denote it

$$u(x,t) \sim \delta G(x,t), \quad t \to \infty. \tag{4.2.3}$$

On the other hand, in the case of $0 < \gamma \le 1$ in (4.2.2), by applying the discussion given in [50], we see that

$$u(x,t) \sim \int_{\mathbb{R}} \frac{c_\gamma(y)G(x-y,t)}{(1+|y|)^\gamma} dy, \quad t \to \infty, \quad c_\gamma(y) \equiv \begin{cases} c_\gamma^+, & y \ge 0, \\ c_\gamma^-, & y < 0, \end{cases} \tag{4.2.4}$$

provided that $\lim_{x\to\pm\infty}(1+|x|)^{\gamma-1}u_0(x) \equiv c_\gamma^\pm$ exist.

Next, for a given parameter $\tau \in \mathbb{R}$, we consider the following initial value problem

$$\begin{aligned} \psi_t - (b\chi\psi)_x - \psi_{xx} = 0, \quad x \in \mathbb{R},\ t > \tau, \\ \psi(x,\tau) = \psi_0(x), \quad x \in \mathbb{R} \end{aligned} \tag{4.2.5}$$

with the following condition:

$$\exists \alpha > 1, \quad \exists C > 0 \ \ s.t.\ \ |\psi_0(x)| \le C(1+|x|)^{-\alpha}, \quad x \in \mathbb{R}. \tag{4.2.6}$$

We set

$$r(x,t) = \int_{-\infty}^{x} \psi(y,t)dy.$$

Then, it follows that

$$\begin{aligned} r_t - b\chi r_x - r_{xx} = 0, \quad x \in \mathbb{R},\ t > \tau, \\ r(x,\tau) = \int_{-\infty}^{x} \psi(y,\tau)dy, \end{aligned} \tag{4.2.7}$$

so that $z(x,t) \equiv \eta(x,t)^{-1}r(x,t)$ satisfies the heat equation

$$\begin{aligned} &z_t - z_{xx} = 0, \quad x \in \mathbb{R},\ t > \tau, \\ &z(x,\tau) = z_0(x) = \eta(x,t)^{-1}\int_{-\infty}^{x} \psi(y,\tau)dy, \end{aligned} \tag{4.2.8}$$

where $\eta(x,t)$ is defined by (2.4.6). Moreover, from (4.2.6) and (4.2.8), we can check the following estimate in the same way to get (3.3.8):

$$|z_0(x)| \le C(1+|x|)^{-(\alpha-1)}, \ y \in \mathbb{R}. \tag{4.2.9}$$

Case (i): $\alpha > 2$ in (4.2.6), i.e. $\alpha - 1 > 1$ in (4.2.9). We have from (4.2.3)

$$z(x,t) \sim M[\psi](\tau)G(x,t-\tau), \quad t \to \infty, \tag{4.2.10}$$

or

$$r(x,t) \sim M[\psi](\tau)G(x,t-\tau)\eta(x,t), \quad t \to \infty, \tag{4.2.11}$$

where $M[\psi](\tau) = \int_{\mathbb{R}} \eta(y,\tau)^{-1}(\int_{-\infty}^{y} \psi(\xi,\tau)d\xi)dy$. Thus, Lemma 2.5.1, (4.2.7) and (4.2.11) lead that

$$\psi(x,t) = U[\psi_0](x,t,\tau) \sim M[\psi](\tau)\partial_x(G(x,t-\tau)\eta(x,t)), \quad t \to \infty. \tag{4.2.12}$$

Case (ii): $1 < \alpha \le 2$ in (4.2.6), i.e. $0 < \alpha - 1 \le 1$ in (4.2.9). We obtain from (4.2.4)

$$z(x,t) \sim \int_{\mathbb{R}} \frac{c_\alpha(y)G(x-y,t)}{(1+|y|)^{\alpha-1}}dy, \quad t \to \infty, \quad c_\alpha(y) \equiv \begin{cases} c_\alpha^+, & y \ge 0, \\ c_\alpha^-, & y < 0, \end{cases} \tag{4.2.13}$$

or

$$r(x,t) \sim \int_{\mathbb{R}} \frac{c_\alpha(y)G(x-y,t)\eta(x,t)}{(1+|y|)^{\alpha-1}}dy, \quad t \to \infty, \tag{4.2.14}$$

provided that the data satisfies $\lim_{x\to\pm\infty}(1+|x|)^{\alpha-1}z_0(x) \equiv c_\alpha^\pm$. Therefore, similarly as before, we see that

$$\psi(x,t) = U[\psi_0](x,t,\tau) \sim \int_{\mathbb{R}} \frac{c_\alpha(y)\partial_x(G(x-y,t-\tau)\eta(x,t))}{(1+|y|)^{\alpha-1}}dy, \quad t \to \infty. \tag{4.2.15}$$

Now, recalling that the solution to (2.6.1) is given by (2.6.4), we have from (4.2.12) that

$$\begin{aligned} v(x,t) &= \int_0^t U\left[\left(\partial_x\left(-\frac{c}{3}\chi^3\right) - k\chi_{xxx}\right)(\tau)\right](x,t,\tau)d\tau \\ &\sim \int_0^t M\left[\partial_x\left(-\frac{c}{3}\chi^3\right) - k\chi_{xxx}\right](\tau)\partial_x(G(x,t-\tau)\eta(x,t))d\tau \\ &= \int_0^t \partial_x(G(x,t-\tau)\eta(x,t)) \int_{\mathbb{R}} \eta(y,\tau)^{-1}\left(-\frac{c}{3}\chi^3 - k\chi_{xx}\right)(y,\tau)dyd\tau \\ &= -d\left(\frac{b^2k}{8} + \frac{c}{3}\right)\int_0^t \partial_x(G(x,t-\tau)\eta(x,t))(1+\tau)^{-1}d\tau, \end{aligned} \tag{4.2.16}$$

where we used the facts

$$\int_{\mathbb{R}} \eta(y,\tau)^{-1}\chi(y,\tau)^3 dy = d(1+\tau)^{-1}, \quad \int_{\mathbb{R}} \eta(y,\tau)^{-1}\chi_{xx}(y,\tau)dy = \frac{b^2d}{8}(1+\tau)^{-1}.$$

Now, we rewrite

$$
\begin{aligned}
&\int_0^t \partial_x(G(x,t-\tau)\eta(x,t))(1+\tau)^{-1}d\tau \\
&= \eta(x,t)\int_0^t \partial_x G(x,t-\tau)(1+\tau)^{-1}d\tau + \partial_x\eta(x,t)\int_0^t G(x,t-\tau)(1+\tau)^{-1}d\tau.
\end{aligned}
$$

To proceed further, let us take $x=0$. Then we have

$$
\begin{aligned}
I(t) &\equiv \int_0^t \partial_x(G(x,t-\tau)\eta(x,t))\mid_{x=0}(1+\tau)^{-1}d\tau \\
&= \frac{b}{2}\chi(0,t)\eta(0,t)\int_0^t G(0,t-\tau)(1+\tau)^{-1}d\tau.
\end{aligned}
$$

Since the factor $(1+\tau)^{-1}$ is irrelevant to the integral when τ is close to 0, we may guess that the leading term of the above integral is

$$
\int_0^t G(0,t)(1+\tau)^{-1}d\tau = \frac{1}{\sqrt{4\pi}}t^{-1/2}\log(1+t).
$$

Actually, it follows that

$$
\begin{aligned}
&\left|\int_0^t G(0,t-\tau)(1+\tau)^{-1}d\tau - \int_0^t G(0,t)(1+\tau)^{-1}d\tau\right| \\
&= \frac{1}{\sqrt{4\pi}}\int_0^t \left(\frac{1}{\sqrt{t-\tau}}-\frac{1}{\sqrt{t}}\right)\frac{1}{1+\tau}d\tau = \frac{1}{\sqrt{4\pi}}\int_0^t \frac{\sqrt{t}-\sqrt{t-\tau}}{\sqrt{t}\sqrt{t-\tau}}\frac{1}{1+\tau}d\tau \\
&= \frac{1}{\sqrt{4\pi}}\int_0^t \frac{1}{\sqrt{t}\sqrt{t-\tau}(\sqrt{t}+\sqrt{t-\tau})}\frac{\tau}{1+\tau}d\tau \\
&\le \frac{1}{\sqrt{4\pi}}\int_0^t \frac{1}{t\sqrt{t-\tau}}d\tau = \frac{1}{\sqrt{\pi}}t^{-1/2}.
\end{aligned}
$$

Therefore, $I(t)$ can be well approximated by

$$
\frac{b}{4\sqrt{\pi}}\chi(0,t)\eta(0,t)t^{-1/2}\log(1+t).
$$

As for the general case, replacing $G(x,t-\tau)$ by $G(x,t+1)$ and restricting the interval of the τ-integral to $(0,t/2)$, from (4.2.16), we see that the leading term would be

$$
\begin{aligned}
v(x,t) &\sim -d\left(\frac{b^2k}{8}+\frac{c}{3}\right)\int_0^{t/2}\partial_x(G(x,1+t)\eta(x,t))(1+\tau)^{-1}d\tau \\
&\sim -d\left(\frac{b^2k}{8}+\frac{c}{3}\right)\partial_x(G(x,1+t)\eta(x,t))\log(1+t) \\
&= V(x,t).
\end{aligned}
$$

On the other hand, for the first term in the right hand side of (3.4.3) with slowly decaying data, we immediately obtain from (4.2.15) that

$$
U[\psi_0](x,t,0) \sim \int_{\mathbb{R}} \frac{c_\alpha(y)\partial_x(G(x-y,t)\eta(x,t))}{(1+|y|)^{\alpha-1}}dy = Z(x,t).
$$

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