博士学位論文

Studies on freely infinitely divisible distributions
(自由無限分解可能分布に関する研究)

植田 優基

北海道大学大学院理学院
数学専攻

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1 Introduction

Histories of free probability theory

In 1980s, free probability theory was initiated by Voiculescu to understand the construction of free group factors. In operator algebra theory, we know that the group algebra $\mathbb{C}G$ for a discrete group G can be extended to the group von Neumann algebra $L(G)$ by taking the closure of the group algebras, acting on $\ell^2(G)$, with respect to the weak topology. To understand the structure of group von Neumann algebra $L(G)$ is one of important investigations in operator algebras. Let $\mathbb{F}_d$ be the free group with d generators and $\tau_{\mathbb{F}_d}$ a state on $\mathbb{F}_d$ defined by

$$\tau_{\mathbb{F}_d}\left(\sum_{g\in\mathbb{F}_d}\alpha_g g\right) := \alpha_e, \tag{1.1}$$

where e is the unit element in $\mathbb{F}_d$. We call $L(\mathbb{F}_d)$ the free group factor. Voiculescu pointed out that there is an important relation (which is called free independence, see Definition 2.4) between free group factors $L(\mathbb{F}_m)$ and $L(\mathbb{F}_n)$ in $(L(\mathbb{F}_{m+n}), \tau_{\mathbb{F}_{m+n}})$. Voiculescu's motivation for introducing the free independence was to attack the (still open) problem of whether $L(\mathbb{F}_m)$ is isomorphic to $L(\mathbb{F}_n)$ or not when $m \neq n$. After Voiculescu's idea, free probability theory contributed much to study the free group factors (e.g. see [54]).

While free probability theory had been evolving to investigate the free group factors, it also found connections with other topics (e.g. classical probability theory, combinatorics, representation theory of symmetric groups and random matrices). Recently, there are a few of relationships between deep learning and free probability theory (see e.g. [41]). A particularly important way of developing free probability theory is to find an analogue of classical probability theory. For example, there are correspondences between central limit theorem and free central limit theorem, cumulants and free cumulants, and convolution and free convolution, and so on. In particular, the free central limit (for short, FCL) theorem is one of most important an analogue of classical probability theory:

Proposition 1.1. (**FCL theorem**) Let $(\mathcal{A}, \varphi)$ be a *-probability space (see Section 2.2). Assume that $X_1, \cdots, X_n \in \mathcal{A}$ are free independent selfadjoint operators in $(\mathcal{A}, \varphi)$ and $\varphi(X_i) = 0$ and $\varphi(X_i^2) = 1$. Then the laws of

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \tag{1.2}$$

in distribution converge to the following probability distribution:

$$\frac{1}{2\pi}\sqrt{4-x^2}\cdot \mathbf{1}_{[-2,2]}(x)dx, \tag{1.3}$$

as $n \to \infty$. The above distribution is called the (standard) semicircle law.

Voiculescu noticed a relation between free probability theory and random matrix theory since it is known that the semicircle law (the limit distribution of FCL theorem) appears as the limit of eigenvalue distributions of Wigner matrices as the size of the random matrices goes to infinity. This observation lead to the following remarkable theorem in 1991: two large size unitarily invariant random matrices have some relation (called *asymptotic freeness*). Moreover, the relation implies that the eigenvalue distribution of sum of two large size random matrices coincides with the free convolution of eigenvalue distributions of two large size random matrices (see [65]).

Moreover, the FCL theorem is very important in a view of a complete analogy of the classical central limit theorem. Motivated by this, many mathematicians in free probability theory had studied the correspondence of limit theorems in classical and free senses (see e.g. [14, 15, 16, 27]). The limit distributions of a most general limit theorem are characterized by infinite divisibility. A probability measure μ on $\mathbb{R}$ is said to be infinitely divisible if for any $n \in \mathbb{N}$ there exists a probability measure μ_n such that $\mu = \mu_n^{*n}$, where $*$ is the classical convolution which means distributions of sum of independent random variables. For example, the normal distribution is ID. The ID distributions are one of most important distributions in probability theory, mathematical statistics and finance and so on. Moreover, it is important since it appears as a limit distribution of limit theorems for random walks (i.e. the sum of independent n random variables at time n). A stochastic process with independent, stationary increments is called a Lévy process and it is distributed as an ID distribution at each time. This stochastic process is important to understand behavior of random phenomena. For example, Brownian motion is a Lévy processes.

In free probability theory, we can also define infinitely divisible distributions and Lévy processes by using free independence and free convolution which is the distribution of sum of freely independent random variables (see [63, 49, 18]). Freely infinitely divisible (for short, FID) distribution is also important since it appears as a limit distribution of limit theorems for "free random walks" (i.e. the sum of freely independent n random variables at time n). Therefore the semicircle law is FID (but this fact can be directly checked by definition). Moreover, free infinite divisibility tells us some distributional information: they have at most one atom and no singular continuous part; the absolutely continuous part has a density function which is real analytic when it is positive (see [18, 7]). The formula of absolutely continuous part was given by [43]. In [29], if $X_1, \cdots, X_n$ are freely independent selfadjoint random variables distributed as symmetric FID distributions and (a_{ij}) is a $n \times n$ Hermitian matrix, then $\sum_{i,j=1}^{n} \alpha_{ij} X_i X_j$ has an FID distribution. The special cases X_1^2 and $i(X_1X_2 - X_2X_1)$ were previously started in [4]. As an application of FID, we know a lot of information (the number of atoms, singular continuous part and absolutely continuous part) on the distributions of $\sum_{i,j=1}^{n} \alpha_{ij} X_i X_j$, X_1^2 and $i(X_1X_2 - X_2X_1)$.

Among the relations between classical and free probability theories, the most remarkable one is the Bercovici-Pata bijection. This bijection plays a role to understand limit theorems in classical and free probability theories (see [15]). For example, the bijection connects the central limit theorem and the free central limit theorem. Furthermore, the Bercovici-Pata bijection helps to construct free Lévy processes, freely selfdecomposable distributions (see [22, 6]) and free subordinators (see [4]) which are important to study free stochastic analysis. In particular, a free Lévy process is of interest in free probability theory. This is a noncommutative stochastic process with freely independent, stationary increments and it is distributed as an FID distribution at each time. Moreover, there are random matrix models which realize free Lévy processes. Roughly speaking, free Lévy processes give a behavior of eigenvalues of large unitary invariant Hermitian matrix-valued processes (see [12]). Therefore, Bercovici-Pata bijection plays a role to understand about ID and FID distributions and Lévy processes and free Lévy processes (i.e. unitary invariant large size random matrices).

Outline of the paper

This paper summarizes results on infinitely divisible distributions in classical and free senses based on [39, 40, 50, 61].

In Section 2, we prepare the basic concepts of free probability theory. Recall the definitions of probability spaces, random variables, independence, (additive/multiplicative) convolution, infinite divisibility and so on. Similarly, we can define these concepts in free probability theory.

In Section 3 - 4, we study a few of properties and examples of freely (additive) infinitely divisible distributions. In particular, the purpose of Section 3 is to give a new class of probability measures, containing many (additive) freely infinitely divisible distributions. The class is called *generalized power distributions with free Poisson term* (for short, GPFP) and it includes free Poisson distributions, free generalized inverse gaussian laws, shifted semicircle laws and beta laws. Moreover, we study powers of random variables which are distributed as freely infinitely divisible GPFP distributions. Motivations come from Bondesson's result: the special class (HCM class) of probability measures are preserved by powers of random variables (see [23]). This section is based on [50].

In Section 4, we study unimodality for distributions of classical (resp. free) Brownian motions with initial distributions, that is, stochastic processes $\{X + B_t\}_{t\geq 0}$ and $\{Y + S_t\}_{t\geq 0}$, where $\{B_t\}_{t\geq 0}$ is the standard Brownian motion, $\{S_t\}_{t\geq 0}$ is the standard free Brownian motion, X is independent of B_t and Y is freely independent of S_t. The distributions of $X + B_t$ and $Y + S_t$ are written by classical (resp. free) convolution with normal (resp. Wigner's semicircle) distribution and probability measure of X (resp. of Y).

In Section 5, we discuss (multiplicative) freely infinitely divisible distributions on the unit circle. Free unitary Brownian motions can be approximated by some unitary matrix valued Brownian motions (see [20]) and the marginal distribution of free unitary Brownian motions is multiplicative FID. In particular, we study the unimodality for distributions of free unitary Brownian motions with inital distribution, that is, the stochastic processes $\{U_tV\}_{t\geq 0}$, where $\{U_t\}_{t\geq 0}$ is free unitary Brownian motions and V is a unitary random variable which is freely independent of U_t. These results are similar to those in Section 4.

In Section 6, we discuss freely max-infinitely divisible distributions, in particular, free extreme values. They are introduced by [11] as distributions of maximum of freely independent random variables with respect to some operator order. These distributions are expected to be related with maximum of eigenvalue distributions of asymptotically freely independent random matrices. In [32], there are more detailed relations between free extreme values and extreme random matrices. Moreover, we explain max-convolution, max-infinitely divisible distributions and extreme values in Boolean case (for details, see [62]). Now, there is no suitable model which realizes Boolean max-notions. However, recently, the author could find a few of relations between additive convolution and max-convolution. The author expects that the relations tell us a suitable model (e.g. random matrix model or operator model) to realize Boolean max-notions. Finally, we give relations between limit theorems for three convolutions (max-classical, max-free and max-Boolean) in the spirit of Bercocivi and Pata's equivalence limit theorems (see [15]).

2 Preliminaries

2.1 Noncommutative probability spaces

We investigate basic concepts of free probability theory. Firstly we give some settings of noncommutative probability theory.

In classical probability theory, we call a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space if Ω is a set, $\mathcal{F}$ is a σ-algebra and $\mathbb{P}$ is a probability measure. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the space $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ of essentially bounded $\mathcal{F}$-measurable functions, equipped with the following linear functional:

$$\mathbb{E}(X) := \int_\Omega X(\omega) d\mathbb{P}(\omega), \qquad X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}). \tag{2.1}$$

The set $\mathcal{A}$ is called a **-algebra* if it is an algebra over $\mathbb{C}$ and is endowed an antilinear operation $\mathcal{A} \ni X \mapsto X^* \in \mathcal{A}$ such that $(X^*)^* = X$ and $(XY)^* = Y^*X^*$ for all $X, Y \in \mathcal{A}$. By setting $X^*(\omega) := \overline{X(\omega)}$ for all $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, the space becomes a *-algebras. Moreover, $\mathbb{E}$ is a state on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, where the linear functional φ is called a *state on a *-algebra* $\mathcal{A}$ if it satisfies that $\varphi(1_\mathcal{A}) = 1$ and $\varphi(X^*X) \geq 0$ for all $X \in \mathcal{A}$, where $1_\mathcal{A}$ is the unit of $\mathcal{A}$.

For $A \in \mathcal{F}$, we have

$$\mathbb{E}(\mathbf{1}_A) = \mathbb{P}(A), \tag{2.2}$$

where $\mathbf{1}_A \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is the characteristic function of $A \in \mathcal{F}$. For $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $f(\omega) = f(\omega)^2 = \overline{f(\omega)}$, $\mathbb{P}$-a.s., we get $f^{-1}(\{1\}) \in \mathcal{F}$ and $f = \mathbf{1}_{f^{-1}(\{1\})}$, $\mathbb{P}$-a.s., and therefore

$$\mathbb{E}(f) = \mathbb{P}(f^{-1}(\{1\})). \tag{2.3}$$

We give settings of noncommutative probability theory as follows.

Definition 2.1. (1) We call a pair $(\mathcal{A}, \varphi)$ a **-probability space* if $\mathcal{A}$ is a unital *-algebra and φ is a state on $\mathcal{A}$. In particular $(\mathcal{A}, \varphi)$ is said to be *tracial* if φ is a trace, that is, $\varphi(XY) = \varphi(YX)$ for all $X, Y \in \mathcal{A}$.

(2) A *-algebra $\mathcal{A}$ is called a *C^*-algebra* if it has a complete norm $\|\cdot\|$ on $\mathcal{A}$, $\|XY\| \leq \|X\|\|Y\|$ and $\|X^*X\| = \|X\|^2$ for all $X, Y \in \mathcal{A}$. The pair $(\mathcal{A}, \varphi)$ is called a *C^*-probability space* if $\mathcal{A}$ is a unital C^*-algebra and φ is a state on $\mathcal{A}$.

(3) The elements in $\mathcal{A}$ are called (non-commutative) *random variables*. In particular,

- X is called *real* if it is selfadjoint, that is, $X = X^*$.

- X is called *unitary* if $X^*X = XX^* = 1_{\mathcal{A}}$.

(4) The *spectrum of* X is defined by $\sigma(X) := \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - X \text{ is not invertible}\}$, where a is said to be *invertible* if there exists $b \in \mathcal{A}$ such that $ab = ba = 1_{\mathcal{A}}$.

For $X \in \mathcal{A}$, a space $\mathbb{C}[X]$ is defined as the algebra of complex polynomials in X.

(5) A *distribution of* $X \in \mathcal{A}$ is given by a linear functional μ_X on $\mathbb{C}[X]$, defined by

$$\mu_X(P) = \varphi(P(X)), \qquad P \in \mathbb{C}[X].$$

In this case, we say that X is *distributed as* μ_X and we write it like $X \sim \mu_X$. In particular, if $(\mathcal{A}, \varphi)$ is a C^*-probability space and X is a real (resp. unitary) random variable, then μ_x extends to a compactly supported probability measure on the real line $\mathbb{R}$ (resp. the unit circle $\mathbb{T}$), that is, there exists a unique probability measure μ_x on $\sigma(X) \subset \mathbb{R}$ (resp. $\subset \mathbb{T}$) such that

$$\varphi(f(X)) = \int_{\mathbb{R}} f(x) d\mu_X(x), \qquad f \in C(\sigma(X)),$$

where the random variable $f(X) \in \mathcal{A}$ is obtained by functional calculus (see [51] for details).

We give a few of examples of *-probability spaces.

Example 2.2. (1) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space. According to the introduction of this section, the pair $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$ is a tracial *-probability space. However, we only deal with genuine random variables that are bounded in this setting. To deal with the most important random variables from usual probability those having a Gaussian random variables, we replace $L^\infty(\Omega, \mathbb{P})$ with the space

$$L^{\infty-}(\Omega, \mathbb{P}) := \bigcap_{1 \le p < \infty} L^p(\Omega, \mathbb{P}). \tag{2.4}$$

For example, the space contains Gaussian random variables and elements in $L^\infty(\Omega, \mathbb{P})$. This is also an abelian unital *-algebra, and therefore the pair $(L^{\infty-}(\Omega, \mathbb{P}), \mathbb{E})$ is also a tracial *-probability space.

(2) Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices. This is a unital *-algebra, where the adjoint A^* of $A = (a_{ij})_{1 \le i,j \le n} \in M_n(\mathbb{C})$ is defined by $A^* := (\overline{a_{ji}})_{1 \le i,j \le n} \in M_n(\mathbb{C})$. The map $\tau_n : M_n(\mathbb{C}) \to \mathbb{C}$ is the normalized trace on $M_n(\mathbb{C})$, that is, for $A = (a_{ij})_{1 \le i,j \le n} \in M_n(\mathbb{C})$,

$$\tau_n(A) := \frac{1}{n} \sum_{i=1}^{n} a_{ii}. \tag{2.5}$$

Then the pair $(M_n(\mathbb{C}), \tau_n)$ is a finite dimensional tracial *-probability space.

(3) Let G be a group with unit $e \in G$. Denote by $\mathbb{C}G$ the group algebra equipped with the following *-operation:

$$\left(\sum_{g \in G} \alpha_g g \right)^* := \sum_{g \in G} \overline{\alpha_g} g^{-1}, \qquad \alpha_g \in \mathbb{C}. \tag{2.6}$$

Moreover we define

$$\tau_G\left(\sum_{g\in G}\alpha_g g\right) := \alpha_e. \tag{2.7}$$

Then τ_G is a trace on $\mathbb{C}G$. The pair $(\mathbb{C}G, \tau_G)$ is a tracial *-probability space.

(4) Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. This is a unital C^*-algebra, where the adjoint X^* of an operator $X \in \mathcal{B}(\mathcal{H})$ is uniquely determined by

$$\langle X\xi, \eta\rangle = \langle \xi, X^*\eta\rangle, \qquad \xi, \eta \in \mathcal{H}. \tag{2.8}$$

Suppose a vector $\xi_0 \in \mathcal{H}$ with $||\xi_0|| := \langle \xi_0, \xi_0\rangle^{1/2} = 1$. We define the linear functional $\varphi_{\xi_0} : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by setting

$$\varphi_{\xi_0}(X) := \langle X\xi_0, \xi_0\rangle, \qquad X \in \mathcal{B}(\mathcal{H}). \tag{2.9}$$

This is a state on $\mathcal{B}(\mathcal{H})$. In particular, φ_{ξ_0} is called a *vector-state* on $\mathcal{B}(\mathcal{H})$. Therefore $(\mathcal{B}(\mathcal{H}), \varphi_{\xi_0})$ is a C^*-probability space.

Moreover, we give one example of distributions of noncommutative random variables.

Example 2.3. (see e.g. [51]) Let $(\mathcal{A}, \varphi)$ be a *-probability space. An element $u \in \mathcal{A}$ is said to be a *Haar unitary* if it is a unitary and $\varphi(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. Then

$$\varphi((u+u^*)^k) = \sum_{j=0}^{k}\binom{k}{j}\varphi(u^j(u^*)^{k-j}) \tag{2.10}$$

$$= \sum_{j=0}^{k}\binom{k}{j}\int_{\mathbb{T}} z^j\overline{z}^{k-j}d\mathbf{h}(z) \text{ (by the definition of a Haar unitary)} \tag{2.11}$$

$$= \int_{\mathbb{T}}(z+\overline{z})^k d\mathbf{h}(z) \tag{2.12}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{i\theta}+e^{-i\theta})d\theta \tag{2.13}$$

$$= \frac{1}{\pi}\int_0^{\pi}(2\cos\theta)^k d\theta \tag{2.14}$$

$$= \frac{1}{\pi}\int_{-2}^{2} t^k\frac{dt}{\sqrt{4-t^2}} \text{ (by the change of variable } 2\cos\theta = t). \tag{2.15}$$

By the Stone-Weierstrass theorem, we can conclude that $\varphi(f(u+u^*)) = \int f(t)\rho(t)dt$ for any $f \in C(\sigma(u+u^*))$. Therefore $u+u^* \sim \rho(t)dt$, where

$$\rho(t) := \frac{1}{\pi\sqrt{4-t^2}}\cdot \mathbf{1}_{(-2,2)}(t) \tag{2.16}$$

is the arcsine density.

In classical probability theory, it is important for us to consider the notion of independence. We know some types of independence in noncommutative probability theory. In particular, we give the definition of free independence as an important role in this paper.

Definition 2.4. Let $(\mathcal{A}, \varphi)$ be a *-probability space and $\{\mathcal{A}_i\}_{i\in I}$ a family of unital *-subalgebras of $\mathcal{A}$. The family is said to be *freely independent* if for any $k \in \mathbb{N}$, $i_1 \neq i_2$, $i_2 \neq i_3$, $\cdots$, $i_{k-1} \neq i_k$ in I and for any $X_s \in \mathcal{A}_{i_s}$ with $\varphi(X_s) = 0$ for $s = 1, \cdots, k$, we have

$$\varphi(X_1 \cdots X_k) = 0 \tag{2.17}$$

holds. In particular, a family $\{X_i\}_{i\in I}$ of random variables in $\mathcal{A}$ is said to be freely independent if the family of unital *-subalgebras generated by X_i are freely independent.

By assuming free independence of random variables, we can calculate the value of the state of mixing random variables.

Example 2.5. Let $X, Y \in \mathcal{A}$ be freely independent random variables. Then

$$\varphi((X - \varphi(X)1_{\mathcal{A}})(Y - \varphi(Y)1_{\mathcal{A}})) = 0. \tag{2.18}$$

Therefore we have $\varphi(XY) = \varphi(X)\varphi(Y)$.

Roughly speaking, we can say that free probability theory is an area based on the noncommutative settings and free independence. In next section, we discuss about the probabilistic tools in free probability theory.

2.2 Free additive convolution

Let $X \sim \mu$ and $Y \sim \nu$ be freely independent real random variables. Then the distribution of $X + Y$ is denoted by $\mu \boxplus \nu$ and the operation $\boxplus$ is called *free additive convolution* (for short, *free convolution*). It was proved in [18] that the distribution $\mu \boxplus \nu$ is well-defined for all probability measures μ, ν on $\mathbb{R}$ (historically, see also [49, 63]). Note that the operation $\boxplus$ is the free analogue of the classical convolution $*$.

We characterize the free additive convolution by analytic tools as follows. Let μ be a probability measure on $\mathbb{R}$. The *Cauchy transform* of μ is defined by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z - x} d\mu(x), \qquad z \in \mathbb{C} \setminus \mathrm{supp}(\mu). \tag{2.19}$$

In particular, if $X \sim \mu$ then we sometimes write G_X as the Cauchy transform of μ. The function G_μ is analytic on the complex upper half plane $\mathbb{C}^+$. Bercovici and Voiculescu ([18, Proposition 5.4]) proved that for any $\gamma > 0$ there exist $\lambda, M, \delta > 0$ such that G_μ is univalent in the truncated cone

$$\Gamma_{\lambda,M} := \{z \in \mathbb{C}^+ : \lambda|\mathrm{Re}(z)| < \mathrm{Im}(z), \mathrm{Im}(z) > M\}, \tag{2.20}$$

and the image $G_\mu(\Gamma_{\lambda,M})$ contains the triangular domain

$$\Lambda_{\gamma,\delta} := \{z \in \mathbb{C}^- : \gamma|\mathrm{Re}(z)| < -\mathrm{Im}(z), \mathrm{Im}(z) > -\delta\}, \tag{2.21}$$

where $\mathbb{C}^-$ is the complex below half plane. Therefore the right inverse function G_μ^{-1} exists on $\Lambda_{\gamma,\delta}$. We define the *Voiculescu transform* of μ by

$$\varphi_\mu(z) := G_\mu^{-1}\left(\frac{1}{z}\right) - z, \qquad \frac{1}{z} \in \Lambda_{\gamma,\delta}. \tag{2.22}$$

It was proved in [18] that for any probability measures μ and ν on $\mathbb{R}$, there exists a unique probability measure λ on $\mathbb{R}$ such that $\varphi_\lambda(z) = \varphi_\mu(z) + \varphi_\nu(z)$ for all z in the intersection of the domains of the three transforms. Then we have $\lambda = \mu \boxplus \nu$. Conversely, we have $\varphi_{\mu\boxplus\nu} = \varphi_\mu + \varphi_\nu$ in the intersection of domains defined three Voiculescu transforms. The *R-transform* of μ is defined by

$$R_\mu(z) := z\varphi_\mu\left(\frac{1}{z}\right), \qquad z \in \Lambda_{\gamma,\delta}. \tag{2.23}$$

In particular, if $X \sim \mu$ then we sometimes write R_X as the R-transform of μ. By the definition, we have that $R_{\mu\boxplus\nu}(z) = R_\mu(z) + R_\nu(z)$ for all z in the intersection of the domains of the three transforms. The transform is the free analogue of $\log\hat{\mu}$.

Next, we give the definition of freely infinitely divisible distributions. A probability measure μ on $\mathbb{R}$ is said to be *freely infinitely divisible* if for each $n \in \mathbb{N}$ there exists a probability measure ρ_n on $\mathbb{R}$ such that $\mu = \rho_n^{\boxplus n}$. Bercovici and Voiculescu ([18, Theorem 5.10]) proved that μ is freely infinitely divisible if and only if the Voiculescu transform φ_μ has an analytic continuation to a map from $\mathbb{C}^+$ taking values in $\mathbb{C}^- \cup \mathbb{R}$, and therefore it has the *Pick-Nevanlinna representation*:

$$\varphi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1 + xz}{z - x} d\sigma(x), \qquad z \in \mathbb{C}^+, \tag{2.24}$$

for some $\gamma \in \mathbb{R}$ and a nonnegative finite measure σ on $\mathbb{R}$. The pair (γ, σ) is uniquely determined and called the *free generating pair*. In this case, we can rewrite the R-transform R_μ as

$$R_\mu(z) = \eta z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - tz} - 1 - tz1_{[-1,1]}(t)\right) d\nu(t), \qquad z \in \mathbb{C}^-, \tag{2.25}$$

where $\eta \in \mathbb{R}$, $a \geq 0$ (called the *semicircular component*) and ν is a nonnegative measure on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge t^2)d\nu(t) < \infty$ (see [6, Proposition 5.2]). The measure ν is called the *free Lévy measure* of μ. The representation (2.25) is called the *free Lévy-Khintchine representation* and it is the free analogue of classical Lévy-Khintchine representation of the Fourier transform $\hat{\mu}$. The triplet (η, a, ν) is uniquely determined by μ and called the *free characteristic triplet*. Moreover the relation between the free generating pair (γ, σ) and the free characteristic triplet (η, a, ν) is given by

$$\begin{aligned} a &= \sigma(\{0\}), \qquad d\nu(t) = \frac{1+t^2}{t^2} \cdot 1_{\mathbb{R}\setminus\{0\}}(t)d\sigma(t), \\ \eta &= \gamma + \int_{\mathbb{R}} t\left(1_{[-1,1]}(t) - \frac{1}{1+t^2}\right) d\nu(t). \end{aligned} \tag{2.26}$$

Example 2.6. (1) The semicircle distribution $S(m,\sigma^2)$ is a probability measure with the PDF:

$$\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-(x-m)^2}1_{(m-2\sigma,m+2\sigma)}(x), \qquad m\in\mathbb{R},\ \ \sigma>0. \tag{2.27}$$

It is known (for example, see [51]) that $R_{S(m,\sigma^2)}(z)=mz+\sigma^2z^2$ for $z\in\mathbb{C}^-$. Therefore $S(m,\sigma^2)$ is freely infinitely divisible and has the free characteristic triplet $(m,\sigma^2,0)$. This distribution appears as the limit of eigenvalue distributions of Wigner matrices as the size of the random matrices goes to infinity.

(2) The free Poisson distribution (or Marchenko-Pastur distribution) $\mathbf{fp}(p,\theta)$ is a probability measure given by

$$\max\{1-p,0\}\delta_0+\frac{\sqrt{\left(\theta(\sqrt{p}+1)^2-x\right)\left(x-\theta(\sqrt{p}-1)^2\right)}}{2\pi x}1_{(\theta(\sqrt{p}-1)^2,\theta(\sqrt{p}+1)^2)}(x)dx \tag{2.28}$$

for $p,\theta>0$. In particular, we write $\mathbf{fp}(p):=\mathbf{fp}(p,1)$. We have that $R_{\mathbf{fp}(p,\theta)}(z)=\frac{p\theta z}{1-\theta z}$ for $z\in\mathbb{C}^-$. Therefore $\mathbf{fp}(p,\theta)$ is freely infinitely divisible. This distribution appears as the limit of eigenvalue distributions of Wishart matrices as the size of the random matrices goes to infinity.

2.3 Free multiplicative convolution

Throughout the paper we identify the unit circle $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$ (often denoted as $[-\pi,\pi)$) and use the notation $\mu(d\theta)$ as well as $\mu(d\xi)$, where $\xi=e^{i\theta}$, for probability measures μ on $\mathbb{T}$. Moreover, we identify the density functions of a Haar absolutely continuous distributions on $\mathbb{T}$ with functions on $\mathbb{R}$ with period 2π. Denote by $\mathbf{p}_{r,\psi}$ the Poisson kernel with the density

$$\frac{d\mathbf{p}_{r,\psi}}{d\theta}(\theta)=\frac{1-r^2}{2\pi|1-re^{i(\theta-\psi)}|^2}, \qquad \theta\in[-\pi,\pi), \tag{2.29}$$

where $r\in[0,1)$ and $\psi\in\mathbb{R}$.

Free multiplicative convolution $\boxtimes$ is a binary operation on the set of probability measures on $\mathbb{T}$. It was introduced in [64] as the distribution of the multiplication of free unitary random variables on a noncommutative probability space. We briefly review on how to compute it, following [17].

For a probability measure μ on $\mathbb{T}$, we define the analytic functions

$$\psi_\mu(z):=\int_{\mathbb{T}}\frac{\xi z}{1-\xi z}\mu(d\xi) \quad\text{and}\quad \eta_\mu(z):=\frac{\psi_\mu(z)}{1+\psi_\mu(z)}, \qquad z\in\mathbb{D}. \tag{2.30}$$

The function η_μ is called the *η-transform of* μ. The probability measure μ can be recovered from ψ_μ via the inversion formula

$$\mu(A)=\lim_{r\uparrow1}\int_A\left(2\Re(\psi_\mu(r\overline{\xi}))+1\right)\mathbf{h}(d\xi) \tag{2.31}$$

for every arc A of $\mathbb{T}$ whose endpoints are continuity points of μ (see [48, Theorem 2]), where $\mathbf{h}$ is the normalized Haar measure on $\mathbb{T}$. There is a short probabilistic proof of it: a direct computation shows that $\int_B (2\Re(\psi_\mu(r\overline{\xi}))+1)\,\mathbf{h}(d\xi) = (\mathbf{p}_{0,r} \circledast \mu)(B)$ for every Borel subset B, and since $\mathbf{p}_{0,r}$ weakly converges to δ_1 as $r \to 1$, $(\mathbf{p}_{0,r} \circledast \mu)(B)$ converges to $\mu(B)$ as long as $\mu(\partial B) = 0$.

It is easy to check that $\eta_\mu(0) = 0$ and $\eta'_\mu(0) = \int_{\mathbb{T}} \xi \, d\mu(\xi)$. If the first moment of μ is nonzero then there exists a function $\eta_\mu^{-1}(z)$ which is analytic in a neighborhood of zero, such that

$$\eta_\mu(\eta_\mu^{-1}(z)) = \eta_\mu^{-1}(\eta_\mu(z)) = z, \tag{2.32}$$

for z in a neighborhood of zero. Then we define the *Σ-transform of μ*

$$\Sigma_\mu(z) := \frac{\eta_\mu^{-1}(z)}{z} \tag{2.33}$$

in the neighborhood of zero where $\eta_\mu^{-1}(z)$ is defined. For any probability measures μ, ν on $\mathbb{T}$ with nonzero first moments, we can find a unique probability measure τ on $\mathbb{T}$ with a nonzero first moment such that

$$\Sigma_\tau(z) = \Sigma_\mu(z)\Sigma_\nu(z), \tag{2.34}$$

for z in a neighborhood of zero where the three functions are defined. We call the probability measure τ the *free multiplicative convolution* of μ and ν, and it is denoted by $\mu \boxtimes \nu$. The Σ-transform is related to the *S-transform* via

$$S_\mu(z) = \Sigma_\mu\left(\frac{z}{1+z}\right) = \frac{1+z}{z}\psi_\mu^{-1}(z). \tag{2.35}$$

For all probability measures μ, ν on $\mathbb{T}$ with nonzero first moments, there exists an analytic map $\eta : \mathbb{D} \to \mathbb{D}$ such that $\eta(0) = 0$ and $\eta_{\mu\boxtimes\nu}(z) = \eta_\mu(\eta(z))$ holds for all $z \in \mathbb{D}$. The function η is called the *subordination function of $\eta_{\mu\boxtimes\nu}$ with respect to η_μ*.

If both μ and ν have the zero first moment, the definition of free independence implies that $\mu \boxtimes \nu = \mathbf{h}$.

A probability measure μ on $\mathbb{T}$ is said to be *$\boxtimes$-infinitely divisible* if for each $n \in \mathbb{N}$ there exists a probability measure μ on $\mathbb{T}$, such that

$$\mu = \overbrace{\mu_n \boxtimes \cdots \boxtimes \mu_n}^{n \text{ times}}. \tag{2.36}$$

In [17, Theorem 6.7], μ is $\boxtimes$-infinitely divisible on $\mathbb{T}$ if and only if there exists a function $u(z)$ which is analytic in $\mathbb{D}$ such that $\Re u(z) \geq 0$ for $z \in \mathbb{D}$ and its Σ-transform is expressed by

$$\Sigma_\mu(z) = \exp[u(z)]. \tag{2.37}$$

Suppose that μ is $\boxtimes$-infinitely divisible on $\mathbb{T}$ and its Σ-transform is expressed by (2.37). Then there exists a $\boxtimes$-semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures on $\mathbb{T}$ such that

$$\Sigma_{\mu_t}(z) = \exp[tu(z)], \qquad t \geq 0, \tag{2.38}$$

in particular $\mu_0 = \delta_1$.

3 Free infinite divisibility for some classes of probability measures

3.1 Introduction

This section is based on [50]. In this section, we give a new subclass of freely infinitely divisible distributions.

In classical probability theory, many people studied infinitely divisible distributions as laws of Lévy processes (e.g. Brownian motions, Poisson processes, Cauchy processes, etc). A probability measure μ on $\mathbb{R}$ is said to be *infinitely divisible* if for each $n \in \mathbb{N}$ there exists a probability measure ρ_n on $\mathbb{R}$ such that $\mu = \rho_n^{*n}$ where the operation $*$ is the classical convolution. We recall that a probability measure μ on $\mathbb{R}$ is infinitely divisible if and only if its characteristic function $\hat{\mu}$ has the Lévy-Khintchine representation:

$$\hat{\mu}(u) = \exp\left(i\eta u - \frac{1}{2}au^2 + \int_{\mathbb{R}} \left(e^{iut} - 1 - iut1_{[-1,1]}(t)\right) d\nu(t)\right), \qquad u \in \mathbb{R}, \tag{3.1}$$

where $\eta \in \mathbb{R}$, $a \geq 0$ (is called the *gaussian component*) and ν is the *Lévy measure* on $\mathbb{R}$, that is, $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge t^2)d\nu(t) < \infty$. The triplet (η, a, ν) is uniquely determined and is called the *characteristic triplet*. For example, the normal distribution $N(m, \sigma^2)$ (with a mean $m \in \mathbb{R}$ and a variance $\sigma > 0$) has the characteristic triplet $(m, \sigma^2, 0)$, (so that it is infinitely divisible). In general, it is very difficult to find the characteristic triplet of probability measures. Many people studied subclasses of probability measures as criteria for infinite divisibility. In particular, these subclasses were studied to understand infinitely divisible distributions which are preserved by powers, products and quotients of independent random variables. We say that the distribution of X belongs to the ME (*mixture of exponential distributions*) class if X is of the form EZ where two random variables E, Z are independent, E follows an exponential distribution and Z is positive. By the Goldie-Steutel theorem, the ME class is a subclass of infinitely divisible distributions (see [31, 56]). The class is important to understand infinitely divisible distributions which is preserved by powers and products of independent random variables, namely, if the distribution of X belongs to the ME class then so is the distribution of X^r when $r \geq 1$, and if X, Y are independent and these distributions belong to the ME class then so is the distribution of XY. As one of the most important subclasses of infinitely divisible distributions, the GGC (*generalized gamma convolution*) class was introduced by Thorin (see [59, 60]). Bondesson (see [23]) proved that the HCM (*hyperbolically completely monotone*) class which is a subclass of the GGC class is preserved by powers (if the distribution of X belongs to the HCM class then so is the distribution of X^r if $|r| \geq 1$), products and quotients of independent random variables. Moreover Bondesson (see [24]) proved that if the distributions of X, Y belong to the GGC class and X, Y are independent then so is the distribution of XY. However, we are not sure about whether the class is preserved by powers of random variables.

In free probability theory, we have the corresponding problems for freely infinitely divisible distributions which was defined in Section 2.3. There are two classes which are important subclasses of freely infinitely divisible distributions, namely, the FR (*free regular*) class and the UI (*univalent inverse Cauchy transform*) class. Firstly, Pérez-Abreu and Sakuma ([52]) introduced the FR class in terms of the Bercovici-Pata bijection. In [4], the FR class was developed as a characterization of nonnegative free Lévy processes. Next, the UI class was introduced by Arizmendi and Hasebe (see [3]) as a subclass of freely infinitely divisible distributions. We will give a precise definition of the UI class in Section 3.3. We are not completely sure about whether the classes FR and UI are closed with respect to powers, products and quotients of free independent random variables. Hasebe ([35]) studied the powers and products for the classes FR and UI by using complex analytic methods.

3.2 Outline of results

We study free infinite divisibility for the *generalized power distributions with free Poisson term* (for short, GPFP distribution). A probability measure on $\mathbb{R}$ is said to be a GPFP distribution if μ is Lebesgue absolutely continuous and its PDF (probability density function) is given by

$$\frac{\sqrt{(b-x)(x-a)}}{x}\sum_{k=1}^{N}\frac{\alpha_k}{x^{l_k}}1_{(a,b)}(x), \tag{3.2}$$

for some $0<a<b$, $N\in\mathbb{N}$, $\alpha=(\alpha_1,\dots,\alpha_N)\in(0,\infty)^N$ satisfying that the function of (3.2) becomes a PDF and $l\in\mathbb{R}^N_<:=\{(l_1,\dots,l_N)\in\mathbb{R}^N : l_1<\cdots<l_N\}$. We write the notation $\mathrm{GPFP}(a,b,N,\alpha,l)$ as the distribution whose PDF is given by (3.2). We prove that if a random variable X follows the GPFP distribution then (i) the distribution of X^r belongs to the UI class if $r\geq 1$ in the case when $l\in[t,t+1]^N_<:=\{(l_1,\dots,l_N)\in\mathbb{R}^N : t\leq l_1<\cdots<l_N\leq t+1\}$ for some $t\geq 0$; (ii) the distribution of X^r belongs to the UI class if $|r|\geq 1$ in the case when $l\in[0,1]^N_<$.

The weak closure of the GPFP class contains the free Poisson distributions which have no atoms at 0, the free positive stable laws with index 1/2, the free Generalized Inverse Gaussian (fGIG) distributions, the shifted semicircle laws and some beta distributions. From the above results (i) and (ii), we obtain free infinite divisibility for powers of random variables which follow the fGIG distributions, the shifted semicircle laws and the beta distributions.

We give examples of GPFP distributions which are not FID. In more detail, we prove that (iii) there is a distribution $\mathrm{GPFP}(a,b,N,\alpha,l)$ with $l\notin[0,1]^N_<$ such that $X\sim\mathrm{GPFP}(a,b,N,\alpha,l)$, but the distribution of X^{-1} is not FID; (iv) there is a distribution $\mathrm{GPFP}(a,b,N,\alpha,l)$ with $l\notin[t,t+1]^N_<$ for any $t\geq 0$ such that it is not FID.

3.3 Univalent Inverse Cauchy transform

Recall that freely infinitely divisible distributions are characterized by R-transform in terms of the free Lévy-Khintchine representation (2.25) (or its Voiculescu transform has the Pick-Nevanlinna representation). However, in general, it is very difficult to find the free characteristic triplet (or free generating pair) of probability measures. Many people studied conditions of free infinite divisibility.

In particular, Arizmendi and Hasebe ([3, Definition 5.1]) defined the UI class as a criterion for free infinite divisibility.

Definition 3.1. A probability measure μ on $\mathbb{R}$ is said to be in the UI (*univalent inverse Cauchy transform*) class if the right inverse function G_μ^{-1}, originally defined in a triangular domain $\Lambda_{\gamma,\delta}$, has univalent analytic continuation to $\mathbb{C}^-$. In particular, if $X \sim \mu$ and $\mu \in$ UI, then we write $X \sim$ UI.

In [3, Proposition 5.2 and p.2763], the UI class has the following property.

Lemma 3.2. *$\mu \in$ UI implies that μ is freely infinitely divisible. The UI class is closed with respect to weak convergence.*

Example 3.3. (1) The semicircle distribution $S(m, \sigma^2)$ is in the UI class. The free Poisson distribution $\mathbf{fp}(p, \theta)$ is also in the UI class (see [35, Section 2.3 (1),(2)]).

(2) The beta distribution $\beta_{p,q}$ is the probability measure with the PDF

$$\frac{1}{B(p,q)} x^{p-1}(1-x)^{q-1} 1_{(0,1)}(x), \qquad p, q > 0. \tag{3.3}$$

The beta distribution $\beta_{p,q}$ is in the UI class if (i) $p, q \geq 3/2$, (ii) $0 < p \leq 1/2$, $q \geq 3/2$ or (iii) $0 < q \leq 1/2$, $p \geq 3/2$ (see [34, Theorem 1.2] and [35, Theorem 3.4]).

A symbol $\tilde{G}_\mu(z)$ denotes the Cauchy transform of a probability measure μ on $\mathbb{R}$ defined in (2.19). In [35, Lemma 3.11], Hasebe mentioned that the Cauchy transform of μ has an analytic continuation to a domain $\mathbb{C}^+ \cup \mathrm{supp}(\mu) \cup \mathbb{C}^-$ in the case when a PDF of μ has some analytic properties. We give an assertion almost the same as [35, Lemma 3.11] as follow.

Lemma 3.4. *Consider $0 < a < b$. Let σ be a probability measure on (a, b) whose PDF is $f(x)$ with respect to Lebesgue measure. Assume that f is analytic in a neighborhood of $(a, b) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ for some $0 < \theta \leq \pi$. Then the Cauchy transform $G_\sigma := \tilde{G}_\sigma|_{\mathbb{C}^+}$ defined on $\mathbb{C}^+$ has an analytic continuation to $\mathbb{C}^+ \cup (a, b) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ (we use the same symbol G_σ), and*

$$G_\sigma(z) = \tilde{G}_\sigma(z) - 2\pi i f(z), \qquad z \in \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}. \tag{3.4}$$

Proof. This proof is very similar to [34, Proposition 4.1], but for readers convenience we include the proof. Since the function f is analytic in a neighborhood of $(a, b) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ for some $0 < \theta \leq \pi$, we have

$$G_\sigma(z) = \int_\gamma \frac{f(w)}{z - w} dw, \qquad z \in \mathbb{C}^+, \tag{3.5}$$

where γ is an arbitrary simple arc contained in $\{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ except its endpoints a, b. Therefore G_σ has an analytic continuation to the domain containing $\mathbb{C}^+$, surrounded by γ and (a, b). Since γ is arbitrary, G_σ has an analytic continuation to the domain $\mathbb{C}^+ \cup (a, b) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$. We prove that the formula (3.4) holds. For $z \in \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$, take a simple arc γ contained in $\{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ with endpoints a, b such that the closed curve $\tilde{\gamma} := \gamma \cup [a, b]$ surrounds the element z. By the residue theorem, we have that

$$\int_{\tilde{\gamma}} \frac{f(w)}{z - w} dw = 2\pi i \mathrm{Res}_{w=z} \left(\frac{f(w)}{z - w} \right) = -2\pi i f(z). \tag{3.6}$$

Since the LHS of (3.6) equals to $G_\sigma(z) - \tilde{G}_\sigma(z)$, the formula (3.4) holds for all $z \in \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$. □

We define the following symbols:

$$x \pm i0 := \lim_{y \to 0\pm} (x + iy), \qquad f(x \pm i0) = \lim_{y \to 0\pm} f(x + iy), \tag{3.7}$$

$$I \pm i0 := \{x \pm i0 : x \in I\} \quad (I : \text{ an interval in } \mathbb{R}) \tag{3.8}$$

Moreover, note that the complex function z^p ($p \in \mathbb{R}$) means the principal value defined on $\mathbb{C} \setminus (-\infty, 0]$. The complex function $\arg(z)$ means the argument of z defined on $\mathbb{C} \setminus (-\infty, 0]$ taking values in $(-\pi, \pi)$.

We prove that a probability measure whose PDF satisfies the following properties is in the UI class and therefore the measure is freely infinitely divisible.

Theorem 3.5. *Suppose that $0 < a < b$. Let μ be a probability measure on (a, b) which has a PDF $f(x)$ where f is real analytic and positive on (a, b) and there exists $\theta \in (0, \pi]$ such that*
(A1) *f analytically extends to a function (is denoted by f) defined in $\{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in (-\theta, 0)\} \cup (a, b)$. Moreover f extends to a continuous function on $\{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [-\theta, 0]\}$;*
(A2) *$Re(f(x - i0)) = 0$ for all $0 < x < a$ and $x > b$;*
(A3) *There exist $\alpha > 0$ and $0 < l < \frac{2\pi - \theta}{\theta}$ such that*

$$f(z) = -\frac{i\alpha}{z^{1+l}}(1 + o(1)), \qquad \text{as } z \to 0, \;\; \arg(z) \in (-\theta, 0); \tag{3.9}$$

(A4) *$Re(f(ue^{-i\theta})) \leq 0$ for all $u > 0$;*
(A5) $\lim_{|z| \to \infty, \arg(z) \in (-\theta, 0)} f(z) = 0$.
(A6) *$G(z) := G_\mu(z)$ has analytic continuation to $\mathbb{C}^+ \cup (a, b) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ (and we use the same symbol G), and $G(z)$ can be extended to a continuous function on $\mathbb{C}^+ \cup [a, b] \cup \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [-\theta, 0)\} \cup ((-\infty, a] \cup [b, \infty) + i0) \cup ([0, a] \cup [b, \infty) - i0)$.*
Then $\mu \in UI$.

Remark 3.6. *Before the proof the authors can make a remark that the existence of analytic continuation of G follows from the above assumption (A1) and Lemma 3.4, so the assumption (A6) is only about the continuous extension.*

Proof. We prove Theorem 3.5. We define the following 8 lines and curves (see Figure 1). In the following $\eta > 0$ is supposed to be large and $\delta > 0$ is supposed to be small.
c_1 : the real line segment from $-\eta + i0$ to $a + i0$;
c_2 : the real line segment from $a - i0$ to $\delta - i0$;
c_3 : the clockwise circle $\delta e^{i\psi}$ where ψ starts from 0 and ends with $-\theta$;
c_4 : the line segment from $\delta e^{-i\theta} + 0$ to $\eta e^{-i\theta} + 0$ if $\theta \in (0, \pi)$ or the line segment from $-\delta - i0$ to $-\eta - i0$ if $\theta = \pi$;
c_5 : the counterclockwise circle $\eta e^{i\psi}$ where ψ starts from $-\theta$ and ends with 0;
c_6 : the real line segment from $\eta - i0$ to $b - i0$;
c_7 : the real line segment form $b + i0$ to $\eta + i0$;
c_8 : the counterclockwise circle $\eta e^{i\psi}$ where ψ starts from 0 and ends with π.

Define the curves $g_k := G(c_k)$ for all $k = 1, \ldots, 8$ (see Figure 2). It is easy to prove that g_1 is contained in the negative real line and g_7 is contained in the positive

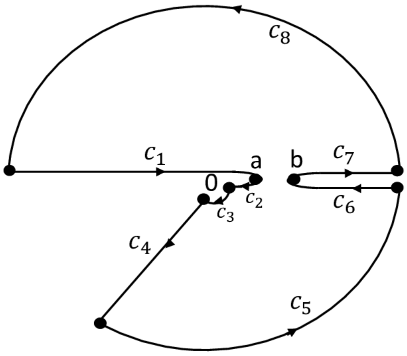

Figure 1: The curves c_k

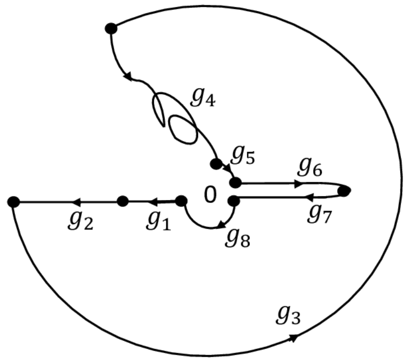

Figure 2: The curves g_k

real line. Since we have

$$G(z) = \tilde{G}(z) = \frac{1}{z}(1 + o(1)), \qquad \text{as } |z| \to \infty, \;\; z \in \mathbb{C}^+, \tag{3.10}$$

the image g_8 is contained in a ball of some small radius and it is the clockwise curve whose argument starts from 0 and ends with $-\pi$ when η is sufficiently large. By our assumption (A6), the curves $g_1 \cup g_2$ and $g_6 \cup g_7$ are continuous since so are $c_1 \cup c_2$ and $c_6 \cup c_7$. By our assumption (A2), we have that $\mathrm{Re}(f(x - i0)) = 0$ for all $0 < x < a$ and $x > b$. Hence $\mathrm{Im}(G(x - i0)) = 0$ for all $0 < x < a$, $x > b$, and therefore g_2 and g_6 are contained in the real line. By our assumptions (A1) and (A4), we have that when $\theta \in (0, \pi)$,

$$\mathrm{Im}(G(ue^{-i\theta} + 0)) = \mathrm{Im}(\tilde{G}(ue^{-i\theta})) - 2\pi \mathrm{Re}(f(ue^{-i\theta})) > 0, \tag{3.11}$$

for all $\delta < u < \eta$ and when $\theta = \pi$,

$$\mathrm{Im}(G(-u - i0)) = \mathrm{Im}(\tilde{G}(-u)) - 2\pi \mathrm{Re}(f(-u)) = -2\pi \mathrm{Re}(f(-u)) \geq 0, \tag{3.12}$$

for all $\delta < u < \eta$. Therefore g_4 is contained in $\mathbb{C}^+ \cup \mathbb{R}$. Note that $\tilde{G}(z) = o(1/z)$ as $z \to 0$, $z \in \mathbb{C}^-$ since $\lim_{z \to 0, z \in \mathbb{C}^-} |\tilde{G}(z)| = \int_a^b \frac{f(t)}{t} dt < \infty$. Since our assumption (A3) and the condition (3.4) hold, we have that

$$G(z) = -\frac{2\pi\alpha}{z^{1+l}}(1 + \varphi(z)), \tag{3.13}$$

where $\varphi(z) = o(1)$ as $z \to 0$, $\arg(z) \in (-\theta, 0)$.

Let $\varepsilon > 0$ be supposed to be small. By the asymptotics of (3.13), we can take $\delta \in (0, (\pi\alpha\varepsilon)^{\frac{1}{1+l}})$ small enough such that $|\varphi(\delta e^{i\psi})| < 1/2$ uniformly on $\psi \in (-\theta, 0)$, and therefore

$$|G(\delta e^{i\psi}) + 2\pi\alpha(\delta e^{i\psi})^{-(1+l)}| = 2\pi\alpha\delta^{-(1+l)}|\varphi(\delta e^{i\psi})| < \pi\alpha\delta^{-(1+l)}, \tag{3.14}$$

$$\tag{3.15}$$

uniformly on $\psi \in (-\theta, 0)$. Hence by the triangle inequality we have

$$|G(\delta e^{i\psi})| \geq |-2\pi\alpha(\delta e^{i\psi})^{-(1+l)}| - |G(\delta e^{i\psi}) + 2\pi\alpha(\delta e^{i\psi})^{-(1+l)}| \tag{3.16}$$

$$> 2\pi\alpha\delta^{-(1+l)} - \pi\alpha\delta^{-(1+l)} = \pi\alpha\delta^{-(1+l)} > \frac{1}{\varepsilon}, \tag{3.17}$$

and therefore the distance between the curve g_3 and 0 is larger than $\frac{1}{\varepsilon}$. Since we have that $G(\delta e^{i\psi}) = \frac{2\pi\alpha}{\delta^{1+l}} e^{(-\pi-\psi(1+l))i}$ as $\delta \to 0$ and the argument ψ starts from 0 and ends with $-\theta$, the image g_3 is the large counterclockwise curve whose radius is larger than $\frac{1}{\varepsilon}$ and the argument of g_3 starts from $-\pi$ and ends with $-\pi + \theta(1+l)$ ($< \pi$ by (A3)) when δ is sufficiently small.

Note that $\tilde{G}(z) = \frac{1}{z}(1+o(1))$ as $|z| \to \infty$, $z \in \mathbb{C}^-$. By our assumption (A5), we have that $|f(z)| \to 0$ as $|z| \to \infty$, $\arg(z) \in (-\theta, 0)$, and therefore

$$|G(z)| \leq |\tilde{G}(z)| + 2\pi|f(z)| \to 0, \tag{3.18}$$

as $|z| \to \infty$, $\arg(z) \in (-\theta, 0)$. Hence there exists $\eta > 0$ large enough such that $|z| \geq \eta$ implies that $|G(z)| < \varepsilon$, that is, g_5 is contained in a small circle with radius $\varepsilon > 0$.

Therefore every point of $D_\varepsilon := \{z \in \mathbb{C}^- : \varepsilon < |z| < \frac{1}{\varepsilon}\}$ is surrounded by the closed curve $g_1 \cup \cdots \cup g_8$ exactly once. By the argument principle, for any $w \in D_\varepsilon$, there exists only one element z in the bounded domain surrounded by the closed curve $c_1 \cup \cdots \cup c_8$, such that $G(z) = w$. Therefore we can define a right inverse function G^{-1} on D_ε. Since $G^{-1}(D_\varepsilon)$ is a connected domain and G is univalent on the domain $G^{-1}(D_\varepsilon)$, the inverse function G^{-1} is univalent on D_ε. We can define a univalent right inverse function (as the same symbol G^{-1}) in $\mathbb{C}^-$ by letting $\varepsilon \to 0$. By the identity theorem, the right inverse function, originally defined in some triangular domain $\Lambda_{\gamma,\delta}$ has a univalent analytic continuation to our function G^{-1} on $\mathbb{C}^-$. Hence we can conclude that $\mu \in \mathrm{UI}$. $\square$

Remark 3.7. *According to Figure 2, it appears that the curves g_1 and g_2 (resp. g_6 and g_7) have empty intersections, but it is not necessarily right. However it does not interfere with the proof of Theorem 3.5 since every point of D_ε is surrounded by the closed curve $g_1 \cup \cdots \cup g_8$ exactly once even if $g_1 \cap g_2 \neq \emptyset$ (resp. $g_6 \cap g_7 \neq \emptyset$).*

3.4 Free infinite divisibility for generalized power distributions with free Poisson term

Consider a random variable X which follows the free Poisson distribution $\mathbf{fp}(p)$ (see Example 2.6 (2)). In [35, Theorem 3.5], if either $p \geq 1$ and $r \in (-\infty, 0] \cup [1, \infty)$ or $0 < p < 1$ and $r \geq 1$, then the distribution of X^r is a FID (UI and FR) distribution. In this section, we study free infinite divisibility for the distribution of X^r when X follows the generalized power distribution with free Poisson term (GPFP distribution). We give a precise definition of the GPFP distributions. Recall that

$$\mathbb{R}^N_< := \{(l_1, \ldots, l_N) \in \mathbb{R}^N : l_1 < \cdots < l_N\} \ \ (N \in \mathbb{N}). \tag{3.19}$$

Definition 3.8. A probability measure μ on $\mathbb{R}$ is said to be the *generalized power distribution with free Poisson term* (for short, *GPFP distribution*) if μ is Lebesgue absolutely continuous and its PDF is given by

$$f_{a,b,N,\alpha,l}(x) := \frac{\sqrt{(b-x)(x-a)}}{x} \sum_{k=1}^{N} \frac{\alpha_k}{x^{l_k}} 1_{(a,b)}(x), \tag{3.20}$$

for some $0 < a < b$, $N \in \mathbb{N}$, $\alpha = (\alpha_1, \ldots, \alpha_N) \in (0, \infty)^N$ with $\int_a^b f_{a,b,N,\alpha,l}(x)dx = 1$ and $l = (l_1, \ldots, l_N) \in \mathbb{R}^N_<$. In this case, the probability measure μ is referred to as the notation $\mathrm{GPFP}(a, b, N, \alpha, l)$.

Remark 3.9. *The GPFP class is closed under talking inverse of random variables, namely, if $X \sim$ GPFP(a, b, N, α, l) with $\alpha = (\alpha_1, \dots, \alpha_N) \in (0,\infty)^N$ and $l = (l_1, \dots, l_N) \in \mathbb{R}^N_<$, then*

$$X^{-1} \sim GPFP\left(\frac{1}{b}, \frac{1}{a}, N, (\alpha_N\sqrt{ab}, \dots, \alpha_1\sqrt{ab}), (1-l_N, \dots, 1-l_1)\right). \tag{3.21}$$

The class of these distributions (namely, the GPFP class) contains the free Poisson distributions and the free Generalized Inverse Gaussian (fGIG) distributions. Moreover the weak closure of the GPFP class contains a lot of examples. We give these examples before the main result (Theorem 3.11).

Example 3.10. (1) Suppose that $p > 1$, $a = (\sqrt{p}-1)^2$, $b = (\sqrt{p}+1)^2$, $N = 1$, $\alpha = \frac{1}{2\pi}$ and $l = 0$. Then we have

$$\begin{aligned} &\mathrm{GPFP}\left(a = (\sqrt{p}-1)^2, b = (\sqrt{p}+1)^2, N = 1, \alpha = \frac{1}{2\pi}, l = 0\right) \\ &= \frac{\sqrt{\left((\sqrt{p}+1)^2 - x\right)\left(x - (\sqrt{p}-1)^2\right)}}{2\pi x} \cdot 1_{((\sqrt{p}-1)^2, (\sqrt{p}+1)^2)}(x)dx \\ &= \mathbf{fp}(p). \end{aligned} \tag{3.22}$$

In this case, the GPFP distribution corresponds to the free Poisson distribution (Marchenko-Pastur distribution) which has no atom at 0.

(2) Consider $\lambda \in \mathbb{R}$ and $\alpha_1, \alpha_2 > 0$. Suppose that $N = 2$, $l = (0,1)$ and $\alpha = \left(\frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi\sqrt{ab}}\right)$ where $0 < a < b$ are the unique solution of

$$\begin{cases} 1 - \lambda + \alpha_1\sqrt{ab} - \alpha_2\frac{a+b}{2ab} = 0 \\ 1 + \lambda + \frac{\alpha_2}{\sqrt{ab}} - \alpha_1\frac{a+b}{2} = 0. \end{cases} \tag{3.23}$$

Then we have

$$\begin{aligned} &\mathrm{GPFP}\left(a, b, N = 2, \alpha = \left(\frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi\sqrt{ab}}\right), l = (0,1)\right) \\ &= \frac{\sqrt{(x-a)(b-x)}}{2\pi x}\left(\alpha_1 + \frac{\alpha_2}{\sqrt{abx}}\right) \cdot 1_{(a,b)}(x)dx =: \mathrm{fGIG}(\alpha_1, \alpha_2, \lambda). \end{aligned} \tag{3.24}$$

In this case, the GPFP distribution corresponds to the fGIG distribution. This distribution was studied by [37] (the free version of the generalized inverse gaussian distributions, free selfdecomposability, free regularity and unimodality for the fGIG distributions) and [57] (the free version of Matsumoto-Yor property, the R-transform of the fGIG distributions).

Next we give examples of distributions contained in the weak closure of the GPFP class.
(3) Consider $n \in \mathbb{N}$, $b > 0$. Define the following constant number:

$$c(n,b) := \left(\int_{\frac{1}{n}}^{b} \frac{\sqrt{(x-1/n)(b-x)}}{2\pi x}dx\right)^{-1}. \tag{3.25}$$

Note that $\lim_{n\to\infty} c(n,b) = \frac{4}{b}$ for each $b > 0$. We define the following GPFP distribution:

$$\begin{aligned}\mu_{n,b}(dx) :&= \mathrm{GPFP}\left(\frac{1}{b},\, n,\, N=1,\, \alpha = \frac{c(n,b)}{2\pi}\sqrt{\frac{b}{n}},\, l=1\right) \\ &= c(n,b)\sqrt{\frac{b}{n}} \cdot \frac{\sqrt{(x-1/b)\,(n-x)}}{2\pi x^2} \cdot 1_{(1/b,n)}(x)dx.\end{aligned} \tag{3.26}$$

We can show that $\mu_{n,b} \to \mu_b$ weakly as $n \to \infty$ for each $b > 0$, where

$$\mu_b(dx) := \frac{4}{b} \cdot \frac{\sqrt{bx-1}}{2\pi x^2} \cdot 1_{(1/b,\infty)}(x)dx. \tag{3.27}$$

In particular, if $b = 4$, then the probability measure μ_b corresponds to the free positive stable law with index $1/2$. The free stable laws were introduced by [15].

(4) Consider $n \in \mathbb{N}$, $b > 0$. We define the following GPFP distribution:

$$\nu_{n,b}(dx) := \mathrm{GPFP}\left(\frac{1}{n}, b, N=1, \alpha = \frac{c(n,b)}{2\pi}, l=0\right), \tag{3.28}$$

where $c(n,b)$ is the constant defined in (3.25). We can show that $\nu_{n,b} \to \nu_b$ weakly as $n \to \infty$ for each $b > 0$, where

$$\nu_b(dx) := \frac{2\sqrt{x(b-x)}}{\pi b x} 1_{(0,b)}(x)dx. \tag{3.29}$$

In particular, if $b = 4$, then the probability measure ν_b corresponds to the free Poisson distribution $\mathbf{fp}(1)$. Note that $X \sim \mu_{n,b}$ if and only if $X^{-1} \sim \nu_{n,b}$ for each $n \in \mathbb{N}$ and $b > 0$ by Remark 3.9.

(5) Consider $n \in \mathbb{N}$ and $l < 1/2$. Define the following constant number:

$$\alpha(n,l) := \left(\frac{1}{B\left(\frac{1}{2}-l, \frac{3}{2}\right)} \int_{\frac{1}{n}}^{1} \frac{\sqrt{(1-x)(x-1/n)}}{x^{1+l}} dx\right)^{-1}, \tag{3.30}$$

where $B(p,q)$ is the Beta distribution. Note that $\lim_{n\to\infty} \alpha(n,l) = 1$ for each $l < 1/2$. For all $n \in \mathbb{N}$ and $l < 1/2$, we consider the following GPFP distribution:

$$\rho_{n,l}(dx) := \mathrm{GPFP}\left(\frac{1}{n},\, 1,\, N=1,\, \frac{\alpha(n,l)}{B(1/2-l, 3/2)},\, l\right). \tag{3.31}$$

We can show that $\rho_{n,l} \to \beta_{1/2-l,3/2}$ weakly as $n \to \infty$ for each $l < 1/2$. Recall that $\beta_{1/2-l,3/2}$ is the beta distribution and it was defined in Example 3.3 (2).

We start to discuss free infinite divisibility for the distribution of X^r where X follows the GPFP distribution. Recall that

$$[t, t+1]_<^N := \{(l_1, \dots, l_N) \in \mathbb{R}^N : t \le l_1 < \cdots < l_N \le t+1\} \ \ (n \in \mathbb{N}, t \in \mathbb{R}). \tag{3.32}$$

Theorem 3.11. *Suppose that $X \sim$ GPFP(a,b,N,α,l) with $l=(l_1,\dots,l_N)\in[t,t+1]^N_<$ for some $t\ge 0$. For $r\ge 1$, $X^r\sim$ UI.*

Proof. Fix a number $N\in\mathbb{N}$. Consider first that $r>1$ and $l=(l_1,\dots,l_N)\in[t,t+1]^N_<$ satisfies $t<l_1<\cdots<l_N<t+1$ for some $t\ge 0$. Assume that $[t]=n-1$ for some $n\in\mathbb{N}$. Define $\tilde{l}_k:=l_k-[t]=l_k-(n-1)\in(0,1)$ for all $k=1,\dots,N$. Then we have

$$X^r \sim \frac{s\sqrt{(B^s-x^s)(x^s-A^s)}}{x}\sum_{k=1}^N \frac{\alpha_k}{x^{s(n-1+\tilde{l}_k)}}\cdot 1_{(A,B)}(x)dx =: h(x)dx, \tag{3.33}$$

where $s=\frac{1}{r}\in(0,1)$, $A=a^r$ and $B=b^r$. We define a function $h_k(x)$ as the k-th term of $h(x)$. Set $n_0:=s(n-1)+1\ge 1$. Note that $s(n-1+\tilde{l}_k)=n_0-1+\tilde{l}_k s$ for each $k=1,\dots,N$. Consider $\theta=\frac{\pi}{n_0}$. Let $G=G_{X^r}$ be the Cauchy transform of X^r. We prove that θ, h and G satisfy the assumptions from (A1) to (A6) in Theorem 3.5.

(A1) Suppose that $z\in\mathbb{C}\backslash\{0\}$ and $\arg(z)\in(-\theta,0)$. One can use similar argument as in the proof of [35, Theorem 3.5] to show that $\arg((B^s-z^s)(z^s-A^s))\in(-\pi,\pi)$. Moreover we have that $\arg(z^{s(n-1+\tilde{l}_k)})\in(-\theta s(n-1+\tilde{l}_k),0)\subset(-\pi,0)$. Thus every h_k analytically extends to a function (is denoted by the same symbol h_k) defined in $\{z\in\mathbb{C}\setminus\{0\}:\arg(z)\in(-\theta,0)\}\cup(A,B)$, so that the assumption (A1) holds.

(A2) For $0<x<A$ we have

$$h_k(x-i0) = -\frac{s\alpha_k\sqrt{(A^s-x^s)(B^s-x^s)}}{x^{1+s(n-1+\tilde{l}_k)}}i. \tag{3.34}$$

For $x>B$ we have

$$h_k(x-i0) = \frac{s\alpha_k\sqrt{(x^s-A^s)(x^s-B^s)}}{x^{1+s(n-1+\tilde{l}_k)}}i. \tag{3.35}$$

Therefore $\mathrm{Re}(h_k(x-i0))=0$ for all $0<x<A$ and $x>B$, so that the assumption (A2) holds.

(A3) For each $k=1,\cdots,N$, we have

$$h_k(z) = -\frac{is\alpha_k\sqrt{(AB)^s}}{z^{1+s(n-1+\tilde{l}_k)}}(1+o(1)), \qquad \text{as } z\to 0,\ \arg(z)\in(-\theta,0)\,. \tag{3.36}$$

Moreover we have that $0<s(n-1+\tilde{l}_k)<\frac{2\pi-\theta}{\theta}$. Therefore the assumption (A3) holds.

(A4) Consider $0<\theta<\pi$ (equivalently, $n>1$). For all $k=1,\dots,N$, $u>0$ we have

$$\begin{aligned}\arg(h_k(ue^{-i\theta})) &= \arg\left(\frac{s\alpha_k\sqrt{(B^s-u^s e^{-is\theta})(u^s e^{-is\theta}-A^s)}}{u^{1+s(n-1+\tilde{l}_k)}e^{-i\theta(1+s(n-1+\tilde{l}_k))}}\right)\\ &\in\left(-\frac{\pi}{2}+\theta(1+s(n-1+\tilde{l}_k)),\frac{\pi-2s\theta}{2}+\theta(1+s(n-1+\tilde{l}_k))\right)\\ &=\left(-\frac{\pi}{2}+\theta(n_0+s\tilde{l}_k),\frac{\pi-2s\theta}{2}+\theta(n_0+s\tilde{l}_k)\right)\\ &=\left(\frac{\pi}{2}+\theta s\tilde{l}_k,\frac{3}{2}\pi-\theta s(1-\tilde{l}_k)\right)\subset\left(\frac{\pi}{2},\frac{3}{2}\pi\right),\end{aligned} \tag{3.37}$$

since we have

$$\arg\left(\sqrt{(B^s - u^s e^{-is\theta})(u^s e^{-is\theta} - A^s)}\right) \in \left(-\frac{\pi}{2}, \frac{\pi - 2s\theta}{2}\right). \tag{3.38}$$

Therefore $\mathrm{Re}(h_k(ue^{-i\theta})) \leq 0$ for all $u > 0$. When $\theta = \pi$ (equivalently, $n = 1$), for all $k = 1, \ldots, N$, $u < 0$ we have that

$$\begin{aligned}
\arg(h_k(u)) &= \arg\left(\frac{s\alpha_k\sqrt{(B^s - (u-i0)^s)((u-i0)^s - A^s)}}{u^{1+s(n-1+\tilde{l}_k)}}\right) \\
&= \arg\left(\frac{s\alpha_k\sqrt{(B^s - (u-i0)^s)((u-i0)^s - A^s)}}{|u|^{1+s\tilde{l}_k}e^{-i\pi(1+s\tilde{l}_k)}}\right) \\
&\in \left(-\frac{\pi}{2} + \pi(1+s\tilde{l}_k), \frac{1-2s}{2}\pi + \pi(1+s\tilde{l}_k)\right) \\
&= \left(\frac{\pi}{2} + \pi s\tilde{l}_k, \frac{3}{2}\pi - \pi s(1-\tilde{l}_k)\right) \subset \left(\frac{\pi}{2}, \frac{3}{2}\pi\right),
\end{aligned} \tag{3.39}$$

since

$$\arg\left(\sqrt{(B^s - (u-i0)^s)((u-i0)^s - A^s)}\right) \in \left(-\frac{\pi}{2}, \frac{1-2s}{2}\pi\right). \tag{3.40}$$

Therefore $\mathrm{Re}(h_k(u)) \leq 0$ for all $u < 0$. Hence we can conclude that the assumption (A4) holds.

(A5) We have

$$h_k(z) = \frac{-is\alpha_k}{z^{1+s(n-2+\tilde{l}_k)}}(1 + o(1)), \qquad \text{as } |z| \to \infty,\ \arg(z) \in (-\theta, 0). \tag{3.41}$$

Hence $\lim_{|z|\to\infty, \arg(z)\in(-\theta,0)} h_k(z) = 0$, so that the assumption (A5) holds.

Since $h(x) = \sum_{k=1}^N h_k(x)$ for all $x \in (A, B)$ and every function h_k satisfies the assumption (A1), the function h also satisfies the assumption (A1) and $h(z) = \sum_{k=1}^N h_k(z)$ for all $z \in \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\} \cup (A, B)$. Finally, we can conclude that the function h satisfies from (A1) to (A5) since so is every function h_k, $(k = 1, \ldots, N)$.

(A6) Since h satisfies the assumption (A1), by applying Lemma 3.4, the Cauchy transform G has an analytic continuation to $\mathbb{C}^+ \cup (A, B) \cup \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$ (and we use the same symbol G) and

$$G(z) = \int_A^B \frac{h(x)}{z - x}dx - 2\pi i h(z) = \sum_{k=1}^N \left(\int_A^B \frac{h_k(x)}{z-x}dx - 2\pi i h_k(z)\right), \tag{3.42}$$

for $z \in \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$. We use the notation $\tilde{G}_k(z)$ as the function $\int_A^B \frac{h_k(x)}{z-x}dx$. To prove that G can be extended to a continuous function on the domain $\mathbb{C}^+ \cup [A, B] \cup \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [-\theta, 0)\} \cup ((-\infty, A] \cup [B, \infty) + i0) \cup ([0, A] \cup [B, \infty) - i0)$, we compute an asymptotic behavior of G at $z = A$ and $z = B$. We define a function $G_k(z)$ as follow:

$$G_k(z) := \tilde{G}_k(z) - 2\pi i h_k(z), \tag{3.43}$$

for $z \in \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$. By the Taylor expansion at $x = A$, there exist $c_k > 0$ and a real analytic function $\psi_k(x)$ on a neighborhood of A which satisfies $\psi_k(x) = o(1)$ as $x \to A + 0$ and has an analytic continuation to $\{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\} \cup (A, B)$ (is denoted by the same symbol ψ_k) such that

$$h_k(x) = c_k(x - A)^{1/2}(1 + \psi_k(x)), \qquad \text{as } x \to A + 0. \tag{3.44}$$

Therefore we have

$$G_k(z) = \beta_k + \gamma_k(-(z - A))^{1/2} + o(|z - A|^{1/2}), \tag{3.45}$$

as $|z-A| \to 0$ with $z \in \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$, where $\beta_k = \lim_{z \to A, z \in \mathbb{C}^-} \tilde{G}_k(z) < 0$ (is finite) and $\gamma_k = 2\pi c_k$. Finally, if we put $\beta := \sum_{k=1}^N \beta_k < 0$ and $\gamma = \sum_{k=1}^N \gamma_k > 0$, then we have

$$G(z) = \sum_{k=1}^N G_k(z) = \beta + \gamma(-(z - A))^{1/2} + o(|z - A|^{1/2}), \tag{3.46}$$

as $|z - A| \to 0$ with $z \in \{z \in \mathbb{C}^- : \arg(z) \in (-\theta, 0)\}$. Moreover $G(A + i0) = \beta = G(A - i0)$. Similarly, we can compute an asymptotic behavior of G at $z = B$. Hence G can be extended to a continuous function on $\mathbb{C}^+ \cup [A, B] \cup \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [-\theta, 0)\} \cup ((-\infty, A] \cup [B, \infty) + i0) \cup ([0, A] \cup [B, \infty) - i0)$. Thus G satisfies the assumption (A6).

By Theorem 3.5, we have $X^r \sim$ UI. By the weak closedness of the UI class (see Lemma 3.2), we have that $X^r \sim$ UI even if $r \geq 1$ and $t \leq l_1 < \cdots < l_N \leq t + 1$. $\square$

Corollary 3.12. *If $X \sim$ GPFP(a, b, N, α, l) and $l \in [0, 1]^N_<$ then $X^r \sim$ UI when $r \leq -1$.*

Proof. Suppose that $r \leq -1$. We have

$$X^r \sim \frac{t\sqrt{(AB)^{-t}}\sqrt{(B^t - x^t)(x^t - A^t)}}{x} \sum_{k=1}^N \frac{\alpha_k}{x^{t(1-l_k)}} \cdot 1_{(A,B)}(x)dx := p(x)dx, \tag{3.47}$$

where $t := -\frac{1}{r} \in (0, 1)$, $A := b^{-1/t}$ and $B := a^{-1/t}$. We set $\theta = \pi$ and a function $p_k(x)$ as the k-th term of $p(x)$. One can use similar argument as in the proof of Theorem 3.11 to show that θ, p_k, p, and $G = G_{X^r}$ satisfy the assumptions from (A1) to (A6) in Theorem 3.5. Therefore $X^r \sim$ UI if $r \leq -1$. $\square$

Finally, if $X \sim$ GPFP(a, b, N, α, l) with $l \in [0, 1]^N_<$ then $X^r \sim$ UI for all $|r| \geq 1$ by applying Theorem 3.11 and Corollary 3.12.

To end this section, we give applications of Theorem 3.11 and Corollary 3.12.

Example 3.13. (1) Consider $p > 1$. If we have

$$X \sim \mathbf{fp}(p) = \text{GPFP}\left(a = (\sqrt{p} - 1)^2,\, b = (\sqrt{p} + 1)^2,\, N = 1,\, \alpha = \frac{1}{2\pi},\, l = 0\right), \tag{3.48}$$

then $X^r \sim$ UI for all $|r| \geq 1$ by applying Theorem 3.11 and Corollary 3.12, (but the result in [35] is stronger than this one).

(2) Consider $\lambda \in \mathbb{R}$ and $\alpha_1, \alpha_2 > 0$. Suppose that $0 < a < b$ are the unique solution of the equation (3.23). If we have

$$X \sim \mathrm{fGIG}(\alpha_1, \alpha_2, \lambda) = \mathrm{GPFP}\left(a,\, b,\, N = 2,\, \alpha = \left(\frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi\sqrt{ab}}\right),\, l = (0, 1)\right), \tag{3.49}$$

then $X^r \sim$ UI for all $|r| \geq 1$ by applying Theorem 3.11 and Corollary 3.12.

(3) Consider $S \sim S(0, 1)$ and $u \neq 0$, where $S(0, 1)$ is the semicircle law with mean 0 and variance 1 (see Example 2.6 (1)). Then the distribution of $(S + u)$ is FID. However, the distribution of $(S + u)^2$ is not FID (see [28]).

If $u > 2$, then we have

$$S + u \sim \mathrm{GPFP}\left(a = u - 2,\, b = u + 2,\, N = 1,\, \alpha = \frac{1}{2\pi},\, l = -1\right) \tag{3.50}$$

$$= \frac{1}{2\pi}\sqrt{4 - (x - u)^2} \cdot 1_{(u-2,u+2)}(x)dx. \tag{3.51}$$

By Remark 3.9, we have

$$(S + u)^{-1} \sim \mathrm{GPFP}\left(a = \frac{1}{u + 2},\, b = \frac{1}{u - 2},\, N = 1,\, \alpha = \frac{1}{2\pi}\sqrt{u^2 - 4},\, l = 2\right). \tag{3.52}$$

By Theorem 3.11, we have that $(S + u)^{-r} = ((S + u)^{-1})^r \sim$ UI for $r \geq 1$. Finally we can conclude that $(S + u)^s \sim$ UI for all $u > 2$ and $s \leq -1$.

(4) Consider a random variable $S_{n,b} \sim \mu_{n,b}$, where $\mu_{n,b}$ is the GPFP distribution given by (3.26) in Example 3.10 (3) for each $n \in \mathbb{N}$ and $b > 0$. Then $S_{n,b}^r \sim$ UI for all $|r| \geq 1$ by applying Theorem 3.11 and Corollary 3.12.

Recall that $\mu_{n,b} \to \mu_b$ weakly as $n \to \infty$, where μ_b is the probability measure given by (3.27) in Example 3.10 (3) for each $b > 0$. If $S_b \sim \mu_b$, then the weak closedness of the UI class implies that $S_b^r \sim$ UI for all $|r| \geq 1$.

(5) Consider a random variable $B_{n,l} \sim \rho_{n,l}$, where $\rho_{n,l}$ is the GPFP distribution given by (3.31) in Example 3.10 (5) for each $n \in \mathbb{N}$ and $l < 1/2$.

If $0 \leq l < 1/2$ then $B_{n,l}^r \sim$ UI for all $|r| \geq 1$ by applying Theorem 3.11 and Corollary 3.12.

By Remark 3.9, we have that

$$B_{n,l}^{-1} \sim \mathrm{GPFP}\left(1,\, n,\, 1,\, \sqrt{\frac{1}{n}} \cdot \frac{\alpha(n, l)}{B(1/2 - l, 3/2)},\, 1 - l\right). \tag{3.53}$$

If $l < 0$ then $B_{n,l}^{-r} = (B_{n,r}^{-1})^r \sim$ UI for all $r \geq 1$ by applying Theorem 3.11 and the condition (3.53). Finally we conclude that if $l < 0$ then $B_{n,l}^s \sim$ UI for all $s \leq -1$.

Recall that $\rho_{n,l} \to \beta_{1/2-l,3/2}$ weakly as $n \to \infty$ for each $l < 1/2$. If $B_l \sim \beta_{1/2-l,3/2}$, then the weak closedness of the UI class implies that (i) if $0 \leq l < 1/2$ then $B_l^r \sim$ UI for all $|r| \geq 1$, and (ii) if $l < 0$ then $B_l^s \sim$ UI for all $s \leq -1$.

3.5 Non free infinite divisibility for GPFP distributions

In this section, we give a few of examples of the GPFP distributions which fail to satisfy the conclusions of Theorem 3.11 or Corollary 3.12 without these assumptions.

Let $(\mathcal{A}, \phi)$ be a C^*-probability space and $X \in \mathcal{A}$ a random variable. The *n-th moment* $m_n = m_n(X)$ of X is defined by the value of $\phi(X^n)$. In particular, if $X \sim \mu$, we write $m_n(\mu)$ as the n-th moment of X (or μ). The *n-th free cumulant* $\kappa_n = \kappa_n(X)$ of X is defined as the n-th coefficient of power series of the R-transform $R_X(z)$. If $X \sim \mu$, we write $\kappa_n(\mu)$ as the n-th free cumulant of X (or μ). We have the *moment-cumulant formula* as follow:

$$\kappa_n = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} m_{|V|} \right) \mu(\pi, 1_n), \qquad n \in \mathbb{N}, \tag{3.54}$$

where $NC(n)$ is the set of non-crossing partitions of the finite set $\{1, \ldots, n\}$, the symbol 1_n is the non-crossing partition which has the block $\{1, \ldots, n\}$, the function $\mu(\pi, \sigma)$ ($\pi \leq \sigma$ in $NC(n)$) is the Möbius function of $NC(n)$, and $|V|$ is the number of elements in a block V of π (see [51] for details). By the moment-cumulant formula, we can compute the free cumulants. For example we have that

$$\begin{aligned}
&\kappa_1 = m_1, \qquad \kappa_2 = m_2 - m_1^2, \qquad \kappa_3 = m_3 - 3m_1m_2 + 2m_1^3, \\
&\kappa_4 = m_4 - 4m_1m_3 - 2m_2^2 + 10m_1^2m_2 - 5m_1^4, \\
&\kappa_5 = m_5 - 5m_1m_4 - 5m_2m_3 + 15m_1^2m_3 + 15m_1m_2^2 - 35m_1^3m_2 + 14m_1^5.
\end{aligned} \tag{3.55}$$

The n-th free cumulant of the free Poisson distribution $\mathbf{fp}(p)$ is given by $\kappa_n(\mathbf{fp}(p)) = p$ for all $n \in \mathbb{N}$ (from Example 2.6 (2)). Then the formula (3.55) yields that

$$\begin{aligned}
&m_1(\mathbf{fp}(p)) = p, \qquad m_2(\mathbf{fp}(p)) = p(p+1), \qquad m_3(\mathbf{fp}(p)) = p(p^2+3p+1), \\
&m_4(\mathbf{fp}(p)) = p(p^3+6p^2+6p+1), \qquad m_5(\mathbf{fp}(p)) = p(p^4+10p^3+20p^2+10p+1).
\end{aligned} \tag{3.56}$$

In particular, we need to compute the moments of $\mathbf{fp}(p)$ of degree $s \in \mathbb{C}$:

$$m_s(\mathbf{fp}(p)) = \int_0^\infty x^s \mathbf{fp}(p)(dx), \qquad s \in \mathbb{C}, \quad p > 1. \tag{3.57}$$

This is analytic in $\mathbb{C}$ as a function of s.

Lemma 3.14. *For $s \in \mathbb{C}$ and $p > 1$, we have that*

$$m_s(\mathbf{fp}(p)) = \frac{m_{-s-1}(\mathbf{fp}(p))}{(p-1)^{-1-2s}}. \tag{3.58}$$

Proof. Consider $s \in \mathbb{C}$ and $p > 1$. Then we have that

$$\begin{aligned}
m_s(\mathbf{fp}(p)) &= \int_{(\sqrt{p}-1)^2}^{(\sqrt{p}+1)^2} x^{s-1} \frac{\sqrt{\left((\sqrt{p}+1)^2 - x\right)\left(x - (\sqrt{p}-1)^2\right)}}{2\pi} dx \\
&= (p-1) \int_{\frac{1}{(\sqrt{p}+1)^2}}^{\frac{1}{(\sqrt{p}-1)^2}} x^{-s-1} \frac{\sqrt{\left(\frac{1}{(\sqrt{p}-1)^2} - x\right)\left(x - \frac{1}{(\sqrt{p}+1)^2}\right)}}{2\pi x} dx \\
&= \frac{1}{(p-1)^{-1-2s}} \int_{(\sqrt{p}-1)^2}^{(\sqrt{p}+1)^2} x^{-s-1} \frac{\sqrt{\left((\sqrt{p}+1)^2 - x\right)\left(x - (\sqrt{p}-1)^2\right)}}{2\pi} dx \\
&= \frac{m_{-s-1}(\mathbf{fp}(p))}{(p-1)^{-1-2s}},
\end{aligned} \tag{3.59}$$

where the second equality holds by changing the variables from x to x^{-1} and the third equality holds by changing the variables from x to $\frac{x}{(p-1)^2}$. □

By Lemma 3.14, we have that for all $p > 1$,

$$m_{-1}(\mathbf{fp}(p)) = \frac{1}{p-1}, \qquad m_{-2}(\mathbf{fp}(p)) = \frac{p}{(p-1)^3}. \tag{3.60}$$

Lemma 3.15. *Consider $p > 1$ and $\alpha_1, \alpha_2 > 0$.*
(1) The following measure

$$\frac{\sqrt{\left(x - \frac{1}{(\sqrt{p}+1)^2}\right)\left(\frac{1}{(\sqrt{p}-1)^2} - x\right)}}{2\pi x^2} \left(\alpha_1 + \frac{\alpha_2}{x}\right) 1_{\left(\frac{1}{(\sqrt{p}+1)^2}, \frac{1}{(\sqrt{p}-1)^2}\right)}(x)dx \tag{3.61}$$

is a probability measure if and only if $\alpha_1 + \alpha_2 p = p - 1$.

(2) The following measure

$$\frac{\sqrt{\left((\sqrt{p}+1)^2 - x\right)\left(x - (\sqrt{p}-1)^2\right)}}{2\pi x} \left(\alpha_1 + \frac{\alpha_2}{x^2}\right) 1_{((\sqrt{p}-1)^2, (\sqrt{p}+1)^2)}(x)dx \tag{3.62}$$

is a probability measure if and only if $\alpha_1 + \frac{p}{(p-1)^3}\alpha_2 = 1$.

Proof. (1) Assume that the measure (3.61) is a probability measure. Suppose that X is a random variable which follows the probability measure (3.61). Then we have

$$X^{-1} \sim \frac{\sqrt{\left(x - (\sqrt{p}-1)^2\right)\left((\sqrt{p}+1)^2 - x\right)}}{2\pi(p-1)x} (\alpha_1 + \alpha_2 x) 1_{((\sqrt{p}-1)^2, (\sqrt{p}+1)^2)}(x)dx. \tag{3.63}$$

By our assumption, the measure (3.63) is also a probability measure. Equivalently, we have

$$1 = \frac{1}{p-1}\left(\alpha_1 m_0(\mathbf{fp}(p)) + \alpha_2 m_1(\mathbf{fp}(p))\right) = \frac{1}{p-1}(\alpha_1 + \alpha_2 p). \tag{3.64}$$

Therefore the relation $\alpha_1 + \alpha_2 p = p - 1$ holds. The converse implication is clear.

(2) Let $\mu_{p,\alpha_1,\alpha_2}$ be the measure (3.62) on the positive real line. It is a probability measure if and only if

$$1 = \int_{(\sqrt{p}-1)^2}^{(\sqrt{p}+1)^2} \mu_{p,\alpha_1,\alpha_2}(dx) = \alpha_1 m_0(\mathbf{fp}(p)) + \alpha_2 m_{-2}(\mathbf{fp}(p)). \tag{3.65}$$

The calculation (3.60) implies our conclusion. $\square$

There is a criterion for free infinite divisibility of probability measures on $\mathbb{R}$ with compact support.

Lemma 3.16. *(See [51, Theorem 13.16]) Let μ be a probability measure on $\mathbb{R}$ with compact support. The following conditions are equivalent.*
(1) μ is FID.
(2) The sequence $\{\kappa_n(\mu)\}_{n\in\mathbb{N}}$ of free cumulants of μ is conditionally positive definite, namely, for all $n \in \mathbb{N}$ and all $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ we have that

$$\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \kappa_{i+j}(\mu) \geq 0. \tag{3.66}$$

In particular, if a compactly supported probability measure μ is FID, then the n-th Hankel determinant $\det((\kappa_{i+j}(\mu))_{i,j=1}^n)$ of the sequence $\{\kappa_n(\mu)\}_{n\in\mathbb{N}}$ is nonnegative for all $n \in \mathbb{N}$.

First we give an example of the GPFP distribution which fails to satisfy the conclusion of Corollary 3.12 without the assumption.

Example 3.17. Consider the following GPFP distribution:

$$\begin{aligned}\sigma_{\alpha_1,\alpha_2}(dx) :&= \mathrm{GPFP}\left(\frac{1}{(\sqrt{2}+1)^2}, \frac{1}{(\sqrt{2}-1)^2}, 2, \left(\frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi}\right), l = (1,2)\right) \\ &= \frac{\sqrt{\left(x - \frac{1}{(\sqrt{2}+1)^2}\right)\left(\frac{1}{(\sqrt{2}-1)^2} - x\right)}}{2\pi x}\left(\frac{\alpha_1}{x} + \frac{\alpha_2}{x^2}\right) 1_{\left(\frac{1}{(\sqrt{2}+1)^2}, \frac{1}{(\sqrt{2}-1)^2}\right)}(x)dx,\end{aligned} \tag{3.67}$$

where $\alpha_1, \alpha_2 > 0$ satisfy the relation $\alpha_1 + 2\alpha_2 = 1$ ($0 < \alpha_2 < 1/2$) by Lemma 3.15 (1). The GPFP distribution $\sigma_{\alpha_1,\alpha_2}$ fails to satisfy the assumption of Corollary 3.12 since $l = (1,2) \notin [0,1]^2_<$.

If $X_{\alpha_1,\alpha_2} \sim \sigma_{\alpha_1,\alpha_2}$, then the distribution of $X_{\alpha_1,\alpha_2}^{-1}$ is not FID for all $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + 2\alpha_2 = 1$. Therefore the GPFP distribution $\sigma_{\alpha_1,\alpha_2}$ does not satisfy the conclusion of Corollary 3.12.

Proof. By Remark 3.9 (or (3.63)), we have

$$\begin{aligned}X_{\alpha_1,\alpha_2}^{-1} &\sim \mathrm{GPFP}\left((\sqrt{2}-1)^2, (\sqrt{2}+1)^2, 2, \left(\frac{\alpha_2}{2\pi}, \frac{\alpha_1}{2\pi}\right), (-1,0)\right) \\ &= \frac{\sqrt{\left(x - (\sqrt{2}-1)^2\right)\left((\sqrt{2}+1)^2 - x\right)}}{2\pi x}(\alpha_1 + \alpha_2 x) 1_{((\sqrt{2}-1)^2, (\sqrt{2}+1)^2)}(x)dx.\end{aligned} \tag{3.68}$$

From the relation $\alpha_1 + 2\alpha_2 = 1$ and the above formulas (3.56), we have that

$$\begin{aligned} m_1(X^{-1}_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_1(\mathbf{fp}(2)) + \alpha_2 m_2(\mathbf{fp}(2)) = 2(1+\alpha_2), \\ m_2(X^{-1}_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_2(\mathbf{fp}(2)) + \alpha_2 m_3(\mathbf{fp}(2)) = 2(3+5\alpha_2), \\ m_3(X^{-1}_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_3(\mathbf{fp}(2)) + \alpha_2 m_4(\mathbf{fp}(2)) = 2(11+23\alpha_2), \\ m_4(X^{-1}_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_4(\mathbf{fp}(2)) + \alpha_2 m_5(\mathbf{fp}(2)) = 2(45+107\alpha_2). \end{aligned} \tag{3.69}$$

From the formula (3.55) and the above calculations (3.69), we have that

$$\begin{aligned} \kappa_2 &:= \kappa_2(X^{-1}_{\alpha_1,\alpha_2}) = -2(2\alpha_2^2 - \alpha_2 - 1), \\ \kappa_3 &:= \kappa_3(X^{-1}_{\alpha_1,\alpha_2}) = 2(8\alpha_2^3 - 6\alpha_2^2 - \alpha_2 + 1), \\ \kappa_4 &:= \kappa_4(X^{-1}_{\alpha_1,\alpha_2}) = -2(2\alpha_2 - 1)(20\alpha_2^3 - 10\alpha_2^2 - 3\alpha_2 + 1), \end{aligned} \tag{3.70}$$

where κ_n is the n-th free cumulant of $X^{-1}_{\alpha_1,\alpha_2}$. By an elementary calculation, the 2nd Hankel determinant $\kappa_2\kappa_4 - \kappa_3^2$ (is a function of α_2) is negative for all $0 < \alpha_2 < 1/2$ with $\alpha_1 + 2\alpha_2 = 1$. By Lemma 3.16 the distribution of $X^{-1}_{\alpha_1,\alpha_2}$ is not FID for all $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + 2\alpha_2 = 1$. □

Next we give an example of the GPFP distribution which fails to satisfy the conclusion of Theorem 3.11 without the assumption.

Example 3.18. Consider the following GPFP distribution:

$$\begin{aligned} \eta_{\alpha_1,\alpha_2}(dx) :&= \mathrm{GPFP}\left((\sqrt{2}-1)^2,\, (\sqrt{2}+1)^2,\, 2,\, \left(\frac{\alpha_1}{2\pi}, \frac{\alpha_2}{2\pi}\right),\, l = (0,2)\right) \\ &= \frac{\sqrt{\left((\sqrt{2}+1)^2 - x\right)\left(x - (\sqrt{2}-1)^2\right)}}{2\pi x}\left(\alpha_1 + \frac{\alpha_2}{x^2}\right) 1_{((\sqrt{2}-1)^2,(\sqrt{2}+1)^2)}(x)dx, \end{aligned} \tag{3.71}$$

where $\alpha_1, \alpha_2 > 0$ satisfy the relation $\alpha_1 + 2\alpha_2 = 1$ by Lemma 3.15 (2). The GPFP distribution η_{α_1,α_2} fails to satisfy the assumption of Theorem 3.11 since $l = (0,2) \notin [t, t+1]^2_<$ for any $t \geq 0$.

There exist $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + 2\alpha_2 = 1$ such that the GPFP distribution η_{α_1,α_2} is not FID. Therefore the GPFP distribution η_{α_1,α_2} does not satisfy the conclusion of Theorem 3.11.

Proof. From the relation $\alpha_1 + 2\alpha_2 = 1$ and the above formulas (3.56) and (3.60), we have that

$$\begin{aligned} m_1(\eta_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_1(\mathbf{fp}(2)) + \alpha_2 m_{-1}(\mathbf{fp}(2)) = 2 - 3\alpha_2, \\ m_2(\eta_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_2(\mathbf{fp}(2)) + \alpha_2 m_0(\mathbf{fp}(2)) = 6 - 11\alpha_2, \\ m_3(\eta_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_3(\mathbf{fp}(2)) + \alpha_2 m_1(\mathbf{fp}(2)) = 2(11 - 21\alpha_2), \\ m_4(\eta_{\alpha_1,\alpha_2}) &= (1-2\alpha_2)m_4(\mathbf{fp}(2)) + \alpha_2 m_2(\mathbf{fp}(2)) = 6(15 - 29\alpha_2). \end{aligned} \tag{3.72}$$

From the formula (3.55) and the above calculations (3.72), we have that

$$\begin{aligned} \kappa_2' &:= \kappa_2(\eta_{\alpha_1,\alpha_2}) = -9\alpha_2^2 + \alpha_2 + 2, \\ \kappa_3' &:= \kappa_3(\eta_{\alpha_1,\alpha_2}) = -54\alpha_2^3 + 9\alpha_2^2 + 6\alpha_2 + 2, \\ \kappa_4' &:= \kappa_4(\eta_{\alpha_1,\alpha_2}) = -405\alpha_2^4 + 90\alpha_2^3 + 34\alpha_2^2 + 10\alpha_2 + 2. \end{aligned} \tag{3.73}$$

This implies that the 2nd Hankel determinant $\kappa_2'\kappa_4' - (\kappa_3')^2$ (is a function of α_2) is negative for $0 < \alpha_2 < \tilde{\alpha}$, where $\tilde{\alpha} \in (0, 1/2)$ is the unique solution α_2 of the equation $\kappa_2'\kappa_4' - (\kappa_3')^2 = 0$. Thus η_{α_1,α_2} is not FID for $0 < \alpha_2 < \tilde{\alpha}$ with $\alpha_1 + 2\alpha_2 = 1$ by Lemma 3.16. $\square$

According to Mathematica, we get that $\tilde{\alpha} \approx 0.157781$ (see Figure 3). For example, the GPFP distribution $\eta_{0.7,0.15}$ is not FID.

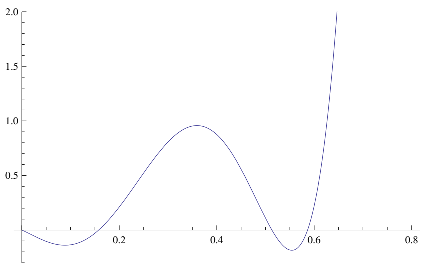

Figure 3: $\kappa_2'\kappa_4' - (\kappa_3')^2$ as a function of α_2. It is negative when $0 < \alpha_2 \preceq 0.157781$.

3.6 Future works

We defined the GPFP distributions as Lebesgue absolutely continuous probability measures and studied the UR property of GPFP distributions. More generally, we consider the following probability measures:

$$c\delta_0 + \frac{\sqrt{(b-x)(x-a)}}{x} \sum_{k=1}^{N} \frac{\alpha_k}{x^{l_k}} 1_{(a,b)}(x)dx, \tag{3.74}$$

where $0 \le c \le 1$, $0 < a < b$, $N \in \mathbb{N}$, $\alpha = (\alpha_1, \cdots, \alpha_N) \in (0, \infty)^N$ with

$$\int_a^b \frac{\sqrt{(b-x)(x-a)}}{x} \sum_{k=1}^{N} \frac{\alpha_k}{x^{l_k}} dx = 1 - c, \tag{3.75}$$

and $l = (l_1, \cdots, l_N) \in \mathbb{R}^N_<$. The probability measures (3.74) include general free Poisson distributions (2.28). We should study UI property and FR property of the above probability measures.

4 Unimodality for classical and free additive Lévy processes with initial distributions

4.1 Introduction

In this section, we discuss about the paper [39] which studies unimodality for classical and free Lévy processes (in particular, Brownian motions, Cauchy processes, positive stable processes with index 1/2) with initial distributions.

A Borel measure μ on $\mathbb{R}$ is *unimodal* if there exist $a \in \mathbb{R}$ and a function $f\colon \mathbb{R} \to [0,\infty)$ which is non-decreasing on $(-\infty, a)$ and non-increasing on (a,∞), such that

$$\mu(dx) = \mu(\{a\})\delta_a + f(x)\,dx. \tag{4.1}$$

The most outstanding result on unimodality is Yamazato's theorem [68] saying that all classical selfdecomposable distributions are unimodal. After this result, in [38], Hasebe and Thorbjørnsen proved the free analog of Yamazato's result: all freely selfdecomposable distributions are unimodal. The unimodality has several other similarity points between the classical and free probability theories [36]. However, it is not true that the unimodality shows complete similarity between the classical and free probability theories. For example, classical compound Poisson processes are likely to be non-unimodal in large time [67], while free Lévy processes with compact support become unimodal in large time [36]. In this section, we mainly focus on the unimodality of classical and free Brownian motions with initial distributions and consider whether the classical and free versions share similarity points or not.

4.2 Outline of results

In free probability theory, it is known that the semicircle distribution is the free analog of the normal distribution. Free Brownian motion is defined as a process with free independent increments, started at 0 and distributed as the centered semicircle distribution $S(0,t)$ at time $t > 0$. Furthermore, we can provide free Brownian motion with initial distribution μ which is a process with free independent increments distributed as μ at $t = 0$ and as $\mu \boxplus S(0,t)$ at time $t > 0$. In [19], Biane gave the density function formula of free Brownian motion with initial distribution (see Section 4.3.1). In our studies, we first consider the symmetric Bernoulli distribution $\mu := \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$ as an initial distribution and compute the density function of $\mu \boxplus S(0,t)$. Then we see that the probability distribution $\mu \boxplus S(0,t)$ is unimodal for $t \geq 4$ (and it is not unimodal for $0 < t < 4$). This computation leads to a natural problem:

Problem 4.1. For which class of probability measures μ on $\mathbb{R}$ does the free Brownian motion with initial distribution μ become unimodal for sufficiently large time?

We then answer to this problem as follows (formulated as Theorem 4.8 and Proposition 4.10 in Section 4.3.3):

Theorem 4.2. (1) Let μ be a compactly supported probability measure on $\mathbb{R}$ and $D_\mu := \sup\{|x-y| : x, y \in \operatorname{supp}(\mu)\}$. Then $\mu \boxplus S(0,t)$ is unimodal for $t \geq 4D_\mu^2$.

(2) Let $f\colon \mathbb{R} \to [0,\infty)$ be a Borel measurable function. Then there exists a probability measure μ on $\mathbb{R}$ such that $\mu \boxplus S(0,t)$ is not unimodal for any $t > 0$ and

$$\int_{\mathbb{R}} f(x)\,d\mu(x) < \infty. \tag{4.2}$$

Note that such a measure μ is not compactly supported by (1).

The function f can grow very fast such as e^{x^2}, and so, a tail decay of the initial distribution does not imply large time unimodality.

The corresponding classical problem is natural, that is, for which class of initial distributions on $\mathbb{R}$ does Brownian motion become unimodal for sufficiently large time $t > 0$? We prove in this direction the following results (formulated as Theorem 4.20 and Proposition 4.19 in Section 4.4, respectively):

Theorem 4.3. (1) Let μ be a probability measure on $\mathbb{R}$ such that

$$\alpha := \int_{\mathbb{R}} e^{\varepsilon x^2}\, d\mu(x) < \infty \tag{4.3}$$

for some $\varepsilon > 0$. Then the distribution $\mu * N(0,t)$ is unimodal for all $t \geq \frac{36\log(2\alpha)}{\varepsilon}$.

(2) There exists a probability measure μ on $\mathbb{R}$ satisfying that

$$\int_{\mathbb{R}} e^{A|x|^p}\, d\mu(x) < \infty \quad \text{for all } A > 0 \text{ and } 0 < p < 2 \tag{4.4}$$

such that $\mu * N(0,t)$ is not unimodal for any $t > 0$.

Thus, in the classical case, the tail decay (4.3) is sufficient and almost necessary to guarantee the large time unimodality. From these results, we conclude that the classical and free versions of the problems of unimodality for Brownian motions with initial distributions share common features as both become unimodal in large time under each assumption.

Further related results in this paper are as follows. In Section 4.3.4 we prove that $\mu \boxplus S(0,t)$ is always unimodal whenever μ is symmetric around 0 and unimodal (see Theorem 4.12). In Section 4.3.5, we define freely strongly unimodal probability measures as a natural free analogue of strongly unimodal probability measures. We conclude that the semicircle distributions are not freely strongly unimodal (Lemma 4.15). More generally, we have the following result (restated as Theorem 4.16 later):

Theorem 4.4. Let λ be a probability measure with finite variance, not being a delta measure. Then λ is not freely strongly unimodal.

On the other hand, there are many strongly unimodal distributions with finite variance in classical probability including the normal distributions and exponential distributions. Thus the notion of strong unimodality breaks the similarity between the classical and free probability theories.

In Section 4.5, we focus on other processes with classical/free independent increments with initial distributions. We consider a symmetric stable process with index 1 and a positive stable process with index 1/2. Then we prove large time unimodality similar to the case of Brownian motion but under different tail decay assumptions.

4.3 Free Brownian motion with initial distribution

4.3.1 Free convolution with semicircle laws

Almost all the materials and methods in this section are following Biane's paper [19]. Recall that $S(0,t)$ is the semicircle law of mean 0 and variance $t > 0$. We then compute its Cauchy transform and its R-transform (see, for example, [51]):

$$G_{S(0,t)}(z) = \frac{z - \sqrt{z^2 - 4t}}{2t}, \quad z \in \mathbb{C}^+, \tag{4.5}$$

where the branch of the square root on $\mathbb{C}\setminus\mathbb{R}^+$ is such that $\sqrt{-1}=\mathrm{i}$. Moreover

$$R_{S(0,t)}(z) = tz, \;\; z \in \mathbb{C}^-. \tag{4.6}$$

Let μ be a probability measure on $\mathbb{R}$ and let $\mu_t = \mu \boxplus S(0,t)$. Then we define the following set:

$$U_t := \left\{ u \in \mathbb{R} \;\middle|\; \int_{\mathbb{R}} \frac{1}{(x-u)^2}\, d\mu(x) > \frac{1}{t} \right\}, \tag{4.7}$$

and the following function from $\mathbb{R}$ to $[0,\infty)$ by setting

$$v_t(u) := \inf \left\{ v \geq 0 \;\middle|\; \int_{\mathbb{R}} \frac{1}{(x-u)^2 + v^2}\, d\mu(x) \leq \frac{1}{t} \right\}. \tag{4.8}$$

Then v_t is continuous on $\mathbb{R}$ and one has $U_t = \{x \in \mathbb{R} \mid v_t(u) > 0\}$. Moreover, for every $u \in U_t$, $v_t(u)$ is the unique solution $v > 0$ of the equation

$$\int_{\mathbb{R}} \frac{1}{(x-u)^2 + v^2}\, d\mu(x) = \frac{1}{t}. \tag{4.9}$$

We define

$$\Omega_{t,\mu} := \{x + \mathrm{i}y \in \mathbb{C} \mid y > v_t(x)\}. \tag{4.10}$$

Then the map $H_t(z) := z + R_{S(0,t)}(G_\mu(z))$, defined on $\mathbb{C}^+$, is a homeomorphism from $\overline{\Omega_{t,\mu}}$ to $\mathbb{C}^+ \cup \mathbb{R}$ and it is conformal from $\Omega_{t,\mu}$ onto $\mathbb{C}^+$. Hence there exists an inverse function $F_t : \mathbb{C}^+ \cup \mathbb{R} \to \overline{\Omega_{t,\mu}}$. By [19, Lemma 1, Proposition 1], the domain $\Omega_{t,\mu}$ is equal to the connected component of the set $H_t^{-1}(\mathbb{C}^+)$ which contains $\mathrm{i}y$ for large $y > 0$ and for all $z \in \mathbb{C}^+$ one has

$$\begin{aligned} G_{\mu_t}(z) &= G_{\mu_t}(H_t(F_t(z))) \\ &= G_{\mu_t}\Big(F_t(z) + R_{S(0,t)}\big(G_\mu(F_t(z))\big)\Big) \\ &= G_\mu(F_t(z)). \end{aligned} \tag{4.11}$$

Moreover G_{μ_t} has a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$. Then μ_t is Lebesgue absolutely continuous by Stieltjes inversion and the density function $p_t \colon \mathbb{R} \to [0,\infty)$ of μ_t is given by

$$p_t(\psi_t(x)) = -\frac{1}{\pi} \lim_{y \to +0} \mathrm{Im}(G_{\mu_t}(\psi_t(x) + \mathrm{i}y)) = \frac{v_t(x)}{\pi t}, \;\; x \in \mathbb{R}, \tag{4.12}$$

where

$$\psi_t(x) = H_t(x + \mathrm{i}v_t(x)) = x + t \int_{\mathbb{R}} \frac{(x-u)}{(x-u)^2 + v_t(x)^2}\, d\mu(u). \tag{4.13}$$

It can be shown that ψ_t is a homeomorphism of $\mathbb{R}$. Furthermore the topological support of p_t is given by $\psi_t(\overline{U_t})$, and p_t extends to a continuous function on $\mathbb{R}$ and is real analytic in $\{x \in \mathbb{R} : p_t(x) > 0\}$.

Example 4.5. Let $\mu := \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$ be the symmetric Bernoulli distribution. Recall that the topological support of the density function p_t is equal to $\psi_t(\overline{U_t})$. If $0 < t \leq 1$, we have

$$U_t = \left\{ u \in \mathbb{R} \;\middle|\; \sqrt{\frac{2 + t - \sqrt{t^2 + 8t}}{2}} < |u| < \sqrt{\frac{2 + t + \sqrt{t^2 + 8t}}{2}} \right\}, \tag{4.14}$$

and if $t > 1$, we have

$$U_t = \left\{ u \in \mathbb{R} \;\middle|\; -\sqrt{\frac{2+t+\sqrt{t^2+8t}}{2}} < u < \sqrt{\frac{2+t+\sqrt{t^2+8t}}{2}} \right\}. \tag{4.15}$$

If $0 < t \le 1$ then the number of connected components of U_t is two, and therefore μ_t is not unimodal. If $t > 1$ then U_t is connected. By the density formula (4.12), the unimodality of μ_t is equivalent to the unimodality of the function $v_t(u)$, the latter being expressed in the form

$$v_t(u) = \sqrt{\frac{-(2u^2+2-t)+\sqrt{t^2+16u^2}}{2}}. \tag{4.16}$$

Calculating its first derivative then shows that μ_t is unimodal if and only if $t \ge 4$; see Figures 4–9.

According to Example 4.5, we have a question about what class of initial distributions implies the large time unimodality of free Brownian motion. We will give a result for this in Section 4.3.3.

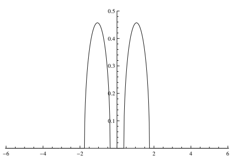

Figure 4: $p_{0.25}(x)$

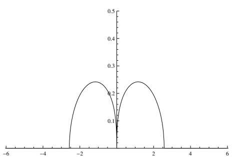

Figure 5: $p_1(x)$

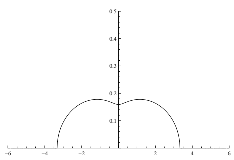

Figure 6: $p_2(x)$

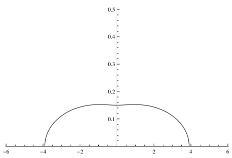

Figure 7: $p_3(x)$

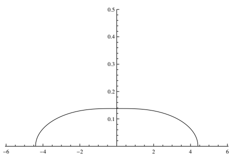

Figure 8: $p_4(x)$

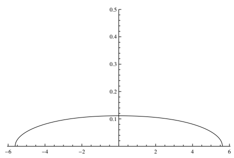

Figure 9: $p_7(x)$

4.3.2 A basic lemma for unimodality

A unimodal distribution is allowed to have a plateau or a discontinuous point in its density such as the uniform distribution or the exponential distribution. If we exclude such cases, then unimodality can be characterized in terms of levels of density.

Lemma 4.6. Let μ be a probability measure on $\mathbb{R}$ that is Lebesgue absolutely continuous. Suppose that its density $p(x)$ extends to a continuous function on $\mathbb{R}$ and is real analytic in $\{x \in \mathbb{R} : p(x) > 0\}$. Then μ is unimodal if and only if, for any $a > 0$, the equation $p(x) = a$ has at most two solutions x.

The proof is just to use the intermediate value theorem and the fact that a real analytic function never has plateaux. Note that the idea of the above lemma was first introduced by Haagerup and Thorbjørnsen [33] to prove that a "free gamma distribution" is unimodal.

4.3.3 Free Brownian motions with compactly supported initial distributions

The semicircular distribution is the free analogue of the normal distribution. Therefore, a process with free independent increments is called *(standard) free Brownian motion with initial distribution* μ if its distribution at time t is given by $\mu \boxplus S(0,t)$. In this section, we firstly prove that free Brownian motion with compactly supported initial distribution is unimodal for sufficiently large time, by using Biane's density formula.

Lemma 4.7. Suppose that $t > 0$. The following statements are equivalent.

(1) $\mu \boxplus S(0,t)$ is unimodal.

(2) For any $R > 0$ the equation

$$\int_{\mathbb{R}} \frac{1}{(x-u)^2 + R^2}\, d\mu(x) = \frac{1}{t} \tag{4.17}$$

has at most two solutions $u \in \mathbb{R}$.

Proof. We adopt the notations $\mu_t = \mu \boxplus S(0,t), v_t, p_t$ and ψ_t in Section 4.3.1. This proof is similar to [38, Proposition 3.8]. Recall that μ_t is absolutely continuous with respect to Lebesgue measure and the density function p_t is continuous on $\mathbb{R}$ since its Cauchy transform G_{μ_t} has a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$, and the density function is real analytic in $\psi_t(U_t)$ and continuous on $\mathbb{R}$ by Section 4.3.1.

Suppose (2) first. Since ψ_t is a homeomorphism of $\mathbb{R}$, by Lemma 4.6 it suffices to show that for any $a > 0$ the equation

$$a = p_t(\psi_t(u)) = \frac{v_t(u)}{\pi t}, \quad u \in \mathbb{R} \tag{4.18}$$

has at most two solutions $u \in \mathbb{R}$. Since $v_t(u)$ is a unique solution of the equation (4.9) if it is positive, then we have

$$\left\{ u \in \mathbb{R} \;\middle|\; a = \frac{v_t(u)}{\pi t} \right\} = \left\{ u \in \mathbb{R} \;\middle|\; \int_{\mathbb{R}} \frac{1}{(x-u)^2 + (\pi a t)^2}\, d\mu(x) = \frac{1}{t} \right\}. \tag{4.19}$$

By the assumption (2), the equation $a = \frac{v_t(u)}{\pi t}$ has at most two solutions in $\mathbb{R}$.

Conversely, suppose that (2) does not hold. Then for some $R > 0$ there are three distinct solutions $u_1, u_2, u_3 \in \mathbb{R}$ to (4.17). By (4.19) this shows that $v_t(u_i) = R$ and hence $p_t(\psi_t(u_i)) = \frac{R}{\pi t}$ for $i = 1, 2, 3$. Again by Lemma 4.6, μ_t is not unimodal. □

Now we are ready to prove Theorem 4.2 (1).

Theorem 4.8. Let μ be a compactly supported probability measure, and let D_μ be the diameter of the support: $D_\mu = \sup\{|x-y| : x, y \in \mathrm{supp}(\mu)\}$. Then $\mu \boxplus S(0,t)$ is unimodal for $t \geq 4D_\mu^2$.

Proof. Applying a shift we may assume that μ is supported on $[-\frac{D_\mu}{2}, \frac{D_\mu}{2}]$. For fixed $R > 0$, we define by

$$\xi_R(u) := \int_{\mathbb{R}} \frac{1}{(x-u)^2 + R^2}\, d\mu(x), \;\; u \in \mathbb{R}. \tag{4.20}$$

Then we have

$$\begin{aligned} \xi_R'(u) &= \int_{-\frac{D_\mu}{2}}^{\frac{D_\mu}{2}} \frac{2(x-u)}{\{(x-u)^2 + R^2\}^2}\, d\mu(x), \\ \xi_R''(u) &= \int_{-\frac{D_\mu}{2}}^{\frac{D_\mu}{2}} \frac{6(x-u)^2 - 2R^2}{\{(x-u)^2 + R^2\}^3}\, d\mu(x). \end{aligned} \tag{4.21}$$

If $u < -\frac{D_\mu}{2}$, then $x - u > -\frac{D_\mu}{2} - (-\frac{D_\mu}{2}) = 0$ for all $x \in \mathrm{supp}(\mu)$, so that $\xi_R'(u) > 0$. If $u > \frac{D_\mu}{2}$, then $x - u < \frac{D_\mu}{2} - \frac{D_\mu}{2} = 0$ for all $x \in \mathrm{supp}(\mu)$, so that $\xi_R'(u) < 0$. We take $t \geq 4D_\mu^2$ and consider the form of $\xi_R(u)$ on $u \in (-\frac{D_\mu}{2}, \frac{D_\mu}{2})$.
If $0 < R < \sqrt{3}D_\mu$, then $(x-u)^2 + R^2 < (\frac{D_\mu}{2} + \frac{D_\mu}{2})^2 + (\sqrt{3}D_\mu)^2 = 4D_\mu^2 \leq t$, so that

$$\xi_R(u) = \int_{\mathbb{R}} \frac{1}{(x-u)^2 + R^2}\, d\mu(x) > \frac{1}{t}, \;\; u \in \Big(-\frac{D_\mu}{2}, \frac{D_\mu}{2}\Big). \tag{4.22}$$

Hence the equation $\xi_R(u) = \frac{1}{t}$ has at most two solutions $u \in \mathbb{R}$ if $t \geq 4D_\mu^2$. If $R \geq \sqrt{3}D_\mu$, then the function $\xi_R(u)$ satisfies the following three conditions:

- $\xi_R'(u) > 0$ for all $u < -\frac{D_\mu}{2}$ and $\xi_R'(u) < 0$ for all $u > \frac{D_\mu}{2}$,
- ξ_R' is continuous on $\mathbb{R}$,
- $\xi_R''(u) < 0$ for all $u \in \Big(-\frac{D_\mu}{2}, \frac{D_\mu}{2}\Big)$.

By the intermediate value theorem, ξ_R' has a unique zero in $[-\frac{D_\mu}{2}, \frac{D_\mu}{2}]$ and hence ξ_R takes a unique local maximum in the interval $[-\frac{D_\mu}{2}, \frac{D_\mu}{2}]$ (and therefore it is a unique global maximum on $\mathbb{R}$). Therefore the equation $\xi_R(u) = \frac{1}{t}$ has at most two solutions if $t \geq 4D_\mu^2$. Hence we have that the free convolution $\mu \boxplus S(0,t)$ is unimodal if $t \geq 4D_\mu^2$ by Lemma 4.7. □

Problem 4.9. What is the optimal universal constant $C > 0$ such that $\mu \boxplus S(0,t)$ is unimodal for all $t \geq CD_\mu^2$ and all probability measures μ with compact support?

We have already shown that $C \leq 4$. Recall from Example 4.5 that if μ is the symmetric Bernoulli distribution on $\{-1, 1\}$, then $\mu \boxplus S(0,t)$ is unimodal if and only if $t \geq 4$. Since the diameter D_μ of the support of μ is 2, we conclude that $2 \leq C \leq 4$.

The next question is whether there exists a probability measure μ such that $\mu \boxplus S(0,t)$ is not unimodal for any $t > 0$ or at least for sufficiently large $t > 0$. Such a distribution must have an unbounded support if it exists. Such an example can be constructed with an idea similar to [44, Proposition 4.13] (see also [36, Example 5.3]).

Proposition 4.10. Let $f\colon \mathbb{R} \to [0,\infty)$ be a Borel measurable function. Then there exists a probability measure μ such that $\mu \boxplus S(0,t)$ is not unimodal at any $t>0$ and

$$\int_{\mathbb{R}} f(x)\, d\mu(x) < \infty. \tag{4.23}$$

Proof. We follow the notations in Section 4.1.1. Let $\{w_n\}_n$ and $\{a_n\}_n$ be sequences in $\mathbb{R}$ satisfying

- $w_n > 0, \sum_{n=1}^{\infty} w_n = 1$,
- $a_{n+1} > a_n,\ \ n \geq 1$,
- $\lim_{n\to\infty} (a_{n+1} - a_n) = \infty$.

Consider the probability measure $\mu := \sum_{n=1}^{\infty} w_n \delta_{a_n}$ on $\mathbb{R}$ and define the function

$$X_\mu(u) := \int_{\mathbb{R}} \frac{1}{(u-x)^2}\, d\mu(x) = \sum_{n=1}^{\infty} \frac{w_n}{(u-a_n)^2}. \tag{4.24}$$

Recall that U_t is the set of $u \in \mathbb{R}$ such that $X_\mu(u) > 1/t$. Set $b_k := \frac{a_{k+1}+a_k}{2}$ for all $k \in \mathbb{N}$. Then we have

$$|b_k - a_n| \geq \frac{a_{k+1}-a_k}{2}, \quad k,n \in \mathbb{N} \tag{4.25}$$

and so

$$X_\mu(b_k) \leq \left(\frac{2}{a_{k+1}-a_k}\right)^2 \sum_{n=1}^{\infty} w_n = \left(\frac{2}{a_{k+1}-a_k}\right)^2 \to 0$$

as $k \to \infty$. This means that for each $t>0$ there exists an integer $K(t)>0$ such that $X_\mu(b_k) < \frac{1}{t}$ for all $k \geq K(t)$. This implies that, for $k \geq K(t)$, the closure of U_t does not contain b_k. Note that the set U_t is an open set. Therefore, there exists a sequence $\{\varepsilon_k(t)\}_{k=K(t)+1}^{\infty}$ of positive numbers such that

$$U_t \cap (b_{K(t)}, \infty) = \bigcup_{k=K(t)+1}^{\infty} (a_k - \varepsilon_k(t), a_k + \varepsilon_k(t)), \tag{4.26}$$

where the closures of the intervals are disjoint. Thus the set $\mathrm{supp}(\mu_t) = \psi_t(\overline{U_t})$ consists of infinitely many connected components for all $t>0$, and hence μ_t is not unimodal for any $t>0$.

In the above construction, the weights w_n are only required to be positive. Therefore, we may take

$$w_n = \frac{c}{n^2 \max\{f(a_n), 1\}}, \tag{4.27}$$

where $c>0$ is a normalizing constant. Then the integrability condition (4.23) is satisfied. □

Proposition 4.10 shows that tail decay estimates of the initial distribution do not guarantee the large time unimodality. In Section 4.4 we shall see that the situation is different for classical Brownian motion.

4.3.4 Free Brownian motion with symmetric initial distribution

This section proves a unimodality result of different flavor. It is well known that if μ and ν are symmetric unimodal distributions, then $\mu * \nu$ is also (symmetric) unimodal (see [53, Exercise 29.22]). The free analogue of this statement is not known.

Conjecture 4.11. Let μ and ν be symmetric unimodal distributions. Then $\mu \boxplus \nu$ is unimodal.

We can give a positive answer in the special case when one distribution is a semicircle distribution.

Theorem 4.12. Let μ be a symmetric unimodal distribution on $\mathbb{R}$. Then $\mu \boxplus S(0,t)$ is unimodal for any $t > 0$.

Remark 4.13. There is a unimodal probability measure μ such that $\mu \boxplus S(0,1)$ is not unimodal (see Lemma 4.15). Hence we cannot remove the assumption of symmetry of μ.

Proof of Theorem 4.12. By Lemma 4.7 it suffices to show that for any $R > 0$ the equation

$$\int_{\mathbb{R}} \frac{1}{(u-x)^2 + R^2} \, d\mu(x) = \frac{1}{t} \tag{4.28}$$

has at most two solutions $u \in \mathbb{R}$. Up to a constant multiplication, the LHS is the density of $\mu * C_R$, which is unimodal since μ and C_R are symmetric unimodal. Since the density of $\mu * C_R$ is real analytic, Lemma 4.6 implies that the equation (4.28) has at most two solutions. This shows that $\mu \boxplus S(0,t)$ is unimodal. □

4.3.5 Freely strong unimodality

In classical probability, a probability measure is said to be *strongly unimodal* if $\mu * \nu$ is unimodal for all unimodal distributions ν on $\mathbb{R}$. A distribution is strongly unimodal if and only if the distribution is Lebesgue absolutely continuous, supported on an interval and a version of its density is log-concave (see [53, Theorem 52.3]). From this characterization, the normal distributions are strongly unimodal. We discuss the free version of strong unimodality.

Definition 4.14. A probability measure μ on $\mathbb{R}$ is said to be *freely strongly unimodal* if $\mu \boxplus \nu$ is unimodal for all unimodal distributions ν on $\mathbb{R}$.

Lemma 4.15. The semicircle distributions are not freely strongly unimodal.

Proof. The Cauchy distribution is not strongly unimodal since its density is not log-concave. Hence, there exists a unimodal probability measure μ such that $\mu * C_1$ is not unimodal. Since the density of $\mu * C_1$ is real analytic on $\mathbb{R}$, by Lemma 4.6 there exists $t > 0$ such that the equation

$$\pi \frac{d(\mu * C_1)}{dx}(x) = \int_{\mathbb{R}} \frac{1}{(x-y)^2 + 1} \, d\mu(y) = \frac{1}{t} \tag{4.29}$$

has at least three distinct solutions $x_1, x_2, x_3 \in \mathbb{R}$. Lemma 4.7 then implies that $S(0,t) \boxplus \mu$ is not unimodal. By changing the scaling, we conclude that no semicircle distributions are freely strongly unimodal. □

Theorem 4.16. Let λ be a probability measure with finite variance, not being a delta measure. Then λ is not freely strongly unimodal.

Proof. We may assume that λ has mean 0. Suppose that λ is freely strongly unimodal. A simple induction argument shows that $\lambda^{\boxplus n}$ is freely strongly unimodal for all $n \in \mathbb{N}$. We derive a contradiction below. By Lemma 4.15 we can take a unimodal probability measure μ such that $\mu \boxplus S(0,1)$ is not unimodal. Our hypothesis shows that the measure $\mathrm{D}_{\sqrt{nv}}(\mu) \boxplus \lambda^{\boxplus n}$ is unimodal, where v is the variance of λ and $\mathrm{D}_c(\rho)$ is the push-forward of a measure ρ by the map $x \mapsto cx$ for $c \in \mathbb{R}$. By the free central limit theorem [49, Theorem 5.2], the unimodal distributions $\mathrm{D}_{1/\sqrt{nv}}(\mathrm{D}_{\sqrt{nv}}(\mu) \boxplus \lambda^{\boxplus n}) = \mu \boxplus \mathrm{D}_{1/\sqrt{nv}}(\lambda^{\boxplus n})$ weakly converge to $\mu \boxplus S(0,1)$ as $n \to \infty$. Since the set of unimodal distributions is weakly closed (see [53, Exercise 29.20]), we conclude that $\mu \boxplus S(0,1)$ is unimodal, a contradiction. $\square$

Problem 4.17. Does there exist a freely strongly unimodal probability measure that is not a delta measure?

4.4 Classical Brownian motions with initial distributions

This section discusses large time unimodality for *classical standard Brownian motion with an initial distribution* μ, which is a process with independent increments distributed as $\mu * N(0,t)$ at time $t \geq 0$. Before going to the general case, one example will be helpful in understanding unimodality for $\mu * N(0,t)$.

Example 4.18. Elementary calculus shows that

$$\frac{1}{2}(\delta_{-1} + \delta_1) * N(0,t) \tag{4.30}$$

is unimodal if and only if $t \geq 1$; see Figures 10-15.

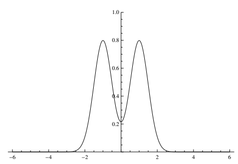

Figure 10: $t = 0.25$

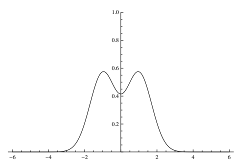

Figure 11: $t = 0.5$

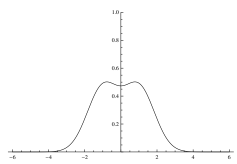

Figure 12: $t = 0.75$

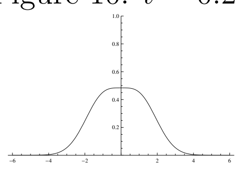

Figure 13: $t = 1$

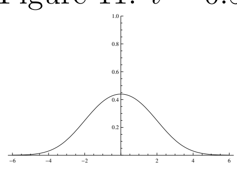

Figure 14: $t = 2$

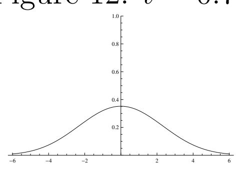

Figure 15: $t = 4$

This simple example suggests that Brownian motion becomes unimodal for sufficiently large time, possibly under some condition on the initial distribution. In some sense, we give an almost optimal condition on the initial distribution for the large time unimodality to hold. We start by providing an example of initial distribution with which the Brownian motion never becomes unimodal.

Proposition 4.19. There exists a probability measure μ on $\mathbb{R}$ such that $\mu * N(0,t)$ is not unimodal at any $t > 0$, and

$$\int_{\mathbb{R}} e^{A|x|^p} d\mu(x) < \infty \tag{4.31}$$

for all $A > 0$ and $0 < p < 2$.

Proof. We consider sequences $\{w_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ and $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\sum_{n=1}^{\infty} w_n = 1, \tag{4.32}$$

$$b_k := \inf_{n\in\mathbb{N}\setminus\{k\}} |a_k - a_n - 1| \to \infty \text{ as } k \to \infty. \tag{4.33}$$

Moreover we assume that for all $t > 0$ there exists $k_0 = k_0(t) \in \mathbb{N}$ such that

$$k \geq k_0(t) \Rightarrow w_k e^{-\frac{1}{2t}} > b_k e^{-\frac{b_k^2}{2t}}. \tag{4.34}$$

Define a probability measure μ by setting $\mu := \sum_{n=1}^{\infty} w_n \delta_{a_n}$ and the following function:

$$f_t(x) := \sqrt{2\pi t} \cdot \frac{d(\mu * N(0,t))}{dx}(x) = \sum_{n=1}^{\infty} w_n e^{-\frac{(x-a_n)^2}{2t}}, \quad x \in \mathbb{R}. \tag{4.35}$$

Then we have

$$f_t'(x) = \sum_{n=1}^{\infty} w_n \Big(-\frac{x-a_n}{t}\Big) e^{-\frac{(x-a_n)^2}{2t}}, \quad x \in \mathbb{R}. \tag{4.36}$$

If needed we may replace $k_0(t)$ by a larger integer so that $b_k > \sqrt{t}$ holds for all $k \geq k_0(t)$. For $k \geq k_0(t)$ we have

$$\begin{aligned} f_t'(a_k - 1) &= \frac{w_k}{t} e^{-\frac{1}{2t}} - \sum_{n\in\mathbb{N}, n\neq k} w_n \Big(\frac{a_k - a_n - 1}{t}\Big) e^{-\frac{(a_k-a_n-1)^2}{2t}} \\ &\geq \frac{w_k}{t} e^{-\frac{1}{2t}} - \sum_{n\in\mathbb{N}, n\neq k} w_n \frac{|a_k - a_n - 1|}{t} e^{-\frac{(a_k-a_n-1)^2}{2t}} \\ &\geq \frac{w_k}{t} e^{-\frac{1}{2t}} - \sum_{n\in\mathbb{N}, n\neq k} w_n \frac{b_k}{t} e^{-\frac{b_k^2}{2t}} \\ &\geq \frac{1}{t}\Big(w_k e^{-\frac{1}{2t}} - b_k e^{-\frac{b_k^2}{2t}}\Big) > 0, \end{aligned} \tag{4.37}$$

where we used the fact that $x \mapsto |x| e^{-\frac{x^2}{2t}}$ takes the global maximum at $x = \pm\sqrt{t}$ on the third inequality and the assumption (4.34) on the last line. This implies that $\mu * N(0,t)$ is not unimodal for any $t > 0$.

Next we take specific sequences $\{w_n\}_n$ and $\{a_n\}_n$ satisfying the conditions (4.32)–(4.34). Set $a_k = a^k$, $k \in \mathbb{N}$ where $a \geq 2$. Note that there exists some constant $c > 0$ such that $b_k \geq ca^k$ for all $k \in \mathbb{N}$. Hence we have that $b_k \to \infty$ as $k \to \infty$. Moreover we set

$$w_k := M e^{-\frac{b_k^2}{k}}, \tag{4.38}$$

where $M > 0$ is a normalized constant, that is, $M = (\sum_{k=1}^{\infty} e^{-\frac{b_k^2}{k}})^{-1}$ (note that the series converges). Since $b_k \to \infty$ as $k \to \infty$, we have

$$\frac{b_k e^{-\frac{b_k^2}{2t}}}{w_k} = \frac{1}{M} b_k e^{\frac{b_k^2}{k} - \frac{b_k^2}{2t}} \to 0, \text{ as } k \to \infty. \tag{4.39}$$

For all $t > 0$ there exists $k_0 = k_0(t) \in \mathbb{N}$ such that $k \geq k_0$ implies that $e^{-\frac{1}{2t}} > b_k e^{-\frac{b_k^2}{2t}} / w_k$. Therefore the sequences $\{w_n\}_n$ and $\{a_n\}_n$ satisfy the condition (4.34). Finally we show that $\mu = \sum_{n=1}^{\infty} w_n \delta_{a_n}$ has the property (4.31). For every $A > 0$ and $0 < p < 2$, using the inequality $b_k \geq ca^k$ shows that

$$\begin{aligned} \int_{\mathbb{R}} e^{A|x|^p} \, d\mu(x) &= \sum_{k=1}^{\infty} w_k e^{A|a_k|^p} = M \sum_{k=1}^{\infty} e^{-k^{-1} b_k^2} e^{A a^{kp}} \\ &\leq M \sum_{k=1}^{\infty} e^{-A a^{pk} (c^2 A^{-1} k^{-1} a^{(2-p)k} - 1)} < \infty. \end{aligned} \tag{4.40}$$

Thus the proof is complete. $\square$

Note that for any sequences $\{w_n\}_n$ and $\{a_n\}_n$ satisfying the conditions (4.32)–(4.34) the distribution $\mu = \sum_{n=1}^{\infty} w_n \delta_{a_n}$ has the following property:

$$\int_{\mathbb{R}} e^{\varepsilon x^2} \, d\mu(x) = \infty \quad \text{for all} \quad \varepsilon > 0. \tag{4.41}$$

Then we have a natural question whether the large time unimodality of $\mu * N(0,t)$ holds or not if the initial distribution μ does not satisfy the condition (4.41). We solve this question as follows.

Theorem 4.20. Let μ be a probability measure on $\mathbb{R}$ such that

$$\alpha := \int_{\mathbb{R}} e^{\varepsilon x^2} \, d\mu(x) < \infty. \tag{4.42}$$

for some $\varepsilon > 0$. Then $\mu * N(0,t)$ is unimodal for $t \geq \frac{36 \log(2\alpha)}{\varepsilon}$.

Remark 4.21. The proof becomes much easier if we assume that μ has a compact support.

Proof of Theorem 4.20. For $x > 0$ we have

$$\mu(|y| > x) = \int_{|y|>x} 1 \, d\mu(y) \leq \int_{|y|>x} \frac{e^{\varepsilon y^2}}{e^{\varepsilon x^2}} \, d\mu(y) \leq \alpha e^{-\varepsilon x^2}. \tag{4.43}$$

Let

$$f_t(x) := \sqrt{2\pi t} \cdot \frac{d(\mu * N(0,t))}{dx}(x) = \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2t}} \, d\mu(y). \tag{4.44}$$

Then we have

$$\begin{aligned} f_t'(x) &= \int_{\mathbb{R}} \Big(\frac{y-x}{t} \Big) e^{-\frac{(y-x)^2}{2t}} \, d\mu(y) \\ &= \int_x^{\infty} \frac{y-x}{t} e^{-\frac{(y-x)^2}{2t}} \, d\mu(y) - \int_{-\infty}^{x} \frac{x-y}{t} e^{-\frac{(x-y)^2}{2t}} \, d\mu(y). \end{aligned} \tag{4.45}$$

For $x > 0$, we have

$$\int_x^\infty \frac{y-x}{t} e^{-\frac{(y-x)^2}{2t}} \, d\mu(y) \le \frac{\sqrt{t}}{t} e^{-\frac{1}{2}} \mu((x,\infty)) \le \frac{\alpha}{\sqrt{t}} e^{-\frac{1}{2}-\varepsilon x^2}. \tag{4.46}$$

For $x \ge \frac{\sqrt{t}}{2}$, we have

$$\begin{aligned} \int_{-\infty}^x \frac{x-y}{t} e^{-\frac{(x-y)^2}{2t}} \, d\mu(y) &\ge \int_{-3x}^{x-\frac{\sqrt{t}}{3}} \frac{x-y}{t} e^{-\frac{(x-y)^2}{2t}} \, d\mu(y) \\ &\ge \frac{1}{t} \min\Big\{ \frac{\sqrt{t}}{3} e^{-\frac{1}{18}}, 4x e^{-\frac{8x^2}{t}} \Big\} \mu\bigg(\Big[-3x, x - \frac{\sqrt{t}}{3} \Big] \bigg). \end{aligned} \tag{4.47}$$

Now the function $g(x) := 4xe^{-\frac{8x^2}{t}}$ has a local maximum at $x = \frac{\sqrt{t}}{4}$, and hence for all $x \ge \frac{\sqrt{t}}{2}$,

$$g(x) \le 2e^{-2}\sqrt{t} < \frac{1}{3} e^{-\frac{1}{18}} \sqrt{t}, \tag{4.48}$$

where $2e^{-2} \approx 0.2706$ and $\frac{1}{3}e^{-\frac{1}{18}} \approx 0.3153$. Hence we have

$$\begin{aligned} \int_{-\infty}^x \frac{x-y}{t} e^{-\frac{(x-y)^2}{2t}} \, d\mu(y) &\ge \frac{4x}{t} e^{-\frac{8x^2}{t}} \mu\bigg(\Big[-3x, x - \frac{\sqrt{t}}{3} \Big] \bigg) \\ &\ge \frac{4x}{t} e^{-\frac{8x^2}{t}} \mu\bigg(\Big[-\frac{\sqrt{t}}{6}, \frac{\sqrt{t}}{6} \Big] \bigg) \\ &\ge \frac{2}{\sqrt{t}} e^{-\frac{8x^2}{t}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}), \end{aligned} \tag{4.49}$$

where the last inequality holds thanks to (4.43) and $x \ge \frac{\sqrt{t}}{2}$. Therefore if $x \ge \frac{\sqrt{t}}{2}$ then

$$\begin{aligned} f_t'(x) &\le \frac{\alpha}{\sqrt{t}} e^{-\frac{1}{2}-\varepsilon x^2} - \frac{2}{\sqrt{t}} e^{-\frac{8x^2}{t}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \\ &= \frac{\alpha}{\sqrt{t}} e^{-\frac{1}{2}-\varepsilon x^2} \bigg\{ 1 - \frac{2e^{\frac{1}{2}}}{\alpha} e^{\varepsilon x^2 - \frac{8x^2}{t}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \bigg\}. \end{aligned} \tag{4.50}$$

If $t \ge \frac{16}{\varepsilon}$ then $e^{\varepsilon x^2 - \frac{8x^2}{t}} \ge e^{\frac{1}{2}\varepsilon x^2} \ge e^{\frac{\varepsilon t}{8}}$. Hence if $t \ge \frac{16}{\varepsilon}$ then

$$f_t'(x) \le \frac{\alpha}{\sqrt{t}} e^{-\frac{1}{2}-\varepsilon x^2} \bigg\{ 1 - \frac{2e^{\frac{1}{2}}}{\alpha} e^{\frac{\varepsilon t}{8}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \bigg\}. \tag{4.51}$$

If $t \ge \frac{36\log(2\alpha)}{\varepsilon}$ then $1 - \alpha e^{-\frac{\varepsilon t}{36}} \ge \frac{1}{2}$ and we have

$$1 - \frac{2e^{\frac{1}{2}}}{\alpha} e^{\frac{\varepsilon t}{8}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \le 1 - e^{\frac{1}{2}} \cdot 2^{\frac{9}{2}} \cdot \alpha^{\frac{7}{2}} < 0. \tag{4.52}$$

By taking $t \ge \max\{\frac{16}{\varepsilon}, \frac{36\log(2\alpha)}{\varepsilon}\} = \frac{36\log(2\alpha)}{\varepsilon}$, we have

$$f_t'(x) < 0 \quad \text{if } x \ge \frac{\sqrt{t}}{2}. \tag{4.53}$$

Similarly, we have

$$f_t'(x) > 0 \quad \text{if } x \le -\frac{\sqrt{t}}{2}. \tag{4.54}$$

Next, we will show that $f_t''(x) < 0$ for all $x \in \mathbb{R}$ with $|x| < \frac{\sqrt{t}}{2}$. We then calculate the following

$$\begin{aligned} f_t''(x) &= \int_{\mathbb{R}} \frac{(x-y)^2 - t}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \\ &= \int_{|x-y|>\sqrt{t}} \frac{(x-y)^2 - t}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \\ &- \int_{|x-y|\le\sqrt{t}} \frac{t-(x-y)^2}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y). \end{aligned} \tag{4.55}$$

Since the function $h(u) := \frac{u^2-t}{t^2}e^{-\frac{u^2}{2t}}$ has a local maximum at $u = \pm\sqrt{3t}$, we have

$$\int_{|x-y|>\sqrt{t}} \frac{(x-y)^2 - t}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \le \frac{2}{t}e^{-\frac{3}{2}}\mu([x-\sqrt{t}, x+\sqrt{t}]^c). \tag{4.56}$$

For all $x \in \mathbb{R}$ with $|x| < \frac{\sqrt{t}}{2}$, we have $[x-\sqrt{t}, x+\sqrt{t}]^c \subset \left[-\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}\right]^c$, and therefore

$$\int_{|x-y|>\sqrt{t}} \frac{(x-y)^2 - t}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \le \frac{2}{t}e^{-\frac{3}{2}}\mu\Big(\Big[-\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}\Big]^c\Big) \le \frac{2}{t}e^{-\frac{3}{2}}\alpha e^{-\frac{\varepsilon t}{4}}. \tag{4.57}$$

Since the function $-h(u)$ is decreasing on $[0, \sqrt{3t}]$, we have

$$\begin{aligned} \int_{|x-y|\le\sqrt{t}} \frac{t-(x-y)^2}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) &\ge \int_{|x-y|\le\frac{2\sqrt{t}}{3}} \frac{t-(x-y)^2}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \\ &\ge \frac{t-\frac{4}{9}t}{t^2} e^{-\frac{\frac{4}{9}t}{2t}} \mu\Big(\Big[x - \frac{2\sqrt{t}}{3}, x + \frac{2\sqrt{t}}{3}\Big]\Big) \\ &= \frac{5}{9t}e^{-\frac{2}{9}}\mu\Big(\Big[x - \frac{2\sqrt{t}}{3}, x + \frac{2\sqrt{t}}{3}\Big]\Big) \\ &\ge \frac{5}{9t}e^{-\frac{2}{9}}\mu\Big(\Big[-\frac{\sqrt{t}}{6}, \frac{\sqrt{t}}{6}\Big]\Big) \\ &\ge \frac{5}{9t}e^{-\frac{2}{9}}(1 - \alpha e^{-\frac{\varepsilon t}{36}}). \end{aligned} \tag{4.58}$$

Therefore

$$\begin{aligned} f_t''(x) &\le \frac{2}{t}e^{-\frac{3}{2}}\alpha e^{-\frac{\varepsilon t}{4}} - \frac{5}{9t}e^{-\frac{2}{9}}(1-\alpha e^{-\frac{\varepsilon t}{36}}) \\ &= \frac{1}{t}\Big\{2e^{-\frac{3}{2}}\alpha e^{-\frac{\varepsilon t}{4}} - \frac{5}{9}e^{-\frac{2}{9}}(1-\alpha e^{-\frac{\varepsilon t}{36}})\Big\}. \end{aligned} \tag{4.59}$$

If $t \ge \frac{36\log(2\alpha)}{\varepsilon}$, then

$$2e^{-\frac{3}{2}}\alpha e^{-\frac{\varepsilon t}{4}} - \frac{5}{9}e^{-\frac{2}{9}}(1-\alpha e^{-\frac{\varepsilon t}{36}}) \le \frac{e^{-\frac{3}{2}}}{(2\alpha)^8} - \frac{5}{18}e^{-\frac{2}{9}} < 0, \tag{4.60}$$

and therefore $f_t''(x) < 0$ if $|x| < \frac{\sqrt{t}}{2}$.

To summarize, if $t \geq \frac{36\log(2\alpha)}{\varepsilon}$, then we have the following properties:

$$\begin{aligned} x \leq -\frac{\sqrt{t}}{2} &\Rightarrow f_t'(x) > 0, \\ x \geq \frac{\sqrt{t}}{2} &\Rightarrow f_t'(x) < 0, \\ |x| < \frac{\sqrt{t}}{2} &\Rightarrow f_t''(x) < 0. \end{aligned} \tag{4.61}$$

Hence $f_t(x)$ has a unique local maximum in $\left(-\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}\right)$ when $t \geq \frac{36\log(2\alpha)}{\varepsilon}$, which is a unique global maximum on $\mathbb{R}$ as well. Therefore $\mu * N(0,t)$ is unimodal for $t \geq \frac{36\log(2\alpha)}{\varepsilon}$. $\square$

Remark 4.22. If μ is unimodal then $\mu * N(0,t)$ is unimodal for all $t > 0$. This is a consequence of the strong unimodality of the normal distribution $N(0,t)$ (see Section 4.3.5), in contrast with the failure of freely strong unimodality of the semicircle distribution (see Lemma 4.15).

We close this section by placing a problem for future research.

Problem 4.23. Estimate the position of the mode of classical Brownian motion with initial distributions satisfying the assumption (4.42). Our proof shows that for $t \geq \frac{36}{\varepsilon}\log(2\alpha)$, the mode is located in the interval $[-\sqrt{t}/2, \sqrt{t}/2]$. How about free Brownian motion?

4.5 Large time unimodality for stable processes with index 1 and index 1/2 with initial distributions

We investigate large time unimodality for stable processes with index 1 (following Cauchy distributions) and index 1/2 (following Lévy distributions) with initial distributions.

4.5.1 Cauchy process with initial distribution

Let $\{C_t\}_{t\geq 0}$ be the symmetric Cauchy distribution

$$C_t(dx) := \frac{t}{\pi(x^2+t^2)} \cdot 1_{\mathbb{R}}(x)\,dx, \quad x \in \mathbb{R}, \qquad C_0 = \delta_0, \tag{4.62}$$

which forms both classical and free convolution semigroups. A *Cauchy process with initial distribution* μ is a process with independent increments that follows the law $\mu * C_t$ at time $t \geq 0$. It is known that the Cauchy distribution satisfies the identity

$$\mu \boxplus C_t = \mu * C_t \tag{4.63}$$

for any μ and $t \geq 0$, and so the distributions $\mu * C_t$ can also be realized as the laws at time $t \geq 0$ of a process with free independent increments with initial distribution μ. The authors do not know a written proof of (4.63) in the literature, so give a proof below. The Cauchy transform of the Cauchy distribution is given by $G_{C_t}(z) = 1/(z + \mathrm{i}t)$ on $\mathbb{C}^+$ (e.g. by the residue theorem), and hence $R_{C_t}(z) = -\mathrm{i}tz$. Then $R_{\mu\boxplus C_t}(z) = R_\mu(z) - \mathrm{i}tz$, and after some computation we can check that $G_{\mu\boxplus C_t}(z) = G_\mu(z + \mathrm{i}t)$. The Stieltjes inversion formula implies that for $t > 0$ the free convolution $\mu \boxplus C_t$ has the density

$$-\frac{1}{\pi}\mathrm{Im}(G_\mu(x + \mathrm{i}t)) = \int_{\mathbb{R}} \frac{t}{\pi((x-y)^2 + t^2)}\, d\mu(y), \tag{4.64}$$

which is exactly the density of $\mu * C_t$.

As in the cases of free and classical BMs, taking μ to be the symmetric Bernoulli $\frac{1}{2}(\delta_{-1} + \delta_1)$ is helpful. By calculus we can show that $\mu * C_t$ is unimodal if and only if $t \geq \sqrt{3}$; see Figures 16-21. Thus it is again natural to expect that a Cauchy process becomes unimodal for sufficiently large time, under some condition on the initial distribution. We start from the following counterexample.

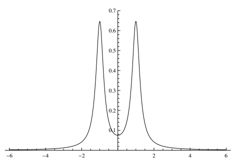
Figure 16: $t = 0.25$

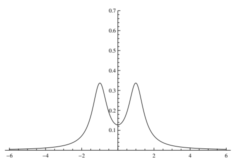
Figure 17: $t = 0.5$

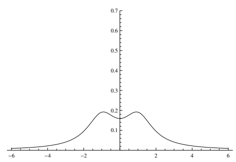
Figure 18: $t = 1$

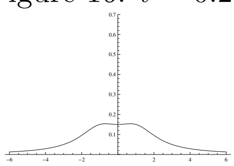
Figure 19: $t = \sqrt{2}$

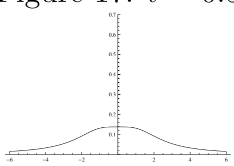
Figure 20: $t = \sqrt{3}$

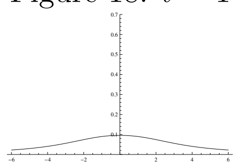
Figure 21: $t = 3$

Proposition 4.24. There exists a probability measure μ on $\mathbb{R}$ such that $\mu * C_t$ is not unimodal for any $t > 0$, and

$$\int_{\mathbb{R}} |x|^p\, d\mu(x) < \infty, \quad 0 < p < 3. \tag{4.65}$$

Proof. Let $\{w_n\}_{n\geq 1}$ be a sequence of positive numbers such that $\sum_{n=1}^\infty w_n = 1$ and $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Consider the probability measure

$$\mu = \sum_{n=1}^\infty w_n \delta_{a_n}. \tag{4.66}$$

Suppose that the sequence

$$b_k = \inf_{n\in\mathbb{N}\setminus\{k\}} |a_k - a_n - 1|, \qquad k \in \mathbb{N} \tag{4.67}$$

satisfies the condition

$$\lim_{k\to\infty} w_k b_k^3 = \infty. \tag{4.68}$$

When $\{a_n\}$ is increasing, this condition means that the distance between a_n and a_{n+1} grows sufficiently fast. Let

$$f_t(x) := \frac{\pi}{t}\frac{d(\mu * C_t)}{dx}(x) = \sum_{n=1}^{\infty} \frac{w_n}{(x-a_n)^2+t^2}. \tag{4.69}$$

Then we obtain

$$f_t'(x) = \sum_{n=1}^{\infty} \frac{-2w_n(x-a_n)}{[(x-a_n)^2+t^2]^2}, \tag{4.70}$$

and so for sufficiently large k such that $b_k > 0$ and for each $t > 0$

$$\begin{aligned} f_t'(a_k-1) &= \frac{2w_k}{(1+t^2)^2} + \sum_{n\geq 1, n\neq k} \frac{-2w_n(a_k-a_n-1)}{[(a_k-a_n-1)^2+t^2]^2} \\ &\geq \frac{2w_k}{(1+t^2)^2} - \sum_{n\geq 1, n\neq k} \frac{2w_n}{|a_k-a_n-1|^3} \\ &\geq \frac{2w_k}{(1+t^2)^2} - \sum_{n\geq 1, n\neq k} \frac{2w_n}{b_k^3} \\ &\geq \frac{2w_k}{(1+t^2)^2} - \frac{2}{b_k^3} = 2w_k\left(\frac{1}{(1+t^2)^2} - \frac{1}{w_k b_k^3}\right). \end{aligned} \tag{4.71}$$

The condition (4.68) shows that $f_t'(a_k-1)$ is positive for sufficiently large $k \in \mathbb{N}$. This shows that $\mu * C_t$ is not unimodal for any $t > 0$.

If we take the particular sequences $a_n = a^n$ and $w_n = cn^r a^{-3n}$, where $a \geq 2, r > 0$ and $c > 0$ is a normalizing constant, then the sequence b_n satisfies $b_n \geq Ca^n$ for some constant $C > 0$ independent of n. Then the conditions (4.68) and (4.65) hold true. $\square$

In the above construction, for any positive weights $\{w_n\}_n$ and any sequence $\{a_n\}_n$ that satisfies (4.68), the third moment of μ is always infinite,

$$\int_{\mathbb{R}} |x|^3 \, d\mu(x) = \infty, \tag{4.72}$$

due to the inequality $|a_k| \geq b_k - |a_1| - 1$ and the condition (4.68). The next question is then whether there exists a probability measure μ with a finite third moment such that $\mu * C_t$ is not unimodal for any $t > 0$, or at least for sufficiently large $t > 0$. The complete answer is given below.

Theorem 4.25. Let μ be a probability measure on $\mathbb{R}$ which has a finite absolute third moment

$$\beta := \int_{\mathbb{R}} |x|^3 \, d\mu(x) < \infty.$$

Then $\mu * C_t$ is unimodal for $t \geq 20\beta^{\frac{1}{3}}$.

Proof. The Markov inequality implies the tail estimate

$$\mu([-x,x]^c) \leq \frac{\beta}{x^3}, \qquad x > 0. \tag{4.73}$$

It suffices to prove that the function

$$f_t(x) := \frac{\pi}{t}\frac{d(\mu * C_t)}{dx}(x) = \int_{\mathbb{R}} \frac{1}{(x-y)^2+t^2}\, d\mu(y) \tag{4.74}$$

has a unique local maximum for large $t > 0$. Suppose first that $x > 0$. The derivative f_t' splits into the positive and negative parts

$$f_t'(x) = \int_{y>x} \frac{2(y-x)}{[(x-y)^2+t^2]^2}\, d\mu(y) - \int_{y\le x} \frac{2(x-y)}{[(x-y)^2+t^2]^2}\, d\mu(y). \tag{4.75}$$

By calculus the function $u \mapsto \frac{2u}{(u^2+t^2)^2}$ takes a global maximum at the unique point $u = t/\sqrt{3}$. Using the tail estimate (4.73) yields the estimate of the positive part

$$\int_{y>x} \frac{2(y-x)}{[(x-y)^2+t^2]^2}\, d\mu(y) \le \int_{y>x} \frac{2\frac{t}{\sqrt{3}}}{(\frac{t^2}{3}+t^2)^2}\, d\mu(y) \le \frac{\beta}{t^3x^3}. \tag{4.76}$$

On the other hand the negative part can be estimated as

$$\int_{y\le x} \frac{2(x-y)}{[(x-y)^2+t^2]^2}\, d\mu(y) \ge \int_{-x<y<x/2} \frac{2(x-y)}{[(x-y)^2+t^2]^2}\, d\mu(y). \tag{4.77}$$

Elementary calculus shows that for $-x < y < x/2$,

$$\frac{2(x-y)}{[(x-y)^2+t^2]^2} \ge \min\left\{\frac{4x}{(4x^2+t^2)^2}, \frac{x}{(x^2/4+t^2)^2}\right\}, \tag{4.78}$$

and if we further restrict to the case $x \ge t/4$, then

$$\begin{aligned} &\min\left\{\frac{4x}{(4x^2+t^2)^2}, \frac{x}{(x^2/4+t^2)^2}\right\} \\ &\quad\ge \min\left\{\frac{4x}{(4x^2+16x^2)^2}, \frac{x}{(x^2/4+16x^2)^2}\right\} \ge \frac{10^{-3}}{x^3}. \end{aligned} \tag{4.79}$$

Thus we obtain

$$\begin{aligned} \int_{y\le x} \frac{2(x-y)}{[(x-y)^2+t^2]^2}\, d\mu(y) &\ge \frac{10^{-3}}{x^3}\mu((-x/2,x/2)) \ge \frac{10^{-3}}{x^3}\left(1-\frac{\beta}{(x/2)^3}\right) \\ &\ge \frac{10^{-3}}{x^3}\left(1-\frac{8^3\beta}{t^3}\right), \qquad x \ge t/4. \end{aligned} \tag{4.80}$$

Comparing (4.76) and (4.80), taking $t \ge 20\beta^{1/3}$ guarantees that the positive part is smaller than the negative part, and hence

$$f_t'(x) < 0, \qquad x \ge \frac{t}{4}. \tag{4.81}$$

Similarly, if $t \ge 20\beta^{1/3}$ then

$$f_t'(x) > 0, \qquad x \le -\frac{t}{4}. \tag{4.82}$$

In order to show that f_t' has a unique zero, it suffices to show that $f_t''(x) < 0$ for $|x| \leq t/4$. Now we have

$$\begin{aligned} f_t''(x) = &\int_{|y-x|>t/\sqrt{3}} \frac{2[3(y-x)^2 - t^2]}{[(x-y)^2+t^2]^3}\, d\mu(y) \\ &- \int_{|y-x|\leq t/\sqrt{3}} \frac{2[t^2 - 3(y-x)^2]}{[(x-y)^2+t^2]^3}\, d\mu(y). \end{aligned} \tag{4.83}$$

The function $u \mapsto 2(3u^2 - t^2)/(u^2+t^2)^3$ attains a global maximum at $u = \pm t$ and a global minimum at $u = 0$. Therefore, the positive part can be estimated as follows:

$$\begin{aligned} \int_{|y-x|>t/\sqrt{3}} \frac{2[3(y-x)^2 - t^2]}{[(x-y)^2+t^2]^3}\, d\mu(y) &\leq \int_{|y-x|>t/\sqrt{3}} \frac{2(3t^2 - t^2)}{(t^2+t^2)^3}\, d\mu(y) \\ &= \frac{1}{2t^4}\mu\left(\left[x - \frac{t}{\sqrt{3}}, x + \frac{t}{\sqrt{3}}\right]^c\right). \end{aligned} \tag{4.84}$$

For all x such that $|x| \leq t/4$, we have the inclusion

$$\left[x - \frac{t}{\sqrt{3}}, x + \frac{t}{\sqrt{3}}\right]^c \subseteq \left[-\frac{t}{5}, \frac{t}{5}\right]^c, \tag{4.85}$$

and hence we obtain

$$\int_{|y-x|>t/\sqrt{3}} \frac{2[3(y-x)^2 - t^2]}{[(x-y)^2+t^2]^3}\, d\mu(y) \leq \frac{1}{2t^4}\mu\left(\left[-\frac{t}{5}, \frac{t}{5}\right]^c\right) \leq \frac{5^3\beta}{2t^7}. \tag{4.86}$$

On the other hand, the negative part has the estimate

$$\int_{|y-x|\leq t/\sqrt{3}} \frac{2[t^2 - 3(y-x)^2]}{[(x-y)^2+t^2]^3}\, d\mu(y) \geq \int_{|y-x|\leq t/2} \frac{2[t^2 - 3(y-x)^2]}{[(x-y)^2+t^2]^3}\, d\mu(y). \tag{4.87}$$

By calculus, the function $u \mapsto (t^2 - 3u^2)/(u^2+t^2)^3$ is decreasing on $[0, t]$, and so

$$\begin{aligned} &\int_{|y-x|\leq t/2} \frac{2[t^2 - 3(y-x)^2]}{[(x-y)^2+t^2]^3}\, d\mu(y) \\ &\quad\geq \int_{|y-x|\leq t/2} \frac{2(t^2 - 3\frac{t^2}{4})}{(\frac{t^2}{4}+t^2)^3}\, d\mu(y) = \frac{32}{125t^4}\mu\left(\left[x - \frac{t}{2}, x + \frac{t}{2}\right]\right) \\ &\quad\geq \frac{32}{125t^4}\mu\left(\left[-\frac{t}{4}, \frac{t}{4}\right]\right) \geq \frac{1}{4t^4}\left(1 - \frac{4^3\beta}{t^3}\right) \end{aligned} \tag{4.88}$$

for all $|x| \leq t/4$. The positive part (4.86) is smaller than the negative part (4.88) if we take t in such a way that $t \geq 10\beta^{1/3}$. Thus $f_t''(x) < 0$ for all $|x| \leq t/4$ and $t \geq 10\beta^{1/3}$. □

4.5.2 Positive stable process with index 1/2 with initial distribution

A positive stable process with index $1/2$ has the distribution

$$L_t(dx) := \frac{t}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{t^2}{2x}}}{x^{3/2}} \cdot 1_{(0,\infty)}(x)\, dx, \quad x \in \mathbb{R}, \tag{4.89}$$

at time $t \geq 0$ which is called the *Lévy distribution*. We restrict to the case where the initial distribution is compactly supported.

Theorem 4.26. If μ is a compactly supported on $\mathbb{R}$ with diameter D_μ then $\mu * L_t$ is unimodal for all $t \geq (90/4)^{1/4} D_\mu^{1/2}$.

Proof. By performing a translation we may assume that μ is supported on $[0,\gamma]$, where $\gamma = D_\mu$. We set the following function:

$$g_{t,y}(x) := \frac{e^{-\frac{t^2}{2(x-y)}}}{(x-y)^{3/2}} 1_{(0,\infty)}(x-y), \qquad x, y \in \mathbb{R}. \tag{4.90}$$

Note that this function is C^∞ with respect to x. Consider the following function

$$f_t(x) := \frac{\sqrt{2\pi}}{t} \cdot \frac{d(\mu * L_t)}{dx}(x) = \int_0^\gamma g_{t,y}(x)\, d\mu(y), \qquad x \in \mathbb{R}, \tag{4.91}$$

which is supported on $(0,\infty)$ and has the derivative

$$\frac{d}{dx} f_t(x) = \int_0^\gamma \frac{t^2 - 3(x-y)}{2(x-y)^{7/2}} e^{-\frac{t^2}{2(x-y)}} 1_{(0,\infty)}(x-y)\, d\mu(y). \tag{4.92}$$

If $0 < x < t^2/3$ then $t^2 - 3(x-y) > t^2 - 3 \cdot t^2/3 = 0$ for all $0 \leq y \leq \gamma$, and therefore $f_t'(x) > 0$. Moreover, if $t^2/3 + \gamma < x$ then we have that $t^2 - 3(x-y) < t^2 - 3(t^2/3 + \gamma) + 3\gamma = 0$ for all $0 \leq y \leq \gamma$, and therefore $f_t'(x) < 0$. For $t^2/3 < x < t^2/3 + \gamma$, the second derivative of f_t is given by

$$f_t''(x) = \int_0^\gamma \frac{15(x-y)^2 - 10t^2(x-y) + t^4}{4(x-y)^{11/2}} e^{-\frac{t^2}{2(x-y)}} 1_{(0,\infty)}(x-y)\, d\mu(y). \tag{4.93}$$

Note that $t^2/3 - \gamma < x - y < t^2/3 + \gamma$. Since for $X \in \mathbb{R}$

$$15X^2 - 10t^2X + t^4 < 0 \ \text{ iff } \ \frac{t^2}{3} - \frac{2t^2}{3\sqrt{10}} < X < \frac{t^2}{3} + \frac{2t^2}{3\sqrt{10}}, \tag{4.94}$$

if $\frac{2t^2}{3\sqrt{10}} \geq \gamma$ then $f_t''(x) < 0$.

To summarize, we have obtained that if $t^2 \geq \frac{3\sqrt{10}}{2}\gamma$ then

- $f_t'(x) > 0, \qquad x < t^2/3,$
- $f_t'(x) < 0, \qquad x > t^2/3 + \gamma,$
- $f_t''(x) < 0, \qquad t^2/3 < x < t^2/3 + \gamma.$

Hence f_t has a unique local maximum in $[t^2/3, t^2/3 + \gamma]$, which is a unique global maximum on $\mathbb{R}$ as well. Therefore $\mu * L_t$ is unimodal for all $t^2 \geq \frac{3\sqrt{10}}{2}\gamma$. □

We give a counterexample for large time unimodality for positive stable processes of index 1/2 when the initial distribution is not compactly supported.

Proposition 4.27. There exists a probability measure μ on $\mathbb{R}$ such that $\mu * L_t$ is not unimodal for any $t > 0$, and

$$\int_{\mathbb{R}} |x|^p\, d\mu(x) < \infty, \qquad 0 < p < \frac{5}{2}. \tag{4.95}$$

Proof. Let $\{w_n\}_{n\geq 1}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} w_n = 1$ and $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that the sequence

$$b_k = \inf_{n\in\mathbb{N}\setminus\{k\}} |a_k - a_n|, \qquad k \in \mathbb{N} \tag{4.96}$$

satisfies

$$\lim_{k\to\infty} w_k b_k^{5/2} = \infty. \tag{4.97}$$

Consider the probability measure

$$\mu = \sum_{n=1}^{\infty} w_n \delta_{a_n}. \tag{4.98}$$

Let

$$f_t(x) := \frac{\sqrt{2\pi}}{t}\frac{d(\mu * L_t)}{dx}(x) = \sum_{n\geq 1, a_n<x} w_n \frac{e^{-\frac{t^2}{2(x-a_n)}}}{(x-a_n)^{3/2}}. \tag{4.99}$$

Then we obtain

$$f_t'(x) = -\sum_{n\geq 1, a_n<x} w_n h(x-a_n), \tag{4.100}$$

where $h(x) = \frac{3x-t^2}{2x^{7/2}} e^{-\frac{t^2}{2x}}$, and for each $k \in \mathbb{N}$ and each $t > 0$

$$f_t'(a_k + t^2/6) = w_k \cdot \frac{54\sqrt{6}}{t^5} e^{-3} - \sum_{\substack{n:n\neq k\\ a_n<a_k+t^2/6}} w_n h(a_k - a_n + t^2/6). \tag{4.101}$$

Since $b_k \to \infty$ as $k \to \infty$, there exists a positive integer $k(t)$ such that $b_k + t^2/6 > \frac{5+\sqrt{10}}{15} t^2$ for all $k \geq k(t)$. For $k \geq k(t)$ and $n \neq k$ such that $a_n < a_k + t^2/6$, we can show that $a_k - a_n \geq b_k$; otherwise, by the definition of b_k it must hold that $a_k - a_n \leq -b_k < -\frac{5+2\sqrt{10}}{30} t^2$, which contradicts $a_n < a_k + t^2/6$. Since the map h is positive and strictly decreasing on $(\frac{5+\sqrt{10}}{15} t^2, \infty)$, for $k \geq k(t)$ we have

$$f_t'(a_k + t^2/6) \geq w_k \cdot \frac{54\sqrt{6}}{t^5} e^{-3} - \sum_{\substack{n:n\neq k\\ a_n<a_k+t^2/6}} w_n h(b_k + t^2/6) \tag{4.102}$$

$$\geq w_k \cdot \frac{54\sqrt{6}}{t^5} e^{-3} - h(b_k + t^2/6) \tag{4.103}$$

$$= w_k \cdot \frac{54\sqrt{6}}{t^5} e^{-3} - \frac{3b_k - t^2/2}{2(b_k + t^2/6)^{7/2}} e^{-\frac{t^2}{2(b_k+t^2/6)}}. \tag{4.104}$$

The condition (4.97) shows that $f_t'(a_k + t^2/6)$ is positive for sufficiently large $k \in \mathbb{N}$. This shows that $\mu * L_t$ is not unimodal for any $t > 0$.

If we take the particular sequences $a_k = a^k$ and $w_k = cka^{-\frac{5}{2}k}$ where $a > 2$ and $c > 0$ is a normalizing constant, then the sequence b_k satisfies $b_k \geq Ca^k$ for some constant $C > 0$. Then the condition (4.97) holds true and

$$\int_{\mathbb{R}} |x|^p \, d\mu(x) = \sum_{k\geq 1} w_k |a_k|^p = c\sum_{k\geq 1} k a^{(p-5/2)k}. \tag{4.105}$$

Hence the above integral is finite if and only if $0 < p < 5/2$. $\square$

In the above construction, for any positive weights $\{w_n\}_n$ and any sequence $\{a_n\}_n$ that satisfies (4.97), the 5/2-th moment of μ is always infinite, that is,

$$\int_{\mathbb{R}} |x|^{5/2}\, d\mu(x) = \infty. \tag{4.106}$$

We conjecture that if the integral in (4.106) is finite then $\mu * L_t$ is unimodal in large time. More generally, considering results on Cauchy processes in Section 4.5.1, it is natural to expect the following.

Conjecture 4.28. Suppose that S_t (resp. T_t) is the law at time $t \geq 0$ of a classical (resp. free) strictly stable process of index $\alpha \in (0, 2)$. If μ is a probability measure such that

$$\int_{\mathbb{R}} |x|^{2+\alpha}\, d\mu(x) < \infty, \tag{4.107}$$

then $\mu * S_t$ (resp. $\mu \boxplus T_t$) is unimodal for sufficiently large $t > 0$.

5 Unimodality for free unitary Brownian motions

5.1 Introduction and outline of results

This section discusses on [40]. This paper gives unimodality for free unitary Brownian motions and Poisson kernels with initial distributions on the unit circle. This result can be contrasted to Section 4.

In this section, we study unimodality for free multiplicative convolution with the free normal distributions on the unit circle. We write λ_t ($t > 0$) for the free normal distribution on the unit circle which is the distribution of free unitary Brownian motion at time $t > 0$ (see [20, 21]). Let $U_t \sim \lambda_t$ ($t > 0$) and $V \sim \mu$ be freely independent unitary operators in a noncommutative probability space. Then we write $\mu \boxtimes \lambda_t$ for the distribution of VU_t for $t > 0$. Recall that the operation $\boxtimes$ is called free multiplicative convolution. It can be defined on the set of general probability measures on $\mathbb{T}$. In [70], Zhong proved that for an arbitrary probability measure μ on $\mathbb{T}$ the probability measure $\mu \boxtimes \lambda_t$ is Haar absolutely continuous and its probability density function is continuous on $\mathbb{T}$ and is analytic wherever it is positive. In this paper, we study the unimodality for $\mu \boxtimes \lambda_t$ by using Zhong's density formula and we conclude the following results:

Theorem 5.1. *(See Theorems 5.22, 5.26, 5.27 and 5.29 below.)*

(1) **Unimodality for** $\mu \boxtimes \lambda_t$: let μ be a symmetric unimodal probability measure on $\mathbb{T}$. Then $\mu \boxtimes \lambda_t$ is symmetric unimodal for all $t > 0$.

(2) **Eventual unimodality for** $\mu \boxtimes \lambda_t$: let μ be a symmetric distribution on $\mathbb{T}$ supported on $\{e^{i\theta} : \theta \in [-\varphi, \varphi]\}$ for some $\varphi \in (0, \pi/2)$. Then $\mu \boxtimes \lambda_t$ is unimodal for all $t \geq \frac{2(1+r_\varphi)}{1-r_\varphi} \log \frac{1}{r_\varphi}$, where $r_\varphi = (\cos\varphi)/4$.

(3) **Non-unimodality for** $\mu \boxtimes \lambda_t$: let $\mathbf{b}$ be the Bernoulli distribution $\frac{1}{2}(\delta_{-1} + \delta_1)$ on $\mathbb{T}$. Then the density of $\mathbf{b} \boxtimes \lambda_t$ attains a strict maximum at ± 1 and hence is not unimodal for any $t > 0$.

(4) **Failure of freely strong unimodality of** λ_t: there exists some $t_0 > 0$ such that λ_t is not freely strongly unimodal at any time $t \in (0, t_0)$.

The above (1) strengthens [70, Corollary 3.13], where the statement is proved for $\mu = \delta_1$. The non-unimodality result (3) contrasts with the additive case; in the latter case the assumption of compact support of the initial distribution implied the eventual unimodality. We do not know whether the failure of freely strong unimodality can be proved for all $t > 0$.

We also prove similar statements for classical multiplicative convolution $\circledast$ on the unit circle, in particular convolution with the Poisson kernel $\mathbf{p}_{r,\psi}$ (see (2.29)).

Theorem 5.2. *(See Theorems 5.13, 5.15, 5.16 and 5.18 below.)*

(1) Let μ and ν be symmetric unimodal probability measures on $\mathbb{T}$. Then so is $\mu \circledast \nu$.

(2) **Eventual unimodality for** $\mathbf{p}_{r,\psi} \circledast \mu$: let μ be a symmetric distribution on $\mathbb{T}$ supported on $\{e^{i\theta} : \theta \in [-\varphi, \varphi]\}$ for some $\varphi \in (0, \pi/2)$. Then $\mathbf{p}_{r,\psi} \circledast \mu$ is unimodal for all $r \in (0, r_\varphi)$, where $r_\varphi = (\cos\varphi)/4$.

(3) **Non-unimdality for** $\mathbf{p}_{r,\psi} \circledast \mu$: let $\mathbf{b}$ be the Bernoulli distribution $\frac{1}{2}(\delta_{-1} + \delta_1)$ on $\mathbb{T}$ as before. Then the density of measure $\mathbf{p}_{r,\psi} \circledast \mathbf{b}$ attains a strict maximum at $\pm 1 \in \mathbb{T}$ and hence is not unimodal at any $r \in (0, 1)$.

(4) **Failure of strong unimodality of** $\mathbf{p}_{r,\psi}$: there exists some $r_0 \in (0, 1)$ such that $\mathbf{p}_{r,\psi}$ is not strongly unimodal for any $r \in (r_0, 1)$ and $\psi \in \mathbb{R}$.

In fact, some statements of Theorem 5.1 are closely related to or directly applied to Theorem 5.2, as well as they are new in classical probability to the authors' knowledge. The reason why classical convolution with Poisson kernels is related to free convolution with free normal distributions can be understood via Lemma 5.12.

After introducing basic known results on free probability in Section 5.2.1- 5.2.3, we introduce and investigate the classes of symmetric and unimodal probability measures on $\mathbb{T}$ in Sections 5.2.4 and 5.3, respectively. The above main results are proved in Section 5.4.

5.2 Preliminaries

5.2.1 Free normal distributions on the unit circle

Recall that the definition and properties of $\boxtimes$-infinitely divisible distributions are given by Section 2.4. For $t > 0$, we define λ_t as the $\boxtimes$-infinitely divisible probability measure on $\mathbb{T}$ whose Σ-transform is given by

$$\Sigma_{\lambda_t}(z) = \exp\left[\frac{t}{2} \cdot \frac{1+z}{1-z}\right], \qquad t > 0. \tag{5.1}$$

We call λ_t the *free normal distribution on the unit circle.* This is the distribution of *free unitary Brownian motion* which was introduced by Biane (e.g, see [20, 21]).

5.2.2 Zhong's density formula

Most of the materials in this section are based on Zhong's papers [69, 70]. As mentioned in Section 2.3, if μ and ν have nonzero first moments, then we can find a subordination function of $\eta_{\mu\boxtimes\nu}$ with respect to η_μ (or η_ν). However, if μ has the zero first moment (and ν has a nonzero first moment), then the subordination function of $\eta_{\mu\boxtimes\nu}$ with respect to η_μ is generally not unique (see [69, Example 3.5]). If we require a subordination function to satisfy additional properties, then it is unique even if μ has the zero first moment [9, 22]. Namely, if $\mu \neq \mathbf{h}$ is a probability measure on $\mathbb{T}$ and ν has a nonzero first moment, then there exists a unique pair of analytic functions $w_1, w_2 : \mathbb{D} \to \mathbb{D}$ such that

- $w_1(0) = w_2(0) = 0$;
- $\eta_{\mu\boxtimes\nu}(z) = \eta_\mu(w_1(z)) = \eta_\nu(w_2(z))$ for all $z \in \mathbb{D}$;
- $w_1(z)w_2(z) = z\eta_{\mu\boxtimes\nu}(z)$ for all $z \in \mathbb{D}$.

We focus on the case when ν is the free normal distribution λ_t which has the nonzero first moment e^{-t}. For a probability measure $\mu \neq \mathbf{h}$ on $\mathbb{T}$ and $t > 0$, let η_t denote the above function w_1. According to [8, Proposition 3.2], for each $t > 0$ there exists a probability measure ρ_t on $\mathbb{T}$, such that $\eta_t(z) = \eta_{\rho_t}(z)$. In [69, Lemma 3.4 and Corollary 3.13], the measures ρ_t are proved to be $\boxtimes$-infinitely divisible on $\mathbb{T}$ and its Σ-transform is expressed by

$$\Sigma_{\rho_t}(z) = \Sigma_{\lambda_t}(\eta_\mu(z)) = \exp\left[\frac{t}{2}\int_{\mathbb{T}} \frac{1+\xi z}{1-\xi z}\mu(d\xi)\right]. \tag{5.2}$$

It is proved in [8] that $\eta_t(z) = \eta_{\rho_t}(z)$ is a conformal map of $\mathbb{D}$ onto a simply connected domain $\Omega_{t,\mu} \subset \mathbb{D}$ and extends to a homeomorphism from $\mathrm{cl}(\mathbb{D})$ to $\mathrm{cl}(\Omega_{t,\mu})$, where cl is the closure of the corresponding sets. To study the measure $\mu \boxtimes \lambda_t$, it is important to describe the domain $\Omega_{t,\mu}$.

According to Zhong's paper (see [70]), we define the open set (of $\mathbb{T}$)

$$U_{t,\mu} := \left\{\theta \in [-\pi,\pi) : \int_{-\pi}^{\pi} \frac{1}{|1-e^{i(\theta+x)}|^2}\mu(dx) > \frac{1}{t}\right\}, \tag{5.3}$$

and also define a function $v_{t,\mu} : [-\pi,\pi) \to (0,1]$ as

$$v_{t,\mu}(\theta) := \sup\left\{r \in (0,1) : \frac{1-r^2}{-2\log r}\int_{-\pi}^{\pi} \frac{1}{|1-re^{i(\theta+x)}|^2}\mu(dx) < \frac{1}{t}\right\}. \tag{5.4}$$

Note that the function $v_{t,\mu}$ is continuous on $[-\pi,\pi)$, analytic in $U_{t,\mu}$ and one has $U_{t,\mu} = \{\theta \in [-\pi,\pi) : v_{t,\mu}(\theta) \neq 1\}$ by the proof of [70, Corollary 3.9]. In particular we have that $v_{t,\mu}(\theta) = v_{t,\mu}(-\theta)$ for all $\theta \in [-\pi,\pi)$ if μ is symmetric. For any $\theta \in U_{t,\mu}$, the value $v_{t,\mu}(\theta)$ is a unique solution $r \in (0,1)$ of the following equation:

$$\frac{1-r^2}{-2\log r}\int_{-\pi}^{\pi} \frac{1}{|1-re^{i(\theta+x)}|^2}\mu(dx) = \frac{1}{t}. \tag{5.5}$$

In [70, Theorem 3.2], it is proved that $\Omega_{t,\mu} = \{re^{i\theta} : 0 \leq r < v_{t,\mu}(\theta), \theta \in [-\pi,\pi)\}$ and $\partial\Omega_{t,\mu} \cap \mathbb{D} = \{v_{t,\mu}(\theta)e^{i\theta} : \theta \in U_{t,\mu}\}$. As the inverse map of η_t, the restriction of the map $\Phi_{t,\mu}(z) := z\Sigma_{\rho_t}(z)$ to $\Omega_{t,\mu}$ is conformal onto $\mathbb{D}$, and it extends to a homeomorphism of $\mathrm{cl}(\Omega_{t,\mu})$ onto $\mathrm{cl}(\mathbb{D})$. We define a homeomorphism $\Psi_{t,\mu} : \mathbb{T} \to \mathbb{T}$ by

$$\Psi_{t,\mu}(e^{i\theta}) := \Phi_{t,\mu}(v_{t,\mu}(\theta)e^{i\theta}), \qquad \theta \in [-\pi, \pi). \tag{5.6}$$

Since $|\Psi_{t,\mu}(e^{i\theta})| = 1$ and $\Phi_{t,\mu}(z) = z\Sigma_{\rho_t}(z)$, we have

$$\arg(\Psi_{t,\mu}(e^{i\theta})) = \theta + t\int_{-\pi}^{\pi} \frac{v_{t,\mu}(\theta)\sin(\theta + x)}{|1 - v_{t,\mu}(\theta)e^{i(\theta+x)}|^2}\mu(dx), \tag{5.7}$$

and hence

$$\Psi_{t,\mu}(e^{i\theta}) = \exp\left[i\left(\theta + t\int_{-\pi}^{\pi} \frac{v_{t,\mu}(\theta)\sin(\theta + x)}{|1 - v_{t,\mu}(\theta)e^{i(\theta+x)}|^2}\mu(dx)\right)\right]. \tag{5.8}$$

By [70, Proposition 3.6, Theorem 3.8], the probability measure $\mu \boxtimes \lambda_t$ is Haar absolutely continuous and its probability density function p_t is given by

$$p_t(\overline{\Psi_{t,\mu}(e^{i\theta})}) = -\frac{\log(v_{t,\mu}(\theta))}{\pi t}, \tag{5.9}$$

and it is analytic wherever it is positive. Moreover we have that

$$\mathrm{supp}(\mu \boxtimes \lambda_t) = \{\overline{\Psi_{t,\mu}(e^{i\theta})} : \theta \in \mathrm{cl}(U_{t,\mu})\}. \tag{5.10}$$

5.2.3 Free multiplicative convolution with Poisson kernels

There is no useful formula for describing the absolutely continuous part of free multiplicative convolution of general probability measures. One special case is the free multiplicative convolution with the free normal distribution, which has some implicit formula for the density as already mentioned. Another computable case is the free multiplicative convolution with the Poisson kernel.

Proposition 5.3. For every probability measure μ, every $r \in [0, 1)$ and every $\psi \in \mathbb{R}$ we have

$$\mathbf{p}_{r,\psi} \circledast \mu = \mathbf{p}_{r,\psi} \boxtimes \mu. \tag{5.11}$$

A proof can be found e.g. in [30, Section 7.1]. In Sections 5.4.2 – 5.4.3, we will discuss unimodality for classical multiplicative convolution (and hence free multiplicative convolution) with Poisson kernels.

5.2.4 Symmetric probability measures on the unit circle

In this section, we define and characterize the symmetric probability measures on $\mathbb{T}$.

Definition 5.4. A probability measure μ on $\mathbb{T}$ is said to be *symmetric* if $\mu(B) = \mu(\overline{B})$ for all Borel sets B in $\mathbb{T}$, where $\overline{B} := \{\overline{\xi} : \xi \in B\}$.

We show that the free multiplicative convolution of two symmetric probability measures on $\mathbb{T}$ is also symmetric. To prove this we characterize the class of symmetric probability measures on $\mathbb{T}$.

Proposition 5.5. For a probability measure μ on $\mathbb{T}$, the following statements are equivalent.

(1) μ is symmetric.

(2) $\psi_\mu(\overline{z}) = \overline{\psi_\mu(z)}$ for $z \in \mathbb{D}$.

(3) All moments of μ are real.

Moreover, if μ has a non-zero mean, then the above conditions are also equivalent to

(4) $\Sigma_\mu(\overline{z}) = \overline{\Sigma_\mu(z)}$ on a neighborhood of 0.

Proof. (1)⇒(2): for a symmetric probability measure μ on $\mathbb{T}$ we have

$$\begin{aligned}\psi_\mu(z) &= \int_{-\pi}^{\pi} \frac{\xi z}{1-\xi z}\mu(d\xi) \\ &= \int_{(0,\pi)} \left(\frac{\overline{\xi} z}{1-\overline{\xi} z} + \frac{\xi z}{1-\xi z}\right)\mu(d\xi) + \frac{z}{1-z}\mu(\{1\}) + \frac{-z}{1+z}\mu(\{-1\}),\end{aligned} \tag{5.12}$$

and therefore we have $\psi_\mu(\overline{z}) = \overline{\psi_\mu(z)}$.

(2)⇒(1): condition (2) implies that $\Re\psi_\mu(z) = \Re\psi_\mu(\overline{z})$. The inversion formula (2.31) shows that

$$\mu(A) = \lim_{r\uparrow 1}\int_A (2\Re(\psi_\mu(r\overline{\xi})) + 1)\,\mathbf{h}(d\xi) = \lim_{r\uparrow 1}\int_A (2\Re(\psi_\mu(r\xi)) + 1)\,\mathbf{h}(d\xi) = \mu(\overline{A})$$

for any arc $A \subset \mathbb{T}$ such that its endpoints are continuity points of μ. Therefore $\mu(B) = \mu(\overline{B})$ holds for all Borel subsets B of $\mathbb{T}$.

Conditions (2), (3) and (4) are equivalent by basic complex analysis. □

Corollary 5.6. If μ and ν are symmetric probability measures on $\mathbb{T}$, then $\mu \boxtimes \nu$ and $\mu \circledast \nu$ are also symmetric.

Proof. Any moment of $\mu \boxtimes \nu$ can be expressed as a polynomial (with real coefficients) of moments of μ and those of ν, and hence is real. The same arguments apply to $\mu \circledast \nu$. □

This corollary implies a part of Theorem 5.1: $\mu \boxtimes \lambda_t$ is symmetric whenever μ is.

5.3 Unimodal probability measures on the unit circle

In this section, we introduce the concept of unimodal probability measures on $\mathbb{T}$. Recall that unimodal distributions on $\mathbb{R}$ are defined in (4.1). In addition, unimodal distributions on $\mathbb{R}$ can be characterized from the viewpoint of geometric function theory: μ is unimodal with some mode if and only if the Cauchy transform G_μ (see (2.19)) is univalent and its range is horizontally convex, namely if z, w are points in the range of G_μ with the same imaginary part then the line segment connecting them is also contained in the range [30, Theorem 6.24].

According to [30], we similarly define the concept of unimodal probability measures on $\mathbb{T}$ as follows.

Definition 5.7. Consider $\phi, \psi \in \mathbb{R}$ with $0 \le \phi - \psi \le 2\pi$. A probability measure μ on $\mathbb{T}$ is said to be *unimodal with mode ϕ and antimode ψ* if μ is written as

$$\mu(d\theta) = c\delta_\phi + f(\theta)\, d\theta, \tag{5.13}$$

for some nonnegative number c and a function $f\colon (\psi, \psi + 2\pi) \to [0, \infty)$ which is non-decreasing on (ψ, ϕ) and non-increasing on $(\phi, \psi + 2\pi)$. In particular, if $\psi = \phi$ (resp. $\psi + 2\pi = \phi$), then we may understand that f is non-increasing (resp. non-decreasing) on $(\psi, \psi + 2\pi)$.

Note that unimodality on $\mathbb{T}$ can be characterized by the univalence (injectivity) of ψ_μ and the "vertical convexity" of the range $\psi_\mu(\mathbb{D})$, namely for any $z, w \in \psi_\mu(\mathbb{D})$ having the same real part and for any $t \in (0,1)$, we have $(1-t)z + tw \in \psi_\mu(\mathbb{D})$; see [30, Remark 7.17].

Example 5.8. The Poisson kernel $\mathbf{p}_{r,\psi}$, defined in (2.29), is unimodal with mode ψ and antimode $-\pi + \psi$. Moreover, the probability measure $p\delta_\psi + (1-p)\mathbf{p}_{r,\psi}$ is also unimodal with mode ψ and antimode $-\pi + \psi$ for $p \in (0,1)$.

For other examples, it is proved in [70, Corollary 3.13] that the free normal distribution λ_t on the unit circle is unimodal with mode 0 and antimode $-\pi$ for all $t > 0$.

It is known that the set of all unimodal probability measures on $\mathbb{R}$ is closed with respect to the weak convergence. A similar result holds for the unit circle by using [30, Lemma 2.11, Theorem 7.16].

Lemma 5.9. The set of all unimodal probability measures on $\mathbb{T}$ is closed with respect to the weak convergence.

Moreover, unimodal distributions can be approximated by smooth densities.

Lemma 5.10. For a unimodal distribution μ there is a sequence of unimodal probability measures $\{\mu_n\}_{n\ge1}$ which have C^∞ densities such that μ_n weakly converges to μ as $n \to \infty$.

Proof. We only sketch the proof which is similar to [38, Lemma 6]. Assume that μ is of the form (5.13). We may assume that the mode ϕ and the antimode ψ are different, and that the antimode ψ is $-\pi$. By approximating δ_ϕ with unimodal Haar absolutely continuous distributions supported on a small arc, we may assume that μ does not have an atom. We regard the density function f as a function on $\mathbb{R}$ supported on $[-\pi, \pi]$. By cutting f near $\pm\pi$, flattening f near ϕ and normalizing, we may assume that $f(x) = 0$ for all $|x| > \pi - \alpha$ for some $\alpha \in (0, \pi)$, and that $f(x)$ is constant for all $|x - \phi| < \beta$ for some $\beta > 0$. Then we apply a mollifier $\varphi_\varepsilon := \varepsilon^{-1}\varphi(\varepsilon^{-1}\cdot)$ with a smooth nonnegative function φ supported on $[-1, 0]$ to get a smooth function $\varphi_\varepsilon * f$ which is unimodal on (ϕ, ∞), as done in [38, Lemma 6]. A similar construction works for the opposite half-line $(-\infty, \phi)$, and combining the two we obtain a unimodal function f_ε on $\mathbb{R}$. Note that if ε is sufficiently small, then f_ε is still supported on $[-\pi, \pi]$, and hence can be regarded as a unimodal density on $\mathbb{T}$. $\square$

Unimodality can be characterized in terms of level sets under some assumptions.

Lemma 5.11. Let μ be a Haar absolutely continuous distribution on $\mathbb{T}$. Suppose that its density p extends to a continuous function on $\mathbb{T}$ and is real analytic in $\{\xi \in \mathbb{T} : p(\xi) > 0\}$. Then μ is unimodal if and only if, for any $a > 0$, the equation $p(\xi) = a$ has at most two solutions $\xi \in \mathbb{T}$.

Proof. Assume that μ is unimodal with mode ϕ and antimode ψ. Since p is continuous we have $0 < \phi - \psi < 2\pi$. We identify the function p on $\mathbb{T}$ with a function on $[\psi, \psi + 2\pi)$. Since p is real analytic wherever it is positive, the function p is strictly increasing on $\{\theta \in (\psi, \phi) : p(\theta) > 0\}$ and strictly decreasing on $\{\theta \in (\phi, \psi + 2\pi) : p(\theta) > 0\}$.

Case 1. If $a > p(\phi)$ or $0 < a < p(\psi)$, then the equation $p(\theta) = a$ has no solutions $\theta \in [\psi, \psi + 2\pi)$.

Case 2. If $a = p(\phi)$, then the equation $p(\theta) = a$ has only one solution, since $p(\phi)$ is the strict maximum.

Case 3. If $a = p(\psi) > 0$, then the equation $p(\theta) = a$ has only one solution since $p(\psi)$ is the strict minimum.

Case 4. If $p(\psi) < a < p(\phi)$, then by the intermediate value theorem and the monotonicity the equation $p(\theta) = a$ has a unique solution in each of $[\psi, \phi)$ and $(\phi, \psi + 2\pi)$.

In any case, we conclude that the equation $p(\theta) = a$ has at most two solutions θ in $[\psi, \psi + 2\pi)$.

Conversely, we assume that μ is not unimodal. Let ψ and ϕ ($0 < \phi - \psi < 2\pi$) be points where p takes a global minimum and a (strict) global maximum, respectively. Then either p is not non-decreasing on (ψ, ϕ) or not non-increasing on $(\phi, \psi + 2\pi)$. Without loss of generality we may focus on the former case. Then there exist $\psi < \theta_1 < \theta_2 < \phi$ such that $p(\psi) < p(\theta_2) < p(\theta_1) < p(\phi)$. For any $a \in (p(\theta_2), p(\theta_1))$, by the intermediate theorem the equation $p(\theta) = a$ has at least one solution in each of the intervals (ψ, θ_1), (θ_1, θ_2) and (θ_2, ϕ). Therefore the equation $p(\theta) = a$ has at least three solutions. $\square$

Combining Zhong's density formula and Lemma 5.11 one can characterized the unimodality for $\mu \boxtimes \lambda_t$ as follows.

Lemma 5.12. Suppose that $t > 0$ and μ is a probability measure on $\mathbb{T}$. The following statements are equivalent.

(1) $\mu \boxtimes \lambda_t$ is unimodal.

(2) For any $r \in (0, 1)$ the equation

$$\frac{d(\mathbf{p}_{0,r} \circledast \mu)}{d\theta}(\theta) = \frac{-\log r}{\pi t} \tag{5.14}$$

has at most two solutions $\theta \in [-\pi, \pi)$.

Proof. This proof is very similar to [38, Proposition 3.8] or [39, Lemma 3.1]. Recall that $\mu \boxtimes \lambda_t$ is Haar absolutely continuous, its density function p_t is continuous on $\mathbb{T}$ and real analytic in $\overline{\Psi_{t,\mu}(U_{t,\mu})}$ by Section 5.2.2. Assume firstly the condition (2). Since $\Psi_{t,\mu}$ is a homeomorphism of $\mathbb{T}$, it suffices to show that for any $a > 0$ the equation

$$a = p_t(\overline{\Psi_{t,\mu}(e^{i\theta})}) = -\frac{\log(v_{t,\mu}(\theta))}{\pi t}, \qquad \theta \in [-\pi, \pi) \tag{5.15}$$

has at most two solutions $\theta \in [-\pi, \pi)$ by Lemma 5.11. Since $v_{t,\mu}(\theta)$ is a unique solution of the equation (5.5) when it is positive, then we have

$$\begin{aligned} &\left\{\theta \in [-\pi,\pi) : a = -\frac{\log(v_{t,\mu}(\theta))}{\pi t}\right\} \\ &\quad = \left\{\theta \in [-\pi,\pi) : \frac{1-(e^{-\pi a t})^2}{-2\log(e^{-\pi a t})} \int_{[-\pi,\pi)} \frac{d\mu(x)}{|1 - e^{-\pi a t} e^{i(x+\theta)}|^2} = \frac{1}{t}\right\} \\ &\quad = \left\{\theta \in [-\pi,\pi) : \frac{d(\mathbf{p}_{0,e^{-\pi a t}} \circledast \mu)}{d\theta}(-\theta) = \frac{-\log(e^{-\pi a t})}{\pi t}\right\} \end{aligned} \tag{5.16}$$

By the assumption (2) the equation $a = -\frac{\log(v_{t,\mu}(\theta))}{\pi t}$ has at most two solutions $\theta \in [-\pi, \pi)$. Conversely, assume that the condition (2) does not hold. Then for some $r \in (0,1)$ there are three distinct solutions $\theta_1, \theta_2, \theta_3 \in [-\pi, \pi)$ to the equation (5.14). By (5.16) this shows that $v_{t,\mu}(-\theta_k) = r$ and hence $p_t(\overline{\Psi_{t,\mu}(e^{-i\theta_k})}) = -\frac{\log r}{\pi t}$ for $a = -\log r/(\pi t)$ and $k = 1, 2, 3$. Therefore $\mu \boxtimes \lambda_t$ is not unimodal by Lemma 5.11 again. $\square$

5.4 Main results

In Sections 5.4.1 – 5.4.3, we prove the main theorem announced as Theorem 5.2, namely: in Section 5.4.1, we study unimodality for the multiplicative convolution of two symmetric probability measures on $\mathbb{T}$; in Section 5.4.2, we discuss when $\mathbf{p}_{r,\psi} \circledast \mu$ is unimodal for sufficiently small $r \in (0,1)$ and when it is not unimodal at any time $t > 0$; in Section 5.4.3, we prove that $\mathbf{p}_{r,\psi}$ is not strongly unimodal for any $r \in (0,1)$ sufficiently close to 1.

In Sections 5.4.4 – 5.4.6, we prove the main theorem announced as Theorem 5.1, namely: in Sections 5.4.4 and 5.4.5 we discuss when $\mu \boxtimes \lambda_t$ is unimodal for all $t > 0$, when it is unimodal for sufficiently large $t > 0$, and when it is not unimodal at any $t > 0$; in Section 5.4.6, we conclude the failure of freely strong unimodality for λ_t at sufficiently small $t > 0$.

5.4.1 Multiplicative convolution of symmetric unimodal distributions

In this section, we study unimodality for the multiplicative convolution of two symmetric unimodal probability measures. It was proved by Wintner that classical additive convolution $*$ preserves the symmetric unimodal probability measures on $\mathbb{R}$ (see [53, Exercise 29.22] or the original article [66, Theorem XIII]). We conjectured a similar property of free additive convolution $\boxplus$ in [39] or the last section, which is still open. We now prove the corresponding property for classical multiplicative convolution $\circledast$ on $\mathbb{T}$.

Theorem 5.13. Let μ and ν be symmetric unimodal probability measures on $\mathbb{T}$. Then so is $\mu \circledast \nu$.

Proof. The symmetry follows from Corollary 5.6 (one can also prove it more directly by formula (5.18) below). We will prove the unimodality. By Lemma 5.9 and Lemma 5.10, we may assume that μ and ν respectively have C^1 densities f and g on $\mathbb{T}$. We extend f and g to even continuous functions on $\mathbb{R}$ which have period 2π, that is, $f(\theta+2\pi) = f(\theta)$ and $g(\theta+2\pi) = g(\theta)$ for all $\theta \in \mathbb{R}$.

Let $a = -f'$ on $(0,\pi)$. Then $a \geq 0$ and

$$f(\theta) = \int_{|\theta|}^{\pi} a(\psi)\, d\psi, \qquad \theta \in (-\pi,\pi). \tag{5.17}$$

Let h be the density of $\mu \circledast \nu$. For $\varphi \in (0,\pi)$ we have

$$\begin{aligned} h(\varphi) &= \int_{-\pi}^{\pi} f(\theta) g(\varphi-\theta)\, d\theta = \int_{-\pi}^{\pi} \left(\int_{|\theta|}^{\pi} a(\psi)\, d\psi \right) g(\varphi-\theta) d\theta \\ &= \int_0^{\pi} a(\psi)\, d\psi \int_{-\psi}^{\psi} g(\varphi-\theta) d\theta. \end{aligned} \tag{5.18}$$

Fix $\psi \in (0,\pi)$ and let $k(\varphi) := \int_{-\psi}^{\psi} g(\varphi-\theta) d\theta = \int_{\varphi-\psi}^{\varphi+\psi} g(\theta)\, d\theta$ for $\varphi \in (0,\pi)$. Then

$$k'(\varphi) = g(\varphi+\psi) - g(\varphi-\psi). \tag{5.19}$$

In order to show $h' \leq 0$ on $(0,\pi)$, we show that $k' \leq 0$ on $(0,\pi)$.

Case 1: $\varphi+\psi \leq \pi$. Then $k' \leq 0$ because $\varphi+\psi \geq |\varphi-\psi|$.

Case 2: $\varphi+\psi \geq \pi$. Then $2\pi-\varphi-\psi \leq \pi$ and

$$k'(\varphi) = g(2\pi-\varphi-\psi) - g(\varphi-\psi), \tag{5.20}$$

since $g(\varphi+\psi) = g(-\varphi-\psi) = g(2\pi-\varphi-\psi)$. We can check that $2\pi-\varphi-\psi \geq |\varphi-\psi|$, and hence $g(\varphi-\psi) \geq g(2\pi-\varphi-\psi)$.

Therefore h is non-increasing on $(0,\pi)$ and also non-decreasing on $(-\pi,0)$ since $\mu \circledast \nu$ is symmetric. Hence $\mu \circledast \nu$ is unimodal. $\square$

So far we found no counterexample to the corresponding conjecture for $\boxtimes$.

Conjecture 5.14. Let μ and ν be symmetric unimodal distributions on $\mathbb{T}$. Then $\mu \boxtimes \nu$ is also unimodal.

Thanks to Proposition 5.3 and Theorem 5.13, this conjecture holds when one of the distributions is the Poisson kernel. In Section 5.4.4 the above conjecture is proved when one of the probability measures is the free normal distribution.

5.4.2 Eventual unimodality and non-unimodality for $\mathbf{p}_{r,\psi} \circledast \mu$

Firstly, we give a class of probability distributions μ on $\mathbb{T}$ such that the distribution $\mathbf{p}_{r,\psi} \circledast \mu$ is unimodal for sufficiently small $r \in (0,1)$.

Theorem 5.15. Let $\varphi \in (0, \pi/2)$ and μ be a symmetric distribution on $\mathbb{T}$ such that $\mathrm{supp}(\mu) \subset \{e^{i\theta} : \theta \in [-\varphi, \varphi]\}$. Then $\mathbf{p}_{r,\psi} \circledast \mu$ is unimodal for $r \in (0, r_\varphi)$ where $r_\varphi := (\cos\varphi)/4$.

Proof. We may assume that $\psi = 0$. The density function f_r of $\mathbf{p}_{r,0} \circledast \mu$ is given by

$$f_r(\theta) = \frac{1-r^2}{2\pi}\left(\int_0^\varphi \frac{d\mu(x)}{1-2r\cos(\theta-x)+r^2} + \int_0^\varphi \frac{d\mu(x)}{1-2r\cos(\theta+x)+r^2}\right), \qquad \theta \in [-\pi, \pi). \tag{5.21}$$

For all $\theta \in (0, \pi)$, we have

$$f_r'(\theta) = -\frac{2r(1-r^2)\sin\theta}{\pi}\int_0^\varphi \frac{L_r(\theta, x)}{(1-2r\cos(\theta-x)+r^2)^2(1-2r\cos(\theta+x)+r^2)^2}d\mu(x), \tag{5.22}$$

where

$$L_r(\theta, x) := \left\{(1+r^2)^2 + 4r^2(1-\sin(\theta-x)\sin(\theta+x))\right\}\cos x - 4r(1+r^2)\cos\theta. \tag{5.23}$$

If $\theta \in (\pi/2, \pi)$, then it is clear that $L_r(\theta, x) > 0$ for all $x \in (0, \varphi)$ and hence $f_r'(\theta) < 0$. If $\theta \in (0, \pi/2)$, then for all $r \in (0, r_\varphi)$ and $x \in (0, \varphi)$, we have

$$\begin{aligned}
L_r(\theta, x) &\geq (1+r^2)^2\cos x - 4r(1+r^2)\cos\theta && (5.24)\\
&\geq (1+r^2)^2\cos x - 4r(1+r^2) && (5.25)\\
&\geq (1+r^2)(\cos x - 4r) && (5.26)\\
&\geq (1+r^2)(\cos\varphi - 4r) && (5.27)\\
&= 4(1+r^2)(r_\varphi - r) > 0. && (5.28)
\end{aligned}$$

Therefore $L_r(\theta, x) > 0$ for all $r \in (0, r_\varphi)$ and $x \in (0, \varphi)$. Thus $f_r'(\theta) < 0$ for $\theta \in (0, \pi/2)$ if $0 < r < r_\varphi$. The above arguments show that f_r is non-increasing on $(0, \pi)$ if $0 < r < r_\varphi$. Since $\mathbf{p}_{r,0} \circledast \mu$ is symmetric, the above result implies that $\mathbf{p}_{r,0} \circledast \mu$ is unimodal for $r \in (0, r_\varphi)$. □

Next we prove that the above theorem does not hold for $\varphi = \pi/2$. In fact, the equally weighted Bernoulli distribution $\mathbf{b}$ on $\{1, -1\} \subset \mathbb{T}$ is a counterexample of the above theorem for $\varphi = \pi/2$.

Theorem 5.16. The density of the measure $\mathbf{p}_{r,\psi} \circledast \mathbf{b}$ attains a strict maximum at the two points $\pm e^{i\psi}$ and hence is not unimodal at any $r \in (0, 1)$.

Proof. We may assume that $\psi = 0$. The density function of $\mathbf{p}_{r,0} \circledast \mathbf{b}$ is computed in the form

$$f_r(\theta) = \frac{1-r^2}{2\pi}\left(\frac{1}{1-2r\cos\theta+r^2} + \frac{1}{1+2r\cos\theta+r^2}\right), \qquad \theta \in [-\pi, \pi). \tag{5.29}$$

For all $\theta \in (0, \pi)$, we have

$$f_r'(\theta) = \frac{1-r^2}{2\pi}\cdot\frac{-8r^2(1+r^2)\sin(2\theta)}{(1-2r\cos\theta+r^2)^2(1+2r\cos\theta+r^2)^2}. \tag{5.30}$$

This formula easily shows that the function f_r has a global maximum at $\theta = 0, \pi$ and a local minimum at $\theta = \pm\pi/2$. Therefore $\mathbf{p}_{r,\psi} \circledast \mathbf{b}$ is not unimodal at any $r \in (0, 1)$. □

5.4.3 Failure of strong unimodality of $\mathbf{p}_{r,\psi}$

We prove that the Poisson kernel $\mathbf{p}_{r,\psi}$ is not strongly unimodal for $r \in (0,1)$ sufficiently close to 1. This is a key fact to prove that the free normal distribution λ_t is not freely strongly unimodal for sufficiently small $t > 0$ in Section 5.4.6. We start from defining the concept of classically and freely strong unimodality (we use the concept of freely strong unimodality in Section 5.4.6 later).

Definition 5.17. A probability measure μ on $\mathbb{T}$ is said to be *strongly unimodal* (resp. *freely strongly unimodal*) if $\mu \circledast \nu$ (resp. $\mu \boxtimes \nu$) is unimodal for every unimodal probability measure ν on $\mathbb{T}$.

Theorem 5.18. There exists $r_0 \in (0,1)$ such that the Poisson kernel $\mathbf{p}_{r,\psi}$ is not strongly unimodal for any $r \in [r_0, 1)$ and any $\psi \in \mathbb{R}$. More strongly, there exists a probability measure μ and a continuous function $h\colon [r_0,1) \to (0,\infty)$ such that $\limsup_{r\to 1} h(r) \in (0,\infty]$ and for each $r \in [r_0,1)$ and $\psi \in \mathbb{R}$ the equation

$$h(r) = \frac{d(\mathbf{p}_{r,\psi} \circledast \mu)}{d\theta}(\theta)$$

has at least three solutions $\theta \in [-\pi, \pi)$.

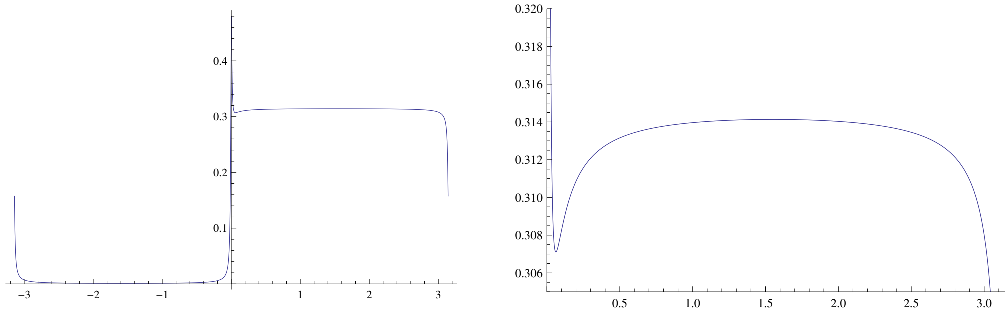

Figure 22: The density of $\mathbf{p}_{0.99,0} \circledast \mu_{0.01}$

Proof. We may assume that $\psi = 0$. Let $\mu = \mu_a$ be the probability measure defined by

$$\mu_a(d\theta) = a\delta_0 + b\mathbf{1}_{(0,\pi)}(\theta)\,d\theta, \tag{5.31}$$

where a, b are fixed positive real numbers such that $a + b\pi = 1$. As we see later, we want a to satisfy $a^2/(4b^2) < 1/27$; for example $a = 1/20$. Let f_r and p_r be the density functions of $\mathbf{p}_{r,0} \circledast \mu_a$ and $\mathbf{p}_{r,0}$, respectively. Then we have

$$f_r(\theta) = ap_r(\theta) + b\int_{-\pi}^{\pi} \mathbf{1}_{(0,\pi)}(\theta - \varphi)p_r(\varphi)\,d\varphi = ap_r(\theta) + b\int_{\theta-\pi}^{\theta} p_r(\varphi)\,d\varphi \tag{5.32}$$

for all $\theta \in (0,\pi)$. Therefore, we have

$$\begin{aligned} f_r'(\theta) &= ap_r'(\theta) + b[p_r(\theta) - p_r(\theta - \pi)] && (5.33)\\ &= \frac{1-r^2}{2\pi}\cdot\frac{g_r(\theta)}{(1+r^2-2r\cos\theta)^2(1+r^2+2r\cos\theta)}, \qquad \theta \in (0,\pi), && (5.34)\end{aligned}$$

where

$$g_r(\theta) = -2ra\sin\theta(1+r^2+2r\cos\theta) + b(1+r^2-2r\cos\theta)(1+r^2+2r\cos\theta) \\ - b(1+r^2-2r\cos\theta)^2. \tag{5.35}$$

In order to prove that g_r has several zeros, we extend the parameter r to the interval $[0,1]$, and then take $r=1$ to simplify g_r:

$$g_1(\theta) = 4[-a\sin\theta(1+\cos\theta) + 2b\cos\theta(1-\cos\theta)]. \tag{5.36}$$

Note that $g_1(0)=0$, $g_1'(0)=-8a<0$ and $g_1(\pi)<0$. It is clear that $g_1<0$ on $[\pi/2,\pi)$. By trigonometry, for $\theta\in(0,\pi/2)$ the inequality $g_1(\theta)<0$ is equivalent to

$$\frac{a^2}{4b^2} > \frac{\cos^2\theta(1-\cos\theta)}{(1+\cos\theta)^3}. \tag{5.37}$$

The function on the RHS increases on $(0,\pi/3)$ and decreases on $(\pi/3,\pi/2)$, and attains the maximum $1/27$ at $\theta=\pi/3$. Thus, if $a^2/(4b^2)<1/27$ (for example $a=1/20$ suffices) then we obtain two solutions α_1,β_1 of the equation

$$\frac{a^2}{4b^2} = \frac{\cos^2\theta(1-\cos\theta)}{(1+\cos\theta)^3}, \qquad \theta\in[0,\pi/2], \tag{5.38}$$

such that $0<\alpha_1<\pi/3<\beta_1<\pi/2$ and $g_1>0$ on (α_1,β_1) and $g_1<0$ on $(0,\alpha_1)\cup(\beta_1,\pi)$. Each of the zeros $0,\alpha_1,\beta_1$ of g_1 has multiplicity one.

Since $g_r(\theta)$ depends continuously on $r\in[0,1]$ and analytically on θ, by the argument principle, there exists $r_1\in(0,1)$ such that the function g_r still has three zeros $-0.1<\gamma_r<\alpha_r<\beta_r<\pi/2+0.1$ for every $r\in(r_1,1)$. This implies that f_r takes local maxima at γ_r and β_r and a local minimum at α_r. The zeros are continuous functions of $r\in[r_1,1]$. Since $g_1'(0)=-8a<0$, $g_1(0)=0$ and $g_r(0)>0$ for every $r\in[0,1)$, we must have $\gamma_r>0$.

The zero β_r depends on $r\in[r_1,1]$ continuously, and in particular $\lim_{r\to1}\beta_r=\beta_1\in(\pi/3,\pi/2)$. Therefore we have $\beta_r\in(\pi/3,\pi/2)$ if r is close enough to 1, and hence

$$b\int_{\beta_r-\pi}^{\beta_r} p_r(\varphi)\,d\varphi \le f_r(\beta_r) \le ap_r\left(\frac{\pi}{3}\right) + b\int_{\beta_r-\pi}^{\beta_r} p_r(\varphi)\,d\varphi. \tag{5.39}$$

Taking the limit $r\to1$ we conclude that

$$\lim_{r\to1} f_r(\beta_r) = b. \tag{5.40}$$

On the other hand, the local maximum at γ_r can be estimated from below as

$$f_r(\gamma_r) \ge f_r(0) \ge ap_r(0) = \frac{a(1+r)}{2\pi(1-r)}. \tag{5.41}$$

Therefore, if r is close enough to 1 then $f_r(\gamma_r)>f_r(\beta_r)$, and so the equation $f_r(\theta)=h$ has at least three solutions $\theta\in(0,\pi)$ for every $h\in(f_r(\alpha_r),f_r(\beta_r))$ (see Figure 22). Thus there exist $r_0\in(0,1)$ and a continuous function $h\colon[r_0,1)\to(0,\infty)$ such that the equation $f_r(\theta)=h(r)$ has at least three solutions $\theta\in(0,\pi)$ for each r, and $h(r)\to b$ as $r\to1$; for example we may define $h(r)=\max\{\frac{1}{2}[f_r(\alpha_r)+f_r(\beta_r)], f_r(\beta_r)-(1-r)\}$. $\square$

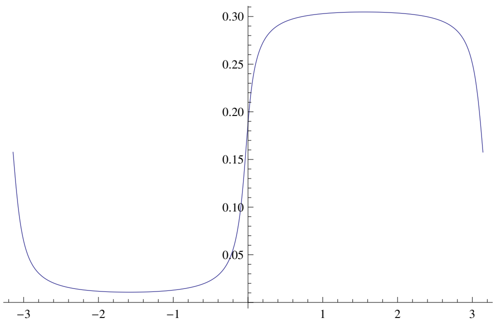

Figure 23: The density of $\mathbf{p}_{0.9,0} \circledast \mu_{0.01}$

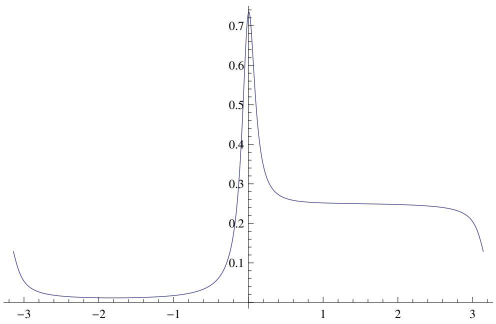

Figure 24: The density of $\mathbf{p}_{0.9,0} \circledast \mu_{0.2}$

Remark 5.19. Numerical simulations suggest that $\mathbf{p}_{0.9,0} \circledast \mu_a$ is already unimodal for all $a \in (0,1)$; see Figures 23 and 24.

We do not know whether $\mathbf{p}_{r,0}$ is strongly unimodal for r smaller than r_0.

Problem 5.20. Does there exist $r \in (0,1)$ such that the Poisson kernel $\mathbf{p}_{r,0}$ is strongly unimodal? Note that $\mathbf{p}_{0,0}$ is the normalized Haar measure and the weak limit $\lim_{r\uparrow 1} \mathbf{p}_{r,0}$ is the delta measure δ_1, and hence both are strongly unimodal.

A more challenging problem is to prove the analogue of Ibragimov's theorem, which characterizes the strongly unimodal distributions on $\mathbb{R}$ by the log concavity of the density (see [53, Theorem 52.3]; the original article is [46]).

Problem 5.21. Characterize the classically strong unimodality on $\mathbb{T}$. In particular, is the classical normal distribution (= the heat kernel) on $\mathbb{T}$ with the density

$$\frac{1}{\sqrt{2\pi t}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{(\theta+2\pi n)^2}{2t}\right), \qquad \theta \in [-\pi,\pi), t>0, \tag{5.42}$$

strongly unimodal?

5.4.4 Unimodality for $\mu \boxtimes \lambda_t$

In this section, we study unimodality for $\mu \boxtimes \lambda_t$ when μ is symmetric unimodal.

Theorem 5.22. If μ is a symmetric unimodal probability measure on $\mathbb{T}$, then $\mu \boxtimes \lambda_t$ is symmetric and unimodal for any $t > 0$.

Proof. Since μ and λ_t are symmetric, the probability measure $\mu \boxtimes \lambda_t$ is also symmetric by Corollary 5.6. To show the unimodality, it suffices by Lemma 5.12 to show that for any $r \in (0,1)$ and $t > 0$ the equation (5.14) has at most two solutions $\theta \in [-\pi,\pi)$. By Theorem 5.13, the measure $\mathbf{p}_{r,0} \circledast \mu$ is unimodal for any $r \in (0,1)$, and by Lemma 5.11 the equation (5.14) has at most two solutions $\theta \in [-\pi,\pi)$ for any $r \in (0,1)$ and $t > 0$. Hence $\mu \boxtimes \lambda_t$ is unimodal for any $t > 0$. $\square$

5.4.5 Eventual unimodality and non-unimodality for $\mu \boxtimes \lambda_t$

We firstly prepare the following functions to describe the density of $\mu \boxtimes \lambda_t$. For $\theta \in [-\pi, \pi)$ and $r \in (0,1)$ define the following two functions on $[-\pi, \pi)$.

$$\Gamma_{\theta,r}(x) := \frac{\sin(\theta + x)}{(1 - 2r\cos(\theta + x) + r^2)^2}, \tag{5.43}$$

$$\Xi_{\theta,r}(x) := \frac{2r(1 + \log r - r^2 + r^2 \log r)\cos(\theta + x) + r^4 - 4r^2 \log r - 1}{(1 - 2r\cos(\theta + x) + r^2)^2}. \tag{5.44}$$

Those functions will play a key role in analyzing the density of $\mu \boxtimes \lambda_t$ as the following lemma shows. Actually, it serves as an alternative of Lemma 5.12; one can choose a convenient one from the two lemmas, according to a problem to be considered.

Lemma 5.23. For arbitrary numbers $\theta \in [-\pi, \pi)$ and $r \in (0,1)$, the function $\Xi_{\theta,r}$ takes negative values on $[-\pi, \pi)$. Moreover, for each $t > 0$ we have

$$\frac{d}{d\theta} v_{t,\mu}(\theta) = \frac{2v_{t,\mu}(\theta)^2(1 - v_{t,\mu}(\theta)^2)\log(v_{t,\mu}(\theta))}{\int_{-\pi}^{\pi} \Xi_{\theta, v_{t,\mu}(\theta)}(x)\,\mu(dx)} \int_{-\pi}^{\pi} \Gamma_{\theta, v_{t,\mu}(\theta)}(x)\,\mu(dx), \qquad \theta \in U_{t,\mu}. \tag{5.45}$$

Proof. By calculus we can check that $1 + \log r - r^2 + r^2 \log r < 0$ for all $r \in (0,1)$. Hence we have

$$\begin{aligned} &2r(1 + \log r - r^2 + r^2 \log r)\cos(\theta + x) + r^4 - 4r^2 \log r - 1 && (5.46)\\ &\le -2r(1 + \log r - r^2 + r^2 \log r) + r^4 - 4r^2 \log r - 1 && (5.47)\\ &= (1 + r)^2(-2r \log r + r^2 - 1). && (5.48) \end{aligned}$$

By calculus we can prove that the RHS is negative, and so is $\Xi_{\theta,r}(x)$.

Recall that the function $v_{t,\mu}$ is analytic in $U_{t,\mu}$ and satisfies $v_{t,\mu}(\theta) \in (0,1)$ and the equation $R(\theta, v_{t,\mu}(\theta)) = 0$ for all $\theta \in U_{t,\mu}$, where

$$R(\theta, r) := \frac{1 - r^2}{-2\log r} \int_{-\pi}^{\pi} \frac{\mu(dx)}{1 - 2r\cos(\theta + x) + r^2} - \frac{1}{t}, \qquad r \in (0,1), \theta \in U_{t,\mu}. \tag{5.49}$$

In order to compute $\frac{d}{d\theta} v_{t,\mu}(\theta)$, we calculate partial derivatives of R which can be expressed with the functions $\Gamma_{\theta,r}$ and $\Xi_{\theta,r}$ as follows:

$$\frac{\partial R}{\partial \theta}(\theta, r) = \frac{r(1 - r^2)}{\log r} \int_{-\pi}^{\pi} \Gamma_{\theta,r}(x)\,\mu(dx), \tag{5.50}$$

and

$$\frac{\partial R}{\partial r}(\theta, r) = -\frac{1}{2r(\log r)^2} \int_{-\pi}^{\pi} \Xi_{\theta,r}(x)\,\mu(dx). \tag{5.51}$$

Since $\frac{\partial R}{\partial r}(\theta, r) > 0$, we can use the differentiation of implicit functions to obtain the desired expression of $v'_{t,\mu}$. $\square$

In Lemma 5.23, the denominator in the expression (5.45) is always strictly negative, that is,

$$\int_{-\pi}^{\pi} \Xi_{\theta, v_{t,\mu}(\theta)}(x)\mu(dx) < 0, \tag{5.52}$$

for all $\theta \in U_{t,\mu}$. Therefore, it is enough to check only the sign of $\int_{-\pi}^{\pi} \Gamma_{\theta, v_{t,\mu}(\theta)}(x)\mu(dx)$ to know the one of the first derivative of $v_{t,\mu}(\theta)$. The sign of $\frac{d}{d\theta} v_{t,\mu}(\theta)$ is used in the proof of Theorems 5.26 and 5.27.

Lemma 5.24. Let $t > 4$. Let $r(t) \in (0,1)$ be the unique solution r to the equation

$$\frac{1+r}{1-r}(-2\log r) = t.$$

Then $v_{t,\mu}(\theta) \leq r(t)$ for every $\theta \in [-\pi, \pi)$ and every probability measure μ on $\mathbb{T}$. In particular, $U_{t,\mu} = \mathbb{T}$ and $\lambda_t \boxtimes \mu$ has a strictly positive real analytic density on $\mathbb{T}$.

Remark 5.25. The opposite inequality of the form $v_{t,\mu}(\theta) \geq a_t$ was obtained in [70, Proposition 3.10] in a similar way.

Proof. By the triangle inequality we obtain

$$\frac{1-r^2}{-2\log r} \int_{\mathbb{T}} \frac{d\mu(\xi)}{|1 - re^{i\theta}\xi|^2} \geq \frac{1-r^2}{-2\log r} \cdot \frac{1}{(1+r)^2} = \frac{1-r}{(1+r)(-2\log r)} =: f(r). \tag{5.53}$$

Elementary calculus shows that f is strictly increasing on $(0,1)$ and $f(+0) = 0$ and $f(1-0) = 1/4$. Therefore, for every $t > 4$ the unique solution $r(t)$ exists. Since $f(r(t)) = 1/t$, the LHS of (5.53) is greater than or equal to $1/t$ at $r = r(t)$, which implies that $v_{t,\mu}(\theta) \leq r(t)$. Since $v_{t,\mu}(\theta) < 1$ the last statement holds. $\square$

We give a class of probability distributions μ on $\mathbb{T}$ such that the distribution $\mu \boxtimes \lambda_t$ is unimodal for sufficiently large time. In fact the class considered in Theorem 5.15 is available.

Theorem 5.26. Let $\varphi \in (0, \pi/2)$ and μ be a symmetric probability measure on $\mathbb{T}$ such that $\mathrm{supp}(\mu) \subset \{e^{i\theta} : \theta \in [-\varphi, \varphi]\}$. Then $\mu \boxtimes \lambda_t$ is unimodal for all $t \geq \frac{2(1+r_\varphi)}{1-r_\varphi} \log \frac{1}{r_\varphi}$, where $r_\varphi = (\cos\varphi)/4$.

Proof. Let $t > 4$. For $\theta \in \mathbb{T} = U_{t,\mu}$, we have

$$\int_{-\pi}^{\pi} \Gamma_{\theta, v_{t,\mu}(\theta)}(x)\, d\mu(x) = \int_0^{\varphi} \left[\Gamma_{\theta, v_{t,\mu}(\theta)}(x) + \Gamma_{\theta, v_{t,\mu}(\theta)}(-x)\right] d\mu(x) \tag{5.54}$$

$$= 2\sin\theta \int_0^{\varphi} \frac{L_{v_{t,\mu}(\theta)}(\theta, x)\, d\mu(x)}{(1 - 2v_{t,\mu}(\theta)\cos(\theta + x) + v_{t,\mu}(\theta)^2)^2 (1 - 2v_{t,\mu}(\theta)\cos(\theta - x) + v_{t,\mu}(\theta)^2)^2}, \tag{5.55}$$

where the function $L_r(\theta, x)$ was defined in (5.23). For $\theta \in [0, \pi]$, we have

$$L_{v_{t,\mu}(\theta)}(\theta, x) \geq 4(1 + v_{t,\mu}(\theta)^2)(r_\varphi - v_{t,\mu}(\theta)). \tag{5.56}$$

by the same calculations from (5.24) to (5.28). Suppose that $r_\varphi \geq v_{t,\mu}(\theta)$ holds for all $\theta \in [0, \pi]$. We can then conclude $L_{v_{t,\mu}(\theta)}(\theta, x) \geq 0$ for all $\theta \in [0, \pi]$ and $x \in [0, \varphi]$, and hence $\int_{-\pi}^{\pi} \Gamma_{\theta, v_{t,\mu}(\theta)}(x)\, d\mu(x) \geq 0$. By Lemma 5.23, we have $\frac{d}{d\theta} v_{t,\mu}(\theta) \geq 0$ for $\theta \in [0, \pi]$, and therefore $\frac{d}{d\theta} v_{t,\mu}(\theta) \leq 0$ for $\theta \in [-\pi, 0]$ since μ is symmetric. Thus $v_{t,\mu}$ is non-increasing in $(-\pi, 0)$ and non-decreasing in $(0, \pi)$. Therefore the density function $p_t(\overline{\psi_t(e^{i\theta})}) = -\frac{\log(v_{t,\mu}(\theta))}{\pi t}$ of $\mu \boxtimes \lambda_t$ is non-decreasing in $(-\pi, 0)$ and non-increasing in $(0, \pi)$, so that $\mu \boxtimes \lambda_t$ is unimodal.

Next we prove $r_\varphi \geq v_{t,\mu}(\theta)$ for large t by using Lemma 5.24. Let $t_\varphi := \frac{1+r_\varphi}{1-r_\varphi}(-2 \log r_\varphi)$, which is greater than 4. For every $t \geq t_\varphi$ we have $r(t) \leq r_\varphi$ since $t \mapsto r(t)$ is strictly decreasing, and hence $r_\varphi \geq v_{t,\mu}(\theta)$. $\square$

Next we prove that the above theorem does not hold for $\varphi = \pi/2$. This result is a complete free analogue of Theorem 5.16 on the classical multiplicative convolution of the Poisson kernel $\mathbf{p}_{r,\psi}$ and the Bernoulli distribution.

Theorem 5.27. The density of the measure $\mathbf{b} \boxtimes \lambda_t$ attains a strict maximum at $\pm 1 \in \mathbb{T}$ and hence is not unimodal at any time $t > 0$.

Proof. By the definition of $U_{t,\mathbf{b}}$ in (5.3), we have

$$U_{t,\mathbf{b}} = \left\{ -\pi \leq \theta < \pi : -\sqrt{\frac{t}{2}} < \sin\theta < \sqrt{\frac{t}{2}} \right\}. \tag{5.57}$$

Therefore, $U_{t,\mathbf{b}}$ and hence the support of $\mathbf{b} \boxtimes \lambda_t$ have two connected components if $0 < t < 2$, and has the single connected component $\mathbb{T}$ if $t \geq 2$. The density function of $\mathbf{b} \boxtimes \lambda_t$ is symmetric for the x-axis since $\mathbf{b}$ is symmetric. For $\theta \in U_{t,\mathbf{b}}$ we have

$$\int_{[-\pi,\pi)} \Gamma_{\theta, v_{t,\mathbf{b}}(\theta)}(x)\, \mathbf{b}(dx) = \frac{1}{2} \left(\frac{\sin\theta}{(1 - 2r\cos\theta + r^2)^2} + \frac{-\sin\theta}{(1 + 2r\cos\theta + r^2)^2} \right) \tag{5.58}$$

$$= \frac{2r(1+r^2)\sin(2\theta)}{(1 - 2r\cos\theta + r^2)^2 (1 + 2r\cos\theta + r^2)^2}. \tag{5.59}$$

By Lemma 5.23 we conclude that $\frac{d}{d\theta} v_{t,\mathbf{b}}(\theta) > 0$ on $(0, \frac{\pi}{2}) \cap U_{t,\mathbf{b}}$ and $\frac{d}{d\theta} v_{t,\mathbf{b}}(\theta) < 0$ on $(\frac{\pi}{2}, \pi) \cap U_{t,\mathbf{b}}$. Since $v_{t,\mathbf{b}}(\theta) = v_{t,\mathbf{b}}(-\theta)$, the function $v_{t,\mathbf{b}}$ has local minima at $\theta = 0, \pi$. By the density formula $p_t(\overline{\Psi_{t,\mathbf{b}}(e^{i\theta})}) = -\frac{\log(v_{t,\mathbf{b}}(\theta))}{\pi t}$, the desired statement holds for any $t > 0$. $\square$

Remark 5.28. We have concluded in [39] that the free additive convolution of the semicircle distribution $S(0, t)$ and a compactly supported probability measure on $\mathbb{R}$ is unimodal for sufficiently large $t > 0$, but a similar statement is not true in the case of the free multiplicative convolution, according to Theorem 5.27.

5.4.6 Failure of freely strong unimodality of λ_t

In this section, combining Lemma 5.12 and Theorem 5.18 we conclude the failure of freely strong unimodality for λ_t for small $t > 0$.

Theorem 5.29. There exists some $t_0 > 0$ such that the free normal distribution λ_t is not freely strongly unimodal for any $t \in (0, t_0)$.

Proof. In order to prove that λ_t is not freely strongly unimodal, according to Lemma 5.12 we need to find a unimodal distribution μ and some $r > 0$ such that equation (5.14) has at least three solutions $\theta \in [-\pi, \pi)$. We take the probability measure μ and the function $h\colon [r_0, 1) \to (0, \infty)$ in Theorem 5.18. Let $t_0 := (-\log r_0)/(\pi h(r_0))$. For each $t \in (0, t_0)$, by the intermediate value theorem there exists some $r_t \in [r_0, 1)$ such that $-\log r_t/(\pi t) = h(r_t)$. For $r = r_t$ equation (5.14) has at least three solutions $\theta \in [-\pi, \pi)$ by Theorem 5.18. $\square$

Problem 5.30. In the case of additive free convolution, once we establish that $S(0,t)$ is not freely strongly unimodal at some $t > 0$ then nor is it at any $t > 0$, because $t > 0$ is just a scaling. However, on the unit circle the parameter $t > 0$ for λ_t is not a scaling. Thus the following problem remains unsolved: does there exist $t > 0$ such that the free normal distribution λ_t is freely strongly unimodal?

5.5 Summary

We conclude this paper by mentioning some relationship between additive and multiplicative free convolutions. Our method of studying unimodality for free multiplicative convolution has been quite similar to the case of free additive convolution in [39]. In particular, the study of the unimodality of $\lambda_t \boxtimes \mu$ was reduced to that of $\mathbf{p}_{r,0} \circledast \mu$ in this paper, while in [39, Lemma 3.1] the study of the unimodality of $S(0,t) \boxplus \mu$ was reduced to that of the classical convolution of μ and Cauchy distributions. Actually, the Poisson kernel $\mathbf{p}_{r,0}$ is the push-forward of the Cauchy distribution by the exponential mapping $x \mapsto e^{ix}$ (see [2, Example 19, Theorem 20]).

Anshelevich and Arizmendi [2] studied the exponential mapping and succeeded in systematically explaining that various results on additive free convolution can be transferred to multiplicative free convolution on $\mathbb{T}$. Therefore, it is tempting to have a proof of some results in this paper by transferring the corresponding results on additive free convolution. However, this seems not possible. The main difficulty is twofold: the push-forward by the exponential mapping does not preserve unimodality; the key class $\mathcal{L}$ of probability measures on $\mathbb{R}$, defined in [2], contains only few unimodal distributions (Cauchy and delta measures).

6 Limit theorems for max-infinitely divisible distributions

6.1 Introduction

This section is based on [61]. We denote by $\mathcal{P}$ and $\mathcal{P}^+$ the set of all probability measures on $\mathbb{R}$ and $[0,\infty)$, respectively. In this section, we give a max-analogue of Bercovici and Pata's result [15] in which there exist bijections between the sets of classical, freely and Boolean infinitely divisible distributions.

Recall that a probability measure μ on $\mathbb{R}$ is said to be $\circ$-infinitely divisible if for any $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{P}$ such that $\mu = \overbrace{\mu_n \circ \cdots \circ \mu_n}^{n \text{ times}}$, where $\circ \in \{*, \boxplus, \uplus\}$. The operation $\uplus$ is said *Boolean convolution* and it means the distribution of sum of Boolean independent random variables. Let $\mathbf{ID}(\circ)$ be the set of all $\circ$-infinitely divisible distributions on $\mathbb{R}$, where $\circ \in \{*, \boxplus, \uplus\}$. Speicher and Woroudi [55, Theorem 3.6] proved that $\mathbf{ID}(\uplus) = \mathcal{P}$.

In [15, Theorem 6.3], we obtained innovated results for three types of infinitely divisible distributions. For any sequences $\{\mu_n\}_n$ in $\mathcal{P}$ and any sequences $\{k_n\}_n$ of positive integers with $k_1 < k_2 < \cdots$, the following conditions are equivalent.

(1) there exists $\mu \in \mathbf{ID}(*)$ such that $\overbrace{\mu_n * \cdots * \mu_n}^{k_n \text{ times}} \xrightarrow{w} \mu$ as $n \to \infty$;

(2) there exists $\nu \in \mathbf{ID}(\boxplus)$ such that $\overbrace{\mu_n \boxplus \cdots \boxplus \mu_n}^{k_n \text{ times}} \xrightarrow{w} \nu$ as $n \to \infty$;

(3) there exists $\lambda \in \mathbf{ID}(\uplus) = \mathcal{P}$ such that $\overbrace{\mu_n \uplus \cdots \uplus \mu_n}^{k_n \text{ times}} \xrightarrow{w} \lambda$ as $n \to \infty$.

According to this result, we can construct the *Bercovici-Pata bijection* $\mathbf{B}_{*\mapsto\boxplus} : \mathbf{ID}(*) \to \mathbf{ID}(\boxplus)$, $\mu \mapsto \nu$ and the *Boolean-free Bercovici-Pata bijection* $\mathbf{B}_{\uplus\mapsto\boxplus} : \mathbf{ID}(\uplus) \to \mathbf{ID}(\boxplus)$, $\lambda \mapsto \nu$. These bijections play an important role to understand limit theorems in three probability theories. For example, $\mathbf{B}_{*\mapsto\boxplus}$ maps the normal distribution to the semicircle law and $\mathbf{B}_{\uplus\mapsto\boxplus}$ sends the symmetric Bernoulli distribution to the semicircle law. The normal distribution is well known as the limit distribution of central limit theorem. The semicircle law is the limit distribution of free central limit theorem (see [51, Theorem 8.10, p.124]). Moreover, the symmetric Bernoulli distribution is known as the limit distribution of Boolean central limit theorem (see [55, Theorem 3.4]). To be precise, we obtain the classical, free and Boolean central limit theorems when $\mu_n := D_{1/\sqrt{n}}(\nu_n)$ in the above conditions (1)-(3), where $\nu_n \in \mathcal{P}$ has mean 0 and a finite variance and $D_c(\mu)(B) := \mu(c^{-1}B)$ for all $c > 0$ and all Borel sets B in $\mathbb{R}$. More generally, $\mathbf{B}_{*\mapsto\boxplus}$ maps stable laws to free stable laws, and $\mathbf{B}_{\uplus\mapsto\boxplus}$ sends Boolean stable laws to free stable laws. Summarizing the above, the Bercovici-Pata bijections connect various limit theorems in classcial, free and Boolean probability theories.

Belinschi and Nica [10, Theorem 1.1] defined the important map to understand free infinite divisibility for probability measures and the Boolean-free Bercovici-Pata bijection as follows:

$$\mathbf{B}_t(\mu) := (\mu^{\boxplus(1+t)})^{\uplus\frac{1}{1+t}}, \qquad t \geq 0, \ \mu \in \mathcal{P}. \tag{6.1}$$

Note that $\mathbf{B}_t \circ \mathbf{B}_s = \mathbf{B}_{t+s}$ for every $t, s \geq 0$. Hence the family $\{\mathbf{B}_t\}_{t\geq 0}$ is a semigroup with respect to the composition of maps and it is called *Belinschi-Nica semigroup.* In [10, Corollary 1.3], we obtained that $\mathbf{B}_t(\mu) \in \mathbf{ID}(\boxplus)$ for every $t \geq 1$ and $\mu \in \mathcal{P}$. Moreover, the map $\mathbf{B}_t$ is a homomorphism on $\mathcal{P}^+$ with respect to free multiplicative convolution $\boxtimes$ (the details of $\boxtimes$ were given by [18]), that is, $\mathbf{B}_t(\mu \boxtimes \nu) = \mathbf{B}_t(\mu) \boxtimes \mathbf{B}_t(\nu)$ for all $\mu, \nu \in \mathcal{P}^+$ (see [10, Theorem 1.1]). Finally, Belinschi and Nica clarified the Boolean-free Bercovici-Pata bijection by using the above semigroup at time one.

Theorem 6.1. (See [10, Theorem 1.2]) The Boolean-free Bercovici-Pata bijection coincides with the map $\mathbf{B}_1$.

Max-probability (extreme value) theory is concerned with maxima of real random variables. Define $X \vee Y := \max\{X, Y\}$ as the maximum of real random variables X

and Y. Denote by F_X the cumulative distribution function of X, that is, $F_X(x) := \mathbb{P}(X \le x)$. If X and Y are independent random variables, then

$$\begin{aligned} F_{X \vee Y}(x) &= \mathbb{P}(X \vee Y \le x) = \mathbb{P}(X \le x, Y \le x) && (6.2) \\ &= \mathbb{P}(X \le x)\mathbb{P}(Y \le x) = F_X(x)F_Y(x), \qquad x \in \mathbb{R}. && (6.3) \end{aligned}$$

According to the above equation, we define *max-convolution* $\vee$ of distribution functions F and G by setting $F \vee G := FG$. The concept of max-infinite divisibility acts like classical convolution case, that is, F is said to be *max-infinitely divisible* if for any $n \in \mathbb{N}$ there exists a distribution function F_n such that $F = \overbrace{F_n \vee \cdots \vee F_n}^{n \text{ times}} = F_n^n$. However, every distribution function F is max-infinitely divisible since the n-th root $F_n := F^{1/n}$ is also distribution function.

A non-trivial distribution function F is called *max-stable law* if $F_{\frac{(X_1 \vee \cdots \vee X_n) - b_n}{a_n}} \xrightarrow{w} F$ for some sequence $\{X_k\}_k$ of independent identically distributed $\mathbb{R}$-valued random variables, a sequence $\{a_k\}_k$ of positive real numbers and a sequence $\{b_k\}_k$ of real numbers. Fisher and Tippett, Fréchet and Gnedenko proved that every max-stable law is an *extreme value distribution* which is one of following types:

- $\mathrm{F}_{\mathrm{I}}^{\mathrm{classical}}(x) = \exp(-\exp(-x)), \quad x \in \mathbb{R}$: *Gumbel distribution*;
- $\mathrm{F}_{\mathrm{II}}^{\mathrm{classical}}(x) = \begin{cases} \exp(-x^{-\alpha}), & x > 0 \\ 0, & x \le 0, \end{cases} \quad \alpha > 0$: *Fréchet distribution*;
- $\mathrm{F}_{\mathrm{III}}^{\mathrm{classical}}(x) = \begin{cases} 0, & x > 0 \\ \exp(-(-x)^{\alpha}), & x \le 0, \end{cases} \quad \alpha > 0$: *Weibull distribution*.

These distribution functions are the most important ones in extreme value theory, and they are used in max-value statistics.

In noncommutative setting, we can realize the maximum of (noncommutative) random variables by using the spectral order (see [1, 11] or Section 6.3). Furthermore, we can establish the concepts of max-convolution, max-infinitely divisible distribution functions and max-stable laws (extreme value distributions) in free and Boolean settings (see Sections 6.4 and 6.5, for details).

6.2 Outline of results

We obtain the limit theorem for freely max-infinitely divisible distributions from Boolean max-limit theorem. First we give a relation of limit theorems for boolean and free max-infinitely divisible distributions.

Theorem 6.2. Let $\{F_n\}_n$ be a sequence in the set Δ_+ of all distribution functions on $[0, \infty)$ and $\{k_n\}_n$ a sequence of positive integers such that $k_1 < k_2 < \cdots$. If there exists $F \in \Delta_+$ such that $\overbrace{F_n \cup\!\!\!\vee \cdots \cup\!\!\!\vee F_n}^{k_n \text{ times}} \xrightarrow{w} F$, then $\overbrace{F_n \boxvee \cdots \boxvee F_n}^{k_n \text{ times}} \xrightarrow{w} \mathbf{B}_1^M(F)$ as $n \to \infty$.

The operations $\boxvee$ and $\cup\!\!\!\vee$ are realized distribution functions of maximum of freely and Boolean independent noncommutative real random variables. They are called *free max-convolution* (see Section 6.4) and *Boolean max-convolution* (see Section 6.5). The map $\mathbf{B}_t^M$ is defined in Section 6.6 and it is an analogue of the Belinschi-Nica semigroup.

Next, we prove the equivalence of limit theorems for classical and Boolean max-infinitely divisible distributions.

Theorem 6.3. Consider a sequence $\{F_n\}_n$ in Δ_+ and a sequence $\{k_n\}_n$ of positive integers with $k_1 < k_2 < \cdots$. The following conditions are equivalent.

(1) There exists $F \in \Delta_+$ such that $\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}} \xrightarrow{w} F$ as $n \to \infty$;

(2) There exists $G \in \Delta_+$ such that $\overbrace{F_n \cup\!\!\!\!\vee \cdots \cup\!\!\!\!\vee F_n}^{k_n \text{ times}} \xrightarrow{w} G$ as $n \to \infty$.

Thus we get the bijection $F \mapsto G$, where F and G are distributions in Theorem 6.3. It is called the *Boolean-classical max-Bercovici-Pata bijection*. Moreover, we find an explicit representation of the bijection.

6.3 Spectral order and distribution functions of noncommutative random variables

Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space, that is, $\mathcal{M}$ is a von Neumann algebra and τ is an ultraweakly continuous faithful tracial state. We may assume that $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$ (e.g. taking a Hilbert space $\mathcal{H} = L^2(\mathcal{M}, \tau)$ with an inner product defined by $\langle X, Y \rangle_{\mathcal{H}} := \tau(Y^*X)$ for all $X, Y \in \mathcal{M}$). Firstly, we define an order $\leq$ of selfadjoint operators in $\mathcal{M}$ as follows.

$$X, Y \in \mathcal{M} \Leftrightarrow Y - X \text{ is a positive operator in } \mathcal{M}. \tag{6.4}$$

The order $\leq$ is called *operator order*. Note that the set of all projections in $\mathcal{M}$ is a complete lattice with respect to $\leq$. For any projections P, Q in $\mathcal{M}$, we define the following projections on $\mathcal{H}$:

$$(P \vee Q)\mathcal{H} := cl(P\mathcal{H} + Q\mathcal{H}); \tag{6.5}$$

$$(P \wedge Q)\mathcal{H} := P\mathcal{H} \cap Q\mathcal{H}. \tag{6.6}$$

In fact, $P \vee Q$ and $P \wedge Q$ are projections in $\mathcal{M}$. Moreover the projections $P \vee Q$ and $P \wedge Q$ are the maximum and the minimum of P and Q with respect to the order $\leq$, respectively.

Next, we discuss the maximum and the minimum of selfadjoint operators in $\mathcal{M}$. However, the set of all (bounded) selfadjoint operators on a Hilbert space $\mathcal{H}$ (in $\mathcal{B}(\mathcal{H})$) does not form a complete lattice with respect to the operator order $\leq$ (see [47]).

For any selfadjoint operator $X \in \mathcal{M}$ and any Borel set T in $\mathbb{R}$, the projection $\mathbf{E}(X; T) \in \mathcal{M}$ denotes the corresponding spectral projection. The *spectral order* on the set of all selfadjoint operators in $\mathcal{M}$ is defined by $X \prec Y \Leftrightarrow \mathbf{E}(X; [t, \infty)) \leq \mathbf{E}(Y; [t, \infty))$ for all $t \in \mathbb{R}$. Note that the spectral order $\prec$ extends to unbounded selfadjoint operators affiliated with $\mathcal{M}$. In [1] ($\mathcal{M}$: matrix algebra) and [11], for any selfadjoint operators X, Y in $\mathcal{M}$, we get well-defined selfadjoint operators $X \vee Y$ and

$X \wedge Y$ in $\mathcal{M}$ as follows:

$$\mathbf{E}(X \vee Y; (t,\infty))) := \mathbf{E}(X; (t,\infty)) \vee \mathbf{E}(Y; (t,\infty)), \qquad t \in \mathbb{R}; \tag{6.7}$$

$$\mathbf{E}(X \wedge Y; [t,\infty)) := \mathbf{E}(X; [t,\infty)) \wedge \mathbf{E}(Y; [t,\infty)), \qquad t \in \mathbb{R}. \tag{6.8}$$

Note that $X \vee Y$ and $X \wedge Y$ are the maximum and the minimum of selfadjoint operators X and Y with respect to the spectral order $\prec$. In [11], since τ is tracial, we have

$$\mathbf{E}(X \vee Y; [t,\infty)) = \mathbf{E}(X; [t,\infty)) \vee \mathbf{E}(Y; [t,\infty)), \qquad t \in \mathbb{R}. \tag{6.9}$$

Moreover, the definitions and properties of $\vee$ and $\wedge$ extend to unbounded selfadjoint operators affiliated with $(\mathcal{M}, \tau)$.

For an (unbounded) selfadjoint operator $X \in \mathcal{M}$, we define a function F_X by setting $F_X(x) := \tau(\mathbf{E}(X; (-\infty, x]))$ for all $x \in \mathbb{R}$. We know that F_X is a distribution function of a probability measure μ_X of an (unbounded) selfadjoint operator $X \in \mathcal{M}$, that is, $\mu_X((-\infty, x]) = F_X(x)$ for all $x \in \mathbb{R}$.

6.4 Theory of freely max-infinitely divisible distributions

6.4.1 Free max-convolution

If X and Y are freely independent real random variables (selfadjoint operators) affiliated with a tracial W^*-probability space $(\mathcal{M}, \tau)$, then we have

$$F_{X \vee Y} = (F_X + F_Y - 1)_+ := \max\{F_X + F_Y - 1, 0\}, \tag{6.10}$$

(see [11, Corollary 3.3] for details). Considering the fact, we can define free max-convolution as follows.

Definition 6.4. Let F and G be distribution functions on $\mathbb{R}$. We define a distribution function $F \boxed{\vee} G$ as a function $(F + G - 1)_+ := \max\{F + G - 1, 0\}$. Moreover we have

$$F^{\boxed{\vee} n} = \overbrace{F \boxed{\vee} \cdots \boxed{\vee} F}^{n \text{ times}} = (nF - (n-1))_+, \qquad n \in \mathbb{N}. \tag{6.11}$$

The operation $\boxed{\vee}$ is called *free max-convolution.*

For a distribution function F on $\mathbb{R}$, we define an interval $[\alpha(F), \omega(F)] \subset [-\infty, \infty]$ by setting

$$\alpha(F) := \sup\{x \in \mathbb{R} : F(x) = 0\}; \tag{6.12}$$

$$\omega(F) := \inf\{x \in \mathbb{R} : F(x) = 1\}. \tag{6.13}$$

For any positive integers $n \geq 2$, we have $\alpha(F^{\boxed{\vee} n}) > -\infty$,

$$\alpha(F^{\boxed{\vee} n}) = \sup\{x \in \mathbb{R} : F(x) \leq 1 - 1/n\}, \tag{6.14}$$

and $\lim_{n\to\infty} \alpha(F^{\boxed{\vee} n}) = \omega(F)$. If F is continuous on $\mathbb{R}$, then there is $u \in \mathbb{R}$ such that $F(u) = 1 - 1/n$, that is, $\alpha(F^{\boxed{\vee} n}) = \sup\{u : F(u) = 1 - 1/n\}$ by applying the intermediate theorem.

In [11, Lemma 6.9], for any distribution functions F on $\mathbb{R}$, we extend free max-convolution $F^{\boxed{\vee}n}$ ($n \in \mathbb{N}$) to distribution functions $F^{\boxed{\vee}t}$ ($t \geq 1$) as follows:

$$F^{\boxed{\vee}t} := (tF - (t-1))_+, \qquad t \geq 1. \tag{6.15}$$

This is a distribution function with $\alpha(F^{\boxed{\vee}t}) = \sup\{x \in \mathbb{R} : F(x) \leq 1 - 1/t\}$ and the map $t \mapsto F^{\boxed{\vee}t}$ is weakly continuous on $[1, \infty)$.

Proposition 6.5. For any distribution function F on $\mathbb{R}$, we have $F^{\boxed{\vee}t} \boxed{\vee} F^{\boxed{\vee}s} = F^{\boxed{\vee}(t+s)}$ for any $t, s \geq 1$.

Proof. Notice that $F^{\boxed{\vee}t} = tF - (t-1)$ on $\{F > 1 - 1/t\}$ for all $t \geq 1$, and we have

$$\begin{aligned} F^{\boxed{\vee}t} \boxed{\vee} F^{\boxed{\vee}s} &= (F^{\boxed{\vee}t} + F^{\boxed{\vee}s} - 1)_+ \\ &= \max\{\max\{tF - (t-1), 0\} + \max\{sF - (s-1), 0\} - 1, 0\}. \end{aligned} \tag{6.16}$$

for all $t \geq s \geq 1$. Consider the following four cases of $t \geq s \geq 1$.
(1) $x \in \{F > 1 - 1/(t+s)\}$; (2) $x \in \{1 - 1/(t+s) > F > 1 - 1/t\}$;
(3) $x \in \{1 - 1/t > F > 1 - 1/s\}$; (4) $x \in \{1 - 1/s > F\}$.

We calculate the RHS of (6.16) for each cases.

(1) If $x \in \{F > 1 - 1/(t+s)\}$, then $F^{\boxed{\vee}t}(x) = tF(x) - (t-1)$ and $F^{\boxed{\vee}s}(x) = sF(x) - (s-1)$. Thus, the RHS of (6.16) equals to $(t+s)F - ((t+s) - 1)$ on $\{F > 1 - 1/(t+s)\}$.

(2) If $x \in \{1 - 1/(t+s) > F > 1 - 1/t\}$, then $F^{\boxed{\vee}t}(x) = tF(x) - (t-1)$ and $F^{\boxed{\vee}s}(x) = sF(x) - (s-1)$, but we have that (RHS of (6.16)) $= \max\{(t+s)F - ((t+s)-1), 0\} = 0$ on $\{1 - 1/(t+s) > F > 1 - 1/t\}$.

(3) If $x \in \{1 - 1/t > F > 1 - 1/s\}$, then $F^{\boxed{\vee}t}(x) = 0$ and $F^{\boxed{\vee}s}(x) = sF(x) - (s-1)$. Then we have that (RHS of (6.16)) $= \max\{sF - s, 0\} = 0$ on $\{1 - 1/t > F > 1 - 1/s\}$.

(4) If $x \in \{1 - 1/s > F\}$, then $F^{\boxed{\vee}t}(x) = 0$ and $F^{\boxed{\vee}s}(x) = 0$. Then we have that (RHS of (6.16)) $= \max\{-1, 0\} = 0$ on $\{1 - 1/s > F\}$.

Consequently, we have $F^{\boxed{\vee}t} \boxed{\vee} F^{\boxed{\vee}s} = F^{\boxed{\vee}(t+s)}$ for any $t, s \geq 1$. $\square$

6.4.2 Freely max-infinitely divisible distributions

In this section, we introduce a concept of freely max-infinitely divisible distributions.

Definition 6.6. A distribution function F on $\mathbb{R}$ is said to be *freely max-infinitely divisible* if for each $n \in \mathbb{N}$, there is a distribution function F_n on $\mathbb{R}$ such that $F = F_n^{\boxed{\vee}n}$.

We give an equivalent property of distribution functions to be freely max-infinitely divisible.

Proposition 6.7. Let F be a distribution function on $\mathbb{R}$. Then F is freely max-infinitely divisible if and only if $\alpha(F) > -\infty$.

Proof. If F is freely max-infinitely divisible, then there is a distribution function F_n such that $F = F_n^{\boxed{\vee}n}$ for each $n \in \mathbb{N}$. Since $\alpha(F_n^{\boxed{\vee}n}) > -\infty$ for every $n \geq 2$, we have that $\alpha(F) > -\infty$. Conversely, if $\alpha(F) > -\infty$, then for each $n \in \mathbb{N}$, we define

$$F_n(x) := \begin{cases} \frac{1}{n}F(x) - \left(\frac{1}{n} - 1\right), & x \geq \alpha(F) \\ 0, & x < \alpha(F). \end{cases} \tag{6.17}$$

For all $x \geq \alpha(F)$, we have

$$F_n^{\boxvee n}(x) = \left(n\left(\frac{1}{n}F(x) - \left(\frac{1}{n} - 1\right)\right) - (n-1)\right)_+ = F(x). \tag{6.18}$$

Furthermore, $F_n^{\boxvee n}(x) = 0$ for all $x < \alpha(F)$. Hence F is freely max-infinitely divisible. $\square$

We give a few of examples of freely max-infinitely divisible distributions.

Example 6.8. (1) (*Compactly supported probability measure*) Let μ be a compactly supported probability measure on $\mathbb{R}$. Then its distribution function F_μ is freely max-infinitely divisible since $-\infty < \alpha(F_\mu)$.

(2) (*Normal distribution*) Let $\mathbf{n}_{m,v}$ be a normal distribution with mean $m \in \mathbb{R}$ and variance $v > 0$. Then its distribution function

$$F_{\mathbf{n}_{m,v}}(x) = \frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{x} \exp\left(-\frac{(t-m)^2}{2v}\right) dt, \tag{6.19}$$

is not freely max-infinitely divisible since $\alpha(F_{\mathbf{n}_{m,v}}) = -\infty$.

To end this section, we give a limit theorem for every distribution function on $\mathbb{R}$ with respect to free max-convolution.

Proposition 6.9. Let F be a distribution function on $\mathbb{R}$. Then there exists a sequence $\{F_n\}_n$ of distribution functions on $\mathbb{R}$ such that $\overbrace{F_n \boxvee \cdots \boxvee F_n}^{n \text{ times}} \xrightarrow{w} F$ as $n \to \infty$.

Proof. If F is freely max-infinitely divisible, then there is nothing to prove. If F is not freely max-infinitely divisible, then we have $\alpha(F) = -\infty$ by Proposition 6.7. For each $n \in \mathbb{N}$, we define a distribution function F_n by setting

$$F_n(x) := \begin{cases} \frac{1}{n}F(x) - \left(\frac{1}{n} - 1\right), & x \geq -n \\ 0, & x < -n. \end{cases} \tag{6.20}$$

Then

$$\overbrace{F_n \boxvee \cdots \boxvee F_n}^{n \text{ times}} = \begin{cases} F(x), & x \geq -n \\ 0, & x < -n \end{cases} \tag{6.21}$$

$$\xrightarrow{n\to\infty} F(x), \tag{6.22}$$

for all continuous points x of F. $\square$

Remark 6.10. In [16], we obtained a free analogue of Khintchine's theorem: Consider a sequence $\{k_n\}_n$ of positive integers with $k_1 < k_2 < \cdots$, an infinitesimal array $\{\mu_{nk}\}_{1\leq k\leq k_n, n\geq 1}$ in $\mathcal{P}$ and a sequence $\{a_n\}_n$ of real numbers. If $\mu_{n1} \boxplus \cdots \boxplus \mu_{nk_n} \boxplus \delta_{a_n}$ weakly converges to some probability measure μ on $\mathbb{R}$, then $\mu \in \mathbf{ID}(\boxplus)$. However, the Khintchine type theorem does not hold in the max-case by Proposition 6.9.

6.4.3 Free max-stable laws

In this section, we study distribution functions of

$$\frac{(X_1 \vee \cdots \vee X_n) - b_n}{a_n}, \qquad n \in \mathbb{N}. \tag{6.23}$$

for some sequences $\{X_n\}_n$ of freely independent identically distributed real random variables, $\{a_n\}_n \subset (0, \infty)$ and $\{b_n\}_n \subset \mathbb{R}$.

Firstly, we define the above distribution function.

Definition 6.11. A non-trivial distribution function G on $\mathbb{R}$ is said to be *freely max-stable* if for each $n \in \mathbb{N}$ there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$G^{\boxed{\vee}n}(a_n \cdot + b_n) = G(\cdot). \tag{6.24}$$

Next we define freely max-domain of attraction.

Definition 6.12. A distribution function F is said to be in the *free max-domain of attraction* of a distribution function G if there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$F^{\boxed{\vee}n}(a_n \cdot + b_n) \xrightarrow{w} G(\cdot), \tag{6.25}$$

as $n \to \infty$. In this case, we write $F \in \mathrm{Dom}_{\boxed{\vee}}(G)$.

In [11, Theorem 6.5] we obtain the following equivalent properties of freely max-stable laws.

Proposition 6.13. The following conditions are equivalent.
(1) G is freely max-stable;
(2) $G \in \mathrm{Dom}_{\boxed{\vee}}(G)$;
(3) $\mathrm{Dom}_{\boxed{\vee}}(G) \neq \emptyset$.

This means that the freely max-stable laws are weak limits of distribution functions of $(X_1 \vee \cdots \vee X_n - b_n)/a_n$, as $n \to \infty$ for some sequence $\{X_n\}_n$ of freely independent identically distributed real random variables, $\{a_n\}_n \subset (0, \infty)$ and $\{b_n\}_n \subset \mathbb{R}$. In [11, Theorem 6.8], the freely max-stable laws are characterized by free extreme value distributions.

Theorem 6.14. A non-trivial distribution function G is freely max-stable if and only if G is of *free extreme value type*, that is, there exist $a > 0$ and $b \in \mathbb{R}$ such that $G(ax+b)$ is one of the following distributions (called *free extreme value distributions*):

- $\mathrm{F}_{\mathrm{I}}^{\mathrm{free}}(x) := (1 - e^{-x})_+, \quad x \in \mathbb{R}$: *exponential distribution*;
- $\mathrm{F}_{\mathrm{II}}^{\mathrm{free}}(x) := \begin{cases} (1 - x^{-\alpha})_+, & x > 0 \\ 0, & x \leq 0, \end{cases} \quad \alpha > 0$: *Pareto distribution*;
- $\mathrm{F}_{\mathrm{III}}^{\mathrm{free}}(x) := \begin{cases} 0, & x > 0 \\ (1 - |x|^{\alpha})_+, & x \leq 0, \end{cases} \quad \alpha > 0$: *Beta law.*

[11, Theorem 6.11-Theorem 6.13] states that the classical max-domains of attraction of the extreme value (Gumbel, Fréchet and Weibull) distributions are corresponding to the free max-domains of attraction of the free extreme value (exponential, Pareto and Beta) distributions. By using a map $\Lambda^\vee$ defined by $\Lambda^\vee(F) = (1 + \log F)_+$, we get a relation between classical extreme value distributions and freely extreme value distributions, that is, $\Lambda^\vee(\mathrm{F}_n^{\mathrm{classical}}) = \mathrm{F}_n^{\mathrm{free}}$ for $n = \mathrm{I}, \mathrm{II}, \mathrm{III}$ in [11, p.2052]. Moreover, the map $\Lambda^\vee$ satisfies

$$\Lambda^\vee(FG) = \Lambda^\vee(F) \boxtimes\!\!\!\!\vee\, \Lambda^\vee(G), \tag{6.26}$$

for all distribution functions F, G. Moreover, $\Lambda^\vee(F)$ is freely max-infinitely divisible for all distribution functions F on $\mathbb{R}$.

In [13, Theorem 3.3], we obtained a random matrix model whose empirical spectral law weakly converges almost surely to a probability measure constructed by $\Lambda^\vee$ as its matrix size goes to infinite.

6.4.4 Free regular max-infinitely divisible distributions

We define a set Δ_+ to be the set of functions $F : [0, \infty) \to [0, 1]$ which are increasing, right-continuous and satisfy $\lim_{x\to\infty} F(x) = 1$. We regard $F \in \Delta_+$ as a distribution function $\tilde{F}$ on $\mathbb{R}$ given by

$$\tilde{F}(x) := \begin{cases} F(x), & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{6.27}$$

In other words, we regard Δ_+ as the set of all distribution functions of positive random variables. By the above identification, we define $\alpha(F) = 0$ when $F(0) > 0$. We can consider in nature free max-convolution of distribution functions in Δ_+. Moreover, we can define a map $\Lambda^\vee$ on Δ_+ by setting

$$\Lambda^\vee(F) := \Lambda^\vee(\tilde{F}) = (1 + \log \tilde{F})_+ = (1 + \log F)_+ \in \Delta_+, \tag{6.28}$$

for any $F \in \Delta_+$. In Section 6.4, we use $\Lambda^\vee$ as the map on Δ_+ defined by (6.28).

In particular, we consider the following subset of Δ_+:

$$\Delta_+^{(0)} := \{F \in \Delta_+ : F(0) = 0\}. \tag{6.29}$$

We can see that $\Delta_+^{(0)}$ is the set of all distribution functions of strictly positive random variables. The set $\Delta_+^{(0)}$ preserves free max-convolution by the definition.

Proposition 6.15. If $F, G \in \Delta_+^{(0)}$, then $F \boxtimes\!\!\!\!\vee\, G \in \Delta_+^{(0)}$.

Finally, we define free max-infinite divisibility of distribution function in $\Delta_+^{(0)}$. A distribution function $F \in \Delta_+^{(0)}$ is said to be *free regular max-infinitely divisible* if for each $n \in \mathbb{N}$, there is $F_n \in \Delta_+^{(0)}$ such that $F = \overbrace{F_n \boxtimes\!\!\!\!\vee \cdots \boxtimes\!\!\!\!\vee\, F_n}^{n \text{ times}}$. Note that for any $F \in \Delta_+^{(0)}$, we have $\alpha(F^{\boxtimes\!\!\!\!\vee\, n}) > 0$. One can use similar argument as in the proof of Proposition 6.7 to get the following proposition.

Proposition 6.16. Consider $F \in \Delta_+^{(0)}$. Then F is free regular max-infinitely divisible if and only if $\alpha(F) > 0$.

6.5 Theory of Boolean max-infinitely divisible distributions

All materials in the following discussions are based on [62]. We can construct two families $(\tilde{X}_i)_{i\in I}$, $(\tilde{Y}_j)_{j\in J}$ of Boolean independent bounded noncommutative random variables (selfadjoint operators) on some Hilbert space $(\mathcal{H},\xi)$ equipped with a unit vector $\xi\in\mathcal{H}$ as follow.

Consider two families $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ of bounded selfadjoint operators on $(\mathcal{H}_1,\xi_1)$ and $(\mathcal{H}_2,\xi_2)$, respectively. If we consider a Hilbert space $(\mathcal{H},\xi)$ equipped with a unit vector $\xi\in\mathcal{H}$, where $\mathcal{H}:=(\mathcal{H}_1\ominus\mathbb{C}\xi_1)\oplus(\mathcal{H}_2\ominus\mathbb{C}\xi_2)\oplus\mathbb{C}\xi$ and identifying isometries $V_1:\mathcal{H}_1\to\mathcal{H}$ and $V_2:\mathcal{H}_2\to\mathcal{H}$, then we have

$$V_1|_{\mathcal{H}_1\ominus\mathbb{C}\xi_1}=I_{\mathcal{H}_1\ominus\mathbb{C}\xi_1},\qquad V_1\xi_1=\xi; \tag{6.30}$$
$$V_2|_{\mathcal{H}_2\ominus\mathbb{C}\xi_2}=I_{\mathcal{H}_2\ominus\mathbb{C}\xi_2},\qquad V_2\xi_2=\xi. \tag{6.31}$$

Consider $\tilde{X}_i:=V_1X_iV_1^*$ and $\tilde{Y}_j:=V_2Y_jV_2^*$ for all $i\in I$ and $j\in J$. Then $(\tilde{X}_i)_{i\in I}$ and $(\tilde{Y}_j)_{j\in J}$ are Boolean independent bounded selfadjoint operators on $(\mathcal{H},\xi)$. Note that

$$\langle(\tilde{X}_i)^n\xi,\xi\rangle_{\mathcal{H}}=\langle(X_i)^n\xi_1,\xi_1\rangle_{\mathcal{H}_1},\qquad \langle(\tilde{Y}_j)^n\xi,\xi\rangle_{\mathcal{H}}=\langle(Y_j)^n\xi_2,\xi_2\rangle_{\mathcal{H}_2}, \tag{6.32}$$

for any $i\in I$, $j\in J$ and $n\in\mathbb{N}$.

This discussion works for unbounded real random variables (selfadjoint operators) by replacing operators to spectral scales. However, $\mathbf{E}(\tilde{X};(-\infty,t])$ and $\mathbf{E}(X;(-\infty,t])$ are not equivalent even if $\tilde{X}$ and X are equivalent, where X and Y are said to be *equivalent* if there exists an isometry V such that $Y=VXV^*$. Actually, we have

$$\mathbf{E}(\tilde{X}_i;(-\infty,t])=V_1\mathbf{E}(X_i;(-\infty,t])V_1^*,\qquad t<0, \tag{6.33}$$
$$\mathbf{E}(\tilde{X}_i;(-\infty,t])=V_1\mathbf{E}(X_i;(-\infty,t])V_1^*+P_{\mathcal{H}_2\ominus\mathbb{C}\xi_2},\qquad t\geq 0. \tag{6.34}$$

Similarly, we have

$$\mathbf{E}(\tilde{Y}_j;(-\infty,t])=V_2\mathbf{E}(Y_j;(-\infty,t])V_2^*,\qquad t<0, \tag{6.35}$$
$$\mathbf{E}(\tilde{Y}_j;(-\infty,t])=V_2\mathbf{E}(Y_j;(-\infty,t])V_2^*+P_{\mathcal{H}_1\ominus\mathbb{C}\xi_1},\qquad t\geq 0. \tag{6.36}$$

Since we have

$$P:=(V_1\mathbf{E}(X_i,(-\infty,t])V_1^*)\wedge(V_2\mathbf{E}(Y_j,(-\infty,t])V_2^*)\leq P_{\mathbb{C}\xi}, \tag{6.37}$$

the projection P is either 0 or $P_{\mathbb{C}\xi}$, and therefore $\langle P\xi,\xi\rangle_{\mathcal{H}}\in\{0,1\}$. On the other hand, the projection

$$Q:=(V_1\mathbf{E}(X_i,(-\infty,t])V_1^*+P_{\mathcal{H}_2\ominus\mathbb{C}\xi_2})\wedge(V_2\mathbf{E}(Y_j,(-\infty,t])V_2^*+P_{\mathcal{H}_1\ominus\mathbb{C}\xi_1}) \tag{6.38}$$

is not trivial, and therefore $\langle Q\xi,\xi\rangle_{\mathcal{H}}\in[0,1]$ takes non-trivial value.

Finally, the non-trivial spectral max works when we choose $X\geq 0$ and $Y\geq 0$. Strictly speaking, the spectral projections $\mathbf{E}(\tilde{X}\vee\tilde{Y};(-\infty,t])$ and $\mathbf{E}(\tilde{X}\wedge\tilde{Y};(-\infty,t])$ are not trivial (neither 0 nor $P_{\mathbb{C}\xi}$) if X and Y are positive. Therefore, we assume positivity of real random variables when we consider the maximum of Boolean independent real random variables.

6.5.1 Boolean max-convolution

Consider a Hilbert space $(\mathcal{H}, \xi)$ equipped with a unit vector $\xi \in \mathcal{H}$. Firstly we define Boolean max-convolution of Boolean independent projections on $(\mathcal{H}, \xi)$. Define the vector state $\varphi(X) := \langle X\xi, \xi \rangle_{\mathcal{H}}$ on the set of all operators on $(\mathcal{H}, \xi)$. If P is a projection on $(\mathcal{H}, \xi)$, then its distribution function F_P is

$$F_P(x) = \begin{cases} 0, & x < 0 \\ p := \varphi(I - P), & 0 \le x < 1 \\ 1, & x \ge 1, \end{cases} \tag{6.39}$$

where $p \in [0, 1]$. In [62, Lemma 3.1], we have the following statement.

Lemma 6.17. If P, Q are Boolean independent projections on $(\mathcal{H}, \xi)$ with $\varphi(P) = 1 - p$ and $\varphi(Q) = 1 - q$, then $\varphi(P \vee Q) = 1 - r$, where $r \in [0, 1]$ satisfies that

$$r^{-1} - 1 = (p^{-1} - 1) + (q^{-1} - 1). \tag{6.40}$$

If either p or q is equal to 0, then we define $r = 0$. Due to Lemma 6.17, we can define a semigroup structure on $[0, 1]$. We define an operation $\veebar$ on $[0, 1]$ by setting

$$(x \veebar y)^{-1} - 1 := (x^{-1} - 1) + (y^{-1} - 1), \qquad x, y \in [0, 1]. \tag{6.41}$$

The notation $\veebar$ means the distribution function of $P \wedge Q$. Strictly speaking, if $\varphi(I - P) = p$ and $\varphi(I - Q) = q$, then $\varphi(P \wedge Q) = \varphi(I - P \vee Q) = r$, where $r^{-1} - 1 = (p^{-1} - 1) + (q^{-1} - 1)$. Two semigroups $([0, 1], \veebar)$ and $([0, 1], \cdot)$ are isomorphic by an order preserving isomorphism $\chi : [0, 1] \to [0, 1]$, where $\chi(x) := \exp(1 - x^{-1})$ and $\chi(0) := 0$ (see [62, Lemma 3.2]).

Considering the above discussion, we define Boolean max-convolution as follow.

Definition 6.18. Consider $F, G \in \Delta_+$. Their *Boolean max-convolution* is defined by

$$(F \uplus G)(x) := F(x) \veebar G(x). \tag{6.42}$$

Recall that Δ_+ is the set of all distribution functions of positive random variables. In [62, Lemma 3.3], if $X, Y \ge 0$ are Boolean independent positive random variables on $(\mathcal{H}, \xi)$, then $F_{X \vee Y} = F_X \uplus F_Y$. Thus, Boolean max-convolution means the distribution function of maximum of Boolean independent positive random variables.

By definition of Boolean max-convolution, for any $F, G \in \Delta_+$, we have

$$F \uplus G = \frac{FG}{F + G - FG} \in \Delta_+, \tag{6.43}$$

and

$$F^{\uplus n} := \overbrace{F \uplus \cdots \uplus F}^{n \text{ times}} = \frac{F}{n - (n-1)F} \in \Delta_+, \qquad n \in \mathbb{N}. \tag{6.44}$$

6.5.2 Boolean max-infinitely divisible distributions

In this section, we introduce Boolean max-infinitely divisible distributions and semi-group of distribution functions with respect to Boolean max-convolution.

Definition 6.19. $F \in \Delta_+$ is said to be *Boolean max-infinitely divisible* if for each $n \in \mathbb{N}$ there exists $F_n \in \Delta_+$ such that

$$F = \overbrace{F_n \cup\!\!\!\!\vee \cdots \cup\!\!\!\!\vee F_n}^{n \text{ times}}. \tag{6.45}$$

We study the class of Boolean max-infinitely divisible distribution functions.

Proposition 6.20. Every distribution function in Δ_+ is Boolean max-infinitely divisible.

Proof. Given $F \in \Delta_+$. For each $n \in \mathbb{N}$, we define the following distribution function $\tilde{F_n}$:

$$\tilde{F_n} := \frac{F}{1/n - (1/n - 1)F} \in \Delta_+. \tag{6.46}$$

Then we have

$$\tilde{F_n}^{\cup\!\!\!\!\vee n} = \frac{\frac{F}{1/n-(1/n-1)F}}{n - (n-1) \times \frac{F}{1/n-(1/n-1)F}} \tag{6.47}$$

$$= \frac{nF}{n(1 + (n-1)F) - (n-1)nF} = F. \tag{6.48}$$

Therefore F is Boolean max-infinitely divisible. $\square$

This proposition is similar to the fact that every probability measure is (additive) Boolean infinitely divisible (see [55, Theorem 3.6]). For any $F \in \Delta_+$ and each $n \in \mathbb{N}$, a choice of the n-th root $\tilde{F_n}$ of F is unique. Denote by $F^{\cup\!\!\!\!\vee 1/n}$ the n-th root $\tilde{F_n}$ for each $F \in \Delta_+$.

For any $t \geq 0$, we define distribution functions F_t as follows:

$$F_t := \frac{F}{t - (t-1)F}, \qquad t > 0 \tag{6.49}$$

$$F_0 := \mathbf{1}_{[\alpha(F),\infty)}. \tag{6.50}$$

Then the map $t \mapsto F_t$ is weakly continuous on $[0, \infty)$ and $F_t \cup\!\!\!\!\vee F_s = F_{t+s}$ for any $t, s \geq 0$. In particular, we have $F_n = F^{\cup\!\!\!\!\vee n}$ and $F_{1/n} = F^{\cup\!\!\!\!\vee 1/n}$ for any $n \in \mathbb{N}$. Hence we denote by $F^{\cup\!\!\!\!\vee t}$ the distribution function F_t for each $t \geq 0$.

6.5.3 Boolean max-stable laws

In this section, we introduce the Boolean max-stable laws. They are the most important to consider Boolean max-probability theory from a reason that it is analogue of classcal and free max-stable (extreme value) laws which are very important to study extreme value statistics.

Definition 6.21. A non-trivial distribution function $G \in \Delta_+$ is said to be *Boolean max-stable* if for some $F \in \Delta_+$ there exists $a_n > 0$ such that $F^{\cup\!\!\!\vee n}(a_n \cdot) \xrightarrow{w} G(\cdot)$.

The Boolean max-stable laws are analogue of classical/freely max-stable laws. This is the weak limit of distribution function of

$$\frac{X_1 \vee \cdots \vee X_n}{a_n}, \qquad n \in \mathbb{N}, \tag{6.51}$$

for some sequences $\{X_n\}_n$ of Boolean independent identically distributed positive random variables, $\{a_n\}_n \subset (0, \infty)$.

Define $\mathcal{X} : \Delta_+ \to \Delta_+$ as an isomorphism $\mathcal{X}(F) := \chi \circ F$, that is, $\mathcal{X}(F) := \exp(1 - F^{-1})$ and $\mathcal{X}(0) := 0$ for all $F \in \Delta_+$. This isomorphism preserves pointwise convergences and satisfies $\mathcal{X}(F) \cdot \mathcal{X}(G) = \mathcal{X}(F \cup\!\!\!\vee G)$ for all $F, G \in \Delta_+$. In other words, this map is connects the set Δ_+ equipped with classical max-convolution $(F, G) \mapsto F \cdot G$ and the set Δ_+ equipped with Boolean max-convolution $(F, G) \mapsto F \cup\!\!\!\vee G$. Looking at the map $\mathcal{X}$ and the classical max-stable laws, we obtain the following important theorem.

Theorem 6.22. (See [62, Theorem 4.1, Corollary 4.1]) A non-trivial distribution function $G \in \Delta_+$ is Boolean max-stable if and only if

$$\mathcal{X}(G)(x) = \exp(-\lambda x^{-\alpha}), \tag{6.52}$$

for some $\lambda, \alpha > 0$. Equivalently,

$$G(x) = (1 + \lambda x^{-\alpha})^{-1}, \tag{6.53}$$

for some $\lambda, \alpha > 0$.

By Theorem 6.22, every Boolean max-stable law is characterized by two parameters $\lambda > 0$ and $\alpha > 0$, so that we write it as

$$\mathbf{S}_{\lambda,\alpha}(x) := (1 + \lambda x^{-\alpha})^{-1}. \tag{6.54}$$

We define Boolean max-domain of attraction of distribution functions.

Definition 6.23. $F \in \Delta_+$ is said to be in the *Boolean max-domain of attraction* of $G \in \Delta_+$ if for some sequence of $a_n > 0$, we have $F^{\cup\!\!\!\vee n}(a_n \cdot) \xrightarrow{w} G(\cdot)$ as $n \to \infty$. In this case we write $F \in \mathrm{Dom}_{\cup\!\!\!\vee}(G)$.

Remark 6.24. For each Boolean max-stable law $G = \mathbf{S}_{\lambda,\alpha}$, we can take $F = G$ and $a_n = n^{1/\alpha}$ in Definition 6.21. From the reason, if G is Boolean max-stable law, then $G \in \mathrm{Dom}_{\cup\!\!\!\vee}(G)$.

Definition 6.25. Consider $\alpha, \beta, p > 0$. The *Dagum distributions* are defined by

$$\mathbf{D}_{\alpha,\beta,p}(x) := \left(1 + (x/\beta)^{-\alpha}\right)^{-p}, \qquad x > 0. \tag{6.55}$$

In particular, we write

$$\mathbf{D}_\alpha := \mathbf{D}_{\alpha,1,1}. \tag{6.56}$$

Note that the Dagum distribution $\mathbf{D}_{\alpha,\beta,1}$ is Boolean max-stable since $\mathbf{D}_{\alpha,\beta,1} = \mathbf{S}_{\beta^\alpha,\alpha}$.

To end this section, we characterize distribution functions which are in the Boolean max-domain of attraction of Dagum distributions by behavior at the tail of distribution function.

For a distribution function G on $\mathbb{R}$, we define a function $\overline{G} := 1-G$. The function is important to study distribution functions of the maximum of random variables since we have to focus on behavior at the tail to look at statistics of the maximum of random variables. Recall the definition of regularly varying functions.

Definition 6.26. A function f is said to be *regularly varying of index* α if for all $t>0$ we have that $f(tx)/f(x) \to t^\alpha$ as $x\to\infty$. In particular, if $\alpha=0$, then f is said to be *slowly varying*.

From [11, Theorem 6.12] and [62, Corollary 4.3], we obtain the following theorem.

Theorem 6.27. The following conditions are equivalent for $\alpha>0$.
(1) $G \in \mathrm{Dom}_{\cup\!\!\!\!\vee}(\mathbf{D}_\alpha)$;
(2) G is in the classical max-domain of attraction of the Fréchet distribution Φ_α:

$$\Phi_\alpha(x) := \mathrm{F}_{\mathrm{II}}^{\mathrm{classical}}(x) = \exp(-x^{-\alpha}), \qquad x>0; \tag{6.57}$$

(3) G is in the free max-domain of attraction of the Pareto distribution $\mathbf{P}_\alpha$ with exponent α:

$$\mathbf{P}_\alpha(x) := \mathrm{F}_{\mathrm{II}}^{\mathrm{free}}(x) = (1-x^{-\alpha})_+, \qquad x>0; \tag{6.58}$$

(4) $\overline{G}$ is regularly varying of index $-\alpha$.

6.5.4 Tails of max-convolution power of distribution functions

Since $\Phi_\alpha(n^{-1/\alpha}\cdot) = \Phi_\alpha^{\vee n}$, $\mathbf{P}_\alpha(n^{-1/\alpha}\cdot) = \mathbf{P}_\alpha^{\Box\!\!\!\!\vee n}$ and $\mathbf{D}_\alpha(n^{-1/\alpha}\cdot) = \mathbf{D}_\alpha^{\cup\!\!\!\!\vee n}$ ($\alpha>0, n\in\mathbb{N}$), the classical, free and Boolean domains of attraction of Φ_α, $\mathbf{D}_\alpha$ and $\mathbf{P}_\alpha$ coincide with ones of $\Phi_\alpha^{\vee n}$, $\mathbf{P}_\alpha^{\Box\!\!\!\!\vee n}$ and $\mathbf{D}_\alpha^{\cup\!\!\!\!\vee n}$, respectively. Therefore, $\Phi_\alpha^{\vee n}$, $\mathbf{P}_\alpha^{\Box\!\!\!\!\vee n}$ and $\mathbf{D}_\alpha^{\cup\!\!\!\!\vee n}$ are regularly varying of index $-\alpha$ by Theorem 6.27. This means that three type max-convolutions preserve tails of corresponding extreme value distributions. More generally, we can conclude that three type max-convolution preserve tails of distribution functions as follows. However, the following proposition has already been obtained in [26] and [42] to study behavior at tails of free and Boolean subexponential distributions, but for readers convenience we include the proof.

Proposition 6.28. Consider $n\in\mathbb{N}$. We have the following conditions.

(1) Let F be a distribution function. Then $\overline{F^{\vee n}} \sim n\overline{F}$ as $x\to\infty$.

(2) Let F be a distribution function. Then $\overline{F^{\Box\!\!\!\!\vee n}} \sim n\overline{F}$ as $x\to\infty$.

(3) Let $F\in\Delta_+$. Then $\overline{F^{\cup\!\!\!\!\vee n}} \sim n\overline{F}$ as $x\to\infty$.

Proof. (1) As $x \to \infty$, we have $F \to 1$. Therefore

$$\overline{F^{\vee n}} = 1 - F^n = n\overline{F}(1 + o(1)), \qquad \text{as } x \to \infty. \tag{6.59}$$

Hence we have $\overline{F^{\vee n}} \sim n\overline{F}$ as $x \to \infty$.

(2) We have

$$\overline{F^{\boxed{\vee} n}} = 1 - \max\{nF - (n-1), 0\} = \min\{n(1-F), 1\}. \tag{6.60}$$

As $x \to \infty$, we may consider $n(1-F) < 1$. Hence we have $\overline{F^{\boxed{\vee} n}} \sim n\overline{F}$ as $x \to \infty$.

(3) We have

$$\overline{F^{\curlyvee n}} = 1 - \frac{F}{n - (n-1)F} = \frac{n(1-F)}{n - (n-1)F}. \tag{6.61}$$

Since $n - (n-1)F \to 1$ as $x \to \infty$, we have $\overline{F^{\curlyvee n}} \sim n\overline{F}$ as $x \to \infty$. $\square$

In the same way to prove Proposition 6.28, it is easy to get a generalization of the above proposition as follows.

Corollary 6.29. We have the following conditions.

(1) Let F be a distribution function and $t > 0$. Then $\overline{F^{\vee t}} \sim t\overline{F}$ as $x \to \infty$.

(2) Let F be a distribution function and $t \geq 1$. Then $\overline{F^{\boxed{\vee} t}} \sim t\overline{F}$ as $x \to \infty$.

(3) Let $F \in \Delta_+$ and $t > 0$. Then $\overline{F^{\curlyvee t}} \sim t\overline{F}$ as $x \to \infty$.

In particular, we get a relation between three type max-convolutions and regularly varying functions by using Corollary 6.29. Note that the following relation has already been obtained in [26, Proposition 2.4] in the case when t is a positive integer.

Corollary 6.30. Consider $\alpha \in \mathbb{R}$. The following conditions are equivalent.

(1) $\overline{F}$ is regularly varying of index α.

(2) $\overline{F^{\vee t}}$ is regularly varying of index α for any $t > 0$.

(3) $\overline{F^{\vee t}}$ is regularly varying of index α for some $t > 0$.

(4) $\overline{F^{\boxed{\vee} t}}$ is regularly varying of index α for any $t \geq 1$.

(5) $\overline{F^{\boxed{\vee} t}}$ is regularly varying of index α for some $t \geq 1$.

(6) $\overline{F^{\curlyvee t}}$ is regularly varying of index α for any $t > 0$.

(7) $\overline{F^{\curlyvee t}}$ is regularly varying of index α for some $t > 0$.

6.6 Max-Belinschi-Nica semigroup

In [10, Proposition 3.1], there is a relation between free (additive) convolution and Boolean (additive) convolution, that is,

$$(\mu^{\boxplus p})^{\uplus q} = (\mu^{\uplus q'})^{\boxplus p'}, \qquad \mu \in \mathcal{P}, \tag{6.62}$$

where $p' = pq/(1-p+pq)$ and $q' = 1-p+pq$. By using free and Boolean max-convolutions, we obtain a completely analogue of the above relation.

Proposition 6.31. For any $F \in \Delta_+$, $p \geq 1$ and $q > 1 - 1/p$, we have

$$(F^{\boxtimes p})^{\cup\!\!\!\!\vee q} = (F^{\cup\!\!\!\!\vee q'})^{\boxtimes p'}, \tag{6.63}$$

where $p' = pq/(1-p+pq)$ and $q' = 1-p+pq$. Note that $p' \geq 1$ and $q' > 0$.

Proof. Note that for any $F \in \Delta_+$, $p \geq 1$ and $q > 1 - 1/p$,

$$\begin{aligned} q + (1-q)(pF - (p-1)) &= pq + pF - (p-1) && (6.64)\\ &> p\left(1 - \frac{1}{p}\right) + pF - (p-1) = pF \geq 0. && (6.65) \end{aligned}$$

Therefore, for any $F \in \Delta_+$, $p \geq 1$ and $q > 1 - 1/p$, we have

$$\begin{aligned} (F^{\boxtimes p})^{\cup\!\!\!\!\vee q} &= \frac{F^{\boxtimes p}}{q + (1-q)F^{\boxtimes p}} = \frac{(pF - (p-1))_+}{q + (1-q)(pF - (p-1))_+} && (6.66)\\ &= \begin{cases} \frac{pF-(p-1)}{q+(1-q)(pF-(p-1))}, & \text{if } pF - (p-1) \geq 0 \\ 0, & \text{if } pF - (p-1) < 0 \end{cases} && (6.67)\\ &= \left(\frac{pF - (p-1)}{q + (1-q)(pF - (p-1))}\right)_+. && (6.68) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (F^{\cup\!\!\!\!\vee q'})^{\boxtimes p'} &= \left(\frac{F}{q' + (1-q')F}\right)^{\boxtimes p'} = \left(\frac{p'F}{q' + (1-q')F} - (p'-1)\right)_+ && (6.69)\\ &= \left(\frac{(1 - q' + p'q')F - (p'-1)q'}{q' + (1-q')F}\right)_+ && (6.70)\\ &= \left(\frac{pF - (p-1)}{q + (1-q)(pF - (p-1))}\right)_+. && (6.71) \end{aligned}$$

Thus the formula of this theorem holds. □

Recall the Belinschi-Nica semigroup in Section 6.1. Similarly, we define a Belinschi-Nica type semigroup for free and Boolean max-convolutions.

Proposition 6.32. For each $t \geq 0$, we define a map $\mathbf{B}_t^M : \Delta_+ \to \Delta_+$ as

$$\mathbf{B}_t^M(F) := \left(F^{\boxtimes(1+t)}\right)^{\cup\!\!\!\!\vee \frac{1}{1+t}}, \qquad F \in \Delta_+. \tag{6.72}$$

The family $\{\mathbf{B}_t^M\}_{t\geq 0}$ is a semigroup with respect to composition and $\alpha(\mathbf{B}_t^M(F)) = \alpha(F^{\boxtimes(1+t)})$. In particular, $\mathbf{B}_t^M(F)$ is free regular max-infinitely divisible for any $F \in \Delta_+^{(0)}$ and $t > 0$.

Proof. Using Proposition 6.31, for any $F \in \Delta_+$, we have

$$\mathbf{B}_s^M \circ \mathbf{B}_t^M(F) = \mathbf{B}_s\left((F^{\boxed{\vee}(1+t)})^{\cup\!\!\!\!\vee \frac{1}{1+t}}\right) = \left[\left((F^{\boxed{\vee}(1+t)})^{\cup\!\!\!\!\vee \frac{1}{1+t}}\right)^{\boxed{\vee}(1+s)}\right]^{\cup\!\!\!\!\vee \frac{1}{1+s}} \tag{6.73}$$

$$= \left[\left((F^{\boxed{\vee}(1+t)})^{\boxed{\vee}\frac{1+t+s}{1+t}}\right)^{\cup\!\!\!\!\vee \frac{1+s}{1+t+s}}\right]^{\cup\!\!\!\!\vee \frac{1}{1+s}} = \left(F^{\boxed{\vee}(1+t+s)}\right)^{\cup\!\!\!\!\vee \frac{1}{1+t+s}} = \mathbf{B}_{t+s}^M(F). \tag{6.74}$$

By using the fact $\alpha(F^{\cup\!\!\!\!\vee t}) = \alpha(F)$ for every $t > 0$, we have $\alpha(\mathbf{B}_t^M(F)) = \alpha(F^{\boxed{\vee}(1+t)})$.

If $F(0) = 0$, then $\mathbf{B}_t^M(F(0)) = \mathbf{B}_t^M(0) = 0$. Hence $\mathbf{B}_t^M(\Delta_+^{(0)}) \subset \Delta_+^{(0)}$ holds. For any $t > 0$, we have $\alpha(F^{\boxed{\vee}(1+t)}) > 0$ by right continuity of F and $F(0) = 0$. Therefore $\mathbf{B}_t^M(F)$ is free regular max-infinitely divisible for any $F \in \Delta_+^{(0)}$ by Proposition 6.16. $\square$

Definition 6.33. The semigroup $\{\mathbf{B}_t^M\}_{t\geq 0}$ is called the *max-Belinschi-Nica semigroup*. For each $t \geq 0$, the map $\mathbf{B}_t^M$ is called the *max-Belinschi-Nica map at time t*.

In [26, Theorem 2.5], we know a relation between behavior at tails of $\mu \in \mathcal{P}$ and one of $\mathbf{B}_t(\mu)$. In the max-case, by applying Corollary 6.30, we can conclude that the map $\mathbf{B}_t^M$ preserves behavior at tails of distribution functions.

Corollary 6.34. Let $F \in \Delta_+$ and $t \geq 0$. Then $\overline{\mathbf{B}_t^M(F)} \sim \overline{F}$ as $x \to \infty$.

Set $\Theta_+ := \Lambda^\vee(\Delta_+)$, where the map $\Lambda^\vee$ was defined in Section 6.4. Recall that $\Lambda^\vee(\Phi_\alpha) = \mathbf{P}_\alpha$, where Φ_α and $\mathbf{P}_\alpha$ are the Fréchet distribution and the Pareto distribution, respectively. We show that the Belinschi-Nica map $\mathbf{B}_1^M$ has rich properties as follow.

Proposition 6.35. The following conditions hold:
(1) We have $\mathbf{B}_1^M = \Lambda^\vee \circ \mathcal{X}$, where the map $\mathcal{X}$ was defined Section 6.5.
(2) $\mathbf{B}_1^M$ is a surjection from Δ_+ to Θ_+.
(3) We have $\mathbf{B}_1^M(\mathbf{D}_\alpha) = \mathbf{P}_\alpha$ for any $\alpha > 0$.
(4) We have $\mathbf{B}_1^M(F \cup\!\!\!\!\vee G) = \mathbf{B}_1^M(F) \boxed{\vee}\, \mathbf{B}_1^M(G)$ for any $F, G \in \Delta_+$.

Proof. (1) For all $F \in \Delta_+$, we have that

$$\mathbf{B}_1^M(F) = \frac{(2F-1)_+}{1/2 + (1/2)(2F-1)_+} = \begin{cases} 2 - \frac{1}{F}, & \text{if } 1/2 \leq F \leq 1 \\ 0, & \text{if } 0 \leq F < 1/2 \end{cases} \tag{6.75}$$

$$= \left(2 - \frac{1}{F}\right)_+ = (1 + \log \mathcal{X}(F))_+ = \Lambda^\vee(\mathcal{X}(F)). \tag{6.76}$$

(2) By (1), the map $\mathbf{B}_1^M$ takes values in Θ_+. For any $G \in \Theta_+$, there is $H \in \Delta_+$ such that $G = (1 + \log H)_+$. Put $F := 1/(1 - \log H) \in \Delta_+$. Then we have

$$\mathbf{B}_1^M(F) = (2F-1)_+^{\cup\!\!\!\!\vee \frac{1}{2}} = \left(\frac{1 + \log H}{1 - \log H}\right)_+^{\cup\!\!\!\!\vee \frac{1}{2}} \tag{6.77}$$

$$= \frac{\left(\frac{1+\log H}{1-\log H}\right)_+}{\frac{1}{2} + \frac{1}{2} \times \left(\frac{1+\log H}{1-\log H}\right)_+} = (1 + \log H)_+ = G. \tag{6.78}$$

Hence $\mathbf{B}_1^M$ is surjective from Δ_+ to Θ_+.

(3) Note that $\mathcal{X}(\mathbf{D}_\alpha) = \Phi_\alpha$ (see Section 6.5). By (1), we have

$$\mathbf{B}_1^M(\mathbf{D}_\alpha) = \Lambda^\vee \circ \mathcal{X}(\mathbf{D}_\alpha) = \Lambda^\vee(\Phi_\alpha) = \mathbf{P}_\alpha. \tag{6.79}$$

(4) By definition of $\Lambda^\vee$ and $\mathcal{X}$, we have

$$\begin{aligned} \mathbf{B}_1^M(F \mathbin{\cup\!\!\!\!\vee} G) &= \Lambda^\vee(\mathcal{X}(F \mathbin{\cup\!\!\!\!\vee} G)) = \Lambda^\vee(\mathcal{X}(F)\mathcal{X}(G)) && (6.80)\\ &= \Lambda^\vee(\mathcal{X}(F)) \mathbin{\Box\!\!\!\!\vee} \Lambda^\vee(\mathcal{X}(G)) = \mathbf{B}_1^M(F) \mathbin{\Box\!\!\!\!\vee} \mathbf{B}_1^M(G), && (6.81) \end{aligned}$$

for any $F, G \in \Delta_+$. $\square$

In addition, we have $\mathbf{B}_1^M(\Delta_+) = \Theta_+$ by Proposition 6.35 (1) and (2). Finally, we conclude that the map $\mathbf{B}_1^M$ connects limit theorems for Boolean and freely max-infinitely divisible distributions.

Theorem 6.36. Let $\{F_n\}_n$ be a sequence in Δ_+, $F \in \Delta_+$ and $\{k_n\}_n$ a sequence of positive integers such that $k_1 < k_2 < \cdots$. If $\overbrace{F_n \mathbin{\cup\!\!\!\!\vee} \cdots \mathbin{\cup\!\!\!\!\vee} F_n}^{k_n \text{ times}} \xrightarrow{w} F$, then $\overbrace{F_n \mathbin{\Box\!\!\!\!\vee} \cdots \mathbin{\Box\!\!\!\!\vee} F_n}^{k_n \text{ times}} \xrightarrow{w} \mathbf{B}_1^M(F)$ as $n \to \infty$.

Proof. Denote by $\mathcal{C}(F)$ the set of all continuous points of F. By our assumption, we have $F_n(x) > 0$ for all $x \in \mathcal{C}(F) \cap \{F > 0\}$, for sufficiently large n. Note that the assumption is equivalent to

$$\frac{k_n F_n(x) - (k_n - 1)}{F_n(x)} - \frac{1}{F_n(x)} \xrightarrow{n\to\infty} 1 - \frac{1}{F(x)}, \qquad x \in \mathcal{C}(F) \cap \{F > 0\}. \tag{6.82}$$

Furthermore, our assumption implies that there is $\varepsilon_x > 0$ such that

$$\frac{F_n(x)}{k_n - (k_n - 1)F_n(x)} > \varepsilon_x, \tag{6.83}$$

and therefore

$$F_n(x) > \frac{\varepsilon_x k_n}{\varepsilon_x(k_n - 1) + 1}, \tag{6.84}$$

for sufficiently large n. Hence $\lim_{n\to\infty} F_n(x) = 1$ for all $x \in \mathcal{C}(F) \cap \{F > 0\}$.

(I) Consider $x \in \mathcal{C}(F) \cap \{F > 1/2\}$. Our assumption implies that

$$\frac{F_n(x)}{k_n - (k_n - 1)F_n(x)} > \frac{1}{2}, \tag{6.85}$$

and therefore $F_n(x) > \frac{k_n}{k_n+1}$ for sufficiently large n. Hence

$$k_n F_n(x) - (k_n - 1) > \frac{1}{k_n + 1} > 0, \tag{6.86}$$

for sufficiently large n. Thus

$$\left|\overbrace{F_n \boxslash \cdots \boxslash F_n}^{k_n \text{ times}}(x) - \left(2 - \frac{1}{F(x)}\right)\right| \tag{6.87}$$

$$= \left|k_n F_n(x) - (k_n - 1) - \left(2 - \frac{1}{F(x)}\right)\right| \tag{6.88}$$

$$\leq |F_n(x)| \left|\frac{k_n F_n(x) - (k_n - 1)}{F_n(x)} - \frac{1}{F_n(x)}\left(2 - \frac{1}{F(x)}\right)\right| \tag{6.89}$$

$$\leq \left|\frac{k_n F_n(x) - (k_n - 1)}{F_n(x)} - \frac{1}{F_n(x)} - \left(1 - \frac{1}{F(x)}\right)\right| + \left|1 - \frac{1}{F_n(x)}\right| \left|1 - \frac{1}{F(x)}\right| \tag{6.90}$$

$$\leq \left|\frac{k_n F_n(x) - (k_n - 1)}{F_n(x)} - \frac{1}{F_n(x)} - \left(1 - \frac{1}{F(x)}\right)\right| + 2\left|1 - \frac{1}{F_n(x)}\right| \tag{6.91}$$

$$\xrightarrow{n\to\infty} 0. \tag{6.92}$$

If $\lim_{x\to 0^+} F(x) > 1/2$, we finish to prove this theorem. If $\lim_{x\to 0^+} F(x) \leq 1/2$, we continue to prove this theorem as follow.

(II) Consider $x \in \mathcal{C}(F) \cap \{F = 1/2\}$. Since $F(x) = 1/2$, the condition (6.82) satisfies

$$\frac{k_n F_n(x) - (k_n - 1)}{F_n(x)} - \frac{1}{F_n(x)} \xrightarrow{n\to\infty} -1. \tag{6.93}$$

Therefore we have

$$|\overbrace{F_n \boxslash \cdots \boxslash F_n}^{k_n \text{ times}}(x)| \leq |k_n F_n(x) - (k_n - 1)| \tag{6.94}$$

$$= |F_n(x)| \left|\frac{k_n F_n(x) - (k_n - 1)}{F_n(x)}\right| \tag{6.95}$$

$$\leq |F_n(x)| \left|\frac{k_n F_n(x) - (k_n - 1)}{F_n(x)} - \frac{1}{F_n(x)} + 1\right| + |1 - F_n(x)| \tag{6.96}$$

$$\leq \left|\frac{k_n F_n(x) - (k_n - 1)}{F_n(x)} - \frac{1}{F_n(x)} + 1\right| + |1 - F_n(x)| \tag{6.97}$$

$$\xrightarrow{n\to\infty} 0. \tag{6.98}$$

(III) Consider $x \in \mathcal{C}(F) \cap \{0 \leq F < 1/2\}$. Our assumption implies that

$$\frac{F_n(x)}{k_n - (k_n - 1)F_n(x)} < \frac{1}{2}, \tag{6.99}$$

for sufficiently large n. Therefore we have $F_n(x) < \frac{k_n}{k_n+1}$. This means that $k_n F_n(x) - (k_n - 1) < \frac{1}{k_n+1}$. Hence $\overbrace{F_n \boxslash \cdots \boxslash F_n}^{k_n \text{ times}}(x) = (k_n F_n(x) - (k_n - 1))_+ \to 0$ as $n \to \infty$.

Finally, by Proposition 6.35, we have

$$\overbrace{F_n \boxtimes\!\!\!\!\!\!\bigvee \cdots \boxtimes\!\!\!\!\!\!\bigvee F_n}^{k_n \text{ times}}(x) \xrightarrow{n\to\infty} \begin{cases} 2-\frac{1}{F(x)}, & x \in \{F \geq 1/2\} \\ 0, & x \in \{0 \leq F < 1/2\} \end{cases} = \mathbf{B}_1^M(F(x)), \tag{6.100}$$

for all $x \in \mathcal{C}(F)$. □

Remark 6.37. We can obtain the above theorem even if we change Δ_+ to $\Delta_+^{(0)}$. In this case, we can interpret that the map $\mathbf{B}_1^M$ makes a limit theorem for free regular max-infinitely divisible distributions from one of Boolean max-infinitely divisible distributions.

Remark 6.38. In Section 6.7, we prove an equivalence of limit theorems for classical and Boolean max-infinitely divisible distributions by using the map $\mathcal{X}$. Moreover, [13] has already proved an implication from the classical max-limit theorem to the free max-limit theorem by using the map $\Lambda^\vee$. Combining these facts gives another proof of Theorem 6.36.

We give three examples of Theorem 6.36.

Example 6.39. (*Uniform distribution*) Let $\mathbf{U}_{(a,b)}$ be a positive random variable distributed as the uniform distribution on $[a,b]$, where $0 < a < b$. By an elementary calculation, we have

$$F_{\mathbf{U}_{(a,b)}}(x) := \mathbb{P}(\mathbf{U}_{(a,b)} \leq x) = \begin{cases} 0, & 0 \leq x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases} = \mathbf{B}_1^M(F(x)), \tag{6.101}$$

where

$$F(x) = \begin{cases} 0, & 0 \leq x < a \\ \frac{b-a}{2b-a-x}, & a \leq x < b \\ 1, & x \geq b. \end{cases} \tag{6.102}$$

On the other hand, if for each $n \in \mathbb{N}$,

$$F_n(x) = \begin{cases} 0, & 0 \leq x < a \\ \frac{n(b-a)}{(n+1)b-na-x}, & a \leq x < b \\ 1, & x \geq b, \end{cases} \tag{6.103}$$

then $\overbrace{F_n \cup\!\!\!\!\!\!\bigvee \cdots \cup\!\!\!\!\!\!\bigvee F_n}^{n \text{ times}} = F$. By Theorem 6.36, we have $\overbrace{F_n \boxtimes\!\!\!\!\!\!\bigvee \cdots \boxtimes\!\!\!\!\!\!\bigvee F_n}^{n \text{ times}} \xrightarrow{w} F_{\mathbf{U}_{(a,b)}}$ as $n \to \infty$.

Example 6.40. (*Free and Boolean max-compound Poisson distribution*) Consider $\lambda \geq 0$ and $\nu \in \mathcal{P}$ with $\nu(\{0\}) = 0$. Let μ_N be a probability measure on $\mathbb{R}$ defined by

$$\mu_N := \left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\nu. \tag{6.104}$$

Let G and F_N be distribution functions of ν and μ_N, respectively. Hence we have

$$F_N(x) := \begin{cases} \frac{\lambda}{N}G(x), & x < 0 \\ \left(1 - \frac{\lambda}{N}\right) + \frac{\lambda}{N}G(x), & x \geq 0. \end{cases} \tag{6.105}$$

Then we have

$$\overbrace{F_N \boxed{\vee} \cdots \boxed{\vee}\, F_N}^{N \text{ times}} = \begin{cases} (1 - N + \lambda G)_+, & x < 0 \\ (1 - \lambda + \lambda G)_+, & x \geq 0 \end{cases} \tag{6.106}$$

$$\xrightarrow{w} \begin{cases} 0, & x < 0 \\ (1 - \lambda + \lambda G)_+, & x \geq 0 \end{cases} =: \Pi^{\boxed{\vee}}_{\lambda, G}, \tag{6.107}$$

as $N \to \infty$. The distribution function $\Pi^{\boxed{\vee}}_{\lambda,G}$ is called the *free max-compound Poisson distribution*.

Next, we consider $\lambda \geq 0$, $\nu \in \mathcal{P}_+$ with $\nu(\{0\}) = 0$ and $\mu_N := (1-\lambda/N)\delta_0 + (\lambda/N)\nu$ for each $N \in \mathbb{N}$. Let $G, F_N \in \Delta_+$ be distribution functions of ν and μ_N, respectively. Then we have

$$\overbrace{F_N \uplus\!\!\!\vee \cdots \uplus\!\!\!\vee F_N}^{N \text{ times}} = \frac{N - \lambda + \lambda G}{N\lambda(1 - G) + N - \lambda + \lambda G} \tag{6.108}$$

$$\xrightarrow{w} \frac{1}{1 + \lambda(1 - G)} =: \Pi^{\uplus\!\!\!\vee}_{\lambda,G} \in \Delta_+, \tag{6.109}$$

as $N \to \infty$. The distribution function $\Pi^{\uplus\!\!\!\vee}_{\lambda,G}$ is called the *Boolean max-compound Poisson distribution*.

It is easy to check that $\mathbf{B}_1^M(\Pi^{\uplus\!\!\!\vee}_{\lambda,G}) = \Pi^{\boxed{\vee}}_{\lambda,G}$ for all $\lambda \geq 0$ and $G \in \Delta_+$. Considering this discussion, the map $\mathbf{B}_1^M$ connects Boolean and free max-compound Poisson type limit theorems.

In particular, if $\nu = \delta_s$ $(s > 0)$, then $G = \mathbf{1}_{[s,\infty)}$. Then

$$\Pi^{\boxed{\vee}}_{\lambda,s} := \Pi^{\boxed{\vee}}_{\lambda,\mathbf{1}_{[s,\infty)}} = (1 - \lambda + \lambda \mathbf{1}_{[s,\infty)})_+ \tag{6.110}$$

$$\Pi^{\uplus\!\!\!\vee}_{\lambda,s} := \Pi^{\uplus\!\!\!\vee}_{\lambda,\mathbf{1}_{[s,\infty)}} = \frac{1}{1 + \lambda(1 - \mathbf{1}_{[s,\infty)})}. \tag{6.111}$$

These distributions are called the *free/Boolean max-Poisson distributions.*

In [45], we have already obtained the bi-free max-compound Poisson distributions. This distribution is a bi-free analogue of the free max-compound Poisson distribution.

Example 6.41. (*Burr type XII distribution*, see [25, 58]) Consider $\alpha, p > 0$. Let $\mathbf{Burr}_{\alpha,p}$ be the *Burr type XII distribution*, that is,

$$\mathbf{Burr}_{\alpha,p}(x) := 1 - (1 + x^\alpha)^{-p} \in \Delta_+. \tag{6.112}$$

In particular, $\mathbf{Burr}_{\alpha,1} = \mathbf{D}_\alpha$. If for each $n \in \mathbb{N}$,

$$F_n(x) := \frac{1 - (1 + x^\alpha)^{-p}}{1 - (1 - 1/n)(1 + x^\alpha)^{-p}} \in \Delta_+, \tag{6.113}$$

then $\overbrace{F_n \uplus \cdots \uplus F_n}^{n \text{ times}} = \mathbf{Burr}_{\alpha,p}$. By Theorem 6.36, we have

$$\overbrace{F_n \boxtimes \cdots \boxtimes F_n}^{n \text{ times}} \xrightarrow{w} \mathbf{B}_1^M(\mathbf{Burr}_{\alpha,p}) = \left(1 - \frac{1}{(1+x^\alpha)^p - 1}\right)_+. \tag{6.114}$$

Of course, if $p = 1$, then the RHS on the above equation coincides with the Pareto distribution $\mathbf{P}_\alpha$.

Next we show the converse claim of Theorem 6.36 under the special case.

Theorem 6.42. Let $\{F_n\}_n$ be a sequence in Δ_+, $F \in \Delta_+$ with $F > 0$ on $[0, \infty)$, and $\{k_n\}_n$ a sequence of positive integers such that $k_1 < k_2 < \cdots$. If $\overbrace{F_n \boxtimes \cdots \boxtimes F_n}^{k_n \text{ times}} \xrightarrow{w} F$, then $\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}} \xrightarrow{w} \frac{1}{2-F} \in \Delta_+$ as $n \to \infty$.

Proof. For any $x \in \mathcal{C}(F) \cap \{F > 0\}$, there is $\varepsilon_x > 0$ such that $k_n F_n(x) - (k_n - 1) > \varepsilon_x$ for sufficiently large n, and therefore $F_n(x) \to 1$ as $n \to \infty$. Then we have

$$\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}}(x) = \frac{F_n(x)}{1 + F_n(x) - (k_n F_n(x) - (k_n - 1))} \tag{6.115}$$

$$\xrightarrow{n \to \infty} \frac{1}{2 - F(x)}. \tag{6.116}$$

Hence $\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}} \xrightarrow{w} \frac{1}{2-F}$ as $n \to \infty$. $\square$

We give two remarks for Theorem 6.36 and Theorem 6.42.

Remark 6.43. (1) In the setting of Theorem 6.36, note that the convergence of $F_n^{\boxtimes k_n}$ to $\mathbf{B}_1^M(F)$ (for some $F \in \Delta_+$) does not necessarily imply the convergence of $F_n^{\uplus k_n}$ to F.

We give an example as follows. Consider $\alpha > 0$. Let F_n be the following distribution function:

$$F_n(x) := \begin{cases} \mathbf{P}_\alpha(n^{-1/\alpha} x), & x \geq 1 \\ 0, & 0 \leq x < 1. \end{cases} \tag{6.117}$$

Then $F_n^{\boxtimes n} \xrightarrow{w} \mathbf{P}_\alpha = \mathbf{B}_1^M(\mathbf{D}_\alpha)$ as $n \to \infty$. However,

$$F_n^{\uplus n} \xrightarrow{n \to \infty} \begin{cases} \mathbf{D}_\alpha, & x \geq 1 \\ 0, & 0 \leq x < 1 \end{cases} \neq \mathbf{D}_\alpha. \tag{6.118}$$

(2) In Theorem 6.42, we assume that $F > 0$ on $[0, \infty)$ to prove the converse claim of Theorem 6.36. However we expect that the converse claim of Theorem 6.36 holds without the special assumption of F.

For example, we consider $F \in \Delta_+$ with $\alpha(F) > 0$. Define the following sequences of distribution functions in Δ_+:

$$F_{1,n} := \begin{cases} 1 - \frac{1}{n} + \frac{F}{n}, & x \geq \alpha(F) \\ \frac{1}{\alpha(F)}\left(1 - \frac{1}{n}\right) x, & 0 \leq x < \alpha(F), \end{cases} \tag{6.119}$$

$$F_{2,n} := \begin{cases} 1 - \frac{1}{n} + \frac{F}{n}, & x \geq \alpha(F) \\ 1 - \frac{1}{n}, & 0 \leq x < \alpha(F), \end{cases} \tag{6.120}$$

$$F_{3,n} := \begin{cases} 1 - \frac{1}{n} + \frac{F}{n}, & x \geq \alpha(F) \\ 1 - \frac{2}{n}, & 0 \leq x < \alpha(F). \end{cases} \tag{6.121}$$

Then $F_{i,n}^{\boxed{\vee} n} \xrightarrow{w} F$ as $n \to \infty$ for $i = 1, 2, 3$. Moreover, we have that

$$F_{1,n}^{\cup\!\!\!\vee n} \xrightarrow{w} \begin{cases} \frac{1}{2-F}, & x \geq \alpha(F) \\ 0, & 0 \leq x < \alpha(F), \end{cases} \tag{6.122}$$

$$F_{2,n}^{\cup\!\!\!\vee n} \xrightarrow{w} \begin{cases} \frac{1}{2-F}, & x \geq \alpha(F) \\ \frac{1}{2}, & 0 \leq x < \alpha(F), \end{cases} \tag{6.123}$$

$$F_{3,n}^{\cup\!\!\!\vee n} \xrightarrow{w} \begin{cases} \frac{1}{2-F}, & x \geq \alpha(F) \\ \frac{1}{3}, & 0 \leq x < \alpha(F), \end{cases} \tag{6.124}$$

as $n \to \infty$. Therefore the convergence of $F_{i,n}^{\cup\!\!\!\vee n}$ to F $(i = 1, 2, 3)$ implies the convergences of $F_{i,n}^{\cup\!\!\!\vee n}$ to some distribution function.

According to the above three examples, for any $x \in \mathcal{C}(F) \cap \{F = 0\}$, the convergence of the sequence of $F_n^{\cup\!\!\!\vee k_n}(x)$ depends on a situation of F_n.

We formulate the conjecture in Remark 6.43 (2) as follows.

Conjecture 6.44. Let $\{F_n\}_n$ be a sequence in Δ_+, $F \in \Delta_+$ and $\{k_n\}_n$ a sequence of positive integers such that $k_1 < k_2 < \cdots$. If $\overbrace{F_n \boxed{\vee} \cdots \boxed{\vee}\, F_n}^{k_n \text{ times}} \xrightarrow{w} F$, then $\overbrace{F_n \cup\!\!\!\vee \cdots \cup\!\!\!\vee F_n}^{k_n \text{ times}} \xrightarrow{w} G$ as $n \to \infty$ for some $G \in \Delta_+$. In particular, we have $G = \frac{1}{2-F}$ on $\{F > 0\}$.

6.7 Limit theorems for classical and Boolean max-infinitely divisible distributions

In this section, we give a relation of limit theorems for classical and Boolean max-infinitely divisible distributions. According to [62, Theorem 4.1] (or Section 6.5), we have $\mathcal{X}(\mathbf{D}_\alpha) = \Phi_\alpha$ and $\mathcal{X}^{-1}(\Phi_\alpha) = \mathbf{D}_\alpha$ for any $\alpha > 0$, where $\mathcal{X}(F) := \exp(1 - F^{-1})$ for any $F \in \Delta_+$. Note that

$$\mathcal{X}^{-1}(F) = \frac{1}{1 - \log F}, \qquad F \in \Delta_+. \tag{6.125}$$

The isomorphism $\mathcal{X}$ is an important key to discuss in this section. We give the following theorem.

Theorem 6.45. Consider a sequence $\{F_n\}_n$ in Δ_+ and a sequence $\{k_n\}_n$ of positive integers with $k_1 < k_2 < \cdots$. The following conditions are equivalent.

(1) There exists $F \in \Delta_+$ such that $\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}} \xrightarrow{w} F$ as $n \to \infty$;

(2) There exists $G \in \Delta_+$ such that $\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}} \xrightarrow{w} G$ as $n \to \infty$.

If either (1) or (2) holds, then we have $\mathcal{X}(G) = F$ and $\mathcal{X}^{-1}(F) = G$.

Proof. Firstly we show the implication (1) $\Rightarrow$ (2). Assume that $\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}} \xrightarrow{w} F$ (i.e., $F_n^{k_n} \xrightarrow{w} F$) as $n \to \infty$. Suppose $x \in \mathcal{C}(F) \cap \{F > 0\}$. Then $F_n(x) > 0$ for sufficiently large n by our assumption. Moreover, $F_n(x) \to 1$ as $n \to \infty$. Indeed, we assume that $F_n(x)$ does not converge to 1 as $n \to \infty$. If necessary, passing to a subsequence, we can find $\varepsilon > 0$ such that $|F_n(x) - 1| \geq \varepsilon$ for sufficiently large n. Thus we have

$$F_n(x)^{k_n} \leq (1-\varepsilon)^{k_n} \xrightarrow{n\to\infty} 0. \tag{6.126}$$

This is a contradiction for that $F_n(x)^{k_n} \xrightarrow{n\to\infty} F(x) > 0$. In addition, the assumption implies that $k_n \log F_n(x) \to \log F(x)$ as $n \to \infty$. Furthermore, $k_n(1 - F_n(x)) \to -\log F(x)$ as $n \to \infty$. Indeed, since $\log(1+z) = z(1+o(1))$ as $z \to 0$ and $F_n(x) \to 1$ as $n \to \infty$, we have

$$k_n(1 - F_n(x)) \sim -k_n \log F_n(x) \xrightarrow{n\to\infty} -\log F(x). \tag{6.127}$$

Hence

$$\begin{aligned}
\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}}(x) &= \frac{F_n(x)}{k_n - (k_n - 1)F_n(x)} &(6.128)\\
&= \frac{F_n(x)}{F_n(x) - k_n \log F_n(x) + \{k_n(1 - F_n(x)) + k_n \log F_n(x)\}} &(6.129)\\
&\xrightarrow{n\to\infty} \frac{1}{1 - \log F(x)} = \mathcal{X}^{-1}(F(x)). &(6.130)
\end{aligned}$$

Suppose that $x \in \mathcal{C}(F) \cap \{F = 0\}$. Assume that $\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}}(x)$ does not converge to 0 as $n \to \infty$. If necessary, passing to a subsequence, we can find $\delta > 0$ such that

$$\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}}(x) = \frac{F_n(x)}{k_n - (k_n - 1)F_n(x)} > \delta, \tag{6.131}$$

equivalently,

$$F_n(x) > \frac{\delta k_n}{(1-\delta) + \delta k_n}, \tag{6.132}$$

for sufficiently large n. Therefore we have

$$\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}}(x) = F_n(x)^{k_n} > \left(\frac{\delta k_n}{(1-\delta)+\delta k_n}\right)^{k_n} \tag{6.133}$$

$$= \frac{1}{\left(1+\left(\frac{1-\delta}{\delta}\right)\frac{1}{k_n}\right)^{k_n}} \xrightarrow{n\to\infty} e^{-\frac{1-\delta}{\delta}} > 0. \tag{6.134}$$

This is a contradiction for that $\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}}(x) \xrightarrow{n\to\infty} 0$. Hence we have

$$\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}}(x) \xrightarrow{n\to\infty} 0 = \mathcal{X}^{-1}(0). \tag{6.135}$$

Finally, we have

$$\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}} \xrightarrow{w} \mathcal{X}^{-1}(F), \qquad \text{as } n \to \infty. \tag{6.136}$$

Next we show the implication (2) $\Rightarrow$ (1). Assume that $\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}} \xrightarrow{w} G$ as $n \to \infty$. Consider $x \in \mathcal{C}(G) \cap \{G > 0\}$. In the proof of Theorem 6.36, we know that $F_n(x) > 0$ for sufficiently large n and $\lim_{n\to\infty} F_n(x) = 1$. Moreover the assumption is equivalent to

$$\frac{k_n(F_n(x)-1)}{F_n(x)} \xrightarrow{n\to\infty} 1 - \frac{1}{G(x)}. \tag{6.137}$$

Since $F_n(x) \to 1$ as $n \to \infty$, we have

$$k_n(F_n(x)-1) \xrightarrow{n\to\infty} 1 - \frac{1}{G(x)}, \tag{6.138}$$

and hence

$$\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}}(x) = F_n(x)^{k_n} = \exp(k_n \log F_n(x)) \sim \exp(k_n(F_n(x)-1)) \tag{6.139}$$

$$\xrightarrow{n\to\infty} \exp\left(1 - \frac{1}{G(x)}\right) = \mathcal{X}(G(x)). \tag{6.140}$$

Suppose that $x \in \mathcal{C}(F) \cap \{G = 0\}$. Then for arbitrary $\varepsilon > 0$, we have

$$\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}} = \frac{F_n(x)}{k_n - (k_n-1)F_n(x)} < \varepsilon, \tag{6.141}$$

equivalently,

$$F_n(x) < \frac{k_n \varepsilon}{(1-\varepsilon)+\varepsilon k_n}, \tag{6.142}$$

for sufficiently large n. Therefore we have

$$F_n(x)^{k_n} < \frac{1}{\left(1+\left(\frac{1-\varepsilon}{\varepsilon}\right)\frac{1}{k_n}\right)^{k_n}} \xrightarrow{n\to\infty} e^{-\frac{1-\varepsilon}{\varepsilon}}. \tag{6.143}$$

Since $\varepsilon > 0$ is arbitrary, we have $F_n(x)^{k_n} \xrightarrow{n\to\infty} 0 = \mathcal{X}(0)$. Finally, we have

$$\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}} \xrightarrow{w} \mathcal{X}(G), \qquad \text{as } n \to \infty. \tag{6.144}$$

Thus, the equivalence of two conditions holds and $\mathcal{X}(G) = F$ and $\mathcal{X}^{-1}(F) = G$. $\square$

According to the above consideration, it is appropriate that the map $\mathcal{X}$ is said the *Boolean-classical max-Bercovici-Pata bijection*. We obtain an application of this bijection.

Example 6.46. Let $\nu \in \mathcal{P}$ with $\nu(\{0\}) = 0$ and $\lambda \geq 0$. Suppose a distribution function G of ν. Then the *classical max-compound Poisson distribution* $\Pi^{\vee}_{\lambda,G}$ is defined as the weak limit of F_N, where

$$F_N(x) = \begin{cases} \frac{\lambda}{N}G(x), & x < 0 \\ \left(1-\frac{\lambda}{N}\right) + \frac{\lambda}{N}G(x), & x \geq 0. \end{cases} \tag{6.145}$$

The distribution is explicitly written by

$$\Pi^{\vee}_{\lambda,G}(x) = \begin{cases} 0, & x < 0 \\ \exp\left(-\lambda(1-G(x))\right), & x \geq 0 \end{cases} \tag{6.146}$$

This distribution has already been mentioned as an example of max-infinitely divisible distributions (e.g. see [5, Example 1]). It is easy to check $\mathcal{X}(\Pi^{\uplus}_{\lambda,G}) = \Pi^{\vee}_{\lambda,G}$ and $\mathcal{X}^{-1}(\Pi^{\vee}_{\lambda,G}) = \Pi^{\uplus}_{\lambda,G}$ for any $\lambda \geq 0$ and distribution functions $G \in \Delta_+$.

Applying discussions in this section and previous section (Section 6.6), we obtain a relation of limit theorems for classical and freely max-infinitely divisible distributions. However, the following corollary has already been proved in [13].

Corollary 6.47. Consider a sequence $\{F_n\}_n$ in Δ_+ and a sequence $\{k_n\}_n$ of positive integers with $k_1 < k_2 < \cdots$. If there exists $F \in \Delta_+$ such that $\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}} \xrightarrow{w} F$, then $\overbrace{F_n \boxvee \cdots \boxvee F_n}^{k_n \text{ times}} \xrightarrow{w} \Lambda^{\vee}(F)$ as $n \to \infty$.

Proof. Assume that $\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}} \xrightarrow{w} F$ as $n \to \infty$. By Theorem 6.45, the assumption is equivalent to $\overbrace{F_n \uplus \cdots \uplus F_n}^{k_n \text{ times}} \xrightarrow{w} \mathcal{X}^{-1}(F)$ as $n \to \infty$. Moreover, by Proposition 6.35 and Theorem 6.36, the above condition implies that

$$\overbrace{F_n \boxvee \cdots \boxvee F_n}^{k_n \text{ times}} \xrightarrow{w} \mathbf{B}_1^M(\mathcal{X}^{-1}(F)) = \Lambda^{\vee} \circ \mathcal{X}(\mathcal{X}^{-1}(F)) = \Lambda^{\vee}(F). \tag{6.147}$$

$\square$

Moreover, the converse claim of the above corollary holds under the special case.

Corollary 6.48. Consider a sequence $\{F_n\}_n$ in Δ_+ and a sequence $\{k_n\}_n$ of positive integers with $k_1 < k_2 < \cdots$. If there exists $F \in \Delta_+$ with $F > 0$ on $[0,\infty)$ such that if $\overbrace{F_n \boxed{\vee} \cdots \boxed{\vee} F_n}^{k_n \text{ times}} \xrightarrow{w} F$, then $\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}} \xrightarrow{w} \Pi^{\vee}_{1,F}$ as $n \to \infty$.

Proof. Assume that $\overbrace{F_n \boxed{\vee} \cdots \boxed{\vee} F_n}^{k_n \text{ times}} \xrightarrow{w} F$ as $n \to \infty$, where $F \in \Delta_+$ with $F > 0$ on $[0,\infty)$. By Theorem 6.42, we have $\overbrace{F_n \mathbin{\cup\!\!\!\!\!\vee} \cdots \mathbin{\cup\!\!\!\!\!\vee} F_n}^{k_n \text{ times}} \xrightarrow{w} \frac{1}{2-F} \in \Delta_+$ as $n \to \infty$. By Theorem 6.45, we have

$$\overbrace{F_n \vee \cdots \vee F_n}^{k_n \text{ times}} \xrightarrow{w} \mathcal{X}\left(\frac{1}{2-F}\right) = \exp(-(1-F)) = \Pi^{\vee}_{1,F}, \tag{6.148}$$

as $n \to \infty$. $\square$

For example, if $F = \Pi^{\boxed{\vee}}_{\lambda,G}$ for some $0 \le \lambda < 1$ and a distribution function G on $\mathbb{R}$ in Corollary 6.48, then we get $\Pi^{\vee}_{1,F} = \Pi^{\vee}_{\lambda,G}$. Therefore we can apply to the classical and free max-compound Poisson type limit theorems by the above corollaries.

From the same reason of Remark 6.43 (1), in general, the convergence of $F_n^{\boxed{\vee} k_n}$ to $\Lambda^{\vee}(F)$ (for some $F \in \Delta_+$) does not necessarily imply the convergence of $F_n^{\vee k_n}$ to F.

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